# Elliptic equations for infinite dimensional probability distributions and Lyapunov functions

**Vladimir Bogachev**[1], **Michael Röckner**[2]

[1]: Department of Mechanics and Mathematics,

Moscow State University, 119899 Moscow, Russia

[2]: Fakultät für Mathematik, Universität Bielefeld,

D–33615 Bielefeld, Germany

**Abstract** – A new concept of a weak elliptic equation for probability distributions on infinite dimensional spaces is introduced. A suitable notion of a Lyapunov function is defined for weak elliptic equations. Sufficient conditions in terms of Lyapunov functions are given for the existence of solutions for weak elliptic equations. Applications to Gibbs measures are discussed.

### Équations elliptiques pour des distributions de probabilités infinidimensionnelles et fonctions de Lyapunov

**Résumé** – Nous introduisons une notion nouvelle d'une equation elliptique faible pour des distributions de probabilités infinidimensionnelles. Une notion de fonction de Lyapunov est définie pour les équations elliptiques faibles. Nous obtenons des conditions suffisantes en terms de fonctions de Lyapunov pour l'existence des solutions des équations elliptiques faibles. Applications aux mesures de Gibbs sont discutées.

**Version française abrégée** – Soient $\{B_n\}$ une collection dénombrable de fonctions boréliennes sur un espace localement convexe et $\{e_n\} \subset X$ une suite de vecteurs. Soit $\mathcal{K}$ une classe de fonctions bornées boréliennes $f$ telles que les dérivées partielles $\partial_{e_n} f$ et $\partial^2_{e_n} f$ existent et sont bornées. On dira que une mesure $\mu \in \mathcal{M}(X)$ satisfait à l'équation elliptique faible (4), si on a (5).

**Théorème 1.** *Soit $\mu$ une mesure de probabilité sur $X$ qui satisfait* (4) *par rapport à $\mathcal{K}$. Supposons que $V$ soit une fonction borélienne non négative sur $X$ telle que $\varphi \circ V \in \mathcal{K}$ pour chaque $\varphi \in C_0^\infty(\mathbb{R}^1)$. Soit $\Theta$ une fonction borélienne non négative sur $X$ qui est $\mu$-intégrable sur les ensembles $\{V \le c\}$. Supposons aussi que $L_B V \le C - \Theta$ $\mu$-p.p. dans le sens suivant: il existent des fonctions $\mu$-intégrables $\lambda_i$ telles que la série $\sum_{n=1}^\infty \lambda_n$ converge en moyenne sur les ensembles $\{V \le c\}$ et on a*

$$\partial^2_{e_n} V(x) + B_n(x)\partial_{e_n} V(x) \le \lambda_n(x) \quad et \quad \sum_{n=1}^\infty \lambda_n(x) \le C - \Theta(x) \quad \mu\text{-}p.p.$$

*Alors on a* $\displaystyle\int_X \Theta\, d\mu \le C.$

Soit $\{l_n\} \subset X^*$ une suite fixée séparante les points de $X$ et soit $\{e_n\} \subset X$ une suite telle que $l_n(e_k) = \delta_{nk}$. Les classes $\mathcal{FC}_b^\infty(\{l_n\})$ et $\mathcal{FC}_b^\infty(X)$ sont définies par (3).

**Théorème 2.** *Supposons que* $\{q_n\} \in l^1$ *soit une collection de nombres positives telle que* $X_0 = \Big\{x\colon\ |x|_0^2 = \sum\limits_{n=1}^{\infty} q_n l_n(x)^2 < \infty\Big\}$ *est un espace de Hilbert et l'injection natuelle* $X_0 \to X$ *est compacte. Soient* $B_n\colon X_0 \to \mathbb{R}^1$ *des fonctions continues sur les boules de* $X_0$ *par rapport à la topologie de* $X$ *telles que* $|B_n(x)| \le C_n + K_n|x|_0^{d_n}$ *sur* $X_0$. *Supposons que* $\sum\limits_{n=1}^{\infty} q_n l_n(x) B_n(x) \le C - \varrho(|x|_0)$ *sur l'enveloppe linéaire de* $\{e_n\}$, *où* $C \in \mathbb{R}^1$ *et* $\varrho$ *est une fonction non négative sur* $\mathbb{R}^1_+$ *telle que* $\lim\limits_{t\to\infty} \varrho(t) = +\infty$. *Alors il existe une mesure de probabilité* $\mu$ *sur* $X_0$ *qui satisfait à* (4) *par rapport à* $\mathcal{FC}_b^\infty(X)$ *et* $|x|_0 \in \bigcap_{1\le p<\infty} L^p(\mu)$.

**Théorème 3.** *Soient* $\mu$ *une mesure de probabilité sur* $X$ *et* $B_n \in L^2(\mu)$, $n \in \mathbb{N}$. *Supposons que* $\mu$ *satisfait* (4) *par raport à* $\mathcal{K} = \mathcal{FC}_b^\infty(\{l_n\})$ *et que* $l_j \in L^2(\mu)$ *pour un nombre* $j$ *quelconque. Alors* $\mu$ *is différentiable lelong* $e_j$ *et* $\beta_{e_j} \in L^2(\mu)$ *si et seulement si* $l_j$ *appartien aux domaine d'opérateur adjoint de*

$$L_B f = \sum_n (\partial_{e_n}^2 f + B_n \partial_{e_n} f), \qquad D(L_B) = \mathcal{FC}_b^\infty(\{l_n\}).$$

*De plus, on a* $\beta_{e_j}^\mu = \frac{L_B^* l_j + B_j}{2}$.

---

We consider weak elliptic equations for measures on infinite dimensional spaces that can be formally written as $L_B^*\mu = 0$ in the sense that

$$\int L_B\psi\, d\mu = 0, \qquad \forall\, \psi \in \mathcal{K}, \tag{1}$$

where $\mathcal{K}$ is a certain class of test functions on $X$ and $L_B$ is formally given by

$$L_B\psi = \sum_i^{\infty} \partial_{e_i}^2 \psi + \sum_{i=1}^{\infty} B_i \partial_{e_i}\psi \tag{2}$$

with some $\mu$-measurable functions $B_i$ and vectors $e_i \in X$. Typical examples of the above situation are elliptic equations for invariant measures of diffusions and integration by parts formulas for measures. In the finite dimensional case, a natural choice of $\mathcal{K}$ is the class $C_0^\infty(\mathbb{R}^n)$ of smooth compactly supported functions (in this case the series becomes a finite sum). In infinite dimensions, there are many natural possibilities to choose $\mathcal{K}$; for example, one can take the class of smooth cylindrical functions or the class of compactly supported functions $\psi$ with bounded partial derivatives $\partial_{e_j}\psi$ and $\partial_{e_j}\partial_{e_j}\psi$. Moreover, the interpretations of (1) and (2) may be different. In this work, we do not define $L_B$ separately: it has only a symbolic meaning in relation to equation (1) to which we give sense in Definition 2. Another novelty in this work is a suitably extended notion of a Lyapunov function, which enables one to get a priori estimates for solutions of the weak elliptic equations and prove existence results for them. As an application, we improve results from [5], [1], [2].

Throughout, $X$ is a locally convex space with Borel $\sigma$-algebra $\mathcal{B}(X)$ and topological dual $X^*$. Let $\mathcal{M}(X)$ denote the class of all signed measures on $\mathcal{B}(X)$ with finite total variation. Given a sequence $\{l_n\} \subset X^*$, by $\mathcal{FC}_b^\infty(\{l_n\})$ we denote the class of all functions $f$ on $X$ of the form

$$f(x) = \psi\big(l_1(x), \dots, l_n(x)\big), \qquad \text{where} \qquad \psi \in C_b^\infty(\mathbb{R}^n). \tag{3}$$

Let $\mathcal{FC}_b^\infty(X)$ be the union of all classes $\mathcal{FC}_b^\infty(\{l_n\})$ with $\{l_n\} \subset X^*$.

**Definition 1.** *Let $\mathcal{K}$ be a certain class of bounded Borel functions on $X$ and let $e \in X$ be such that the partial derivatives $\partial_e f$ exist and are bounded for all $f \in \mathcal{K}$. We say that $\mu \in \mathcal{M}(X)$ is differentiable (in the sense of Fomin) along $e$ with respect to the class $\mathcal{K}$ if there exists a $\mu$-measurable function $\beta_e^\mu$ such that for all $f \in \mathcal{K}$ one has $f\beta_e^\mu \in L^1(\mu)$ and $\int_X \partial_e f\, d\mu = -\int_X f\, \beta_e^\mu\, d\mu$. The function $\beta_e^\mu$ is called the partial logarithmic derivative of $\mu$ along $e$ with respect to $\mathcal{K}$.*

Let $\{e_n\} \subset X$ and let $\mathcal{K}$ be a certain class of bounded Borel functions $\psi$ on $X$ with bounded partial derivatives $\partial_{e_i}\psi$, $\partial_{e_i}^2\psi$ for all $i \in \mathbb{N}$.

**Definition 2.** *Let $B = (B_n)$ be a sequence of Borel functions on $X$. We shall say that $\mu \in \mathcal{M}(X)$ satisfies the weak elliptic equation*

$$L_B^*\mu = 0 \tag{4}$$

*with respect to the class $\mathcal{K}$ if, for every $\psi \in \mathcal{K}$, one has $B_n\partial_{e_n}\psi \in L^1(\mu)$ for all $n$ and*

$$\sum_{n=1}^{\infty} \int_X \left[\partial_{e_n}^2\psi + B_n\partial_{e_n}\psi\right] d\mu = 0, \quad \forall\, \psi \in \mathcal{K}. \tag{5}$$

As in the symmetric case (see below) analyzed in [1], [2], this definition enables one to consider equation (4) without the global integrability assumption on $B_n$ made in our previous work [3]. We note that the functions $B_n$ may not correspond to any "drift" $B : X \to X$.

Given a Borel mapping $B\colon X \to X$, we set $B_n = \langle l_n, B\rangle$. Then (4) becomes the characteristic equation for the invariant measures of the diffusion with drift $B/2$. In order to make sense of $L_B$ it suffices to assume that $B_n \in L^1(\mu)$ and take for $\mathcal{K}$ the class $\mathcal{FC}_b^\infty(\{l_n\})$ with $l_n(e_k) = \delta_{nk}$, on which $L_B$ is defined by its natural expression. However, there is a lot of examples where $L_B$ is not defined on $\mathcal{K}$ and (4) becomes a symbolic expression for (5) which still makes sense, since $\sum$ is in front of the integral .

Note that equation (4) is trivially satisfied if the $\beta_{e_n}^\mu$'s exist with respect to the class $\mathcal{K}$, $\partial_{e_n}\psi \in \mathcal{K}$ for all $\psi \in \mathcal{K}$ and $n \in \mathbb{N}$, and $B_n = \beta_{e_n}^\mu$ $\mu$-a.e. Indeed, then every term in the series in (5) vanishes separately. We shall now state some technical results that enable us to derive the integration by parts formula and elliptic equation for measures with differentiable conditional measures. The next lemma is a modification of a result in [7], where it was proved for globally integrable logarithmic derivatives. Later special cases of that result were derived in the context of "the integration by parts characterization of Gibbs measures" (cf. [6]).

**Lemma 1.** *Let $X = \mathbb{R}^n \times Y$, where $(Y, \mathcal{F})$ is a measurable space, let $\mu$ be a signed measure with finite total variation on $\mathcal{B} = \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}$, having regular conditional measures $\mu^y$ on $\mathbb{R}^n \times \{y\}$, and let $\nu$ be the projection of $|\mu|$ to $Y$. Suppose that $\mathcal{K}$ is a class of bounded $\mathcal{B}$-measurable functions that satisfies the following conditions:*

(i) *for every $\psi \in \mathcal{K}$ and $y \in Y$, the function $x \mapsto \psi(x, y)$ is continuously differentiable and $\nabla_x\psi$ is bounded;*

(ii) *$(x, y) \mapsto \psi(x + v, y) \in \mathcal{K}$ and $\varphi \circ \psi \in \mathcal{K}$ whenever $\psi \in \mathcal{K}$, $v \in \mathbb{R}^n$, $\varphi \in C_0^\infty(\mathbb{R}^1)$, $\varphi(0) = 0$;*

(iii) *the class $\mathcal{K}$ separates the measures on $\mathcal{B}$.*

*Let $\beta\colon X \to \mathbb{R}^n$ be a $\mu$-measurable mapping such that, for every $\psi \in \mathcal{K}$ and $v \in \mathbb{R}^n$, one has $\psi|\beta| \in L^1(\mu)$ and $\int_X \langle\nabla_x\psi, v\rangle_{\mathbb{R}^n}\, d\mu = -\int_X \psi\langle\beta, v\rangle_{\mathbb{R}^n}\, d\mu$, where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}^n}$ is the*

*Euclidean inner product on $\mathbb{R}^n$. Then, for $\nu$-a.e. $y$, $\mu^y$ admits a density $f^y$ on the fibre $\mathbb{R}^n \times \{y\}$ such that $f^y \in W^{1,1}_{loc}(\mathbb{R}^n)$ and $\beta(x,y) = \nabla_x f^y(x)/f^y(x)$.*

**Corollary 1.** *Let $\mu$ be the same as in Lemma 1 and let $f$ be a $\mu$-measurable function such that, for every $y \in Y$, the function $x \mapsto f(x,y)$ is in $W^{1,1}_{loc}(\mathbb{R}^n)$. Suppose that $|\nabla_x f|$, $f|\beta| \in L^1(\mu)$. Then $\int_X \langle \nabla_x f, v\rangle_{\mathbb{R}^n}\, d\mu = -\int_X f\langle v, \beta\rangle_{\mathbb{R}^n}\, d\mu$, $\forall\, v \in \mathbb{R}^n$. The same is true if instead of $f(\,\cdot\,, y) \in W^{1,1}_{loc}(\mathbb{R}^n)$, one has that the usual partial derivatives $\partial_{x_i} f(x,y)$ exist for every $x$.*

**Remark 1.** The separation assumption (iii) on $\mathcal{K}$ can be weakened; e.g., it would be enough to replace it by the following condition:

(iii$'$) there exists a measurable linear subspace $X_0 \subset X$ so that all its images under the shifts $(x,y) \mapsto (x+v, y)$, $v \in \mathbb{R}^n$, have full $\mu$-measure and $\mathcal{K}$ separates the measures on $X_0$.

**Example 1.** Let $\mu \in \mathcal{M}(X)$. Suppose that $(X_0, \|\cdot\|_0) \subset X$ is a continuously embedded separable reflexive Banach space of full $\mu$-measure. Let $h \in X_0$. Suppose that there exists a $\mu$-measurable function $\beta$ such that $\beta$ is $\mu$-integrable on all $\|\cdot\|_0$-bounded sets and $\int_{X_0} \partial_h \psi\, d\mu = -\int_{X_0} \psi\beta\, d\mu$ for every function $\psi$ from the class $C^1_0(X_0)$ of all continuously Fréchet differentiable functions with bounded supports on $X_0$. Let $X = \mathbb{R}^1 h \otimes Y$, where $Y$ is a closed linear subspace. Then $\mu$ has regular conditional measures $\mu^y$ on the lines $y + \mathbb{R}^1 h$, $y \in Y$, that have locally absolutely continuous densities $f^y$ with $\big(f^y(t)\big)'/f^y(t) = \beta(y+th)$. If $X_0$ is a Hilbert space, then the same is true with $C^\infty_0(X_0)$ replacing $C^1_0(X_0)$.

In the next theorem we consider a natural generalization of Lyapunov functions suitable for the study of weak elliptic equations.

**Theorem 1.** *Let $\{e_n\} \subset X$ and let $\mu$ be a probability measure on $X$ satisfying equation (4) with respect to $\mathcal{K}$. Suppose that $V$ is a nonnegative Borel function on $X$ such that $\varphi \circ V \in \mathcal{K}$ for every $\varphi \in C^\infty_0(\mathbb{R}^1)$. Let $\Theta$ be a nonnegative Borel function on $X$ that is $\mu$-integrable on the sets $\{V \le c\}$. Assume, in addition, that there exists $C \in \mathbb{R}^1$ such that $L_B V \le C - \Theta$ $\mu$-a.e. in the following sense: there exist $\mu$-integrable functions $\lambda_i$ such that the series $\sum_{n=1}^\infty \lambda_n$ converges in $L^1(\mu)$ on the sets $\{V \le c\}$ and one has*

$$\partial^2_{e_n} V(x) + B_n \partial_{e_n} V(x) \le \lambda_n(x) \quad \textit{and} \quad \sum_{n=1}^\infty \lambda_n(x) \le C - \Theta(x) \quad \mu\textit{-a.e.}$$

*Then one has $\int_X \Theta\, d\mu \le C$.*

Let us consider an example which extends a result from [1], [2], where the gradient case (i.e., the case of symmetric operator) was considered and more restrictive assumptions on $B_i$ were used.

Let $J = (J_{s,t})_{s,t\in S}$ be an infinite matrix with nonnegative entries $J_{s,t} = J_{t,s}$ indexed by a countable set $S$ and let $q = (q_s)$, $s \in S$, be positive numbers such that $\sum_{s\in S} q_s < \infty$. We denote by $l^1(q)$ the weighted $l^1$-space of all families $x = (x_s)_{s\in S}$ with $\|x\|_{l^1(q)} = \sum_{s\in S} q_s |x_s| < \infty$.

**Corollary 2.** *Let $X = \mathbb{R}^S$ and let $X_0$ be the weighted Banach space of all sequences $x = (x_s)_{s\in S}$ such that*

$$|x|_0 = \Big(\sum_{s\in S}^{\infty} q_s|x_s|^\alpha\Big)^{1/\alpha},$$

*where $\alpha \geq 2$. Suppose that the operator norm of $J$ on $l^1(q)$ satisfies the estimate $\|J\|_{\mathcal{L}(l^1(q))} \leq \lambda$ and that $\sum_{t\in S} J_{s,t} \leq \lambda$. Let $B = (B_s)_{s\in S}$ be a collection of Borel functions on $X_0$ with*

$$x_s B_s(x) \leq c - (\lambda+\varepsilon)|x_s|^\alpha + \sum_{t\in S}^{\infty} J_{s,t}|x_t|^\alpha, \qquad \forall\, s \in S,$$

*for some $c$, $\varepsilon \in (0,+\infty)$. Let $\mu$ be a probability measure on $\mathbb{R}^S$ such that $\mu(X_0) = 1$. If the $B_s$'s are $\mu$-integrable on all balls in $X_0$ and (4) is satisfied with respect to $C_0^2(X_0)$, then, for every $N \in \mathbb{N}$, there exists $C_N > 0$ such that $\int_X |x_s|^N\, d\mu \leq C_N$, $\forall\, s \in S$.*

Uniform estimates for finite dimensional equations together with standard weak compactness arguments yield the following existence results. We shall assume that $\{l_n\}$ is a sequence of continuous linear functionals separating the points in $X$ and that $\{e_n\} \subset X$ is such that $l_n(e_k) = \delta_{nk}$.

**Theorem 2.** *Suppose that $\{q_n\} \in l^1$ is a sequence of positive numbers such that $X_0 = \Big\{x \in X\colon\ |x|_0^2 = \sum_{n=1}^{\infty} q_n l_n(x)^2 < \infty\Big\}$ is a separable Hilbert space compactly embedded into $X$. Let $B_n\colon X_0 \to \mathbb{R}^1$ be continuous on balls in $X_0$ for the topology induced from $X$ and satisfy the estimates $|B_n(x)| \leq C_n + K_n|x|_0^{d_n}$ on $X_0$. Assume that $\sum_{n=1}^{\infty} q_n l_n(x) B_n(x) \leq C - \varrho(|x|_0)$ on the linear span of the $e_n$'s, where $\varrho$ is a nonnegative function on $\mathbb{R}^1_+$ with $\lim_{t\to\infty} \varrho(t) = +\infty$. Then there exists a probability measure $\mu$ on $X_0$ satisfying equation (4) with respect to $\mathcal{FC}_b^\infty(X_0)$. In addition, $|x|_0 \in \bigcap_{1\leq p<\infty} L^p(\mu)$.*

The above theorem yields the existence of solutions in the situation of Corollary 2.

**Theorem 3.** *Suppose that the hypotheses of Theorem 2 are fulfilled and let $E_n$ be the linear span of $\{e_1, \dots, e_n\}$. Assume, in addition, that there exist continuously differentiable functions $G_n$ on $E_n$ such that $B_i = \partial_{e_i} G_n$ on $E_n$ for each $i \leq n$. Then $\mu$ is differentiable along the $e_n$'s with respect to $\mathcal{FC}_b^\infty(X)$ and $\beta^\mu_{e_n} = B_n$. Moreover, the global bound on $B_n$ can be dropped, but in this case the logarithmic derivatives exist with respect to $C_0^2(X_0)$.*

Theorems 2 and 3 yield a priori estimates and existence of Gibbs measures on $\mathbb{R}^S$, $S = \mathbb{Z}^d$, with one dimensional conditional distributions for $n \in \mathbb{Z}^d$ given by the densities $p(x_n|x_n^c) = c(x_n^c)\exp\Big(-V_n(x_n) - \sum_{j\in S} W_{n,j}(x_n, x_j)\Big)$, $x = (x_n, x_n^c) \in \mathbb{R}^S$, where $V_n$ and $W_{n,j}$ are continuously differentiable and satisfying the conditions $A_1) - A_3)$ in [2], but with the coercivity estimate in $A_3)$ replaced by the following weaker condition:

$$-x_n V_n'(x_n) \leq C - (3\lambda+\varepsilon)|x_n|^N,$$

where $\lambda$ and $\varepsilon$ are the same as in Corollary 2. Hence this result extends the corresponding results from [1], [2].

**Theorem 4.** *Let $\mu$ be a probability measure satisfying* (4) *with respect to $\mathcal{F}\mathcal{C}_b^\infty(X)$. Suppose that $(B_1, \dots, B_n) = \nabla G_n(l_1, \dots, l_n) + D_n$, where $G_n$ is a continuously differentiable function on $\mathbb{R}^n$, $D_n$ is a Borel mapping with values in $\mathbb{R}^n$, and $\sup_n \|D_n\|_{L^2(\mu,\mathbb{R}^n)} < \infty$. Suppose that $B_n \in L^2(\mu)$ for every $n$. Then $\mu$ is differentiable along each $e_n$ and $\beta^\mu_{e_n} \in L^2(\mu)$.*

*If $\mathbb{E}_n$ stands for the conditional expectation with respect to the $\sigma$-field generated by $l_1, \dots, l_n$, and one has $\lim_{n\to\infty} \|\mathbb{E}_n D_n\|_{L^2(\mu,\mathbb{R}^n)} = 0$, then $\beta^\mu_{e_n} = B_n$ $\mu$-a.e.*

The last claim means that, under the corresponding assumption, any solution $\mu$ of (4) is symmetrizing for $L_B$. In probabilistic terms, this means that any invariant probability for the associated diffusion is Gibbsian. The next result can be informally interpreted in terms of the time-reversal of the corresponding diffusions.

**Theorem 5.** *Let $\mu$ be a probability measure on $X$ and let $B_n \in L^2(\mu)$, $n \in \mathbb{N}$. Suppose that $\mu$ satisfies* (4) *with respect to $\mathcal{K} = \mathcal{F}\mathcal{C}_b^\infty(\{l_n\})$. Assume also that $l_j \in L^2(\mu)$ for some $j$. Then $\mu$ is differentiable along $e_j$ with $\beta^\mu_{e_j} \in L^2(\mu)$ precisely when $l_j$ belongs to the domain of the adjoint operator for $L_B f = \sum_n (\partial^2_{e_n} f + B_n \partial_{e_n} f)$ with $D(L_B) = \mathcal{F}\mathcal{C}_b^\infty(\{l_n\})$. In addition, $\beta^\mu_{e_j} = \frac{L_B^* l_j + B_j}{2}$.*

Detailed proofs of all results above can be found in [4].

This work has been supported in part by the RFBR projects 96–15–96865 and 97–01–00932, the INTAS Grant 94–378, the INTAS-RFBR Grant 95-0099, the DFG Grant 436 RUS 113/343/0(R), and the SFB–343, as well as EU–TMR Project No. ERB–FMRX CT 96–0075.

# References

[1] Albeverio S., Kondratiev Yu.G., Röckner M., Tsykalenko T., A-priori estimates and existence of Gibbs measures: a simplified proof, C. R. Acad. Sci. Paris (1999), to appear.

[2] Albeverio S., Kondratiev Yu., Röckner M., Tsykalenko T., A-priori estimates on symmetrizing measures and their applications to Gibbs states, BiBoS Preprint (1999).

[3] Bogachev V.I., Röckner M., Regularity of invariant measures on finite and infinite dimensional spaces and applications, J. Funct. Anal. 133 (1995), 168–223.

[4] Bogachev V.I., Röckner M., Elliptic equations for measures on infinite dimensional spaces and applications to Gibbs measures, Preprint SFB 343 (1999).

[5] Kirillov A.I., Infinite-dimensional analysis and quantum theory as semimartingale calculi, Uspekhi Mat. Nauk 49 (1994), no. 3, 43–92 (in Russian); English transl.: Russian Math. Surveys 49 (1994).

[6] Roelly S., Zessin H., Une caractérisation des mesures de Gibbs sur $C(0,1)^{Z^d}$ par le calcul des variationes stochastiques, Ann. Inst. H. Poincaré 29 (1993), 327–338.

[7] Yamasaki Y., Hora A., Differentiable shifts for measures on infinite dimensional spaces, Publ. RIMS, Kyoto Univ. 23 (1987), 275–296.