Standardly stratified algebras and cellular algebras

by

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1 Introduction

Let A be an artin algebra. Then there are finitely many non-isomorphic simple A-modules. Suppose $S_1, S_2, ..., S_n$ form a complete list of all non-isomorphic simple A-modules and we fix this ordering of simple modules once for all. Let P_i and Q_i be the projective cover and the injective envelope of S_i respectively. With this order of simple modules we define for each i a standard module $\Delta(i)$ to be the maximal quotient of P_i with composition factors S_j with $j \leq i$. Let Δ be the set of all these standard modules $\Delta(i)$. We denote by $\mathcal{F}(\Delta)$ the subcategory of A-mod whose objects are the modules M which have a Δ-filtration, namely there is a finite chain

$$0 = M_0 \subset M_1 \subset M_2 \subset ... \subset M_t = M$$

of submodules of M such that M_i/M_{i-1} is isomorphic to a module in Δ for all i. The modules in $\mathcal{F}(\Delta)$ are called Δ-good modules. Dually, we define the costandard module $\nabla(i)$ to be the maximal submodule of Q_i with composition factors S_j with $j \leq i$ and denote by ∇ the collection of all costandard modules. In this way, we have also the subcategory $\mathcal{F}(\nabla)$ of A-mod whose objects are these modules which have a ∇-filtration. Of course, we have the notion of ∇-good modules. Note that $\Delta(n)$ is always projective and $\nabla(n)$ is always injective.

From the definition, we have the following properties of standard modules:
(1) $\mathrm{Hom}_A(\Delta(i), \Delta(j)) = 0$ if $i > j$,
(2) $\mathrm{Ext}^1_A(\Delta(i), \Delta(j)) = 0$ if $i \geq j$.

For the fixed order of simple modules, beside the subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$, there arise the following subcategories of A-mod:

- $\mathcal{Y}(\Delta) = \{Y \in A-\mathrm{mod} \mid \mathrm{Ext}^1(\mathcal{F}(\Delta), Y) = 0\}$,
- $\mathcal{F}(\Delta) \cap \mathcal{Y}(\Delta)$,
- $\mathcal{W}(\nabla) = \{W \in A-\mathrm{mod} \mid \mathrm{Ext}^1(W, \mathcal{F}(\nabla)) = 0\}$,
- $\mathcal{W}(\nabla) \cap \mathcal{F}(\nabla)$.

If ${}_AA \in \mathcal{F}(\Delta)$ then A is said to be (left) **standardly stratified** (see [7],[8] or [1]). A standardly stratified algebra is called a **quasi-hereditary algebra** if the endomorphism ring of each standard module is semisimple (here the definitions are always with respect to the ordering of simple modules). Quasi-hereditary algebras were introduced in [6] to deal with certain categories in the representation theory of Lie algebras and algebraic

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groups. For a further generalization of standardly stratified algebras one may see the paper [7].

Now let us recall some definitions from [5]. A morphism $f : M \longrightarrow N$ in A-mod is said to be **right minimal** if an endomorphism $g : M \longrightarrow M$ is an automorphism whenever $f = gf$. Note that the morphism f is right minimal if and only if the restriction of f to any direct summand of M is non-zero. A subcategory $\mathcal{X}$ of A-mod is called **contravariantly finite** in A-mod if for each module C there is a right $\mathcal{X}$-approximation, that is, there is a morphism $f : X \longrightarrow C$ with $X \in \mathcal{X}$ such that the induced sequence $\mathrm{Hom}_A(X', X) \longrightarrow \mathrm{Hom}_A(X', C) \longrightarrow 0$ is exact for all X' in $\mathcal{X}$. A right $\mathcal{X}$-approximation $f : X \longrightarrow C$ is said to be a minimal right $\mathcal{X}$-approximation if f is right minimal. Dually, one has the notions of left minimal morphisms, left $\mathcal{X}$-approximations and covariantly finite subcategory in A-mod. A subcategory $\mathcal{X}$ in A-mod is called **functorially finite** in A-mod provided it is both contravariantly finite and covariantly finite in A-mod.

Note that the subcategory $\mathcal{F}(\Delta)$ is always functorially finite in A-mod and $\mathcal{Y}(\Delta)$ is covariantly finite in A-mod. Thus, as was proved in [16], the subcategory $\mathcal{F}(\Delta)$ is closed under direct summands and has (relative) almost split sequences.

In this paper we are mainly interested in studying these subcategories. First, we retreat the materials of [3] in our context and add some new results, here the point is that we want to get results only by working on the category $\mathcal{F}(\Delta)$ without knowing the modules in $\mathcal{Y}(\Delta)$. In fact, this paper comes from an understanding of standardly stratified algebras, especially the results in [3]. Second, we discuss some subcategories arising from cellular algebras. In detail the paper is organized as follows: we collect in section 2 some elementary facts and reprove that $\mathcal{F}(\Delta)$ is closed under kernels of surjections. This implies that if A is a standardly stratified algebra then $\mathcal{F}(\Delta)$ is a resolving subcategory. In section 3 we characterize the quasi-hereditary algebras inside the class of standardly stratified algebras in terms of the above subcategories, our result shows that a standardly stratified algebra is quasi-hereditary if and only if $\mathcal{F}(\nabla) = \mathcal{Y}(\Delta)$, and the latter happens if and only if the global dimension of A is finite. Section 4 is devoted to considering the subcategory $\omega(\Delta) = \mathcal{F}(\Delta) \cap \mathcal{Y}(\Delta)$. Note that for a standardly stratified algebra, the subcategory $\omega(\Delta)$ is completely determined by a tilting module [3]. In general, this tilting module is not a cotilting module. We prove that it is a cotilting module if and only if the algebra A is a Gorenstein algebra. The endomorphism algebra of the tilting module is discussed in section 5. In the last section we make some applications of our methods to cellular algebras. The main interest is the studying of the cohomology of cell modules. We show that the first cohomology groups between certain cell modules vanish. Using this, we establish a connection between certain class of cellular algebras and that of right standardly stratified algebras. In this section, new homological characterizations of quasi-hereditary algebras in terms of the cohomology groups of cell modules are provided. Also some results from the literature on cohomology groups of modules over Hecke algebras can be obtained in the view of cellular algebras.

Throughout the paper we denote by A an artin algebra and by A-mod (respectively, mod-A) the category of all finitely generated left (respectively, right) A-modules. By a module we mean usually a left module. Given two homomorphisms $f : L \longrightarrow M$ and $g : M \longrightarrow N$, we denote the composition of f and g by fg which is a homomorphism from L to N.

2 Elementary facts on $\mathcal{F}(\Delta)$

In this section, we collect some preliminary facts needed later on in the paper and give a different proof of the known fact that $\mathcal{F}(\Delta)$ is closed under kernels of surjective homomorphisms. In particular, if ${}_AA \in \mathcal{F}(\Delta)$, then $\mathcal{F}(\Delta)$ is a resolving subcategory in A-mod. First of all, let us recall the definition of a resolving subcategory.

Definition 2.1 *A subcategory $\mathcal{X}$ of A-mod is said to be a resolving subcategory if the following three conditions hold :*

(1) it is closed under extensions, that is, if $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence in A-mod with $L, N \in \mathcal{X}$, then the middle term M is in $\mathcal{X}$,

(2) it is closed under kernels of surjections, that is, If M and N are two modules in $\mathcal{X}$ and f is a surjective homomorphism from M to N, then the kernel of f is in $\mathcal{X}$,

(3) all projective modules belong to $\mathcal{X}$.

The following result is proved in [10]. We give here a different proof which can be adapted in other situation (see section 6 below).

Proposition 2.2 *$\mathcal{F}(\Delta)$ is closed under kernels of surjections.*

Proof. First, we have the following fact:

Let $1 \leq t \leq n$. Then each module M in $\mathcal{F}(\Delta)$ has a unique maximal submodule $M_1 \in \mathrm{add}(\Delta(t) \oplus \Delta(t+1) \oplus ... \oplus \Delta(n))$ such that $M/M_1 \in \mathcal{F}(\Delta(1), \Delta(2), ..., \Delta(t-1))$. (This is the consequence of the Ext^1-property of standard modules.)

Second, let us introduce some notations. Given a module M in $\mathcal{F}(\Delta)$, we denote by $[M : \Delta(i)]$ the multiplicity of $\Delta(i)$ in a Δ-filtration of M. Note that this number is independent of the choice of the filtration. By $S_\Delta(M)$ we denote the set of all numbers i in $\{1, 2, ..., n\}$ with $[M : \Delta(i)] \neq 0$. This set is called the Δ-support of M.

Now suppose M and N are in $\mathcal{F}(\Delta)$ and f is a surjection from M to N. We shall use induction on the cardinality of $S_\Delta(M)$ to prove that the kernel K of f is still in $\mathcal{F}(\Delta)$.

Suppose $n = 1$. In this case, $M = \Delta(i)^m$ and $N = \Delta(i)^s$ with $m \geq s$. We shall show that $\mathrm{Ext}^1(\Delta(i), K) = 0$ and then K is in $\mathcal{F}(\Delta)$. By definition of the $\Delta(i)$, we have the following exact sequence

$$0 \longrightarrow U(i) \longrightarrow P_i \longrightarrow \Delta(i) \longrightarrow 0,$$

where $U(i)$ is the sum of all images of the homomorphisms $g : P_j \longrightarrow P_i$ with $j > i$. Suppose that h is a non-zero homomorphism from $U(i)$ to K. Then there is an element $x \in U(i)$ such that the image of x under h is not zero and there is a homomorphism $g : P_j \longrightarrow P_i$ with $j > i$ such that $x \in \mathrm{Im}(g)$. Hence the composition $P_j \xrightarrow{g} \mathrm{Im}(g) \subseteq U(i) \xrightarrow{h} K$ is not zero. This means that K has composition factor S_j with $j > i$, and thus M has a composition factor isomorphic to S_j, a contradiction. Hence we have proved that $\mathrm{Hom}_A(U(i), K) = 0$. Now applying $\mathrm{Hom}_A(-, K)$ to the above exact sequence, we get the following exact sequence

$$... \longrightarrow \mathrm{Hom}(P_i, K) \longrightarrow \mathrm{Hom}(U(i), K) \longrightarrow \mathrm{Ext}^1(\Delta(i), K) \longrightarrow 0.$$

This shows that $\mathrm{Ext}^1(\Delta(i), K) = 0$, as we desired.

Suppose the theorem is proved for M with the cardinality of $S_\Delta(M)$ smaller than s. Now consider the case that the cardinality of $S_\Delta(M)$ is s. Let t be the largest number in

$S_\Delta(M)$. Then, by the fact stated at the beginning of the proof, there is a submodule M_1 of M and a submodule N_1 of N such that $M/M_1, N/N_1 \in \mathcal{F}(\Delta(1), \Delta(2), ..., \Delta(t-1))$ and both M_1 and N_1 are in $\mathrm{add}(\Delta(t))$.

Let us consider the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M_1 \arrow[r, "f'"] \arrow[d, "\mu"] & N_1 \arrow[d, "\nu"] & \\
0 \arrow[r] & \mathrm{Ker}(f) \arrow[r] & M \arrow[r, "f"] \arrow[d, "p"] & N \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & K' \arrow[r] & M/M_1 \arrow[r] \arrow[d] & N/N_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Note that f' exists since $\mu f \pi = 0$ and that f' is surjective since f is surjective and t is the maximal number in the Δ-support of M. So the kernel of f' lies in $\mathcal{F}(\Delta)$. Since the modules M/M_1 and N/N_1 have the Δ-supports smaller than that of M, the kernel K' of the canonical homomorphism $M/M_1 \longrightarrow N/N_1$ is already in $\mathcal{F}(\Delta)$ by induction. Then the induced exact sequence

$$0 \longrightarrow \mathrm{Ker}(f') \longrightarrow \mathrm{Ker}(f) \longrightarrow K' \longrightarrow 0$$

shows that $\mathrm{Ker}(f)$ is in $\mathcal{F}(\Delta)$ since $\mathcal{F}(\Delta)$ is closed under extensions. This finishes the proof.

Remark. If ${}_AA \in \mathcal{F}(\Delta)$, then we may use induction on the number of simple modules to prove 2.2. The argument in this case is based on the following fact, which is used later on in the paper.

Lemma 2.3 *If ${}_AA \in \mathcal{F}(\Delta)$, then for the module ${}_AA$ we have $A_1 = AeA$, where A_1 is the maximal submodule of ${}_AA$ such that A_1 lies in add$(\Delta(n))$ with $\Delta(n) = Ae$ for a primitive idempotent e in A and A/A_1 is in $\mathcal{F}(\Delta(1), \Delta(2), ..., \Delta(n-1))$.*

Proof. In fact, we have $A_1 \subseteq AeA$ since the ideal AeA is the trace of Ae in A. On one hand, if AeA/A_1 is not zero, then it has at least one composition factor isomorphic to S_n. On the other hand, since AeA/A_1 is a submodule of A/A_1 and A/A_1 is in $\mathcal{F}(\Delta(1), \Delta(2), ..., \Delta(n-1))$ whose objects have no composition factor isomorphic to S_n, the quotient AeA/A_1 has no composition factor ismorphic to S_n, too. Hence we must have the equality $AeA = A_1$.

Dually, we have the following result on the subcategory $\mathcal{F}(\nabla)$.

Proposition 2.4 *$\mathcal{F}(\nabla)$ is closed under cokernels of injections.*

As an immediate consequence, we have the following result.

Proposition 2.5 *(1) If ${}_AA \in \mathcal{F}(\Delta)$, then $\mathcal{F}(\Delta)$ is a resolving subcategory in A-mod.*
(2) If $DA \in \mathcal{F}(\nabla)$, then $\mathcal{F}(\nabla)$ is a coresolving subcategory of A-mod.

Recall that a subcategory $\mathcal{X}$ of A-mod is said to be **coresolving** if it is closed under extensions and the cokernels of injections, and it contains all injective modules.

Note that ${}_AA \in \mathcal{F}(\Delta)$ does not imply that $DA \in \mathcal{F}(\nabla)$, in general. An easy example is the following one: The algebra is given by the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "x"] & 2 \arrow[l, "\alpha"'] \arrow[loop right, "y"] \end{tikzcd}$$

with relations $x^2 = \alpha x = y\alpha = y^2 = 0$. (The composition of two arrows α and x is written in the way that α comes first and then x follows).

Given a subcategory $\mathcal{X}$ in A-mod, we have defined the subcategory $\mathcal{Y}$ whose objects are the A-modules Y with $\mathrm{Ext}^1(X, Y) = 0$ for all $X \in \mathcal{X}$. We denote by $\mathcal{Y}(\Delta)$ the corresponding subcategory $\mathcal{Y}$ for the category $\mathcal{X} = \mathcal{F}(\Delta)$.

As a consequence, we have the following corollary.

Corollary 2.6 *If ${}_AA \in \mathcal{F}(\Delta)$, then $\mathrm{Ext}^i_A(X, Y) = 0$ for all $X \in \mathcal{F}(\Delta)$ and $Y \in \mathcal{Y}(\Delta)$ and $i \geq 1$.*

Proof. Take a module M in $\mathcal{F}(\Delta)$, and let $P \longrightarrow M \longrightarrow 0$ be a projective cover of M. By Proposition 2.2, the first syzygy K of M is in $\mathcal{F}(\Delta)$. Thus $\mathrm{Ext}^{i+1}(X, Y) = \mathrm{Ext}^i(K, Y) = 0$ by induction on i. Thus the corollary follows.

As another corollary we have the following result originally proved in [16].

Corollary 2.7 *If A is a quasi-hereditary algebra or a standardly stratified algebra (with the given order for simple modules), then $\mathcal{F}(\Delta)$ is a resolving subcategory of A-mod.*

The following proposition shows the relation of $\mathcal{F}(\nabla)$ and $\mathcal{Y}(\Delta)$.

Proposition 2.8 *(1) If ${}_AA \in \mathcal{F}(\Delta)$, then $\mathcal{Y}(\Delta)$ is a coresolving subcategory of A-mod.*
(2) $\mathcal{F}(\nabla) \subseteq \mathcal{Y}(\Delta)$.

Proof. (1) $\mathcal{Y}(\Delta)$ is closed under extensions and contains all injective modules. Suppose $f : M \longrightarrow N$ is an injective morphism with M and N in $\mathcal{Y}(\Delta)$. Denote the cokernel of f by C. Applying $\mathrm{Hom}_A(\Delta(i), -)$ to the exact sequence $0 \to M \to N \to C \to 0$, we get then

$$\mathrm{Ext}^1_A(\Delta(i), N) \longrightarrow \mathrm{Ext}^1_A(\Delta(i), C) \longrightarrow \mathrm{Ext}^2_A(\Delta(i), M).$$

By 2.6, the both end terms vanish, thus the middle term vanishes, too. This means that C is in $\mathcal{Y}(\Delta)$.

(2) follows from Lemma 1.3 in [10].

The dual statement of the above result is the following.

Proposition 2.9 *(1) If $DA \in \mathcal{F}(\nabla)$, then $\mathcal{W}(\nabla)$ is resolving subcategory in A-mod.*
(2) $\mathcal{F}(\Delta) \subseteq \mathcal{W}(\nabla)$.

Remark. The above inclusion can be proper. Let us illustrate this by examples. We denote by k a field in the following examples.

(1) Consider the algebra $A = k[x]/(x^2)$. This is a local algebra with only one simple module. We have $\mathcal{F}(\Delta) = \mathcal{F}(\nabla) = \mathrm{add}(A)$. But $\mathcal{Y}(\Delta)$ is the whole module category which contains properly the subcategory $\mathcal{F}(\nabla)$.

(2) In quasi-hereditary case, the intersection of $\mathcal{F}(\Delta)$ with $\mathcal{F}(\nabla)$ is not zero, but for the standardly stratified algebra the situation is different. Consider the algebra $A = k[x,y]/(x,y)^2$. It is clear that $\mathcal{F}(\Delta)$ is just the projective modules and that $\mathcal{F}(\nabla)$ is just the injective modules. Thus the intersection of $\mathcal{F}(\Delta)$ with $\mathcal{F}(\nabla)$ is zero. Of course, we also have a proper inclusion of $\mathcal{F}(\nabla)$ into $\mathcal{Y}(\Delta)$.

For the quasi-hereditary algebras, we know that $\mathcal{F}(\nabla) = \mathcal{Y}(\Delta)$, as was shown in [16]. In the next section we will see when this equality holds true for standardly stratified algebras.

3 Global dimensions

Whether $\mathcal{F}(\nabla)$ is equal to $\mathcal{Y}(\Delta)$, this is related to the global dimension of the artin algebra A. In this section we show that the equality holds if and only if the global dimension of the algebra A is finite. Throughout this section, we assume that A is standardly stratified, that is, ${}_AA \in \mathcal{F}(\Delta)$.

First, we prove

Proposition 3.1 *If $\mathcal{F}(\nabla) = \mathcal{Y}(\Delta)$, then $gl.dim(A) < \infty$.*

Proof. We prove this by induction on n, the number of simple modules. If $n = 1$, then $\mathcal{F}(\nabla) = \mathrm{add}(\nabla(1))$ and $\mathcal{Y}(\Delta) = A$-mod. Thus each A-module is injective, and in particular, the simple module is injective. This means that the algebra is simple. Thus the global dimension is finite.

Suppose the proposition is true for algebras with $n-1$ simple modules. Now let A be an artin algebra with n simple modules and $\mathcal{F}(\nabla) = \mathcal{Y}(\Delta)$. Note that $\Delta(n) = Ae$ for some primitive idempotent element $e \in A$ and that the ideal AeA is a projective left A-module. We shall show that the assumption holds true for the quotient algebra A/AeA. Let Y be an A/AeA-module with $\mathrm{Ext}^1_{A/AeA}(\Delta(i), Y) = 0$ for all $i \leq n-1$. Then $Y \in \mathcal{Y}(\Delta)$ since $\mathrm{Ext}^1_A(\Delta(i), Y) = \mathrm{Ext}\ ^1_{A/AeA}(\Delta(i), Y) = 0$ for $i \leq n-1$ and $\mathrm{Ext}^1_A(\Delta(n), Y) = 0$. By our assumption, Y has a ∇-filtration. Since Y is an A/AeA-module, any ∇-filtration of Y has no ∇-factor isomorphic to $\nabla(n)$. This indicates that $Y \in \mathcal{F}(\nabla(1), ..., \nabla(n-1))$. So we have proved that for the algebra A/AeA the condition $\mathcal{F}(\nabla) = \mathcal{Y}(\Delta)$ holds true. By induction, gl.dim $(A/AeA) < \infty$. Since $\mathcal{Y}(\Delta) = \mathcal{F}(\nabla)$, we get that each module in $\mathcal{Y}(\Delta)$ has finite injective dimension. Indeed, if $Y \in \mathcal{Y}(\Delta)$, then Y has a ∇-filtration

$$0 = Y_0 \subset Y_1 \subset Y_2 \subset ... \subset Y_n = Y$$

such that Y_{n-1} is an A/AeA-module and $Y/Y_{n-1} \cong \nabla(n)^m$. Note that Y_{n-1} has finite injective dimension and that $\nabla(n)$ is injective. Hence Y has finite injective dimension. Then, by [4] , Prop. 5.3, p.140, we know that for each module X in A-mod, we have an exact sequence

$$0 \longrightarrow C_m \longrightarrow ... \longrightarrow C_1 \longrightarrow C_0 \longrightarrow X \longrightarrow 0$$

with $C_i \in \mathcal{F}(\Delta)$. (Here we need $\mathcal{F}(\Delta)$ to be a resolving subcategory.) With a dual argument, we can show that each module in $\mathcal{F}(\Delta)$ has finite projective dimension. Hence each module X has finite projective dimension. In particular, the simple modules have finite projective dimension. This implies that the global dimension of A is finite.

Now we prove the converse of the above proposition.

Proposition 3.2 *If gl.dim* $(A) < \infty$, *then* $\mathcal{F}(\nabla) = \mathcal{Y}(\Delta)$.

Proof. This proposition can be proved by using the result that a standardly stratified artin algebra is quasi-hereditary if and only if the global dimension is finite. But we prefer to giving a more elementary proof here.

We use induction on n. If $n = 1$, then the algebra A is either a simple algebra or a local algebra with non-zero radical. The latter can not happen since A has finite global dimension. Hence we have $\mathcal{F}(\nabla) = \mathcal{Y}(\Delta)$. Suppose the proposition is true for $n-1$. We show that it is true for algebra with n simple modules. Put $\Delta(n) = Ae$ and $I = AeA$. Then I is a projective left module, and by [9], the global dimension of A/I is finite. Hence we can apply the induction to the algebra A/I and get that $\mathcal{F}(\nabla(1), ..., \nabla(n-1))$ $= \{Y \in A/I - \text{mod} \mid \text{Ext}^1_{A/I}(\mathcal{F}(\Delta(1), ..., \Delta(n-1)), Y) = 0\}$.

Let $Y \in \mathcal{Y}(\Delta)$. Suppose Y' is the maximal A/I-submodule of Y, then Y/Y' can be embedded into a direct sum of copies of $\nabla(n)$, say $Y/Y' \hookrightarrow \nabla(n)^m$. This gives us an exact sequence

$$0 \longrightarrow Y/Y' \longrightarrow \nabla(n)^m \longrightarrow Z \longrightarrow 0$$

with Z an A/I-module. We shall show that $Y' \in \mathcal{F}(\nabla)$. For this we need only to show that $Y' \in \mathcal{Y}(\Delta(1), ..., \Delta(n-1))$. It follows from

$$0 \longrightarrow \text{Hom}_A(\Delta(i), Y/Y') \longrightarrow \text{Hom}_A(\Delta(i), \nabla(n)^m)$$

that $\text{Hom}_A(\Delta(i), Y/Y') = 0$ for all $1 \leq i \leq n-1$ since $\text{Hom}_A(\Delta(i), \nabla(n)) = 0$. Now we have the following exact sequence

$$\text{Hom}_A(\Delta(i), Y/Y') \longrightarrow \text{Ext}^1(\Delta(i), Y') \longrightarrow \text{Ext}^1(\Delta(i), Y).$$

Since $Y \in \mathcal{Y}(\Delta)$, the last term vanishes. Hence $\text{Ext}^1(\Delta(i), Y') = 0$ for all $i \leq n-1$. This means that $Y' \in \mathcal{Y}(\Delta(1), ..., \Delta(n-1))$ as we claimed. Thus Y' has a ∇-filtration. To finish the proof, we show that $Z = 0$. It follows from 2.6 that $\text{Ext}^2(\Delta(i), Y') = 0$ for all i. So we get from the following exact sequence

$$\text{Ext}^1(\Delta(i), Y) \longrightarrow \text{Ext}^1(\Delta(i), Y/Y') \longrightarrow \text{Ext}^2(\Delta(i), Y')$$

that $\text{Ext}^1(\Delta(i), Y/Y') = 0$ for all $i < n$. Now we apply $\text{Hom}(\Delta(i), -)$ to the exact sequence $0 \longrightarrow Y/Y' \longrightarrow \nabla(n)^m \longrightarrow Z \longrightarrow 0$. This yields an exact sequence

$$\text{Hom}_A(\Delta(i), \nabla(n)^m) \longrightarrow \text{Hom}_A(\Delta(i), Z) \longrightarrow \text{Ext}^1(\Delta(i), Y/Y').$$

Since the both end terms vanish for $i \leq n-1$, we obtain $\text{Hom}(\Delta(i), Z) = 0$ for all $i \leq n-1$. But this is true for $i = n$ since Z has no composition factor isomorphic to S_n. Thus we have proved that $\text{Hom}(\Delta(i), Z) = 0$ for all i. This imples that $Z = 0$, and therefore Y has a ∇-filtration.

Combining these results with that in [8], we have a characterisation of quasi-hereditary algebras in terms of subcategories.

Corollary 3.3 *Let A be a standardly stratified algebra. Then the following are equivalent:*

(1) A is quasi-hereditary;

(2) $\mathcal{F}(\nabla) = \mathcal{Y}(\Delta)$;

(3) gl.dim $A < \infty$.

4 Tilting modules and cotilting modules

In this section we consider the intersection of the subcategory $\mathcal{F}(\Delta)$ with $\mathcal{Y}(\Delta)$, which is denoted by $\omega(\Delta)$. This subcategory is determined by a tilting module. We shall prove that this tilting module is cotilting if and only if the algebra A is Georenstein.

Definition 4.1 *Let A be an artin algebra. A module T in A-mod is called a* **tilting module** *if*

(1) $Ext^i_A(T,T) = 0$ for all $i > 0$, and

(2) the projective dimension of T is finite, and

(3) there is an exact sequence

$$0 \longrightarrow {}_AA \longrightarrow T_0 \longrightarrow T_1 \longrightarrow ... \longrightarrow T_s \longrightarrow 0$$

such that T_i belongs to $add(T)$ for all i.

Dually, we have also the concept of cotilting module: An A-module T in A-mod is said to be a cotilting module if it satisfies (1), and

(2') the injective dimension of T is finite, and

(3') there is an exact sequence

$$0 \longrightarrow T_s \longrightarrow ... \longrightarrow T_1 \longrightarrow T_0 \longrightarrow DA \longrightarrow 0$$

such that T_i belongs to $add(T)$ for all i. Here we use D to denote the usual duality for artin algebras.

Now let us begin with the following general result proved in [4], Prop. 3.4.

Lemma 4.2 *Let $\mathcal{X}$ be a resolving contravariantly finite subcategory of A-mod.Denote by $\mathcal{Y}$ the subcategory whose objects are the modules C with $Ext^1(\mathcal{X}, C) = 0$. Then $\omega = \mathcal{X} \cap \mathcal{Y}$ has the following properties:*

(1) ω is selforthogonal, that is, $Ext^i(X,Y) = 0$ for all $X, Y \in \omega$ and all $i > 0$.

(2) For each X in $\mathcal{X}$ there is an exact sequence $0 \longrightarrow X \longrightarrow W \longrightarrow X' \longrightarrow 0$ with W in ω and X' in $\mathcal{X}$.

(3) For each Y in $\mathcal{Y}$ there is an exact sequence $0 \longrightarrow Y' \longrightarrow W \longrightarrow Y \longrightarrow 0$ with W in ω and Y' in $\mathcal{Y}$.

The following theorem is proved in [3]. We include here a short proof.

Theorem 4.3 *Let A be a standardly stratified algebra. Then there is a tilting module T (unique up to multiplicity of indecomposable direct summands) such that $add(T) = \omega(\Delta)$.*

Proof. It is easy to see that proj.dim $X \leq n-1$ for all $X \in \mathcal{F}(\Delta)$ (see the proof of 3.1 or [2]). Let $X_{-1} = X$. By 4.2, we can construct exact sequences $\epsilon_i : \quad 0 \longrightarrow X_{i-1} \longrightarrow W_i \longrightarrow X_i \longrightarrow 0$ with $W_i \in \omega(\Delta)$ and $X_i \in \mathcal{F}(\Delta)$. Applying $\mathrm{Hom}_A(X_{n-1}, -)$ to ϵ_i, we get that $\mathrm{Ext}^j(X_{n-1}, X_i) \cong \mathrm{Ext}^{j+1}(X_{n-1}, X_{i-1})$. This yields that $\mathrm{Ext}^1(X_{n-1}, X_{n-2}) = \mathrm{Ext}^2(X_{n-1}, X_{n-3}) = ... = \mathrm{Ext}^n(X_{n-1}, X_{-1}) = 0$ since proj.dim $X_{n-1} \leq n-1$. Hence the exact sequence ϵ_{n-1} splits and therefore the module X_{n-2} is a direct summand of W_{n-1} and in $\omega(\Delta)$. So we have an exact sequence

$$0 \longrightarrow X \longrightarrow W_0 \longrightarrow W_1 \longrightarrow ... \longrightarrow W_{n-2} \longrightarrow W_{n-1} \longrightarrow 0$$

with $W_{n-1} = X_{n-2}$.

Since $A \in \mathcal{F}(\Delta)$, we have an exact sequence

$$0 \longrightarrow {}_A A \longrightarrow W_0 \longrightarrow W_1 \longrightarrow ... \longrightarrow W_{n-2} \longrightarrow W_{n-1} \longrightarrow 0$$

with $W_i \in \omega(\Delta)$. Put $T = \bigoplus_j W_j$. Then T is a tilting module. Moreover, if $M \in \omega(\Delta)$, then $T \oplus M$ is a tilting module. By tilting theory, we must have $M \in \mathrm{add}(T)$. Therefore $\omega(\Delta) = \mathrm{add}(T)$. This finishes the proof.

Comparing this result with that for quasi-hereditary algebras, however, one cann't hope that the module T with add(T)= $\omega(\Delta)$ would be always a cotilting module. Let us see an example. Put $A = k[x,y]/(x,y)^2$. Then $\mathcal{F}(\Delta) = \mathrm{add}(A)$ and $\mathcal{Y}(\Delta) = A$-mod. Hence $\mathcal{F}(\Delta) = \mathcal{Y}(\Delta) \cap \mathcal{F}(\Delta)$. Since A is a local algebra, we know that the A-modules of finite injective dimension are just the injective modules. This shows that the injective dimension of ${}_A A$ is infinite and therefore there is no cotilting module T with $\omega(\Delta) = \mathrm{add}(T)$.

For quasi-hereditary algebra, we know from [16] that there is a cotilting module T such that $\omega(\Delta) = \mathrm{add}(T)$. The following easy corollary indicates that in some other cases we can get a cotilting module T, too.

Corollary 4.4 *Let A be a standardly stratified algebra. Then the following are equivalent:*
(1) $\omega(\Delta) = add(DA)$,
(2) $DA \in \mathcal{F}(\Delta)$.

Proof. If $DA \in \mathcal{F}(\Delta)$, then $DA \in \omega(\Delta)$. Since the number of non-isomorphic indecomposable modules in $\omega(\Delta)$ is the number of non-isomorphic simple modules, we must have that $\mathrm{add}(DA) = \omega(\Delta)$.

In particular, if the standardly stratified algebra is self-injective, then $\omega(\Delta) = \mathrm{add}(DA)$. Note that a self-injective algebra is never quasi-hereditary unless it is semisimple (see [18]).

The condition that $\omega(\Delta) = \mathrm{add}(DA)$ gives us an interesting class of algebras. Recall that an artin algebra is called a **Gorenstein algebra** if inj.dim${}_A A < \infty$ and inj.dim$A_A < \infty$. We have the following easy observation.

Corollary 4.5 *Let A be a standardly stratified algebra. If $\omega(\Delta) = add(DA)$, then A is a Gorenstein algebra.*

Proof. We know from $\omega(\Delta) = \mathrm{add}(DA)$ that $DA \in \mathcal{F}(\Delta)$, this implies that proj.dim $DA < \infty$, that is, inj.dim $A_A < \infty$. By [2], the finitistic dimension of A is finite. (Recall that the finitistic dimension of A is by definition the supremum of projective dimensions of all modules with finite projective dimension). This yields together with Prop. 6.10 in [4] that inj.dim ${}_AA < \infty$. Thus A is Gorenstein.

It would be interesting to know how the algebra A looks like when the module T with $\omega(\Delta) = \mathrm{add}(T)$ is cotilting. The rest of this section is devoted to a discussion of this and provides an answer to the question.

Proposition 4.6 *Let A be a standardly stratified algebra and $\omega(\Delta) = add(T)$. The following are equivalent:*
(1) T is a cotilting module,
(2) inj.dim $A_A < \infty$,
(3) A is a Gorenstein algebra.

Proof. If T is a cotilting module, then there is an exact sequence

$$0 \longrightarrow T_s \longrightarrow \dots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow DA \longrightarrow 0$$

with $T_i \in \omega(\Delta)$. Since each T_i in $\mathcal{F}(\Delta)$ has finite projective dimension, we see that DA has finite projective dimension, this is equivalent to saying that A_A has finite injective dimension.

Conversely, if inj.dim $A_A < \infty$, then proj.dim $DA < \infty$. By 4.2, we can construct a series of short exact sequences

$$\epsilon_i: \quad 0 \longrightarrow Y_i \longrightarrow W_i \longrightarrow Y_{i-1} \longrightarrow 0$$

with $Y_{-1} = DA, Y_i \in \mathcal{Y}(\Delta)$ and $W_i \in \omega(\Delta)$. Assume that proj.dim $DA = m < \infty$. We apply $\mathrm{Hom}_A(-, Y_m)$ to the exact sequence ϵ_i and get the following exact sequence

$$\mathrm{Ext}^j(W_i, Y_m) \longrightarrow \mathrm{Ext}^j(Y_i, Y_m) \longrightarrow \mathrm{Ext}^{j+1}(Y_{i-1}, Y_m) \longrightarrow \mathrm{Ext}^{j+1}(W_i, Y_m).$$

For $j \geq 1$ we have that $\mathrm{Ext}^j(W_i, Y_m) = 0 = \mathrm{Ext}^{j+1}(W_i, Y_m)$ since $W_i \in \mathcal{F}(\Delta)$ and $Y_i \in \mathcal{Y}(\Delta)$. This yields that $\mathrm{Ext}^j(Y_i, Y_m) \cong \mathrm{Ext}^{j+1}(Y_{i-1}, Y_m)$ for $j \geq 1$ and all i. In particular, $\mathrm{Ext}^1(Y_{m-1}, Y_m) = \mathrm{Ext}^2(Y_{m-2}, Y_m) = \dots = \mathrm{Ext}^{m+1}(Y_{-1}, Y_m) = 0$ since proj.dim $DA = m$. Thus the sequence ϵ_m splits and $Y_{m-1} \in \omega(\Delta)$. This implies that for DA we have an exact sequence

$$0 \longrightarrow W_m \longrightarrow \dots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow DA \longrightarrow 0$$

with $W_i \in \omega(\Delta)$. Now we show that $T' := \bigoplus_j W_j$ is a cotilting module. It is sufficient to prove that the injective dimension of T' is finite. Since inj.dim $A_A < \infty$ and the finitistic dimension of A is finite, we see that inj.dim ${}_AA < \infty$ by [4], Prop.6.10. This shows that A is Gorenstein. It follows now from [4], Prop.6.9 that each module with finite projective dimension has also finite injective dimension. Thus T' has finite injective dimension. By 4.3, the number of non-isomorphic indecomposable modules in $\omega(\Delta)$ is the number of non-isomorphic simple A-modules, hence $\omega(\Delta) = \mathrm{add}(T') = \mathrm{add}(T)$. This finishes the proof.

Clearly, $\mathcal{F}(\Delta) \subseteq {}^{\perp}\omega(\Delta)$ for any standardly stratified algebra, where ${}^{\perp}\omega(\Delta)$ is the subcategory of A-mod whose objects are the X with $\mathrm{Ext}^i(X, \omega(\Delta)) = 0$ for all $i > 0$. In general, this inclusion is proper, one can know this from an example like $A = k[x]/(x^2)$. It would be interesting to know under which conditions on the standardly stratified algebra A we could have $\mathcal{F}(\Delta) = {}^{\perp}\omega(\Delta)$. Of course, quasi-hereditary algebras have the property. The following is another example of this kind of algebras.

Let A be a local algebra which is not self-injective, then A is automatically a standardly stratified algebra with $\mathcal{F}(\Delta) = \mathrm{add}(A) = {}^{\perp}\omega(\Delta)$.

Remark. We note that in [3] the intersection of $\mathcal{F}(\Delta)$ with $\mathcal{Y}(\Delta)$ is discussed, and it was shown that $\mathcal{F}(\Delta)\cap\mathcal{Y}(\Delta) = \mathrm{add}(T)$ for a cotilting module T with $\mathcal{F}(\Delta) = {}^{\perp}(\mathrm{add}(T))$ if and only if the algebra is quasi-hereditary.

5 Endomorphism algebras of tilting modules

We have seen that if A is a standardly stratified algebra then there is a tilting module T such that $\omega(\Delta) = \mathrm{add}(T)$. In this section we consider the endomorphism algebra of this module T.

First let us describe the indecomposable modules in $\omega(\Delta)$. We have the following proposition.

Proposition 5.1 *Let A be a standardly stratified algebra. Then, for each i, there is an exact sequence*

$$0 \longrightarrow \Delta(i) \xrightarrow{\beta} T(i) \longrightarrow X(i) \longrightarrow 0$$

with β a left minimal $\mathcal{Y}(\Delta)$-approximation and $X(i) \in \mathcal{F}(\Delta(1), ..., \Delta(n-1))$. Moreover, the module $T(i)$ is indecomposable and $\omega(\Delta) = add(\bigoplus_j T(j))$.

Proof. By 4.2, or Lemma 4 in [16], there is an exact sequence

$$0 \longrightarrow \Delta(i) \xrightarrow{\beta'} T'(i) \longrightarrow X'(i) \longrightarrow 0$$

with $T'(i) \in \omega(\Delta)$ and $X'(i) \in \mathcal{F}(\Delta(1), ..., \Delta(i-1))$. Clearly, the β' is a left $\mathcal{Y}(\Delta)$-approximation for $\Delta(i)$. Thus we may choose a minimal left $\mathcal{Y}(\Delta)$-approximation for $\Delta(i)$:

$$0 \longrightarrow \Delta(i) \xrightarrow{\beta} T(i) \xrightarrow{\pi} X(i) \longrightarrow 0,$$

where $T(i)$ lies in $\mathcal{Y}(\Delta)$ and $X(i)$ in $\mathcal{F}(\Delta(1), ..., \Delta(i-1))$. Now we show that $T(i)$ is indecomposable.

Suppose that $T(i) = T_1 \oplus T_2$, $\beta = (\beta_1, \beta_2)$ and $\pi = (\pi_1, \pi_2)^t$, with $\beta_j : \Delta(i) \longrightarrow T_j$ and $\pi_j : T_j \longrightarrow X(i)$ for $j = 1, 2$. Since β is minimal, both β_1 and β_2 are non-zero. Now we consider the following pullback diagram:

$$\begin{array}{ccc} \Delta(i) & \xrightarrow{\beta_1} & T_1 \\ {\scriptstyle\beta_2}\big\downarrow & & \big\downarrow{\scriptstyle -\pi_1} \\ T_2 & \xrightarrow{\pi_2} & X(i). \end{array}$$

Since $X(i)$ has no composition factor isomorphic to S_i, we see that $\mathrm{Hom}_A(\Delta(i), X(i)) = 0$. This implies that $\beta_1\pi_1 = \beta_2\pi_2 = 0$. It follows then from the property of pullback diagram and $\beta_1(-\pi_1) = 0\pi_2 = 0$ that there is a homomorphism $\alpha : \Delta(i) \longrightarrow \Delta(i)$ such that $\beta_1 = \alpha\beta_1$ and $0 = \alpha\beta_2$. Since $\mathrm{End}\Delta(i)$ is a local algebra, we know from $0 = \alpha\beta_2$ that α is nilpotent, and therefore $\beta_1 = \alpha\beta_1 = \alpha^m\beta_1$ for all m, thus β_1 is zero, a contradiction. Hence $T(i)$ is indecomposable. Since the composition factors of $T(i)$ are of the form

S_j with $j \leq i$ and S_i occurs at least once, the modules $T(i)$, $1 \leq i \leq n$, are pairwise non-isomorphic, and therefore $\omega(\Delta) = \mathrm{add}(\bigoplus_j T(j))$.

We also need the following fact:

Lemma 5.2 *Suppose A is standardly stratified. Let $T = \oplus_j T(j)$ and $B = End(_AT)$. Then*

(1) for each module $X \in \mathcal{F}(\Delta)$, the evaluation map

$$e_X : \quad {}_AX \cong_A Hom_B(Hom_A(X,T)_B, T).$$

is an isomorphism of A-modules.

(2) The contravariant functor $Hom_A(-,T)$ is an equivalence between the category $\mathcal{F}(\Delta)$ and its image which is a subcategory of mod-B.

Proof. The isomorphism is true for $X = T$, so it is true for direct summands of T and therefore for all modules X in $\mathrm{add}(T)$. For a general X in $\mathcal{F}(\Delta)$, we have an exact sequence

$$0 \longrightarrow X \longrightarrow T_0 \xrightarrow{f_0} T_1 \longrightarrow \dots \xrightarrow{f_{s-1}} T_s \longrightarrow 0.$$

such that the image X_i of f_i is in $\mathcal{F}(\Delta)$ and T_i in $\mathrm{add}(T)$. So we have the following exact sequences

$$\epsilon_i : \quad 0 \longrightarrow X_{i-1} \longrightarrow T_i \longrightarrow X_i \longrightarrow 0$$

for $1 \leq i \leq s-1$ with $X_{-1} = X$ and $X_s = T_s$, and the following commutative diagram (for simplicity, we denote $\mathrm{Hom}(X,T)$ just by X^*):

$$\begin{array}{ccccccc}
0 \longrightarrow & X_{s-2} & \longrightarrow & T_{s-1} & \longrightarrow & T_s & \longrightarrow 0 \\
 & \big\downarrow e_{X_{s-2}} & & \big\downarrow e_{T_{s-1}} & & \big\downarrow e_{T_s} & \\
0 \longrightarrow & X_{s-2}^{**} & \longrightarrow & T_{s-1}^{**} & \longrightarrow & T_s^{**} &
\end{array}$$

Note that $\mathrm{Hom}_A(-,T)$ is exact on $\mathcal{F}(\Delta)$. Since $e_{T_{s-1}}$ and e_{T_s} are isomorphisms, we see that $e_{X_{s-2}}$ is an isomorphism. Inductively, we can show that e_X is an isomorphism. This proves (1).

(2) follows from (1) and the canonical isomorphism

$$\mathrm{Hom}_R({}_RM, \mathrm{Hom}_S(N_S, {}_RU_S)) \cong \mathrm{Hom}_S(N_S, \mathrm{Hom}_R({}_RM, {}_RU_S))$$

for two rings R and S, and the modules M, N and U. This finishes the proof.

Now we prove the following theorem.

Theorem 5.3 *Let A be a standardly stratified algebra. Then the endomorphism algebra of T is right standardly stratified algebra (the order is just the opposite order of that for the algebra A).*

Proof. We define $i' = n - i + 1$ and denote the projective right B-module $\mathrm{Hom}_A(T(i'),T)$ by $P'(i)$. Now we define $\Delta'(i) = \mathrm{Hom}_A(\Delta(i'),T)$. Let Δ' be the collection of $\Delta'(1), \Delta'(2), ..., \Delta'(n)$. Since $\Delta(1)$ is in $\omega(\Delta)$, we see that $\Delta'(n)$ is a projective right B-module. It is clear that each $P'(i)$ has a Δ'-filtration since the functor $F = \mathrm{Hom}_A(-,T)$ is exact on $\mathcal{F}(\Delta)$. To finishes the proof, we need to show that $\Delta'(i)$ is obtained from

$P'(i)$ by factoring out all homomorphic images $\mathrm{Im}(g)$ with $g : P'(j) \longrightarrow P'(i)$ with $j > i$. It follows from 5.1 that we have the following exact sequence of right B-modules:

$$0 \longrightarrow F(X(n-i+1)) \longrightarrow F(T(n-i+1)) \longrightarrow F(\Delta(n-i+1)) \longrightarrow 0.$$

If g is a right B-module homomorphism from $P'(j)$ to $P'(i)$ with $j > i$, then, by 5.2, there is a left A-module homomorphism h from $T(i')$ to $T(j')$ such that $g = F(h)$. Let $\beta_{i'}$ denote the minimal morphism in 5.1, then $\beta_{i'} h = 0$ since $i' > j'$ and the composition factors of $T(j')$ are of the form S_t with $t \leq j'$. Thus h factors through the morphism $T(i') \longrightarrow X(i')$ and then the image of g belongs to $FX(i')$. This shows that $\Delta'(i)$ is a maximal quotient module of $P'(i)$ with composition factors of indices at most i. This completes the proof.

If we start from the right standardly stratified algebra B and make the similar construction as we did for the left standardly stratified algebra A, then we get a tilting right B-module T'_B for the corresponding orthogonal category $\omega(\Delta')$. The question now gives rise: Is the endomorphism algebra of T'_B isomorphic to the algebra A ? Before we answer this question, we deduce a lemma from 5.2 which is needed.

Lemma 5.4 *Suppose that A is left stardardly stratified. Then, for modules X, Y in $\mathcal{F}(\Delta)$, there holds $Ext^1_B(FY, FX) \cong Ext^1_A(X, Y)$, where F is the contravariant functor $Hom_A(-, T)$.*

Proof. By 4.2, we have an exact sequence

$$0 \longrightarrow Y \longrightarrow T_0 \longrightarrow Y_0 \longrightarrow 0$$

with $T_0 \in \omega(\Delta)$ and $Y_0 \in \mathcal{F}(\Delta)$. This provides the following exact sequences:

$$0 \longrightarrow FY_0 \longrightarrow FT_0 \longrightarrow FY \longrightarrow 0$$

and

$$\mathrm{Hom}_B(FT_0, FX) \longrightarrow \mathrm{Hom}_B(FY_0, FX) \longrightarrow \mathrm{Ext}^1_B(FY, FX) \longrightarrow \mathrm{Ext}^1_B(FT_0, FX).$$

Note that the last term in the last sequence vanishes because FT_0 is a projective right B-module. Now it follows from 5.2 and the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_B(FT_0, FX) & \longrightarrow & \mathrm{Hom}_B(FY_0, FX) & \longrightarrow & \mathrm{Ext}^1_B(FY, FX) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_A(X, T_0) & \longrightarrow & \mathrm{Hom}_A(X, Y_0) & \longrightarrow & \mathrm{Ext}^1_A(X, Y) & \longrightarrow & 0.
\end{array}$$

that the lemma is true since the first two vertical maps are bijective by 5.2.

Theorem 5.5 *Let A be a standardly stratified algebra and T the tilting module such that add(T)= $\omega(\Delta)$. Let B be the endomorphism algebra of T. For the right standardly stratified algebra B, we denote by T' the right tilting B-module with add(T')= $\omega(\Delta')$. Then the endomorphism algebra of T' is Morita equivalent to the opposite algebra of A.*

Proof. We show that FP_i is in $\omega(\Delta')$. Once we have done this, then the theorem follows immediately from 5.2. Since P_i has a Δ-filtration, FP_i is in $\mathcal{F}(\Delta')$. But the above lemma says that FP_i belongs also to $\mathcal{Y}(\Delta')$. Thus it is in $\omega(\Delta')$, as was desired.

6 Subcategories from cellular algebras

In this section we study some subcategories arising from the cell modules over a cellular algebra. We first recall the definitions of cellular algebras given in [11] and [13], and then we give a new homological characterization of quasi-hereditary algebras inside the class of cellular algebras in terms of the cell modules. We also study the extension closed subcategory $\mathcal{F}(W(\Lambda_0))$ of cell modules. For a further information on the study of cellular algebras one may refer to the survey paper [14].

Let k be an arbitrary field.

Definition 6.1 *(Graham and Lehrer, [11]) An associative k–algebra A is called a* **cellular algebra** *with cell datum (I, M, C, i) if the following conditions are satisfied:*

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has a k–basis $C^{\lambda}_{S,T}$ where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is a k–linear anti–automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U,S)C^{\lambda}_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly greater than λ, and where the coefficients $r_a(U,S) \in k$ do not depend on T.

Note that our partial order on Λ is just the opposite one used in [11]. Typical examples of cellular algebras are Hecke algebras of type A_n, Brauer algebras, partition algebras and many others (see [11] and [14]).

In the following we shall call a k-linear anti-automorphism i of A with $i^2 = id$ an involution of A.

For each $\lambda \in \Lambda$, there is a **cell module** $W(\lambda)$ with a k-basis $\{C_S \mid S \in M(\lambda)\}$, the module structure is given by

$$aC_S = \sum_{T \in M(\lambda)} r_a(T,S)C_T$$

where the coefficients $r_a(T,S)$ are defined as in the definition 6.1. We have also a right cell module $i(W(\lambda))$ which is defined dually. For a cell module, one has also defined a bilinear form $\Phi_\lambda : W(\lambda) \times W(\lambda) \to k$ by

$$C^{\lambda}_{S,S}C^{\lambda}_{T,T} \equiv \Phi_\lambda(C_S, C_T)C^{\lambda}_{S,T}$$

modulo the ideal generated by all basis elements with upper index greater than λ. We denote this ideal by $J^{>\lambda}$ in the sequel.

We need also the following equivalent definition of cellular algebras in [13].

Definition 6.2 *([13]) Let A be a k–algebra. Assume there is an involution i on A. A two–sided ideal J in A is called a* **cell ideal** *if and only if $i(J) = J$ and there exists a left ideal $W \subset J$ such that W has finite k–dimension and that there is an isomorphism of A–bimodules $\alpha : J \simeq W \otimes_k i(W)$ (where $i(W) \subset J$ is the i–image of W) making the following diagram commutative:*

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & W \otimes_k i(W) \\ i \big\downarrow & & \big\downarrow \; x \otimes y \mapsto i(y) \otimes i(x) \\ J & \xrightarrow{\alpha} & W \otimes_k i(W) \end{array}$$

The algebra A (with the involution i) is called **cellular** *if and only if there is a vector space decomposition $A = J_1' \oplus J_2' \oplus \ldots \oplus J_n'$ (for some n) with $i(J_j') = J_j'$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J_l'$ gives a chain of two sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J_j' = J_j / J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.*

The modules $W(j), 1 \leq j \leq n$, which are obtained from the sections J_j/J_{j-1} of the chain, are the cell modules. The above chain in A is called a **cell chain**. Note that the cell modules are called standard modules in [13, 14].

The following result is shown in [11].

Lemma 6.3 *let A be a cellular algebra with cell datum (I, M, C, i). Then*

(1) the simple modules are parameterized by $\Lambda_0 := \{\lambda \in \Lambda \mid \Phi_\lambda \neq 0\}$. For $\lambda \in \Lambda_0$, we denote by S_λ the simple module corresponding to λ, which is isomorphic to the top of the cell module $W(\lambda)$.

(2) The multiplicity $[W(\lambda) : S_\mu]$ of a simple module S_μ ($\mu \in \Lambda_0$) in a cell module $W(\lambda)$ is given by

$$[W(\lambda) : S_\mu] = \begin{cases} 0, & \textit{unless} \quad \mu \leq \lambda, \\ 1, & \lambda = \mu. \end{cases}$$

In particular, for $\lambda, \mu \in \Lambda_0$, $Hom_A(W(\lambda), W(\mu)) = 0$ unless $\lambda \leq \mu$, and $End_A(W(\lambda)) = k$.

Thus the cell module $W(\lambda)$ with $\lambda \in \Lambda_0$ can be described as the quotient module of P_λ modulo the submodule $U(\lambda) = \sum_{\mu \not\leq \lambda, f} \mathrm{Im}(f : P_\mu \longrightarrow P_\lambda) + \sum_{g, \mathrm{Im}(g) \subseteq \mathrm{rad}(P_\lambda)} \mathrm{Im}(g : P_\lambda \longrightarrow P_\lambda)$.

Before we give our criteria for quasi-hereditary algebras, we first point out the following fact.

Proposition 6.4 *Let A be a cellular algebra. If $gl.dim(A) = \infty$, then there is a cell module $W(\lambda)$ with $\lambda \in \Lambda_0$ such that the projective dimension of $W(\lambda)$ is infinite.*

Proof. We prove the following statement: If proj.dim $W(\mu) < \infty$ for all $\mu \in \Lambda_0$, then proj.dim $S_\mu < \infty$ for all $\mu \in \Lambda_0$ and gl.dim $(A) < \infty$.

Indeed, if μ is a minimal element in Λ_0, then $W(\mu)$ is a simple module since the composition factors of $W(\mu)$ are of the form S_λ with $\lambda \leq \mu$ and $[W(\mu) : S_\mu] = 1$. Hence proj.dim $S_\mu =$ proj.dim $W(\mu) < \infty$. Suppose that μ is not a minimal element in Λ_0. Then the radical of $W(\mu)$ has composition factors of the form S_λ with $\lambda < \mu$ by 6.3, and by induction, for those λ we have that proj.dim $S_\lambda < \infty$. It follows now from the exact sequence $0 \to \mathrm{rad}(W(\mu)) \to W(\mu) \to S_\mu \to 0$ that proj.dim $S_\mu < \infty$. Hence gl.dim$(A) < \infty$.

The next result gives a criterion for quasi-heredity in terms of first cohomology groups of cell modules.

Theorem 6.5 *For a cellular algebra A the following are equivalent:*
(1) A is quasi-hereditary;
(2) $Ext^1_A(W(\lambda), Di(W(\mu))) = 0$ for all $\lambda, \mu \in \Lambda$, where D stands for $Hom_k(-, k)$.

Proof. If A is quasi-hereditary, then we know that the cardinality of Λ is the number of the non-isomorphic simple A-modules and that the cell modules are just the standard modules. Hence the statement (2) holds true.

Conversely, assume that (2) holds. For the given cell datum, we have a cell chain

$$0 \subset J_1 \subset J_2 \subset ... \subset J_m = A.$$

Note that the cell modules are obtained from the sections of this chain. If J_1 is a heredity ideal in A, then, by induction on the length of the cell chain, we can show that A is a quasi-hereditary algebra. Since J_1 is either a heredity ideal or a nilpotent ideal in A by [13], the remainder case to be considered is that J_1 is nilpotent, that is, $J_1^2 = 0$. We shall prove that this is impossible unless $J_1 = 0$. Let $J = J_1$ and $B = A/J$. Then $J \cong W \otimes_k i(W)$, where W is a left cell module and $i(W)$ is the right cell module. This means that J_A is a direct sum of copies of $i(W)$. The canonical exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

shows that $J \otimes_A J \cong \mathrm{Tor}_2^A(B, B) \cong \mathrm{Tor}_1^A(\oplus i(W), B)$ by [13], Prop.6.1. From the definition of the cell chain, we may assume that $J_j/J_{j-1} \cong W(j) \otimes_k i(W(j))$, where $W(j)$ is the cell module for all j and $W = W(1)$. Now it follows from the canonical isomorphism $\mathrm{DExt}^j_A(X, Y) \cong \mathrm{Tor}^A_j(DY, X)$ that $\mathrm{Tor}^A_j(i(W(s)), W(t)) \cong \mathrm{Ext}^j_A(W(t), Di(W(s))) = 0$ for $j = 1$ and all s, t. Now we apply $i(W) \otimes_A -$ to the exact sequences

$$0 \longrightarrow J_j/J \longrightarrow J_{j+1}/J \longrightarrow W(j+1) \otimes_k i(W(j+1)) \longrightarrow 0,$$

with $j = 2, 3, ..., m-1$, and we get that $\mathrm{Tor}_1^A(i(W), B) = 0$. Thus $J \otimes_A J = 0$. However, the corollary 6.2 in [13] says that if J is not zero, then $J \otimes_A J$ is never zero. Hence we must have J to be zero , and the proof is finished.

Combining the results in [14], we have the following corollary.

Corollary 6.6 *For a cellular algebra A, the following are equivalent:*
(1) A is quasi-hereditary;
(2) $Ext^1(W(\lambda), Di(W(\mu))) = 0$ for all $\lambda, \mu \in \Lambda$;
(3) proj.dim $W(\mu) < \infty$ for all $\mu \in \Lambda_0$;
(4) the Cartan determinant of A is 1;
(5) there is a cell chain which is a heredity chain;
(6) every cell chain in A is a heredity chain.

Recall that a chain $0 = J_0 \subset J_1 \subset J_2 \subset ... \subset J_m = A$ of ideals in A is called a **heredity chain** if for each j the ideal J_j/J_{j-1} is idempotent, left projective as an A/J_{j-1}-module and the endomorphism algebra of the A/J_{j-1}-module J_j/J_{j-1} is semisimple.

For a subset $\Phi \subseteq \Lambda$, we put $W(\Phi) = \{W(\mu) \mid \mu \in \Phi\}$ and $W(\Phi)^* = \{Di(W(\mu)) \mid \mu \in \Phi\}$. With these notations, the above theorem can be reformulated as follows:

Let A be a cellular algebra. Then A is quasi-hereditary if and only if $\mathcal{F}(W(\Phi)^*) = \mathcal{Y}(W(\Lambda))$.

Thus the first cohomology groups of cell modules may play a role in the study of cellular algebras. In this direction, we have the following result on cohomology groups of cell modules.

Proposition 6.7 *Let A be a cellular algebra and $\lambda, \mu \in \Lambda_0$.*

(1) If $\lambda > \mu$, then

(i) $Ext^1(W(\lambda), W(\mu)) = 0$,

(ii) $Ext^1(W(\lambda), S_\mu) = 0$;

(2) if X is an A-module with $Ext^1_A(X, W(\Lambda)^*) = 0$, then $X \in \mathcal{F}(W(\Lambda))$, that is, $\mathcal{W}(W(\Lambda)^*) \subseteq \mathcal{F}(W(\Lambda))$;

(3) if $Ext^1_A(W(\lambda), W(\mu)^*) \neq 0$, then either $\lambda = \mu$, or λ and μ are incomparable.

Proof. For (1), we just give a proof of the statement (i) since one can prove (ii) in a similar way. The description of $W(\lambda)$ provides us an exact sequence

$$0 \longrightarrow U(\lambda) \longrightarrow P_\lambda \longrightarrow W(\lambda) \longrightarrow 0.$$

By applying $\mathrm{Hom}_A(-, W(\mu))$ to this sequence, we get the following exact sequence:

$$\mathrm{Hom}_A(P_\lambda, W(\mu)) \to \mathrm{Hom}_A(U(\lambda), W(\mu)) \to \mathrm{Ext}^1(W(\lambda), W(\mu)) \to 0.$$

We shall show that $\mathrm{Hom}_A(U(\lambda), W(\mu)) = 0$. Suppose that $h : U(\lambda) \longrightarrow W(\mu)$ is a non-zero homomorphism. Then there is an element u in $U(\lambda)$ such that the image of u under h is not zero. By the definition of $U(\lambda)$, we may assume that u lies in $\mathrm{Im}(f)$ for some $f : P_\gamma \longrightarrow P_\lambda$, where either $\gamma \not\leq \lambda$ or $\gamma = \lambda$ with $\mathrm{Im}(\gamma) \subseteq \mathrm{rad}(P_\lambda)$. In both cases we have a non-zero morphism from P_γ to $W(\mu)$. Thus, by 6.3, $\gamma \leq \mu$ and $\gamma < \lambda$. This contradicts the choice of $\gamma \not\leq \lambda$. Thus we have proved our claim and therefore the statement (i).

(2) Suppose that X is an A-module with $\mathrm{Ext}^1(X, W(\Lambda)^*) = 0$. Let

$$0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_m = A$$

be a cell chain which gives the cell modules $W(\Lambda)$. Assume that $J_j/J_{j-1} \cong W(j) \otimes_k iW(j)$ for all j. Thus $J_j/J_{j-1} \cong W(j)^{m_j}$ as a left module, where $m_j = \dim_k W(j)$. We have the following exact sequence

$$\mathrm{Tor}_1^A(iW(j)^{m_j}, X) \to J_{j-1} \otimes_A X \to J_j \otimes_A X \to W(j) \otimes_k iW(j) \otimes_A X \to 0.$$

The first term vanishes since $\mathrm{Tor}_1^A(iW(j), X) \cong \mathrm{Ext}^1_A(X, W(j)^*) = 0$. This implies that we have an chain of submodules of X :

$$0 = J_0 \otimes_A X \subset J_1 \otimes_A X \subset J_2 \otimes_A X \subset \ldots \subset J_m \otimes_A X = X$$

with the sections being cell modules. Hence X is in $\mathcal{F}(W(\Lambda))$.

(3) Since $\mathrm{Ext}^1(W(\lambda), W(\mu)^*) \neq 0$, there is a composition factor S_γ of $W(\mu)$ with $\gamma \leq \mu$ such that $\mathrm{Ext}^1(W(\lambda), S_\gamma) \neq 0$. By (1)(ii), λ is not bigger than γ, thus we have either $\lambda \leq \gamma \leq \mu$; or λ and γ are incomparable and $\gamma \leq \mu$. Since in the latter case we can not have $\lambda > \mu$, there are two possibilities for λ and μ : either $\lambda \leq \mu$, or λ and μ are incomparable. Dually, we have that $\mathrm{Ext}^1(W(\mu), W(\lambda)^*) \neq 0$ (since A is a cellular algebra). This implies that $\mu \leq \lambda$ or they are incomparable. Hence, summarizing all cases together, we must have either $\lambda = \mu$, or λ and μ are incomparable. This finishes the proof.

As an application of 6.7, we just mention the following fact for Hecke algebra $H_n(q)$ over a field k. For the definitions, one may read the book [12], or [15]. (Of course, one may apply the result to Birman-Wenzl algebra, too, because it is proved in [19] that Birman-Wenzl algebras are also cellular.)

Corollary 6.8 *Let $H_n(q)$ be the Hecke algebra of type A_n over a field k with $0 \neq q \in k$ and e the smallest positive integer such that $1 + q + q^2 + ... + q^{e-1} = 0$. For a e-regular partition λ of n, we denote by $W(\lambda)$ the corresponding left cell representation of $H_n(q)$. If λ and μ are two e-regular partitions of n such that λ is smaller than μ under the dominant order, then $Ext^1(W(\lambda), W(\mu)) = 0$.*

Remarks. (1) If $\lambda = \mu$ in 6.7, then $\mathrm{Ext}^1(W(\lambda), W(\mu))$ may not vanish, this can be seen from the example $A = k[x]/(x^m)$ for $m \geq 2$.

(2) The higher cohomology groups may not vanish even if $\lambda > \mu$. For instance, the algebra A is given by the following quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[loop right, "\gamma"] \end{tikzcd}$$

with relations $\alpha\beta\alpha = \beta\alpha\beta = \alpha\gamma = \gamma\alpha = 0$, $\beta\alpha = \gamma^2$. An involution on A can be defined by fixing the vertices and the arrow γ, but interchanging the arrows β and α. We put $\Lambda = \{1 < 2 < 3 < 4\}$, and define $C^4_{1,1} = \gamma^2$, $C^3_{1,1} = \gamma$, $C^2_{1,1} = e_2, C^2_{1,2} = \beta$, $C^2_{2,1} = \alpha, C^2_{2,2} = \alpha\beta$ and $C^1_{1,1} = e_1$. Then one can check that this basis satisfies all conditions in 6.1. An easy computation shows that $W(1)$ is the simple module corresponding to the vertex 1 and $W(2)$ is a uniserial module of length two with S_2 as its top and S_1 as its socle. For this algebra, we have that $\mathrm{Ext}^2(W(2), W(1)) \neq 0$.

In contrary to 2.8, we have the following fact for cellular algebras, which is the dual statement (2) of Proposition 6.7.

Proposition 6.9 *Let A be a cellular algebra. Then $\mathcal{Y}(W(\Lambda)) \subseteq \mathcal{F}(W(\Lambda)^*)$. In particular, if gl.dim$(A) = \infty$, then $\mathcal{Y}(W(\Lambda))$ is properly contained in $\mathcal{F}(W(\Lambda)^*)$.*

In the following we assume that A is a cellular algebra with an involution and that Λ_0 with the partial order of Λ restricted to Λ_0 is a total order. In this case, we have the following result.

Proposition 6.10 *Let $W(\Lambda_0)$ be the collection of all cell modules $W(\lambda)$ with $\lambda \in \Lambda_0$. Then $\mathcal{F}(W(\Lambda_0))$ is closed under kernels of surjections.*

Proof. The proof is similar to that of 2.2, we just indicate the necessary modification in the argument. For each module M in $\mathcal{F}(W(\Lambda_0))$, we can define the $W(\Lambda_0)$-support of M and there is a unique submodule M_1 of M such that M_1 is in $\mathcal{F}(W(\lambda_0))$ with λ_0 the largest element in the $W(\Lambda_0)$-support of M and that $M/M_1 \in \mathcal{F}(W(\Lambda_0))$ does not have λ_0 in its $W(\Lambda_0)$-support . We use induction on the cardinality of the $W(\Lambda_0)$-support of M. If the support of M has one element, we know from [17] p.114 that $\mathcal{F}(M)$ is an abelian category, so it is closed under kernels of surjections. Suppose the statement is true for a module M in $\mathcal{F}(W(\Lambda_0))$ with a smaller $W(\Lambda_0)$-support. Now assume that M is a module with a bigger $W(\Lambda_0)$-support. So we have a commutative diagram as in the proof of 2.2. To show that f' is surjective, we just note that the module K' in the diagram has no composition factor isomorphic to S_{λ_0}, this means that f' must be surjective. Thus we can get the desired result without any problem.

As a consequence, we have the following result which tell us the relationship between certain class of cellular algebras and the class of standardly stratified algebras

Proposition 6.11 *$\mathcal{F}(W(\Lambda_0))$ is a resolving subcategory of A-mod if and only if A is a right standardly stratified.*

Proof. By 6.3, we have that $\mathrm{End}W(\lambda) \cong k$ for all λ in Λ_0. Note that $W(\lambda)$, with λ in Λ_0, is the maximal quotient module of P_λ such that the composition factors of $W(\lambda)$ are of the form S_μ with $\mu \leq \lambda$ and $\mathrm{End}W(\lambda)$ is semisimple. Thus $W(\lambda)$ is the socalled proper standard module in terms of [2]. It follows from Prop.1.2 in [2] that ${}_AA \in \mathcal{F}(W(\Lambda_0))$ is equivalent to saying that A is right standardly stratified algebra.

Now let us consider the second cohomology groups of cell modules. Comparing with the homological definition of quasi-hereditary algebras, the following question arises naturally.

Question. Let A be a cellular algebra (without any restriction on Λ_0). Are the following statements equivalent:
(1) A is quasi-hereditary;
(2) $\mathrm{Ext}^1_A(W(\Lambda), W(\Lambda)^*) = 0$;
(3) $\mathrm{Ext}^2_A(W(\Lambda), W(\Lambda)^*) = 0$.

Our answer to this question is the following theorem.

Theorem 6.12 *Let A be a cellular algebra. Then A is quasi-hereditary if and only if $Ext^2_A(W(\Lambda), W(\Lambda)^*) = 0$.*

Proof. We need only to show the "*if*" part. Let

$$0 \subset J_1 \subset J_2 \subset ... \subset J_m = A$$

be a cell chain which produces the cell modules W(Λ). It follows from $\mathrm{Ext}^2_A(W(\Lambda), W(\Lambda)^*) = 0$ that $\mathrm{Ext}^1_A(J_j, W(\Lambda)^*) = \mathrm{Ext}^2_A(A/J_j, W(\Lambda)^*) = 0$ since A/J_j has a $W(\Lambda)$-filtration. Now we show that J_1 is a heredity ideal. If this is done, then we can use induction to get the desired statement.

Since a cell ideal J is either a heredity ideal or $J^2 = 0$, what we have to do is just to exclude the case $J_1^2 = 0$. Now the proof is similar to that of 6.5. The condition that $\mathrm{Ext}^1_A(J_j, W(\Lambda)^*) = 0$ can be interpreted as $\mathrm{Tor}^A_1(i(W(\Lambda)), J_j) = 0$ by the canonical isomorphism $\mathrm{DExt}^j_A(X, Y) \cong \mathrm{Tor}^A_j(DY, X)$. Since as a right A-module A/J_1 has an $i(W(\Lambda))$-filtration, we know that $\mathrm{Tor}^A_1(A/J_1, J_1) = 0$. Suppose that J_1 is non-zero with $J_1^2 = 0$. Then we have

$$J_1 \otimes_A J_1 \cong \mathrm{Tor}^A_2(A/J_1, A/J_1) \cong \mathrm{Tor}^A_1(A/J_1, J_1) = 0.$$

This implies that J_1 must be zero, a contradiction. Thus J_1 must be a heredity ideal in A. This finishes the proof.

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References

[1] I.Ágoston, V.Dlab and E.Lukács Stratified algebras. Math. Report of Academy of Science, Canada 20 (1998), 22-28.

[2] I.Ágoston, D.Happel, E. Lukács and L.Unger, Finitistic dimension of standardly stratified algebras. Preprint, 1998.

[3] I.Ágoston, D.Happel, E.Lukács and L.Unger, Standardly stratified algebras and tilting. Preprint, 1998.

[4] M.Auslander and I.Reiten, Applications of contravariantly finite subcategories. Adv. Math. 86 (1991), 111-152.

[5] M.Auslander and S.O.Smalø, Almost split sequences in subcategories. J. Algebra 69 (1981), 426-454.

[6] E.Cline, B.J.Parshall and L.L.Scott, Finite dimensional algebras and highest weight categories. J. reine angew. Math. 391 (1988), 85-99.

[7] E.Cline, B.J.Parshall and L.L.Scott, Stratifying endomorphism algebras. Memoirs Amer. Math. Soc. 591 (1996).

[8] V.Dlab, Quasi-hereditary algebras revisited. An.St.Univ. Ovidius Constantz 4 (1996), 43-54.

[9] V.Dlab and C.M.Ringel, Quasi-hereditary algebras. Illinois J. Math. 35 (1989), 280-291.

[10] V.Dlab and C.M.Ringel, The module theoretical approach to quasi-hereditary algebras. London Math. Soc. Lect. Note Ser. 168, (1992), 200- 224.

[11] J.Graham and G.Lehrer, Cellular algebras. Invent. Math. 123 (1996), 1-34.

[12] J.E.Humphreys, Reflection groups and Coxeter groups. University Press, Cambridge, 1990.

[13] S.König and C.C.Xi, On the structure of cellular algebras. In: I.Reiten, S.Smalø and Ø.Solberg (Eds.): Algebras and Modules II. Canadian Mathematical Society Conference Proceedings Vol. 24 (1998), 365–386.

[14] S.König and C.C.Xi, Cellular algebras and quasi-hereditary algebras - a comparison. Electronic Research Announcements of the Amer. Math. Soc. 5 (1999), 71-75. (http://www.ams.org/era).

[15] A.Mathas, Hecke algebras and Schur algebras of symmetric groups. SFB-343, Ergaenzungsreihe 98-005, Bielefeld, 1998.

[16] C.M.Ringel, The category of Δ-good modules over a quasi-hereditary algebra has almost split sequences. Math. Z. 208 (1991), 209-223.

[17] C.M.Ringel, Tame algebras and integral quadratic forms. Lecture Notes in Mathematics 1099, Springer-Verlag, 1984.

[18] C.C.Xi, On the structure of Schur algebras $S(n,p)$ with $n \geq p$. Canad. J. Math. 44(1992), No. 3, 665-672.

[19] C.C.Xi, On the quasi-heredity of Birman-Wenzl algebras. Preprint, 1999.

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