FINITENESS FOR PARABOLIC GROUP ACTIONS IN CLASSICAL GROUPS

T. Brüstle, L. Hille, and G. Röhrle*

1. Introduction

Let G be a reductive algebraic group defined over an algebraically closed field k. Let P be a parabolic subgroup of G with unipotent radical P_u. We denote the Lie algebra of P_u by $\mathfrak{p}_u$. The group P acts on any member $P_u^{(l)}$ of the descending central series of P_u for $l \geq 0$, likewise for $\mathfrak{p}_u^{(l)}$.

The modality of the action of P on $\mathfrak{p}_u^{(l)}$, denoted by $\operatorname{mod}(P : \mathfrak{p}_u^{(l)})$, is the maximal number of parameters upon which a family of P-orbits on $\mathfrak{p}_u^{(l)}$ depends, see Section 2 for a precise definition. Note that the modality is zero precisely when there is only a finite number of orbits under the action.

All instances of parabolic subgroups P of classical algebraic groups with a finite number of orbits on $\mathfrak{p}_u = \mathfrak{p}_u^{(0)}$ were determined in [3, Theorem 1.1]. In this paper we extend this classification to the action of parabolic subgroups P on $\mathfrak{p}_u^{(l)}$ for $l \geq 1$ in all classical groups.

Theorem 1.1. *Let G be a simple classical algebraic group and P a parabolic subgroup of G. Suppose that* $\operatorname{char} k$ *is either zero or a good prime for G. Then* $\operatorname{mod}(P : \mathfrak{p}_u^{(l)}) = 0$ *for $l \geq 1$ precisely if one of the following conditions holds:*

(i) *G is of type A_r and $\ell(P_u) \leq 5 + 2l$;*
(ii) *G is of type B_r and $\ell(P_u) \leq 4 + 2l$;*
(iii) *G is of type C_r and $\ell(P_u) \leq 5 + 2l$;*
(iv) *G is of type D_r, either $\ell(P_u) \leq 4+2l$, or $\ell(P_u) = 5+2l$ and $\tau P \neq P$, where τ is the graph automorphism of G of order 2.*

The following result is proved in [2] for an arbitrary infinite field k.

Theorem 1.2. *Let V be a finite-dimensional k-vector space. Let P be a parabolic subgroup of $GL(V)$. Then for $l \geq 0$ the number of P-orbits on $\mathfrak{p}_u^{(l)}$ (or on $P_u^{(l)}$) is finite if and only if $l = 0$ and $\ell(P_u) \leq 4$, or $l \geq 1$ and $\ell(P_u) \leq 5 + 2l$.*

Since k is algebraically closed, the same statement of Theorem 1.2 also holds for $SL(V)$ instead of $GL(V)$, cf. [3], and thus part (i) of Theorem 1.1 follows.

The remaining parts of Theorem 1.1 follow directly from Propositions 3.1 and 4.1 below.

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2. Recollection and preliminary Results

For an algebraic group G we denote its Lie algebra by $\operatorname{Lie} G$ or simply by the corresponding lower case Gothic letter $\mathfrak{g}$. The identity component of G is denoted by G^0 and its unipotent radical by G_u. The Lie algebra of G_u is denoted by $\mathfrak{g}_u$.

Suppose that G acts morphically on an algebraic variety X. For x in X the G-orbit in X through x is denoted by $G \cdot x$ and for $Z \subset X$ a subset $\cup_{z\in Z} G \cdot z$ by $G \cdot Z$. Recall from [4] that the *modality* of the action of G on X is defined as $\operatorname{mod}(G : X) := \max_Z \min_{z\in Z} \operatorname{codim}{}_Z G^0 \cdot z$, where Z runs through all irreducible G^0-invariant subvarieties of X. In case X is an irreducible variety let $K(X)^G$ denote the field of G-invariant rational functions on X. As $\min_{x\in X} \operatorname{codim}{}_X G\cdot x = \operatorname{trdeg} K(X)^G$ (e.g. see [5, 2.3]), $\operatorname{mod}(G : X)$ measures the maximal number of parameters upon which a family of G-orbits on X depends. The modality of the action of G on X is 0 precisely when G admits only a finite number of orbits on X.

Lemma 2.1. *Let G be a linear algebraic group and $Q \subseteq P$ parabolic subgroups of G. Then for each $l \geq 0$ we have* $\operatorname{mod}(P : \mathfrak{p}_u^{(l)}) \leq \operatorname{mod}(Q : \mathfrak{q}_u^{(l)})$.

Proof. Since $Q \subseteq P$, any irreducible P-invariant subvariety Z of $\mathfrak{p}_u^{(l)}$ is also Q-invariant and $\operatorname{codim}{}_Z P \cdot z \leq \operatorname{codim}{}_Z Q \cdot z$ for any z in Z. Consequently, since $\mathfrak{p}_u^{(l)} \subseteq \mathfrak{q}_u^{(l)}$, $\operatorname{mod}(P : \mathfrak{p}_u^{(l)}) \leq \operatorname{mod}(Q : \mathfrak{p}_u^{(l)}) \leq \operatorname{mod}(Q : \mathfrak{q}_u^{(l)})$, by the definition of modality.

For the differential of an automorphism Θ of G we write $\theta = d\Theta_e$. Assume that P is a Θ-invariant parabolic subgroup of G. Then, since $\mathfrak{p}_u^{(l)}$ is a characteristic subalgebra of $\mathfrak{p}$ for each $l \geq 0$, it is θ-invariant.

Lemma 2.2. *Let G be a linear algebraic group, Θ a semisimple automorphism of G, and P a Θ-invariant parabolic subgroup of G. Then for each $l \geq 0$ we have* $\operatorname{mod}(P^\Theta : (\mathfrak{p}_u^\theta)^{(l)}) \leq \operatorname{mod}(P : \mathfrak{p}_u^{(l)})$.

Proof. Since $(\mathfrak{p}_u^\theta)^{(l)} \subseteq (\mathfrak{p}_u^{(l)})^\theta$ and by Corollary 3.4 in [7], we conclude that $\operatorname{mod}(P^\Theta : (\mathfrak{p}_u^\theta)^{(l)}) \leq \operatorname{mod}(P^\Theta : (\mathfrak{p}_u^{(l)})^\theta) \leq \operatorname{mod}(P : \mathfrak{p}_u^{(l)})$, as desired.

Proposition 2.3. *Let G be a reductive algebraic group and suppose that* $\operatorname{char} k$ *is either zero or a good prime for G. Let H be a closed reductive subgroup of G normalized by a maximal torus T of G and P a parabolic subgroup of G containing T. Set $Q := P \cap H$. Then* $\operatorname{mod}(Q : \mathfrak{q}_u^{(l)}) \leq \operatorname{mod}(P : \mathfrak{p}_u^{(l)})$.

Proof. The result follows from the last lemma and Theorem 3.12 in [7].

Lemma 2.4. *Let $G = SL(V)$. Then each of the classical groups $SO(V)$ and $Sp(V)$ can be realized as a fixed point subgroup of G for a suitable semisimple automorphism Θ of G, i.e., G^Θ equals $SO(V)$ or $Sp(V)$. Moreover, each parabolic subgroup of G^Θ can be realized as a fixed point subgroup P^Θ for some Θ-invariant parabolic subgroup P of G. In case G^Θ is $SO(V)$ and $\dim V$ is even, P^Θ is only determined up to equivalence under the graph automorphism of G^Θ.*

For a proof of the assertion on G^Θ see Steinberg [8, §11 p. 169]. The remaining statements follow easily from the explicit description of Θ in *loc. cit.*

Remark 2.5. Suppose G is a reductive algebraic group and P is a parabolic subgroup of G and suppose that $\operatorname{char} k$ is zero or a good prime for G. Then

$\operatorname{mod}(P : P_u^{(l)}) = \operatorname{mod}(P : \mathfrak{p}_u^{(l)})$. This follows from [7, Corollary 4.2], since the characteristic restrictions ensure that $\operatorname{Lie} P_u^{(l)} = \mathfrak{p}_u^{(l)}$. Thus in our situation we obtain analogous statements for the action of P on $P_u^{(l)}$.

3. Finite Q-actions on $\mathfrak{q}_u^{(l)}$ for $l \geq 1$

We now formulate a first consequence of Theorem 1.1(i):

Proposition 3.1. *Let H be a simple algebraic group and Q a parabolic subgroup of H. Suppose $\operatorname{char} k \neq 2$. Then $\operatorname{mod}(Q : \mathfrak{q}_u^{(l)}) = 0$ for $l \geq 1$ provided one of the following holds:*

(i) *H is of type B_r and $\ell(Q_u) \leq 4 + 2l$;*
(ii) *H is of type C_r and $\ell(Q_u) \leq 5 + 2l$;*
(iii) *H is of type D_r, either $\ell(Q_u) \leq 4 + 2l$, or $\ell(Q_u) = 5 + 2l$ and $\tau Q \neq Q$, where τ is the graph automorphism of H of order 2.*

Proof. Let G be of type A_{2r} or of type A_{2r-1} and Θ the graph automorphism of G as in Lemma 2.4 such that $G^\Theta = H$. The characteristic restriction ensures that Θ is semisimple. We denote the simple roots of H by $\beta_1, \ldots, \beta_r$ following the labeling of the Dynkin diagram of H according to [1] and we assume that Q is standard. We show that in each instance the desired finiteness result is obtained from Theorem 1.1(i) and Lemmas 2.1, 2.2, and 2.4, and Proposition 2.3.

Suppose H is of type B_r. Every parabolic subgroup Q of H can be realized as a fixed point subgroup under Θ of a parabolic subgroup P of G such that either $\ell(Q_u) = \ell(P_u)$ or $\ell(Q_u) = \ell(P_u) - 1$. The latter case occurs precisely when $-\beta_1 \in \Psi(Q)$. Thus if $\ell(Q_u) \leq 4 + 2l$, then P satisfies $\ell(P_u) \leq 5 + 2l$ and $\operatorname{mod}(P : \mathfrak{p}_u^{(l)}) = 0$ by Theorem 1.1(i). Whence by Lemma 2.2 we have $\operatorname{mod}(Q : \mathfrak{q}_u^{(l)}) = 0$, as claimed.

Now suppose H is of type C_r. Every parabolic subgroup Q of H can be realized as a fixed point subgroup under Θ of a parabolic subgroup P of G such that $\ell(Q_u) = \ell(P_u)$. The result follows readily from Theorem 1.1(i) and Lemma 2.2.

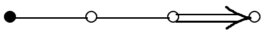
Finally, suppose that H is of type D_r and that Q is a standard parabolic subgroup of H with $\ell(Q_u) = 5 + 2l$ and $\tau Q \neq Q$ as in (iii) above. This implies that $-\beta_1 \in \Psi(Q)$. In this instance Q can be realized (up to conjugacy by τ) as a fixed point subgroup under Θ of a parabolic subgroup P of G with $\ell(Q_u) = \ell(P_u)$. Again, the desired result follows from Theorem 1.1(i) and Lemma 2.2.

It follows from this particular D_r case, Lemma 2.1, and Proposition 2.3 that $\operatorname{mod}(Q : \mathfrak{q}_u^{(l)}) = 0$ in the remaining instances as well. Each of these cases can also be treated directly with the same "folding" technique as before. We omit the details and illustrate some examples below.

Examples 3.2. Let P be a standard parabolic subgroup of G. The colored nodes indicate the simple roots in $\Psi(P)$ and likewise for $\Psi(G^\Theta)$. In each case we consider the action of P on $\mathfrak{p}_u'$ and likewise for Q on $\mathfrak{q}_u'$, i.e., for $l = 1$. For an explicit description of the automorphisms consult [8, §11].

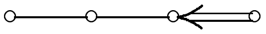
In our last example we show a folding for a τ-invariant instance in D_5. The same finiteness result can be obtained by realizing the group of type D_5 as a suitable regular subgroup in the D_6 case below. Namely, let $\{\alpha_1, \ldots, \alpha_6\}$ denote the simple roots of D_6 and let Q be the parabolic subgroup of D_6 as in the third example. Let S be the simple regular subgroup of D_6 of type D_5 with $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4+\alpha_5, \alpha_4+\alpha_6\}$

as a base for the simple roots. Then the parabolic subgroup $Q \cap S$ of S is precisely the one shown in the last example, cf. the proof of [6, Theorem 4.2]. The finiteness statement follows in this instance from Proposition 2.3.



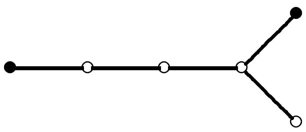
$G^{\Theta} = SO_9(k)$, $\ell(P_u^{\Theta}) = 6$

$G = SL_9(k)$, $\ell(P_u) = 6$



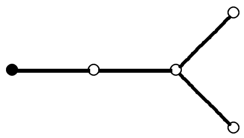
$G^{\Theta} = Sp_8(k)$, $\ell(B_u^{\Theta}) = 7$

$G = SL_8(k)$, $\ell(B_u) = 7$



$G^{\Theta} = SO_{12}(k)$, $\ell(P_u^{\Theta}) = 7$

$G = SL_{12}(k)$, $\ell(P_u) = 7$



$G^{\Theta} = SO_{10}(k)$, $\ell(P_u^{\Theta}) = 6$

$G = SL_{10}(k)$, $\ell(P_u) = 6$

Figure 1.

4. Infinite Q-actions on $\mathfrak{q}_u^{(l)}$ for $l \geq 1$

The goal of this section is the following

Proposition 4.1. *Let G be a simple classical algebraic group and P a parabolic subgroup of G. Suppose that* $\operatorname{char} k$ *is either zero or a good prime for G. Then* $\operatorname{mod}(P : \mathfrak{p}_u^{(l)}) > 0$ *for $l \geq 1$ provided one of the following conditions holds:*

(i) *G is of type B_r and $\ell(P_u) \geq 5 + 2l$;*
(ii) *G is of type C_r and $\ell(P_u) \geq 6 + 2l$;*
(iii) *G is of type D_r, either $\ell(P_u) > 5+2l$, or $\ell(P_u) = 5+2l$ and $\tau P = P$, where τ is the graph automorphism of G of order 2.*

Before starting the proof of Proposition 4.1 we recall a particular family of matrices from [2]. Here the dots represent 0 entries and the columns and rows of $*$'s represent block submatrices whose entries are 0 of variable size and $\lambda \in k$. Depending on the size of these submatricies this family lies in $\mathfrak{gl}(V)$ for some suitable V:

$$z_\lambda := \begin{pmatrix} . & . & * & . & 1 & \lambda & * & . & . \\ . & . & * & 1 & . & 1 & * & . & . \\ * & * & * & * & * & * & * & * & * \\ . & . & * & . & . & . & * & . & 1 \\ . & . & * & . & . & . & * & . & 1 \\ . & . & * & . & . & . & * & 1 & . \\ * & * & * & * & * & * & * & * & * \\ . & . & * & . & . & . & * & . & . \\ . & . & * & . & . & . & * & . & . \end{pmatrix}$$

for $\lambda \in k$. Suppose that $\operatorname{char} k \neq 2$. Then z_λ is conjugate under $GL(V)$ to each of the following matrices

$$x_\mu = \begin{pmatrix} . & . & * & . & 1 & \mu & * & . & . \\ . & . & * & 1 & . & 1 & * & . & . \\ * & * & * & * & * & * & * & * & * \\ . & . & * & . & . & . & * & -1 & -\mu \\ . & . & * & . & . & . & * & . & -1 \\ . & . & * & . & . & . & * & -1 & . \\ * & * & * & * & * & * & * & * & * \\ . & . & * & . & . & . & * & . & . \\ . & . & * & . & . & . & * & . & . \end{pmatrix} \qquad x'_\mu = \begin{pmatrix} . & . & * & . & 1 & \mu & * & . & . \\ . & . & * & 1 & . & 1 & * & . & . \\ * & * & * & * & * & * & * & * & * \\ . & . & * & . & . & . & * & 1 & \mu \\ . & . & * & . & . & . & * & . & 1 \\ . & . & * & . & . & . & * & 1 & . \\ * & * & * & * & * & * & * & * & * \\ . & . & * & . & . & . & * & . & . \\ . & . & * & . & . & . & * & . & . \end{pmatrix},$$

where $\mu \in k$. Note that x_μ and $x_{-\mu}$ lie in the same $GL(V)$-orbit; likewise for x'_μ and $x'_{-\mu}$. Note that x_μ lies in $\mathfrak{so}(V)$. We also consider the 'inflated' family

$$y_\mu = \begin{pmatrix} . & . & * & . & 1 & 1 & \mu & * & . & . \\ . & . & * & 1 & . & . & 1 & * & . & . \\ * & * & * & * & * & * & * & * & * & * \\ . & . & * & . & . & . & . & * & 1 & \mu \\ . & . & * & . & . & . & . & * & . & 1 \\ . & . & * & . & . & . & . & * & . & 1 \\ . & . & * & . & . & . & . & * & 1 & . \\ * & * & * & * & * & * & * & * & * & * \\ . & . & * & . & . & . & . & * & . & . \\ . & . & * & . & . & . & . & * & . & . \end{pmatrix}$$

which is obtained from x'_μ by 'doubling' the center row and column. Note that y_μ lies in $\mathfrak{sp}(V)$.

Proof of Proposition 4.1. First suppose that G is of type B_r. Suppose that l is the size of the blocks represented by the $*$'s in x_μ. Then $x_\mu \in \mathfrak{so}_{2l+7}(k)$ for every $\mu \in k$. If B is the standard Borel subgroup of $SO_{2l+7}(k)$, then we see that $x_\mu \in \mathfrak{b}_u^{(l)}$. Since $\{x_\mu | \mu \in k\}$ describes a family of nilpotent matricies in $\mathfrak{so}_{2l+7}(k)$ which are not conjugate under the Borel subgroup of $GL_{2l+7}(k)$ by [2], this family affords distinct B-orbits on $\mathfrak{b}_u^{(l)}$. Therefore, $\operatorname{mod}(B : \mathfrak{b}_u^{(l)}) > 0$. Note that $\ell(B_u) = 2l + 5$. Using the construction from Proposition 2.3, we derive that for groups of type B_r

the statement on modality holds for the Borel subgroups. If $P \subset G$ is a parabolic subgroup so that $\ell(P_u) \geq 2l+5$, then we construct a subgroup H of G such that H is of type $B_{r'}$ for some $r' < r$, so that $Q := H \cap P$ is the Borel subgroup of H, and $\ell(Q_u) = \ell(P_u)$ or $\ell(P_u) - 1$; the latter case only occurs when $\ell(P_u)$ is even, so that $\ell(P_u) \geq 2l+6$ in that instance. At any rate, for Q we have $\ell(Q_u) \geq 2l+5$, and so $\mathrm{mod}(Q : \mathfrak{q}_u^{(l)}) > 0$. The desired result for P now follows readily from Proposition 2.3.

Next let G be of type D_r and $P \subset G$ satisfies the condition of the statement. Let $H := G^\tau$. Then H is of type B_{r-1}. First suppose that $\tau P = P$. Then $\ell(P_u) = \ell(P_u^\tau) \geq 2l+5$. Thus by the result for B_r just established and Lemma 2.2, we get $\mathrm{mod}(P : \mathfrak{p}_u^{(l)}) \geq \mathrm{mod}(P^\tau : (\mathfrak{p}_u^{d\tau})^{(l)}) > 0$.

If $\tau P \neq P$, then, by our assumptions, $\ell(P_u) \geq 2l+6$. We can embed P into a parabolic subgroup R of G so that $\ell(R_u) = \ell(P_u) - 1 \geq 2l+5$ and $\tau R = R$. Then by Lemma 2.1 and the τ-stable case just proved, $\mathrm{mod}(P : \mathfrak{p}_u^{(l)}) \geq \mathrm{mod}(R : \mathfrak{r}_u^{(l)}) > 0$.

Finally, we address the case when G is of type C_r. By Lemma 2.1 it suffices to only consider such parabolic subgroups P of G where $\ell(P_u) = 2l+6$.

Consider the family of matrices y_μ. Suppose that the size of the blocks given by the $*$'s is l. Then $y_\mu \in \mathfrak{so}_{2l+8}(k)$ for each $\mu \in k$. Despite the fact that the family y_μ no longer corresponds to a family of indecomposable objects in the associated category of modules over a certain quasi-hereditary algebra [2], in that category each y_μ decomposes as $y_\mu = x'_\mu \oplus s$, where s is a simple object. Consequently, y_μ does define a family of nilpotent matrices in $\mathfrak{so}_{2l+8}(k)$ which are non-conjugate under the action of $GL_{2l+8}(k)$, thus in particular, they are not conjugate under the action of $Sp_{2l+8}(k)$. Note that $Sp_{2l+8}(k)$ is of rank $l+4$. Now let P be the standard parabolic subgroup of G whose semisimple subsystem of roots is defined by the long simple root of G. Then P is of semisimple rank 1 and $\ell(P_u) = 2(r-1) = 2l+6$. Observe that $y_\mu \in \mathfrak{p}_u^{(l)}$ for each $\mu \in k$. Consequently, $\mathrm{mod}(P : \mathfrak{p}_u^{(l)}) > 0$, as desired. This was the only case left to prove in type C_r.

This completes the proof of the proposition.

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Authors' addresses:

T. Brüstle, G. Röhrle
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
Germany

e-mail: bruestle@mathematik.uni-bielefeld.de
e-mail: roehrle@mathematik.uni-bielefeld.de

L. Hille
Mathematisches Seminar
der Universität Hamburg
Bundesstr. 55
20146 Hamburg
Germany

e-mail: hille@math.uni-hamburg.de