Elliptic equations for invariant measures on Riemannian manifolds: existence and regularity of solutions

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Abstract – We obtain sufficient conditions in terms of Lyapunov functions for the existence of invariant measures for diffusions on finite dimensional manifolds and prove some global regularity results for such measures. These results are extended to countable products of finite dimensional manifolds. A new concept of a weak elliptic equation for measures on infinite dimensional manifolds is introduced. As an application, we obtain some a priori estimates for Gibbs measures on countable products of manifolds and prove a new existence result for such measures.

Equations elliptiques pour des mesures invariantes sur des variétés et applications aux mesures de Gibbs

Résumé – Nous obtenons des conditions suffisantes en terms de fonctions de Lyapunov pour l'existence des mesures invariantes de diffusions sur une variété Riemannienne de dimension finie et démontrons un resultat de régularité global pour telles mesures. Ses resultats sont généralisés pour le cas d'un produit dénombrable de variétés de dimension finie. Nous introduisons une notion nouvelle d'une equation elliptique faible pour des mesures sur une variété de dimension infinie et appliquons nos resultats aux mesures de Gibbs dans le cas òu l'espace des états est une variété riemannienne; en particulier, nous obtenons quelques estimations a priori pour telles mesures de Gibbs et démontrons un resultat d'existence qui s'appliquet à une classe large de modèles.

Version française abrégée –

Soit M une variété riemannienne complète de dimension d. Étant donné un champ vectoriel Z sur M, nous posons $Lf = L_Z f = \Delta f + \langle \nabla f, Z\rangle$, où Δ est l'opérateur de Beltrami–Laplace. Nous fixons $o \in M$ et désignons par ϱ la distance à o. Notons que $L_Z(F\circ\varrho) = F'(\varrho)L_Z\varrho + F''(\varrho)$, $F\in C^2(\mathbb{R}^1)$.

Théorème 1. *Soit Z un champ vectoriel sur M tel que $|Z| \in L^{\alpha}_{loc}(\lambda_M)$, $\alpha > d$. Supposons qu'il existe une fonction $F \in C^2[0,\infty)$ telle que on a $\lim\limits_{r\to\infty} F(r) = +\infty$ et $\lim\limits_{\varrho\to\infty}[F'(\varrho)L_Z\varrho + F''(\varrho)] = -\infty$ lorsque la limite est pris dehors cut(o). Alors il* *existe une mesure de probabilité μ à densité $p \in W^{\alpha,1}_{loc}(\lambda_M)$ qui est une solution de l'equation $L^*\mu = 0$ au sense de (2).*

Étant donné un ensemble dénombrable S et une collection de variétes riemanniennes M^n, $n \in S$, nous désignons par M^S le produit $\bigotimes_{n\in S} M^n$. Soit $\{q_n\}_{n\in S}$ une collection de nombres tels que $q_n > 0$ et $\sum_{n\in S} q_n < \infty$, et soit $A = (a_{i,j})$ une matrice infinie symétrique telle que $a_{ij} \geq 0$, $\|A\|_{l^1(q)} \leq \lambda$ et $\sum\limits_{i\in S} a_{n,i} \leq \lambda$ pour chaque n. Soit $\Psi(x) = \sum\limits_{n\in S} q_n G_n(x_n)$, òu $G_n \in C(M^n)$ et $G_n(o_n) = 0$, òu $o_n \in M^n$ sont fixés. Supposons que l'ensemble $\Omega = \{\Psi < \infty\}$ soit muni d'une topologie complètement régulière τ telle que l'injection naturelle $(\Omega, \tau) \to M^S$ soit continue.

Théorème 2. *Supposons que pour chaque $n \in S$ il existent des fonctions nonnegatives compactes $V_n \in C^2(M^n)$ telles que pour quelques c, $\varepsilon > 0$, on a*

$$\begin{aligned} L_n V_n(x) :&= \Delta_n V_n(x_n) + \langle \nabla_n V_n(x_n), Z_n(x)\rangle \\ &\leq c - (\lambda+\varepsilon) G_n(x_n) + \sum_j a_{n,j} G_j(x_j). \end{aligned} \tag{1}$$

Soit Ψ être donné par (8). Supposons que les ensembles $\{\sum_{n\in S} C_n q_n G_n \leq r\}$ soient τ-compactes quel que soit $C_n \uparrow +\infty$ et que les fonctions Z_n soient τ-continues sur ces ensembles. Alors il existe une mesure $\mu \in \mathcal{P}(M^S)$ telle que $\int G_n \mathrm{d}\mu \leq c/\varepsilon$ et $L^*\mu = 0$ par rapport à la classe $\mathcal{K}(\Psi)$ (voir Definition 1). De plus, on peut remplacer V_n par $F_n \circ \varrho_n$, où ϱ_n est la distance à o_n et la fonction $F_n \in C^2[0,\infty)$ est telle que $\lim\limits_{r\to\infty} F_n(r) = +\infty$ et on a (1) dehors cut(o_n).

Let M be a complete Riemannian manifold of dimension d with the Riemannian volume λ_M. Let ϱ denote the distance from a fixed point $o \in M$. By Vec$\,(M)$ we denote the collection of all vector fields on M and by $\mathcal{P}(M)$ the space of all Borel probability measures on M. The Sobolev class of all functions $f \in L^2(\lambda_M)$ such that $|\nabla f| \in L^2(\lambda_M)$, where ∇f is the gradient in the distribution sense, is denoted by $H^{2,1}(\lambda_M)$. The local Sobolev classes defined by analogy with the case of $\mathbb{R}^d$ are denoted by $W^{p,1}_{loc}(\lambda_M)$. A function F is called compact if the sets $\{F \leq r\}$ have compact closures. Given a vector field Z on M, we set $L_Z\psi = \Delta\psi + \langle\nabla\psi, Z\rangle$, where Δ is the Laplace–Beltrami operator.

Theorem 1. *Assume that the Ricci curvature of M is bounded below and*

$$\inf_x \lambda_M\big(B(x,r)\big) > 0 \quad \textit{for all } r > 0,$$

where $B(x,r)$ is the closed geodesic ball with centre x and radius r. Let $\mu \in \mathcal{P}(M)$ be such that $|Z| \in L^2(\mu)$ and $L_Z^*\mu = 0$ in the following sense:

$$\int_M L_Z\psi \,\mathrm{d}\mu = 0, \qquad \forall\, \psi \in C_0^\infty(M). \tag{2}$$

Then $\mu = f\,\lambda_M$ with $\sqrt{f} \in H^{2,1}(\lambda_M)$ and $\displaystyle\int_M \frac{|\nabla f|^2}{f}\,\mathrm{d}\lambda_M \leq \int_M |Z|^2\,\mathrm{d}\mu$. In addition, $\nabla f/f$ is the orthogonal projection of Z onto the closure of $\{\nabla\psi,\ \psi \in C_0^\infty(M)\}$ in $L^2\big(\mu, \mathrm{Vec}(M)\big)$.

Theorem 2. *Let $|Z| \in L^\alpha_{loc}(\lambda_M)$, $\alpha > d$. Suppose that there exists a function $F \in C^2[0,\infty)$ such that $\lim_{r\to\infty} F(r) = \infty$ and $\lim_{\varrho\to\infty}[F'(\varrho)L\varrho + F''(\varrho)] = -\infty$ provided the limit is taken outside the cut locus of o. Then there exists a probability measure μ which satisfies the equation $L_Z^*\mu = 0$ with density $p \in W^{\alpha,1}_{loc}(\lambda_M)$.*

Related results in the case of a smooth compact Lyapunov function have been obtained in [5].

Example 1. Suppose that $\mathrm{Ric} \geq -k$, $k \geq 0$. If $|Z| \in L^\alpha_{loc}(\lambda_M)$ with $\alpha > d$ and $\limsup_{\varrho\to\infty}\langle Z, \nabla\varrho\rangle < -\sqrt{k(d-1)}$, where the limit is taken outside the cut locus of o, then the assertion of Theorem 2 is valid. It suffices to take $F(r) = r^2$ and note that $\limsup_{\varrho\to\infty} \Delta\varrho \leq \sqrt{k(d-1)}$.

Let us present a curvature condition for the existence and uniqueness of invariant measures. Given a C^1 vector field Z, we set

$$k(r) = \inf\Big\{\mathrm{Ric}(X,X)(x) - \langle\nabla_X Z, X\rangle(x)\colon\ X \in \mathrm{Vec}^0(M),\ |X(x)| = 1,\ \varrho(x) \geq r\Big\},$$

$$\overline{k}(r) = \inf\Big\{-\langle\nabla_X Z, X\rangle(x)\colon\ \ X \in \mathrm{Vec}^0(M),\ |X(x)| = 1,\ \varrho(x) \geq r\Big\},$$

Let us define k_ϱ and $\overline{k}_\varrho$ as k and $\overline{k}$, respectively, with X replaced by $\nabla\varrho$.

Theorem 3. *Assume that Z is a C^1 vector field such that*

$$\int_0^\infty k_\varrho(r)\mathrm{d}r = \infty. \tag{3}$$

Then there exists an invariant probability measure μ that satisfies $L_Z^*\mu = 0$ in the sense of (2). *If* (3) *holds for k in place of k_ϱ, then the L_Z-diffusion process is ergodic and has a unique invariant probability measure μ. Finally, if the Ricci curvature is bounded below, then the same assertions are true provided that* (3) *holds for $\overline{k}_\varrho$ or $\overline{k}$ in place of k_ϱ or k, respectively.*

The existence and uniqueness of invariant measures have been proved by Cruzeiro and Malliavin [8] under stronger assumptions: $\inf k > 0$, $\|\nabla^j R\|_\infty + \|\nabla^{j+1}Z\|_\infty < \infty$ for $j = 0, \ldots, 3$, where R is the curvature tensor.

Let S be a countable set, e.g., let $S = \mathbb{N}$ be the set of natural numbers. For each $i \in S$, let M^i be a complete connected finite dimensional Riemannian manifold. For $\Lambda \subset S$, let $M^\Lambda = \Pi_{i\in\Lambda} M^i$ be equipped with the product Borel σ-field $\mathcal{B}_\Lambda$. We denote by ϱ_i the distance function on M^i from a fixed point o_i. For every $x \in M^S$ and $\Lambda \subset S$, let $x_\Lambda \in M^\Lambda$ be the natural projection of x. We set

$$\mathcal{FC}_0^\infty = \Big\{\psi\colon\ \ \psi \in C_0^\infty(M^\Lambda) \text{ for some finite } \Lambda \subset S\Big\},$$

where here and below, we regard a function on M^Λ as a cylindrical function on M^S in the natural way.

Let $Z = (Z_n)_{n\in S}$ be a collection of Borel vector fields on M^S such that $Z_n(x) \in T_{x_n}M^n$ and let

$$L = L_Z = \Delta + Z := \sum_{i\in S}\Delta_i + \sum_{i\in S} Z_i,$$

where Δ_i is the Laplacian on M^i. In which sense the sum is understood will be explained later. In particular, $L\psi$ makes sense if $\psi \in \mathcal{FC}_0^\infty$.

Definition 1. *Let $\mathcal{K}$ be a certain class of bounded $\mathcal{B}_\Lambda$-measurable functions. We shall say that a bounded signed measure μ on $(M^S, \mathcal{B}_\Lambda)$ satisfies the elliptic equation*

$$L_Z^* \mu = 0 \tag{4}$$

with respect to the class $\mathcal{K}$ if for every $\psi \in \mathcal{K}$, one has $\Delta_n \psi$, $\langle \nabla \psi, Z_n \rangle \in L^1(\mu)$ and

$$\sum_{n \in S} \int_M \Big[\Delta_n \psi + \langle \nabla_n \psi, Z_n \rangle \Big] \, \mathrm{d}\mu = 0. \tag{5}$$

For example, one can consider (4) with respect to $\mathcal{K} = \mathcal{F}\mathcal{C}_0^\infty$ (then the series becomes a finite sum). Another possibility is to consider the class $\mathcal{K}_c(M^S)$ of all bounded Borel functions ψ on M^S such that $x_n \mapsto \psi(x_1, \dots, x_n, \dots)$ is a smooth compactly supported function on M^n. One of the motivations for the study of equation (4) is that Gibbs distributions on M^S satisfy (4) under very broad assumptions.

Let v be a fixed smooth compactly supported vector field on a Riemannian manifold M and let U_t^v, $t \in \mathbb{R}^1$, be the corresponding flow, i.e., $U_t^v(x)$ solves the ordinary differential equation $x'(t) = v\big(x(t)\big)$, $x(0) = x$. The next lemma links elliptic equations and Gibbs measures. Its proof is based on the same ideas as the one given in [10] in the linear case.

Lemma 1. *Let $X = M \times Y$, where M is a finite dimensional Riemannian manifold and $(Y, \mathcal{F})$ is a measurable space. Let μ be a bounded signed measure on $\mathcal{B} = \mathcal{B}(M) \otimes \mathcal{F}$ with the conditional measures μ^y on $M \times \{y\}$, and let ν be the projection of $|\mu|$ to Y. Suppose that $\mathcal{K}$ is a class of bounded $\mathcal{B}$-measurable functions that satisfies the following conditions:* (i) *for every $\psi \in \mathcal{K}$ and $y \in Y$, the function $x \mapsto \psi(x, y)$ is continuously differentiable and $\nabla_x \psi$ is bounded;* (ii) $(x, y) \mapsto \psi(U_t^v(x), y) \in \mathcal{K}$ *and $\varphi \circ \psi \in \mathcal{K}$ whenever $\varphi \in C_0^\infty(\mathbb{R}^1)$, $\varphi(0) = 0$, $\psi \in \mathcal{K}$, $t \in \mathbb{R}^1$, and $v \in \mathrm{Vec}_0^\infty(M)$;* (iii) *the class $\mathcal{K}$ separates the measures on $\mathcal{B}$.*

Let $\beta \colon X \to \mathrm{Vec}(M)$ be a μ-measurable mapping such that $\psi\beta \in L^1(\mu)$ for every $\psi \in \mathcal{K}$ and

$$\int \langle \nabla_x \psi, v \rangle \, \mathrm{d}\mu = -\int \psi \operatorname{div} v \, \mathrm{d}\mu - \int \psi \langle \beta, v \rangle \, \mathrm{d}\mu \tag{6}$$

for every $\psi \in \mathcal{K}$ and every smooth compactly supported vector field v on M. Then μ admits conditional densities f^y on the fibres $M \times \{y\}$ such that

$$f^y \in W^{1,1}_{loc}(M) \quad \text{and} \quad \beta(x, y) = \nabla f^y(x) / f^y(x) \tag{7}$$

Remark 1. (i) The separation assumption (iii) on $\mathcal{K}$ can be replaced by the following condition: (iii$'$) there exists a measurable set $\Omega \subset X$ of full measure with respect to all shifts $(\mu)_t$ generated by the fields v as above such that $\mathcal{K}$ separates the measures on the set Ω. In particular, this is the case when Ω has full μ-measure and is mapped by the transformations U_t^v into itself.

(ii) If μ is a Gibbs measure on M^S specified by conditional densities $f^\Lambda \in W^{1,1}_{loc}$ on the fibres $M^\Lambda \times \{y\}$, $y \in M^{S \setminus \Lambda}$ such that $\nabla_\Lambda f^\Lambda / f^\Lambda$ is locally λ_{M^Λ}-integrable, then, as in [3], the previous lemma yields an "integration by parts characterization" of μ, i.e., every probability measure μ' on M^S that satisfies the analog of (6) with every $\beta_\Lambda = \nabla_\Lambda f^\Lambda / f^\Lambda$ in place of β has the same conditional measures as μ. This follows from the fact that a probability measure on M is uniquely determined by its logarithmic gradient if it is locally λ_M-integrable (cf. [4, Sect. 6]).

Let $J = (J_{i,j})_{i,j\in S}$ be an infinite symmetric matrix with $J_{i,j} \geq 0$. Given a collection of positive numbers $q = (q_i)_{i\in S}$, we denote by $l^1(q)$ the space L^1 with respect to the discrete measure on S that assigns q_i to i. In the next theorem we shall consider S as a union of an increasing sequence of finite sets Λ_i; a typical example is $S = \mathbb{Z}^d$, $\Lambda_n = \{(z_1,\dots,z_d)\colon |z_i| \leq n\}$. Given a family $(C_i)_{i\in S}$ of real numbers, we write $C_i \uparrow +\infty$ if $\lim\limits_{n\to\infty} \inf\limits_{i\notin\Lambda_n} C_i = +\infty$.

Let us introduce a class of test functions that will be used below. Given positive functions $G_n \in C(M^n)$, $n \in S$, we set

$$\Psi(x) = \sum_j q_j G_j(x). \tag{8}$$

Suppose that the set $\Omega = \{\Psi < \infty\}$ is equipped with some completely regular topology τ such that the natural embedding $(\Omega,\tau) \to M^S$ is continuous. We denote by $\mathcal{K}_\Psi$ the class of all bounded τ-continuous functions f with support in one of the sets $S_r := \{\Psi \leq r\}$ having partial derivatives of all orders such that the $L_n f$'s are τ-continuous, and the series $\sum_{n\in S} L_n f$ converges uniformly on S_r.

Theorem 4. *Let $\{q_n\}_{n\in S}$ be a collection of positive numbers with $\sum_{n\in S} q_n < \infty$, and let $J = (J_{i,j})$ be as above such that $\|J\|_{l^1(q)} \leq \lambda$ and $\sum\limits_{i\in S} J_{n,i} \leq \lambda$ for all n. Assume that for any $n \in S$, there exist nonnegative compact functions $V_n \in C^2(M^n)$ and $G_n \in C(M^n)$ such that $G_n(o_n) = 0$ and, for some c, $\varepsilon > 0$, one has*

$$\begin{aligned} L_n V_n(x) : &= \Delta_n V_n(x_n) + \langle \nabla_n V_n(x_n), Z_n(x)\rangle \\ &\leq c - (\lambda+\varepsilon) G_n(x_n) + \sum_j a_{n,j} G_j(x_j). \end{aligned} \tag{9}$$

Let Ψ be given by (8). Suppose that the sets $\{\sum_{n\in S} C_n q_n G_n \leq r\}$ are τ-compact whenever $C_n \uparrow +\infty$ and that the functions Z_n are τ-continuous on them. Then there exists $\mu \in \mathcal{P}(M^S)$ such that $\int G_n \mathrm{d}\mu \leq c/\varepsilon$ and $L^*\mu = 0$ with respect to the class $\mathcal{K}_\Psi$. Moreover, V_n can be replaced by $F_n \circ \varrho_n$, where $F_n \in C^2[0,\infty)$ is such that $\lim\limits_{r\to\infty} F_n(r) = +\infty$ and (9) holds outside of cut(o_n).

Corollary 1. *Let q, A, V_n, G_n, and Z_n be as in Theorem 4 such that (9) holds. Suppose that for every $n \in S$, there exist $\gamma_n \in C[0,\infty)$ and $\psi_n \in C(M)$ such that $\lim\limits_{r\to\infty} \gamma_n(r)/r = 0$ and $|Z_n(x)| \leq \psi_n(x_n) + \gamma_n\Big(\sum_j q_j G_j(x_j)\Big)$. Let $h \in C^2[0,\infty)$ be such that $h' \geq 0$, $h'' \leq 0$, $h(\infty) = \infty$, $\sup|\nabla h(V_n)| < \infty$. Then there exists $\mu \in \mathcal{P}(M^S)$ such that $\int G_n \mathrm{d}\mu \leq c/\varepsilon$ and $L^*\mu = 0$ with respect to the class $\mathcal{FC}_0^\infty$.*

The next corollary extends a part of the results obtained in the linear case in [1], [2], [6], and [9].

Corollary 2. *Suppose that in the situation of the previous corollary, for all finite sets $\Lambda \subset S$, there exist continuously differentiable functions U_Λ on M^Λ such that*

$$Z_\Lambda(x_\Lambda \times o^{\Lambda^c}) = \nabla_\Lambda U_\Lambda(x_\Lambda).$$

Then L is symmetric on $L^2(\mu)$, and for every $\psi \in \mathcal{FC}_0^\infty$ and every $v \in \mathrm{Vec}_b^1(M^n)$, one has $\displaystyle\int \langle \nabla_{x_n}\psi, v\rangle \,\mathrm{d}\mu = -\int \psi \langle v, Z_n\rangle \,\mathrm{d}\mu - \int \psi \,\mathrm{div}\, v \,\mathrm{d}\mu.$

Theorem 5. *Suppose that in the situation of Theorem 4, $\mu \in \mathcal{P}(M^S)$ satisfies $L^*\mu = 0$ with respect to the class $\mathcal{K}$ such that $\varphi \circ V \in \mathcal{K}$ for every $\varphi \in C_0^\infty(\mathbb{R}^1)$, where $V = \sum_{n\in S} q_n V_n$ is finite μ-a.e. Assume also that $G_n \le V_n$. Then*

$$\int G_n \, \mathrm{d}\mu \le \frac{c}{\varepsilon} \quad \textit{and} \quad \int \sum_{n\in S} q_n G_n \, \mathrm{d}\mu \le \frac{c}{\varepsilon} \sum_{n\in S} q_n. \tag{10}$$

The above results extend in part results obtained in the linear case in [6]. They yield in particular the existence of Gibbs measures in many models. All proofs and a number of applications are contained in [7].

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