On simple mixed modules over the Virasoro algebra

Volodymyr Mazorchuk*

Abstract

We describe the support of a simple weight module over the Virasoro Lie algebra and classified all those modules, whose support is one element less than the weight lattice. We also prove more general result that there do not exist simple weight modules having both finite and infinite dimensional weight spaces under the condition that the kernels of e_1 and e_{-1} have finite supports.

1 Introduction and preliminaries

Let $\mathcal{V}$ denote the Virasoro algebra over the complex field $\mathbb{C}$. Recall that $\mathcal{V}$ is generated by a central element c and elements e_i, $i \in \mathbb{Z}$ and has the following Lie bracket:

$$[e_i, e_j] = (j-i)e_{i+j} + \delta_{i,-j}\frac{i^3-i}{12}c.$$

The elements c and e_0 span the Cartan subalgebra $\mathfrak{H}$ of $\mathcal{V}$ and any $\mathfrak{H}$-diagonalizable $\mathcal{V}$-module is usually called a *weight module*. If, additionally, all weight spaces of a weight $\mathcal{V}$-module are finite-dimensional, this module is called a *Harish-Chandra module* ([M]). All simple Harish-Chandra modules have been already classified ([MP1, MP2, M]) and are exhausted by simple highest weight modules, simple lowest weight modules and simple modules from the *intermediate series*. The last one contains all subquotients of modules $V(a,b)$, $a,b \in \mathbb{C}$ defined as follows: $V(a,b)$ has a basis v_i, $i \in \mathbb{Z}$ in which $e_i \cdot v_j = (ai+b+j)v_{i+j}$, $c \cdot v_j = 0$.

Using this classification one can easily answer the following question: what does the support of a simple Harish-Chandra $\mathcal{V}$-module look like? Let $\Delta \in \mathfrak{H}^*$ denote the root system of $\mathcal{V}$ and let α be the root corresponding to e_1. It happens that for a simple Harish-Chandra $\mathcal{V}$-module V only one of the five possibilities, listed below, can occur.

1. $\operatorname{Supp} V = \{0\}$. In this case V is the trivial $\mathcal{V}$-module. It is a simple highest weight module, a simple lowes weight module and a simple module from the intermediate series simultaneously.

*Alexander von Humboldt fellow at Bielefeld University

2. There exists $\lambda \in \mathfrak{H}^*$ such that $\operatorname{Supp} V = \{\lambda - k\alpha, k \in \mathbb{Z}_+\}$ (to be complete, in this case $\lambda \neq 0$ and all other λ really occur). Such V is the (unique) simple highest weight module with the highest weight λ.

3. There exists $\lambda \in \mathfrak{H}^*$ such that $\operatorname{Supp} V = \{\lambda + k\alpha, k \in \mathbb{Z}_+\}$ ($\lambda \neq 0$ and all other λ really occur). Such V is the (unique) simple lowest weight module with the lowest weight λ.

4. There exists $\lambda \in \mathfrak{H}^*$ such that $\operatorname{Supp} V = \{\lambda + k\alpha, k \in \mathbb{Z}\}$. All λ really occur and in this case V is a simple module from the intermediate series.

5. There exists $\lambda \in \mathfrak{H}^*$ and $m \in \mathbb{Z}$ such that $\operatorname{Supp} V = \{\lambda + k\alpha, k \in \mathbb{Z} \setminus \{m\}\}$. In this case $\lambda \in \mathbb{Z}\alpha$ and $\lambda + m\alpha = 0$. Such V is a simple module from the intermediate series.

Using the terminology, proposed in [Ma1], we call a weight $\mathcal{V}$-module *cut* if $\operatorname{Supp} V \subset \lambda \pm \mathbb{Z}_+\alpha$ for some λ, *dense* if $\operatorname{Supp} V = \lambda + \mathbb{Z}\alpha$ for some λ and *pinned* if $\operatorname{Supp} V = \lambda + \mathbb{Z}_{\neq 0}\alpha$ for some λ. Now, the above description of the support of a simple Harish-Chandra $\mathcal{V}$-module can be roughly collected in the following statement: Any simple Harish-Chandra $\mathcal{V}$-module is either cut or pinned or dense.

It is not difficult to find out that the requirement for V to be a Harish-Chnadra module (here we mean to have finite-dimensional weight spaces) is too restrictive for the last statement. In Section 2 we prove the analogous statement for any weight $\mathcal{V}$-module and this proof will be much easier than one for the analogous statement in the case of a higher rank Virasoro algebra (compare with the main result in [Ma1]).

As soon as we know this result, there arise three natural problems: classify all simple cut, all simple pinned and all simple dense modules. The first one is easy. Indeed, any simple cut module is necessarily a highest wieght module or a lowest weight module, thus is a Harish-Chandrs module. The third one seems to be very difficult and not much is known here. For instance, there are examples of simple dense $\mathcal{V}$-modules all weight spaces of which are infinite-dimensional ([Z]). Our motivation for this paper is to determine all simple pinned modules. In fact, we will prove that they are exhausted by the simple pinned modules from the intermediate series.

The above result can be considered as the aim of this paper, however the arguments we present work in a much more general situation, which is closely related to the old conjecture stating that any simple pointed $\mathcal{V}$-module (i.e. a weight module having a one-dimensional weight space) is a Harish-Chandra module (see [X]). A counterexample to this conjecture (if any) as good as a new simple pinned module (if any) should belong to the special class of $\mathcal{V}$-module, which we call *mixed*. If V is a simple weight $\mathcal{V}$-module, one has that for some $\lambda \in \mathfrak{H}^*$ holds $\operatorname{Supp} V \subset \lambda + \mathbb{Z}\alpha$. We call such V a *mixed* module provided there exist $k, m \in \mathbb{Z}$ such that $\dim V_{\lambda+k\alpha} = \infty$ and $\dim V_{\lambda+m\alpha} < \infty$, or, in other words, $\operatorname{Supp} V$ is a mixture of infinite-dimensional and finite-dimensional weight spaces. We note, that $\dim V_{\lambda+m\alpha}$ in the definition above may be 0 and thus formally $\lambda + m\alpha \notin \operatorname{Supp} V$. Hence, the definition above is not equivalent to the requirement that V has both finite-dimensional and infinite-dimensional weight spaces.

The main result of this paper is the following Theorem.

Theorem 1. *There do not exist simple mixed $\mathcal{V}$-modules in which the kernels of e_1 and e_{-1} have finite supports.*

Two immediate consequences of Theorem 1 are the following.

Corollary 1. *Any simple pinned $\mathcal{V}$-module is a Harish-Chandra module and hence is a module from the intermediate series.*

Corollary 1 completely solves the second problem stated above: it completes the classification of all simple pinned $\mathcal{V}$-modules. Moreover, it also reduces the problem of classification of simple weight $\mathcal{V}$-modules to that of classification of simple dense $\mathcal{V}$-modules.

Corollary 2. *Any simple pointed $\mathcal{V}$-module in which the kernels of e_1 and e_{-1} have finite supports is a Harish-Chandra module.*

Corollary 2 gives a partial result towards the proof of the conjecture, stated in [X].

2 The support of a weight $\mathcal{V}$-module

We start with a description of the support of a weight $\mathcal{V}$-module. This result is much easier than the corresponding result for some higher rank Virasoro algebras, obtained in [Ma1, Ma2], but we present it since we have never seen it published.

Theorem 2. *Any simple weight $\mathcal{V}$-module is either cut or pinned or dense.*

Proof. Let V be a simple weight $\mathcal{V}$-module. Clearly, there exists $\lambda \in \mathfrak{H}^*$ such that $\operatorname{Supp} V \subset \lambda + \mathbb{Z}\alpha$. Assume that V is neither dense nor pinned. We have to show that in this case V is a highest or lowest weight module (and hence, cut). As V is neither dense nor pinned there exist at least two $k < l \in \mathbb{Z}$ such that $\lambda + k\alpha, \lambda + l\alpha \notin \operatorname{Supp} V$. Up to the standard involution (which interchange highest and lowest weight modules) we can assume that $\operatorname{Supp} V$ contains $\lambda + m\alpha$ for some $m < k$. Consider the following cases.

Case 1. If we can choose $l = k + 1$, then any non-zero element in $V_{\lambda+m\alpha}$, where m is maximal such that $m < k$ and $\lambda + m\alpha \in \operatorname{Supp} V$, is annihilated by e_1 and e_2 and thus generates a highest weight submodule of V. This means that V is a highest weight module.

Case 2. Assume that we are outside Case 1 and we can choose $l = k + n$, $n > 2$. Then $\lambda + (k+1)\alpha \in \operatorname{Supp} V$ since we ore outside Case 1 and for any $v \in V_{\lambda+(k+1)\alpha}$ holds $e_{-1}v = 0$ and $e_{n-1}v = 0$. Commuting e_{-1} with e_{n-1} we obtain that $e_i v = 0$ for any $1 \leqslant i \leqslant n-1$ and hence $e_1 v = e_2 v = 0$ since $n > 2$. As in Case 1 we now have that V is a highest weight module.

Case 3. Assume that we are outside Case 1 and Case 2. This means that we can choose $l = k + 2$ and $\lambda + m\alpha \in \operatorname{Supp} V$ for any $m \in \mathbb{Z} \setminus \{k, l\}$. Consider a non-zero

$v \in V_{\lambda+(k-1)\alpha}$. We have $e_1 v = e_3 v = 0$ and hence $e_i v = 0$ for any $i \geqslant 3$. If $w = e_2 v = 0$, then v generates a highest weight submodule of V and hence V is a highest weight module. Assume that $w \neq 0$. For any $i \geqslant 3$ we still have $e_i w = e_i e_2 v = e_2 e_i v + (2-i)e_{2+i} v = 0$. Moreover, $e_{-1} w = 0$ and, as in Case 2, we obtain that $e_1 w = e_2 w = 0$ implying that V is a highest weight module. This completes the proof of the Theorem. □

3 Auxiliary Lemmas

We begin with the following Lemma which we will often use in our arguments.

Lemma 1. *Let V be a simple $\mathcal{V}$-module. Assume that there is $0 \neq v \in V$ and $n \in \mathbb{N}$ such that $e_l v = 0$ for all $l \geqslant n$. Then for any $w \in V$ there exists $n(w) \in \mathbb{N}$ such that $e_l w = 0$ for all $l \geqslant n(w)$.*

Proof. As V is simple, we have $V = U(\mathcal{V})v$. Choosing a monomial PBW-basis in $U(\mathcal{V})$ we see that it is sufficient to prove the statement for $w = e_{i_1} \dots e_{i_s} v$, hence, by induction, it is sufficient to prove the statement for $w = e_i v$, $i \in \mathbb{Z}$. In this case set $n(w)$ to be the maximum between n and $n - i$. For $l \geqslant n(w)$ we have $e_l w = e_l e_i v = e_i e_l v +\ (i-l)e_{i+l} v$. Now $e_l v = 0$ since $l \geqslant n(w) \geqslant n$ and $e_{i+l} v = 0$ since $i + l \geqslant i + n(w) \geqslant i + (n - i) = n$. We have $e_l w = 0$ completing the proof. □

Denote by $\mathcal{W}_+$ (resp. $\mathcal{W}_-$) the Lie subalgebra of $\mathcal{V}$ generated by e_{-1} and e_i, $i \in \mathbb{Z}_+$ (resp. e_1 and e_{-i}, $i \in \mathbb{Z}_+$). Denote also by $\mathfrak{A}$ the $sl(2,\mathbb{C})$-subalgebra of $\mathcal{V}$ generated by $e_{\pm 1}$. Throughout Sections 3-4, if nothing is mentioned, we assume that V is a simple mixed $\mathcal{V}$-module. The next Lemma can be viewed as a generalization of Theorem 2.

Lemma 2. *Let V be a simple mixed $\mathcal{V}$ module and* $\operatorname{Supp} V \subset \lambda + \mathbb{Z}\alpha$, $\lambda \in \mathfrak{H}^*$. *Then there exists exactly one $k \in \mathbb{Z}$ such that* $\dim V_{\lambda+k\alpha} < \infty$.

Proof. Assume that such k is not unique. Without loss of generality we can assume that $\dim V_\lambda = \infty$, $\dim V_{\lambda+\alpha} < \infty$ and $\dim V_{\lambda+k\alpha} < \infty$ for some $k \geqslant 2$. If we can choose $k = 2$, then as V_λ is infinite-dimensional there exists $0 \neq v \in V_\lambda$ such that $e_1 v = e_2 v = 0$ and V is a highest weight module, hence not mixed. Thus $\dim V_{\lambda+2\alpha} = \infty$. If we can choose $k > 3$, then $\dim V_{\lambda+2\alpha} = \infty$ implies that there exists $0 \neq v \in V_{\lambda+2\alpha}$ such that $e_{-1} v = e_{k-2} v = 0$, $k - 2 > 1$, which means that $e_1 v = e_2 v = 0$ and V is again highest weight (see proof of Theorem 2). So, we have that the only possibility for k is 3. Consider the infinite-dimensional subspace V_λ^0 of V_λ, which contains all elements annihilated by both e_1 and e_3 and hence by all e_i, $i \geqslant 3$. By Lemma 1, for any element $v \in V_{\lambda+2\alpha}$ there exists $n(v) \in \mathbb{N}$ such that $e_l v = 0$ for all $l \geqslant n(v)$. But $V_{\lambda+2\alpha}$ is infinite-dimensional whereas $V_{\lambda+\alpha}$ is finite-dimensional. Hence there exists $0 \neq v \in V_{\lambda+2\alpha}$ such that $e_{-1} v = 0$ and we obtain $e_i v = 0$ for any $i \in \mathbb{N}$. Hence V is a highest weight module. The contradictions obtained in all cases prove our Lemma. □

By Lemma 2, fixing a simple mixed $\mathcal{V}$-module V with $\operatorname{Supp} V \subset \lambda + \mathbb{Z}\alpha$ we can assume $\dim V_\lambda < \infty$. If $\dim V_\lambda = 0$, the module V is pinned and if $\dim V_\lambda > 0$, the module V is dense. Set $k = \dim V_\lambda$. Our firt step is to analize the structure of V as an $\mathfrak{A}$-module. Denote by $\mathfrak{p}$ the natural projection of $\mathfrak{H}^*$ onto $\langle e_0 \rangle^*$. Formally, one should distinguish the weights of $\mathcal{V}$ and $\mathfrak{A}$ which are $\mathfrak{H}^*$ and $\mathfrak{p}(\mathfrak{H}^*)$ correspondingly, but we usually will talk about elements from $\mathfrak{H}^*$ as $\mathfrak{A}$-weights having $\mathfrak{p}$ in mind.

Lemma 3. *Let $M(\lambda - \alpha)$ (resp. $\hat{M}(\lambda + \alpha)$) denote the Verma $\mathfrak{A}$-module with the highest weight $\mathfrak{p}(\lambda - \alpha)$ (resp. with the lowest weight $\mathfrak{p}(\lambda + \alpha)$). Then $M(\lambda - \alpha)$ (resp. $\hat{M}(\lambda + \alpha)$) is simple.*

Proof. As there is a standard involution on $\mathcal{V}$ sending e_i to e_{-i}, $i \in \mathbb{Z}$, it is sufficient to prove only one statement, say, for $M(\lambda - \alpha)$. Since the kernel W of e_1 acting on $V_{\lambda - \alpha}$ is infinite-dimensional, there is a natural $\mathfrak{A}$-homomorphism f from $T = \oplus_{i \in \mathbb{N}} M(\lambda - \alpha)^{(i)}$, where $M(\lambda - \alpha)^{(i)} \simeq M(\lambda - \alpha)$ for all i, in V, whose image contains W. Denote by S^1 the $\mathfrak{A}$-submodule of V generated by V_λ. As V_λ is finite-dimensional and $\mathfrak{A}$ is a simple finite-dimensional Lie algebra, all weight spaces of S^1 are also finite-dimensional. Assume that $M(\lambda - \alpha)$ is not simple and denote by K the kernel of f.

There are two possibilities. First one: for any weight μ of K the codimension of K_μ in T_μ is infinite (in particular, this contains the case $K = 0$). In this case there is an injective $\mathfrak{A}$-homomorphism f' from T to V (it is possible, that W is not contained in the image of f'). Further, $M(\lambda - \alpha)$ is not simple and hence there exists a weight, $\mu \neq \lambda$, of T such that $e_1(T_\mu) = 0$. As f' is injective, the kernel W_1 of e_1 acting on V_μ is infinite-dimensional. Let $\mu = \lambda - l\alpha$, $\lambda \in \mathbb{N}$. The image of e_l acting on W_1 is finite-dimensional, thus there exists an infinite-dimensional subspace W_2 of W_1 such that $e_l(W_2) = 0$. From $e_1 W_2 = e_l W_2 = 0$ it follows that $e_i W_2 = 0$ for any $j \geqslant l$. Using Lemma 1 we find an element $0 \neq v \in V_{\lambda + \alpha}$ such that $e_{-1} v = 0$ and $e_i v = 0$ for all i big enough. Hence V is a highest weight module. This completes the proof for the first possibility.

The second possibility is that there exists a weight, $\mu = \lambda - l\alpha$, of K such that the codimension of K_μ in T_μ is finite. Then both $f(T_\mu)$ and S^1_μ are finite-dimensional. Assume that l is minimal with this property and denote by W_1 the (infinite-dimensional) kernel of e_{-1} acting on $V_{\mu+\alpha}$. We again have that there exists an infinite-dimensional subspace W_2 of W_1 such that $e_{l-1} W_2 = 0$ and thus V is a trivial module if $l > 2$. It remains only to consider the case $l = 2$. We have $e_{-1} W_2 = e_1 W_2 = 0$. If there exists a non-zero element $v \in W_2$ such that $e_{-2} v = 0$, it generates a lowest weight submodule of V. Otherwise $W_3 = e_{-2} W_2$ is infinite-dimensional and thus contains an infinite-dimensional subspace W_4 such that $e_3 W_4 = 0$. Moreover for any $w \in W_4$ we have $w = e_{-2} w'$, $w' \in W_2$ and $e_1 w = e_1 e_{-2} w' = e_{-2} e_1 w' - 3 e_{-1} w' = 0$. Hence $e_1 W_4 = e_3 W_4 = 0$ and we complete the proof by the same arguments as for the first possibility. □

Corollary 3. *All highest weight (resp. lowest weight) Verma $\mathfrak{A}$-modules $M(\lambda - k\alpha)$ (resp. $\hat{M}(\lambda + k\alpha)$), $k \in \mathbb{N}$ are simple.*

Proof. This follows from the standard $sl(2, \mathbb{C})$-theory. □

Corollary 4. *If the homomorphism f described in the proof of Lemma 3 induces a bijection between $T_{\lambda-\alpha}$ and W then f is injective.*

Proof. As $M(\lambda-\alpha)$ is simple, any submodule of T has a non-trivial intersection with $T_{\lambda-\alpha}$. Hence f is injective since its restricition to $T_{\lambda-\alpha}$ is injective. □

Clearly, there is an analogue of Corollary 4 for the homomorphism $\hat{f}:\hat{T}\to\hat{W}$, where $\hat{T}\simeq\oplus_{i\in\mathbb{N}}\hat{M}(\lambda+\alpha)^{(i)}$ and $\hat{W}$ is the kernel of e_{-1} acting on $V_{\lambda+\alpha}$. So, we can also assume that $\hat{f}$ is injective.

Lemma 4. *The quotient V^1 of V over $f(T)+\hat{f}(\hat{T})$ is an $\mathfrak{A}$-module with finite-dimensional weight spaces.*

Proof. Suppose that the statement is false. Certainly V^1_λ and $V^1_{\lambda\pm\alpha}$ are finite dimensional. Without loss of generality we can assume that there is $k\in\mathbb{N}$, $k>1$ such that $V_{\lambda-k\alpha}$ is infinite-dimensional. Choose k minimal possible. Then we can find an infinite-dimensional subspace W of $V^1_{\lambda-k\alpha}$ such that $e_1W=0$. Since e_1 acts bijectively between $f(T)_{\lambda-k\alpha}$ and $f(T)_{\lambda-(k-1)\alpha}$, we can lift W to an infinite-dimensional subset W' of $V_{\lambda-k\alpha}$ such that $e_1W'=0$. Now we can find an infinite-dimensional subset W_1 of W' such that $e_kW_1=0$ and complete the proof by the same arguments as in Lemma 3. □

Proposition 1. *As an $\mathfrak{A}$-module, V can be decomposed as $V\simeq V_-\oplus V_0\oplus V_+$ such that the dimensions of the weight spaces of V_0 are finite and uniformly bounded, V_+ is a direct sum of simple highest weight Verma modules and V_- is a direct sum of simple lowest weight Verma modules.*

Proof. This follows from the standard $sl(2,\mathbb{C})$-folklore. To be complete we present the proof in Section 5. □

Since $V_\pm$ is a direct sum of simple Verma modules, we can further decompose it $V_\pm\simeq V^0_\pm\oplus V^1_\pm$, where V^0_- is a direct sum of copies of $M(\lambda-\alpha)$, V^0_+ is a direct sum of copies of $\hat{M}(\lambda+\alpha)$ and $(V^1_-)_{\lambda-\alpha}=(V^1_+)_{\lambda+\alpha}=0$. By Lemma 4, $V^1_\pm$ has finite-dimensional weight spaces.

Now we can sum up our knowledge about a simple mixed $\mathcal{V}$-module V. First of all, $\dim V_\lambda<\infty$ and $\dim V_{\lambda+n\alpha}=\infty$ for all $n\in\mathbb{Z}\setminus\{0\}$. Further, as an $\mathfrak{A}$-module V can be decomposed into a direct sum of five modules V_0, $V^0_\pm$ and $V^1_\pm$ such that V^0_- is a direct sum of copies of the simple highest weight Verma $\mathfrak{A}$-module $M(\lambda-\alpha)$, V^0_+ is a direct sum of copies of the simple lowest weight Verma $\mathfrak{A}$-module $\hat{M}(\lambda+\alpha)$, V^1_- is a direct sum of copies of simple highest weight Verma $\mathfrak{A}$-modules $M(\lambda-n\alpha)$, $n\geqslant 2$ and has finite-dimensional weight spaces, V^1_+ is a direct sum of copies of simple lowest weight Verma $\mathfrak{A}$-modules $M(\lambda+n\alpha)$, $n\geqslant 2$ and has finite-dimensional weight spaces, and, finally, V_0 is a *bounded module*, i.e. the dimensions of its weight spaces are uniformly bounded. We also note that all the modules $V^{0,1}_\pm$ are completely reducible, i.e. each submodule in any of these modules has a direct complement. Our next step will be to show that without loss of generality under our conditions on the kernels of e_1 and e_{-1} we can assume $V^1_\pm=0$, or, equivalently, $V^1_\pm$ are bounded.

Lemma 5. *Let V be a simple mixed module. Assume that the kernels of e_1 and e_{-1} on V have finite supports. Then modules $V_{\pm}^1$ are bounded, in particular, we can assume $V_{\pm}^1 = 0$.*

Proof. As the kernels of e_1 and e_{-1} on V have finite supports the modules $V_{\pm}^1$ have finite length and hence are bounded. Replacing V_0 with $V_0 \oplus V_+^1 \oplus V_-^1$ we can assume $V_{\pm}^1 = 0$. $\square$

Now we are ready to prove our main Theorem.

4 Proofs of the main results

Proof of Theorem 1. We retain all the notations from the previous Section.

Step 1. Decompose $V = V_0 \oplus V_+ \oplus V_-$ as an $\mathfrak{A}$-module, where V_0 is bounded, V_- is a direct sum of copies of simple $M(\lambda - \alpha)$ and V_+ is a direct sum of copies of simple $\hat{M}(\lambda+\alpha)$. For a moment choose such a decomposition with a minimal V_0. Under such choice $V_{\pm}$ are not closed under the action of $e_{\pm 2}$ in general, but we can choose direct summunds $\hat{V}_{\pm}$ of $V_{\pm}$, closed under $e_{\pm 2}$ and such that the corresponding complements in $V_{\pm}$ are bounded. Indeed, let us do it for V_-. We use the identity $3e_{-2}e_1 - 3e_1e_{-2} = e_{-1}$. For $v \in (V_-)_{\lambda-\alpha}$ one has $-3e_1e_{-2}v = e_{-1}v$ and hence $e_{-2}v = v_1 + v_2$, where v_1 is proportional to $e_{-1}^2 v$ and $e_1 v_2 = 0$. As the kernel of e_1 acting on $V_{\lambda-3\alpha}$ is finite-dimensional, we can find a subspace W of finite codimension in $(V_-)_{\lambda-\alpha}$ such that $e_{-2}v$ is proportional to $e_{-1}^2 v$ for any $v \in W$. As V_0 is bounded, the kernel of e_1 acting on V_0 is finite-diemnsional and one constructs $\hat{V}_-$ by induction starting from $U(\mathfrak{A})W$. So, we can assume that $e_{\pm 2}V_{\pm} \subset V_{\pm}$. As the kernels of e_1 and e_{-1} acting on V_0 are finite-dimensional, we can find bounded submodules $W_{\pm} \subset V_{\pm}$ satisfying the following condition: for any $v \in (V_0)_{\lambda+n\alpha}$, $n \in \mathbb{Z}$ with e_1e_{-1} or $e_{-1}e_1$ acting nilpotently on v holds $e_2^l v, e_{-2}^l v \in V_0 \oplus W_+ \oplus W_-$ for any $-|n| - 2 \leqslant l \leqslant |n| + 2$. Set $T = V_0 \oplus W_+ \oplus W_-$;

Step 2. There exist bounded submodules $W_1^{\pm} \subset V_{\pm}$ such that $T_1^{\pm} = T \oplus W_1^{\pm}$ is a $\mathcal{W}_{\pm}$-module. We prove the existence of W_1^- and for W_1^+ everything will follow from the canonical involution on $\mathcal{V}$. We let W_1^- be generated by the projection of $e_{-2}(T)_{\lambda+\alpha}$ on V_- with respect to T. Clearly, T_1^- is closed under the action of $e_{\pm 1}$ by construction and we have to prove that $e_{-2}T_1^- \subset T_1^-$ only. First, $e_{-2}W_1^- \subset W_1^-$ by properties of V_-. We also note that, by definition of V_0, we have $e_{-2}(T_1^-)_{\lambda+2\alpha} \subset T_1^-$. For $v \in (T_1^-)_{\lambda+\alpha}$ we have $e_{-2}v \in W_1^- \subset T_1^1$ and by induction $e_{-2}(T_1^-)_{\lambda-n\alpha} \subset T_1^-$ for any $n \in \mathbb{Z}_+$. Further, from $3e_{-2}(e_1 v) = 3e_1e_{-2}v + e_{-1}v$ we have that for $v \in T_1^-$ from $e_{-2}v \in T_1^-$ it follows that $e_{-2}e_1 v \in T_1^1$ and it is sufficient to consider only $v \in (T_1^-)_{\lambda+n\alpha}$, $n \geqslant 3$ which do not belong to the image of e_1. But then $(e_1e_{-1})^l v = 0$ for some l and $e_{-2}v \in T_1^-$ by definition of W_-.

Step 3. If $e_{\pm 2}(W_1^{\mp})_{\lambda \mp 1} \subset T_1^{\pm}$ then $T_1 = T \oplus T_1^+ \oplus T_1^-$ is a $\mathcal{V}$-module. Indeed, by induction this means that $e_{\pm 2}W_1^{\mp} \subset T_1$ and hence $e_{\pm 2}T_1 \subset T_1$. In this case $T_1 = V$ since V is simple and we have that V is a bounded module, which is impossible. Thus at least one of the following holds: $e_2(W_1^-)_{\lambda-1} \not\subset T_1^+$ or $e_{-2}(W_1^+)_{\lambda+1} \not\subset T_1^-$.

Step 4. Let $W_2^{\pm}$ be generated by the projection of $e_{\pm 2}(W_1^{\mp})_{\lambda \mp \alpha}$ on $V_{\pm}$ with respect to T_1. From the properties of $V_{\pm}$ it follows that $W_2^{\pm}$ is a module over $\mathcal{W}_{\pm}$. Set $T_2 = T_1 \oplus W_2^+ \oplus W_2^-$. As in Step 3 we have that T_2 is not a $\mathcal{V}$-module, hence at least one of the following holds: $e_2(W_2^-)_{\lambda-1}\not\subset T_2$ or $e_{-2}(W_2^+)_{\lambda+1}\not\subset T_2$. So we proceed constructing $W_i^{\pm}$ and T_i for any $i\in\mathbb{N}$. Without loss of generality we can assume $e_{(-1)^i 2}(W_i^{(-1)^{i+1}})_{\lambda+(-1)^{i+1}\alpha}\cap T_i=0$, $i\geqslant n\in\mathbb{N}$. But in this case for each $m\geqslant n$ we obtain that $\oplus_{i\geqslant m}W_i^{(-1)^{i+1}}$ is a $\mathcal{V}$-submodule of V. This is a contradiction with the assumption that V is simple. This completes the proof. □

Proof of Corollary 1. We have only to show that the kernels of e_1 and e_{-1} have finite supports in any pinned $\mathcal{V}$-module V. Let V be a pinned module, $\operatorname{Supp} V\subset\lambda+\mathbb{Z}\alpha$ and $\dim V_\lambda=0$. We show that from $e_1v=0$, $v\neq 0$ follows $v\in V_{\lambda\pm\alpha}$. One can use the same arguments to prove that from $e_{-1}v=0$, $v\neq 0$ follows $v\in V_{\lambda\pm\alpha}$.

Assume $0\neq v\in V_{\lambda-n\alpha}$, $e_1v=0$. Then $e_nv=0$. If $n\in\mathbb{N}$, $n\geqslant 2$ we have $e_lv=0$ for any $l\geqslant n$. By Lemma 1, for any $0\neq w\in V_{\lambda+\alpha}$ there exists $n(w)\in\mathbb{N}$ such that $e_lw=0$ for any $l\geqslant n(w)$. From $e_{-1}w=0$ we get $e_1w=e_2w=0$ and V is a highest weight module, which contradicts our assumptions. If $n\in-\mathbb{N}$, $n\leqslant -2$ we have $e_nv=e_1v=0$ and hence $e_{-2}v=e_{-1}v=0$ obtaining that V is a lowest weight module, which again contradicts our assumptions. Finally, n can be only ± 1. □

Proof of Corollary 2. As any pointed non-Harish-Chandra $\mathcal{V}$-module is mixed, the statement is a direct consequence of Theorem 1. □

5 Some $sl(2,\mathbb{C})$-theory

Let $\mathfrak{G}$ denote the Lie algebra $sl(2,\mathbb{C})$ with the standard basis e,f,h. In this Section by a weight module we will mean h-diagonalizable $\mathfrak{G}$-module. We identify the space, dual to the Cartan subalgebra, with $\mathbb{C}$ using the eigenvalues of h. Under this identification, the root system of $\mathfrak{G}$ is $\{2,-2\}$. Fix $\lambda\in\mathbb{C}$ and $k\in\mathbb{N}$ and consider a full subcategory $K(\lambda,k)$ of the category of all $\mathfrak{G}$-modules, which consists of all $\mathfrak{G}$-modules M satisfying the following conditions:

1. M is a weight module with $\operatorname{Supp} M\subset\lambda+2\mathbb{Z}$;
2. $\dim M_\lambda<k$.

We retain the notion of a bounded module from Section 3. The aim of this Section is to prove Proposition 1, so we restrict our consideration to those $K(\lambda,k)$, which can physically appear in this Proposition. To determine such λ and k we recall Corollary 3, which says that λ should satisfy the following conditions: $M(\lambda-2)$ and $\hat{M}(\lambda+2)$ are simple. From the first condition we get $\lambda\neq 2,3,4,\dots$, from the second one we get $\lambda\neq -2,-3,-4,\dots$. Hence, either $\lambda\notin\mathbb{Z}$ or $\lambda=-1,0,1$. The statement of Proposition 1 follows immediately from the following Theorem with $B=U(\mathfrak{A})V_\lambda$.

Theorem 3. *Let $M\in K(\lambda,k)$ and B be a bounded submodule in M. Then there exists a bounded direct summand B' in M containing B.*

We start with a description of all simple modules in $K(\lambda, k)$. Let $\mathfrak{c} = (h+1)^2 + 4fe$ denote the Casimir operator. For $\gamma \in \mathbb{C}$ consider the $\mathfrak{G}$-module $V(\gamma)$ with the basis v_i, $i \in \mathbb{Z}$ and the action of $\mathfrak{G}$ given by

$$fv_i = v_{i-1}, \quad hv_i = (\lambda + 2i)v_i, \quad ev_i = \frac{1}{4}\left(\gamma - (\lambda + 2i + 1)^2\right) v_{i+1}.$$

Lemma 6. *Any simple module in $K(\lambda, k)$ is a subqoutient of some $V(\gamma)$.*

Proof. Let U_0 denote the centralizer of the Cartan subalgebra in $U(\mathfrak{G})$. Fortunately, in our case U_0 is generated by h and $\mathfrak{c}$ and thus is commutative. By [DOF, Lemma 3] any weight space of a simple weight $\mathfrak{G}$-module is a simple U_0-module and any simple U_0-module can be uniquely extended to a simple $\mathfrak{G}$-module. The algebra U_0 is commutative and generated by h and $\mathfrak{c}$. So any simple U_0-module is one-dimensional and is determined by two parameters μ and χ, where μ is the eigenvalue of h and χ is the eigenvalue of $\mathfrak{c}$. To complete the proof we have only to mention that any simple U_0-module with $\mu \in \lambda + 2\mathbb{Z}$ occures as a weight space in some $V(\gamma)$. $\square$

Now we can compute all simple subquotients of $V(\gamma)$.

Lemma 7. *1. If $(\gamma - (\lambda + 2i + 1)) \neq 0$ for all $i \in \mathbb{Z}$ then $V(\gamma)$ is simple.*

2. If there is a unique $i \in \mathbb{Z}$ such that $(\gamma - (\lambda + 2i + 1)) = 0$ then the length of $V(\gamma)$ is 2 and the simple subquotients of $V(\gamma)$ are: simple highest weight Verma module $M(\lambda + 2i)$ (here $\lambda + 2i$ is the highest weight) and simple lowest weight Verma module $\hat{M}(\lambda + 2(i+1))$ (here $\lambda + 2(i+1)$ is the lowest weight).

3. If there are two diffrent integers $i^+ > i^-$ such that $(\gamma - (\lambda + 2i^{\pm} + 1)) = 0$ then γ is the square of an integer, the length of $V(\gamma)$ is 3 and the simple subquotients of $V(\gamma)$ are: simple highest weight Verma module $M(\lambda + 2i^-)$, simple lowest weight Verma module $\hat{M}(\lambda + 2(i^+ + 1))$ and simple finite-dimensional module $L(\lambda + 2i^+)$ (the unique simple quotient of $M(\lambda + 2i^+)$).

Proof. Follows from the definition of $V(\gamma)$ by direct calculation. $\square$

Lemma 8. *The center of $\mathfrak{G}$ acts locally finite on any $M \in K(\lambda, k)$. In particular, $K(\lambda, k) = \oplus_\chi K(\lambda, k, \gamma)$, where $K(\lambda, k, \gamma)$ consists of all M such that for any $v \in M$ there exists $n \in \mathbb{N}$ with $(\mathfrak{c} - \gamma)^n v = 0$.*

Proof. Let $M \in K(\lambda, k)$ and $S = U(\mathfrak{G})M_\lambda$. Then S has finite-dimensional weight spaces and hence the action of $\mathfrak{c}$ on it is locally finite. Let $N \simeq M/S$. We have $N_\lambda = 0$ and hence $N \simeq N_- \oplus N_+$, where $N_\pm \simeq \oplus_{n \in \pm\mathbb{N}} N_{\lambda + 2n}$. We will prove that $\mathfrak{c}$ acts locally finite on N_- (for N_+ the proof is analogous). Let μ be the highest weight of N_-. By induction it is enough to prove that $\mathfrak{c}$ acts locally finite on $(N_-)_\mu$. But as μ is the highest weight, the action of $\mathfrak{c}$ is diagonalizable with the scalar $(\mu+1)^2$ on $\mathfrak{c}$, which completes the proof. $\square$

Proof of Theorem 3. By Lemma 8 we can prove Theorem 3 for each $K(\lambda, k, \gamma)$ independently. So, we fix $K(\lambda, k, \gamma)$ and some M in it.

Case 1. Assume that $V(\gamma)$ is simple. Then it is dense and $\dim V(\gamma)_{\lambda+2n} = 1$ for all $n \in \mathbb{Z}$. From $M \in K(\lambda, k, \gamma)$ we immediately obtain $\dim M_{\lambda+2n} = \dim M_\lambda < k$, hence M itself is bounded and we can take $B' = M$ completing the proof.

Case 2. Assume that $V(\gamma)$ has length 2. Then there are two simple modules in $K(\lambda, k, \gamma)$: $M(\lambda + 2i)$ and $\hat{M}(\lambda + 2(i + 1))$ for some fixed $i \in \mathbb{Z}$. If $i \geqslant 0$ we obtain $\dim M_{\lambda-2n} = \dim M_\lambda < k$ for any $n \in \mathbb{Z}_+$. If $i < 0$ we obtain $\dim M_{\lambda+2n} = \dim M_\lambda < k$ for any $n \in \mathbb{Z}_+$. Up to the Chevalley involution these cases are the same, so we consider the first one: $i \geqslant 0$. Moreover, we can assume $i = 0$. Denote by W_1 the kernel of f acting on $M_{\lambda+2}$. Clearly, W_1 is a subspace of a finite codimension $< k$. Let W_2 be the image of e acting on M_λ, which is also finite-dimensional of dimension $< k$. Denote by W_3 any complement to $W_1 \cap W_2$ in W_1. Then W_3 has a finite codimension $< 2k$. Consider the $\mathfrak{G}$-submodule $N = U(\mathfrak{G})W_3$ of M. From the definition it follows that N is a (possibly infinite) direct sum of copies of $\hat{M}(\lambda + 2)$. We claim that N is a direct summand of M. Indeed, consider any complement W_4 to W_1 in $V_{\lambda+2}$ and set $N' = U(\mathfrak{G})(V_\lambda + W_4)$. Then $(N+N')_\lambda \supset V_\lambda$ and $(N+N')_{\lambda+2} \supset W_4+W_2+W_3 = V_{\lambda+2}$ and hence $N+N' = V$. We want to show that $N \cap N' = 0$. Consider the $\mathfrak{A}$-submodule $N \cap N'$ of V. From the definitions of W_1, W_2, W_3 and W_4 it follows that $(N \cap N')_{\lambda+2n} \neq 0$ implies $n \geqslant 2$. But the supports of both simple modules in $K(\lambda, k, \gamma)$ are not subsets in $\{\lambda + 2n : n \in \mathbb{N}, n \geqslant 2\}$ and we obtain that $\mathrm{Supp}(N \cap N') = \varnothing$ implying $N \cap N' = 0$. As any submodule of N has a direct complement in N, we can take as B' the direct sum of N' with the projection of B on N with respect to N'.

Case 3. Finally assume that $V(\gamma)$ has length 3 and integers $i^+ > i^-$ are choosen such that the list of simple modules in $K(\lambda, k, \gamma)$ is $M(\lambda+2i^-)$, $L(\lambda+2i^+)$ and $\hat{M}(\lambda+2(i^++1))$. First of all we note that this means $\lambda \in \mathbb{Z}$ and hence $\lambda = -1, 0, 1$ by our restrictions. From this it follows that $i^- < 0 \leqslant i^+$, in other words, $\lambda \in \mathrm{Supp}\, L(\lambda + 2i^+)$. From this we immediately obtain $[M : L(\lambda + 2i^+)] < k$ and $\dim M_{\lambda+2i} < k$ for all $i^- < i \leqslant i^+$. We proceed as in Case 2. Denote by W_1 the kernel of f acting on $M_{\lambda+2(i^++1)}$. Clearly, W_1 is a subspace of a finite codimension $< k$. Let W_2 be the image of e acting on $M_{\lambda+2i^+}$, which is also finite-dimensional of dimension $< k$. Denote by W_3 any complement to $W_1 \cap W_2$ in W_1. Then W_3 has a finite codimension $< 2k$. Consider the $\mathfrak{G}$-submodule $N_1 = U(\mathfrak{G})W_3$ of M. Denote by W_4 the kernel of e acting on $M_{\lambda+2i^-}$. Clearly, W_4 is a subspace of a finite codimension $< k$. Let W_5 be the image of f acting on $M_{\lambda+2(i^-+1)}$, which is also finite-dimensional of dimension $< k$. Denote by W_6 any complement to $W_4 \cap W_5$ in W_4. Then W_6 has a finite codimension $< 2k$. Consider the $\mathfrak{G}$-submodule $N_2 = U(\mathfrak{G})W_4$ of M and set $N = N_1 \oplus N_2$ (clearly, $N_1 \cap N_2 = 0$). From definition it follows that N is a (possibly infinite) direct sum of copies of $M(\lambda + 2i^-)$ and $\hat{M}(\lambda + 2(i^+ + 1))$. We again claim that N is a direct summand of M. Indeed, consider any complement W_7 (resp. W_8) to W_1 (resp. W_4) in $V_{\lambda+2(i^++1)}$ (resp. $V_{\lambda+2i^-}$) and set $N' = U(\mathfrak{G})(V_\lambda + W_7 + W_8)$. Then $(N + N')_\lambda \supset V_\lambda$, $(N+N')_{\lambda+2(i^++1)} \supset W_7+W_2+W_3 = V_{\lambda+2(i^++1)}$ and $(N+N')_{\lambda+2i^-} \supset W_8+W_5+W_6 = V_{\lambda+2i^-}$. Hence $N + N' = V$. Again, we show that $N \cap N' = 0$. Consider the $\mathfrak{A}$-submodule $N \cap N'$ of V. From the definitions of all W_i it follows that $(N \cap N')_{\lambda+2n} \neq 0$ implies $n < i^-$

or $n > i^+ + 2$. But the support of any simple module in $K(\lambda, k, \gamma)$ is not contained in $\{\lambda + 2n : n \in \mathbb{Z}, n < i^- \text{ or } n > i^+ + 2\}$ and we obtain that $\mathrm{Supp}(N \cap N') = \varnothing$ implying $N \cap N' = 0$. As any submodule of N has a direct complement in N, we can take as B' the derect sum of N' with the projection of B on N with respect to N'. This completes the proof. $\square$

Acknowledgments

The research was done during the visit of the author to Bielefeld University as an Alexander von Humboldt fellow. The financial support of Humboldt Foundation and the hospitality by Bielefeld University are gratefully acknowledged.

References

[DOF] *Yu.A.Drozd, S.A.Ovsienko, V.M.Futornyi*, The Harish-Chandra S-homomorphism and $\mathfrak{G}$-modules generated by a semiprimitive element. (Russian) Ukrain. Mat. Zh. 42 (1990), no. 8, 1031–1037; translation in Ukrainian Math. J. 42 (1990), no. 8, 919–924 (1991).

[F] *V.M.Futorny*, Imaginary Verma modules for affine Lie algebras. Canad. Math. Bull. 37 (1994), no. 2, 213–218.

[MP1] *C.Martin, A.Piard*, Indecomposable modules over the Virasoro Lie algebra and a conjecture of V. Kac. Comm. Math. Phys. 137 (1991), no. 1, 109–132.

[MP2] *C.Martin, A.Piard*, Classification of the indecomposable bounded admissible modules over the Virasoro Lie algebra with weightspaces of dimension not exceeding two. Comm. Math. Phys. 150 (1992), no. 3, 465–493.

[M] *O.Mathieu*, Classification of Harish-Chandra modules over the Virasoro Lie algebra. Invent. Math. 107 (1992), no. 2, 225–234.

[Ma1] *V.Mazorchuk*, Futorny theorem for generalized Witt algebras of rank 2. Comm. Algebra 25 (1997), no. 2, 533–541.

[Ma2] *V.Mazorchuk*, Classification of simple Harish-Chandra modules over $\mathbb{Q}$-Virasoro algebra. Preprint 97-019, Bielefeld University, to appear in Math. Nach.

[Ma3] *V.Mazorchuk*, Verma modules over generalized Witt algebras. Compositio Math. 115 (1999), no. 1, 21–35.

[X] *X.Xu*, Pointed representations of Virasoro algebra. A Chinese summary appears in Acta Math. Sinica 40 (1997), no. 3, 479. Acta Math. Sinica (N.S.) 13 (1997), no. 2, 161–168.

[Z] *H.Zhang*, A class of representations over the Virasoro algebra. J. Algebra 190 (1997), no. 1, 1–10.

Department of Mechanics and Mathematics, Kyiv Taras Shevchenko University, 64, Volodymyrska st., 252033, Kyiv, Ukraine, e-mail: `mazorchu@uni-alg.kiev.ua`

Address till 31 December 1999: Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501, Bielefeld, Germany, e-mail: `mazor@mathematik.uni-bielefeld.de`