ULTRACONTRACTIVITY AND SUPERCONTRACTIVITY OF MARKOV SEMIGROUPS

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Abstract. By using perturbation arguments, a sufficient condition is presented for the ultracontractivity of symmetric diffusion semigroups. As a consequence, a result suggested by D. Stroock is proved: let P_t be generated by $L = \Delta + \nabla V$ with $V = -\alpha\nabla\rho^\delta(\alpha > 0, \delta > 2)$ on a complete connected Riemannian manifold M, where ρ is the Riemannian distance function from a fixed point, then P_t is ultracontractive provided the Ricci curvature is bounded below. Furthermore, $\|P_t\|_{1\to\infty} \le \exp[\lambda_1 + \lambda_2 t^{-\delta/(\delta-2)}]$ for some $\lambda_1, \lambda_2 > 0$. Next, it is shown that, for a diffusion semigroup (not necessarily symmetric) with an invariant probability measure, if the curvature of its generator is bounded from below, then the ultracontractivity is equivalent to $\|P_t \exp[\lambda\rho^2]\|_\infty < \infty$ for any $\lambda, t > 0$. Especially, the above estimate of $\|P_t\|_{1\to\infty}$ holds if $L\rho^2 \le c_1 - c_2\rho^\delta$ for some $c_1, c_2 > 0$ and $\delta > 2$. This estimate is sharp as is shown by examples given at the end of the paper. Corresponding results are proven for supercontractive semigroups, from which one obtains explicit examples to show that ultracontractivity is strictly stronger than supercontractivity. Ultracontractivity and supercontractivity are also studied by using Sobolev type inequalities and isoperimetric inequalities. Finally, some concentration results are obtained for reversible measures of supercontractive semigroups.

1. Introduction

Let E be a locally compact, second countable Hausdorff space. Consider the symmetric Markov semigroup P_t on $L^2(E; \mathrm{d}\mu)$ in the sense of [13] (Chapter 2), where μ is a Borel measure on E. We call P_t ultracontractive (respectively, supercontractive) if $\|P_t\|_{1\to\infty} < \infty$ (respectively, $\|P_t\|_{2\to 4} < \infty$) for any $t > 0$, where $\|\cdot\|_{p\to q}$ denotes the operator norm from $L^p(E; \mathrm{d}\mu)$ to $L^q(E; \mathrm{d}\mu)$. We recall that this definition of ultracontractivity coincides with the one in [13] and [18] since $\|P_t\|_{1\to\infty} < \infty$ for all $t > 0$ is equivalent to $\|P_t\|_{2\to\infty} < \infty$ for all $t > 0$, and in this case (see e.g. [13; Lemma 2.1.2])

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$$\|P_t\|_{1\to\infty} = \|P_{t/2}\|_{2\to\infty}^2, \qquad t > 0. \tag{1.0}$$

There is a large number of papers concerning contractivity properties of symmetric Markov semigroups, including hypercontractivity, supercontractivity and ultracontractivity. In many of them contractivity properties were derived by establishing various inequalities, such as log-Sobolev inequalities, Sobolev inequalities, Nash type inequalities and isoperimetric inequalities, see for instances [3], [11], [13], [19] and references therein.

Our first purpose is to present some explicit sufficient conditions for ultracontractivity and supercontractivity for symmetric diffusion semigroups. See [23] for some results on ultracontractivity on $\mathbb{R}^d$, and we refer to [7], [11] and [15] for intrinsic ultracontractivity on domains in $\mathbb{R}^d$.

Let, for instance, P_t be the semigroup of the diffusion process generated by $L = \Delta - \alpha\nabla\rho^\delta$ $(\alpha, \delta > 0)$ on a complete connected Riemannian manifold M, where $\rho(\cdot) := \rho(\cdot, o)$ denotes the Riemannian distance function from a fixed point $o \in M$ and $\nabla\rho^\delta$ is taken to be zero on the cut locus of o. It was shown recently in [28] and [36] that if $\delta \geq 2$, then P_t is hypercontractive (in the sense that $\|P_t\|_{2\to4} \leq 1$ for some $t > 0$) provided the Ricci curvature is bounded below, namely, $\mathrm{Ric}(X, X) \geq -K\langle X, X\rangle$ for some $K \geq 0$ and any $X \in TM$. It was also conjectured by D. Stroock in [28] that P_t is ultracontractive for $\delta > 2$ under the same curvature condition. This conjecture is proved in the paper as a consequence of the main result in the next section (Corollary 2.3), see also [38] for an alternative proof using Poincaré type inequalities.

Moreover, some other criteria are presented in section 3 by using the Harnack inequality of semigroups proved in [33]. For instance, if the curvature of a diffusion operator is bounded from below and the invariant measure μ is finite, then the ultracontractivity of the corresponding semigroup P_t is equivalent to $\|P_t \exp[\lambda\rho^2]\|_\infty < \infty$ for any $t, \lambda > 0$ (Theorem 3.3). Consequently, a diffusion semigroup is ultracontractive if the curvature of its generator L is bounded below and $L\rho^2 \leq c_1 - c_2\rho^\delta$ for some $c_1, c_2 > 0$ and $\delta > 2$. See Corollary 3.5 for the concrete estimate. A necessary and sufficient condition is also presented for supercontractivity, from which one sees that there exist supercontractive semigroups which are not ultracontractive (see Example 6.4). Moreover, results in this part remain true for nonsymmetric diffusion semigroups.

Another motivation of the paper comes from the basic theorem due to Gross [18]: the hypercontractivity in the sense that $\|P_t\|_{2\to4} < \infty$ for some $t > 0$ is equivalent to the (defective) log-Sobolev inequality

$$\mu(f^2 \log f^2) \leq c_1 D(f, f) + c_2, \qquad f \in \mathcal{D},\ \mu(f^2) = 1, \tag{1.1}$$

where $(D, \mathcal{D})$ is the corresponding symmetric Dirichlet form of P_t. It is well-known by [14] that the ultracontractivity is equivalent to a somehow stronger inequality of the same type as (1.1). We remark that the ultracontractivity has been also studied in [9], [12] and [29] by using Nash type inequalities. Our purpose (section 4) is to study ultracontractivity and supercontractivity by using so-called F-Sobolev inequalities which are well related to isoperimetric inequalities as was shown in [6] and [35].

Let $F \in C[0,\infty)$ be an increasing function with $F(\infty) := \lim_{r\to\infty} F(r) > 0$. The F-Sobolev inequality with "dimension" $p \in (2,\infty]$ reads

$$\mu(|f|^{2p/(p-2)}F(f^2)) \le \{c_1 D(f,f) + c_2\}^{p/(p-2)}, \qquad f \in \mathcal{D},\ \mu(f^2)=1. \tag{1.2}$$

Since for $F \equiv 1$ it is well-known that the inequality with finite dimension p is equivalent to $\|P_t\|_{1\to\infty} \le ct^{-p/2}$ for some $c>0$ and all $t \in (0,1]$ (see for example [13; Corollary 2.4.3]), we shall only consider the infinite dimensional case for which the inequality becomes

$$\mu(f^2F(f^2)) \le c_1 D(f,f) + c_2, \qquad f \in \mathcal{D},\ \mu(f^2)=1. \tag{1.3}$$

Actually, (1.3) is already wide enough to recover also the classical Sobolev inequalities of finite dimensions (see Corollary 4.3): (1.2) holds for some $p>2$ and $F \equiv 1$ if and only if (1.3) holds for $F(r) = r^{2/p}$. We simply call (1.3) the F-Sobolev inequality in analogy with the usual log-Sobolev inequality. We assume that $\lim_{r\to\infty} F(r)/(\log r) = \infty$ so that (1.3) possibly implies the ultracontractivity. We may also assume that $F \ge 1$ for simplicity, since (1.2) and (1.3) remain true (with different constants) for any increasing function $\bar F \ge 1$ with $\bar F(r) \le cF(r)$ for some $c>0$ and big r.

Let us recall the isoperimetric inequality corresponding to (1.3) for diffusion processes presented in [6] (see [35] for jump processes). Let $E = M$ be a d-dimensional complete connected Riemannian manifold possibly with boundary $\partial M \ne \emptyset$, and let $D(f,f) = \mu(|\nabla f|^2)$ with $\mathrm{d}\mu = \mathrm{e}^V \mathrm{d}x$ for some $V \in C(M)$, where $\mathrm{d}x$ denotes the Riemannian volume element. Define

$$k(r) = \inf_{A:\mu(A)\le r} \frac{\mu_\partial(\partial A \setminus \partial M)}{\mu(A)\sqrt{F(\mu(A)^{-1})}}, \qquad r>0,$$

where $A \subset M$ runs over all smooth open domains, and $\mu_\partial(\partial A\setminus\partial M)$ is the $(d-1)$-dimensional measure on $\partial A \setminus \partial M$ induced by μ. By [41], $k(r)$ does not change if test sets in its definition are assumed additionally to be connected.

Theorem 1.1([6; Theorem 4.1]). *Assume that* $\overline{\lim}_{r\to\infty} \sup \frac{rF'(r)}{F(r)} < 1$. *Then* $k(r)>0$ *for some* $r>0$ *is equivalent to*

$$\mu\Big(f\sqrt{F(f)}\Big) \le c_1\mu(|\nabla f|) + c_2, \quad f \ge 0,\ f \in \mathcal{D},\ \mu(f)=1, \tag{1.4}$$

which implies (1.3).

The next result follows from [14; Theorem 4.7] (see also Lemma 2.2.6 and Theorem 2.2.7 in [13]), which takes a very important role in the study of ultracontractivity.

Theorem 1.2 (Davies and Simon). *If there exists a positive decreasing function $\beta \in C(0,\infty)$ such that*

$$\mu(f^2 \log f^2) \le rD(f,f) + \beta(r), \quad r > 0,\ f \in \mathcal{D}, \mu(f^2) = 1. \tag{1.5}$$

Then $\|P_t\|_{1\to\infty} \le \exp[m(t)]$ with

$$t = \int_2^\infty \frac{\gamma(r)}{r}\, dr, \quad m(t) = 2\int_2^\infty \frac{\beta(\gamma(r))}{r^2}\, dr$$

for any strictly positive continuous function γ such that $t, m(t) < \infty$. On the other hand, $\|P_t\|_{1\to\infty} \le \exp[m(t)]$ implies (1.5) for any decreasing $\beta \in C(0,\infty)$ with $\beta(r) \ge \inf_{t\in(0,r]} m(t)$.

According to [14], (1.5) is equivalent to the supercontractivity of P_t. By (1.0) it is easily seen that indeed Theorem 1.2 is an equivalent statement of the original one by Davies and Simon.

Finally, we follow Gross and Rothaus [20] to study the concentration of reference measures for supercontractive semigroups. The concentration of measures has been proved in several papers if the Poincaré inequality or the log-Sobolev inequality holds by using a modified Herbst's argument, see for instance [1], [14], [25] and [27]. Our result shows that the general results by Gross and Rothaus are sharp (see Corollary 5.4 and Example 6.3).

2. Ultracontractivity by perturbations

In this and the next sections, we consider the case where $E = M$ is a complete connected Riemannian manifold with boundary ∂M either convex or empty. Denote by $\mathrm{d}x$ the Riemannian volume element. Let V and W be two locally Lipschitz continuous functions, and let $\mathrm{d}\mu = \exp[V]\mathrm{d}x, \mathrm{d}\nu = \exp[W]\mathrm{d}x$. Assume that the following inequality holds for ν with some decreasing positive function $\beta \in C(0,\infty)$:

$$\nu(f^2 \log f^2) \le r\nu(|\nabla f|^2) + \beta(r),\ \nu(|\nabla f|^2) < \infty,\ \nu(f^2) = 1,\ r > 0. \tag{2.1}$$

Since $C_0^\infty(M)$ is dense in the Sobolev space $\{f : \nu(f^2) + \nu(|\nabla f|^2) < \infty\}$ under the Sobolev norm $\|f\|_S = \sqrt{\nu(f^2) + \nu(|\nabla f|^2)}$, (2.1) is equivalent to the same inequality for all $f \in C_0^\infty(M)$. When $\partial M \ne \emptyset$, it is also equivalent to the inequality for all $f \in C_0^\infty(M)$ satisfying the Neumann boundary condition. Actually, for any $f \in C_0^\infty(M)$, there exists a sequence of smooth functions with Neumann boundary condition which converges to f in the Sobolev norm, see e.g. [39] (the proof of Lemma 2.3).

It is easy to see that (2.1) follows from the F-Sobolev inequality (1.3) for ν with $\beta(r) = \inf_{t\in(0,r]}\big\{\frac{tc_2}{c_1} + \sup_{s>0}[s\log s - \frac{ts}{c_1}F(s)]\big\}$.

Proposition 2.1. *Let μ and ν be as in above. Assume that* (2.1) *holds. Let $\eta \in C(M)$ be such that $\Delta e^V \geq e^V \eta$ in the distribution sense, i.e.,*

$$\int e^V \Delta f \, dx \geq \int f \eta \, d\mu, \quad f \geq 0, \ f \in C_0^\infty(M), \tag{2.2}$$

where f satisfies Neumann boundary conditions if $\partial M \neq \emptyset$. Finally, assume that there exists decreasing $\beta_1 \in C(0, \infty)$ such that

$$\beta_1(r) \geq \inf_{\delta \in [0,1]} \mathrm{ess}_\mu\text{-}\sup_M \Big\{ W - V + \frac{r(1+\delta)}{4} |\nabla(V-W)|^2 + \frac{r(1-\delta)}{4} \big(|\nabla W|^2 - 2\eta\big) \Big\}.$$

Then we have

$$\mu(f^2 \log f^2) \leq r\mu(|\nabla f|^2) + (\beta + \beta_1)(r/2), \quad r > 0, \ f \in \mathcal{D}, \ \mu(f^2) = 1. \tag{2.3}$$

Proof. For any $f \in C_0^\infty(M)$ with $\mu(f^2) = 1$ which satisfies Neumann boundary condition when $\partial M \neq \emptyset$, applying (2.1) to $f \exp[(V-W)/2]$, we have

$$\mu(f^2 \log f^2) \leq \mu(f^2(W-V)) + r\nu(|\nabla(f \exp[(V-W)/2])|^2) + \beta(r). \tag{2.4}$$

It is easy to see that

$$\nu(|\nabla(f \exp[(V-W)/2])|^2) \leq 2\mu(|\nabla f|^2) + \frac{1}{2}\mu(f^2 |\nabla(V-W)|^2). \tag{2.5}$$

On the other hand, by (2.2) and using the integral by parts formula,

$$\begin{aligned}
&\nu(|\nabla(f \exp[(V-W)/2])|^2) \\
&= \mu(|\nabla f|^2) + \frac{1}{4}\mu(f^2|\nabla(V-W)|^2) + \frac{1}{2}\nu(\langle \nabla f^2, \nabla \exp[V-W] \rangle) \\
&= \mu(|\nabla f|^2) + \frac{1}{4}\mu(f^2|\nabla(V-W)|^2) - \frac{1}{2}\mu(\Delta f^2) - \frac{1}{2}\mu(\langle \nabla f^2, \nabla W \rangle) \\
&\leq 2\mu(|\nabla f|^2) + \frac{1}{4}\mu(f^2[|\nabla(V-W)|^2 - 2\eta + |\nabla W|^2]).
\end{aligned} \tag{2.6}$$

By combining (2.4) with (2.5) and (2.6), we obtain

$$\mu(f^2 \log f^2) \leq 2r\mu(|\nabla f|^2) + (\beta + \beta_1)(r), \quad r > 0.$$

This is equivalent to (2.3). $\square$

Corollary 2.2. *Consider the situation of Proposition* 2.1*. Let P_t be the symmetric Markov semigroup on $L^2(\mu)$ generated by $L = \Delta + \nabla V$ (with Neumann boundary condition whenever $\partial M \neq \emptyset$). If there exists a strictly positive $\gamma \in C[0,\infty)$ such that*

$$\int_2^\infty \left[\frac{\gamma(r)}{r} + \frac{\beta(s\gamma(r))}{r^2}\right] dr < \infty, \quad s > 0, \tag{2.7}$$

then $\|P_t\|_{1\to\infty} \le \exp[m(t)]$ with

$$m(t) = 2\int_2^\infty \frac{1}{r^2}\Big\{\beta + \beta_1\Big\}\Big(\frac{t\gamma(r)}{2c}\Big) dr \in (0,\infty], \quad c := \int_2^\infty \frac{\gamma(r)}{r} dr. \tag{2.8}$$

Proof. For any $t > 0$, we have $t = \int_2^\infty r^{-1}[\frac{t}{c}\gamma(r)]\mathrm{d}r$. By (2.3) and Theorem 1.2, we complete the proof. $\square$

Remark. By using the inequality $\mu(f^2 g) \le \mu(f^2\log f^2) + \mu(\exp[g])$, we may also define β_1 in Theorem 2.1 by using an exponential integral instead of the supermum. Actually, by (2.4), (2.5) and (2.6), we have

$$\begin{aligned}\mu(f^2\log f^2) &\le 2r\mu(|\nabla f|^2) + \mu(f^2 g_{r\delta}) + \beta(r)\\ &\le 2r\mu(|\nabla f|^2) + \varepsilon\mu(f^2\log f^2) + \mu(\exp[g_{r\delta}/\varepsilon]) + \beta(r),\end{aligned}$$

where $\varepsilon \in (0,1)$ and

$$g_{r\delta} = W - V + \frac{r(1+\delta)}{4}|\nabla(V-W)|^2 + \frac{r(1-\delta)}{4}(|\nabla W|^2 - 2\eta).$$

Let $\beta_\varepsilon(r) = \inf_{\delta\in[0,1]}\mu(\exp[g_{r\delta}/\varepsilon])$, the proof of Corollary 2.2 implies that P_t is ultracontractivity if

$$\int_2^\infty \Big[\frac{\gamma(r)}{r} + \frac{(\beta+\beta_\varepsilon)(s\gamma(r))}{r^2}\Big]\mathrm{d}r < \infty$$

for some positive $\gamma \in C[0,\infty), \varepsilon \in (0,1)$ and any $s > 0$. We did not recommend this argument in Proposition 2.1 since β_ε might be too big to get a good estimate of $\|P_t\|_{1\to\infty}$. But from here we obtain a weaker sufficient condition for supercontractivity: $\mu(\exp[g_{r\delta}/\varepsilon]) < \infty$ for some $\varepsilon \in (0,1), \delta \in [0,1]$ and all $r > 0$.

Corollary 2.3. *Assume that the Ricci curvature is bounded below by $-K$ for some $K \ge 0$, namely,* $\mathrm{Ric}(X,X) \ge -K|X|^2, X \in TM$*. Let $c > \sqrt{K(d-1)}$, where d is the dimension of M, and let ρ be the Riemannian distance function from a fixed point $o \in M\setminus\partial M$. If there exists a strictly positive $\gamma \in C[0,\infty)$ such that*

$$\int_2^\infty \Big[\frac{\gamma(r)}{r}+\frac{\beta_1(s\gamma(r))}{r^2}\Big]dr<\infty,\quad s>0, \tag{2.9}$$

where $\beta_1(r)=\sup\{c\rho-V+\frac{r}{4}[c^2+(c+|\nabla V|)^2-2\eta]\}, r>0$, *then* P_t *is ultracontractive. Especially, if* $V=-\alpha\rho^\delta(\alpha>0,\delta>2)$, *then*

$$\|P_t\|_{1\to\infty}\le \exp\big[\lambda_1+\lambda_2 t^{-\delta/(\delta-2)}\big] \tag{2.10}$$

for some $\lambda_1,\lambda_2>0$ *and all* $t>0$.

Proof. Let $W=c\rho$ and $\nu(\mathrm{d}x)=\exp[W]\mathrm{d}x$. By [36; Theorem 1.5] (see also the Appendix in the present paper), we have

$$\nu(|f|^{2p/(p-2)})^{(p-2)/p}\le c_1\nu(|\nabla f|^2)+c_2,\quad \nu(f^2)=1$$

for some $c_1,c_2>0$ and $p>2$. Equivalently (see for instance [3]),

$$\nu(f^2\log f^2)\le \frac{p}{2}\log\big[c_1\nu(|\nabla f|^2)+c_2\big],\quad \nu(f^2)=1.$$

Then (2.1) holds with

$$\beta(r)=\sup_{s\ge 0}\Big\{\frac{p}{2}\log(c_1 s+c_2)-rs\Big\}\le \frac{p}{2}\log\Big(\frac{c_1 p}{2r}+c_2\Big),\qquad r>0.$$

Let $\bar\gamma(r)=\gamma(r)\vee r^{-1}$. Since β and β_1 are decreasing, it is easy to see that (2.9) implies

$$\int_2^\infty \Big[\frac{\bar\gamma(r)}{r}+\frac{(\beta+\beta_1)(s\bar\gamma(r))}{r^2}\Big]\mathrm{d}r<\infty,\quad s>0.$$

Then P_t is ultracontractive by Corollary 2.2.

Next, in the case that $V=-\alpha\rho^\delta(\alpha>0,\delta>2)$, by the argument in [40; Appendix], we have (2.2) with $\eta=\alpha^2\delta^2\rho^{2(\delta-1)}-\alpha c_3(1+\rho^{\delta-1})$ for some $c_3>0$ since the Ricci curvature is bounded below. Then we may take

$$\beta_1(r)=\sup_M\Big\{-\frac{r}{4}\alpha^2\delta^2\rho^{2(\delta-1)}+c_4+2\alpha\rho^\delta\Big\}\le c_5+c_6 r^{-\delta/(\delta-2)}$$

for some $c_4,c_5,c_6>0$ and all $r\in(0,1]$. Taking $\gamma(r)=r^{-\varepsilon},\varepsilon=\frac{\delta-2}{2\delta}$. By Corollary 2.2, $\|P_t\|_{1\to\infty}\le\exp[m(t)]$ with

$$m(t)=2\int_2^\infty\frac{(\beta+\beta_1)(c_7 t\gamma(r))}{r^2}\mathrm{d}r\le c_8+c_9 t^{-\delta/(\delta-2)}$$

for some $c_7,c_8,c_9>0$ and all $t\in(0,1/c_7]$. This proves (2.10) for some $\lambda_1,\lambda_2>0$. $\square$

Remark. (i) Let $\partial M = \emptyset$. By Croke's isoperimetric inequality (see for instance [10]), if the injectivity radius is infinite, or if the Ricci curvature is bounded below and the injectivity radius is positive, then the Sobolev inequality holds for $\mathrm{d}x$. Therefore in these cases, the assertions in Corollary 2.3 are true for $c = 0$.

(ii) The estimate (2.10) was also obtained in [38] using Poincaré-type inequalities. Another method to prove (2.10) was suggested by Stroock in [28] and meanwhile realized by Z.M. Ahmed (private communication). This method is a modification of the one in [28], hence is, in particular, based on [23].

3. Criteria for diffusion semigroups with curvature bounded below

Let $E = M$ be as in the last section. Let P_t be the semigroup of the (reflecting) diffusion process generated by L, where $L = \Delta + X$ for some C^1-vector field X satisfying

$$\mathrm{Ric}(Y,Y) - \langle \nabla_Y X, Y\rangle \geq -K\langle Y,Y\rangle \tag{A}$$

for some $K \geq 0$ and all $Y \in TM$. Assume that P_t has an invariant probability measure μ. In particular, this is the case when $X = \nabla V$ with $\int \exp[V]\mathrm{d}x = 1$. Since X is locally bounded, we have (by [8]) $\mathrm{d}\mu = \phi\mathrm{d}x$ with $\phi \in W^{\alpha,1}_{loc}(\mathrm{d}x)$ for any $\alpha > 0$. In particular, ϕ is Hölder continuous. Moreover, we see that $L^*\phi = 0$ in the distribution sense, where L^* is the adjoint of L in $L^2(M;\mathrm{d}\mu)$. By [30; Corollaries 5.3 and 5.5], ϕ is positive everywhere whenever $\partial M = \emptyset$. Since $C_0^\infty(M)$ is dense in $\mathcal{D}(D) := W^{2,1}(M;\mathrm{d}\mu)$, it suffices to consider test functions in $C_0^\infty(M)$ for Sobolev type inequalities. Let ρ be the Riemannian distance function from a fixed point $o \in M \setminus \partial M$.

We first prove the following version of Gross theorem for our present nonsymmetric case.

Theorem 3.1. *Assume (A).*

(1) *If there exist $C > 0$ and $q > p > 1$ such that $\|P_t\|_{p\to q} \leq C$, then*

$$\mu(f^2\log f^2) \leq \frac{2(\exp[Kt]-1)p(q-1)}{K(q-p)}\mu(|\nabla f|^2) + \frac{pq}{q-p}\log C \tag{3.1}$$

holds for any $f \in \mathcal{D}$ with $\mu(f^2) = 1$.

(2) *If there exist $C_1, C_2 > 0$ such that*

$$\mu(f^2\log f^2) \leq C_1\mu(|\nabla f|^2) + C_2, \quad f \in \mathcal{D},\ \mu(f^2) = 1, \tag{3.2}$$

then

$$\|P_t\|_{p\to q} \leq \exp\left[4C_2\Big(\frac{1}{p} - \frac{1}{q}\Big)\right]$$

for $t > 0$ and $q > p > 1$ satisfying $\exp[4t/C_1] \geq (q-1)/(p-1)$.

Proof. It suffices to prove (1) since (2) follows immediately from Proposition 3.1 and Theorem 3.2 in [3]. For the symmetric case, the key point of the proof is to use Stein interpolation theorem and then apply Gross theorem. This interpolation theorem does not apply to our present case since $P_z := \exp[-zL]$ may be not analytic in $\{0 < \mathrm{Re} z < 1\}$. Our proof is based on the following semigroup log-Sobolev inequality (see [5], [17], [21] and [31] for different proofs)

$$P_t(f^2 \log f^2) \leq \frac{2(\exp[Kt]-1)}{K} P_t|\nabla f|^2 + (P_t f^2)\log(P_t f^2). \tag{3.3}$$

Since μ is an invariant measure, (3.3) yields

$$\mu(f^2 \log f^2) \leq \frac{2(\exp[Kt]-1)}{K} \mu(|\nabla f|^2) + \mu((P_t f^2)\log(P_t f^2)). \tag{3.4}$$

For any $s \in (0, (p-1)/p)$, let

$$r = \frac{ps}{p-1}, p_s = \frac{1}{1-s}, q_s = \frac{1}{1-\delta s}, \delta = \frac{p(q-1)}{q(p-1)}.$$

Then $r \in (0,1)$ and

$$\frac{1}{p_s} = \frac{r}{p} + 1 - r, \quad \frac{1}{q_s} = \frac{r}{q} + 1 - r.$$

By Riesz-Thorin's interpolation theorem, we have

$$\|P_t\|_{p_s \to q_s} \leq C^r = C^{ps/(p-1)}.$$

Therefore, for any $f \geq 0$ with $\mu(f^2) = 1$,

$$\int (P_t f^{2(1-s)})^{q_s} \mathrm{d}\mu \leq \|P_t\|_{p_s \to q_s}^{q_s} \leq C^{ps/(p-1)(1-\delta s)}. \tag{3.5}$$

Observing that the equalities in (3.5) also hold for $s = 0$, we obtain

$$\delta\mu(P_t f^2 \log P_t f^2) - \mu(f^2 \log f^2) = \frac{\mathrm{d}}{\mathrm{d}s}\mu\big((P_t f^{2(1-s)})^{q_s}\big)\big|_{s=0} \leq \frac{p}{p-1}\log C.$$

By combining this with (3.4), we prove (3.1).

Theorem 3.1 enables us to present the following interpolation theorem.

Corollary 3.2. *Assume (A). If $\|P_t\|_{p \to q} \leq C$ for some $t, C > 0$ and $q > p > 1$, then*

$$\|P_s\|_{p_s \to q_s} \leq C^{4pq(q_s - p_s)/p_s q_s (q-p)}$$

for any $s > 0$ and $q_s > p_s > 1$ satisfying

$$s \geq \frac{(\exp[Kt]-1)p(q-1)}{2K(q-p)} \log \frac{q_s - 1}{p_s - 1}.$$

Theorem 3.3. *Assume* (A). *We have*

(1) $\|P_t\|_{2\to\infty} < \infty$ *for any* $t > 0$ *if and only if* $\|P_t \exp[\lambda\rho^2]\|_\infty < \infty$ *for any* $\lambda, t > 0$.

(2) P_t *is supercontractive if and only if* $\mu(\exp[\lambda\rho^2]) < \infty$ *for any* $\lambda > 0$.

(3) *If there exists* $\lambda > K/2$ *such that* $\mu(\exp[\lambda\rho^2]) < \infty$, *then* $\|P_t\|_{2\to 4} < \infty$ *for some* $t > 0$ *(equivalently,* (1.1) *holds).*

Proof. By Theorem 3.1, supercontractivity is equivalent to (1.5). Then $\|P_t\|_{2\to\infty} < \exp[m(t)]$ for $t > 0$ implies (1.5) and hence $\mu(\exp[\lambda\rho^2]) < \infty$ for any $\lambda > 0$ (see Corollary 5.5 below). Then $\|P_t \exp[\lambda\rho^2]\|_\infty \le \|P_t\|_{2\to\infty}\|\exp[\lambda\rho^2]\|_2 < \infty$.

On the other hand, assume that $\|P_t \exp[\lambda\rho^2]\|_\infty < \infty$ for any $t, \lambda > 0$. For any f with $\mu(f^2) = 1$, we have (see [33; Lemma 2.1] and its proof)

$$|P_s f(x)|^2 \le P_s f^2(y) \exp\Big[\frac{K\rho(x,y)^2}{1-\exp[-2Ks]}\Big], \tag{3.6}$$

where $\rho(x,y)$ is the Riemannian distance between x and y (recall that $\rho(x) := \rho(x,o)$). This implies that

$$\begin{aligned}
1 &\ge |P_s f(x)|^2 \int \exp\Big[-\frac{K\rho(x,y)^2}{1-\exp[-2Ks]}\Big]\mu(\mathrm{d}y)\\
&\ge |P_s f(x)|^2 \mu(B_o(1)) \exp\Big[-\frac{K(\rho(x)+1)^2}{1-\exp[-2Ks]}\Big],
\end{aligned}$$

where $B_o(r)$ is the geodesic ball with center o and radius r. Then there exist $c_1, c_2 > 0$ such that

$$|P_s f| \le \exp\big[(c_1 + c_2\rho^2)/s\big], \quad s \in (0,1]. \tag{3.7}$$

By combining this with (3.6) we obtain

$$\|P_t\|_{2\to\infty} \le \|P_{t/2}\exp[2(c_1+c_2\rho^2)/t]\|_\infty < \infty, \quad t \in (0,1], \tag{3.8}$$

and the proof of (1) is completed.

Next, (2) follows from (3.7) and Corollary 5.5 below immediately by noting that the supercontractivity of P_t is equivalent to (1.5).

Finally, if there exists $\lambda > K/2$ such that $\mu(\exp[\lambda\rho^2]) < \infty$, then there exists $t > 0$ and $q > p \ge 2$ such that $\|P_t\|_{p\to q} < \infty$. This can be proved by the argument leading to (3.7), just consider f with $\|f\|_p = 1$ instead of $\|f\|_2 = 1$, and apply the Harnack inequality in [33; Lemma 2.1] with power p, see [36] for details. Then, by Riesz-Thorin's interpolation theorem, we have $\|P_t\|_{2\to q} < \infty$ for some $t > 0$ and $q > 2$. In particular, we may take $p = 4$ (for possibly bigger t) by the semigroup property. $\square$

Corollary 3.4. *Assume (A). If $X = \nabla V$, then P_t is ultracontractive if and only if $\|P_t \exp[\lambda\rho^2]\|_\infty < \infty$ for any $\lambda, t > 0$. Furthermore,*

$$\|P_t\|_{1\to\infty} \le \exp[\lambda_1/t]\|P_{t/4}\exp[\lambda_2\rho^2/t]\|_\infty^2$$

for some $\lambda_1, \lambda_2 > 0$.

Proof. Noting that $\|P_t\|_{1\to\infty} = \|P_{t/2}\|_{2\to\infty}^2$ as we have indicated in section 1, the proof is completed by Theorem 3.3 (1) and (3.8). □

Theorem 3.3 enables us to present the following result which is still valid even when the Ricci curvature is not bounded below (see Example 6.1 below).

Corollary 3.5. *Assume that (A) holds and there exists a strictly positive increasing $\gamma \in C(0,\infty)$ such that $\lim_{r\to\infty}\frac{\gamma(r)}{r} = \infty, g_\lambda(r) := r\gamma(\lambda\log r)$ is convex on $[1,\infty)$ for any $\lambda > 0$, and*

$$L\rho^2(x) \le c - \gamma(\rho^2(x)) \tag{3.9}$$

holds for some $c > 0$ and all $x \notin \mathrm{cut}(o)$, the cut locus of o. If

$$\int_2^\infty \frac{dr}{r\gamma(\lambda\log r)} < \infty, \quad \lambda > 0, \tag{3.10}$$

then $\|P_t\|_{2\to\infty} < \infty$ for any $t > 0$. If in particular (3.9) holds for $\gamma(r) = \alpha r^\delta (\alpha > 0, \delta > 1)$, then

$$\|P_t\|_{2\to\infty} \le \exp\left[ct^{-\delta/(\delta-1)}\right] \tag{3.11}$$

for some $c > 0$ and all $t \in (0,1]$. The above assertions are also true for $\|P_t\|_{1\to\infty}$ whenever P_t is symmetric.

Proof. By the proof of Theorem 3.3, it suffices to prove (3.6) for the first assertion. Let x_t be the L-diffusion process (with reflecting boundary if $\partial M \ne \emptyset$). We have (by [22] and the assumption on ∂M)

$$\mathrm{d}\exp[\lambda\rho^2(x_t)] = 2\lambda\sqrt{2}\rho(x_t)\exp[\lambda\rho^2(x_t)]\mathrm{d}b_t + L\exp[\lambda\rho^2](x_t)\mathrm{d}t - \mathrm{d}L_t, \tag{3.12}$$

where b_t is a one-dimensional Brownian motion and L_t is an increasing process with support contained in $\{t \ge 0 : x_t \in \mathrm{cut}(o) \cup \partial M\}$. On the other hand, by (3.9) we have

$$\begin{aligned} L\exp[\lambda\rho^2] &= \lambda\exp[\lambda\rho^2]L\rho^2 + 4\lambda^2\rho^2\exp[\lambda\rho^2] \le \lambda\exp[\lambda\rho^2](c - \gamma(\rho^2) + 4\lambda\rho^2) \\ &\le \lambda\exp[\lambda\rho^2](c_1 - \gamma(\rho^2)/2), \end{aligned} \tag{3.13}$$

where $c_1 = \sup\{c - \frac{1}{2}\gamma(\rho^2) + 4\lambda\rho^2\} < \infty$. For fixed $t > 0$ and $x \in M$, let $h(s) = E^x \exp[\lambda\rho^2(x_s)]$, $s \in [0, t]$. By (3.12) and (3.13),

$$\begin{aligned} h'(s) &\le \lambda c_1 h(s) - \frac{\lambda}{2} E^x\Big\{\exp[\lambda\rho^2(x_s)]\gamma(\rho^2(x_s))\Big\} \\ &\le \lambda c_1 h(s) - \frac{\lambda}{2} h(s)\gamma(\lambda^{-1}\log h(s)) \end{aligned} \tag{3.14}$$

since g_λ is convex on $[1, \infty)$ for any $\lambda > 0$. Taking $c_2 > 1$ such that $\gamma(\lambda^{-1}\log c_2) \ge 4c_1$, we obtain

$$\lambda c_1 h(s) - \frac{\lambda}{2} h(s)\gamma(\lambda^{-1}\log h(s)) \le -\frac{\lambda}{4} h(s)\gamma(\lambda^{-1}\log h(s)), \quad h(s) \ge c_2. \tag{3.15}$$

By (3.14) we see that if $h(0) \le c_2$, then $h(t) \le c_2$. Next, for the case that $h(t) > c_2$, we have $h(0) > c_2$ and $h'(s) \le 0$ for all $s > 0$. Then (3.15) holds for all $s > 0$. Hence, (3.14) implies that

$$h'(s) \le -\frac{\lambda}{4} h(s)\gamma(\lambda^{-1}\log h(s)), \quad \text{if } h(t) > c_2. \tag{3.16}$$

By (3.10), we may define $G(r) = \int_r^\infty \frac{\mathrm{d}r}{r\gamma(\lambda^{-1}\log r)}$ which is finite for $r > 1$. It then follows from (3.16) that, for the case $h(t) > c_2$,

$$G(h(t)) = \int_{h(t)}^\infty \frac{\mathrm{d}r}{r\gamma(\lambda^{-1}\log r)} \ge \int_{h(t)}^{h(0)} \frac{\mathrm{d}r}{r\gamma(\lambda^{-1}\log r)} \ge \frac{\lambda}{4}t.$$

Therefore

$$h(t) \le G^{-1}(\lambda t/4) \vee c_2 \tag{3.17}$$

since G is decreasing, where $G^{-1}(r) = 1$ if $r \ge \sup_{(1,\infty)} G$. This implies (3.6) since t and λ are arbitrary. The first assertion then follows from Theorem 3.3 (1) since x, t and λ are arbitrary.

If (3.9) and (3.10) hold for $\gamma(r) = \alpha r^\delta (\alpha > 0, \delta > 1)$, then $G(r) = \frac{\lambda^\delta}{\alpha(\delta-1)}(\log r)^{1-\delta}$ and

$$c_1 = \sup\{c - \gamma(\rho^2)/2 + 4\lambda\rho^2\} = c_3 + c_4\lambda^{\delta/(\delta-1)}$$

for some $c_3, c_4 > 0$. Then, we may take

$$c_2 = \exp\left[(4c_1/\alpha)^{1/\delta}\lambda\right] \le \exp\left[c_5 + c_6\lambda^{\delta/(\delta-1)}\right]$$

for some $c_5, c_6 > 0$. By (3.17),

$$\begin{aligned}\|P_{t/2}\exp[\lambda\rho^2]\|_\infty &\le G^{-1}(\lambda t/8)\vee c_2\\ &\le \exp\Big[\big(\alpha(\delta-1)t/(8\lambda^{\delta-1})\big)^{1/(1-\delta)}\vee\big(c_5+c_6\lambda^{\delta/(\delta-1)}\big)\Big].\end{aligned}$$

This implies (3.11) by taking $\lambda=\lambda_2/t$ in (3.8). $\square$

We may also obtain a necessary condition corresponding to (3.9). We call a function f compact, if $\{f\le r\}$ is relatively compact for any $r\in\mathbb{R}$.

Corollary 3.6. *Assume (A). If there exists a compact function $\bar\rho\in C^2(M)$ (satisfying Neumann boundary condition whenever $\partial M\ne\emptyset$) such that $\bar\rho>1, \big|\nabla\sqrt{\bar\rho}\big|\le 1$, and*

$$L\bar\rho\ge -c\bar\rho\log\bar\rho \tag{3.18}$$

for some $c>0$, then $\|P_t\|_{2\to\infty}=\infty$ for small $t>0$.

Proof. For any $n\ge 2$, take $h_n\in C_0^\infty[0,\infty)$ such that $0\le h_n\le 1, h_n(r)=1$ for $r\in[0,n], h_n(r)=0$ for $r\ge 2n, -\frac{2}{n}\le h_n'\le 0$. Let $f_n(r)=\int_0^r h_n(s)\mathrm{d}s$ and $F_n=f_n(\exp[\bar\rho])$. We have $|\nabla\bar\rho|^2=4\bar\rho\big|\nabla\sqrt{\bar\rho}\big|^2\le 4\bar\rho, r\log r\le 4f_n(r)\log f_n(r)$ for $r\in[1,2n]$ and $rh_n'(r)\ge -4$. It then follows from (3.18) that

$$\begin{aligned}LF_n &= h_n(\exp[\bar\rho])\exp[\bar\rho]L\bar\rho+\big(h_n(\exp[\bar\rho])+h_n'(\exp[\bar\rho])\exp[\bar\rho]\big)\exp[\bar\rho]|\nabla\bar\rho|^2\\ &\ge -1_{\{\exp[\bar\rho]<2n\}}(c\log(2n)+16)\exp[\bar\rho]\bar\rho\ge-\alpha(n)F_n\log F_n,\end{aligned} \tag{3.19}$$

where $\alpha(n)=4(c\log(2n)+16)$. By (3.3),

$$\begin{aligned}P_t(F_n\log F_n) &\le (P_tF_n)\log(P_tF_n)+\frac{2(\exp[Kt]-1)}{K}P_t|\nabla F_n|^2\\ &\le (P_tF_n)\log(P_tF_n)+c_2tP_t(F_n\log F_n)\end{aligned}$$

for some $c_2>0$ and all $t\in[0,1]$. Let $t\in[0,1/(2c_2)]$, then

$$P_t(F_n\log F_n)\le 2(P_tF_n)\log(P_tF_n).$$

By combining this with (3.19), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}P_tF_n\ge -2\alpha(n)(P_tF_n)\log(P_tF_n),\qquad t\in[0,1/(2c_2)].$$

This implies $P_tF_n\ge\exp\big\{(\log F_n)\exp[-2\alpha(n)t]\big\}$ for $t\le 1/(2c_2)$. Therefore,

$$\begin{aligned}\|P_t\exp[\bar\rho]\|_\infty &\ge \|P_tF_n\|_\infty \ge \exp\big\{(\log n)\exp[-8t(c\log\log(2n)+16)]\big\}\\ &= \exp\big\{\exp[\log\log n - 8tc\log\log(2n) - 128t]\big\},\ t\le 1/(2c_2).\end{aligned}$$

By letting $n\uparrow\infty$ we obtain $\|P_t\exp[\bar\rho]\|_\infty=\infty$ for small $t>0$. The proof is now completed by Theorem 3.3 (2) since $\sqrt{\bar\rho}\le\sqrt{\bar\rho(o)}+\rho$. $\square$

Correspondingly to Corollary 3.6, we have the following result for supercontractivity and hypercontractivity.

Corollary 3.7. *Assume (A).*
(1) *If* $\lim_{\rho\to\infty}\frac{L\rho^2}{\rho^2}=-\infty$, *then* P_t *is supercontractive.*
(2) *If* $\lim_{\rho\to\infty}\sup\frac{L\rho^2}{\rho^2}<-2K$, *then* $\|P_t\|_{2\to4}=\infty$ *for some* $t>0$.
Here limits are taken outside of $\mathrm{cut}(o)$.

Proof. For $\lambda>0$, we have $L\exp[\lambda\rho^2]=\lambda\exp[\lambda\rho^2](L\rho^2+4\lambda\rho^2)$. If $\lim_{\rho\to\infty}\sup\frac{L\rho^2}{\rho^2}<-4\lambda$, then there exist $c_1,c_2>0$ such that $L\exp[\lambda\rho^2]\le c_1-c_2\exp[\lambda\rho^2]$ outside of $\mathrm{cut}(o)$. By this and the proof of Corollary 3.5, we obtain

$$P_t\exp[\lambda\rho^2]\le c_1/c_2+\exp[-c_2t+\lambda\rho^2]. \tag{3.20}$$

This implies that $\mu(\exp[\lambda\rho^2])\le c_1/c_2$. Actually, let $\rho_n=\rho\wedge n$ for $n\ge1$. It follows from (3.20) that

$$\begin{aligned}\mu(\exp[\lambda\rho_n^2]) &= \mu(P_t\exp[\lambda\rho_n^2])\le\mu(\exp[\lambda n^2]\wedge P_t\exp[\lambda\rho^2])\\ &\le \mu(\exp[\lambda n^2]\wedge(c_1/c_2+\exp[-c_2t+\lambda\rho^2])).\end{aligned}$$

The desired assertion follows immediately by first letting $t\uparrow\infty$ then $n\uparrow\infty$. The proof is now completed by Theorem 3.3 (2) and (3). $\square$

Remark. (i) From the proofs we see that Corollaries 3.5 and 3.7 remain true with ρ replaced by any smooth function $\bar\rho\ge\lambda\rho$ for some $\lambda>0$ such that $\langle\nabla\bar\rho,N\rangle\ge 0$ whenever $\partial M\ne\emptyset$, where N denotes the outward normal vector field of ∂M. Corollary 3.7 (1) is sharp in the since that P_t is not supercontractive if $\partial M=\emptyset$, o is a pole and $\lim_{\rho\to\infty}\inf\frac{L\rho^2}{\rho^2}>-\infty$. Actually, in this case $\mu(\exp[\lambda\rho^2])=\infty$ for big enough λ (see the proof of [37; Corollary 1.4]).

(ii) By Theorem 3.1, $\|P_t\|_{2\to4}<\infty$ for some $t>0$ implies the (defective) log-Sobolev inequality (1.1) for μ. It is easy to see from (1.1) that there exist $r,c(r)>0$ such that

$$\mu(f^2)\le c(r)\mu(|\nabla f|^2),\quad f\in C_0^\infty(M), f|_{B_o(r)}=0.$$

Then the Poincaré inequality holds (see [37]), and hence the strict inequality (i.e. (1.1) with $c_2 = 0$) holds. Furthermore, according to [6] and [37], the F-Sobolev inequality implies that the essential spectrum of $L' := \Delta + \nabla \log \phi$ is empty provided L' is essentially self-adjoint on $L^2(\mu)$, where $\phi = \frac{\mathrm{d}\mu}{\mathrm{d}x}$.

(iii) By Theorem 3.3 (2) and (3), it is easy to construct examples to show that supercontractivity is strictly stronger than hypercontractivity. Moreover, from Theorem 3.3 (2) and Corollary 3.6 we see that ultracontractivity is strictly stronger than supercontractivity (see Example 6.4). On the other hand, however, these three contractivity properties are equivalent if the following curvature-dimension condition holds:

$$|\mathrm{Hess}_f|^2 + \mathrm{Ric}(\nabla f, \nabla f) - \langle \nabla_{\nabla f} Z, \nabla f\rangle \geq -K|\nabla f|^2 + \frac{1}{n}(Lf)^2$$

for some $n, K > 0$ and all $f \in C^\infty(M)$. Actually, by the proof of [5; Theorem 2.3], if the above condition holds, then hypercontractivity implies a classical Sobolev inequality with finite dimension, and hence also implies ultracontractivity.

4. Ultracontractivity and supercontractivity by the F-Sobolev inequality and isoperimetric inequalities

Theorem 4.1. *Assume that* $\lim_{r\to\infty} \frac{F(r)}{\log r} = \infty$ *and define* $\beta(r) = \sup_{s>0}\{\log s - rF(s)\}, r > 0$. *Let* $\gamma \in C[0,\infty)$ *be a positive function such that* $c := \int_2^\infty \frac{\gamma(r)}{r}\, dr < \infty$. *If the F-Sobolev inequality* (1.3) *holds, then* $\|P_t\|_{1\to\infty} \leq \exp[m(t)]$ *with*

$$m(t) = 2\int_2^\infty \frac{1}{r^2}\Big\{\frac{c_2 t\gamma(r)}{cc_1} + \beta\Big(\frac{t\gamma(r)}{cc_1}\Big)\Big\}\, dr \in [0,\infty], \quad t > 0.$$

Proof. Since $f^2 \log f^2 \leq rf^2F(r^2) + \beta(r)f^2$, (1.3) implies that

$$\mu(f^2\log f^2) \leq c_1 r\mu(|\nabla f|^2) + c_2 r + \beta(r), \quad r > 0,\ \mu(f^2) = 1.$$

Equivalently,

$$\mu(f^2\log f^2) \leq r\mu(|\nabla f|^2) + c_2 r/c_1 + \beta(r/c_1), \quad r > 0,\ \mu(f^2) = 1.$$

Hence the assertion follows from Theorem 1.2. $\square$

We remark that by combining Theorem 4.1 with [35], one can easily construct examples for infinite Markov chains which are ultracontractive, see for instance Example 6.5 below. The next step is to prove the F-Sobolev inequality from supercontractivity.

Theorem 4.2. *Assume that* (1.5) *holds. Let*

$$\xi(t) = \sup_{r>0} \frac{[\log t - \beta(r)]^+}{r}, \qquad t > 0.$$

Then, for any $\varepsilon \in (0,1)$, (1.3) holds for

$$F(r) = \frac{1}{r}\int_0^r \xi(\varepsilon t)\,dt.$$

Consequently, for any $\varepsilon \in (0,1)$, (1.5) implies (1.3) with

$$F(r) = \frac{\log r}{\beta^{-1}(\varepsilon \log r)} + \frac{1}{r}\int_0^r \frac{ds}{\beta^{-1}(\log s - \varepsilon)}, \tag{4.1}$$

where $\beta^{-1}(\lambda) := \inf\{r \geq 0 : \beta(r) \leq \lambda\}$ and $\beta^{-1}(\lambda) = \infty$ if $\lambda \leq \inf \beta$.

Proof. For any nonnegative $f \in \mathcal{D}$ with $\mu(f^2) = 1$ and any $\delta > 1$, let $A_n = \{\delta^{n+1} > f^2 \geq \delta^n\}$ and $f_n = (f - \delta^{n/2})^+ \wedge (\delta^{(n+1)/2} - \delta^{n/2}), n \geq 0$. By [26; Theorem I-4.12], $f_n \in \mathcal{D}, n \geq 0$. Moreover, it is easy to check that for any $N \geq 1$ and any $x, y \in E$,

$$\sum_{n=0}^N |f_n(x)| = \sum_{n=0}^N f_n(x) = (f(x) - 1)^+ \wedge (\delta^{(N+1)/2} - 1) \leq |f(x)|,$$
$$\sum_{n=0}^N |f_n(x) - f_n(y)| = \left|\sum_{n=0}^N (f_n(x) - f_n(y))\right| \leq |f(x) - f(y)|.$$

Then, following the proofs of Theorem I-4.12 and Corollary I-4.13 in [26], we obtain

$$\sum_{n=0}^N D(f_n, f_n) \leq D(f, f), \quad N \geq 1. \tag{4.2}$$

Next, by (1.5),

$$\mu(f_n^2 \log f_n^2) - \mu(f_n^2)\log \mu(f_n^2) \leq rD(f_n, f_n) + \beta(r)\mu(f_n^2). \tag{4.3}$$

Let $p_n = \mu(f^2 \geq \delta^n)$ and $\mu_n(A) = \mu(A \cap \{f \geq n\delta\})/p_n$, we have

$$\mu(f_n^2 \log f_n^2) = p_n \mu_n(f_n^2 \log f_n^2) \geq \mu(f_n^2)\log[\mu(f_n^2)/p_n].$$

Noting that $p_n \leq \delta^{-n}$, (4.3) yields that

$$\begin{aligned} D(f_n, f_n) &\geq \xi(\delta^n)\mu(f_n^2) \geq \xi(\delta^n)\mu(f^2 \geq \delta^{n+1})(\delta^{(n+1)/2} - \delta^{n/2})^2 \\ &\geq \frac{(\sqrt{\delta} - 1)^2}{1 - \delta^{-1}}\int_{\delta^{n-1}}^{\delta^n} \xi(t)\mu(f^2 \geq \delta^2 t)\mathrm{d}t. \end{aligned}$$

Combining this with (4.2) and noting that $\xi(t) = 0$ for $t \leq 1$, we obtain

$$D(f, f) \geq c(\delta)\int_0^\infty \xi(\delta^{-2}t)\mu(f^2 \geq t)\mathrm{d}t = c(\delta)\mu(f^2 F(f^2))$$

for some $c(\delta) > 0$ and $F(r) = \frac{1}{r}\int_0^r \xi(\delta^{-2}t)\mathrm{d}t$. This proves the first assertion.

Next, taking $r = \beta^{-1}(\sqrt{\varepsilon}\log t)$, we see that

$$\xi(t) \geq \frac{(1-\sqrt{\varepsilon})\log t}{\beta^{-1}(\sqrt{\varepsilon}\log t)}.$$

Then

$$\frac{1}{r}\int_0^r \xi(\varepsilon t)\mathrm{d}t \geq \frac{1}{r}\int_{\varepsilon t}^t \frac{(1-\sqrt{\varepsilon})\log(\varepsilon t)}{\beta^{-1}(\sqrt{\varepsilon}\log(\varepsilon t))}\mathrm{d}t \geq c_1 + \frac{c_2\log r}{\beta^{-1}(\varepsilon\log r)}$$

for some $c_1, c_2 > 0$ (depending on ε). Therefore, (1.3) holds for $F(r) = \frac{\log(\varepsilon r)}{\beta^{-1}(\varepsilon\log r)}$.

Finally, taking $r = \beta^{-1}(\log t - \varepsilon)$, it follows that $\xi(t) \geq \varepsilon/[\beta^{-1}(\log t - \varepsilon)]$. Similarly, we prove (1.3) for

$$F(r) = \frac{1}{r}\int_0^r \frac{\mathrm{d}t}{\beta^{-1}(\log t - 2\varepsilon)}. \quad \square$$

To see that the result in Theorem 4.2 is sharp, we present the following simple consequences.

Corollary 4.3. (1) *The F-Sobolev inequality holds for $F(r) = [\log(r+e)]^\delta (\delta > 1)$ if and only if* (1.5) *holds for $\beta(r) = \lambda_1 r^{-1/(\delta-1)} + \lambda_2$.*

(2) *For any $p \geq 1$, the F-Sobolev inequality holds for $F(r) = r^{2/p}$ if and only if* (1.5) *holds for $\beta(r) = \frac{p}{2}\log[c(1+r^{-1})]$ and some $c > 0$. They are also equivalent to* (1.2), *which in turn is equivalent to* (*see* [3; *section* 4]):

$$\mu(f^2\log f^2) \leq \frac{p}{2}\log[c_1 D(f,f) + c_2], \quad f \in \mathcal{D},\ \mu(f^2) = 1. \tag{4.4}$$

Proof. The sufficiency of (1) follows from Theorem 4.2 by taking $F(r) = \frac{\log r}{\beta^{-1}(r/2)}$, while the necessity follows from the choice of β specified in Theorem 4.1. Next, it is easy to see that (4.4) is equivalent to (1.5) for $\beta(r) = \frac{p}{2}\log[c(1+r^{-1})]$ and some $c > 0$. They imply (1.3) for $F(r) = r^{2/p}$ by taking F as the second term in (4.1). Noting that

$$\sup\{t\log r - r^{2/p} : r > 0\} = \frac{p}{2}t\log[t + p/(2\mathrm{e})], \quad t \geq 0,$$

(1.3) with $F(r) = r^{2/p}$ then implies

$$t\mu(f^2\log f^2) - \frac{p}{2}t\log[t + p/(2\mathrm{e})] \leq c_1 D(f,f) + c_2, \quad f \in \mathcal{D},\ \mu(f^2) = 1$$

for some $c_1, c_2 > 0$. This yields (1.5) for the desired β. $\square$

Moreover, (1.5) may also imply an isoperimetric inequality which is equivalent to the stronger F-Sobolev inequality (1.4). To this end, we adopt Ledoux's trick [24] to establish the corresponding isoperimetric inequalities. We shall use the following gradient estimate to apply Ledoux's argument:

$$\|\nabla P_t f\|_\infty \le \frac{c}{\sqrt{t}}\|f\|_\infty, \qquad t\in(0,1], \tag{4.5}$$

where P_t is the semigroup of the (reflecting) L-diffusion process with $L=\Delta+X$ for some C^1-vector field X. If condition (A) holds and ∂M is either convex or empty, then (4.5) holds (see [32] for detailed estimates). Our next result is an analogy of [6; Theorem 4.2] which treated the case where $p<\infty$ in (1.2).

Theorem 4.4. *Let P_t be as in above with $X=\nabla V$ for some $V\in C^2(M)$. Assume that (4.5) holds. If (1.5) holds for some decreasing positive function $\beta\in C(0,\infty)$, then for any $c\in(0,1/16)$, there exists $r_0\in(0,1)$ such that*

$$\inf_{\mu(A)\le r_0}\frac{\mu_\partial(\partial A\setminus\partial M)\sqrt{\beta^{-1}(c\log\mu(A)^{-1})}}{\mu(A)\sqrt{\log\mu(A)^{-1}}}>0, \tag{4.6}$$

where $\mu=\exp[V]dx$ and β^{-1} is as in Theorem 4.2. Consequently, (1.4) hold for $D(f,f)=\mu(|\nabla f|^2)$ and $F(r)=\frac{\log(r+\mathrm{e})}{\beta^{-1}(c\log(r+\mathrm{e}))}$ provided $\lim_{r\to\infty}\sup\frac{-c}{r\beta'(r)}<1$.

Proof. By Gross Theorem [18] (see also [16; Theorem 6.1.14]), (1.5) implies that

$$\|P_t\|_{p_t\to 2}\le\exp\big[4\beta(r)(1/p_t-1/2)\big], \quad p_t=1+\exp[-4t/r]. \tag{4.7}$$

According to [24], (4.5) implies that

$$2c\sqrt{t}\mu(|\nabla f|)\ge\mu(|f-P_tf|).$$

By taking test functions $f_n\in C^\infty(M)$ such that $f_n(x)=1$ for $x\in A$ with $\mathrm{dist}(x,\partial A\setminus\partial M)\ge 1/n$, $f_n(x)=0$ for $x\notin A$, and $|\nabla f_n|\le 1/n+1/n^2$, and then letting $n\uparrow\infty$ in the above inequality, we obtain

$$2c\sqrt{t}\mu_\partial(\partial A\setminus\partial M)\ge 2\mu(A)-2\mu(1_AP_t1_A)=2\mu(A)-2\mu((P_{t/2}1_A)^2).$$

Combining this with (4.7), we obtain

$$c_1\sqrt{t}\mu_\partial(\partial A\setminus\partial M)\ge\mu(A)\big\{1-\mu(A)^{(2-p_t)/p_t}\exp\big[8\beta(r)(2-p_t)/(2p_t)\big]\big\} \tag{4.8}$$

for some $c_1>0$, any $t\in[0,1]$ and any connected smooth domain A with $\mu(A)\le 1$. Let

$$r=\beta^{-1}(c\log\mu(A)^{-1}), \quad t=\frac{r\log 2}{\log\mu(A)^{-1}-16\beta(r)}=\frac{r\log 2}{(1-16c)\log\mu(A)^{-1}}.$$

Then, there exists $r_0 \in (0,1)$ such that $t \le \frac{r}{4} \le 1$ if $\mu(A) \le r_0$ (note that $\beta(0) = \infty$ so that $r \to 0$ as $\mu(A) \to 0$).

On the other hand, we have $\frac{2-p_t}{2p_t} \le \frac{2t}{r}$ and $\frac{2-p_t}{p_t} \ge \frac{t}{r}$ if $t \le \frac{r}{4}$. Then, for $\mu(A) \le r_0$,

$$\mu(A)^{(2-p_t)/p_t} \exp\left[8\beta(r)(2-p_t)/(2p_t)\right] \le \mu(A)^{t/r} \exp[16\beta(r)t/r] = \frac{1}{2}.$$

By combining this with (4.8), we prove the first assertion.

Next, let $c' \in (c, 1/16)$ and $r_1 \in (0, r_0)$ be such that $c' \log r_1^{-1} \ge c \log(r_1^{-1} + \mathrm{e})$ and (4.6) holds with c, r_0 replaced by c', r_1 respectively. We have

$$\begin{aligned} k(r_1) &:= \inf_{\mu(A)\le r_1} \frac{\mu_\partial(\partial A \setminus \partial M)}{\mu(A)\sqrt{F(\mu(A)^{-1})}} \\ &\ge \inf_{\mu(A)\le r_1} \frac{\mu_\partial(\partial A \setminus \partial M)\sqrt{c\beta^{-1}(c' \log \mu(A)^{-1})}}{\mu(A)\sqrt{c' \log \mu(A)^{-1}}} > 0. \end{aligned}$$

The second assertion then follows from Theorem 1.1 by noting that

$$\limsup_{r\to\infty} \frac{rF'(r)}{F(r)} = \limsup_{r\to 0} \frac{-c}{r\beta'(r)}. \quad \square$$

Remark. The condition $\lim_{r\to\infty} \frac{-c}{r\beta'(r)} < 1$ in Theorem 4.4 is trivial when the limit exists. Actually, let $r > 0$ so that $B_o(r)$ is convex and denote by P_t^D the Dirichlet semigroup of L on $B_o(r)$. Then, there exist $\lambda_1, \lambda_2 > 0$ such that (see [34; Theorem 2.1])

$$\exp[\beta(\lambda_1 t)] \ge \|P_t\|_{2\to 4} \ge \|P_t^D\|_{2\to 4} \ge \lambda_2 t^{-d/2}, \quad t \in (0,1]. \tag{4.9}$$

If $\lim_{r\to 0} \frac{-c}{r\beta'(r)} \ge 1$, then $-\beta'(r) \le 1/(16r), r \in (0, r_0]$ for some $r_0 > 0$. This implies that $\beta(r) \le \beta(r_0) - \frac{1}{16}\log\frac{r}{r_0}$ for $r \in (0, r_0)$ since $c < 1/16$, which is a contradiction to (4.9) by taking $r = \lambda_1 t$ and letting $t \to 0$.

To see that Theorems 4.1 and 4.4 are sharp, we consider once again the concrete case where $F(r) = [\log(1+r)]^\delta, \delta > 1$.

Corollary 4.5. *Let $\delta > 1$. Then the F-Sobolev inequality* (1.3) *holds for $F(r) = [\log(\mathrm{e}+r)]^\delta$ is equivalent to*

$$\|P_t\|_{1\to\infty} \le \exp\left[ct^{-1/(\delta-1)}\right], \quad t \in (0,t] \tag{4.10}$$

for some $c > 0$. Especially, in the situation of Theorem 4.4, (1.3) *and* (1.4) *are equivalent for this concrete F.*

Proof. The first assertion is a direct consequence of Theorems 1.2 and 4.1 and Corollary 4.3, as well as the second assertion follows from Theorems 1.1 and 4.4. $\square$

Corollary 4.6. *Under the assumptions of Theorem* 4.4 *and Corollary* 2.3, *let* $V = -\alpha\rho^\delta(\alpha > 0, \delta > 2)$. (1.3) *and* (1.4) *hold for* $F(r) = [\log(\mathrm{e}+r)]^\lambda$ *if and only if* $\lambda \le 2(\delta-1)/\delta$.

Proof. By Corollary 2.3, we have (2.10) which implies (4.10) for some $c > 0$. The proof is completed by Corollary 4.5 and Corollary 5.4. □

Theorem 4.4 also enables us to estimate the decay of the reference measures for ultracontractive diffusion semigroups, which is related to the concentration property studied in the next section.

Corollary 4.7. *Under the assumptions of Theorem* 4.4 *let* $o \in M$ *be fixed. If* μ *is finite and* (1.5) *holds, then*

$$\mu(B_o(r)^c) \le \exp\big[-G^{-1}(\lambda_1 r - \lambda_2)\big] \tag{4.11}$$

for some $\lambda_1, \lambda_2 > 0$ *and all* $r > 0$, *where*

$$G(r) = \int_{r_1}^r \sqrt{\beta^{-1}(cs)/s}\, ds$$

for some $r_1 > 1$ *and* $c \in (0, 1/16)$ *with* $cr_1 > \inf m$. *Consequently, if* $\beta(t) = \alpha t^{-\delta}(t \in (0,1])$ *for some* $\alpha > 0$ *and* $\delta \ge 1$, *then there exist* $\lambda_4, \lambda_5 > 0$ *such that*

$$\mu(B_o(r)^c) \le \begin{cases} \exp[\lambda_4 - \lambda_5 r^{2\delta/(\delta-1)}], & \text{if } \delta > 1 \\ \exp[\lambda_4 - \exp(\lambda_5 r)], & \text{if } \delta = 1. \end{cases} \tag{4.12}$$

Finally, if $G(\infty) < \infty$(*which is the case when* $\delta < 1$), *then* M *is compact.*

Proof. Let $f(r) = \mu(B_o(r)^c)$, then $f(\infty) = 0$ since μ is finite. We may take $r_0 \in (0,1)$ such that $-c\log r_0 > \inf\beta$ and (4.6) holds. Then there exist $c_1 > 0$ and $r_1 > 0$ such that $f(r_1) \le r_0$ and

$$\frac{-f'(r)\sqrt{\beta^{-1}(-c\log f(r))}}{f(r)\sqrt{-\log f(r)}} \ge c_1, \quad r \ge r_1.$$

This implies that $\frac{\mathrm{d}}{\mathrm{d}r}G(-\log f(r)) \ge c_1$ for $r \ge r_1$. Hence

$$G(-\log f(r)) \ge c_1(r - r_1) + G(-\log f(r_1)), \quad r \ge r_1$$

which proves (4.11) for $r \ge r_1$ and hence for all $r > 0$ with proper choices of $\lambda_1, \lambda_2 > 0$.

Next, if $\beta(t) = \alpha(1 \vee t^{-\delta})$ for some $\alpha > 0$ and $\delta \ge 1$, then $G(r) = c_2 r^{(\delta-1)/(2\delta)} - c_3$ for $\delta > 1$ and $= c_4 \log r$ for $\delta = 1$, for some $c_2, c_3, c_4 > 0$ and all $r \ge r_1$. (4.12) then follows from (4.11) immediately.

Finally, if $G(\infty) < \infty$, then we can define

$$h(r)=\int_r^\infty \sqrt{\beta^{-1}(cs)/s}\,\mathrm{d}s,\quad r\ge r_1.$$

For any $x\in M$, let $f_x(r)=\mu(B_x(r))$, i.e. the measure of the geodesic ball with center x and radius r. By (4.6) we have $\frac{\mathrm{d}}{\mathrm{d}r}h(-\log f_x(r))\ge c_5$ for some $c_5>0$ and all $r>0$ satisfying $f_x(r)\le r_0\wedge\exp[-r_1]$. Then $h(-\log f_x(r))\ge c_5 r$ if $f_x(r)\le r_0\wedge\exp[-r_1]$. Therefore,

$$\inf_x \mu(B_x(r))\ge \exp[-h^{-1}(c_5 r)]\wedge r_0\wedge\exp[-r_1]>0,\quad r>0.$$

Hence M is compact since μ is finite. $\square$

5. Concentration of reference measures

In this section, we study the concentration property of reversible measures for supercontractive semigroups. Let us recall some general results by Gross and Rothaus [20], see also [1], [14], [25] and [27] for earlier results.

Let P_t be a symmetric Markov semigroup on $L^2(E;\mathrm{d}\mu)$ with μ a probability measure on E. Denote by $(D,\mathcal{D})$ the corresponding Dirichlet form. For $f\in\mathcal{D}$ with $f^2\in\mathcal{D}$, define

$$\Gamma(f)^2=\sup\{D(gf,f)-D(g,f^2)/2: g\in\mathcal{D}\text{ such that } fg\in\mathcal{D}\text{ and }\mu(|g|)=1\}.$$

See [2] for a natural extension of Γ to a bigger domain. Note that $\Gamma(f)=\|\nabla f\|_\infty$ for diffusion semigroups. Let $a=1$ for diffusion semigroups and $=2$ otherwise.

If (1.5) holds, by [20; Theorem 3.1] we have $\mu(\exp[\lambda\rho])<\infty$ for any $\lambda>0$ and $\rho\ge0$ with $\Gamma(\rho)\le1$. Moreover, let $c=\log\mu(\exp[\rho])$, then

$$\mu(\exp[\lambda\rho])\le\exp\left[c\lambda+\lambda\int_1^\lambda\Big\{a\gamma(r)+\frac{\beta(\gamma(r))}{r^2}\Big\}\mathrm{d}r\right]\tag{5.1}$$

for any positive $\gamma\in C[0,\infty)$ and any $\lambda\ge1$. Next, let $\phi:[0,\infty)\to[0,\infty)$ be left-continuous and increasing with $\phi(0)=0$. Let $\Phi(\lambda)=\int_0^\lambda\phi(r)\mathrm{d}r,\Psi(\lambda)=\int_0^\lambda\phi^{-1}(r)\mathrm{d}r$, where $\phi^{-1}(\lambda):=\inf\{s\ge0:\phi(s)\ge\lambda\}$.

Theorem 5.1([20; Theorem 4.3]). *Assume that* (1.5) *holds and* $\rho\ge0$ *with* $\Gamma(\rho)\le1$. *Let* Φ *and* Ψ *be defined as above such that* $\Psi\in C^2[R,\infty)$ *with* $\kappa:=\sup_{[R,\infty)}\Psi''<\infty$ *and* $\phi^{-1}\circ\phi(s)=s$ *on* $[R,\infty)$ *for some* $R>1$. *For any* $r_0>R$ *with* $\phi(r_0)\ge R$, *we have*

$$\mu\big(\exp[\Phi(\rho)1_{\{\rho\ge r_0\}}]\big)\le\exp[\kappa/2]\int_1^\infty\mu(\exp[\lambda\rho])\exp[-\Psi(\lambda)]d\lambda.\tag{5.2}$$

By combining (5.1) and (5.2) and taking a concrete ϕ, we obtain the following result which is sharp as is shown in Example 6.3 below.

Theorem 5.2. *Assume that* (1.5) *holds and let* c, a *and* ρ *be in* (5.1) *and* (5.2). *Let* $H(r) = \frac{\beta(r)}{r}, r > 0$, *and define*

$$\gamma(r) = \begin{cases} \frac{c+1}{2a}, & \text{if } r \in [0,1], \\ H^{-1}(ar^2), & \text{otherwise}, \end{cases}$$

$$\psi(\lambda) = 2a\lambda\gamma(\lambda) + 2a\int_0^\lambda \gamma(r)\,dr, \quad \lambda > 0.$$

We have $\psi'(\lambda) > 0$ *for* $\lambda > 0$. *Let* $\Phi(\lambda) = \int_0^\lambda \psi^{-1}(r)\,dr$, *where* $\psi^{-1}(\lambda) = \infty$ *for* $r \geq \psi(\infty)$. *We have* $\mu(\exp[\Phi(\rho)]) < \infty$. *Consequently,* $\|\rho\|_\infty \leq \psi(\infty)$.

Proof. Assume that $c > 0$, otherwise we have $\rho = 0$, μ-a.e. Let $\Psi(\lambda) = \int_0^\lambda \psi(r)\mathrm{d}r$. It is easy to see that $a\gamma(r) = \beta(\gamma(r))/r^2$ for $r > 1$ and

$$\Psi(\lambda) = 2a\lambda\int_0^\lambda \gamma(r)\mathrm{d}r = (c+1)\lambda + 2a\lambda\int_1^\lambda \gamma(r)\mathrm{d}r, \quad \lambda > 0.$$

By (5.1),

$$\mu(\exp[\lambda\rho]) \leq \exp\left[c\lambda + 2a\lambda\int_1^\lambda \gamma(r)\mathrm{d}r\right] = \exp[-\lambda + \Psi(\lambda)], \quad \lambda > 0. \tag{5.3}$$

Noting that γ is constant on $[0,1]$ and for $r > 1$,

$$\begin{aligned} \gamma'(r) &= \frac{2ar}{H'(\gamma(r))} = \frac{-2ar\gamma(r)^2}{\beta(\gamma(r)) - \gamma(r)\beta'(\gamma(r))} > \frac{-2ar\gamma(r)^2}{\beta(\gamma(r))} \\ &= \frac{-2ar\gamma(r)}{H(\gamma(r))} = \frac{-2\gamma(r)}{r}, \end{aligned}$$

we see that $\psi'(r) = 4a\gamma(r) + 2ar\gamma'(r) > 0, r > 0$. Since γ is non-increasing, we have $0 \leq \Psi''(r) \leq 4a\gamma(1)$ for $r \geq 1$. If $\psi(\infty) = \infty$, we see that $\phi = \psi^{-1}$ is finite in $(0, \infty)$ and $\psi \circ \phi(s) = s, \phi(0) = 0, \phi(\infty) = \infty$. The first assertion then follows from Theorem 5.1.

For the case that $\psi(\infty) < \infty$, let $\psi_\varepsilon(r) = \psi(r) + \varepsilon r$ and $\Phi_\varepsilon(\lambda) = \int_0^\lambda \psi_\varepsilon^{-1}(r)\mathrm{d}r, \varepsilon > 0$. Since $\psi_\varepsilon^{-1} \uparrow \phi$ as $\varepsilon \downarrow 0$ and $\phi(\psi(\infty)) = \infty$, we have $\psi_\varepsilon^{-1}(R) > R$ for some $R \in (1, \psi(\infty))$ and small $\varepsilon > 0$, recall that $\psi(\infty) > \psi(1) > 1$. By (5.2) and (5.3),

$$\mu\big(\exp[\Phi_\varepsilon(\rho)]1_{\{\rho \geq R\}}\big) \leq \exp[2a\gamma(1) + \varepsilon/2]\int_1^\infty \exp[-\lambda]\mathrm{d}\lambda.$$

By letting $\varepsilon \downarrow 0$ we obtain that $\mu(\exp[\Phi(\rho)]) < \infty$. $\square$

Our choice of γ in Theorem 5.2 is optimal in terms of (5.1) and (5.2) in the sense that for any positive $\bar\gamma$,

$$a\bar\gamma(r)+\frac{\beta(\bar\gamma(r))}{r^2}\geq \frac{a}{2}\gamma(r)+\frac{\beta(\gamma(r))}{2r^2}=a\gamma(r),\quad r>1.$$

This follows from the facts that $a\gamma(r)=\beta(\gamma(r))/r^2$ for $r>1$ and β is decreasing.

The following is an alternative result to Theorem 5.2, which is simpler and is sharp in several concrete cases (see Section 6 below).

Theorem 5.3. *Assume that* (1.5) *holds and let* ρ, H *be as in Theorem* 5.2. *Define*

$$\xi(s)=\inf\Big\{\lambda\geq 1: 2a\int_1^\lambda H^{-1}(ar^2)\,dr\geq s\Big\},\quad s>0.$$

We have

$$\mu\big(\exp[\delta(\rho-c-\sigma)\xi(\varepsilon\rho)]\big)<\infty,\quad \textit{for any } \sigma>0,\ \varepsilon\in(0,1),\ \delta\in(0,1-\varepsilon). \tag{5.4}$$

Consequently, $\|\rho\|_\infty<\infty$ *if* $\int_1^\infty H^{-1}(ar^{-2})\,dr<\infty$.

Proof. Let

$$h(\lambda)=\exp\Big[-(c+\sigma)\lambda-2a\lambda\int_1^\lambda H^{-1}(ar^2)\mathrm{d}r\Big],\quad \lambda>0.$$

By (5.1) with $\gamma(r)=H^{-1}(ar^2)$ we have

$$\int \mathrm{d}\mu\int_1^\infty \exp[\lambda\rho]h(\lambda)\mathrm{d}\lambda=\int_1^\infty \mu(\exp[\lambda\rho])h(\lambda)\mathrm{d}\lambda<\infty. \tag{5.5}$$

For any $\varepsilon\in(0,1)$ and $\delta\in(0,1-\varepsilon)$, there exists $c(\varepsilon,\delta)>0$ such that

$$\begin{aligned}\int_1^\infty \exp[\lambda\rho]h(\lambda)\mathrm{d}\lambda&\geq\int_1^{\xi(\varepsilon\rho)}\exp\big[((1-\varepsilon)\rho-c-\sigma)\lambda\big]\mathrm{d}\lambda\\&\geq c(\varepsilon,\delta)\exp[\delta(\rho-c-\sigma)\xi(\varepsilon\rho)]-\frac{1}{c(\varepsilon,\delta)},\quad \rho>c-\sigma.\end{aligned}$$

Therefore, (5.5) implies (5.4). $\square$

Theorems 5.2 and 5.3 enable us to extend (4.12) to a general setting.

Corollary 5.4. *Assume that* (1.5) *holds for* $\beta(r) = \lambda_1 + \lambda_2 r^{-\delta}(\lambda_1, \lambda_2, \delta > 0)$. *Let* $\rho \geq 0$ *be such that* $\Gamma(\rho) \leq 1$. *Then there exists* $\alpha = \alpha(\delta) > 0$ *such that*

$$\begin{cases} \mu\big(\exp[\alpha\rho^{2\delta/(\delta-1)}]\big) < \infty, & \text{if } \delta > 1, \\ \mu\big(\exp[\exp[\alpha\rho]]\big) < \infty, & \text{if } \delta = 1, \\ \|\rho\|_\infty < \infty, & \text{if } \delta < 1. \end{cases} \tag{5.6}$$

Proof. We use the notation in Theorems 5.2 and 5.3. Take $r_0 > 0$ such that $H(r_0) := \frac{\beta(r_0)}{r_0} \leq 1$. Let $c_1 > \lambda_2$ be such that $\beta(r) \leq c_1 r^{-\delta}$ for $r \in (0, r_0]$. We have $r \leq r_0$ for $H(r) \geq 1$ and hence

$$\begin{aligned} \gamma(\lambda) &:= H^{-1}(a\lambda^2) = \sup\{r \geq 0 : H(r) \geq a\lambda^2\} \\ &\leq \sup\{r \geq 0 : c_1 r^{-1-\delta} \geq a\lambda^2\} = [c_1/(a\lambda^2)]^{1/(1+\delta)}, \quad \lambda > 1. \end{aligned}$$

By Theorem 5.3 we have $\|\rho\|_\infty < \infty$ if $\delta < 1$. Next, it is easy to see that there exists $c_2 > 0$ such that

$$\xi(\lambda) \geq \begin{cases} c_2\lambda^{(\delta+1)/(\delta-1)}, & \delta > 1, \\ \exp[c_2\lambda], & \delta = 1. \end{cases}$$

Then the proof is completed by Theorem 5.3.

To apply Theorem 5.2, we observe that there exists $c_3 > 0$ such that

$$\psi(\lambda) \leq \begin{cases} c_3\lambda^{(\delta-1)/(\delta+1)}, & \text{if } \delta \neq 1, \\ c_3 \log(1+\lambda), & \text{if } \delta = 1; \lambda > 0. \end{cases}$$

For the case that $\delta < 1$, one has $\psi(\infty) < \infty$, then $\|\rho\|_\infty < \infty$ by Theorem 5.2. If $\delta > 1$, we have

$$\begin{aligned} \phi(\lambda) &:= \psi^{-1}(r) = \inf\{r \geq 0 : \psi(r) \geq \lambda\} \\ &\geq \inf\{r \geq 0 : c_3 r^{(\delta-1)/(\delta+1)} \geq \lambda\} = (\lambda/c_3)^{(\delta+1)/(\delta-1)}. \end{aligned}$$

Then $\Phi(\lambda) \geq c_4\lambda^{2\delta/(\delta-1)}$ for some $c_4 > 0$ and the first inequality in (5.6) follows from Theorem 5.2 immediately. The proof of the case $\delta = 1$ is similar. □

Corollary 5.5. *If* (1.5) *holds, then* $\mu(\exp[\lambda\rho^2]) < \infty$ *for any* $\lambda > 0$ *and* $\rho \geq 0$ *with* $\Gamma(\rho) \leq 1$.

Proof. By Theorem 5.2, it suffices to prove the case where $\psi(\infty) = \infty$. Since $\gamma(r) \downarrow 0$ as $r \uparrow \infty$, we have $\psi(\lambda) \leq \varepsilon\lambda$ for any $\varepsilon > 0$ and big enough λ. Then, there exists $\lambda_\varepsilon > 0$ such that $\phi(\lambda) := \psi^{-1}(\lambda) \geq \lambda/\varepsilon$ for $\lambda \geq \lambda_\varepsilon$, hence $\Phi(\rho) \geq \frac{1}{2\varepsilon}(\rho^2 - r_\varepsilon^2)$. The proof is completed by Theorem 5.2, since ε is arbitrary. □

6. Some examples

Below we present five examples to illustrate our results. They are also interesting by themselves with respect to sharp conditions in concrete cases.

Example 6.1. Let M be a complete connected Riemannian manifold with pole o and non-positive sectional curvatures. By Hessian comparison theorem, $-\mathrm{Hess}_V \geq \alpha\delta(\delta-1)\rho^{\delta-2}$ for $V=-\alpha\rho^\delta$, where $\alpha>0, \delta>2$ and ρ is the Riemannian distance function from o. If $\mathrm{Ric}_x \geq -c-\alpha\delta(\delta-1)\rho^{\delta-2}(x)$ for some $c>0$ and all $x\in M$, then $\mathrm{Ric}-\mathrm{Hess}_V$ is bounded below. Let $k(r)=\sqrt{(c+\alpha\delta(\delta-1)r^{\delta-2})/(d-1)}$, where d is the dimension of M. Then $\mathrm{Ric}\geq -k(\rho)^2(d-1)$ on $B_o(\rho)$. By Bishop's comparison theorem (see for instance [10; Theorems 3.8 and 3.9]), we have $\mathrm{volume}(B_o(\rho)) \leq \lambda\rho^d\exp[(d-1)\rho k(\rho)]$ for some $\lambda>0$ and

$$L\rho^2=\Delta\rho^2-2\alpha\delta\rho^\delta \leq 2+2(d-1)\rho k(\rho)\coth(\rho k(\rho))-2\alpha\delta\rho^\delta.$$

Therefore, there exist $c_1,c_2>0$ such that

$$L\rho^2\leq c_1-c_2\rho^\delta$$

and $\int\exp[V+\varepsilon\rho^\delta]\mathrm{d}x<\infty$ for $\varepsilon<\alpha$. Then, by Corollary 3.5, the semigroup is ultracontractive and

$$\|P_t\|_{1\to\infty}\leq \exp[ct^{\delta/(2-\delta)}]$$

for some $c>0$ and all $t\in(0,1]$. Consequently, by Corollary 4.5, the F-Sobolev inequality (1.3) (also (1.4)) holds for $F(r)=[\log(\mathrm{e}+r)]^{2(\delta-1)/\delta}$. Moreover, by Corollary 4.5 and Example 6.3 below, (1.3) and (1.4) hold for $F(r)=(\log(\mathrm{e}+r))^2$ if $V=-\exp[\alpha\rho]$ for some $\alpha>0$ and $\mathrm{Ric}-\mathrm{Hess}_V$ is bounded below.

Finally, the next example shows that the F-Sobolev inequality does not hold for finite μ on noncompact M and $F(r)=[\log(\mathrm{e}+r)]^\lambda$ with $\lambda>2$.

Example 6.2. Let $L=\Delta+\nabla V$ on a complete connected Riemannian manifold M, with $\mathrm{d}\mu=\exp[V]\mathrm{d}x$ a probability measure. If the F-Sobolev inequality (1.3) holds for $F(r)=[\log(\mathrm{e}+r)]^\lambda$, or more weakly (Corollary 4.5) $\|P_t\|_{1\to\infty}\leq\exp[c_1+c_2t^{-1/(\lambda-1)}]$ for some $\lambda>2, c_1,c_2>0$, then M is compact. This follows from Corollary 5.4 by letting ρ be the Riemannian distance from a fixed point.

Example 6.3. Let M be a noncompact, connected complete Riemannian manifold with Ricci curvature bounded below. Let V be as in Example 6.1 with ρ the distance function from a fixed point. By Corollary 2.3 and Corollary 5.4, there exist $c_1,c_2>0$ such that $\|P_t\|_{1\to\infty}\leq\exp[\lambda_1+\lambda_2t^{-\lambda}]$ (necessarily, $\lambda\geq1$ by Example 6.2) if and only if $\delta\geq 2\lambda/(\lambda-1)$.

Next, if $V=-\exp[\alpha\rho]$ for some $\alpha>0$, we have the above estimate of $\|P_t\|_{1\to\infty}$ for $\lambda=1$. Actually, we may replace ρ by $h(\rho)$ for some $h\in C^\infty[0,\infty)$ satisfying $h'(0)=h(0)=0, h'\geq0$ and $h(r)=r$ for $r\geq0$. Then $\Delta h(\rho)$ is locally bounded at o. Let $\overline{V} = \exp[-\alpha h(\rho)]$. It is easy to see that $\Delta \exp[\overline{V}] \geq \exp[\overline{V}]\eta$ in the distribution sense with $\eta = c_1 \exp[2\alpha\rho] - c_2$ for some $c_1, c_2 > 0$. Let β be as in the proof of Corollary 2.3 and β_1 be defined as in Proposition 2.1 with the above η and $W = c\rho$ for big enough $c > 0$. Then $\beta_1 \leq c_3(1 + r^{-1})$ for some $c_3 > 0$ and (2.8) holds with $\mathrm{d}\mu$ replaced by $\mathrm{d}\bar{\mu} = \exp[\overline{V}]\mathrm{d}x$. Noting that $\|V - \overline{V}\|_\infty < \infty$, we see that there exists $c_4 > 0$ such that

$$\mu(f^2 \log f^2) \leq r\mu(|\nabla f|^2) + c_4(1 + r^{-1}), \qquad r > 0,$$

By taking $\gamma(r) = r^{-1/2}$, Corollary 2.2 implies $\|P_t\|_{1\to\infty} \leq \exp[m(t)]$ with

$$m(t) = c_5 \int_2^\infty \left(\frac{1}{r^2} + \frac{1}{tr^{3/2}}\right)\mathrm{d}r \leq \lambda_1 + \lambda_2 t^{-1}$$

for some $c_5, \lambda_1, \lambda_2 > 0$.

We now present the following example to show that ultracontractivity is strictly stronger than supercontractivity.

Example 6.4. Let M be a noncompact, connected complete Riemannian manifold without boundary. Assume that the Ricci curvature is bounded below and the sectional curvatures are non-positive. Let $o \in M$ be a pole and let P_t be the diffusion semigroup generated by $L = \Delta + \nabla V$ with $V = -\alpha\rho^2[\log(\rho^2 + \mathrm{e})]^\delta, \alpha, \delta > 0$. It is easy to see from the curvature conditions that $\mathrm{Ric}-\mathrm{Hess}_V$ is bounded below. Then, by Theorem 3.3 (2), P_t is supercontractive.

On the other hand, we claim that P_t is ultracontractive if and only if $\delta > 1$. Actually, it is easy to check that (3.9) holds for $\gamma(r) = \alpha r[\log(r + \mathrm{e})]^\delta$. Then the sufficiency follows from Corollary 3.5. Next, for $\delta \leq 1$,

$$L\rho^2 = \Delta\rho^2 - \langle \nabla V, \nabla\rho^2\rangle \geq -8\rho^2 \log(\rho^2 + \mathrm{e}).$$

By Corollary 3.9 with $\bar{\rho} = \rho^2 + \mathrm{e}$, we see that P_t is not ultracontractive.

Finally, the next example shows that our results are also sharp for infinite Markov chains.

Example 6.5. Consider the birth-death process with a_i(death rate)$=$ b_i(birth rate)$=$ $i^s[\log(1+i)]^\alpha (i \geq 1), s > 1, \alpha \in \mathbb{R}$ and $a_0 = 0, b_0 = 1$. We have $\mu_0 = c, \mu_i = ci^{-s}[\log(1+i)]^{-\alpha}$, where $c > 0$ is such that μ is a probability measure on $\{0, 1, \cdots\}$. By [31; Example 3.1], the F-Sobolev inequality (1.2) holds with $p \in (2, \infty]$ and $F(r) = \log(r + \mathrm{e})^\delta (\delta > 0)$ if and only if either $s > 2$, or $s = 2$ and $\alpha \geq \delta$. Then, we claim that the corresponding semigroup is ultracontractive if and only if either $s > 2$ or $s = 2$ and $\alpha > 1$.

Actually, the sufficiency follows from Corollary 4.5 immediately. On the other hand, if $s < 2$ or $s = 2$ but $\alpha < 1$, then the log-Sobolev inequality does not hold, so the semigroup is not ultracontractive. We now consider the case that $s=2$ and $\alpha=1$. For any f and ρ, we have

$$D(f\rho,\rho)-\frac{1}{2}D(f,\rho^2)=\sum_{i,j}\Big[(f_i\rho_i-f_j\rho_j)(\rho_i-\rho_j)-\frac{1}{2}(f_i-f_j)(\rho_i^2-\rho_j^2)\Big]q_{ij}\mu_i$$

$$=\frac{1}{2}\sum_{i,j}(f_i+f_j)(\rho_i-\rho_j)^2q_{ij}\mu_i$$

$$=\frac{c}{2}(f_0+f_1)(\rho_0-\rho_1)^2+\frac{c}{2}\sum_{i=1}^{\infty}\Big[(f_i+f_{i+1})(\rho_i-\rho_{i+1})^2+(f_i+f_{i-1})(\rho_i-\rho_{i-1})^2\Big],$$

where $q_{ij}=b_i$ for $j=i+1;=a_i$ for $j=i-1;=0$ otherwise. Let $\rho(i)=\sqrt{\log(1+i)}$, it is easy to see from above that $\Gamma(\rho)<\infty$. But for $\lambda>1$,

$$\mu(\exp[\lambda\rho^2])\ge\sum_{i=1}^{\infty}ci^{-2}[\log(1+i)]^{-1}\exp[\lambda\log(1+i)]=\infty.$$

By Corollary 5.5, the semigroup is not ultracontractive.

Appendix

In this section, for completeness, we present the proof of [6; Theorem 4.1] (which recovers Theorem 1.1 in this paper). Moreover, [36; Theorem 1.5] (used in the proof of Corollary 2.3) is also proved for the same purpose.

Let $E=M$ be a d-dimensional complete connected Riemannian manifold, possibly with boundary. Denote by $FS(p)$ and $TFS(p)$, respectively, the inequality (1.2) with $c_2>0$ and $c_2=0$. We will assume the following

$$F(0)=1,\ F'\ge 0,\ \text{and}\ \lim_{r\to\infty}\sup\frac{(p-2)rF'(r)}{2pF(r)}<\frac{p-1}{p}. \tag{1}$$

Consider the following related inequality for $p\in(1,\infty]$:

$$\mu\big(f^{p/(p-1)}F(f)^{(p-2)/2(p-1)}\big)^{(p-1)/p}\le C_1+C_2\mu(|\nabla f|),\quad f\ge 0,\ \mu(f)=1.$$

We simply denote it by $FS(p)'$ (and $TFS(p)'$ for $C_1=0$).

Next, we define the following isoperimetric constants:

$$k_p(r)=\inf_{\mu(A)\in(0,r]}\frac{\mu_\partial(\partial A\setminus\partial M)}{\mu(A)^{(p-1)/p}F(\mu(A)^{-1})^{(p-2)/2p}},\quad r>0,$$
$$k_p(0)=\lim_{r\to 0+}k_p(r),\quad k_p(\infty)=\lim_{r\to\infty}k_p(r),$$

where A runs over all open smooth domains, equivalently, all connected open smooth domains.

Theorem 1([6; Theorem 4.1]). *Assume that* (1) *holds and* $F(\infty)=\infty$ *whenever* $FS(\infty)$ *or* $FS(\infty)'$ *is considered. For any* $p\in(1,\infty]$, $FS(p)'$ (*resp.* $TFS(p)'$) *holds if and only if* $k_p(0)>0$ (*resp.* $k_p(\infty)>0$). *Consequently, it implies* $FS(p)$ (*resp.* $TFS(p)$) *for* $p\in(2,\infty]$.

Proof. a) For any connected smooth domain A (with small $\mu(A)$ for $FS(p)'$), let $f_n\in C^\infty(M)$ be such that $f_n(x)=1$ for $x\in A$ with $\operatorname{dist}(x,\partial A\setminus\partial M)\geq 1/n$, $f_n(x)=0$ for $x\notin A$ and $|\nabla f_n|\leq 1/n+1/n^2$. The sufficiency in the first assertion follows immediately by taking these test functions and letting $n\uparrow\infty$. We now go to prove the necessity.

Assume that $k_p(r)>0$ for some $r>0$. Let $f\geq 0$ be a smooth function with compact support and $\mu(f)=1$. Define $A_t=\mu_\partial(\{f=t\}\setminus\partial M), h(t)=\mu(f>t)$. By the coarea formula, and noting that $h(t)\leq 1/t$, we have

$$\begin{aligned}\mu(|\nabla f|)^{p/(p-1)}&=\Big\{\int_0^\infty A_t\mathrm{d}t\Big\}^{p/(p-1)}\\&\geq\Big\{k_p(r)\int_{r^{-1}}^\infty h(t)^{(p-1)/p}F(h(t)^{-1})^{(p-2)/2p}\Big\}^{p/(p-1)}\\&=c_1\int_{r^{-1}}^\infty\Big\{\int_{r^{-1}}^t h(s)^{(p-1)/p}F(h(s)^{-1})^{(p-2)/2p}\mathrm{d}s\Big\}^{1/(p-1)}\\&\qquad\qquad\cdot h(t)^{p/(p-1)}F(h(t)^{-1})^{(p-2)/2p}\mathrm{d}t. \qquad (2)\end{aligned}$$

Let $\phi(s)=s^{(p-1)/p}F(s^{-1})^{(p-2)/2p}$. We have

$$\phi'(s)=s^{-1/p}F(s^{-1})\Big(\frac{p-1}{p}-\frac{(p-2)F'(s^{-1})}{2psF(s^{-1})}\Big).$$

By (1), $\phi'(s)>0$ for small or big s. Then there exists $c>0$ such that $\phi(t)\geq c\phi(s)$ for all $t\geq s\geq 0$. Noting that $h(t)$ is decreasing in t, we have

$$\int_{r^{-1}}^t\phi(h(s))\mathrm{d}s\geq c(t-r^{-1})\phi(h(t)),\quad t\geq r^{-1}.$$

By substituting this into (2) and noting that $h(t)\leq 1/t$, we obtain

$$\mu(|\nabla f|)^{p/(p-1)}\geq c_2\int_{r^{-1}}^\infty(t-r^{-1})^{1/(p-1)}F(t)^{(p-2)/2(p-1)}h(t)\mathrm{d}t. \qquad (3)$$

Obviously, there exists $\beta(r)\geq 0$ such that $\beta(\infty)=0$ and

$$(t-r^{-1})^{1/(p-1)}F(t)^{(p-2)/2(p-1)}\geq\frac{1}{2}t^{1/(p-1)}F(t)^{(p-2)/2(p-1)}-\beta(r),\quad t\geq r^{-1}.$$

Then, we have

$$\begin{aligned}\mu(|\nabla f|)^{p/(p-1)} &\geq \frac{c_2}{2}\int_{r^{-1}}^{\infty} t^{1/(p-1)}F(t)^{(p-2)/2(p-1)}h(t)\mathrm{d}t - c_2\beta(r)\\ &\geq \frac{c_2}{2}\int_0^{\infty} t^{1/(p-1)}F(t)^{(p-2)/2(p-1)}h(t)\mathrm{d}t - c(r),\end{aligned} \tag{4}$$

where $c(r) = c_2\beta(r) + c_2 \max_{[0,r^{-1})} t^{1/(p-1)}F(t)^{(p-2)/2(p-1)}$ which is equal to zero when $r = \infty$.

Let $\gamma(t) = t^{p/(p-1)}F(t)^{(p-2)/2(p-1)}$. By (1), we have

$$\gamma'(t) = \frac{\gamma(t)}{t}\left\{\frac{p}{p-1} + \frac{tF'(t)}{F(t)}\right\} \leq c_3\frac{\gamma(t)}{t}$$

for some $c_3 > 0$. By substituting this into (4) and using Fubini's theorem, we obtain

$$\begin{aligned}\mu(|\nabla f|)^{p/(p-1)} &\geq c_4\int_0^{\infty}\gamma'(t)h(t)\mathrm{d}t - c(r) = c_4\int_M \mathrm{d}\mu\int_0^f \gamma'(t)\mathrm{d}t - c(r)\\ &= c_4\mu(\gamma(f)) - c(r).\end{aligned}$$

This proves the necessity (recall that $c(\infty) = 0$).

b) We now go to prove the second assertion. When $F \equiv 1$, the proof is very simple by applying $S(p)'$ and $TS(p)'$ to the test function $g = f^{2(p-1)/(p-2)}$. But for the general case, we have to go back to the isoperimetric inequality.

Assume that $k_p(r) > 0$ for some $r > 0$. Let $\psi(t) = t^{(p-1)/(p-2)}\sqrt{F(t)}$ which is strictly increasing in t since $p > 2$. For a smooth nonnegative function f with compact support and $\mu(f^2) = 1$, let $h(t) = \mu(f^2 > t), g = \psi(f^2)$ and $\bar{h}(t) = \mu(g > t)$. We then have $\bar{h}(t) = h(\psi^{-1}(t)) \leq 1/\psi^{-1}(t) \leq r$ whenever $t \geq \psi(r^{-1})$. Now by changing f and r^{-1} into g and $\psi(r^{-1})$ respectively, the argument in a) implies

$$\mu(|\nabla g|)^{p/(p-1)} \geq c_5\int_{\psi(r^{-1})}^{\infty}(t-\psi(r^{-1}))^{1/(p-1)}F(t)^{(p-2)/2(p-1)}\bar{h}(t)\mathrm{d}t$$

for some $c_5 > 0$. By taking the integral transformation $t \to \psi(t)$, we obtain

$$\begin{aligned}\mu(|\nabla g|)^{p/(p-1)} &\geq c_5\int_{r^{-1}}^{\infty}(\psi(t)-\psi(r^{-1}))^{1/(p-1)}F(\psi(t))^{(p-2)/2(p-1)}\psi'(t)h(t)\mathrm{d}t\\ &\geq \frac{c_5}{2}\int_0^{\infty}\psi(t)^{1/(p-1)}F(\psi(t))^{(p-2)/2(p-1)}\psi'(t)h(t)\mathrm{d}t - c(r)\end{aligned}$$

for some $c(r)$ with $c(\infty) = 0$. Noting that $\psi'(t) \geq \frac{p-1}{p-2}t^{1/(p-2)}\sqrt{F(t)}$ and

$$F(\psi(t)) = F\Big(t^{(p-1)/(p-2)}\sqrt{F(t)}\Big) \geq \frac{F(t)}{\max_{[0,1]} F},$$

we have (using (1) in the second step)

$$\mu(|\nabla g|)^{p/(p-1)} + c(r) \geq c_6 \int_0^\infty t^{2/(p-2)} F(t)h(t)\mathrm{d}t$$
$$\geq c_7 \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\Big\{t^{p/(p-2)}F(t)\Big\}h(t)\mathrm{d}t = c_7\mu(f^{2p/(p-2)}F(f^2)). \tag{5}$$

On the other hand, by (1), $\psi'(t) \leq c_8 t^{1/(p-2)}\sqrt{F(t)}$ for some $c_8 > 0$. Hence

$$\mu(|\nabla g|) \leq c_9\mu\Big(|\nabla f|f^{p/(p-2)}\sqrt{F(f^2)}\Big) \leq c_9\sqrt{\mu(|\nabla f|^2)\mu(f^{2p/(p-2)}F(f^2))}. \tag{6}$$

The proof is now completed by combining (5) with (6). □

Theorem 2 ([36; Theorem 1.5]). *Assume that ∂M is either empty or convex and the Ricci curvature is bounded below by $-K$ for some $K \geq 0$. Let $d\nu = \exp[W]dx$ with $W = c\rho$ for some $c > \sqrt{K(d-1)}$, where ρ is the Riemannian distance function from a fixed point o and d is the dimension of M. Then*

$$\nu(|f|^{2p/(p-2)})^{(p-2)/p} \leq c_1\nu(|\nabla f|^2) + c_2, \quad f \in W^{2,1}(M; d\nu),\ \nu(f^2) = 1 \tag{7}$$

holds for $p \in [d,\infty) \cap (2,\infty)$.

Proof. By Green-Wu's approximation theorem (Ann. Scient. Éc. Norm. Sup. Série 4, 12(1979), 47–84), there exists $\bar\rho \in C^\infty(M)$ such that $|\rho - \bar\rho| < 1$ and $|\nabla\bar\rho| < 2$. Let $\mathrm{d}\bar\nu = \mathrm{e}^{c\bar\rho}\mathrm{d}x$. Then it suffices to prove (7) for $\bar\nu$. Let $P_t^{(1)}$ be the semigroup of the (reflecting) diffusion process generated by $\Delta + c\nabla\bar\rho$. By Davies [13; Corollary 2.4.3], (7) for $\bar\nu$ is equivalent to the upper bound of $P_t^{(1)}$

$$(P_t^{(1)}f)^2 \leq c_3 t^{-p/2}, \quad t \in (0,1],\ \bar\nu(f^2) = 1. \tag{8}$$

Let $P_t^{(2)}$ be the (Neumann) diffusion semigroup generated by Δ. By [33; Lemma 2.1] we have

$$[P_t^{(2)}f(x)]^\alpha \leq P_t^{(2)}f^\alpha(y)\exp\left[\frac{\alpha K\rho(x,y)^2}{2(\alpha-1)(1-\mathrm{e}^{-2Kt})}\right] \tag{9}$$

for any $\alpha > 1$, any nonnegative $f \in C_b(M)$, and any $x, y \in M$. On the other hand, it is easy to prove that

$$[P_t^{(i)}f]^\alpha \leq [P_t^{(j)}f^\alpha]\exp[c^2\alpha t/(4(\alpha-1)], \quad i,j = 1,2,\ f \geq 0, f \in C_b(M). \tag{10}$$

Actually, letting $\psi(s) = \log P_s^{(j)}(P_{t-s}^{(i)} f)^\alpha$, we see that $\psi'(s) \geq -c^2\alpha/4(\alpha-1), s \in [0,t]$, which implies (10). Let $\gamma = 2^{1/3}$. For $f \geq 0$ with $\bar{\nu}(f^2) = 1$, it follows from (9) and (10) that

$$
\begin{aligned}
[P_t^{(1)} f(x)]^2 &\leq c_4 [P_t^{(2)} f^\gamma(x)]^{\gamma^2} \leq c_4 [P_t^{(2)} f^{\gamma^2}(y)]^\gamma \mathrm{e}^{c_5 \rho(x,y)^2/t} \\
&\leq c_6 P_t^{(1)} f^2(y) \mathrm{e}^{c_5\rho(x,y)^2/t}, \quad x, y \in M, \ t \in (0,1].
\end{aligned}
$$

Then

$$
\begin{aligned}
c_6 = c_6 \int P_t^{(1)} f^2(y) \bar{\nu}(\mathrm{d}y) &\geq [P_t^{(1)} f(x)]^2 \int \mathrm{e}^{-c_5\rho(x,y)^2/t} \bar{\nu}(\mathrm{d}y) \\
\geq [P_t^{(1)} f(x)]^2 \bar{\nu}\big(B\big(x, \sqrt{t}\big)\big) \mathrm{e}^{-c_5}, \quad x \in M.
\end{aligned}
$$

¿From this we see that (8) follows from the volume lower bound

$$
\bar{\nu}(B_x(r)) \geq c_7 r^p, \quad r \in [0,1], \ x \in M. \tag{11}
$$

But by Cheeger-Gromov-Taylor's volume comparison theorem (J. Diff. Geom. 17(1982), 15–53), we have

$$
\frac{|B_o(1)|}{|B_x(r)|} \leq \frac{|B_x(\rho(x)+1)|}{|B_x(r)|} \leq \frac{(\rho(x)+1)^d}{r^d} \exp\Big[\sqrt{K(d-1)}(\rho(x)+1)\Big], \ r \in [0,1],
$$

where $|\cdot|$ denotes the Riemannian volume. Noting that $\bar{\rho} \geq \rho - 1$, we obtain

$$
\bar{\nu}(B_x(r)) \geq \mathrm{e}^{c(\rho(x)-2)} |B_x(r)| \geq c_7 r^d
$$

for some $c_7 > 0$ and all $r \in [0,1]$. Therefore, (9) holds for $p \geq d$. $\square$

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