SOME PROPERTIES OF TRACES FOR STOCHASTIC AND DETERMINISTIC PARABOLIC WEIGHTED SOBOLEV SPACES

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ABSTRACT. The results presented in the article, in particular, provide an information on the behavior near $x^1 = 0$ for fixed $t > 0$ in the sense of L_p-spaces with weights for functions $u(t) = u(t, x^1, x')$ $(x^1 \geq 0, x' \in \mathbb{R}^{d-1}$, $t \geq 0)$ satisfying stochastic partial differential equations of type $du(t) = (\Delta u(t) + f(t))\,dt + g^k(t)\,dw_t^k$ with zero boundary condition on $x^1 = 0$.

1. INTRODUCTION

Fix an integer $d \geq 1$ and let $\mathbb{R}^d_+ = \{x = (x^1, x') : x^1 > 0, x' \in \mathbb{R}^{d-1}\}$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(\mathcal{F}_t, t \geq 0)$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$ containing all P-null subsets of Ω, and $\mathcal{P}$ be the predictable σ-field generated by $(\mathcal{F}_t, t \geq 0)$. Let $\{w_t^k; k = 1, 2, ...\}$ be a family of independent one-dimensional $\mathcal{F}_t$-adapted Wiener processes defined on $(\Omega, \mathcal{F}, P)$.

The main purpose of this article is to develop some analytical tools in order to treat linear and nonlinear equations like

$$du(t, x) = (a^{ij}(t, x)u_{x^i x^j}(t, x) + f(u, t, x))\,dt + g^k(u, t, x)\,dw_t^k, \tag{1.1}$$

$t > 0, x \in \mathbb{R}^d_+$, with zero boundary condition on $x^1 = 0$. The functions f and g are assumed to be predictable and regular in (t, x) with respect to certain integral norms.

Such equations arise quite naturally in many applications of probability theory, such as filtering problem of partially observable diffusion processes, quantum field theory, populational genetics and many others (see, for instance, [16]). There are various approaches to investigating (1.1) in various situations. One of these approaches is based on the theory of semigroups (see [1], [2], [3], [14]) and works well for the equations with *nonrandom* coefficients *independent* of time. Although this situation is indeed important it does not cover, say filtering equations in which the coefficients are random and time-dependent and the stochastic term contains first order derivatives of u with respect to x.

For equations in $\mathbb{R}^d$ instead of $\mathbb{R}^d_+$ there is a quite satisfactory solvability and regularity theory in "usual" stochastic Sobolev spaces (see, for instance, [8]).

It turns out that, if we are interested in L_p-properties of derivatives of solutions in domains, the natural scale of Banach spaces in which equations (1.1) should be considered is the scale of so-called weighted Sobolev spaces. Otherwise one is constraint either to compatibility conditions (see [3]) or to regularity results which do not extend to higher order derivatives (see [1], [3], [14]).

1991 *Mathematics Subject Classification.* 60H15, 35K20.

Key words and phrases. Embedding theorems, Sobolev spaces with weights.

The work was partially supported by NSF Grant DMS–9876586.

Weighted stochastic Sobolev spaces were introduced and the corresponding solvability theory in domains is developed in [5] ($p=2$, general smooth domains) and [11], [12] ($p\geq 2$, equations in $\mathbb{R}^d_+$). This theory is basically linear. If one wants to extend it to nonlinear equations, then regularity properties of solutions for fixed t become very important (see, for instance, the treatment of space-time white noise driven (1.1) with $g^k=u^\gamma\phi^k(x)$ in [8] and [7]). This issue is also important even in the linear case as shown in [21] for divergence form equations. Articles [7], [8], and [21] deal with equations in the whole space. In [15] the theory from [11], [12] is carried over to equations in smooth domains and some deep results on the properties of t-traces of solutions are obtained.

The main purpose of this article now can be narrowed down to extending in several directions the results from [15] bearing on t-traces. We present extensions for the case of "mixed" norms when the powers of summability with respect to x and t are different. A necessity to have such extensions can be seen, for instance, from article [14] which treats a case in a sense "orthogonal" to the one from [15]. In [14] the traces are investigated in a very particular case of equations with nonrandom coefficients independent of t and with L_p-norms of $g(t)$ been bounded with respect to t *and* ω. We present "a bridge" between the results of [15] and [14] which "essentially" covers the extreme cases considered in [15] and [14] (see Remarks 4.6 and 4.8). We write essentially in quotes because, actually, our results bear not on solutions of equations like (1.1) but rather on elements of weighted stochastic Sobolev spaces. Considering a more general setting allows us to get rid of any restriction on the range of weights controlling the rate with which the derivatives with respect to x are allowed to blow near the boundary. In terms of our Definition 3.1, in article [15] it is assumed that $d-1<\theta<d-1+p$ (which is the most important range in applications) and in [14] that $\theta=d$.

Even in the case of the same power of integrability and restricted range of θ our estimates are sharper than those in [15]. In particular, we obtain the estimates in a half-space and not only in bounded domains, we do not have some extra terms from [15], the dependence of constants in our estimates on T is exhibited explicitly and our estimates have the form of multiplicative inequalities, which turns out to be important in establishing a link between estimates for functions on $\mathbb{R}^d$ and $\mathbb{R}^d_+$ (see Section 6) and may be important in investigations of nonlinear equations.

Up to what is said above about the quotation marks, our estimates extend those from [14] to the case in which the x-norms of f and g are summable with respect to (ω,t) to some powers instead of just been bounded, the x-norms can be weighted Sobolev norms with fractional positive or negative number of derivatives, and the traces are investigated not only in the Hölder space C^α, $\alpha\in(0,1)$, but also in more general spaces which can be better or worse that C^α. As opposed to [14], our estimates as well as those from [15] are applicable to the very popular one-dimensional equations driven by the space-time white noise (see [8], [15], [19]). On the other hand, we did not try to include in our theory the case from [14] of divergence form equations (1.1) with only measurable coefficients *independent of* ω *and* t. Another case which is not covered is the one from [1] where the driving operator can be a higher order elliptic operator and power of summability with respect to t is 2. The latter does not allow one to estimate the modulus of continuity of solutions with respect to t in natural spaces. However unlike [1] we are able to handle derivatives of solutions with respect to x of any order. We say several words about possible extensions of our results in Remark 5.4.

The article is organized as follows. In Section 2 we briefly recall some definitions and facts concerning Sobolev spaces with weights. In Section 3 we give a modification of the definition of stochastic Sobolev spaces from [12]. This modification allows us in the future using any parameter θ controlling the weights. Section 4 contains the statements of our main results. Some of them are proved there apart from the basic Theorem 4.1, which is proved in Section 5. In turn the proof of Theorem 4.1 is based on Lemma 5.1, the proof of the latter being ultimately based on Theorem 8.3 which is proved in Section 8. The proof of Theorem 8.3 involves some interpolation theorems for Sobolev spaces with weights which were usually avoided in [11], [12], [15]. That is why in a very short Section 6 we give a different and more traditional proof of the most important Corollary 4.3 of Theorem 4.1 for the case of the same power of summability in t and in x. Here the multiplicative inequalities turn out to be instrumental. We want to emphasize that, by using our estimates in $\mathbb{R}^d_+$, we even obtain some multiplicative estimates for equations in the whole $\mathbb{R}^d$ with sharp dependence of constants on T.

Finally, notwithstanding the main topics of our investigation being stochastic partial differential equations, in principle we allow g in (1.1) to be identically zero. Then we are dealing with deterministic functions solving, say the heat equation. It turns out that for deterministic equations the range of parameters, we are dealing with, can be extended even further. We present the corresponding results in Section 7. As far as we know, our results are the first ones in this situation.

Several words about notation. Casually we assume that the functions under consideration are complex valued. We use the following standard notation

$$D_i = \partial/\partial x^i, \quad Du = u_x = (D_1 u, ..., D_d u).$$

For a multi–index $\alpha = (\alpha_1, ..., \alpha_d)$ where α_i's are nonnegative integers, we denote

$$D^\alpha = D_1^{\alpha_1} ... D_d^{\alpha_d}, \quad |\alpha| = \alpha_1 + ... + \alpha_d.$$

By $H^\gamma_p = H^\gamma_p(\mathbb{R}^d)$ we denote the space of Bessel potentials ($= (1-\Delta)^{-\gamma/2} L_p$) with norm $||\cdot||_{\gamma,p}$. For $\gamma = 0$, we have $H^0_p = L_p$ and we denote $||\cdot||_p = ||\cdot||_{0,p}$.

Any function given on $\mathbb{R}_+ := \mathbb{R}^1_+$ is also considered as a function on $\mathbb{R}^d_+$ independent of x'. We define M^α as an operator of multiplying by $(x^1)^\alpha$, $M = M^1$.

2. Sobolev spaces with weights

Here we collect some definitions and facts from [9] and [10]. By $\mathcal{D}(R^d_+)$ we denote the space of all distributions on $\mathbb{R}^d_+$ that is of all continuous linear functionals on $C^\infty_0(\mathbb{R}^d_+)$, the latter being the space of all infinitely differentiable functions on $\mathbb{R}^d_+$ with compact support belonging to $\mathbb{R}^d_+$.

Definition 2.1. Take and fix a nonnegative function $\zeta \in C^\infty_0(\mathbb{R}_+)$ such that

$$\sum_{n=-\infty}^{\infty} \zeta(e^{x-n}) \geq 1 \quad \forall x \in \mathbb{R}.$$

For $\gamma, \theta \in \mathbb{R}$, and $p \in (1,\infty)$ let $L_{p,\theta} = H^0_{p,\theta}$ and let $H^\gamma_{p,\theta}$ be the set of all $u \in \mathcal{D}(\mathbb{R}^d_+)$ such that

$$||u||^p_{\gamma,p,\theta} := \sum_{n=-\infty}^{\infty} e^{n\theta} ||u(e^n \cdot)\zeta||^p_{\gamma,p} < \infty.$$

It is easy to check that $L_{p,d} = L_p(\mathbb{R}^d_+)$. Also it turns out that using different functions ζ yields the same spaces with equivalent norms and the set $C_0^\infty(\mathbb{R}^d_+)$ is dense in $H^\gamma_{p,\theta}$. The following less trivial properties of $H^\gamma_{p,\theta}$ will be often used in the future.

Lemma 2.2 (See [9]). *(i) For any $a > 0$ and $\alpha \in \mathbb{R}$,*

$$||u(a\,\cdot)||^p_{\gamma,p,\theta} \leq a^{-\theta} N ||u||^p_{\gamma,p,\theta} \leq N ||u(a\,\cdot)||^p_{\gamma,p,\theta},$$

$$||M^\alpha u||_{\gamma,p,\theta} \leq N ||u||_{\gamma,p,\theta+p\alpha} \leq N ||M^\alpha u||_{\gamma,p,\theta}.$$

(ii) There is a sequence $\eta_k \in C_0^\infty(\mathbb{R}^d_+)$ such that, for any $u \in H^\gamma_{p,\theta}$, $||\eta_k u||_{\gamma,p,\theta} \leq N||u||_{\gamma,p,\theta}$ and $||\eta_k u - u||_{\gamma,p,\theta} \to 0$ as $k \to \infty$.

(iii) Let $\mu \leq \gamma$ and $q \geq p$ be such that $\gamma - d/p = \mu - d/q$. Denote $\tau = \theta q/p$ (so that $\tau/q = \theta/p$). Then for any $u \in H^\gamma_{p,\theta}$ we have

$$u \in H^\mu_{q,\tau}, \quad ||u||_{\mu,q,\tau} \leq N ||u||_{\gamma,p,\theta}.$$

(iv) Assume $\gamma p > d$ and represent $\gamma - d/p$ as $k + \varepsilon$, where k is an integer and $\varepsilon \in (0,1]$. Let i, j be multi–indices such that $|i| \leq k, |j| = k$. Then for any $u \in H^\gamma_{p,\theta}$, we have

$$M^{|i|+\theta/p} D^i u \in C(\mathbb{R}^d_+), \quad M^{k+\varepsilon+\theta/p} D^j u \in \mathcal{C}^\varepsilon_{loc}(\mathbb{R}^d_+),$$

$$||M^{|i|+\theta/p} D^i u||_{C(\mathbb{R}^d_+)} \leq N ||u||_{\gamma,p,\theta}, \quad [M^{k+\varepsilon+\theta/p} D^j u]_{\mathcal{C}^\varepsilon(\mathbb{R}^d_+)} \leq N ||u||_{\gamma,p,\theta},$$

where $\mathcal{C}^\varepsilon(\mathbb{R}^d_+)$ is the Zygmund space (coinciding with $C^\varepsilon(\mathbb{R}^d_+)$ if $\varepsilon \in (0,1)$).

Furthermore, the above constants N are independent of u, a, λ, and k.

The following fact is due to S.V. Lototsky.

Theorem 2.3 (See [10]). *Take $p \in (1,\infty)$, $\gamma, \theta \in \mathbb{R}$ and define p', γ', and θ' so that*

$$1/p + 1/p' = 1, \quad \gamma' = -\gamma, \quad \theta/p + \theta'/p' = d.$$

For $\phi, \psi \in C_0^\infty(\mathbb{R}^d_+)$ define (ϕ, ψ) as the scalar product of ϕ and ψ in $L_2(\mathbb{R}^d_+)$. Then

$$||\phi||_{\gamma,p,\theta} \leq N \sup_{\psi \neq 0} \frac{(\phi,\psi)}{||\psi||_{\gamma',p',\theta'}} \leq N ||\phi||_{\gamma,p,\theta},$$

where the constants N are independent of ϕ. Moreover, the relation (ϕ,ψ) can be extended by continuity on all $\phi \in H^\gamma_{p,\theta}$ and $\psi \in H^{\gamma'}_{p',\theta'}$ and then it identifies the dual to $H^\gamma_{p,\theta}$ with $H^{\gamma'}_{p',\theta'}$. In particular, the space $H^\gamma_{p,\theta}$ is reflexive.

The following is an interpolation theorem.

Theorem 2.4 (See [10]). *Let $\kappa \in (0,1)$, $p, q \in (1,\infty)$, $\gamma_i, \nu_i, \theta_i, \sigma_i \in \mathbb{R}$, $i = 0, 1$, and*

$$\theta = \kappa\theta_1 + (1-\kappa)\theta_0, \quad \gamma = \kappa\gamma_1 + (1-\kappa)\gamma_0,$$

$$\nu = \kappa\nu_1 + (1-\kappa)\nu_0, \quad \sigma = \kappa\sigma_1 + (1-\kappa)\sigma_0.$$

Let S be an operator defined on $C_0^\infty(\mathbb{R}^d_+)$ satisfying

$$||Su||_{\gamma_i,p,\theta_i} \leq N_i ||u||_{\nu_i,q,\sigma_i}$$

for $i = 0, 1$ with N_i independent of u. Then

$$||u||_{\gamma,p,\theta} \leq N ||u||^\kappa_{\gamma_1,p,\theta_1} ||u||^{1-\kappa}_{\gamma_0,p,\theta_0}, \quad ||Su||_{\gamma,p,\theta} \leq N N_1^\kappa N_0^{1-\kappa} ||u||_{\nu,q,\sigma},$$

where N is independent of N_i and u.

For real b and c we denote

$$\mathcal{L}_{b,c} = M^2\Delta + bMD_1 - c$$

and define $\mathcal{L}^*_{b,c}$ as the formal adjoint to $\mathcal{L}_{b,c}$. It turns out that for any real b and sufficiently large $c > 0$, the fractional powers $(-\mathcal{L}_{b,c})^{\nu/2}$ are well defined.

Theorem 2.5 (See [10]). *(i) Let $\nu, b \in \mathbb{R}$ and c be sufficiently large. Then, for any $\psi, \phi \in C_0^\infty(\mathbb{R}^d_+)$ we have $\phi(-\mathcal{L}^*_{b,c})^{\nu/2}\psi \in C_0^\infty(\mathbb{R}^d_+)$.*

(ii) Let $\nu, b \in \mathbb{R}$ and c be sufficiently large. Then, for any $u \in \mathcal{D}(\mathbb{R}^d_+)$ and any $\phi \in C_0^\infty(\mathbb{R}^d_+)$ we have $(-\mathcal{L}_{b,c})^{\nu/2}(\phi u) \in \mathcal{D}(\mathbb{R}^d_+)$ and, for any $\psi \in C_0^\infty(\mathbb{R}^d_+)$,

$$(\psi, (-\mathcal{L}_{b,c})^{\nu/2}(\phi u)) = (\phi(-\mathcal{L}^*_{b,c})^{\nu/2}\psi, u).$$

(iii) Let $\gamma, \nu, \theta, b, \alpha \in \mathbb{R}$ and c be sufficiently large. Then $(-\mathcal{L}_{b,c})^{\nu/2}$ is a one-to-one bounded operator from $H^\gamma_{p,\theta}$ onto $H^{\gamma-\nu}_{p,\theta}$ and its inverse is also bounded.

(iv) Let $\nu, b \in \mathbb{R}$ and c be sufficiently large. Then

$$(-\mathcal{L}_{b,c})^{\nu/2}M^\alpha = M^\alpha(-\mathcal{L}_{b+2\alpha, c-\alpha b-\alpha(\alpha-1)})^{\nu/2}.$$

3. Deterministic and stochastic parabolic Sobolev spaces

As usual, for stopping time σ, τ, we define $(\sigma, \tau]\!] = \{(\omega, t) : \sigma(\omega) < t \le \tau(\omega)\}$. Similarly we define $[\![\sigma, \tau]\!]$ and $[\![\sigma, \tau)\!)$. Also, for $p, q \in (1, \infty)$ and $\gamma, \theta \in \mathbb{R}$, let

$$\mathbb{H}^{\gamma,q}_{p,\theta}(\tau) = L_q((0, \tau]\!], \mathcal{P}, H^\gamma_{p,\theta}).$$

We use the same notation $H^\gamma_{p,\theta}$ and $\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$ for l_2-valued functions on $\mathbb{R}^d_+$ and $(0, \tau]\!]$, respectively, defining

$$||u||_{H^\gamma_{p,\theta}} = ||\,|(1-\Delta)^{\gamma/2}u|_{l_2}||_p.$$

It is worth noting that if $q = p$ the spaces $\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$ were introduced in [12] and denoted $\mathbb{H}^\gamma_{p,\theta}(\tau)$ there.

We also need the spaces of initial data. We write $u \in U^{\gamma,q}_{p,\theta}$ if and only if u is $\mathcal{F}_0$-measurable and $M^{2/q-1}u \in L_q(\Omega, H^{\gamma-2/q}_{p,\theta})$ and denote

$$||u||^q_{U^{\gamma,q}_{p,\theta}} = E||M^{2/q-1}u||^q_{\gamma-2/q,p,\theta}.$$

Definition 3.1. For a $\mathcal{D}(\mathbb{R}^d_+)$-valued function u defined on $[\![0, \tau]\!] \cap [\![0, \infty)\!)$ with $u(0, \cdot) \in U^{\gamma,q}_{p,\theta}$ we write $u \in \mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ if and only if $M^{-1}u \in \mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$ and there exist $f \in \mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)$ and $g \in \mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)$ such that, for any $\phi \in C_0^\infty(\mathbb{R}^d_+)$, with probability one, we have

$$(u(t, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^t (M^{-1}f(s, \cdot), \phi)\,ds + \sum_{k=1}^\infty \int_0^t (g^k(s, \cdot), \phi)\,dw^k_s \tag{3.1}$$

for all $t \in [0, \tau] \cap [0, \infty)$, where the series is assumed to converge uniformly on each finite interval $[0, \tau \wedge T]$ in probability. In this situation we also write $M^{-1}f = \tilde{\mathbb{D}}u$, $g = \tilde{\mathbb{S}}u$,

$$du = M^{-1}f\,dt + g^k\,dw^k_t$$

and define

$$[|u|]_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)} = ||M^{-1}u||_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)} + ||f||_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)} + ||g||_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)}, \tag{3.2}$$

$$||u||_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)} = [|u|]_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)} + ||u(0, \cdot)||_{U^{\gamma,q}_{p,\theta}}.$$

For $p=q$, we write $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ instead of $\mathfrak{H}^{\gamma,p}_{p,\theta}(\tau)$.

Definition 3.2. For $T\in(0,\infty)$, by $H^{\gamma,q}_{p,\theta}(T)$ we denote the space of all functions from $\mathfrak{H}^{\gamma,q}_{p,\theta}(T)$ independent of ω. For such functions the last term on the right in (3.1) disappears and we define $u_t:=M^{-1}f$.

Remark 3.3. By Lemma 2.2 (i) we have $u\in\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ if and only if $M^{(\theta-d)/p}u\in\mathfrak{H}^{\gamma,q}_{p,d}(\tau)$. The same holds for the spaces $H^{\gamma,q}_{p,\theta}(T)$ and $\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)$.

Remark 3.4. Although the space of initial values does not play almost any role in this paper, it is worth mentioning that the space $U^{\gamma,q}_{p,\theta}$ is very close to a natural space of initial values for functions with finite seminorm $[|\cdot|]_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)}$. This is seen from Remark 4.2.

We will also use similar spaces of functions defined on $\mathbb{R}^d$ instead of $\mathbb{R}^d_+$.

Definition 3.5. Denote

$$\mathbb{H}^{\gamma,q}_{p}(\tau)=L_q((0,\tau]\!],\mathcal{P},H^{\gamma}_{p}).$$

In Definition 3.1 replace $\mathbb{R}^d_+$ with $\mathbb{R}^d$ and drop all θ, M^{-1}, and $M^{2/q-1}$ (in the definition of $U^{\gamma,q}_{p,\theta}$). Then what we get we call the definition of the space $\mathcal{H}^{\gamma,q}_{p}(\tau)$, the operators $\mathbb{D}$ and $\mathbb{S}$, and the norm in $\mathcal{H}^{\gamma,q}_{p}(\tau)$. If $p=q$, we write $\mathcal{H}^{\gamma}_{p}(\tau)$ (the space coinciding with the one introduced in [8] if τ is a bounded stopping time, see Theorem 3.7 of [8]) instead of $\mathcal{H}^{\gamma,p}_{p}(\tau)$. By $H^{\gamma,q}_{p}(T)$ we denote the space of all functions from $\mathcal{H}^{\gamma,q}_{p}(T)$ independent of ω. We write $H^{\gamma}_{p}(T)=H^{\gamma,p}_{p}(T)$.

Remark 3.6. By inspecting the argument in Remark 3.2 of [8], we see that, if $p,q\in[2,\infty)$, the series in (3.1) converges uniformly on each finite time interval $\subset[0,\tau]$ in probability for any l_2-valued $g\in\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)$.

Remark 3.7. In what concerns $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$, the only difference between Definition 2.2 of [12] and Definition 3.1 is that in (3.2) we now take the $\mathbb{H}^{\gamma}_{p,\theta}(\tau)$-norm of $M^{-1}u$ in place of $\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$-norm of u_x. It turns out that if $\theta\neq d-1+p$ and $p\in[2,\infty)$, the spaces $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ introduced in Definition 2.2 of [12] and Definition 3.1 coincide and their respective norms are equivalent.

Indeed, if $u\in\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$, then by Remark 2.2 of [9] we have $(MM^{-1}u)_x=u_x\in\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$ with a corresponding estimate of the $\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$-norm of u_x (for any θ). On the other hand, if $u_x\in\mathbb{H}^{\gamma-1}_{p,\theta}(\tau)$, by Corollary 2.12 (i) of [9] we have

$$||M^{-1}u||_{H^{\gamma}_{p,\theta}}\le N||u||_{H^{\gamma}_{p,\theta-p}}\le N||Mu_x||_{H^{\gamma}_{p,\theta-p}}\le N||u_x||_{H^{\gamma}_{p,\theta}}$$

for $\theta\neq d-1+p$, so that $M^{-1}u\in\mathbb{H}^{\gamma}_{p,\theta}(\tau)$.

Remark 3.8. The spaces $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ for $p,q\in[2,\infty)$ and the spaces $H^{\gamma,q}_{p,\theta}(T)$ for $p,q\in(1,\infty)$ are Banach spaces.

Indeed, the statement concerning the spaces $H^{\gamma,q}_{p,\theta}(T)$ is almost obvious and very well known. In the case of $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$, it suffices to show that if u_n is a Cauchy sequence in $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$, then it converges to an element in $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$. Obviously $M^{-1}u_n$, $M\tilde{\mathbb{D}}u_n$, $\tilde{\mathbb{S}}u_n$, and $u_n(0)$ converge in corresponding spaces to some limits, say $M^{-1}u$, f, g, and $u(0)$ respectively. Therefore, we only need to show that one can pass to the limit in (3.1) after possibly modifying u. While doing so, without loss of generality, we may assume that τ is bounded.

Take a function $\psi \in C_0^\infty(\mathbb{R}^d_+)$ which equals 1 on the support of ϕ. Then $u_n\psi$ is a Cauchy sequence in $\mathcal{H}_2^\gamma(\tau)$ which is a Banach space by Theorem 3.7 of [8]. Therefore, $u_n\psi$ converges in the $\mathcal{H}_2^\gamma(\tau)$-norm to a function $v \in \mathcal{H}_2^\gamma(\tau)$. Of course, $v = u\psi$, $\mathbb{D}v = f\psi$, $\mathbb{S}v = g\psi$, and $v(0) = u(0)\psi$ for almost all (ω, t, x) satisfying $t \leq \tau(\omega)$. This implies that a modification of $u\psi$ belongs to $\mathcal{H}_2^\gamma(\tau)$ and satisfies (3.1) for our particular ϕ. Expanding the support of ψ easily leads to the conclusion that a modification of u satisfies (3.1) for any $\phi \in C_0^\infty(\mathbb{R}^d_+)$.

The following theorem provides a convenient way to move along the scale of spaces $\mathfrak{H}_{p,\theta}^{\gamma,q}(\tau)$.

Theorem 3.9. *(i) Let $p, q \in [2, \infty)$, $\gamma, \nu, \theta, b \in \mathbb{R}$, and c be sufficiently large. Then $(-\mathcal{L}_{b,c})^{\nu/2}$ is a one-to-one bounded operator from $\mathfrak{H}_{p,\theta}^{\gamma,q}(\tau)$ onto $\mathfrak{H}_{p,\theta}^{\gamma-\nu,q}(\tau)$ with bounded inverse. In addition, $(-\mathcal{L}_{b,c})^{\nu/2}\tilde{\mathbb{D}}u = \tilde{\mathbb{D}}(-\mathcal{L}_{b,c})^{\nu/2}u$ and $(-\mathcal{L}_{b,c})^{\nu/2}\tilde{\mathbb{S}}u = \tilde{\mathbb{S}}(-\mathcal{L}_{b,c})^{\nu/2}u$.*

(ii) Let $p, q \in (1, \infty)$ and $\gamma, \nu, \theta, b \in \mathbb{R}$ and c be sufficiently large. Then $(-\mathcal{L}_{b,c})^{\nu/2}$ is a one-to-one bounded operator from $H_{p,\theta}^{\gamma,q}(T)$ onto $H_{p,\theta}^{\gamma-\nu,q}(T)$ with bounded inverse.

Proof. (i) By Theorem 2.5 and Lemma 2.2 we have

$$||M^{-1}(-\mathcal{L}_{b,c})^{\nu/2}u||_{\mathbb{H}_{p,\theta}^{\gamma-\nu,q}(\tau)} \sim ||(-\mathcal{L}_{b,c})^{\nu/2}u||_{\mathbb{H}_{p,\theta-p}^{\gamma-\nu,q}(\tau)}$$

$$\sim ||u||_{\mathbb{H}_{p,\theta-p}^{\gamma,q}(\tau)} \sim ||M^{-1}u||_{\mathbb{H}_{p,\theta}^{\gamma,q}(\tau)},$$

where by $\sim$ we mean that each term estimates the others with a constant independent of u (q and τ). In the same way $(-\mathcal{L}_{b,c})^{\nu/2}$ acts on $\tilde{\mathbb{D}}u$ and $\tilde{\mathbb{S}}u$.

By remembering that $(-\mathcal{L}_{b,c})^{\nu/2}$ is invertible, we see that it only remains to prove that

$$d(-\mathcal{L}_{b,c})^{\nu/2}u = (-\mathcal{L}_{b,c})^{\nu/2}\tilde{\mathbb{D}}u\,dt + (-\mathcal{L}_{b,c})^{\nu/2}\tilde{\mathbb{S}}^k u\,dw_t^k \tag{3.3}$$

whenever $u \in \mathfrak{H}_{p,\theta}^{\gamma,q}(\tau)$.

Take $\phi \in C_0^\infty(\mathbb{R}^d_+)$ and η_n from Lemma 2.2 and substitute $\eta_n(-\mathcal{L}^*_{b,c})^{\nu/2}\phi$ (which is in $C_0^\infty(\mathbb{R}^d_+)$ by Theorem 2.5 (i)) in place of ϕ in (3.1). Then from Theorem 2.5 (ii) we see that $(-\mathcal{L}_{b,c})^{\nu/2}(\eta_n u) \in \mathfrak{H}_{p,\theta}^{\gamma-\nu,q}(\tau)$ and

$$d(-\mathcal{L}_{b,c})^{\nu/2}(\eta_n u) = (-\mathcal{L}_{b,c})^{\nu/2}(\eta_n\tilde{\mathbb{D}}u)\,dt + (-\mathcal{L}_{b,c})^{\nu/2}(\eta_n\tilde{\mathbb{S}}^k u)\,dw_t^k.$$

Here, for instance,

$$||M(-\mathcal{L}_{b,c})^{\nu/2}(\eta_n\tilde{\mathbb{D}}u) - M(-\mathcal{L}_{b,c})^{\nu/2}\tilde{\mathbb{D}}u||_{\mathbb{H}_{p,\theta}^{\gamma-\nu-2,q}(\tau)}$$

$$\leq N||(-\mathcal{L}_{b,c})^{\nu/2}(\eta_n\tilde{\mathbb{D}}u) - (-\mathcal{L}_{b,c})^{\nu/2}\tilde{\mathbb{D}}u||_{\mathbb{H}_{p,\theta+p}^{\gamma-\nu-2,q}(\tau)} \to 0$$

as $n \to \infty$. It follows, from relations like this, that $(-\mathcal{L}_{b,c})^{\nu/2}(\eta_n u)$ converges in the $\mathfrak{H}_{p,\theta}^{\gamma-\nu,q}(\tau)$-norm to $(-\mathcal{L}_{b,c})^{\nu/2}u$ and since $\mathfrak{H}_{p,\theta}^{\gamma-\nu,q}(\tau)$ is a Banach space, $(-\mathcal{L}_{b,c})^{\nu/2}u \in \mathfrak{H}_{p,\theta}^{\gamma-\nu,q}(\tau)$. At the same time we get (3.3). This proves assertion (i) and since (ii) is proved quite similarly, the theorem is proved.

The following result allows us to approximate any $u \in \mathfrak{H}_{p,\theta}^{\gamma,q}(\tau)$ with "much simpler" functions.

Lemma 3.10. *Let $p,q\in[2,\infty)$, and $\tau\le T$, where T is a finite constant. Let $u\in\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ with $du=M^{-1}f\,dt+g^k\,dw^k_t$. Denote $u_0=u(0)$. Then there exist stopping times $\tau^i_n\le\tau^{i+1}_n\le T$ and functions $f^i_n,g^{ik}_n\in C^\infty_0(\mathbb{R}^d_+)$ defined for $n,i,k=1,2,3,...$ and functions $u_{n0}\in L_q(\Omega,C^n_0(\mathbb{R}^d_+))$ such that, for the functions*

$$f_n(\omega,t,x):=\sum_{i=1}^n I_{(\tau^i_n,\tau^{i+1}_n]\!]}(\omega,t)f^i_n(x),$$

$$g^k_n(\omega,t,x):=\begin{cases}\sum_{i=1}^n I_{(\tau^i_n,\tau^{i+1}_n]\!]}(\omega,t)g^{ik}_n(x) & \text{if}\quad k\le n,\\ 0 & \text{if}\quad k>n,\end{cases}$$

$$u_n(t,x):=u_{n0}(x)+\sum_{i=1}^n M^{-1}f^i_n(x)(\tau^{i+1}_n\wedge t-\tau^i_n\wedge t)$$
$$+\sum_{i,k\le n}g^{ik}_n(x)(w^k_{\tau^{i+1}_n\wedge t}-w^k_{\tau^i_n\wedge t}),\quad (3.4)$$

we have $f_n\in\mathbb{H}^{\gamma-2,q}_{p,\theta}(T)$, $g_n\in\mathbb{H}^{\gamma-1,q}_{p,\theta}(T)$, $u_n\in\mathfrak{H}^{\gamma,q}_{p,\theta}(T)$, $du_n=M^{-1}f_n+g^k_n\,dw^k_t$ and

$$||M^{-1}(u-u_n)||_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)}+||f-f_n||_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)}$$
$$+||g-g_n||_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)}+E||M^{2/p-1}(u_0-u_{n0})||^q_{H^{\gamma-2/q}_{p,\theta}}\to0\quad (3.5)$$

as $n\to\infty$.

Similar statement holds for $H^{\gamma,q}_{p,\theta}(T)$ if $p,q\in(1,\infty)$.

Proof. By multiplying u by functions of class $C^\infty_0(\mathbb{R}^d_+)$, as in the proof of Theorem 2.9 of [12], we convince ourselves that we can assume that $u\in\mathcal{H}^{\gamma,q}_p(\tau)$. After this again as in the proof of Theorem 2.9 of [12] we mollify u and reduce further the general situation to the one in which, in addition,

$$u=u\eta,\quad u,\mathbb{D}u,\mathbb{S}u\in L_q((0,\tau]\!],C^\mu(\mathbb{R}^d)),\quad u_0\in L_q(\Omega,C^\mu(\mathbb{R}^d)),\quad (3.6)$$

where $\eta\in C^\infty_0(\mathbb{R}^d_+)$ and μ is as large as we like. Now it is clear how to finish the proof in the case of $H^{\gamma,q}_{p,\theta}(T)$.

To deal with the spaces $\mathfrak{H}^{\gamma,q}_{p,\theta}(T)$, we want to apply some results from [13]. Since they are proved for finite number of Wiener processes, first we approximate u with functions constructed on the basis of finite number of Wiener processes. Define $\tilde f=\mathbb{D}u-\Delta u$ and observe that $\tilde f\in\mathbb{H}^{\mu-2}_q(\tau)$ and

$$du=(\Delta u+\tilde f)\,dt+g^k\,dw^k_t.$$

It follows from Theorem 5.7 and Remark 5.9 of [8] that the equations

$$du_m=(\Delta u_m+\tilde f)\,dt+\sum_{k\le m}g^k\,dw^k_t$$

with initial data $u_{m0}=u_0$ have unique solutions in $\mathcal{H}^\mu_q(\tau)$ and they converge to u in the norm of $\mathcal{H}^\mu_q(\tau)$ as $m\to\infty$. Certainly, $u_n\eta\to u\eta=u$ in $\mathcal{H}^\mu_q(\tau)$. In particular,

$$I_m:=||u-u_m\eta||_{\mathbb{H}^\mu_q(\tau)}+||\mathbb{D}u-\mathbb{D}(u_m\eta)||_{\mathbb{H}^{\mu-2}_q(\tau)}+||\mathbb{S}u-\mathbb{S}(u_m\eta)||_{\mathbb{H}^{\mu-1}_q(\tau)}\to0.$$

If μ is large enough, by Sobolev embedding theorems ($\eta H^\mu_q\subset H^\gamma_p$), we get

$$||M^{-1}(u-u_m\eta)||_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)}+||M\tilde{\mathbb{D}}u-M\tilde{\mathbb{D}}(u_m\eta)||_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)}$$

$$+||\tilde{\mathbb{S}}u-\tilde{\mathbb{S}}(u_m\eta)||_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)}\leq NI_m\to 0.$$

It follows that in the rest of the proof, without loss of generality, we can assume that g^k is nonzero only for finitely many k and (3.6) holds.

By Theorem 3.10 of [8], there exist $\hat{f}_n$, $\hat{g}_n$, and $\hat{u}_{n0}$ as in the assertion of the lemma such that

$$||f-\hat{f}_n||_{\mathbb{H}^\mu_q(\tau)}+||g-\hat{g}_n||_{\mathbb{H}^\mu_q(\tau)}+E||u_0-\hat{u}_{n0}||^q_{H^\mu_q}\to 0$$

as $n\to\infty$. Since $u=u\eta$, we obviously have $f=f\eta$, $g=g\eta$, and $u_0=u_0\eta$, so that

$$||f-\hat{f}_n\eta||_{\mathbb{H}^\mu_q(\tau)}+||g-\hat{g}_n\eta||_{\mathbb{H}^\mu_q(\tau)}+E||u_0-\hat{u}_{n0}\eta||^q_{H^\mu_q} \tag{3.7}$$

$$\leq N(||f-\hat{f}_n||_{\mathbb{H}^\mu_q(\tau)}+||g-\hat{g}_n||_{\mathbb{H}^\mu_q(\tau)}+E||u_0-\hat{u}_{n0}||^q_{H^\mu_q})\to 0.$$

Now define $f_n=\hat{f}_n\eta$, $g_n=\hat{g}_n\eta$, $u_{n0}=\hat{u}_{n0}\eta$, and introduce u_n according to (3.4). The fact that $du_n=M^{-1}f_n\,dt+g^k_n\,dw^k_t$ for $t\leq T$ in the sense of Definition 3.1 is obvious. Also obviously

$$d(u_n-u)(t\wedge\tau)=I_{t\leq\tau}M^{-1}(f_n-f)\,dt+I_{t\leq\tau}(g^k_n-g^k)\,dw^k_t.$$

By Theorem 3.1 of [13] applied to the above "equation" considered as a completely degenerate one, it follows that, if μ is an integer ≥ 2, then

$$||u-u_n||^q_{\mathbb{H}^\mu_q(\tau)}\leq N(||M^{-1}(f-f_n)||^q_{\mathbb{H}^{\mu-2}_q(\tau)}$$

$$+||g-g_n||^q_{\mathbb{H}^{\mu-2}_q(\tau)}+E||u_0-u_{n0}||^q_{H^{\mu-2}_q}),$$

where the right-hand side goes to zero due to (3.7). Upon remembering again that the supports of all functions involved belong to that of η and again applying Sobolev embedding theorems, we finally get (3.5). The lemma is proved.

The following result is an immediate corollary of Lemma 2.2 (i), (iii), (iv).

Theorem 3.11. *(i) Let $\mu\leq\gamma$ and r $(\geq p)$ be such that*

$$\gamma-\mu=\frac{d}{p}-\frac{d}{r}.$$

Denote $\sigma=\theta r/p$ (so that $\sigma/r=\theta/p$). Then

$$E\int_0^\tau ||M^{-1}u(t)||^q_{\mu,r,\sigma}\,dt\leq N||M^{-1}u||^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)},$$

where N is independent of τ and u, for any u for which the right-hand side is defined.

(ii) Let $\gamma p>d$ and represent $\gamma-d/p$ as $k+\varepsilon$, where $k\in\{0,1,2,...\}$ and $\varepsilon\in(0,1]$. Then for any multi-indices i and j such that $|i|\leq k$ and $|j|=k$ we have

$$E\int_0^\tau \big(||M^{\delta+|i|}D^iu(t)||^q_{C(\mathbb{R}^d_+)}+[|M^{\delta+|j|+\varepsilon}D^ju(t)|]^q_{C^\varepsilon(\mathbb{R}^d_+)}\big)\,dt\leq N||M^{-1}u||^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)},$$

where $\delta=\theta/p-1$ and N is independent of τ and u, for any u for which the right-hand side is defined.

4. Main results for stochastic spaces

Throughout this section we assume that $p,q\in[2,\infty)$. For $\kappa\in(0,1)$, a Banach space X, and a set $A\subset\mathbb{R}^d$ by $C^\kappa(A,X)$ we mean Hölder's space of continuous X-valued functions on A with finite norm $||\cdot||_{C^\kappa(A,X)}$ defined by

$$[|u|]_{C^\kappa(A,X)}=\sup_{s,t\in A}|t-s|^{-\kappa}|u(t)-u(s)|_X,\quad ||u||_{C(A,X)}=\sup_{t\in A}|u(t)|_X,$$

$$||u||_{C^\kappa(A,X)}=[|u|]_{C^\kappa(A,X)}+||u||_{C(A,X)}.$$

Theorem 4.1. *Let $\gamma,\theta\in\mathbb{R}$, $T\in(0,\infty)$, $\tau\le T$, $u\in\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$, and*

$$2/q<\alpha<\beta\le 1. \tag{4.1}$$

Then

$$M^{\beta-1}u(t)-M^{\beta-1}u(0)\in C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{p,\theta})$$

(a.s.) and for any constant $a>0$, stopping time $\eta\le\tau$, and $0\le s<t<\infty$,

$$E||M^{\beta-1}(u(t\wedge\eta)-u(s\wedge\eta))||^q_{\gamma-\beta,p,\theta}\le N_1|t-s|^{q\beta/2-1}a^{\beta-1}I(a), \tag{4.2}$$

$$E[|M^{\beta-1}u|]^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{p,\theta})}\le N_2T^{(\beta-\alpha)q/2}a^{\beta-1}I(a), \tag{4.3}$$

where

$$I(a):=a||M^{-1}u||^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)}+a^{-1}||M\tilde{\mathbb{D}}u||^q_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)}+||\tilde{\mathbb{S}}u||^q_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)} \tag{4.4}$$

If, in addition, $M^{\beta-1}u(0)\in L_q(\Omega,H^{\gamma-\beta}_{p,\theta})$, then

$$E||M^{\beta-1}u||^q_{C([0,\tau],H^{\gamma-\beta}_{p,\theta})}\le N_3E||M^{\beta-1}u(0)||^q_{H^{\gamma-\beta}_{p,\theta}}+N_3T^{\beta q/2-1}a^{\beta-1}I(a). \tag{4.5}$$

Furthermore, the constants N_i are independent of a, τ, T, and u.

Remark 4.2. The restriction on the parameters α,β in Theorem 4.1 is natural in the following sense.

On the one hand, for $q=p=2$ solutions of the equation $du=\Delta u\,dt+g^k\,dw^k_t$ in one space dimension with zero initial and boundary condition belong to $\mathfrak{H}^1_{2,d}(T)$ if $g\in\mathbb{H}^0_{2,d}(T)$ (see, for instance, [5]) and admit an explicit representation from which it can be derived quite easily that, generally, $E||u(t)||^2_{\varepsilon,2,d}=\infty$ if $t,\varepsilon>0$. This shows that we have to assume that $\beta\le1$.

On the other hand, let $V^{\gamma,q}_p$ be the set of all $u(1)$ such that $u\in H^{\gamma,q}_{p,d}(2)$. Take $u\in V^{\gamma,q}_p$. Upon defining $v(t)$ as a solution of the heat equation for $t\in[1,2]$ and of the reverse heat equation for $t\in[0,1]$ with the condition $v(1)=u$ and with zero condition on $x^1=0$, one can show that $v\in H^{\gamma,q}_{p,d}(2)$. Therefore, $V^{\gamma,q}_p$ is just the set of *all* functions extendible to functions of class $H^{\gamma,q}_{p,d}(1)$. Theorem 4.1 asserts that, for small $\delta>0$ and $u\in H^{\gamma,q}_{p,d}(2)$ with $u(0)=0$, we have $u(1)\in M^{1-2/q-\delta}H^{\gamma-2/q-\delta}_{p,d}$, which can be rewritten as

$$V^{\gamma,q}_p\subset M^{1-2/q-\delta}H^{\gamma-2/q-\delta}_{p,d}.$$

This result is close to be optimal, which is seen from Theorem 8.6 saying that

$$M^{1-2/q+\delta}H^{\gamma-2/q+\delta}_{p,d}\subset V^{\gamma,q}_p.$$

Furthermore, $M^{1-2/q+\delta}H^{\gamma-2/q+\delta}_{p,d}\ne V^{\gamma,q}_p$ no matter which $\delta\ge0$ we take. The obstacle here is not that we have weights but that even for $q=p$ the sharp space of traces of elements in $H_p^\gamma(T)$ is a Besov space larger than $H_p^{\gamma-2/p}$ (see [20], where general p and q are treated). Therefore, in Theorem 4.1 we cannot take $\beta\leq 2/q$ even in the case of deterministic functions.

To discuss Theorem 4.1 further, notice that by Lemma 2.2, for

$$\check\beta=\beta+(1/p-1/q)d,\quad \check\theta=d+(\theta-d)q/p,$$

we have

$$||M^{\check\beta-1}u||_{\gamma-\check\beta,q,\check\theta}\leq N||M^{\beta-1}u||_{\gamma-\check\beta,q,\theta q/p}\leq N||M^{\beta-1}u||_{\gamma-\beta,p,\theta}$$

provided $\check\beta\geq\beta$ that is $q\geq p$. By combining this with Theorem 4.1, we get the following.

Corollary 4.3. *Let $\gamma,\theta\in\mathbb{R}$, $q\geq p$, $T\in(0,\infty)$, $\tau\leq T$, $u\in\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$,*

$$\tfrac{2}{q}+(\tfrac{1}{p}-\tfrac{1}{q})d<\alpha<\beta\leq 1+(\tfrac{1}{p}-\tfrac{1}{q})d, \tag{4.6}$$

$$\hat\alpha=\alpha-(\tfrac{1}{p}-\tfrac{1}{q})d,\quad \hat\beta=\beta-(\tfrac{1}{p}-\tfrac{1}{q})d,\quad \tfrac{\hat\theta-d}{q}=\tfrac{\theta-d}{p}.$$

Then

$$M^{\beta-1}u(t)-M^{\beta-1}u(0)\in C^{\hat\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{q,\hat\theta})$$

(a.s.) and for any constant $a>0$, stopping time $\eta\leq\tau$, and $0\leq s<t<\infty$,

$$E||M^{\beta-1}(u(t\wedge\eta)-u(s\wedge\eta))||^q_{\gamma-\beta,q,\hat\theta}\leq N_1|t-s|^{q\hat\beta/2-1}a^{\hat\beta-1}I(a), \tag{4.7}$$

$$E[|M^{\beta-1}u|]^q_{C^{\hat\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{q,\hat\theta})}\leq N_2T^{(\hat\beta-\hat\alpha)q/2}a^{\hat\beta-1}I(a), \tag{4.8}$$

where $I(a)$ is introduced in (4.4). If, in addition, it holds that

$$M^{\beta-1}u(0)\in L_q(\Omega,H^{\gamma-\beta}_{q,\hat\theta}),$$

then

$$E||M^{\beta-1}u||^q_{C([0,\tau],H^{\gamma-\beta}_{q,\hat\theta})}\leq N_3E||M^{\beta-1}u(0)||^q_{H^{\gamma-\beta}_{q,\hat\theta}}+N_3T^{q\hat\beta/2-1}a^{\hat\beta-1}I(a). \tag{4.9}$$

Furthermore, the constants N_i are independent of a, τ, T, and u.

Remark 4.4. It is worth noting that if $\theta=d$, then $\hat\theta=d$, and that, actually, this is the main case in Corollary 4.3. Indeed, the general case follows from this one just by Remark 3.3.

Remark 4.5. If $q=p=2$, there is no possible values of β consistent with (4.6). This case is covered by Theorems 2.11 of [11] and [12] which say that (4.9) also holds for $\beta=1$ and any $p=q\geq 2$. Actually, in [11] and [12] there is a general restriction on values of θ. But this restriction is not used in the proofs of Theorems 2.11 of [11] and [12]. Also it is assumed there that $u(0)=0$, but the proof can be easily carried through for general $u(0)\in L_p(\Omega,H^{\gamma-1}_{p,\theta})$ as well.

Remark 4.6. Corollary 4.3 is part of the bridge result we mentioned in Introduction. If $q=p$ and $d-1<\theta<d-1+p$, one can find in [15] a version of Corollary 4.3 for bounded domains without specified dependence of constants on T and with some additional terms entering $I(a)$. The relation of Corollary 4.3 to some results of [14] we discuss in Remark 4.8 below.

Simple combination of Lemma 2.2, and Corollary 4.3, after replacing $\hat{\alpha}$ and $\hat{\beta}$ with α and β, respectively, proves the following.

Theorem 4.7. *Let $\gamma,\theta\in\mathbb{R}$, $q\geq p$, $T\in(0,\infty)$, $\tau\leq T$. Assume*

$$2/q<\alpha<\beta\leq 1,\quad \gamma-\beta-d/p=k+\varepsilon,$$

where $k\in\{0,1,2,...\}$ and $\varepsilon\in(0,1]$. Denote

$$\delta=\beta-1+\theta/p.$$

Then for any $u\in\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ and any multi-indices i and j such that $|i|\leq k$ and $|j|=k$, we have that

(i) on the set of probability one, the functions $D^iu(t,x)$ are well defined and continuous for $(t,x)\in[0,\tau]\times\mathbb{R}^d$ and

$$M^{\delta+|i|}D^iu(t)-M^{\delta+|i|}D^iu(0)\in C^{\alpha/2-1/q}([0,\tau],C(\mathbb{R}^d_+));$$

(ii) for any constant $a>0$,

$$E\sup_{t,s\leq\tau}|t-s|^{1-q\alpha/2}\big(||M^{\delta+|i|}D^i(u(t)-u(s))||^q_{C(\mathbb{R}^d_+)}$$

$$+[|M^{\delta+|j|+\varepsilon}D^j(u(t)-u(s))|]^q_{\mathcal{C}^{\varepsilon}(\mathbb{R}^d_+)}\big) \tag{4.10}$$

$$\leq NT^{(\beta-\alpha)q/2}a^{\beta-1}\big(a||M^{-1}u||^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)}+a^{-1}||M\tilde{\mathbb{D}}u||^q_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)}+||\tilde{\mathbb{S}}u||^q_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)}\big),$$

where N is independent of u, τ, and T.

Remark 4.8. If $q\geq p$, $\tau\leq T$, and

$$\kappa_0:=1-2/q-d/p>0,$$

then, for any $\kappa\in(0,\kappa_0)$ and $u\in\mathfrak{H}^{1,q}_{p,d}(\tau)$ with $u(0)=0$, we have

$$E\big|\sup_{t\leq\tau}\sup_{x,y\in\mathbb{R}^d}|u(t,x)-u(t,y)|/|x-y|^{\kappa}\big|^q<\infty, \tag{4.11}$$

$$E\big|\sup_{x\in\mathbb{R}^d}\sup_{s,t\leq\tau}|u(t,x)-u(s,x)|/|t-s|^{\kappa/2}\big|^q<\infty. \tag{4.12}$$

Indeed, for $\gamma=1$, $\theta=d$, $j=0$, and $\beta=\kappa_0-\kappa+2/q$ we have $2/q<\beta<1$, $\varepsilon=\gamma-\beta-d/p=1-\beta-d/p=\kappa=-\delta$, $\delta+|j|+\varepsilon=0$, and (4.10) along with $u(0)=0$ implies (4.11). To get (4.12), it suffices to take $i=0$, $\alpha=\kappa+2/q$, $\beta=1-d/p$ and to notice that $2/q<\alpha<\beta<1$, $\delta+|i|=0$, and $1-q\alpha/2=-q\kappa/2$.

Estimates similar to (4.11) and (4.12) are derived in [14] for solutions of equations

$$du=Lu\,dt+g^k\,dw^k_t \tag{4.13}$$

in smooth bounded domains with zero initial and boundary condition, where L is a uniformly elliptic second-order operator with smooth coefficients depending only on x and with only finitely many nonzero g^k. In a subsequent paper the author intends to show that even if the number of nonzero g^k is infinite and the coefficients of L are predictable functions of (ω,t) (which is typical for filtering problems), for $d-1<\theta<d-1+p$ and $q\geq p$, equation (4.13) in a smooth bounded domain G with zero initial and boundary condition has a unique solution in $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau,G)$ provided that $g\in\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau,G)$, where the spaces $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau,G)$ and $\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau,G)$ are introduced for domains G similarly to $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ and $\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)$ as it is done for $q=p$ in [15], where the corresponding solvability result is proved for $q=p$. By the

way, the fact that the number of nonzero g^k can be infinite allows one treating the equations driven by space-time white noise (see [8], [15]) when $u \in \mathfrak{H}^{\gamma,q}_{p,\theta}(\tau,G)$ with γ slightly less than $1/2$. In this case, the results like Theorem 4.7 still allow one to get continuity properties of u (see [8], [15]).

Coming back to [14] it is worth noticing that, although the estimates like (4.11) and (4.12) are obtained there only for $g \in \mathbb{H}^{0,\infty}_{p,d}(T,G)$ and with no specific expressions on the right instead of just ∞, they are also derived for the situation when L is a divergence form operator with measurable coefficients. Our results are not directly applicable to such operators (see more abou this in Remark 5.4).

In Theorem 3.11 the power of integration with respect to t is always q. Sometimes (see, for instance, [21]) one needs to get a higher power on the left.

Theorem 4.9. *Let $r \geq p$, $\kappa \in [0,1]$,*

$$2/q < \beta \leq 1, \quad \frac{\sigma}{r} = \frac{\theta}{p}, \quad \mu = \gamma + \frac{d}{r} - \frac{d}{p} - (1-\kappa)\beta.$$

Let $T \in (0,\infty)$ and $\tau \leq T$. Then for any $u \in \mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$ satisfying $M^{\beta-1}u(0) \in L_q(\Omega, H^{\gamma-\beta}_{p,\theta})$, we have

$$E\Big(\int_0^\tau ||M^{(1-\kappa)(\beta-1)+\kappa(-1)}u(t)||^{q/\kappa}_{\mu,r,\sigma}\,dt\Big)^\kappa$$
$$\leq N\big(T^{q\beta/2-1}[|u|]^q_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)} + E||M^{\beta-1}u(0)||^q_{H^{\gamma-\beta}_{p,\theta}}\big)^{1-\kappa}||M^{-1}u||^{q\kappa}_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)}, \tag{4.14}$$

where N is independent of u,τ, and T. In particular, for $u(0) = 0$,

$$E\Big(\int_0^\tau ||M^{(1-\kappa)(\beta-1)+\kappa(-1)}u(t)||^{q/\kappa}_{\nu,r,\sigma}\,dt\Big)^\kappa$$
$$\leq NT^{(q\beta/2-1)(1-\kappa)}[|u|]^{q(1-\kappa)}_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)}||M^{-1}u||^{q\kappa}_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)},$$

whenever

$$\nu \leq \gamma + \frac{d}{r} - \frac{d}{p} - (1-\kappa)\beta.$$

Proof. Denote

$$\mu_1 = \mu, \quad \sigma_1 = \sigma + r[(1-\kappa)(\beta-1) - \kappa],$$
$$\mu_2 = \mu_1 + d/p - d/r, \quad \sigma_2 = \sigma_1 p/r, \quad \mu_3 = \gamma, \quad \sigma_3 = \theta - p.$$

Observe that

$$\mu_2 = (1-\kappa)(\gamma-\beta) + \kappa\mu_3, \quad \sigma_2 = (1-\kappa)[\theta + (\beta-1)p] + \kappa\sigma_3.$$

Therefore by Lemma 2.2 and Theorem 2.4 we get

$$||M^{(1-\kappa)(\beta-1)-\kappa}u||_{\mu,r,\sigma} \leq N||u||_{\mu_1,r,\sigma_1} \leq N||u||_{\mu_2,p,\sigma_2}$$
$$\leq N||u||^{1-\kappa}_{\gamma-\beta,p,\theta+(\beta-1)p}||u||^\kappa_{\mu_3,p,\sigma_3} \leq N||M^{\beta-1}u||^{1-\kappa}_{\gamma-\beta,p,\theta}||M^{-1}u||^\kappa_{\gamma,p,\theta}.$$

Hence the left-hand side of (4.14) is less than

$$NE\sup_{t\leq\tau}||M^{\beta-1}u(t)||^{q(1-\kappa)}_{\gamma-\beta,p,\theta}\Big(\int_0^\tau ||M^{-1}u(t)||^q_{\gamma,p,\theta}\Big)^\kappa,$$

and to finish the proof it only remains to apply Hölder's inequality and Theorem 4.1. The theorem is proved.

Remark 4.10. For $\kappa = 0$ and $r = p$ Theorem 4.9 gives the last statement of Theorem 4.1, whereas for $\kappa = 1$ it gives Theorem 3.11 (i).

Remember that, if a constant $\varepsilon > 0$ and $u \in H^\gamma_p$ and $u(x) = 0$ for $x^1 \notin [\varepsilon, \varepsilon^{-1}]$, then $u \in H^\gamma_{p,d}$ and the norms of u in H^γ_p and $H^\gamma_{p,d}$ are comparable. Taking into account this and the translation invariance of the H^γ_p-norms, from Theorem 4.1 we get the following

Theorem 4.11. *Let $\gamma, \theta \in \mathbb{R}$, $T \in (0, \infty)$, $\tau \le T$,*

$$2/q < \alpha < \beta \le 1,$$

and let $\varepsilon > 0$ be a constant. Then for any function $u \in \mathcal{H}^{\gamma,q}_p(\tau)$ such that $u(t,x) = 0$ for $x^1 \notin [-\varepsilon, \varepsilon]$, we have

$$u \in C^{\alpha/2-1/q}([0,\tau], H^{\gamma-\beta}_q)$$

(a.s.) and for any constant $a > 0$, stopping time $\eta \le \tau$, and $0 \le s < t < \infty$,

$$E||u(t\wedge\eta) - u(s\wedge\eta)||^q_{\gamma-\beta,p} \le N|t-s|^{q\beta/2-1}a^{\beta-1}I(a),$$

$$E[|u|]^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_p)} \le NT^{(\beta-\alpha)q/2}a^{\beta-1}I(a),$$

$$E||u||^q_{C([0,\tau],H^{\gamma-\beta}_p)} \le NE||u(0)||^q_{H^{\gamma-\beta}_p} + NT^{q\beta/2-1}a^{\beta-1}I(a),$$

where

$$I(a) := a||u||^q_{\mathbb{H}^{\gamma,q}_p(\tau)} + a^{-1}||\mathbb{D}u||^q_{\mathbb{H}^{\gamma-2,q}_p(\tau)} + ||\mathbb{S}u||^q_{\mathbb{H}^{\gamma-1,q}_p(\tau)} \tag{4.15}$$

and the constants N are independent of a, γ, τ, T, and u.

Theorem 4.11 has a corollary which is a counterpart of Corollary 4.3. This corollary of Theorem 4.11 holds without the assumption that u has compact support with respect to x^1.

Corollary 4.12. *Let $\gamma \in \mathbb{R}$, $q \ge p$, $T \in (0,\infty)$, $\tau \le T$. Assume (4.6). Then for any function $u \in \mathcal{H}^{\gamma,q}_p(\tau)$, we have*

$$u \in C^{\hat\alpha/2-1/q}([0,\tau], H^{\gamma-\beta}_q)$$

(a.s.) and for any constant $a > 0$, stopping time $\eta \le \tau$, and $0 \le s < t < \infty$,

$$E||u(t\wedge\eta) - u(s\wedge\eta)||^q_{\gamma-\beta,q} \le N|t-s|^{q\hat\beta/2-1}a^{\hat\beta-1}I(a), \tag{4.16}$$

$$E[|u|]^q_{C^{\hat\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_q)} \le NT^{(\hat\beta-\hat\alpha)q/2}a^{\hat\beta-1}I(a), \tag{4.17}$$

$$E||u||^q_{C([0,\tau],H^{\gamma-\beta}_q)} \le NE||u(0)||^q_{H^{\gamma-\beta}_q} + NT^{q\hat\beta/2-1}a^{\hat\beta-1}I(a), \tag{4.18}$$

where $I(a)$ is taken from (4.15) and the constants N are independent of a, γ, τ, T, and u.

Proof. We observe that shifting the origin shows that Theorem 4.11 is true for any function u which is zero for x^1 outside a fixed interval of unit length and use it to consider the case of general u.

Take a function $\eta \in C^\infty_0(\mathbb{R})$ such that $\eta(x^1) = 0$ for $|x^1| \ge 1/2$ and

$$\sum_{n=-\infty}^{\infty} \eta^q_n(x) \ge 1 \quad \forall x \in \mathbb{R}^d,$$

where $\eta_n := \eta(x+n)$. We use the facts that sup of a sum is less than the sum of sup's and (remember $q \geq p$)

$$\sum_i \nu_i^q \leq \big(\sum_i \nu_i^p\big)^{q/p}, \quad \nu_i \geq 0. \tag{4.19}$$

We also apply Theorem 2.1 of [6] (which is essentially covered by Theorem 2.4.7 of [18]). Then, for $T = a = 1$, we get

$$E[|u|]^q_{C^{\hat\alpha/2-1/q}([0,\tau],H_q^{\gamma-\beta})} \leq N \sum_n E[|u\eta_n|]^q_{C^{\hat\alpha/2-1/q}([0,\tau],H_q^{\gamma-\beta})}$$

$$\leq N \sum_n \big[E\int_0^\tau ||u(t)\eta_n||^q_{\gamma,p}\,dt + E\int_0^\tau ||\mathbb{D}u(t)\eta_n||^q_{\gamma-2,p}\,dt$$

$$+E\int_0^\tau ||\mathbb{S}u(t)\eta_n||^q_{\gamma-1,p}\,dt\big] \leq N\big[E\int_0^\tau \big(\sum_n ||u(t)\eta_n||^p_{\gamma,p}\big)^{q/p}\,dt$$

$$+E\int_0^\tau \big(\sum_n ||\mathbb{D}u(t)\eta_n||^p_{\gamma-2,p}\big)^{q/p}\,dt + E\int_0^\tau \big(\sum_n ||\mathbb{S}u(t)\eta_n||^p_{\gamma-1,p}\big)^{q/p}\,dt\big]$$

$$\leq N\big(||u||^q_{\mathbb{H}_p^{\gamma,q}(\tau)} + ||\mathbb{D}u||^q_{\mathbb{H}_p^{\gamma-2,q}(\tau)} + ||\mathbb{S}u||^q_{\mathbb{H}_p^{\gamma-1,q}(\tau)}\big).$$

This proves (4.17) in the case $T = a = 1$, to which we have confined ourselves just for simplicity of notation. Inequality (4.16) is proved in the same way and (4.18) is a trivial consequence of (4.17). The corollary is proved.

Remark 4.13. If $q = p$ (so that $\hat\alpha = \alpha, \hat\beta = \beta$), Corollary 4.12 is an improvement of Theorem 7.2 (i) of [8], which does not specify the dependence of constants on T and has $a = 1$. By the way, the fact that in Corollary 4.12 we have the parameter a will be instrumental in Section 6. One also can find a somewhat weaker version of Corollary 4.12 in [15] for $q = p$ and functions with compact support with respect to x^1.

Remark 4.14. If $q = p > 2$ and $\beta = 1$ $(= \hat\beta)$, assertion (4.18) is basically the same as Lemma 2.12 of [12]. The difference is that Lemma 2.12 of [12] gives an estimate which is more detailed in a certain sense. Notice that although Lemma 2.12 of [12] is only proved under the condition $u(0) = 0$, its proof can be easily adjusted to cover the case of general $u(0) \in L_p(\Omega, H_p^{\gamma-1})$.

If $q = p = 2$, there is no possible values of β consistent with (4.6). Therefore, it is worth noting that Corollary 4.12 also holds if $p = q = 2$ and $\alpha = \beta = 1$. This fact follows from Lemma 2.12 of [12] combined with the well-known statement that almost surely $u(t)$ is an $H_2^{\gamma-1}$-valued continuous function (see, for instance [8] or [16]).

5. Proof of Theorem 4.1

Observe that (4.5) is a straightforward consequence of (4.3). The proof of assertions (4.2) and (4.3) is based on the following lemma which is proved later in this section.

Lemma 5.1. *The assertions of Theorem 4.1 hold if we additionally assume that $\gamma = \beta$, $\theta = d$, $T = a = 1$.*

Proof of Theorem 4.1. It turns out that, if (4.2) is true for $a=1$, then (4.2) holds for any $a>1$. Indeed, let $du=M^{-1}f\,dt+g^k\,dw^k_t$, $u_a(t,x):=u(t,ax)$. Then $du_a=M^{-1}\tilde f\,dt+\tilde g^k\,dw^k_t$, where

$$\tilde f(t,x)=a^{-1}f(t,ax),\quad \tilde g(t,x)=g(t,ax).$$

Hence by using the scaling properties of the $H^\gamma_{p,\theta}$-norms, we find

$$a^{-q(\beta-1)-\theta q/p}E||M^{\beta-1}(u(t\wedge\eta)-u(s\wedge\eta))||^q_{\gamma-\beta,p,\theta}$$

$$\leq NE||M^{\beta-1}(u_a(t\wedge\eta)-u_a(s\wedge\eta))||^q_{\gamma-\beta,p,\theta}$$

$$\leq NN_1|t-s|^{q\beta/2-1}\big(||M^{-1}u_a||^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)}+||\tilde f||^q_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)}+||\tilde g||^q_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)}\big)$$

$$\leq NN_1|t-s|^{p\beta/2-1}\big(a^{q-\theta q/p}||M^{-1}u||^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)}$$

$$+a^{-q-\theta q/p}||f||^q_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)}+a^{-\theta q/p}||g||^q_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)}\big),$$

which yields (4.2) with a^q in place of a. The same is true for (4.3). Therefore, we may only concentrate on the case $a=1$.

Furthermore, if assertions (4.2) and (4.3) are true for $T=1$, then they are true for any T. To show this, notice that if $du=M^{-1}f\,dt+g^k\,dw^k_t$, then, for any constant $c>0$, the function $u_c(t,x)=u(c^2t,cx)$ satisfies

$$du_c(t,x)=M^{-1}\hat f(t,x)\,dt+\hat g^k(t,x)\,d\hat w^k_t,$$

where

$$\hat f(t,x)=cf(c^2t,cx),\quad \hat g(t,x)=cg^k(c^2t,cx),\quad \hat w^k_t=c^{-1}w^k_{c^2t}.$$

Observe that $\hat w^k_t$ are independent Wiener processes. Therefore, if (4.3) is true for $c^{-2}T$ in place of T, then, by using the homogeneity of the norms in $H^\gamma_{p,\theta}$ and seminorms in C^α, we find

$$c^{-\theta q/p}E[|M^{\beta-1}u|]^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{p,\theta})}$$

$$\leq Nc^{q(\beta-1)}E[|M^{\beta-1}(u(\cdot,c\,\cdot))|]^q_{C^{\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{p,\theta})}$$

$$=Nc^{2-q+q(\beta-\alpha)}E[|M^{\beta-1}u_c|]^q_{C^{\alpha/2-1/q}([0,c^{-2}\tau],H^{\gamma-\beta}_{p,\theta})}$$

$$=Nc^{2-q+q(\beta-\alpha)}(||M^{-1}u_c||^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(c^{-2}\tau)}+||\hat f||^q_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(c^{-2}\tau)}+||\hat g||^q_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(c^{-2}\tau)})$$

$$\leq Nc^{q(\beta-\alpha)-\theta q/p}(||M^{-1}u||^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\tau)}+||f||^q_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\tau)}+||g||^q_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)}),$$

where N are independent of c. By choosing $c^2=T$, we see that it suffices to prove (4.3) only for $\tau\leq T=1$. The same holds for (4.2).

Finally, owing to Theorems 2.5 and 3.9 assertions (4.2) and (4.3) in the case $a=T=1$ follow directly from Lemma 5.1 and Remark 3.3. The theorem is proved.

To prove Lemma 5.1 we need the following simple fact the proof of which is quite standard although somewhat long.

Lemma 5.2. *Let $r > 0$, $p \geq 2$. Let $g^k(\omega, t, x)$ be defined for $k = 1, 2, ...$, $\omega \in \Omega$, $t \in [0, T]$, and $x \in \mathbb{R}^d$. Assume that $g^k \equiv 0$ if k is large, g^k is predictable for any x, has bounded variation with respect to t for any ω, x, and belongs to L_p for any ω, t. Assume that for any k, ω, x*

$$\operatorname{Var}_{[0,T]} (g^k(\cdot, x) - g^k(\cdot, y)) \to 0$$

as $y \to x$ and

$$\int_{\mathbb{R}^d} |\operatorname{Var}_{[0,T]} g(\cdot, x)|^p \, dx < \infty.$$

Then, for any x, the stochastic integral

$$u(t, x) = \int_0^t g^k(s, x) \, dw_s^k$$

admits a modification which is continuous in (t, x) for each ω. Keep the same notation $u(t, x)$ for this modification. Then

$$E \sup_{t \leq T} ||u(t)||_p^r \leq N E(\int_0^T ||g(t)||_p^2 \, dt)^{r/2}, \tag{5.1}$$

where N is independent of g and T.

Proof. As easy to see, the modification in question is given by

$$g^k(t, x) w_t^k - \int_{[0,t]} w_s^k \, d_s g(s, x),$$

which is obtained by integrating by parts. From this representation we also see that

$$|u(t, x)| \leq 2 \sup_{t \in [0,T]} |w_t| \operatorname{Var}_{[0,T]} g(\cdot, x).$$

Next, by Itô's formula we have

$$|u(t, x)|^p = p(p-1)/2 \int_0^t |u(s, x)|^{p-2} |g(s, x)|^2 \, ds$$
$$+ p \int_0^t |u(s, x)|^{p-2} u(s, x) g^k(s, x) \, dw_s^k.$$

Since we are dealing with processes continuous with respect to x and

$$\int_{\mathbb{R}^d} \sup_{s \leq T} (|u(s, x)|^{p-1} |g(s, x)|) \, dx < \infty,$$

by Fubini's theorem

$$||u(t)||_p^p = N \int_0^t \big(\int_{\mathbb{R}^d} |u(s, x)|^{p-2} |g(s, x)|^2 \, dx \big) \, ds$$
$$+ N \int_0^t \big(\int_{\mathbb{R}^d} |u(s, x)|^{p-2} u(s, x) g^k(s, x) \, dx \big) \, dw_s^k. \tag{5.2}$$

Hence by the Burkholder-Davis-Gundy inequalities

$$E \sup_{[0,T]} ||u(t)||_p^r \leq N_1 E\big(\int_0^T \big(\int_{\mathbb{R}^d} |u(s, x)|^{p-2} |g(s, x)|^2 \, dx \big) \, ds \big)^{r/p}$$
$$+ N_2 E\big(\int_0^T \sum_k \big(\int_{\mathbb{R}^d} |u(s, x)|^{p-1} g^k(s, x) \, dx \big)^2 \, dt \big)^{r/(2p)} =: N_1 I_1 + N_2 I_2.$$

Here, by Hölder's inequality

$$\sum_k \Big(\int_{\mathbb{R}^d} |u(s,x)|^{p-1} g^k(s,x)\,dx\Big)^2$$

$$\leq \int_{\mathbb{R}^d} |u(s,x)|^p\,dx \int_{\mathbb{R}^d} |u(s,x)|^{p-2}|g(s,x)|^2\,dx \leq ||u(s)||_p^{2(p-1)}||g(s)||_p^2,$$

$$I_2 \leq E\sup_{[0,T]} ||u(t)||_p^{r(p-1)/p}\Big(\int_0^T ||g(s)||_p^2\,ds\Big)^{r/(2p)}$$

$$\leq (1/4)N_2^{-1}E\sup_{[0,T]}||u(t)||_p^r + NE\Big(\int_0^T ||g(s)||_p^2\,ds\Big)^{r/2}.$$

Also

$$\int_{\mathbb{R}^d} |u(s,x)|^{p-2}|g(s,x)|^2\,dx \leq ||u(s)||_p^{p-2}||g(s)||_p^2,$$

$$I_1 \leq E\sup_{[0,T]} ||u(t)||_p^{r(p-2)/p}\Big(\int_0^T ||g(s)||_p^2\,ds\Big)^{r/p}$$

$$\leq (1/4)N_1^{-1}E\sup_{[0,T]}||u(t)||_p^r + NE\Big(\int_0^T ||g(s)||_p^2\,ds\Big)^{r/2}.$$

Therefore,

$$E\sup_{[0,T]}||u(t)||_p^r \leq (1/2)E\sup_{[0,T]}||u(t)||_p^r + NE\Big(\int_0^T ||g(s)||_p^2\,ds\Big)^{r/2}.$$

This implies (5.1) if the left-hand side of (5.1) is finite. Actually, using the fact that, by (5.2), the norm $||u(t)||_p$ is continuous with respect to t and using appropriate stopping times one easily get rid of this extra assumption. The lemma is proved.

Proof of Lemma 5.1. By bearing in mind that $H^\gamma_{p,d}$ are Banach spaces, we convince ourselves that we only need to prove the lemma for functions of type u_n described in Lemma 3.10. We take such a function u and closely follow the proof of Theorem 7.2 of [8].

First we prove (4.2). Let us use the notation $||\cdot||_p$ for the norm in $L_p(\mathbb{R}^d_+)$. Notice that, due to (4.1), we have $q\alpha/2 > 1$, and by Lemma 7.4 of [8], for $\hat{t} := t\wedge\eta$, $\hat{s} := s\wedge\eta$, and any ω, we have

$$||M^{\beta-1}(u(t\wedge\eta) - u(s\wedge\eta))||_p^q = ||M^{\beta-1}(u(\hat{t}) - u(\hat{s}))||_p^q$$

$$\leq N(\hat{t}-\hat{s})^{q\alpha/2-1}\int_0^{\hat{t}-\hat{s}} \frac{d\kappa}{\kappa^{1+q\alpha/2}}\int_{\hat{s}}^{\hat{t}-\kappa} ||M^{\beta-1}(u(r+\kappa)-u(r))||_p^q\,dr.$$

$$\leq N(t-s)^{q\alpha/2-1}\int_0^{t-s} \frac{d\kappa}{\kappa^{1+q\alpha/2}}\int_{s}^{t-\kappa} I_{r+\kappa<\eta}||M^{\beta-1}(u(r+\kappa)-u(r))||_p^q\,dr. \tag{5.3}$$

To estimate the right-hand side define

$$h = M(\tilde{\mathbb{D}}u - \Delta u)I_{(0,\tau]\!]}, \quad g = \tilde{\mathbb{S}}uI_{(0,\tau]\!]},$$

extend u, f, g for $x^1 \leq 0$ letting $u(t,x) = u(t,|x^1|,x')\operatorname{sign}x^1$,

$$h(t,x) = h(t,|x^1|,x'), \quad g(t,x) = g(t,|x^1|,x')\operatorname{sign}x^1,$$

and observe that, for $t \leq \tau$ and $x \in \mathbb{R}^d$,

$$u(t,x) = u(0,x) + \int_0^t (\Delta u(s,x) + M^{-1}h(s,x))\,ds + \int_0^t g^k(s,x)\,dw_s^k,$$

where due to the specified structure of g the summation with respect to k involves only finitely many terms and each stochastic integral is just a finite sum. From the proof of Theorem 4.2 of [8] exactly for our case of step functions h and g it is easily seen that the following natural representation holds

$$u(r+\kappa,x) - u(r,x) = (T_\kappa - 1)u(r,x) + \int_r^{r+\kappa} T_{r+\kappa-\rho}M^{-1}h(\rho,x)\,d\rho + \int_r^{r+\kappa} T_{r+\kappa-\rho}g^k(\rho,x)\,dw_\rho^k$$

on the set $\{\omega : r+\kappa \leq \tau\}$ (a.s.) for any $x \in \mathbb{R}^d$, r, and κ, where T_t is the semigroup associated with the operator Δ in $\mathbb{R}^d$.

Accordingly,

$$EI_{r+\kappa<\eta}||M^{\beta-1}(u(r+\kappa) - u(r))||_p^q \leq N\sum_{i=1}^3 A_i(r,\kappa),$$

where

$$A_1(r,\kappa) = E||I_{r+\kappa<\eta}M^{\beta-1}(T_\kappa - 1)u(r)||_p^q,$$

$$A_2(r,\kappa) = E||\int_r^{r+\kappa} M^{\beta-1}T_{r+\kappa-\rho}M^{-1}h(\rho)\,d\rho||_p^q,$$

$$A_3(r,\kappa) = E||\int_r^{r+\kappa} M^{\beta-1}T_{r+\kappa-\rho}g^k(\rho)\,dw_\rho^k||_p^q.$$

By the way, due to oddness of the functions $u, M^{-1}h$, and g, one can replace $T_\cdot$ with $\tilde{T}_\cdot$ in the above formulas, where $\tilde{T}_t$ is the semigroup defined on functions on $\mathbb{R}_+^d$ associated with the operator Δ with zero condition on $x^1 = 0$. We see that

$$E||M^{\beta-1}(u(t\wedge\eta) - u(s\wedge\eta))||_p^q \leq N(t-s)^{q\alpha/2-1}\sum_{i=1}^3 J_i(t,s), \tag{5.4}$$

with

$$J_i(t,s) = \int_0^{t-s} \frac{d\kappa}{\kappa^{1+q\alpha/2}} \int_s^{t-\kappa} A_i(r,\kappa)\,dr.$$

By Corollary 8.5 (applied for $q = p$ and $\theta = d$) using $\alpha < \beta$, we get

$$A_1(r,\kappa) \leq N\kappa^{q\beta/2}EI_{r<\tau}||M\Delta u(r)||_{\beta-2,p,\theta}^q,$$

$$J_1(t,s) \leq N\int_0^{t-s} \frac{d\kappa}{\kappa^{1+q(\alpha-\beta)/2}} E\int_0^\tau ||M\Delta u(r)||_{\beta-2,p,\theta}^q\,dr$$

$$\leq N(t-s)^{q(\beta-\alpha)/2}||M\Delta u||_{\mathbb{H}_{p,\theta}^{\beta-2,q}(\tau)}^q \leq N(t-s)^{q(\beta-\alpha)/2}||M^{-1}u||_{\mathbb{H}_{p,\theta}^{\beta,q}(\tau)}^q.$$

Furthermore, notice that, for $q' = q/(q-1)$, we have $(\beta/2 - 1)q' > -1$, since $\beta/2 > 1/q$, and by Hölder's inequality and Theorem 8.3 (even if $\beta \in (1,2]$ or $p \in (1,2]$)

$$A_2(r,\kappa) \leq E\Big(\int_0^\kappa ||M^{\beta-1}\tilde{T}_{\kappa-\rho}M^{-1}h(r+\rho)||_p\,d\rho\Big)^q$$

$$\leq \Big(\int_0^\kappa (\kappa-\rho)^{(\beta/2-1)q'}\,d\rho\Big)^{q-1} E\int_0^\kappa (\kappa-\rho)^{(1-\beta/2)q}||M^{\beta-1}\tilde{T}_{\kappa-\rho}M^{-1}h(r+\rho)||_p^q\,d\rho$$

$$\leq N\kappa^{q\beta/2-1}E\int_0^\kappa ||h(r+\rho)||^q_{\beta-2,p,\theta}\,d\rho,$$

$$J_2(t,s)\leq N\int_0^{t-s}\frac{d\kappa}{\kappa^{2+q(\alpha-\beta)/2}}\int_0^\kappa d\rho\, E\int_0^1 ||h(r)||^q_{\beta-2,p,\theta}\,dr$$
$$\leq N(t-s)^{q(\beta-\alpha)/2}||h||^q_{\mathbb{H}^{\beta-2,q}_{p,\theta}(\tau)}.$$

To estimate $J_3(t,s)$, we use Lemma 5.2 and the inequality $(\beta-1)q''>-1$ (due to $\beta>2/q$), where $q''=q/(q-2)$. Then we obtain

$$A_3(r,\kappa)\leq NE\Big(\int_0^\kappa \rho^{\beta-1}\rho^{1-\beta}||M^{\beta-1}\tilde{T}_\rho g(r+\kappa-\rho)||_p^2\,d\rho\Big)^{q/2}$$

$$\leq N\kappa^{q\beta/2-1}E\int_0^\kappa \rho^{(1-\beta)q/2}||M^{\beta-1}\tilde{T}_\rho g(r+\kappa-\rho)||_p^q\,d\rho.$$

Furthermore, by Theorem 8.3, due to

$$-1<\beta+1\leq 2,\quad d-1-p(\beta+1)<\theta-p<d-1+p,$$

it holds that

$$||M^{\beta-1}\tilde{T}_\rho g(r+\kappa-\rho)||_p\leq N||M^\beta\tilde{T}_\rho g(r+\kappa-\rho)||_{0,p,\theta-p}$$
$$\leq N\rho^{(\beta-1)/2}||Mg(r+\kappa-\rho)||_{\beta-1,p,\theta-p}\leq N\rho^{(\beta-1)/2}||g(r+\kappa-\rho)||_{\beta-1,p,\theta}.$$

Therefore,

$$A_3(r,\kappa)\leq N\kappa^{q\beta/2-1}E\int_0^\kappa ||g(r+\kappa-\rho)||^q_{\beta-1,p,\theta}\,d\rho,$$

$$J_3(t,s)\leq NE\int_0^{t-s}\frac{d\kappa}{\kappa^{2+(\alpha-\beta)q/2}}\int_s^{t-\kappa}dr\int_0^\kappa ||g(r+\kappa-\rho)||^q_{\beta-1,p,\theta}\,d\rho$$
$$\leq NE\int_0^{t-s}\frac{d\kappa}{\kappa^{2+(\alpha-\beta)q/2}}\int_0^\kappa d\rho\int_0^\tau ||g(r)||^q_{\beta-1,p,\theta}\,dr$$
$$=N(t-s)^{(\beta-\alpha)q/2}E\int_0^\tau ||g(r)||^q_{\beta-1,p,\theta}\,dr.$$

The above estimates of J_i and (5.4) finish proving (4.2) for $\beta=\gamma,\theta=d,T=a=1$. Interestingly enough the above proof goes exactly the same way for any bounded τ. However, while proving (4.3), it is very convenient to use $\tau\leq 1$. In this case it suffices to repeat the above proof taking $\eta=\tau$ and observing that

$$\sup_{0\leq s<t\leq\tau}(t-s)^{1-q\alpha/2}||M^{\beta-1}(u(t\wedge\eta)-u(s\wedge\eta))||_p^q$$
$$\leq N\int_0^1\frac{d\kappa}{\kappa^{1+q\alpha/2}}\int_0^{1-\kappa}I_{r+\kappa<\tau}||M^{\beta-1}(u(r+\kappa)-u(r))||_p^q\,dr.$$

The lemma is proved.

Remark 5.3. It makes sense to consider $H^{\gamma,q}_{p,\theta}(T)$ (see definition 3.2) for all $q,p>1$ and not only for $q,p\geq 2$. In connection with this observe that the inequalities $\beta\leq 1$ and $q,p\geq 2$ were only used in the proof of Lemma 5.1 while estimating J_3. Without this term, applying Theorem 8.3 is possible even if $1<\beta\leq 2$ or $1<p\wedge q\leq 2$. It follows that, if $u\in H^{\gamma,q}_{p,\theta}(T)$, then (4.2) holds for any γ,θ,β, and any $p,q>1$ satisfying $2/q<\beta\leq 2$.

Remark 5.4. The technique of estimating Hölder norms on the basis of relations like (5.3) is applicable not only to the spaces $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$, which are mostly suitable for concidering second order parabolic equations. In the $\mathfrak{H}^{\gamma,q}_{p,\theta}(\tau)$-norm the gap in the number of derivatives of u, $\tilde{\mathbb{S}}u$, and $\tilde{\mathbb{D}}u$ equals one, and this made concidering the heat semigroup natural. One can introduce a space with a different gap and then use powers of Δ or other elliptic operators as generators of semigroups, which can be suitable for pseudo-differential SPDEs. One can find some examples of such equations in [1] where $q=2$.

One can also use this technique with no connection to scales of spaces, say to investigate properties of stochastic convolutions of type

$$u(t)=\int_0^t U_{t-s}g(s)\,dw_s,$$

where $U(s)$ is a semigroup of operators acting in a space of measurable functions given in a domain and w_s is, say a one-dimensional Wiener process. If one knows appropriate smoothing properties of $U(t)$ in terms of Hölder spaces, then, by replacing the L_p-norm in relations like (5.3) with H^γ_p-norm for small $\gamma>0$ an repeating the proof of Lemma 5.1, one can get that u is smooth also in terms of Sobolev spaces. The author believes that on this way one can get the result of [14] for equations with second order elliptic operator in divergence form.

6. An alternative proof of Corollary 4.3

The above derivation of Corollary 4.3 uses the fact that the scale $H^\gamma_{p,\theta}$ is generated by the operators $(-\mathcal{L}_{b,c})^{\gamma/2}$. Here we present a different proof based on somewhat simpler arguments. However, we do not know how to adjust these arguments in the case of Theorem 4.1. The basic fact which we will be using is that Theorem 4.1 holds if $\theta=d$ and $\gamma=\beta$. This is the statement of Lemma 5.1 apart from the restriction $T=a=1$ in its formulation. One get rid of this restriction by using the scaling properties of $H^\gamma_{p,\theta}$ as in Section 5.

Next we show an alternative proof of Corollary 4.12. Since the operators $(1-\Delta)^\mu$ map isometrically H^γ_p onto $H^{\gamma-\mu}_p$, one only needs prove the corollary for any particular γ. We choose $\gamma=\beta$. In that case, if $u(t,x)=0$ for $x^1\notin[-\varepsilon,\varepsilon]$, we immediately get the assertions of the corollary from the above mentioned modification of Lemma 5.1 in the same way as Theorem 4.11. For general u it suffices to repeat the proof of Corollary 4.12 given in Section 4.

Now we proceed as in the proof of Theorem 2.11 of [12] noticing again that without loss of generality we may assume $T=a=1$. Define $f=M\tilde{\mathbb{D}}u$, $g=\tilde{\mathbb{S}}u$ and notice that

$$\begin{aligned}
I:=E[|M^{\beta-1}u|]^q_{C^{\hat\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_{q,\hat\theta})}\\
\le\sum_{n=-\infty}^{\infty}e^{n\hat\theta}E[|(M^{\beta-1}u)(\cdot,e^n\cdot)\zeta|]^q_{C^{\hat\alpha/2-1/q}([0,\tau],H^{\gamma-\beta}_q)}\\
\le N\sum_{n=-\infty}^{\infty}e^{n(\hat\theta+q(\beta-1))}E[|u(\cdot,e^n\cdot)\zeta|]^q_{C^{\hat\alpha/2-1/p}([0,\tau],H^{\gamma-\beta}_q)}=:N\sum_{n=-\infty}^{\infty}e^{nq\theta/p}I_n,
\end{aligned}$$

where by Corollary 4.12 with $a=e^{-nq}$

$$I_n\le N\big(e^{-nq}||u(\cdot,e^n\cdot)\zeta||^q_{\mathbb{H}^{\gamma,q}_p(\tau)}$$

$$+e^{nq}||(M^{-1}f)(\cdot,e^n\cdot)\zeta||^q_{\mathbb{H}^{\gamma-2,q}_p(\tau)}+||g(\cdot,e^n\cdot)\zeta||^q_{\mathbb{H}^{\gamma-1,q}_p(\tau)}\big).$$

Hence, using (4.19), we conclude

$$I\le N\big(||u||^q_{\mathbb{H}^{\gamma,q}_{p,\theta-p}(\tau)}+||M^{-1}f||^q_{\mathbb{H}^{\gamma,q}_{p,\theta+p}(\tau)}+||g||^q_{\mathbb{H}^{\gamma-1,q}_{p,\theta}(\tau)}\big)$$

and to finish proving (4.8) (with $T=a=1$), it only remains to use Lemma 2.2. In the same way one proves (4.7), whereas (4.9) follows from (4.8) as always.

7. Trace theorems for deterministic spaces

In this section we assume that $p,q\in(1,\infty)$ and remember that the spaces $H^{\gamma,q}_{p,\theta}(T)$ are introduced in Definition 3.2. For the case of deterministic functions u the range of parameters p,q,β automatically extends as pointed out in Remark 5.3. For the sake of brevity we will not formulate the deterministic counterparts of Theorems 4.7 and 4.9 and rather confine ourselves to the discussion of the deterministic counterparts for Theorems 4.1 and 4.11, Corollary 4.3, and Corollary 4.12.

The following is just a restatement of Remark 5.3.

Theorem 7.1. *Let $\gamma,\theta\in\mathbb{R}$, $2/q<\beta\le 2$, $T\in(0,\infty)$. Then for any $u\in H^{\gamma,q}_{p,\theta}(T)$, for any constant $a>0$, and $0\le s\le t\le T$, we have*

$$||M^{\beta-1}(u(t)-u(s))||_{\gamma-\beta,p,\theta}$$
$$\le N|t-s|^{\beta/2-1/q}a^{\beta-1}\big(a||M^{-1}u||_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)}+a^{-1}||Mu_t||_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(T)}\big),\quad(7.1)$$

where N is independent of T, u, s, t, and a, which upon minimizing with respect to $a>0$ yields

$$||M^{\beta-1}(u(t)-u(s))||_{\gamma-\beta,p,\theta}\le N|t-s|^{\beta/2-1/p}||M^{-1}u||^{1-\beta/2}_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)}||Mu_t||^{\beta/2}_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(T)}.$$

In the same way as Corollary 4.3 is derived from Theorem 4.1, we obtain the following.

Corollary 7.2. *Let $\gamma,\theta\in\mathbb{R}$, $q\ge p$, $T\in(0,\infty)$, $u\in H^{\gamma,q}_{p,\theta}(T)$,*

$$\tfrac{2}{q}+(\tfrac{1}{p}-\tfrac{1}{q})d<\beta\le 2+(\tfrac{1}{p}-\tfrac{1}{q})d,$$

$$\hat\beta=\beta-(\tfrac{1}{p}-\tfrac{1}{q})d,\quad \tfrac{\hat\theta-d}{q}=\tfrac{\theta-d}{p}.$$

Then for any constant $a>0$, $0\le s<t\le T$,

$$||M^{\beta-1}(u(t)-u(s))||_{\gamma-\beta,q,\hat\theta}$$
$$\le N|t-s|^{\hat\beta/2-1/q}a^{\hat\beta-1}\big(a||M^{-1}u||_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)}+a^{-1}||Mu_t||_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(T)}\big),$$

where N is independent of T, s, t, a, and u.

In Section 6 we gave a proof of Corollary 4.3 on the basis of Lemma 5.1 which is ultimately based on Lemmas 8.1 and 8.2 and interpolation Theorem 2.4 for the spaces $H^\gamma_{p,\theta}$. It turns out that in the deterministic case one can avoid using these tools and prove Corollary 7.2 differently. This is what we want to show here.

First we show how to prove differently the following counterpart of Theorem 4.11 without restrictions on the support of u.

Theorem 7.3. *Let $\gamma \in \mathbb{R}$, $T \in (0,\infty)$, and $2/q < \beta \leq 2$. Then for any function $u \in H^{\gamma,q}_p(T)$ and for any constant $a > 0$ and $0 \leq s < t \leq T$, we have*

$$||u(t)-u(s)||_{\gamma-\beta,p} \leq N|t-s|^{\beta/2-1/q}a^{\beta-1}\big(a||u||_{\mathbb{H}^{\gamma,q}_p(T)} + a^{-1}||u_t||_{\mathbb{H}^{\gamma-2,q}_p(T)}\big), \quad (7.2)$$

where N is independent of T, s, t, a, and u.

Proof. First, if (7.2) is true for any T and $a=1$, then upon taking $a>0$ and introducing $u_a(t,x)=u(at,x)$, we get

$$\begin{aligned} ||u(t)-u(s)||_{\gamma-\beta,p} &= ||u_a(t/a)-u_a(s/a)||_{\gamma-\beta,p} \\ &\leq N|t/a-s/a|^{\beta/2-1/q}\big(||u_a||_{\mathbb{H}^{\gamma,q}_p(T)} + ||u_{at}||_{\mathbb{H}^{\gamma-2,q}_p(T)}\big) \\ &= N|t-s|^{\beta/2-1/q}a^{1/q-\beta/2}\big(||u(a\,\cdot,\cdot)||_{\mathbb{H}^{\gamma,q}_p(T)} + a||u_t(a\,\cdot,\cdot)||_{\mathbb{H}^{\gamma-2,q}_p(T)}\big) \\ &= N|t-s|^{\beta/2-1/q}a^{1/q-\beta/2}\big(a^{-1/q}||u||_{\mathbb{H}^{\gamma,q}_p(T)} + a^{1-1/q}||u_t||_{\mathbb{H}^{\gamma-2,q}_p(T)}\big), \end{aligned}$$

which is equivalent to (7.2) with $a^{-1/2}$ in place of a. This shows that one only needs prove (7.2) with $a=1$.

Next, if (7.2) is true with $a=1$ and additional assumption that $|t-s|\leq 1$, then in order to prove (7.2) for $T \geq t \geq s+1 \geq s \geq 0$, it suffices to observe that

$$\begin{aligned} ||u(t)-u(s)||_{\gamma-\beta,p} \leq \int_{t-1}^{t} ||u(t)-u(t_1)||_{\gamma-\beta,p}\,dt_1 + \int_{t-1}^{t} ||u(t_1)||_{\gamma-\beta,p}\,dt_1 \\ + \int_{s}^{s+1} ||u(s_1)||_{\gamma-\beta,p}\,ds_1 + \int_{s}^{s+1} ||u(s)-u(s_1)||_{\gamma-\beta,p}\,ds_1, \end{aligned}$$

where, for instance,

$$\begin{aligned} \int_{t-1}^{t} ||u(t_1)||_{\gamma-\beta,p}\,dt_1 &\leq N\int_{t-1}^{t} ||u(t_1)||_{\gamma,p}\,dt_1 \\ &\leq \big(\int_{t-1}^{t} ||u(t_1)||^q_{\gamma,p}\,dt_1\big)^{1/q} \leq ||u||_{\mathbb{H}^{\gamma,q}_p(T)}. \end{aligned}$$

Finally, for $|t-s|\leq 1$ one can always shift the origin of t-axis and assume that $T=1$. In that case one gets (7.2) either by repeating the proof Theorem 7.2 of [8] (which is done in [8] for $q=p$) observing that in the deterministic case the range of parameters automatically extends (cf. Remark 5.3) or from much deeper results from [20] or from Chapter 5 of [17] bearing on sharp trace theorems describing the traces in terms of the Besov spaces. Thereby we end our proof of the theorem.

Remark 7.4. There are several places in the above proof that we do not know how to carry over to spaces $\mathcal{H}^{\gamma,q}_p(\tau)$. In Section 6 we have seen that, even if we are only interested in proving Corollary 4.3 for $T=1$, we need to have a right dependence of our estimates on the parameter a. The above proof shows that this is equivalent to obtaining the estimates for all T and $a=1$ with correct dependence of constants on T. The hardest point now is that while estimating seminorms in C^α with respect to t we have sup's inside the expectation sign and the reduction to the case $|t-s|\leq 1$ does not make the problem easier because still t can be any point on $[0,\tau]$ probably depending on ω.

An approach how to avoid these difficulties can be found in [15], where for $q=p$ functions with compact support in x^1 are concidered and some extra terms on the right appear.

Next, we state an analogue of Corollary 4.12 which now follows immediately from Theorem 7.3 by embedding theorems (we use $H_p^{\gamma-\beta} \subset H_q^{\gamma-\beta-d/p+d/q}$ if $q \geq p$).

Corollary 7.5. *Let $\gamma \in \mathbb{R}$ $q \geq p$, $T \in (0,\infty)$. Assume $2/q + (1/p-1/q)d < \beta \leq 2+(1/p-1/q)d$ and define $\hat{\beta} = \beta-(1/p-1/q)d$. Then for any function $u \in H_p^{\gamma,q}(\tau)$ and for any constant $a > 0$ and $0 \leq s < t \leq T$, we have*

$$||u(t) - u(s)||_{\gamma-\beta,q} \leq N|t-s|^{\hat{\beta}/2-1/q}a^{\hat{\beta}-1}\big(a||u||_{\mathbb{H}_p^{\gamma,q}(T)} + a^{-1}||u_t||_{\mathbb{H}_p^{\gamma-2,q}(T)}\big),$$

where N is independent of T, s, t, a, and u.

Now to finish proving Corollary 7.2 , it suffices to repeat the very short computations given in Section 6.

Theorem 7.3 has the following corollary which will be used in a subsequent paper of V. Bogachev, M. Röckner, and the author. The proof of this corollary is almost identical to that of Theorem 4.9 and is left to the reader.

Corollary 7.6. *Let $\gamma \in \mathbb{R}$, $T \in (0,\infty)$, $2/q < \beta \leq 2$, $r \geq p$, $\kappa \in [0,1]$, and*

$$\mu \leq \gamma + \frac{d}{r} - \frac{d}{p} - (1-\kappa)\beta.$$

Then, for any $u \in H_p^{\gamma,q}(T)$ and $a > 0$, we have

$$\int_0^T ||u(t)||_{\mu,r}^{q/\kappa}\,dt \leq N\big[||u(0)||_{\gamma-\beta,p}$$

$$+T^{\beta/2-1/q}a^{\beta-1}(a||u||_{\mathbb{H}_p^{\gamma,q}(T)} + a^{-1}||u_t||_{\mathbb{H}_p^{\gamma-2,q}(T)})\big]^{q/\kappa-q}||u||^q_{\mathbb{H}_p^{\gamma,q}(T)},$$

where N is independent of u, T, and a.

8. Appendix: Some properties of the operator $\tilde{T}_t$

We start with an auxiliary result.

Lemma 8.1. *Let $g(x,y)$ be a function on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$ satisfying*

$$|g(x,y)| \leq (1+y^1)^{1-\alpha}(1+x^1)^{\beta-1}e^{-\varepsilon|x-y|^2}, \tag{8.1}$$

where $\varepsilon > 0$, $\alpha,\beta \in \mathbb{R}$ are some constants. Let $1 < p \leq q < \infty$, $\theta \in \mathbb{R}$, and $\delta := (\theta-d)q/p - q(\beta-\alpha)$ Assume

$$d - p/q + p(\beta-\alpha) < \theta < d-1+p$$

(so that automatically $\delta > -1$ and $\alpha-\beta > 1/p-1/q-1$) Then, for any $u \in L_{p,\theta}$,

$$\int_{\mathbb{R}^d_+}(x^1)^\delta\Big|\int_{\mathbb{R}^d_+} g(x,y)u(y)\,dy\Big|^q dx \leq N||u||^q_{0,p,\theta}. \tag{8.2}$$

where N is independent of u.

Proof. By using Hölder's inequality we see that

$$\Big|\int_{\mathbb{R}^d_+} g(x,y)u(y)\,dy\Big| \leq (1+x^1)^{\beta-1}I_1^{1/p}(x)I_2^{(p-1)/p}(x),$$

where

$$I_1(x) = \int_{\mathbb{R}^d_+} |y^1|^{\theta-d}|u(y)|^p e^{-\varepsilon|x-y|^2}\,dy$$

$$\leq \int_{\mathbb{R}^{d-1}} \Big(\int_0^\infty |y^1|^{\theta-d} |u(y)|^p \, dy^1 \Big) e^{-\varepsilon |x'-y'|^2} \, dy' = J(x'),$$

$$I_2(x) = I_2(x^1) = \int_{\mathbb{R}^d_+} |y^1|^{(d-\theta)/(p-1)} (1+y^1)^{(1-\alpha)p/(p-1)} e^{-\varepsilon|x-y|^2} \, dy$$

$$= N \int_0^\infty |y^1|^{(d-\theta)/(p-1)} (1+y^1)^{(1-\alpha)p/(p-1)} e^{-\varepsilon|x^1-y^1|^2} \, dy^1,$$

where the last integral converges due to $(d-\theta)/(p-1) > -1$. Also, as easy to see, this integral behaves like $(x^1)^{(d-\theta)/(p-1)+(1-\alpha)p/(p-1)}$ when $x^1 \to \infty$. Therefore,

$$(1+x^1)^{\beta-1} I_2^{(p-1)/p}(x) \leq N(1+x^1)^{\beta-\alpha+(d-\theta)/p},$$

$$(x^1)^\delta \Big| \int_{\mathbb{R}^d_+} g(x,y) u(y) \, dy \Big|^q \leq N(x^1 \wedge 1)^\delta I_1^{q/p}(x).$$

Thus the left-hand side of (8.2) is less than

$$\int_{\mathbb{R}^{d-1}} J^{q/p}(x') \, dx' + \int_{\mathbb{R}^d} I_1^{q/p}(x) \, dx.$$

To estimate these integrals we notice that I_1 and J are convolutions and the $L_{q/p}$-norm of a convolution is less than the L_1-norm of one function times the $L_{q/p}$-norm of the other. Then we obviously come to (8.2). The lemma is proved.

Remember that $\tilde{T}_t$ is the semigroup associated with the operator Δ in $\mathbb{R}^d_+$ with zero boundary condition on $\{x^1 = 0\}$. Define

$$S_t = M^{-1} \tilde{T}_t M^{-1}.$$

Lemma 8.2. *Let $1 < p \leq q < \infty$, $\alpha, \theta \in \mathbb{R}$, $\sigma := d + (\theta - d)q/p + q\alpha$. Assume $u \in \mathcal{D}(\mathbb{R}^d_+)$,*

$$d - p/q - p\alpha < \theta < d - 1 + p.$$

(i) Let $1/p - 1/q - 1 < \alpha \leq 2$. Then

$$||S_1 u||_{0,q,\sigma} \leq N ||u||_{0,p,\theta},$$

where N is independent of u.

(ii) Let $1/p - 1/q - 1 < \alpha \leq 0$. Then

$$||S_1 u||_{2,q,\sigma} \leq N ||u||_{0,p,\theta}, \quad ||S_1 u||_{0,q,\sigma} \leq N ||u||_{-2,p,\theta} \tag{8.3}$$

where N is independent of u.

Proof. (i) Define

$$p(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/(4t)}, \quad Ax = (-x^1, x^2, ..., x^d),$$

$$p(t,x,y) = p(t, x-y) - p(t, x - Ay), \quad g(x,y) = \frac{1}{x^1 y^1} p(1,x,y).$$

It is well known that

$$\tilde{T}_t u(x) = \int_{\mathbb{R}^d_+} p(t,x,y) u(y) \, dy.$$

Therefore,

$$S_1 u(x) = \int_{\mathbb{R}^d_+} g(x,y) u(y) \, dy.$$

and assertion (i) will follow from Lemma 8.1 if we show that assumption (8.1) is satisfied, perhaps with a multiplicative constant on the right, for $\alpha \leq 2$ and $\beta = 0$. Obviously, we only need to check (8.1) for $\alpha = 2$ and $\beta = 0$.

We are going to use some properties of p and q, one of which is their symmetry with respect to x, y. Also obviously, $p(1, x, y) = 0$ if $x^1 = 0$. Therefore, for $x, y \in \mathbb{R}^d$,

$$g(x,y) = \frac{1}{y^1} \int_0^1 p_{x^1}(1, rx^1, x', y)\, dr. \tag{8.4}$$

In addition, $p(1, x, y) = 0$ if $y^1 = 0$. Therefore, $p_{x^1}(1, x, y) = 0$ if $y^1 = 0$ and, for any $x, y \in \mathbb{R}^d$,

$$g(x,y) = \int_0^1 \int_0^1 p_{x^1 y^1}(1, rx^1, x', sy^1, y')\, drds. \tag{8.5}$$

It follows that $q(x, y)$ is an infinitely differentiable function in $\mathbb{R}^d \times \mathbb{R}^d$. Furthermore, from (8.5) and estimates of the type

$$|p(1,x,y)| + |p_{y^i}(1,x,y)| + |p_{x^i y^j}(1,x,y)| \leq N e^{-|x-y|^2/8}, \quad x, y \in \mathbb{R}^d_+ \tag{8.6}$$

we get that

$$|g(x,y)| \leq N e^{-|x-y|^2/8} \quad 0 \leq x^1 \leq 1, 0 \leq y^1 \leq 1. \tag{8.7}$$

For $y^1 \geq 1$ and $0 \leq x^1 \leq 1$ it follows from (8.4) that

$$|g(x,y)| \leq N \frac{1}{y^1} e^{-((y^1)^2 + |x'-y'|^2)/8} \leq N \frac{1}{(1+y^1)(1+x^1)} e^{-|x-y|^2/16},$$

and, by (8.7) the inequality between the extreme terms actually holds for all $y^1 > 0$ if $0 \leq x^1 \leq 1$. The symmetry of $g(x, y)$ shows that this inequality is also true if $\min(x^1, y^1) \leq 1$. The fact that this inequality holds for $\min(x^1, y^1) \geq 1$ follows immediately from the definition of g. This proves (i).

(ii) It follows by integrating by parts that, if $u \in C_0^\infty(\mathbb{R}^d_+)$, then

$$S_1 \mathcal{L}_{2,c} u(x) = \int_{\mathbb{R}^d_+} \bar{g}(x,y) u(y)\, dy,$$

where

$$\bar{g}(x,y) = \frac{y^1}{x^1} \Delta_y p(1,x,y) - cg(x,y) =: g_2(x,y) - cg(x,y).$$

For $\alpha \leq 0$ we have

$$|g_2(x,y)| \leq (1+y^1)^{1-\alpha} (x^1)^{-1} \Delta_y p(1,x,y).$$

If $x^1 \geq 1$, then (8.6) shows that g_2 satisfies an estimate of type (8.1) with $\beta = 0$. The same is true for $0 < x^1 \leq 1$ as well, as can be easily seen from the fact that

$$(x^1)^{-1} \Delta_y p(1,x,y) = \int_0^1 \Delta_y p_{x^1}(1, rx^1, x', y)\, dr.$$

From Lemma 8.1 and from assertion (i) we conclude that

$$||S_1 \mathcal{L}_{2,c} u||_{0,q,\sigma} \leq N ||u||_{0,p,\theta},$$

$$||S_1 v||_{0,q,\sigma} \leq N ||(\mathcal{L}_{2,c})^{-1} v||_{0,p,\theta} \leq N ||v||_{-2,p,\theta}$$

if $u, (\mathcal{L}_{2,c})^{-1} v \in C_0^\infty(\mathbb{R}^d_+)$ and c is sufficiently large. Since $C_0^\infty(\mathbb{R}^d_+)$ is everywhere dense in $H^\gamma_{p,\theta}$, we have thus proved the second inequality in (8.3). One can prove the first one again by using Lemma 8.1 this time taking there $\alpha = 0$ and $\beta = 2$.

There is also a different proof based on duality (Theorem 2.3). From the second inequality in (8.3) by duality we get

$$||S_1 u||_{2,p',\theta'} \leq N||u||_{0,q',\sigma'}, \tag{8.8}$$

where

$$1/p + 1/p' = 1, \quad \theta/p + \theta'/p' = d, \quad 1/q + 1/q' = 1, \quad \sigma/q + \sigma'/q' = d. \tag{8.9}$$

To finish proving the first inequality in (8.3) we only have to show that p', q' can be taken arbitrarily in $(1,\infty)$, with $p' \geq q'$, that σ' can be taken arbitrarily in $(d - q'/p' - q'\alpha, d - 1 + q')$ and that

$$\theta' = d + (\sigma' - d)p'/q' + p'\alpha. \tag{8.10}$$

To do this take any p', q' in $(1,\infty)$, with $p' \geq q'$ and $\sigma' \in (d - q'/p' - q'\alpha, d - 1 + q')$. Define θ' by (8.10) and find p, q, θ, and σ from (8.9). Then simple calculations show that

$$p \leq q, \quad \sigma = d + (\theta - d)q/p + q\alpha, \quad d - p/q - p\alpha < \theta < d - 1 + p,$$

so that (8.8) holds. The lemma is proved.

The following theorem is used in several places in this article and has an obvious application to proving interior in time smoothness of solutions to the heat equation.

Theorem 8.3. *Let $1 < p \leq q < \infty$, $\alpha, \theta \in \mathbb{R}$,*

$$-1 + 1/p - 1/q < \alpha \leq 2, \quad d - p/q - p\alpha < \theta < d - 1 + p.$$

Then, for $\alpha_+ = \max(0, \alpha)$, $v \in \mathcal{D}(\mathbb{R}^d_+)$,

$$\hat{\alpha} = \alpha - (1/p - 1/q)d, \quad (\hat{\theta} - d)/q = (\theta - d)/p,$$

we have

$$||M^{\alpha-1}\tilde{T}_t v||_{0,q,\hat{\theta}} \leq N t^{\hat{\alpha}/2-1}||Mv||_{\alpha_+-2,p,\theta}, \tag{8.11}$$

$$||M^{\alpha-1}\tilde{T}_t v||_{2-\alpha_+,q,\hat{\theta}} \leq N t^{\hat{\alpha}/2-1}||Mv||_{0,p,\theta}, \tag{8.12}$$

where N is independent of v and t.

Proof. Denote $u = Mv$, $\sigma = d + (\theta - d)q/p + q\alpha$. Then as easy to see (8.11) and (8.12) become

$$||S_t u||_{0,q,\sigma} \leq N t^{\hat{\alpha}/2-1}||u||_{\alpha_+-2,p,\theta}, \tag{8.13}$$

$$||S_t u||_{2-\alpha_+,q,\sigma} \leq N t^{\hat{\alpha}/2-1}||u||_{0,p,\theta}, \tag{8.14}$$

respectively. Now, remember that if $u(t,x)$ is a solution of the heat equation, then for any constant $c > 0$ the function $u(c^2 t, cx)$ is also a solution of the same equation. It follows that $\tilde{T}_t u(c\,\cdot)(x) = \tilde{T}_{c^2 t} u(cx)$,

$$S_t[u(c\,\cdot)](x) = cM^{-1}\tilde{T}_t[(M^{-1}u)(c\,\cdot)](x)$$
$$= cM^{-1}[(\tilde{T}_{c^2 t}M^{-1}u)(c\,\cdot)](x) = c^2[M^{-1}\tilde{T}_{c^2 t}M^{-1}u](cx),$$
$$S_t u(x) = c^2 S_{c^2 t}[u(c^{-1}\cdot)](cx).$$

By taking $c = t^{-1/2}$ and adding to this the homogeneity property of $H^\gamma_{p,\theta}$-norms and denoting $v(x) = u(t^{1/2}x)$, we get that, if (8.13) is true for $t = 1$, then

$$||S_t u||_{0,q,\sigma} = t^{-1}||[S_1 v](t^{-1/2}\cdot)||_{0,q,\sigma} \leq N t^{\sigma/(2q)-1}||S_1 v||_{0,q,\sigma}$$

$$\leq Nt^{\sigma/(2q)-1}||u(t^{1/2}\cdot)||_{\alpha_+-2,p,\theta} = Nt^{\sigma/(2q)-1-\theta/(2p)}||u||_{\alpha_+-2,p,\theta}.$$

Similar relations hold for (8.14). Hence we may and will only concentrate on the case $t=1$.

Next we notice that (8.13) and (8.14) are dual to each other, so that (8.13) can be derived from (8.14) as in the end of the proof of Lemma 8.2. This lemma also shows that (8.14) holds for $\alpha \leq 0$, so that in the remaining part of the proof we assume $\alpha \geq 0$. Finally, by Lemma 8.2

$$||S_1u||_{0,q,\sigma_0} \leq N||u||_{0,p,\theta_0}, \quad ||S_1u||_{2,q,\sigma_1} \leq N||u||_{0,p,\theta_1},$$

whenever

$$\sigma_0 = d + (\theta_0 - d)q/p + 2q, \quad d - p/q - 2p < \theta_0 < d - 1 + p,$$
$$\sigma_1 = d + (\theta_1 - d)q/p, \quad d - p/q < \theta_1 < d - 1 + p.$$

It follows by interpolation (Theorem 2.4) that (8.14) holds if we can find $\kappa \in [0,1]$ and θ_i as above so that

$$\theta = \kappa\theta_1 + (1-\kappa)\theta_0, \quad 2 - \alpha = \kappa 2, \quad \sigma = \kappa\sigma_1 + (1-\kappa)\sigma_0. \tag{8.15}$$

Of course, we take $\kappa = 1 - \alpha/2$. With this choice, the last equation in (8.15) follows from the first one. It only remains to notice that, $\kappa\theta_1 + (1-\kappa)\theta_0$ spans the interval $(d - p/q - p\alpha, d - 1 + p)$ as θ_1 and θ_0 run through $(d - p/q, d - 1 + p)$ and $(d - p/q - 2p, d - 1 + p)$ respectively. The theorem is proved.

Remark 8.4. We use this theorem for complex and Hilbert space valued functions v.

Corollary 8.5. *Under the assumptions of Theorem 8.3 let $\hat{\alpha} > 0$ and $M^{-1}u \in H^2_{p,\theta}$. Then*

$$||M^{\alpha-1}(\tilde{T}_t - 1)u||_{0,q,\hat{\theta}} \leq Nt^{\hat{\alpha}/2}||M\Delta u||_{\alpha-2,p,\theta},$$
$$||M^{\alpha-1}(\tilde{T}_t - 1)u||_{2-\alpha,q,\hat{\theta}} \leq Nt^{\hat{\alpha}/2}||M\Delta u||_{0,p,\theta}.$$

Indeed, if $u \in C_0^\infty(\mathbb{R}^d_+)$,

$$(\tilde{T}_t - 1)u = \int_0^t \tilde{T}_r \Delta u\, dr.$$

Hence, for instance,

$$||M^{\alpha-1}(\tilde{T}_t - 1)u||_{2-\alpha,q,\hat{\theta}} \leq \int_0^t ||M^{\alpha-1}\tilde{T}_r\Delta u||_{2-\alpha,q,\hat{\theta}}\, dr$$
$$\leq N\int_0^t r^{\hat{\alpha}/2-1}\, dr||M\Delta u||_{0,p,\theta} \leq Nt^{\hat{\alpha}/2}||M\Delta u||_{0,p,\theta}.$$

Similarly we get the other inequality for $u \in C_0^\infty(\mathbb{R}^d_+)$. Since $C_0^\infty(\mathbb{R}^d_+)$ is dense in the spaces $H^\gamma_{p,\theta}$ and $||M\Delta u||_{\gamma,p,\theta} \leq N||M^{-1}u||_{\gamma+2,p,\theta}$, our assertion follows.

The following theorem is discussed in Remarks 3.4 and 4.2. Although we believe that its statement is not far from being sharp and it catches the most important cases for applications, it is worth noting that probably one can get sharper results in terms of the Besov spaces on the basis of methods in [20] and [4].

Theorem 8.6. *Let $\gamma, \theta \in \mathbb{R}$, $p, q \in (1,\infty)$, $\delta \in (0, 2/q)$, $T \in (0,\infty)$. Then there exists a bounded operator $P : M^{1-2/q+\delta}H^{\gamma-2/q+\delta}_{p,\theta} \to H^{\gamma,q}_{p,\theta}(T)$ such that $Pu|_{t=0} = u$. If in addition, $q = p$, then the assertion holds with $\delta = 0$ and $T = \infty$ as well.*

Proof. First let $\gamma = 2/q - \delta$, $\theta = d$. Define $\alpha = 2 - 2/q + \delta$ and let $P_0 u(t) = \tilde{T}_t u$. By observing that $\alpha \in (0, 2)$ and $d - 1 - p\alpha < d - p\alpha < d - 1 + p$ and applying (8.12) with $p = q$, we get

$$||M^{-1} P_0 u(t)||_{\gamma,p,d} = ||M^{-1} \tilde{T}_t u||_{2-\alpha,p,d} \le N ||M^{\alpha-1} \tilde{T}_t u||_{2-\alpha,p,d-p\alpha}$$

$$\le N t^{-1/q+\delta/2} ||Mu||_{0,p,d-p\alpha} = N t^{-1/q+\delta/2} ||M^{2/q-1-\delta} u||_{0,p,d}.$$

Hence

$$\int_0^T ||M^{-1} P_0 u(t)||^q_{\gamma,p,d} \, dt \le N ||M^{2/q-1-\delta} u||_{0,p,d}.$$

In addition, $(P_0 u(t))_t = \Delta \tilde{T}_t u$ and

$$||M(P_0 u(t))_t||_{\gamma-2,p,d} = ||M \Delta \tilde{T}_t u||_{\gamma-2,p,d} \le N ||M^2 \Delta \tilde{T}_t u||_{\gamma-2,p,d-p}$$

$$\le N ||\tilde{T}_t u||_{\gamma,p,d-p} \le N ||M^{-1} \tilde{T}_t u||_{\gamma,p,d} \le N t^{-1/q+\delta/2} ||M^{2/q-1-\delta} u||_{0,p,d},$$

so that $M(P_0 u(t))_t$ also belongs to $\mathbb{H}^{\gamma,q}_{p,\theta}(T)$. This proves the first assertion of the theorem if $\gamma = 2/q - \delta$ and $\theta = d$. Under these conditions the second assertion of the theorem is actually a very particular case of Theorem 5.6 of [9].

To construct the operator P for general γ and θ, it suffices to take a large constant c, let

$$P = M^{-\theta/p} (-\mathcal{L}_{0,c})^{-\gamma/2+1/q-\delta/2} P_0 (-\mathcal{L}_{0,c})^{\gamma/2-1/q+\delta/2} M^{\theta/p},$$

and use Lemma 2.2 and Theorems 2.5 and 3.9. The theorem is proved.

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