Functional Inequalities for Uniformly Integrable Semigroups and Applications *

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Abstract

Let $(E, \mathcal{F}, \mu)$ be a probability space, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ a (not necessarily symmetric) Dirichlet form on $L^2(\mu)$, and P_t the associated sub-Markov semigroup. The equivalence of the following eight properties is studied: (i) the L^2-uniform integrability of the unit ball in the Sobolev space; (ii) the super-Poincaré inequality (1.2); (iii) the F-Sobolev inequality (1.3); (iv) the L^2-uniform integrability of P_t; (v) the L^2-uniform integrability of the associated resolvents; (vi) the compactness of P_t; (vii) the compactness of the associated resolvents; (viii) empty essential spectrum of the associated generator.

The main results can be summarized as follows. In general, (i), (ii) and (iii) are equivalent to each other, and they imply (iv) which is equivalent to (v). If P_t has transition density and $\mathcal{F}$ is μ-separable, then the first seven properties from (i) to (vii) are equivalent. If in addition $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is symmetric, then all the above eight properties are equivalent.

Moreover, the essential spectrum of the generator is estimated by using weaker functional inequalities.

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1 Introduction

Let $(E, \mathcal{F}, \mu)$ be a probability space, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ a Dirichlet form on (real) $L^2(\mu)$. Denote by R_λ, P_t and L respectively the associated resolvent, sub-Markov semigroup and generator. It is well known that, when $\mathcal{E}$ is symmetric, L has a spectral gap if and only if the following Poincaré inequality holds:

$$\mu(f^2) \le c\mathcal{E}(f,f) + \mu(f)^2, \tag{1.1}$$

where and throughout the paper, the test functions in an inequality are in $\mathcal{D}(\mathcal{E})$.

To describe the essential spectrum of L, the following super-Poincaré inequality has been introduced in the recent work [20]:

$$\mu(f^2) \le r\mathcal{E}(f,f) + \beta(r)\mu(|f|)^2, \quad r > 0, \tag{1.2}$$

where $\beta \in C(0,\infty)$ is a positive decreasing function. It was shown in [20] that, for Dirichlet forms of symmetric diffusions on a Riemannian manifold, (1.2) is equivalent to that the essential spectrum of L is empty, they are also equivalent to the following F-Sobolev inequality:

$$\mu(f^2 F(f^2)) \le \mathcal{E}(f,f), \quad \mu(f^2) = 1, \tag{1.3}$$

where $F \in C(0,\infty)$ is such that $\sup_{r\in(0,1]} |rF(r)| < \infty$ and $\lim_{r\to\infty} F(r) = \infty$. In general (Theorem 3.1 in [20]), (1.3) implies (1.2). Some estimates of β in (1.2) and F in (1.3) are also available in [20].

The first aim of this paper is to extend the above results to general framework. Our another motivation comes from the famous work [10] (see also [11], [2] and [6]), which proved the equivalence of the log-Sobolev inequalities and the hypercontractivity of corresponding semigroups. It is then very natural for us to study semigroup properties for the above general inequalities (1.2) and (1.3). To see what are possibly the desired semigroup properties, we first recall the following result claimed in [21] (theorems 5 and 6). We say that $A \subset L^p(\mu)$ is L^p-uniformly integrable, if $\sup_{f\in A} \mu(|f|^p 1_{\{|f|\ge r\}}) \to 0$ as $r \to \infty$.

Theorem 1.1. *Let E be a Polish space with $\mathcal{F}$ the Borel σ-field and μ a probability measure. Let P_t be the semigroup of a Markov process on E and L its generator in $L^2(\mu)$. If P_t is symmetric in $L^2(\mu)$, then the L^2-uniform integrability of $\{f \in \mathcal{D}(L) : \mu(f^2) = 1, -\mu(fLf) \le r\}$ for all $r > 0$ is equivalent to that of P_t for all $t > 0$, i.e., $\{P_t f : \mu(f^2) \le 1\}$ is uniformly integrable in $L^2(\mu)$ for any $t > 0$. If in addition P_t has transition density for each $t > 0$, then they are also equivalent to the compactness of P_t for all $t > 0$.*

It is easy to see that the L^2-uniform integrability of the Sobolev unit ball $B := \{f : \mu(f^2) + \mathcal{E}(f,f) \le 1\}$ is equivalent to that of $A_r := \{f : \mu(f^2) = 1, \mathcal{E}(f,f) \le r\}$

for all $r > 0$. Actually, it suffices to show the L^2-uniform integrability of B from that of A_r for all $r > 0$. For any $\varepsilon \in (0,1)$, we have $\{f/\sqrt{\mu(f^2)} : f \in B, \mu(f^2) \geq \varepsilon\} \subset A_{\varepsilon^{-1}}$ which is L^2-uniformly integrable. Since $f^2/\mu(f^2) \geq f^2$ for $f \in B$, we see that $\{f : f \in B, \mu(f^2) \geq \varepsilon\}$ is also L^2-uniformly integrable. Then, there exists $n_\varepsilon > 0$ such that $\mu(f^2 1_{\{f^2 \geq n_\varepsilon\}}) \leq \varepsilon$ for any $f \in B$. This means that B is L^2-uniformly integrable.

Since the F-Sobolev inequality implies the L^2-uniform integrability of A_r for any $r > 0$, it also implies, in the situation of Theorem 1.1, the L^2-uniform integrability of P_t for any $t > 0$. Actually, in this paper, we prove that all of them are equivalent in a general setting. From this, the equivalence of the above functional inequalities, the compactness of semigroups (and resolvents), and the empty essential spectrum of generators is also proved. The main results are summarized as follows.

Theorem 1.2. *If $\mathcal{F}$ is μ-separable, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is symmetric, and P_t has transition density w.r.t. μ for any $t > 0$, then the following eight statements are equivalent to each other:*

(i) *The Sobolev unit ball is uniformly integrable in $L^2(\mu)$;*
(ii) *The super-Poincaré inequality* (1.2) *holds;*
(iii) *The F-Sobolev inequality* (1.3) *holds;*
(iv) *P_t is uniformly integrable in $L^2(\mu)$ for each $t > 0$;*
(v) *R_λ is uniformly integrable in $L^2(\mu)$ for each λ in the resolvent set;*
(vi) *P_t is compact in $L^2(\mu)$ for each $t > 0$;*
(vii) *R_λ is compact in $L^2(\mu)$ for each λ in the resolvent set;*
(viii) *The essential spectrum of L is empty.*

More precisely, for any probability space $(E, \mathcal{F}, \mu)$ and any Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mu)$, we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v), *where "(a) $\Leftrightarrow$ (b)" means "(a) is equivalent to (b)" while "(a) $\Rightarrow$ (b)" means "(a) implies (b)". If in addition P_t has transition density for each $t > 0$ and $\mathcal{F}$ is μ-separable, then the first seven statements are equivalent. If moreover L is normal, i.e., $L^*L = LL^*$, where L^* is the adjoint of L, then all the above eight statements are equivalent.*

Obviously, the framework in Theorem 1.2 is more general than that in Theorem 1.1. Especially, Theorem 1.2 holds without the existence of Markov processes. The proof of Theorem 1.2 is contained in the next two sections, which consist of more precise results and their proofs.

Next, in section 4 we estimate the essential spectrum of L by using the following weaker inequality:

$$\mu(f^2) \leq r\mathcal{E}(f,f) + \beta(r)\mu(|f|)^2, \quad r > r_0, \tag{1.4}$$

where $r_0 \geq 0$ and $\beta \in C(r_0, \infty)$ is positive and decreasing. Obviously, for $\beta \equiv 1$ in (r_0, ∞), (1.4) is equivalent to the Poincaré inequality (1.1) with $c = r_0$. Actually, (1.4) with $\beta \equiv 1$ implies (1.1) for nonnegative $f \in \mathcal{D}(\mathcal{E})$, and hence for any $f \in \mathcal{D}(\mathcal{E})$ by first replacing f with $(f+n)^+$ and then letting $n \to \infty$.

It has been shown in [20] that, for Dirichlet forms of symmetric diffusions on a Riemannian manifold and a class of jump processes, (1.4) is equivalent to $\sigma_{\text{ess}}(-L) \subset [r_0^{-1}, \infty)$, where $\sigma_{\text{ess}}(-L)$ denotes the essential spectrum of $-L$. The proof is based on Donnely-Li's decomposition principle (see [7]). Recently, such sort of principle has been established for more general diffusions. Let E be a locally compact Hausdorff topological space, $\mathcal{F}$ the Borel σ-field, and μ a probability measure on $(E, \mathcal{F})$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local, regular, irreducible symmetric Dirichlet form on $L^2(\mu)$ (see [9] for detailed definitions). Assume that the metric induced by $\mathcal{E}$ is equivalent to the original topology (see [9] for the definition of the metric). Then

$$\inf \sigma_{\text{ess}}(-L) = \sup_{A \text{ is compact}} \inf\{\mathcal{E}(f,f) : f \in \mathcal{D}(\mathcal{E}), \mu(f^2) = 1, \operatorname{supp} f \subset A^c\}.$$

Combining this with the proof of Theorem 2.1 in [20], we see that in this case, (1.4) also implies that $\sigma_{\text{ess}}(-L) \subset [r_0^{-1}, \infty)$. Such results are proved in section 4 for general Dirichlet forms with P_t having transition density. Especially, we have the following result which follows immediately from Corollary 4.2 and Theorem 4.3.

Theorem 1.3. *Assume that $(E, \mathcal{F}, \mu)$ is a probability space and $\mathcal{F}$ is μ-separable. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric closed form on $L^2(\mu)$. If P_t has transition density for some $t > 0$, then* (1.4) *is equivalent to $\sigma_{\text{ess}}(-L) \subset [r_0^{-1}, \infty)$.*

Finally, some further discussions, including methods to prove the functional inequalities and an example in infinite dimension, are presented in the last section.

2 Poincaré-Sobolev type inequalities and uniform integrability

In this section, we prove the equivalence of the first five statements in Theorem 1.2. The proof follows from the following four theorems which include more precise results.

From now on, we use $\|\cdot\|_p$ and $\|\cdot\|_{p\to q}$ to denote, respectively, the norm in $L^p(\mu)$ and the norm of an operator from $L^p(\mu)$ to $L^q(\mu)$. Moreover, let $\langle f, g\rangle_\mu = \mu(gf) = \int_E fg \mathrm{d}\mu$ for $f, g \in L^2(\mu)$. If there is no specific explanation, $(E, \mathcal{F}, \mu)$ is a probability space and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form on $L^2(\mu)$.

Theorem 2.1. *Let $(E, \mathcal{F}, \mu)$ be a measure space and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ a bilinear form on $L^2(\mu)$. Then* (iii) *implies* (i). *If* (i) *holds, then for any $\varepsilon \in (0,1)$,* (1.2) *holds with*

$$\beta(r) = \frac{1}{(1-\varepsilon)^2} \inf\big\{s > 0 : \mu(f^2 1_{\{f^2 \geq s\}}) \leq \varepsilon \textit{ for any } f \in A_{r^{-1}}\big\}. \tag{2.1}$$

Proof. The first assertion is obvious. Denote by $n(r)$ the left-hand side of (2.1). We see that $n(r)$ is finite for any $r > 0$ since $A_{r^{-1}}$ is uniformly integrable in $L^2(\mu)$. Then, for any $f \in A_{r^{-1}}$,

$$
\begin{aligned}
1 = \mu(f^2) &= \int_{\{f^2 \geq (1-\varepsilon)^2 n(r)\}} f^2 \mathrm{d}\mu + \int_{\{f^2 < (1-\varepsilon)^2 n(r)\}} f^2 \mathrm{d}\mu \\
&\leq \varepsilon + (1-\varepsilon)\sqrt{n(r)}\mu(|f|).
\end{aligned}
$$

Therefore,

$$
\mu(f^2) = 1 \leq n(r)\mu(|f|)^2, \qquad f \in A_{r^{-1}}. \tag{2.2}
$$

On the other hand, if $f \notin A_{r^{-1}}$ and $\mu(f^2) = 1$, then $\mu(f^2) \leq r\mathcal{E}(f,f)$. The proof is completed by combining this with (2.2). $\square$

To prove that (ii) implies (iii), we need the following lemma.

Lemma 2.2. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric Dirichlet form on $L^2(\mu)$, where $(E, \mathcal{F}, \mu)$ is a measure space. For any $N \geq 1$ and any $f \in \mathcal{D}(\mathcal{E})$, if $f_0, \cdots, f_N \in L^2(\mu)$ satisfy*

$$
\sum_{n=0}^{N} |f_n(x) - f_n(y)| \leq |f(x) - f(y)|, \quad \sum_{n=0}^{N} |f_n(x)| \leq |f(x)|, \quad \mu\text{- a.e. } x, y \in E, \tag{2.3}
$$

then $f_n \in \mathcal{D}(\mathcal{E})$ for each n and $\sum_{n=0}^{N} \mathcal{E}(f_n, f_n) \leq \mathcal{E}(f,f)$.

Proof. By Proposition I-4.11 in [14], (2.3) implies that $f_n \in \mathcal{D}(\mathcal{E})$, $n = 0, \cdots, N$. We then modify the proof of Theorem I-4.12 in [14]. Let $\{R_\alpha : \alpha > 0\}$ be the resolvents of L. Then R_α is symmetric and bounded in $L^2(\mu)$ for each $\alpha > 0$. Put

$$
\mathcal{E}^\alpha(h, g) = \alpha \langle h - \alpha R_\alpha h, g \rangle_\mu, \quad h, g \in L^2(\mu).
$$

By Theorem I-2.13 in [14],

$$
\mathcal{E}(g,g) = \lim_{\alpha \to \infty} \mathcal{E}^\alpha(g,g), \quad g \in \mathcal{D}(\mathcal{E}).
$$

Therefore, it suffices to show that

$$
\sum_{n=0}^{N} \mathcal{E}^\alpha(f_n, f_n) \leq \mathcal{E}^\alpha(f,f), \quad \alpha > 0. \tag{2.4}
$$

Noting that the space of simple functions is dense in $L^2(\mu)$ and R_α is bounded, we need only to prove (2.4) for simple functions. Let

$$
f_n = \sum_{i=1}^{m} \alpha_{ni} 1_{A_i}, \quad f = \sum_{i=1}^{m} \alpha_i 1_{A_i}
$$

for some $m \geq 1, \alpha_{ni}, \alpha_i \in \mathbb{R}$ and measurable sets $\{A_i\}$ with $A_i \cap A_j = \emptyset$ for any $i \neq j$ and $0 < \lambda_i := \mu(A_i) < \infty$ for any i. Then (2.3) implies that

$$\sum_{n=0}^{N} |\alpha_{ni} - \alpha_{nj}| \le |\alpha_i - \alpha_j|, \quad \sum_{n=0}^{N} |\alpha_{nj}| \le |\alpha_j|, \quad 1 \le i, j \le m. \tag{2.5}$$

Let $b_{ij} = \mathcal{E}^\alpha(1_{A_i}, 1_{A_j})$, we have

$$\mathcal{E}^\alpha(f_n, f_n) = \alpha \sum_{i,j=1}^{m} \alpha_{ni}\alpha_{nj} b_{ij}, \quad \mathcal{E}^\alpha(f, f) = \alpha \sum_{i,j=1}^{m} \alpha_i \alpha_j b_{ij}. \tag{2.6}$$

Next, let $a_{ij} = \alpha \langle 1_{A_j}, R_\alpha 1_{A_i} \rangle_\mu$, then $b_{ij} = \lambda_j \delta_{ij} - a_{ij}$ and $a_{ij} = a_{ji} \ge 0$. Since αR_α is sub-Markovian,

$$m_j := \lambda_j - \sum_{i+1}^{m} a_{ij} = \lambda_j - \alpha \sum_{i=1}^{m} \langle R_\alpha 1_{A_i}, 1_{A_j} \rangle_\mu \ge 0.$$

Then, for any $z_1, \cdots, z_m \in \mathbb{R}$,

$$\sum_{i,j=1}^{m} b_{ij} z_i z_j = \sum_{i<j} a_{ij}(z_i - z_j)^2 + \sum_{j=1}^{m} m_j z_j^2.$$

By combining this with (2.5) and (2.6), we obtain

$$\begin{aligned}
\sum_{n=0}^{N} \mathcal{E}^\alpha(f_n, f_n) &= \alpha \sum_{i<j} a_{ij} \sum_{n=0}^{N} (\alpha_{ni} - \alpha_{nj})^2 + \alpha \sum_{j=1}^{m} m_j \left(\sum_{n=0}^{N} \alpha_{nj}^2 \right) \\
&\le \alpha \sum_{i<j} a_{ij} \left(\sum_{n=0}^{N} |\alpha_{ni} - \alpha_{nj}| \right)^2 + \alpha \sum_{j=1}^{m} m_j \left(\sum_{n=0}^{N} |\alpha_{nj}| \right)^2 \\
&\le \alpha \sum_{i<j} a_{ij} (\alpha_i - \alpha_j)^2 + \alpha \sum_{j=1}^{m} m_j \alpha_j^2 = \mathcal{E}^\alpha(f, f).
\end{aligned}$$

This proves (2.4). □

Lemma 2.3 enables us to extend Theorem 3.2 in [20] to general case.

Theorem 2.3. *Let $(E, \mathcal{F}, \mu)$ be a measure space and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ a Dirichlet from on $L^2(\mu)$. If (1.2) holds, then (1.3) holds with*

$$F(r) = \frac{c_1(\varepsilon)}{r} \int_0^r \xi(\varepsilon t) \mathrm{d}t - c_2(\varepsilon) \tag{2.7}$$

for any $\varepsilon \in (0, 1)$ and some $c_1(\varepsilon), c_2(\varepsilon) > 0$, where

$$\xi(t) = \sup_{r>0} \left(\frac{1}{r} - \frac{\beta(r)}{rt} \right), \quad t > 0. \tag{2.8}$$

Proof. Since the inequalities (1.2) and (1.3) depend only on the symmetric part of $\mathcal{E}$, we may assume that $\mathcal{E}$ itself is symmetric. Let $f \in \mathcal{D}(\mathcal{E})$ with $\mu(f^2) = 1$. For any $\delta > 1$ and any $n \geq 0$, define

$$f_n = (|f| - \delta^{n/2})^+ \wedge (\delta^{(n+1)/2} - \delta^{n/2}).$$

Obviously, for any $x, y \in E$, we have $f_n(x) \geq f_n(y)$ (resp. $f_n(x) \leq f_n(y)$) for each n if $|f(x)| \geq |f(y)|$ (resp. $|f(x)| \leq |f(y)|$). Then, for any $N \geq 1$,

$$\begin{aligned}\sum_{n=0}^{N} |f_n(x) - f_n(y)| &= \left|\sum_{n=0}^{N} [f_n(x) - f_n(y)]\right| \\ &= \big|(|f(x)| - 1)^+ \wedge (\delta^{(N+1)/2} - 1) - (|f(y)| - 1)^+ \wedge (\delta^{(N+1)/2} - 1)\big| \\ &\leq |f(x) - f(y)|\end{aligned}$$

and

$$\sum_{n=0}^{N} |f_n(x)| = \sum_{n=0}^{N} f_n(x) = (|f(x)| - 1)^+ \wedge (\delta^{(N+1)/2} - 1) \leq |f(x)|.$$

By Lemma 2.2, we have

$$\mathcal{E}(f, f) \geq \sum_{n=0}^{N} \mathcal{E}(f_n, f_n). \tag{2.9}$$

Now, if (1.2) holds, then

$$\mu(f_n^2) \leq r\mathcal{E}(f_n, f_n) + \beta(r)\mu(|f_n|)^2, \quad r > 0. \tag{2.10}$$

Noting that

$$\mu(|f_n|)^2 = \big(\mu(f_n 1_{\{f^2 > \delta^n\}})\big)^2 \leq \mu(f_n^2)\mu(f^2 > \delta^n) \leq \delta^{-n}\mu(f_n^2),$$

it follows from (2.10) that

$$\mathcal{E}(f_n, f_n) \geq \sup_{r>0} \left(\frac{1}{r} - \frac{\beta(r)}{r\delta^n}\right)\mu(f_n^2) = \xi(\delta^n)\mu(f_n^2).$$

Combining this with (2.9), we obtain

$$
\begin{aligned}
\mathcal{E}(f,f) &\geq \sum_{n=0}^{\infty}\xi(\delta^n)\mu(f_n^2) \geq \sum_{n=0}^{\infty}\xi(\delta^n)\mu(f^2\geq\delta^{n+1})[\delta^{(n+1)/2}-\delta^{n/2}]^2\\
&\geq \frac{(\sqrt{\delta}-1)^2}{1-\delta^{-1}}\sum_{n=0}^{\infty}\int_{\delta^{n-1}}^{\delta^n}\xi(t)\mu(f^2\geq\delta^2 t)\mathrm{d}t\\
&\geq c_1(\delta)\int_0^{\infty}\xi(t)\mu(f^2\geq\delta^2 t)\mathrm{d}t - c_2(\delta)\\
&= \delta^2 c_1(\delta)\int_0^{\infty}\xi(\delta^{-2}t)\mu(f^2\geq t)\mathrm{d}t - c_2(\delta)\\
&= \delta^2 c_1(\delta)\int_E \mathrm{d}\mu\int_0^{f^2}\xi(\delta^{-2}t)\mathrm{d}t - c_2(\delta)
\end{aligned}
$$

for some $c_1(\delta), c_2(\delta) > 0$. This proves the Theorem since $\delta > 1$ is arbitrary. □

Theorem 2.4. *Let $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ be a coercive closed form on $L^2(\mu)$.*

(1) *If* (1.3) *holds, then P_t is uniformly integrable in $L^2(\mu)$ for any $t>0$. More precisely, let $\overline{F}(r) = \inf_{s\geq r}F(s)$, then*

$$\sup_{\mu(f^2)\leq 1}\mu\big((P_tf)^2\overline{F}((P_tf)^2)\big) < \infty, \quad t>0. \tag{2.11}$$

(2) *The statement* (iv) *is equivalent to* (v).

Proof. a) For a linear operator $(T,\mathcal{D}(T))$ on $L^2(\mu)$, we denote by $(T^{\mathbb{C}},\mathcal{D}(T^{\mathbb{C}}))$ its complexification on the complex L^2 space $L^2_{\mathbb{C}}(\mu)$, see e.g. [13] and [14] for details. Since $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a coercive closed form, by the proof of Corollary I-2.21 in [14], $L' := (1-L)^{\mathbb{C}}$ satisfies the strong sector condition (see pages 15 and 16 in [14]). By Theorem I-2.20 and Corollary I-2.21 in [14], we have $\mathrm{e}^{-t}P_t^{\mathbb{C}}L^2_{\mathbb{C}}(\mu)\subset\mathcal{D}(L')$ and

$$\|L'\mathrm{e}^{-t}P_t^{\mathbb{C}}u\|_2 \leq \frac{c}{t}\|u\|_2 \tag{2.12}$$

for some $c>0$, any $t>0$ and any $u\in L^2_{\mathbb{C}}(\mu)$. Since $\mathcal{D}(L') = \mathcal{D}(L^{\mathbb{C}}) = \{u : \operatorname{Im} u, \operatorname{Re} u\in\mathcal{D}(L)\}$ and $L^2(\mu)\subset L^2_{\mathbb{C}}(\mu)$, for any $f\in L^2(\mu)$, we have $P_tf\in\mathcal{D}(L)$. It then follows from (2.12) that

$$\|(L-1)P_tf\|_2 \leq \frac{c\mathrm{e}^t}{t}\|f\|_2, \quad t>0.$$

Noting that $\mathcal{D}(L)\subset\mathcal{D}(\mathcal{E})$ and $\mathcal{E}(f,f) = -\mu(fLf)$ for $f\in\mathcal{D}(L)$, we obtain

$$
\begin{aligned}
\mathcal{E}(P_tf,P_tf) &= \langle(1-L)P_tf,P_tf\rangle_\mu - \mu((P_tf)^2)\\
&\leq \frac{c\mathrm{e}^t}{t}\|f\|_2\cdot\|P_tf\|_2 \leq \frac{c\mathrm{e}^t}{t}\mu(f^2).
\end{aligned}
$$

Combining this with the F-Sobolev inequality, we obtain for any $t > 0$ and any f with $\mu(f^2) \le 1$ and $\mu((P_t f)^2) > 0$,

$$\begin{aligned}\mu\big((P_t f)^2\overline{F}((P_t f)^2)\big) &\le \mu\Big((P_t f)^2\overline{F}\big((P_t f)^2/\mu((P_t f)^2)\big)\Big)\\ &\le \mu\Big((P_t f)^2 F\big((P_t f)^2/\mu((P_t f)^2)\big)\Big)\\ &\le \mathcal{E}(P_t f, P_t f) \le \frac{c\mathrm{e}^t}{t}.\end{aligned}$$

This means that P_t is uniformly integrable in $L^2(\mu)$ for any $t > 0$.

b) It remains to prove the equivalence of (iv) and (v). Observing that the under the L^2-operator norm of a sequence of L^2-uniformly integrable linear operators is still L^2-uniformly integrable. If (iv) holds, then $\int_0^N \mathrm{e}^{-\alpha t}P_t \mathrm{d}t$ is uniformly integrable in $L^2(\mu)$ for any $N > 0$. Since for any $\alpha > 0$,

$$R_\alpha = \lim_{N\to\infty}\int_0^N \mathrm{e}^{-\alpha t}P_t \mathrm{d}t$$

in the L^2-operator norm (see e.g. the proof of Proposition I-1.10 in [14]), R_α is uniformly integrable in $L^2(\mu)$ (and hence in $L^2_{\mathbb{C}}(\mu)$) for any $\alpha > 0$. For any $\lambda \in \rho(L)$, the resolvent set of L, by the resolvent equation

$$R_\lambda = R_\alpha(1 + (\lambda-\alpha)R_\lambda)$$

and noting that R_λ is bounded, we prove that R_λ is uniformly integrable in $L^2_{\mathbb{C}}(\mu)$.

Conversely, assume that R_λ is L^2-uniformly integrable for each $\lambda \in \rho(L)$. Let R'_λ be the resolvent of $-L'$, then $R'_\lambda = R_{\lambda+1}$. Since L' satisfies the strong sector condition, the spectrum of L' is contained in a sector set

$$S(K) = \{z \in \mathbb{C} : |\mathrm{Im}\, z| \le K \mathrm{Re}\, z\}$$

for some $K > 0$. For any $t > 0$ we have (see the first paragraph on page 490 in [13])

$$\mathrm{e}^{-t}P_t^{\mathbb{C}} = \frac{1}{2\pi\sqrt{-1}}\int_\Gamma \mathrm{e}^{-\lambda t}R'_\lambda \mathrm{d}\lambda$$

in the L^2-operator norm topology, where Γ is a positively oriented closed curve in $\mathbb{C} \setminus S(K)$ which encloses the spectrum of L'. Therefore, for any $t > 0$, $\mathrm{e}^{-t}P_t^{\mathbb{C}}$ is uniformly integrable in $L^2_{\mathbb{C}}(\mu)$. Consequently, P_t is L^2-uniformly integrable for each $t > 0$. □

Theorem 2.5. *If* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is a symmetric closed form on* $L^2(\mu)$*, then* (v) *implies* (i).

Proof. Since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is symmetric, $-L$ is nonnegative and self-adjoint. Moreover, $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\sqrt{-L})$ and $\mathcal{E}(f,f) = \langle\sqrt{-L}f, \sqrt{-L}f\rangle_\mu$, see e.g. page 28 in [14]. By the proof of Theorem V-3.49 in [13] (just replace “ compact” by “uniformly integrable”), we prove that (v) implies the L^2-uniform integrability of R''_α, the resolvent of $\sqrt{-L}$. Observing that for any $f \in \mathcal{D}(\mathcal{E}) = \mathcal{D}(\sqrt{-L})$ and any $\alpha < 0$,

$$\left\|\left(\alpha - \sqrt{-L}\right) f\right\|_2^2 \le 2\left[\alpha^2 \|f\|_2^2 + \left\|\sqrt{-L} f\right\|_2^2\right],$$

we obtain for any $r > 0$ and $\alpha < 0$,

$$\{f : \mu(f^2) \le 1, \mathcal{E}(f,f) \le r\} \subset \{R''_\alpha g : \mu(g^2) \le 2(\alpha^2 + r)\}$$

which is uniformly integrable in $L^2(\mu)$. $\square$

3 Applications to the compactness of P_t and the essential spectrum of L

The aim of this section is to complete the proof of Theorem 1.2. We first present the following lemma which is an extension of a result due to Wu (see Proposition 3 in [21] and Theorem 2.3 in [22]). This lemma enables one to prove the compactness of semigroups on a Hilbert space-valued L^p-space.

Lemma 3.1. *Assume that $\mathcal{F}$ is μ-separable and let $p \in [1, \infty)$ be fixed. Let H be a separable Hilbert space, and $L^p_H(\mu)$ the L^p-space (w.r.t. μ) of H-valued functions. Let P be a bounded linear operator on $L^p_H(\mu)$ with kernel $p(x,y)$, i.e., for μ-a.e. $x, y \in E$, $p(x,y)$ is a bounded linear operator on H such that*

$$Pf(x) = \int_E p(x,y) f(y) \mu(\mathrm{d}y), \quad f \in L^p_H(\mu), \quad \mu\text{-a.e. } x \in E.$$

Suppose that for μ-a.e. $x \in E$ and an orthonormal basis $\{e_j\}$ of H,

$$\sum_k \left(\int_E \sqrt{\sum_j \langle p(x,y)e_j, e_k\rangle_H^2}\, \mu(\mathrm{d}y)\right)^2 < \infty \tag{3.1}$$

and $\{|Pf|_H^p : \|f\|_\infty \le 1\}$ is uniformly integrable w.r.t. μ. Then for any L^p-uniformly integrable $A \subset L^p_H(\mu)$, $P(A) := \{Pf : f \in A\}$ is relatively compact in $L^p_H(\mu)$.

Proof. If $P(A)$ is not relatively compact in $L^p_H(\mu)$, then there exist $\varepsilon > 0$ and a sequence $\{f_n\} \subset A$ such that $\|Pf_n - Pf_m\|_p \ge \varepsilon$, $n \ne m$. Since P is bounded in $L^p_H(\mu)$ and $\{|f|_H^p : f \in A\}$ is uniformly integrable, we may choose $K > 0$ such that

$$\|Pf_{n,K} - Pf_{m,K}\|_p \ge \frac{\varepsilon}{2}, \quad n \ne m, \tag{3.2}$$

where $f_{n,K} = f_n 1_{\{|f_n|_H \le K\}}$. Since $\mathcal{F}$ is μ-separable, $L^p_H(\mu)$ is separable. By Bourbaki theorem (see page 11 in [3]), the set $\{f : \|f\|_\infty \le K\} \subset L^\infty_H(\mu)$ is weakly compact and metrisable w.r.t. the weak topology $\sigma(L^\infty_H(\mu); L^1_H(\mu))$. Therefore, there exist

$f \in L_H^\infty(\mu)$ and a subsequence $\{f_{n_i,K}\}$ such that $f_{n_i,K} \to f$ weakly as $i \to \infty$. On the other hand, for μ-a.e. $x \in E$,

$$\begin{aligned}|Pf(x) - Pf_{n_i,K}(x)|_H^2 &= \sum_k \langle Pf_{n_i,K}(x) - Pf(x), e_k\rangle_H^2 \\ &= \sum_k \bigg(\int_E \langle f_{n_i,K}(y) - f(y), \sum_j \langle p(x,y)e_j, e_k\rangle_H e_j\rangle_H \mu(\mathrm{d}y)\bigg)^2.\end{aligned}$$

Noting that

$$\begin{aligned}&\bigg(\int_E \langle f_{n_i,K}(y) - f(y), \sum_j \langle p(x,y)e_j, e_k\rangle_H e_j\rangle_H \mu(\mathrm{d}y)\bigg)^2 \\ &\le 4K^2\bigg(\int_E \sqrt{\sum_j \langle p(x,y)e_j, e_k\rangle_H^2}\,\mu(\mathrm{d}y)\bigg)^2\end{aligned}$$

which is sumable w.r.t. k according to (3.1), by the weak convergence of $f_{n_i,K}$ and using the dominated convergence theorem, we prove that $Pf_{n_i,K}(x) \to Pf(x)$ in H for μ-a.e. $x \in E$. Since $\{|Pf_{n_i,K}|_H^p : i \ge 1\}$ is uniformly integrable, $Pf_{n_i,K} \to Pf$ in $L_H^p(\mu)$. This is a contradiction to (3.2). □

Theorem 3.2. *Assume that $\mathcal{F}$ is μ-separable and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a coercive closed form on $L^2(\mu)$. If P_t has transition density w.r.t. μ, then* (iv) $\Rightarrow$ (vi) $\Leftrightarrow$ (vii) $\Rightarrow$ (viii) *and* (i). *Consequently, if in addition $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form, then the first seven statements in Theorem* 1.2 *are equivalent.*

Proof. Noting that the essential spectrum of L is empty if R_λ is compact (see e.g. Theorem III-6.29 in [13]), it suffices to prove that (iv) $\Rightarrow$ (vi) $\Leftrightarrow$ (vii) $\Rightarrow$ (i). The proof of (vi) $\Leftrightarrow$ (vii) is similar to that of (iv) $\Leftrightarrow$ (v), and (iv) $\Rightarrow$ (vi) follows immediately from Lemma 3.1 by taking $H = \mathbb{R}, P = P_{t/2}$ and $A = \{P_{t/2}f : \mu(f^2) \le 1\}$. Hence we need only to show that (vii) $\Rightarrow$ (i).

Let $\tilde{L}$ be the self-adjoint operator associated to the symmetric part of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, and let $L' = (1-L)^{\mathbb{C}}$. Then L' is sectorial and $\mathrm{Re}L' = (1-\tilde{L})^{\mathbb{C}}$ (see pages 309 and 310 in [13]). By (vii), the resolvents of L' are compact. Then (see Theorem VI-3.3 in [13]), the resolvents of $\mathrm{Re}L'$ are compact and hence so are those of $\tilde{L}$. Therefore, the resolvents of $\tilde{L}$ are L^2-uniformly integrable. By Theorem 2.5, we prove (i). Finally, the second assertion follows from the first one and Theorems 2.1, 2.3 and 2.4. □

Theorem 3.3. *Assume that L is normal, i.e., $L^*L = LL^*$, then* (viii) *implies* (vi).

Proof. Let $P_t^{\mathbb{C}}$ be the semigroup generated by $L^{\mathbb{C}}$. Since L is normal, so is $L^{\mathbb{C}}$. By the Note after Theorem 13.38 in [17], we have

$$P_t^{\mathbb{C}} = \int_{\sigma(L)} \mathrm{e}^{\lambda t}\mathrm{d}E_\lambda, \quad t > 0 \tag{3.3}$$

under the operator norm topology, where $\sigma(L)$ denotes the spectrum of L, and (E_λ) the corresponding resolution of the identity. Let $\{\lambda_i : i \geq 1\}$ be all the eigenvalues (including multiplicity) of L, and $\{f_i : i \geq 1\}$ the corresponding normalized eigenfunctions. It follows from (3.3) that

$$P_t^{\mathbb{C}} = \sum_{i=1}^{\infty} \mathrm{e}^{t\lambda_i} \langle \cdot, f_i \rangle_\mu f_i, \quad f \in L^2_{\mathbb{C}}(\mu),\ t > 0,$$

where the series converges in the operator norm. Since $\sum_{i=1}^{N} \mathrm{e}^{t\lambda_i} \langle \cdot, f_i \rangle_\mu f_i$ is compact for any $N \geq 1$, $P_t^{\mathbb{C}}$ is compact too. Therefore, P_t is compact for any $t > 0$. □

4 Estimates of the essential spectrum

The aim of this section is to prove the equivalence of (1.4) and $\sigma_{\text{ess}}(-L) \subset [r_0^{-1}, \infty)$. More general, let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a closed form on $L^2_H(\mu)$, where H is a separable Hilbert space. Consider the inequality

$$\mu(|f|_H^2) \leq r\mathcal{E}(f,f) + \beta(r)\mu(|f|_H)^2, \quad f \in \mathcal{D}(\mathcal{E}),\ r > r_0 \tag{4.1}$$

for some $r_0 \geq 0$ and positive decreasing $\beta \in C(r_0, \infty)$.

Theorem 4.1. *Assume that $\mathcal{F}$ is μ-separable, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a coercive closed form on $L^2_H(\mu)$ and the generator L is normal. Let $\theta \in [0, \frac{\pi}{2})$ be such that*

$$\mathcal{E}_1(f,g) \leq (1 + \tan\theta)\mathcal{E}_1(f,f)^{1/2}\mathcal{E}_1(g,g)^{1/2}, \quad f, g \in \mathcal{D}(\mathcal{E}), \tag{4.2}$$

where $\mathcal{E}_1(f,g) = \mathcal{E}(f,g) + \mu(\langle f, g\rangle_H)$. If P_t has kernel satisfying (3.1) *for some $t > 0$, then* (4.1) *implies*

$$\sigma_{\text{ess}}(-L) \subset \left\{ z \in \mathbb{C} : |\mathrm{Im}\, z| \leq \theta(1 + \mathrm{Re}\, z), \mathrm{Re}\, z \geq \frac{r_0 + 1}{r_0\sqrt{1+\theta^2}} - 1 \right\}. \tag{4.3}$$

Proof. From pages 310 and 311 in [13] we know that (4.2) implies

$$\sigma(1 - L) \subset \{z \in \mathbb{C} : |\mathrm{Im}\, z| \leq \theta \mathrm{Re}\, z\}, \tag{4.4}$$

where $\sigma(1 - L)$ denotes the spectrum of $1 - L$. For any $\delta > 0$, let $D_\delta = \{z \in \sigma(1 - L) : \mathrm{Re}\, z \leq \delta\}$. By (4.4), D_δ is bounded and closed, and

$$D_\delta \subset \{z \in \mathbb{C} : |\mathrm{Im}\, z| \leq \theta \mathrm{Re}\, z, \mathrm{Re}\, z \leq \delta\}. \tag{4.5}$$

Put

$$Q_\delta = \int_{D_\delta} \mathrm{d}E'_\lambda = \int_{\sigma(1-L)} 1_{D_\delta}(\lambda)\mathrm{d}E'_\lambda,$$

where E'_λ is the resolution of the identity of L', the complexification of $1-L$ on $L^2_{H^\mathbb{C}}(\mu)$, where $H^\mathbb{C}$ is the complexification Hilbert space of H. Similarly to the last paragraph on page 374 in [17], we have

$$L'Q_\delta = \int_{\sigma(L')} \lambda 1_{D_\delta}(\lambda)\mathrm{d}E'_\lambda = \int_{D_\delta} \lambda \mathrm{d}E'_\lambda \tag{4.6}$$

which is bounded and normal, and $Q_\delta L' \subset L'Q_\delta$. Next, by the remark after Theorem 13.25 in [17], we have $P'_t Q_\delta = Q_\delta P'_t$, where $P'_t = \mathrm{e}^{-t}P^\mathbb{C}_t$ is the semigroup generated by L'. Then, for $\mathcal{H}_\delta := Q_\delta(L^2_{H^\mathbb{C}}(\mu))$, we have $P'_t\mathcal{H}_\delta \subset \mathcal{H}_\delta$ for any $t \geq 0$ and $(P'_t|_{\mathcal{H}_\delta})_{t\geq 0}$ is the semigroup on $\mathcal{H}_\delta$ generated by $\overline{L'|_{\mathcal{H}_\delta}}$, the closure of $L'|_{\mathcal{H}_\delta}$. From (4.6) we see (c.f. Theorem 12.21 and the last paragraph on page 374 in [17])

$$\|L'|_{\mathcal{H}_\delta}\|_{2\to 2} = \|L'Q_\delta\|_{2\to 2} \leq \sup_{\lambda\in D_\delta} |\lambda| \leq \delta\sqrt{1+\theta^2}. \tag{4.7}$$

Moreover, since $Q_\delta L' \subset L'Q_\delta$, we have $Q_\delta(\mathcal{D}(L')) \subset \mathcal{D}(L')$. Then $L'|_{\mathcal{H}_\delta}$ is densely defined and hence is well defined on the whole space $\mathcal{H}_\delta$ since it is bounded. For any $f \in \mathcal{H}_\delta$, we have

$$\mathcal{E}_1(\mathrm{Re}\, f, \mathrm{Re}\, f) = \mu(\langle \mathrm{Re}\, f, (1-L)\mathrm{Re}\, f\rangle_H) \leq \frac{s}{2}\mu(|\mathrm{Re}\, f|_H^2) + \frac{1}{2s}\mu(|L'\mathrm{Re}\, f|_H^2), \quad s>0.$$

The same holds for $\mathrm{Im}\, f$. Then, it follows from (4.7) that

$$\mathcal{E}_1(\mathrm{Re}\, f, \mathrm{Re}\, f) + \mathcal{E}_1(\mathrm{Im}\, f, \mathrm{Im}\, f) \leq \frac{1}{2}[s + s^{-1}\delta^2(1+\theta^2)]\mu(|f|^2_{H^\mathbb{C}}), \quad s>0.$$

Taking $s = \sqrt{\delta^2(1+\theta^2)}$ and applying (4.1) to $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$, we obtain

$$\mu(|f|^2_{H^\mathbb{C}}) \leq \frac{\delta r\sqrt{1+\theta^2}}{1+r}\mu(|f|^2_{H^\mathbb{C}}) + \frac{2\beta(r)}{1+r}\mu(|f|_{H^\mathbb{C}})^2, \quad r>r_0, f\in\mathcal{H}_\delta. \tag{4.8}$$

If $\delta < \frac{r_0+1}{r_0\sqrt{1+\theta^2}}$, then there exists $r_\delta > r_0$ such that $\frac{\delta r_\delta\sqrt{1+\theta^2}}{1+r_\delta} < 1$. Therefore, (4.8) implies that the L^1-unit ball in $\mathcal{H}_\delta$ is L^1-uniformly integrable and that the L^1-norm and the L^2-norm are equivalent on $\mathcal{H}_\delta$. By Lemma 3.1, $P'_t|_{\mathcal{H}_\delta}$ is L^1-compact and hence also L^2-compact. By Theorems 1 and 2 in [12], the essential spectrum of $L'|_{\mathcal{H}_\delta}$ is empty. By (4.6) and noting that $(L'Q_\delta)|_{\mathcal{H}_\delta} = \overline{L'|_{\mathcal{H}_\delta}}$ since Q_δ is a projection, we have

$$\emptyset = \sigma_{\mathrm{ess}}(L'|_{\mathcal{H}_\delta}) = \sigma_{\mathrm{ess}}(L') \cap D_\delta.$$

Therefore,

$$\sigma_{\rm ess}(1-L) = \sigma_{\rm ess}(L') \subset D_\delta^c, \qquad \delta < \frac{r_0+1}{r_0\sqrt{1+\theta^2}}.$$

By Combining this with (4.4), we complete the proof. $\square$

Corollary 4.2. *Assume that $\mathcal{F}$ is μ-separable, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a symmetric closed form on $L^2_H(\mu)$, and P_t has kernel satisfying (3.1) for some $t>0$. If (4.1) holds, then $\sigma_{\rm ess}(-L) \subset [r_0^{-1}, \infty)$.*

Proof. In this case we have $\theta = 0$, then the assertion follows from Theorem 4.1 immediately. Below we present a different proof using Weyl's criterion.

Assume that there exists $\lambda \in [0, r_0^{-1}) \cap \sigma_{\rm ess}(-L)$. Let $r_1 > r_0$ be such that $r_1\lambda < 1$ and take

$$\varepsilon = \min\left\{\frac{1-r_1\lambda}{2\sqrt{2}\,r_1},\ \frac{{\rm e}^{-2\lambda t}(1-\lambda r_1)}{8t\beta(r_1)}\right\}$$

which is positive. By Weyl's criterion (see [15], Theorem VII.12 and comments on page 264), there exists a sequence $\{f_n\} \subset \mathcal{D}(L)$ such that $\|f_n\|_2 = 1$, $\mu(\langle f_n, f_m\rangle_H) = 0$ for $n \neq m$, and

$$\|(\lambda + L)f_n\|_2 \le \varepsilon, \qquad n \ge 1. \tag{4.9}$$

For any $m, n \ge 1$, let $h(s) = \|P_s(f_n - f_m) - {\rm e}^{-\lambda s}(f_n - f_m)\|_2^2$, $s \ge 0$. By (4.9),

$$\begin{aligned} h'(s) &= 2\mu(\langle (P_s - {\rm e}^{-\lambda s})(f_n - f_m), (P_sL + \lambda {\rm e}^{-\lambda s})(f_n - f_m)\rangle_H) \\ &= -2\lambda h(s) + 2\mu(\langle (P_s - {\rm e}^{-\lambda s})(f_n - f_m), P_s(\lambda + L)(f_n - f_m)\rangle_H) \\ &\le 2\varepsilon\sqrt{2h(s)} < 8\varepsilon. \end{aligned} \tag{4.10}$$

Then

$$\|P_t(f_n - f_m) - {\rm e}^{-\lambda t}(f_n - f_m)\|_2 \le \sqrt{8\varepsilon t}, \quad n, m \ge 1.$$

Therefore,

$$\begin{aligned} \|P_t(f_n - f_m)\|_1 &\ge {\rm e}^{-\lambda t}\|f_n - f_m\|_1 - \|P_t(f_n - f_m) - {\rm e}^{-\lambda t}(f_n - f_m)\|_2 \\ &\ge {\rm e}^{-\lambda t}\|f_n - f_m\|_1 - \sqrt{8\varepsilon t}. \end{aligned} \tag{4.11}$$

Next, by (4.1) and (4.9), for $n \neq m$ we have

$$\begin{aligned} 2 = \|f_n - f_m\|_2^2 &\le r_1\mu(\langle f_n - f_m, L(f_m - f_n)\rangle_H) + \beta(r_1)\|f_n - f_m\|_1^2 \\ &= 2r_1\lambda + r_1\mu(\langle f_n - f_m, (L+\lambda)(f_m - f_n)\rangle_H) + \beta(r_1)\|f_n - f_m\|_1^2 \\ &\le 2r_1\lambda + 2\sqrt{2}r_1\varepsilon + \beta(r_1)\|f_n - f_m\|_1^2. \end{aligned}$$

This implies

$$\|f_n - f_m\|_1 \ge \sqrt{\frac{2(1-r_1\lambda) - 2\sqrt{2}r_1\varepsilon}{\beta(r_1)}} \ge \sqrt{\frac{1-r_1\lambda}{\beta(r_1)}}, \quad n \ne m. \tag{4.12}$$

This means that $A = \{f_n : n \ge 1\}$ is L^1-uniformly integrable. By Lemma 3.1, there exists $n_i \uparrow \infty$ such that

$$\lim_{i\to\infty} \|P_t(f_{n_i} - f_{n_{i+1}})\|_1 = 0.$$

But combining (4.10) with (4.11), we obtain

$$\|P_t(f_n - f_m)\|_1 \ge \mathrm{e}^{-\lambda t}\sqrt{\frac{1-r_1\lambda}{\beta(r_1)}} - \sqrt{8\varepsilon t} > 0$$

for any $n \ne m$, this is a contradiction. □

The following is a converse result of Theorem 4.1 and Corollary 4.2.

Theorem 4.3. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a closed form on $L^2_H(\mu)$ such that L is normal. If $\sigma_{\rm ess}(-L) \subset \{z \in \mathbb{C} : \mathrm{Re}\, z \ge r_0^{-1}\}$, then (4.1) holds.*

Proof. For any $r > r_0$, let $r_1 = \frac{1}{2}(r + r_0)$. Then $\sigma_{\rm ess}(-L) \cap \{z \in \mathbb{C} : \mathrm{Re}\, z \le r_1^{-1}\} = \emptyset$. Let $\{\lambda_1, \cdots, \lambda_{n_r}\}$ be all the eigenvalues of $-L$ (including multiplicity) satisfying $\lambda_i \le r_1^{-1}$, and $\{f_1, \cdots, f_{n_r}\}$ the corresponding normalized eigenfunctions. Let $\mathcal{H}_r = \mathrm{spn}\{f_1, \cdots, f_{n_r}\}$, we have $H_r = Q_r(L^2_{H^{\mathbb{C}}}(\mu))$, where $Q_r = \int_{\{\mathrm{Re}\, z \le r_1^{-1}\}} \mathrm{d}E_\lambda$ and E_λ is the resolution of the identity of $-L$. For any $f \in \mathcal{D}(L)$, let $g = Q_r(f)$ and $h = f - g$. We have $g = \sum_{i=1}^{n_r} \mu\langle f, f_i\rangle_{H^{\mathbb{C}}} f_i \in \mathcal{D}(L^{\mathbb{C}})$ and hence $h \in \mathcal{D}(L^{\mathbb{C}})$. It is easy to see that

$$\mu(|h|^2_{H^{\mathbb{C}}}) \le -r_1\mu(\langle h, L^{\mathbb{C}}h\rangle_{H^{\mathbb{C}}}) \le -r_1\mu(fLf) = r_1\mathcal{E}_1(f,f). \tag{4.13}$$

Next, since $\mathcal{H}_r$ is a finite-dimensional space,

$$\beta_1(r) := \sup_{g\in\mathcal{H}_r} \frac{\mu(|g|^2_{H^{\mathbb{C}}})}{\mu(|g|_{H^{\mathbb{C}}})^2} \in [1, \infty).$$

We have

$$\mu(|g|^2_{H^{\mathbb{C}}}) \le \beta_1(r)\mu(|g|_{H^{\mathbb{C}}})^2. \tag{4.14}$$

Let $\varepsilon_r = \frac{r - r_1}{2r\beta_1(r)n_r^2}$, and let $c_r > 0$ be the smallest constant such that

$$\mu(|f_i|^2_{H^{\mathbb{C}}} 1_{\{|f_i|_{H^{\mathbb{C}}} \ge c_r\}}) \le \varepsilon_r, \quad i = 1, \cdots, n_r.$$

Then

$$
\begin{aligned}
\mu(|g|_{H^{\complement}}) &\le \sum_{i=1}^{n_r} \mu(|f_i|_{H^{\complement}} \cdot |f|_H) \le \sum_{i=1}^{n_r} \left(c_r \mu(|f|_H) + \mu(|f|_H \cdot |f_i|_{H^{\complement}} 1_{\{|f_i|_{H^{\complement}} \ge c_r\}})\right) \\
&\le n_r c_r \mu(|f|_H) + n_r \sqrt{\varepsilon_r \mu(|f|_H^2)}.
\end{aligned}
$$

Therefore, (4.13) implies that

$$
\mu(|g|_{H^{\complement}}^2) \le 2\beta_1(r) n_r^2 \varepsilon_r \mu(|f|_H^2) + 2\beta_1(r) n_r^2 c_r^2 \mu(|f|_H)^2.
$$

Combining this with (4.12), we obtain

$$
\begin{aligned}
\mu(|f|_H^2) &= \mu(|g|_{H^{\complement}}^2) + \mu(|h|_{H^{\complement}}^2) \\
&\le r_1 \mathcal{E}(f,f) + 2\beta_1(r) n_r^2 c_r^2 \mu(|f|_H)^2 + 2\beta_1(r) n_r^2 \varepsilon_r \mu(|f|_H^2).
\end{aligned}
$$

This proves (4.1) with decreasing $\beta \in C(r_0, \infty)$ such that

$$
\beta(r) \ge \frac{2\beta_1(r) n_r^2 c_r^2 r}{r_1} = \frac{4\beta_1(r) n_r^2 c_r^2 r}{r_0 + r}.
$$

□

One may also study the essential spectrum and semigroup properties on $L_H^p(\mu)$ using the F-Sobolev inequality. In this case, $|f|$ and f^2 in the literature are understood respectively as $|f|_H$ and $|f|_H^2$.

Theorem 4.4. *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a coercive closed form on $L_H^2(\mu)$. If* (1.3) *holds, which obviously implies* (i) *and* (ii), *then P_t (equivalently, the R_λ) is uniformly integrable in $L_H^2(\mu)$ for each $t > 0$(each $\lambda \in \rho(L)$). If in addition $\mathcal{F}$ is μ-separable and $P = P_t$ satisfies the condition in Lemma* 3.1 *for each $t > 0$, then P_t (equivalently, the resolvents) is compact for each $t > 0$, and hence the essential spectrum of L is empty.*

Proof. The first assertion follows from the proof of Theorem 2.4 with f^2 replaced by $|f|_H^2$. The second assertion follows from Lemma 3.1 and Theorem VI-3.3 in [13]. □

As an application of Theorems 4.1, 4.3 and 4.4, we consider the second-order differential operators on differential forms. Let M be a n-dimensional complete Riemannian manifold. We fix a point $o \in M$ and let $\{X_1, \cdots, X_n\}$ a family of smooth vector fields outside of cut(o), the cut locus of o, such that for each $x \notin$ cut(o), $\{X_1(x), \cdots, X_n(x)\}$ is an orthonormal basis of the tangent vector space at x. Let ω_i be the dual one-form of $X_i, 1 \le i \le n$, then for each $1 \le p \le n$, $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} : 1 \le i_1 < \cdots < i_p \le n\}$ is an orthonormal basis of the space of differential p-forms Ω^p outside of cut(o). Since the volume of cut(o) is zero, we may consider the space of μ-square integrable differential p-forms as a dense sub-space of $L_{\mathbb{R}^d}^2(\mu)$ via the above basis, where μ is a probability measure on M with positive density w.r.t. the volume element, and $d = \frac{n!}{p!(n-p)!}$. Then we are able to apply Theorems 4.1, 4.3 and 4.4 to study second order differential operators on the space of L^2-integrable differential p-forms which are generators of coercive closed forms.

Finally, we remark that if P_t is compact for some $t > 0$, then the essential spectrum of L is empty. Actually, this result is true for any strongly continuous semigroup with $\|P_t\| \le c_1 e^{c_2 t}$ for some $c_1, c_2 > 0$ and all $t > 0$, see Theorems 1 and 2 in [12]. Therefore, to ensure the empty essential spectrum, it suffices to prove the uniform integrability of P_t for some $t > 0$ according to Lemma 3.1. This can be done by using Harnack inequalities of P_t, see the next section. It would be also useful to note that the L^p-uniform integrability of the sub-Markov operator P_t on $L^2(\mu)$ for some $p \in (1, \infty)$ is equivalent to that for any $p \in (1, \infty)$. Then one may also prove the empty essential spectrum on $L^p(\mu)$ for $p \in (1, \infty)$.

5 Further discussions

We first recall some known criteria for the super-Poincaré inequality. It has been proved in [20] for $L = \Delta + \nabla V$ on a Riemannian manifold with $\mu(\mathrm{d}x) = e^V \mathrm{d}x$ a probability measure, the super-Poincaré inequality (1.2) holds provided

$$\lim_{\rho \to \infty} L\rho = -\infty,$$

where ρ is the Riemannian distance function from a fixed point, and the limit is taken outside of the cut-locus of this point. More generally, if $\overline{\lim}_{\rho \to \infty} L\rho = -c < 0$, then by Cheeger's inequality and Donnely-Li's decomposition principle, we have $\sigma_{\mathrm{ess}}(-L) \subset [\frac{c^2}{4}, \infty)$. By Theorem 4.3, (1.4) holds for $r_0 = \frac{4}{c^2}$. The assertions remain true if ρ is a smooth compact function with $|\nabla \rho| \le 1$. By "compact" we mean that $\{\rho \le r\}$ is compact for any $r > 0$. An analogue sufficient condition for (1.2) was also presented there for a class of jump processes, including irreducible countable Markov chains.

An alternative way to prove (1.2) is to use the isoperimetric inequalities. Consider once again the above L for symmetric diffusion, then (1.2) holds provided (Theorem 3.4 in [20])

$$k(r) := \inf_{\mu(A) \le r} \frac{\mu_\partial(\partial A)}{\mu(A)} \to \infty \quad \text{as } r \downarrow 0,$$

where μ_∂ is the $(d-1)$-dimensional measure induced by μ, and A runs over all connected smooth domains. The converse result holds if the curvature of L is bounded from below. Corresponding results were also proved in [19] for general jump processes.

By Theorem 1.2, we may prove the super-Poincaré inequality by checking the uniform integrability of P_t. For this it is convenient to apply a Harnack inequality. We shall only consider the above L on a manifold, but the Harnack inequalities are also available for other cases, see e.g. [1] for diffusion semigroups on abstract Wiener spaces and [16] for generalized Mehler semigroups.

Let $L=\Delta+Z$ for some C^1-vector field Z. Assume that $\mathrm{Ric}-\langle\nabla.Z,\cdot\rangle$ is bounded from below. By the proof of Lemma 2.1 in [18], for any $t>0$ there exists $c_t>0$(see the original statement for the specific choice of c_t) such that

$$(P_tf(x))^2\le P_tf^2(y)\exp[c_t\rho(x,y)^2], \tag{5.1}$$

where $\rho(x,y)$ denotes the Riemannian distance between x and y. This implies the following result.

Proposition 5.1. *Let P_t be the diffusion semigroup generated by L, and let $B(x,r)$ denote the geodesic ball with center x and radius r. If there exists a positive measurable function γ such that*

$$\int_M\frac{\exp[c_t\gamma(x)^2]}{\mu(B(x,\gamma(x))}\mu(\mathrm{d}x)<\infty, \tag{5.2}$$

then P_t is uniformly integrable in $L^2(\mu)$. Especially, it is the case if

$$\int_M\frac{\mu(\mathrm{d}x)}{\mu(B(x,r))}<\infty\quad \text{for some } r>0.$$

Proof. Let f be such that $\mu(f^2)=1$. By (5.1),

$$\begin{aligned}1=\int_M P_tf^2(y)\mu(\mathrm{d}y)&\ge[P_tf(x)]^2\int_M\exp[-c_t\rho(x,y)^2]\mu(\mathrm{d}y)\\&\ge[P_tf(x)]^2\mu(B(x,\gamma(x))\exp[-c_t\gamma(x)^2].\end{aligned}$$

Then

$$[P_tf(x)]^2\le\frac{\exp[c_t\gamma(x)^2]}{\mu(B(x,\gamma(x))}.$$

Therefore, if (5.2) holds, then P_t is uniformly integrable in $L^2(\mu)$. □

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Finally, we present below an example to show that our results apply also to some infinite-dimensional cases.

Let T_t be a strongly continuous semigroup on a separable Hilbert space H with $\|T_t\|\to0$ as $t\to\infty$. Let Q be a continuous linear self-adjoint nonnegative operator on H. Define

$$Q_t=\int_0^t T_sQT_s^*\mathrm{d}s,\qquad t>0.$$

We assume that $\sup_{t>0}\mathrm{Tr}Q_t<\infty$ and $\ker(Q_T)=\{0\}$ for any $t>0$. Put

$$Q_\infty = \int_0^\infty T_t Q T_t^* \mathrm{d}s,$$

then Q_∞ is well-defined with finite trace. Consider the generalized Mehler semigroup

$$P_t f(x) = \int_H f(y) N(T_t x, Q_t)(\mathrm{d}y), \quad x \in H, \ t \geq 0,$$

where $N(T_t x, Q_t)$ denotes the Gaussian measure on H with mean $T_t x$ and covariance operator Q_t. Then the unique invariant measure of P_t is $\mu = N(0, Q_\infty)$. According to [5], P_t has transition density if and only if

$$\mathrm{Range}(T_t) \subset \mathrm{Range}(Q_t^{1/2}), \quad t > 0. \tag{5.3}$$

Proposition 5.2. *Assume that* (5.3) *holds. If P_t is uniformly integrable in $L^2(\mu)$ for some $t > 0$, then P_s is compact in $L^2(\mu)$ for any $s > t$ and the essential spectrum of the generator is empty. Especially, it is the case if there exists $t > 0$ such that* $\|Q_\infty^{-1/2} T_t Q_\infty^{1/2}\| < 1$.

Proof. The first assertion follows immediately from Lemma 3.1 and [12], while the second one follows from [4] and [8] which proved that P_t is hypercontractive if $\|Q_\infty^{-1/2} T_t Q_\infty^{1/2}\| < 1$. $\square$

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