# Parabolic decomposition for projectively stratified algebras

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**Abstract**

We generalize the machinery of exact Borel subalgebras of quasi-hereditary algebras on projectively stratified algebras.

## 1 Introduction

The representation theories of a block of category $\mathcal{O}$ and of the degree $r$ polynomial representations of the algebraic groups correspond to representation theories of quasi-hereditary algebras, defined in [CPS1]. In these cases the so-called *standard modules* (Verma, resp. Weyl modules) arise via induction from Borel subalgebras. A natural question is whether this induction works at the finite-dimensional algebra level; that is, can we find a subalgebra of the quasi-hereditary algebra such that tensor induction of one-dimensional representations gives the standard modules. Such considerations led S.König to study *exact Borel subalgebras* in quasi-hereditary algebras ([K1, K2, K3]).

In recent years, the study of representations of complex Lie algebras has extended to the study of representations induced from simple (not necessarily finite-dimensional) modules over parabolic subalgebras (see, for example [FKM1] and references therein). Similar to what occurs in category $\mathcal{O}$, there is a block decomposition and the representation theory of each block corresponds to the representation theory of a finite-dimensional algebra. Here the algebras, in general, are no longer quasi-hereditary, but are often so-called *left projectively, or standardly, stratified* ([FKM1, FKM2, FKM3]). Being so closely related to the quasi-hereditary algebras, we naturaly asked which properties are retained by the projectively stratified algebras.

Our goal in this paper is to generalize the results of König on triangular decomposition (or Cartan decomposition) in quasi-hereditary algebras ([K1, K2, K3]). After presenting some preliminaries in Section 2, we give in Section 3 a module-theoretic characterization of projectively stratified algebras in terms of standard and costandard modules analogous to the quasi-hereditary situation. In the case of quasi-hereditary algebras, the existence of standard modules and standard flags of projectives is sufficient to guarantee the existence

*Alexander von Humboldt fellow at Bielefeld University of costandard modules and costandard flags of injectives. In the case of left projectively stratified algebras, it was already shown by Cline, Parshall and Scott ([CPS2]) that the fact that projective modules have standerd flags does not necessarily imply that the injective modules have constandard flags.

Central to the study of triangular decomposition of quasi-hereditary algebras is a semisimple algebra contained in the exact Borel and $\Delta$-subalgebras. In the current case, this must be replaced by a so-called quasi-local algebra which is the direct sum of local algebras; that is, of algebras with a unique simple module. Section 4 is a study of quasi-directed algebras (algebras whose simple modules can be ordered so that $\mathrm{Ext}(L_i, L_j) \neq 0$ implies $L_i \geqslant L_j$) which are projective as left and right modules over their maximal quasi-local algebras. It turns out that this is a convenient condition for the exact Borel and $\Delta$-subalgebras.

Section 6 generalizes the classical Brauer-Humphreys reciprocities. The next Section establishes that the notions of exact Borel and $\Delta$-subalgebras are dual to each other. Section 8 proves the fact that parabolic decomposition, that is, the existence of an exact Borel subalgebra, $B$, and a $\Delta$-subalgebra, $C$ such that $A \simeq C \otimes_S B$ as $C - B$ bimodule and $S$ is the maximal quasi-local subalgebra in both $B$ and $C$, is necessarily projectively stratified and gives a partial converse. In Section 9, we give a method for constructing new projectively stratified algebras from old ones. We finish, in Section 9, with an example from parabolic induction in complex Lie algebras.

# 2 Preliminaries

We let $A$ be a finite dimensional algebra over $\mathbf{k}$, an algebraically closed field. When we want to make clear over which algebra we are taking a module we will give an indication via subscripts. Let $J = AeA$ be a two-sided ideal in $A$, generated by a primitive idempotent $e$. $J$ is called *left projectively stratifying (resp. projectively stratifying)* if it is a projective left (resp. two-sided) $A$-module. If we can order the equivalence classes $e_1, \dots, e_n$ of primitive idempotents of $A$ such that for each $l$ the idempotent $e_l$ generates a left projectively stratifying (resp. projectively stratifying) ideal in the quotient algebra $A/ < e_n, \dots, e_{l+1} >$, then $A$ is called *left projectively stratified (resp. projectively stratified)* (see [FKM1, Section 5]). We will indicate the (left) projectively stratified structure on an algebra, $A$, by the pair $(A, \leqslant)$ with $\leqslant$ the above order on the idempotents. Left stratified algebras have already been studied by Cline, Parshall and Scott ([CPS2]) under the name of *standardly stratified algebras.*

Two lemmas follow immediately from this definition.

**Lemma 1.** *Let $(A, \leqslant)$ be a projectively stratified algebra. Then $(A^{op}, \leqslant)$ is projectively stratified (with the same order on the isoclasses of primitive idempotents).*

**Lemma 2.** *Let $A$ be an algebra with an order $\leqslant$ on the isomorphism classes of primitive idempotents. Then $(A, \leqslant)$ is projectively stratified if and only if both $(A, \leqslant)$ and $(A^{op}, \leqslant)$ are left projectively stratified.*

First, we remark that that the definition of projectively stratified algebra in [FKM1] coressponds to our definition of left projectively startified algebra. As an example of a left projectively stratified algebra which is not projectively stratified, consider the algebra of the quiver $\Gamma$ with two vertices $\{a,b\}$, three arrows $\{\alpha : a \to b,\ \beta : b \to a,\ \gamma : a \to a\}$ and relations $\gamma^2 = 0$, $\gamma\alpha = 0$, $\beta\gamma = 0$, and $\beta\alpha\beta = 0$.

We also note that a left projectively stratified algebra is a stratifying endomorphism algebra in the sense of Cline, Parshall and Scott [CPS2] and all quasi-hereditary algebras are projectively stratified. It is also the case that a left projectively stratified algebra is quasi-hereditary if and only if it has finite global dimension [AHLU, CPS2].

Let $(A, \leqslant)$ be a (left) projectively stratified algebra. In what follows we will denote by $L(\lambda)$ the simple $A$-module which corresponds to $e_\lambda$, and will call $\lambda$ a *weight*. We will also denote by $P(\lambda)$ (resp. $I(\lambda)$) the corresponding projective cover (resp. injective envelope). Following [CPS2], for a simple $A$-module $L$, corresponding to the idempotent $e_\lambda$, we define the *standard* module $\Delta(\lambda)$ as $A/_{\langle e_n,\dots,e_{\lambda+1}\rangle} e_\lambda$ and *costandard* module $\nabla(\lambda)$ as the largest submodule of $I(\lambda)$ having factors $L(\mu)$ with $\mu \leqslant \lambda$. We note that each projective has a *standard flag*, i.e. a filtration whose quotients are standard modules.

Let $(B, \preceq)$ be a finite-dimensional algebra with $\preceq$ a partial order on the set of equivalence classes of simple modules. $(B, \preceq)$ is called *quasi-directed* if $\mathrm{Ext}^k_B(L, L') \neq 0$ for some $k$ implies $L' \preceq L$. By a *quasi-local algebra* we will mean a direct sum of local algebras.

For a quasi-directed algebra $B$ call an indecomposable module $M$ *local* if all its simple composition factors are isomorphic. Call it *projectively local* if it is projective over the maximal quasi-local subalgebra of $B$ (whose existence will be proved in Section 4).

Let $S$ be a quasi-local subalgebra of an algebra $B$. We say $B$ is $S$*-diagonalizable* if $B$ is projective as left and right $S$-module. A quasi-directed algebra diagonalizable over its maximal quasi-local subalgebra will be called *pyramidal*.

**Definition 1.** *Let $(A, \leqslant)$ be a projectively stratified algebra and $B$ and $C$ subalgebras of $A$.*

1. *We will call $B$ an* exact Borel subalgebra *of $A$ if*
   - *there is a one-to-one correspondence between the simples of $B$ and the simples of $A$;*
   - *$(B, \succeq)$ is pyramidal with the opposite order induced from the simples of $A$;*
   - *the tensor induction functor $A \otimes_B \_$ is exact;*
   - *$A \otimes_B \_$ sends the projectively local $B$-module $V$ to the standard $A$-module of the same weight;*
   - *$[A \otimes_B L_B(i) : L_A(i)] = 1$.*
2. *We will call $C$ a* $\Delta$-subalgebra *of $A$ if*
   - *there is a one-to-one correspondence between the simples of $C$ and the simples of $A$;*

- $(C, \preceq)$ *is pyramidal with the order induced from the simples of A;*
- *for each weight $i$ the indecomposable projective $Ae_i$ occurs exactly once in the decomposition of the projective A-module $A \otimes_C Ce_i$;*
- *fixing epimorphisms $\kappa(i) : A \otimes_C Ce_i \to \Delta(i)$, one has isomorphisms $\kappa(i)|_{Ce_i}$ : $1 \otimes Ce_i \xrightarrow{\simeq} \Delta(i)$.*

The importance of the last condition on exact Borel subalgebras can be seen from the example $A = \mathbf{k}[x]/(x^4)$ and $B = \mathbf{k}[x^2]/(x^4) \subset A$. In this example the unique standard objects are the algebras and induction clearly doubles the length of modules.

In the remainder of the paper, set $i \preceq j$ if and only if $i \leqslant j$. The first symbol will indicate quasi-directedness and the second will indicate a projectively stratified structure.

**Definition 2.** *Let $A$ be an algebra and $\leqslant$ be a total order on the set of equivalence classes of simple A-modules. Let $B$ and $C$ be subalgebras of $A$ such that $B \cap C = S$ is a quasi-local subalgebra of $A$ containing at least one representative from the classes of isomorphisms of primitive idempotents, maximal in both $B$ and $C$. Assume that $(B, \succeq)$ and $(C, \preceq)$ are pyramidal. Call $(B, C)$ a* parabolic decomposition *of $A$ provided that the multiplication in $A$ induces isomorphisms $C \otimes_S B \simeq A$ as left C-modules and right B-modules.*

We have chosen this terminology to reflect the fact that the major example we know comes from tensor induction from a parabolic subalgebra of a Lie algebra.

# 3 Module-theoretic characterization of projectively stratified alghebras

A quasi-hereditary algebra $(A, \leqslant)$, can be characterized by the existence of standard modules $\Delta(i)$ having simple subquotients $L(k)$ with $k \leqslant i$ and $L(i)$ occuring once, and such that the projective $P(j)$ has a standard flag with sections $\Delta(i)$ with $j \leqslant i$ among which $\Delta(j)$ occurs exactly once. Left projectively stratified algebras ([CPS2, Section 2.2]) have the same module-theoretic characterization except that the multiplicity of $L(i)$ in $\Delta(i)$ may exceed one.

**Theorem 1.** $(A, \leqslant)$ *is left projectively stratified if and only if there exist modules $\Delta(i)$ such that*

*(i) there is a surjection $\varphi_i : \Delta(i) \to L(i)$ whose kernel has composition factors $L(j)$, $j \leqslant i$.*

*(ii) $P(i)$ surjects onto $\Delta(i)$ and the kernel of this map has a standard flag with sections $\Delta(j)$, $j \nleqslant i$.*

*Proof.* Let $A$ be left projectively stratified and $\Delta(i)$ be the standard modules defined in Section 2. Then $(i)$ is automatically satisfied since the standard module, $\Delta(i)$ is defined over the algebra $A/\langle e_n, \ldots, e_{i+1}\rangle$ and this algebra only has simple modules of the required type. To prove $(ii)$ we proceed by induction. Let $e = e_n$ be the maximal (with respect to $\leqslant$) primitive idempotent of $A$. We have $P(n) = \Delta(n)$ and by [DR, Statment 7] we have a bijection

$$Ae \otimes_{eAe} eA \to AeA.$$

Now $Ae \otimes_{eAe} eA \simeq Ae \otimes_{eAe} \mathrm{Hom}(Ae, A)$ and so if $Ae_i$, $i \neq n$, appears as a direct summand of $AeA$ then there is a surjection of $Ae$ onto $Ae_i$ and this contradicts the indecomposability of $Ae$. This completes the proof in one direction.

To prove the converse, suppose we have standard modules satisfying conditions $(i)$ and $(ii)$. Let $e = e_n$ be the maximal primitive idempotent. Then $Ae = P(n) = \Delta(n)$ and we have a surjection

$$Ae \otimes_{eAe} eA \simeq Ae \otimes_{eAe} \mathrm{Hom}(Ae, A) \to Ae_nA.$$

Now, $Ae \otimes_{eAe} \mathrm{Hom}(Ae, A) \simeq Ae \otimes_{eAe} \oplus_i \mathrm{Hom}(Ae, Ae_i)$. So, the image of $Ae \otimes_{eAe} eA$ is the direct sum of its images in each projective of $A$. We show that the image of $Ae \otimes_{eAe} \mathrm{Hom}(Ae, Ae_i)$ in $P(i)$ is a direct sum of copies of $Ae$. This will follow if we prove the existence of a short exact sequence,

$$0 \to M' = \oplus\Delta(n) \to P(i) \to M \to 0,$$

with $M$ having a standard flag with sections $\Delta(j)$, $j < n$. From the projectivity of $\Delta(n)$ we have that any $N$ with $0 \to N' \to N \to \Delta(n) \to 0$ is isomorphic to $N' \oplus \Delta(n)$ thus by induction on the length of the standard flag we have the result. The image of $Ae \otimes_{eAe} \mathrm{Hom}(Ae, Ae_i)$ in $P(i)$ certainly contains $M'$. Since there are no non-zero maps $\Delta(n) \to M$, $M'$ is this image. So, $AeA$ is a sum of projective modules and, hence, projective and left projectively stratifying. The projectives of $A/AeA$ are the projectives of $A$ modulo the trace of $P(n) = \Delta(n)$ and so have flags satisfying $(ii)$ and with the exception of $n$ all standard modules remain unchanged. By induction in the number of idempotents, we are then done. $\square$

**Corollary 1.** *Let $(A, \leqslant)$ be an algebra. Then $(A, \leqslant)$ is quasi-directed if and only if $(A, \leqslant)$ is left projectively stratified with projective standard modules.*

*Proof.* If $(A, \leqslant)$ is left projectively stratified with projective standard modules then $(A, \leqslant)$ is quasi-directed by definition. Now assume that $(A, \leqslant)$ is quasi-directed. Define $\Delta(i) = P(i)$. Condition $(ii)$ is trivial and condition $(i)$ follows by directedness. $\square$

One can clearly obtain the notion of *right* projectively stratified algebras by requiring right projectivity of the stratifying ideals.

**Corollary 2.** $(A, \leqslant)$ *is right projectively stratified if and only if there exist a choice of costandard modules $\nabla(i)$ such that*

*(i) there is an injection $\varphi_i : L(i) \to \nabla(i)$ whose cokernel has composition factors $L(j)$, $j \leqslant i$.*

*(ii)* $\nabla(i)$ *injects into* $I(i)$ *and the cokernel of this map has a costandard flag with sections* $\nabla(j)$, $j \nleqslant i$.

*Proof.* Pass to the opposite algebra. The functor $\mathrm{Hom}(\_, \mathbf{k})$ carries injectives into projectives and costandard modules into standard modules. Now apply Theorem 1. ☐

**Corollary 3.** $(A, \leqslant)$ *is projectively stratified if and only if there exist a choice of standard modules* $\Delta(i)$ *and costandard modules* $\nabla(i)$ *satisfying necessary conditions from Theorem 1 and Corollary 2.*

# 4 Pyramidal algebras as projectively stratified algebras

¿From the theory of quasi-hereditary algebras we have that an algebra $(A, \leqslant)$ is *directed* (i.e. $\mathrm{Ext}(L, L') \neq 0$ implies $L \nleqslant L'$) if and only if it is quasi-hereditary with projective standard modules. We have already seen (Corollary 1) that the same relationship holds between quasi-directed and left projectively stratified algebras. In this Section, we examine pyramidal algebras; in fact, we prove that all pyramidal algebras are projectively stratified.

**Lemma 3.** *Let* $(B, \preceq)$ *be a quasi-directed algebra. Then the maximal quasi-local subalgebra* $S$ *of* $B$ *is isomorphic to* $\oplus_i e_i B e_i$.

*Proof.* Clearly, $S' = \oplus_i e_i B e_i$ is a subalgebra of $B$. Because of the directedness of $B$ no endomorphisms of $Be_i$ can factor through a non-isomorphic projective, so each $e_i B e_i$ is local, and hence $S'$ is quasi-local. Now let $P$ be an indecomposable summand of $S$. Then $P$ equals $Se_i$ is local and so $e_i S e_i = Se_i$ is a subalgebra of $e_i B e_i$ and hence $S \subset S'$. ☐

**Lemma 4.** *Let* $(B, \preceq)$ *be quasi-directed. Then the projectively local module* $K(i)$ *is isomorphic to* $Be_i/N$*, where* $N$ *is the trace of all* $P(j)$ *with* $j \preceq i$.

*Proof.* Because of the directedness of $B$ we have

$$Be_i = e_i B e_i \oplus \sum_{j<i} e_j B e_i$$

as a vectorspace. For each element of $e_j B e_i$ there is a map for $P(j)$ to $P(i)$ covering it, so $\sum_{j<i} e_j B e_i \subset N$. But $[Be_i : L(i)] = [e_i B e_i : L(i)]$ by directedness of $B$. This completes the proof. ☐

**Proposition 1.** *Let* $(B, \preceq)$ *be a pyramidal algebra. Then* $(B, \leqslant)$ *is projectively stratified.*

*Proof.* $(B, \leqslant)$ is left projectively stratified by Corollary 1. Now consider $B^{op}$. Then the algebra $(B^{op}, \succeq)$ is quasi-directed. This implies that a left projective $B^{op}$-module $P(i)$ has only $L(j)$ with $i \leqslant j$ as composition subquotients. Since $B^{op}$ is pyramidal, each $P(i)$ has a projectively local flag, whose subquotients satisfy the same order condition. Let $S$ be the maximal quasi-local subalgebra of $B$. Choose for $B^{op}$, $\Delta(i) = e_i S^{op} e_i$. By Lemma 4 these are the projectively local modules. Then these standards clearly satisfy conditions $(i)$ and $(ii)$ of Theorem 1, and so $(B^{op}, \leqslant)$ is left projectively stratified. And now, by Lemma 2, $(B, \leqslant)$ is projectively stratified. □

Thus, a pyramidal algebra $(B, \preceq)$ has both $(B, \leqslant)$ and $(B, \geqslant)$ projectively stratified structures. To finish this Section we give necessary and sufficient conditions for a quasi-directed algebra to be projectively stratified. We begin with the following Lemma.

**Lemma 5.** *Let $e$ be a primitive idempotent in a quasi-directed algebra $(B, \preceq)$ and $X$ an $eAe$-module. Then $X \otimes_{eBe} eB$ is right $B$-projective if and only if $X$ is right $eBe$-projective.*

*Proof.* If $X_{eBe}$ is projective, then $X$ is free over $eBe$ and hence $X \otimes_{eBe} eB$ is right $B$-projective. On the other hand, suppose $X \otimes_{eBe} eB$ is right $B$-projective. Let $L$ be the maximal local top of $eB$. $X$ has top $(\hat{L})^n$, where $\hat{L}$ is the unique simple $eBe$-module. We have an exact sequence

$$0 \to T \xrightarrow{\varphi} (eBe)^n \to X \to 0.$$

Inducing to $B$ we have

$$T \otimes_{eBe} eB \xrightarrow{\beta} (eBe)^n \otimes_{eBe} eB \xrightarrow{\alpha} X \otimes_{eBe} eB \to 0.$$

Since $X$ surjects on $(\hat{L})^n$, we have the following chain of surjections

$$X \otimes_{eBe} eB \to (\hat{L} \otimes_{eBe} eB)^n \to (\hat{L} \otimes_{eBe} L)^n \to L^n.$$

$X \otimes_{eBe} eB$ is projective and must contain, as a direct summand, the projective cover, $(eB)^n$, of $L^n$. This implies that $\alpha$ is an isomorphism, $\beta = 0$ and, last, $\beta \cdot e = 0$. But $\beta \cdot e = \varphi : T \otimes_{eBe} eBe \to (eBe)^n \otimes_{eBe} eBe$ is non-zero if $X$ is not projective. This contradiction proves the Lemma. □

And this Lemma allows us to give the following characterization of when a quasi-directed algebra is projectively stratified.

**Proposition 2.** *Let $(B, \preceq)$ be quasi-directed. Then the following conditions are equivalent.*

*(i) $B$ is projectively stratified.*

*(ii) the maximal quasi-local subalgebra $S$ of $B$ is an exact Borel subalgebra.*

*(iii) $Be$ is $eBe$-projective for every primitive idempotent $e$.*

*Proof.* ($(i) \Leftrightarrow (iii)$) By [DR, Statment 7] we have $Be \otimes_{eSe} eB \simeq BeB$, where $e$ is the maximal primitive idempotent. So $BeB$ is right $B$-projective if and only if $Be$ is right $eSe$-projective (Lemma 5).

($(ii) \Rightarrow (iii)$) Since $B \otimes_S \_$ is exact, $B$ is a right flat, hence right projective ([F, 11.31]) $S$-module. Then $Be$ is a projective $S$ module and hence projective $eSe$-module.

($(iii) \Rightarrow (ii)$) We have $B \otimes_S \_ = \oplus_i Be_i \otimes_{e_i S e_i} \_$ and hence exact since each $Be_i$ is $e_i S e_i$-projective. Now we just need to prove that this functor carries $\Delta_S(i) = e_i S e_i$ to $\Delta_B(i)$. Set $e = e_i$. We have $B \otimes_S eSe = B \otimes_{eSe} eSe = Be \otimes_{eSe} eSe = Be = \Delta_B(i)$. □

# 5 Brauer-Humphreys reciprocities

We will now prove an appropriate generalization of the classical Brauer-Humphreys reciprocities; this will be the refinement of [ADL, Theorem 2.5] in our situation. As is usual, $[M : \Delta(i)]$ (resp. $[M : \nabla(i)]$) is the number of occurrences of $\Delta(i)$ (resp. $\nabla(i)$) in a standard (resp. costandard) flag of $M$, should it exist. Similarly, for a simple $L$, $[M : L]$ means the corresponding composition multiplicity.

**Theorem 2.** *Let $(A, \leqslant)$ be a projectively stratified algebra. Then for any pair of weights $i$ and $j$ we have*

$$\dim_{\mathbf{k}}(\mathrm{End}(\Delta(j)))[P(i) : \Delta(j)] = [\nabla(j) : L(i)],$$
$$\dim_{\mathbf{k}}(\mathrm{End}(\nabla(j)))[I(i) : \nabla(j)] = [\Delta(j) : L(i)].$$

*Proof.* By duality it is sufficient to prove the first. By induction it is sufficient to prove it for maximal $j = n$. In this case $P(i) = Ae_i$, $P(n) = \Delta(n) = Ae_n$ and $I(n) = \nabla(n)$. Set $l = [P(i) : \Delta(n)]$. From the proof of Theorem 1, this is equal to

$$l = \frac{\dim_{\mathbf{k}} \mathrm{Hom}_A(P(n), P(i))}{\dim_{\mathbf{k}} \mathrm{End}_A(P(n))};$$

that is $Ae_i \cap Ae_n A = (Ae_n)^l$. So, it remains to show that $\dim_{\mathbf{k}} \mathrm{Hom}_A(P(n), P(i)) = [I(n) : L(i)]$. Passing to the opposite algebra we have

$$\begin{aligned}\dim_{\mathbf{k}} \mathrm{Hom}_A(P(n), P(i)) &= \dim_{\mathbf{k}}(e_n A e_i) = \dim_{\mathbf{k}}(e_i A^{op} e_n)\\ &= \dim_{\mathbf{k}} \mathrm{Hom}_A(A^{op} e_i, A^{op} e_n) = [A^{op} e_n : L(i)] = [I(n) : L(i)].\end{aligned}$$

□

And as corollary we get a weakened form of [FKM1, Theorem 4].

**Corollary 4.** *Assume that $A$ is projectively stratified and has a duality (i.e. a contravariant exact equivalence on the category of $A$-modules, which preserves simples). Then*

$$\dim_{\mathbf{k}}(\mathrm{End}(\Delta(j)))[P(i) : \Delta(j)] = [\Delta(j) : L(i)].$$

*Proof.* The duality carries standards to costandards and we have $[\Delta(i) : L(j)] = [\nabla(i) : L(j)]$, which completes the proof. □

# 6 Duality between exact Borel and $\Delta$-subalgebras

In this Section we explore the left-right symmetry of projectively stratified algebras and their exact Borel and $\Delta$-subalgebras. For the remainder of the paper we will assume that the projectively stratified structures on exact Borel and $\Delta$-subalgebras are given by the same order on the idempotents of the algebra. With this projectively stratified structure the fourth condition of an exact Borel subalgebra can be rephrased by: tensor induction carries standard modules to standard modules.

**Lemma 6.** *Let* $(A, \leqslant)$ *be a projectively stratified algebra. Then*

$$\mathrm{Ext}_A^k(\Delta(i), \nabla(j)) = 0$$

*unless* $k = 0$ *and* $i = j$.

*Proof.* Assume $k > 0$. Let $m$ be the maximum of $i$ and $j$. Put $e = \sum_{s=m+1}^{n} e_s$ and put $A' = A/AeA$. Over this algebra $\Delta(m)$ is projective and $\nabla(m)$ injective. Thus we have for $k \neq 0$ $\mathrm{Ext}_A^k(\Delta(i), \nabla(j)) = \mathrm{Ext}_{A'}^k(\Delta(i), \nabla(j)) = 0$ ([CPS2, 2.1.2]). If $k = 0$, $i \neq j$ then the image of any map $\varphi : \Delta(i) \to \nabla(j)$ must have $L(i)$ as top and other composition factors $L(j)$, $j \leqslant i$, since it is a quotient of $\Delta(i)$. If this map is non-zero then $L(i)$ is a composition factor of $\nabla(j)$ that is $i < j$. But then every submodule of $\nabla(j)$ has composition factor $L(j)$ which forces $j < i$. □

Let $(B, \succeq)$ be an exact Borel subalgebra of a projectively stratified algebra $(A, \leqslant)$. In the case that $A$ is quasi-hereditary the standard objects are induced from simple $B$-modules. In the general case, this no longer holds; however, these modules, $\tilde{\Delta}_A(i) = A \otimes_B L_B(i)$, which we will call *substandard modules*, continue to play an important role (see also [AHLU]). In particular, $\Delta(i)$ has a $\tilde{\Delta}(i)$-flag.

**Lemma 7.** *Let* $(A, \leqslant)$ *be a projectively stratified algebra with an exact Borel subalgebra* $B$. *Then*

$$\mathrm{Ext}_A^k(\tilde{\Delta}(i), \nabla(j)) = 0,$$

*unless* $k = 0$ *and* $i = j$.

*Proof.* For $k = 0$, $\tilde{\Delta}(i)$ is an image of $\Delta(i)$ and the statement follows from Lemma 6. Consider $k \neq 0$. Let $l$ be the maximal of $i$ and $j$. Then $\tilde{\Delta}(i)$ and $\nabla(j)$ are modules over $A/\langle e_{k+1}, \dots, e_n\rangle$ and so (by [CPS2, 2.1.2]) we may assume that $l = n$. If $j = n$, then $\nabla(j)$ is injective and the current Lemma clearly holds. So now assume $i = n$ and $j < i$ and consider the exact sequence

$$0 \to N \to \Delta(i) \to \tilde{\Delta}(i) \to 0.$$

Apply $\mathrm{Hom}_A(\_, \nabla(j))$ and pass to the long exact sequence. We get $\mathrm{Ext}_A^1(\tilde{\Delta}(i), \nabla(j)) = 0$ and $\mathrm{Ext}_A^{l+1}(\tilde{\Delta}(i), \nabla(j)) \simeq \mathrm{Ext}_A^l(N, \nabla(j))$. And the Lemma follows from the standard dimension shift arguments. □

**Lemma 8.** *Let* $(B, \succeq)$ *be an exact Borel subalgebra of a projectively stratified algebra,* $(A, \leqslant)$. *Then for all weights,* $\dim \mathrm{End}_A(\Delta_A(\lambda)) = \dim \mathrm{End}_B(\Delta_B(\lambda))$.

*Proof.* We prove the Lemma by induction. Let $n$ be the maximal weight. We have

$$\dim \mathrm{End}_X(\Delta_X(n)) = \dim \mathrm{Hom}_X(P_X(n), \Delta_X(n)) = [\Delta_X(n) : L_X(n)]$$

for both $X = A$ or $X = B$. By the last condition for an exact Borel subalgebra we have $[\Delta_A(n) : L_A(n)] = [\Delta_B(n) : L_B(n)]$ and the statment follows for the maximal weight. Induction is clear. □

**Proposition 3.** *Let $(A, \leqslant)$ be a projectively stratified algebra and $(B, \succeq)$ a pyramidal subalgebra of $A$ with the same poset of isoclasses of primitive idempotents. Then $B$ is an exact Borel subalgebra of $A$ if and only if for each weight $i$, restriction from $A$ to $B$ induces an isomorphism $\nabla_A(i) \simeq \nabla_B(i)$ as $B$-modules.*

*Proof.* Assume that $B$ is an exact Borel subalgebra of $A$. We compare dimensions of $\nabla_A(i)$ and $\nabla_B(i)$. Using Theorem 2 we have

$$\dim_{\mathbf{k}} \nabla_X(i) = \sum_j \dim_{\mathbf{k}} L_X(j)[\nabla_X(i) : L_X(j)] =$$
$$= \sum_j \dim \mathrm{End}(\Delta_X(i)) \dim_{\mathbf{k}} L_X(j)[P_X(j) : \Delta_X(i)] = \dim \mathrm{End}(\Delta_X(i))[X : \Delta_X(i)]$$

for both $X = A$ and $X = B$. By the exactness of induction, $[A : \Delta_A(i)] = [B : \Delta_B(i)]$. By Lemma 8, $\dim \mathrm{End}(\Delta_A(i)) = \dim \mathrm{End}(\Delta_B(i))$ and so $\dim_{\mathbf{k}} \nabla_A(i) = \dim_{\mathbf{k}} \nabla_B(i)$.

¿From Lemma 7 it follows that the functor $\mathrm{Hom}_A({}_-, \nabla_A(j))$ is exact on $A$-modules having a substandard flag. Thus $\mathrm{Hom}_B({}_-, \nabla_A(j))$ is exact on the category of $B$-modules. So, $\nabla_A(j)$ containing $\nabla_B(j)$ is an injective $B$-module. The previous dimension count says that they are, in fact, equal.

Now assume that for each weight $i$, restriction from $A$ to $B$ induces an isomorphism $\nabla_A(i) \simeq \nabla_B(i)$ as $B$-modules. Since $(B, \succeq)$ is pyramidal, $(B, \leqslant)$ is projectively stratified by Lemma 1 with injective costandard modules (Corollary 1 and Corollary 3). We want to prove that $A$ is right projective over $B$ implying $A \otimes_B {}_-$ is exact. The right standard modules for $A$ (and both right standard and projective over $B$) are $\nabla_A(i)^*$, so, as a right projective $A$-module, $A$ has a $\nabla_A(i)^*$-flag, the last being a direct sum decomposition over $B$.

We are finished when we show that $A \otimes_B {}_-$ sends projectively local $B$-modules to standard $A$-modules. Let $K(i)$ be a projectively local $B$-module corresponding to the weight $i$. First we show $\dim_{\mathbf{k}}(A \otimes_B K(i)) = \dim_{\mathbf{k}} \Delta_A(i)$. We have $d := \dim_{\mathbf{k}}(A \otimes_B K(i)) = \dim_{\mathbf{k}}(\mathrm{Hom}_B(K(i), A^*))^*$. Since $A$ is a right projective $B$-module, $A^*$ is left injective and as such decomposes into a direct sum of injective $B$-modules, $\nabla_B(i)$. On each summand we have

$$\dim_{\mathbf{k}} \mathrm{Hom}_B(K(i), \nabla_B(j)) = \begin{cases} 0, & i \neq j; \\ \dim_{\mathbf{k}} \mathrm{End}_B(K(i)), & i = j; \end{cases}$$

this follows from the fact that $\nabla_B(i) = (e_i B)^*$ implies $[\nabla_B(i) : L_B(i)] = [K(i) : L_B(i)]$. Now $\dim \mathrm{End}_B(K(i)) = [K(i) : L(i)]$ because $K(i)$ is projective in the category of $B$-modules filtered by $L_B(i)$. Thus $d$ equals $\dim_{\mathbf{k}} \mathrm{End}_B(K(i)) \cdot [A^* : \nabla_B(i)] = \dim_{\mathbf{k}} \mathrm{End}_B(K(i)) \cdot [A^* : \nabla_A(i)]$. Further, $\dim_{\mathbf{k}} \mathrm{End}_A(\nabla_A(i)) = \dim_{\mathbf{k}} \mathrm{End}_{A^{op}}(\Delta_{A^{op}}(i)) = \dim_{\mathbf{k}} \mathrm{End}_{B^{op}}(K(i)^*) = \dim_{\mathbf{k}} \mathrm{End}_B(K(i))$ by Lemma 8. Applying Brauer-Humphreys reciprocity we get

$$d = \dim_{\mathbf{k}} \mathrm{End}_B(K(i)) \cdot [A^* : \nabla_A(i)] =$$
$$= \sum_j \dim_{\mathbf{k}} \mathrm{End}_A(\nabla_A(i)) \cdot [I_A(j) : \nabla_A(i)] \cdot \mathrm{mult}_{A^*}(I_A(j)) =$$
$$= \sum_j [\Delta_A(i) : L_A(j)] \cdot \mathrm{mult}_A(P_A(j)) = \dim_{\mathbf{k}}(\Delta_A(i)).$$

¿From the quasi-directedness of $B$ and adjunction we have that $A \otimes_B P_B(i)$ has $P_A(i)$ as a direct summand exactly once and all other direct summands (if any) are of the form $P_A(j)$, $j > i$. So, for the largest weight we have $\Delta_A(n) = P_A(n) = A \otimes_B P_B(n) = A \otimes_B K(n)$. Now we proceed by induction. We have an exact sequence:

$$0 \to V \to P_B(i) \to K(i) \to 0$$

with $V$ filtered by $K(j)$, $j > i$ because $\Delta_B(j) = K(j)$. By exactness of $A \otimes_B \_$ we obtain

$$0 \to A \otimes_B V \to A \otimes_B P_B(i) \to A \otimes_B K(i) \to 0,$$

and by the inductive hypothesis, $A \otimes_B V$ is filtered by $\Delta_A(j)$ with $j > i$. Now, since $P_A(i)$ occurs as a summand of $A \otimes_B P_B(i)$, there is a surjection of $A \otimes_B P_B(i)$ onto $\Delta_A(i)$ and the kernel $V'$ is the the sum of the images of all possible maps from $P_A(j)$ for $j > i$. Hence $V \subset V'$ and thus $A \otimes_B K(i)$ surjects onto $\Delta_A(i)$ and the isomorphism follows from the dimension count. □

**Corollary 5.** *$B$ is an exact Borel subalgebra of a projectively stratified algebra $(A, \leqslant)$ if and only if $B^{op}$ is a $\Delta$-subalgebra of $(A^{op}, \leqslant)$.*

*Proof.* Follows from Proposition 3, its proof and the duality of the conditions for a $\Delta$-subalgebra and the equivalent conditions for an exact Borel subalgebra. □

# 7 Projectively stratified structure of algebras with parabolic decomposition

In this Section we prove a Theorem relating parabolic decomposition to projectively stratified algebras. It generalizes the corresponding result for quasi-hereditary algebras [K2, Theorem 4.1]. The proof closely follows the ideas of the proof there.

**Theorem 3.** *Let $A$ be a finite-dimensional algebra and $\leqslant$ be a total order on the set of isomorphism classes of simple $A$-modules. Assume that $(B, \succeq)$ and $(C, \preceq)$ are pyramidal basic subalgebras, whose intersection $B \cap C = S$ is the maximal quasi-local subalgebra of both $B$ and $C$. The following statments are equivalent.*

*(i) The algebra $A$ is projectively stratified with an exact Borel subalgebra $B$ and a $\Delta$-subalgebra $C$.*

*(ii)* $(B, C)$ *is a parabolic decomposition of* $A$.

*Proof.* Assume we have listed the idempotents in $A$ (and hence in $B$ and $C$) with respect to the natural total order.

($(i) \Rightarrow (ii)$) We proceed by induction on the number of direct summands in $S$ (the number of weights). If $S$ is local, then $A = B = C = S$ and we are done. Assume $S$ is not local and $e = e_n$ is the maximal primitive idempotent in $A$. Since $A$ is projectively stratified, $AeA$ is a projectively stratifying ideal and hence is projective, as left $A$-module. In particular, by [DR, Statement 7], we have that the multiplication in $A$ induces a bijection

$$Ae \otimes_{eAe} eA \to AeA.$$

We also have $eAe = eSe$, since $e$ is the maximal primitive idempotent. We have the identifications

$$\Delta_A(n) \simeq \Delta_C(n) \simeq Ce$$

by the definition of $\Delta$-subalgebra and Corollary 1, and

$$\nabla_A(n)^* \simeq \nabla_B(n)^* \simeq eB$$

by dual arguments (Corollary 5). We get a bijection

$$Ce \otimes_{eSe} eB \simeq Ae \otimes_{eAe} eA \simeq AeA,$$

compatible with left $C$ and right $B$ multiplication. Continuing by induction (see arguments in [K2, Theorem 4.1]) we see that $\mathbf{k}$-dimensions of both sides of

$$C \otimes_S B \to A$$

are equal. We are done.

($(ii) \Rightarrow (i)$) Let $e = e_n$ be the maximal primitive idempotent. We wish to show that $AeA$ is projectively stratifying. Since $B$ and $C$ are quasi-directed, we have $eC = eCe = eSe$ and $Be = eBe = eSe$ are projectively local modules. First we show that $eSe = eAe$ (this says $\mathrm{End}_A(\Delta_A(n)) \simeq \mathrm{End}_C(\Delta_C(n))$). We have

$$eAe \simeq eC \otimes_S Be \simeq eSe \otimes_S eSe \simeq eSe \otimes_{eSe} eSe \simeq eSe.$$

Analogously,

$$Ae \simeq C \otimes_S Be \simeq C \otimes_S eSe \simeq C \otimes_{eSe} eSe \simeq Ce$$

and

$$eA \simeq eC \otimes_S B \simeq eSe \otimes_S B \simeq eSe \otimes_{eSe} B \simeq eB.$$

Now, since $B$ and $C$ are pyramidal they are projective over $S$ both as left and right modules. So $Ae = Ce$ (resp. $eB = eA$) is a right (resp. left) projective $eSe$-module. Thus they are free over $eSe$ and so $Ae \otimes_{eSe} eA$ is projective as left and right $A$-module. And the Theorem follows from standard induction. $\square$

# 8 Construction of algebras with given parabolic decomposition

In this Section we give a general construction of a projectively stratified algebra having given exact Borel and $\Delta$-subalgebras. The central Theorem of this Section allows us to construct a projectively stratified algebra as an extension of a projectively stratified algebra with a pyramidal algebra. In the case that the projectively stratified algebra has a parabolic decomposition, the extension will as well.

To state the Theorem we assume the following set-up: Given a semi-local algebra $S$ by an $S$-*algebra* we will mean an algebra $T_S(M)/I$, where $M$ is an $S$-bimodule, finite-dimensional over $\mathbf{k}$. Let $(A, \leqslant)$ be a basic projectively stratified algebra with an exact Borel subalgebra $B$. Let $(D, \preceq)$ be a basic pyramidal algebra. Assume that $B$ and $D$ have isomorphic maximal quasi-local subalgebras, $S$ (in particular, they have the same set of idempotents). To fix notation then $A \simeq T_S(M_A)/I_A$, $B \simeq T_S(M_B)/I_B$ and $D \simeq T_S(M_D)/I_D$. Let $A' = T_S(M_A \oplus M_D)/(I_A + I_D + \langle a \otimes_S d \,|\, a \in M_A, d \in M_D\rangle)$.

**Theorem 4.** *Let $A$, $B$, $D$ and $A'$ be defined as above. $A'$ is projectively stratified and isomorphic to $D \otimes_S A$ as left $D$-module and right $A$-module. $B$ is an exact Borel subalgebra of $A'$ via the embedding $b \mapsto 1 \otimes b$. If $C \simeq T_S(M_C)$ is a $\Delta$-subalgebra of $A$ containing $S$, then $A'$ has a $\Delta$-subalgebra $C' \simeq T_S(M_C \oplus M_D)/(I_C + I_D + \langle c \otimes_S d \,|\, c \in M_C, d \in M_D\rangle)$. Last, $C' \simeq D \otimes_S C$ as left $D$-module and right $C$-module.*

*Proof.* By construction, we have appropriate module isomorphisms: $A' \simeq D \otimes_S A$ (and when relevant $C' \simeq D \otimes_S C$). Let $e$ be the maximal primitive idempotent. Then $e = e \otimes_S e$ ($= 1 \otimes_S e$). So, $J = A'eA' = D \otimes_S C \cdot e \otimes_S e \cdot D \otimes_S C = D \otimes_S AeA$. Since $D$ (resp. $AeA$) is a two-sided projective $D$ (resp. $A$)-module, $J$ is a two-sided projective $A'$-module and $A'$ is projectively stratified by induction.

We prove $B$ is an exact Borel subalgebra of $A'$. To begin, $D \otimes_S A \otimes_B \_$ is exact, since $A \otimes_B \_$ is exact and $D$ is pyramidal and hence flat over $S$. By induction, $D \otimes_S A \otimes_B \_$ sends standard $B$-modules to standard $A'$-modules. Indeed, let $e$ be the maximal primitive idempotent. $D \otimes_S A \otimes_B Be \simeq D \otimes_S Ae \simeq (D \otimes_S A)(1 \otimes_S e)$, which is $A'$-standard. It remains to show that $[A' \otimes_B L_B(i) : L_{A'}(i)] = 1$. Now $A' \otimes_B L_B(i) = D \otimes_S A \otimes_B L_B(i)$. Since $B$ is an exact Borel subalgebra of $A$, we have $[A \otimes_B L_B(i) : L_A(i)] = 1$ and $[A \otimes_B L_B(i) : L_A(j)] \neq 0$ implies $j \leqslant i$. We are done, if we show $[D \otimes_S L_A(j) : L_{A'}(j)] = 1$ and $[D \otimes_S L_A(j) : L_{A'}(k)] \neq 0$ implies $k \leqslant j$, for this would clearly imply that $[A' \otimes_B L_B(i) : L_{A'}(i)] = 1$. But $D \otimes_S L_A(j) \simeq \sum_{k \leqslant j} e_k D e_j \otimes_S L_A(j)$ and $[\sum_{k \leqslant j} e_k D e_j \otimes_S L_A(j) : L_S(m)] \neq 0$, $m \leqslant j$ and so $[D \otimes_S L_A(j) : L_{A'}(k)] \neq 0$ implies $k \leqslant j$. Further, $[\sum_{k \leqslant j} e_k D e_j \otimes_S L_A(j) : L_S(j)] = 1$ implies $[D \otimes_S L_A(j) : L_{A'}(j)] = 1$. The remaining statements follow arguments already seen. ☐

**Corollary 6.** *In the case that $C = A$ we have that $A' \simeq C \otimes_S B$ is a parabolic decomposition.*

# 9 Parabolic decomposition of projectively stratified algebras attached to blocks of $\mathcal{O}(\mathcal{P}, \Lambda)$

We have mentioned that our motivation stems from the representation theory of complex Lie algebras and algebraic groups. In this Section we give a parabolic decomposition for algebras arising from generalizations of classical category $\mathcal{O}$. We will work with projectively stratified algebras whose module category is equivalent to a block of the category $\mathcal{O}(\mathcal{P}, \Lambda)$, studied in [FKM2]. Let us give an overview of the set-up.

Let $\mathfrak{A}$ denote the Lie algebra $sl(2, \mathbb{C})$ with a fixed root basis $e = X_\alpha$, $f = X_{-\alpha}$, $h = H_\alpha$, where $\alpha$ is the root of $\mathfrak{A}$. For any $\gamma \in \mathbb{C}$ and $\lambda \in \mathbb{C}/2\mathbb{Z}$ there is a unique weight $\mathfrak{A}$-module, $V(\lambda, \gamma)$, satisfying the following conditions ([FM1]):

1. $\lambda$ is the support of $V(\lambda, \gamma)$ and all weight spaces of $V(\lambda, \gamma)$ are one-dimensional,
2. $\gamma$ is the unique eigenvalue of the Casimir operator, $\mathfrak{c} = (h + 1)^2 + 4fe$, on $V(\lambda, \gamma)$,
3. $V(\lambda, \gamma)$ is *admissible*, i.e. $f$ acts bijectively.

Let $\tilde{\Lambda} = \tilde{\Lambda}(V(\lambda, \gamma))$ denote the full subcategory of $\mathfrak{A}$-modules having as objects all admissible submodules and quotients of modules of the form $V(\lambda, \gamma) \otimes F$, as $F$ varies over all finite-dimensional $\mathfrak{A}$-modules. It has been shown ([FKM2, Section 2]) that $\tilde{\Lambda}$ is an abelian category.

Now let $\mathfrak{G}$ be a complex simple finite-dimensional Lie algebra and $\mathcal{P}$ a parabolic subalgebra of $\mathfrak{G}$, such that $\mathcal{P}$ has Levi factor $\mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$ ($\mathfrak{A}$ as above) and nilpotent radical $\mathfrak{N}$. Construct the full subcategory $\Lambda = \Lambda(V(\lambda, \gamma))$ of the category of $\tilde{\mathfrak{A}} = \mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$-modules satisfying:

1. any $M \in \Lambda$ belongs to $\tilde{\Lambda}$, when viewed as an $\mathfrak{A}$-module,
2. any $M \in \Lambda$ is $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable,
3. for any $M \in \Lambda$ and any $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable finite-dimensional $\tilde{\mathfrak{A}}$-module, $F$, the module $M \otimes F$ decomposes into a direct sum of indecomposable modules from $\Lambda$.

Define $\mathcal{O}(\mathcal{P}, \Lambda)$ to be the full subcategory of the category of $\mathfrak{G}$-modules, whose objects are finitely generated and $\mathfrak{N}$-finite $\mathfrak{G}$-modules, which decompose into a direct sum of modules from $\Lambda$, when wieved as $\tilde{\mathfrak{A}}$-modules. By [FKM1, Section 4], $\mathcal{O}(\mathcal{P}, \Lambda)$ has a block decomposition (with finitely many simples in each block) and by [FKM1, Section 10], this decomposition can be choosen such that each block is equivalent to the module category over a left projectively stratified finite-dimensional algebra. By [FKM1, Section 12], $\mathcal{O}(\mathcal{P}, \Lambda)$ has a duality (see Corollary 4). Thus by Corollary 2, these algebras are also right projectively stratified and hence projectively stratified. The main result of this Section is the following.

**Theorem 5.** *For every block of $\mathcal{O}(\mathcal{P}, \Lambda)$ there is an algebra, $A$, with parabolic decomposition, whose module category is equivalent to this block.*

We will prove this by explicit construction of $A$ and its exact Borel and $\Delta$-subalgebras. We require more terminology. For the rest of the section we fix a block, $\mathcal{O}_i$, of $\mathcal{O}(\mathcal{P},\Lambda)$ assumed to have finitely many simples.

Let $\mathcal{Q}_+$ be the set of positive roots of $\mathfrak{G}$. $\mathcal{P}$ uniquely identifies a copy of $\mathfrak{A}$ in $\mathfrak{G}$. Assume that the root $\alpha$ of $\mathfrak{A}$ is contained in $\mathcal{Q}_+$. A weight, $\lambda$, of a weight $\mathfrak{G}$-module $V$ will be called an *$\alpha$-highest weight* provided that $\lambda+\beta$ is not a weight of $V$ for any $\beta\in\mathcal{Q}_+\setminus\{\alpha\}$. For a weight $\lambda$ and a weight $\mathfrak{G}$-module $V$ set $V[\lambda]=\oplus_{k\in\mathbb{Z}}V_{\lambda+k\alpha}$, which is closed under the action of $\mathfrak{A}$.

It is known that indecomposable modules from $\Lambda$ have the form $V(a,b)$ or $\tilde{V}(a,b)$, which is a self-extension of $V(a,b)$ (see [FKM1, Section 10]). For simple $V\in\Lambda$ denote by $\hat{V}$ its projective cover (in $\Lambda$), which is either $V$ itself or its self-extension. Given $M\in\mathcal{O}_i$, a weight $\lambda$ and a $b\in\mathbb{C}$ set

$$M_{\lambda,b}=\{v\in M_\lambda\,|\text{ there exists } k\in\mathbb{N} \text{ such that } (\mathfrak{c}-b)^k v=0\}.$$

Since, as an $\mathfrak{A}'$-module, $M$ decomposes into a direct sum of objects from $\Lambda$, one has that $M_{\lambda,b}=\{v\in M_\lambda\,|\,(\mathfrak{c}-b)^2v=0\}$. For a simple module $L\in\mathcal{O}_i$ denote $\mu_L$ its $\alpha$-highest weight. Then $L[\mu_L]\simeq V(a_L,b_L)$ for some $a_L$ and $b_L$. Put $V_L=V(a_L,b_L)$.

**Lemma 9.** *Let $L$ be a simple module in $\mathcal{O}_i$. There exists a projective module $P^L$ such that* $\mathrm{Hom}_{\mathfrak{G}}(P^L,M)\simeq M_{\mu_L,b_L}$ *for any $M\in\mathcal{O}_i$.*

*Proof.* Set $\mu=\mu_L$, $a=a_L$ and $b=b_L$. We can pick $k\in\mathbb{N}$ big enough such that $\mathfrak{N}^kM_\mu=0$ for any $M\in\mathcal{O}_i$ and consider the $\mathfrak{G}$-module

$$\hat{P}^L=U(\mathfrak{G})\bigotimes_{U(\mathcal{P})}\left((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k))\otimes\hat{V}_L\right).$$

Now we can take $P^L$ to be the $\mathcal{O}_i$-projection of $\hat{P}^L$. That $\mathrm{Hom}_{\mathfrak{G}}(P^L,M)\simeq M_{\mu,b}$ for any $M\in\mathcal{O}_i$ is identical to the classical argument in [BGG, Theorem 2]. □

Since the $\alpha$-highest weight of $L$ is unique up to shifts by $\alpha$, $P^L$ is independent of the choice of this $\alpha$-highest weight. Now we take

$$A=\mathrm{End}_{\mathfrak{G}}\left(\bigoplus_{L\text{ simple in }\mathcal{O}_i}P^L\right).$$

Clearly, the category of $A$-modules is equivalent to $\mathcal{O}_i$.

Let $\mathfrak{G}=\mathfrak{N}_-\oplus\mathfrak{H}\oplus\mathfrak{N}_+$ be the standard triangular decomposition of $\mathfrak{G}$ and $\mathfrak{N}^-$ be the image of $\mathfrak{N}$ under the Chevalley involution. Consider an $\mathfrak{A}'$-submodule $V^L=1\otimes 1\otimes\hat{V}_L$ in $P^L$ and set $M_L=U(\mathfrak{N}_-)V^L$. Let $\Delta(L)$ denote the standard module associated with $L$ ([FKM1, Section 3]). We have $\Delta(L)\simeq U(\mathfrak{G})\otimes_{U(\mathcal{P})}\hat{V}_L$.

For any simple $L\in\mathcal{O}_i$ fix the canonical generator, $p(L)=1\otimes 1\otimes v$, of $P^L$, where $v$ is a canonical generator of $\hat{V}_L$; then the map $\varphi\mapsto\varphi(p(L))$ is a $\mathbb{C}$-isomorphism between $\mathrm{Hom}_{\mathfrak{G}}(P^L,M)$ and $M_{\mu_L,b_L}$ for any $M\in\mathcal{O}_i$.

**Lemma 10.** *Any surjection $P^L \to \Delta(L)$ induces an $U(\mathfrak{N}_-)$-isomorphism $M_L \to \Delta(L)$.*

*Proof.* Let $\varphi : P^L \to \Delta(L)$ be a surjection. It carries $p(L)$ to a generator of $\Delta(L)$ and hence it induces an $X_{-\alpha}$-isomorphism $M_L[\mu_L] \to \Delta(L)[\mu_L]$. Now the statment follows from the fact that $U(\mathfrak{N}_-) = U(\mathfrak{N}^-) \otimes_{\mathbb{C}} U(X_{-\alpha})$ and that both $M_L$ and $\Delta(L)$ are $U(\mathfrak{N}^-)$-free ([FM2, Proposition 1]). $\square$

**Lemma 11.** *$M_L$ is an $\mathfrak{A}'$-module.*

*Proof.* Follows from the construction of $P^L$ and the definition of $M_L$. $\square$

**Lemma 12.** *Assume that $\varphi : P^{L_j} \to P^{L_k}$ is a homomorphism and $\varphi(p(L_j)) \in M_{L_k}$. Then $\varphi(M_{L_j}) \subset M_{L_k}$.*

*Proof.* By definition, $p(L_j)$ generates $V^{L_j}$ as $\mathfrak{A}'$-module. Since $\varphi$ is an $\mathfrak{A}'$-homomorphism, Lemma 11 says that $\varphi(V^{L_j}) \subset M^{L_k}$ and the statment follows from the fact that $M_{L_j} = U(\mathfrak{N}_-)V^{L_j}$ and the fact that $M_{L_k}$ is stable under left $U(\mathfrak{N}_-)$-multiplication. $\square$

*Proof of Theorem 5.* First we prove the existence of a $\Delta$-subalgebra in $A$. Denote by $I$ an indexing poset of simple modules in $\mathcal{O}_i$. Put

$$C = \bigoplus_{j,k \in I} \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_k}) \mid \varphi(p(L_j)) \in M_{L_k}\}.$$

$C \subset A$ is a vectorspace, which is a subalgebra by Lemma 12. By Lemma 10, $C$ has trivial intersection with the kernel of the projection $A \to \oplus_{j \in I} \Delta_A(i)$. Clearly, $C$ is quasi-directed and contains a maximal quasi-local subalgebra which is isomorphic to $\oplus_{j \in I} \mathrm{End}(\Delta(j))$. Now, we have to prove that the vectorspace $Ce_j$ is large enough, i.e. $\dim_{\mathbb{C}}(Ce_j) = \dim_{\mathbb{C}}(\Delta_A(j))$. Let $t = \dim_{\mathbb{C}}(\Delta_A(j))$. By the definition of $A$ and by Lemma 9, we have

$$t = \sum_{k \in I} \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathfrak{G}}(P^{L_k}, \Delta(L_j))) = \sum_{k \in I} \dim_{\mathbb{C}}(\Delta(L_j)_{\mu_{L_k}, b_{L_k}}) =$$
$$= \sum_{k \in I} \dim_{\mathbb{C}}((M_{L_j})_{\mu_{L_k}, b_{L_k}}) = \sum_{k \in I} \dim_{\mathbb{C}}(e_k C e_j) = \dim_{\mathbb{C}} Ce_j.$$

So, we have only to show that $C$ is pyramidal. The maximal quasi-local subalgebra of $C$ is

$$S = \bigoplus_{j \in I} \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_j}) \mid \varphi(p(L_j)) \in M_{L_j}\}.$$

We will show that $C$ is right $S$-projective. Left projectivity can be proved analogously. In fact, we will show that for any $j, k \in I$

$$e_j C e_k = \{\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_k}) \mid \varphi(p(L_j)) \in M_{L_k}\}$$

is a free right $e_k S e_k$-module. Recall that $M_{L_k}$ maps bijectively onto $\Delta(L_k)$ for any surjection from $P^{L_k}$ to $\Delta(L_k)$. Let $M_{\mathcal{P}}(V_{L_k})$ be the generalized Verma module associated with $L_k$

([FKM1, Section 2]). It follows from the description of $\Lambda$ that either $M_{\mathcal{P}}(V_{L_k}) \simeq \Delta(L_k)$ or $\Delta(L_k)$ is a self-extension of $M_{\mathcal{P}}(V_{L_k})$. Let $M^k$ denote a vectorsubspace of $M_{L_k}$, which maps bijectively on $M_{\mathcal{P}}(V_{L_k})$ under any composition $P^{L_k} \to \Delta(L_k) \to M_{\mathcal{P}}(V_{L_k})$. Such $M^k$ clearly exists. Now, an $e_kSe_k$ basis of $e_jCe_k$ is given by any linear basis of the vectorspace of all maps $\varphi \in \mathrm{Hom}_{\mathfrak{G}}(P^{L_j}, P^{L_k})$, such that $\varphi(p(L_j)) \in M^k$. Hence, $e_jCe_k$ is $e_kSe_k$-free.

Since $A$ has a $\Delta$-subalgebra and there is a duality on $\mathcal{O}_i$ ([FKM1, Section 12]), one has that $A$ has an exact Borel subalgebra; the statement follows. $\square$


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