ARE BRAID GROUPS LINEAR GROUPS?

Remarks on a paper of S. Bachmuth

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1. Introduction

Having attended Bachmuth's talk on the linearity of the braid groups during the conference Groups St. Andrews/Galway '93 without gathering much insight, we decided afterwards to work through Bachmuth's preprint (in the version accepted for publication by Advances in Mathematics[1]). Various versions of this preprint have been widely circulated since 1991 without convincing the mathematical community of the validity of Bachmuth's arguments. [2] Bachmuth claims to prove the linearity of the braid groups B_n by showing that the so called Gassner representation, a certain representation of the finite index subgroup P_n consisting of all pure braids of B_n, is faithful for every n.[3] The result of our analysis is that Bachmuth's paper does in fact not answer this question, and that the problem of deciding whether the Gassner representation is faithful or not has to be considered completely open. In Section 3 of this note we present a counter–example to Bachmuth's main assertion [3, Theorem C], which – if true – would have implied the faithfulness of the Gassner representation. This counter–example is supplemented, in Section 4, by a detailed analysis of Bachmuth's proof strategy, including the variations sketched in [3, Sect. 3, Remarks (1), (2)]. The outcome is a complete breakdown of his arguments, none of which sheds any light (in whatever direction) on the question as to whether the Gassner representation is faithful or not.

At an early stage (1993/94) we communicated our arguments and conclusions to Bachmuth and the editors of Advances in Mathematics expecting that the paper in question would be withdrawn, which would have freed us from the obligation of publishing our observations.

2. Some background

For $n \geq 2$, the braid group B_n may be defined by its standard presentation[4]

$$B_n = \big\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_\nu\sigma_{\nu+1}\sigma_\nu = \sigma_{\nu+1}\sigma_\nu\sigma_{\nu+1} \ (1 \leq \nu \leq n-2), \\ \sigma_\nu\sigma_\mu = \sigma_\mu\sigma_\nu \ (|\nu - \mu| \geq 2)\big\rangle.$$

The group B_n has several interesting interpretations; for instance it can be identified with a certain group of automorphisms of the free group F_n of rank n freely generated

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[1]Apart from some minor modifications occurring in the remarks at the end of Bachmuth's Section 3, it is this version which has meanwhile appeared; cf. [3].

[2]Cf. for instance Birman's recent review [4] of Bachmuth's paper.

[3]Remarks concerning the background of this problem can be found in the next section.

[4]See Artins papers [1] and [2].

by $\xi_1, \ldots, \xi_n$, by letting σ_ν act on the ξ_μ via

$$\sigma_\nu(\xi_\mu) = \begin{cases} \xi_{\nu+1}, & \mu = \nu \\ \xi_{\nu+1}^{-1}\xi_\nu\xi_{\nu+1}, & \mu = \nu+1 \\ \xi_\mu, & \mu \neq \nu, \nu+1 \end{cases} \qquad (\nu = 1, 2, \ldots, n-1).$$

Denote by P_n the subgroup of pure braids, i.e., the kernel of the canonical map of B_n onto the symmetric group S_n, and let R_{n-1} be the subgroup of P_n generated by the elements

$$\tau_i = (\sigma_1 \ldots \sigma_{i-1})^i\,(\sigma_{i+1} \ldots \sigma_{n-1})^{i-n}, \quad 1 \leq i \leq n-1.$$

By a result of Magnus, R_{n-1} is free of rank $n-1$; cf. [10] or [11, Sect. 3.7, Theorem N9]. In terms of the above representation of B_n as a group of automorphisms of F_n, the subgroup P_n is generated by a set of automorphisms A_{ik} for $1 \leq i < k \leq n$, where

$$A_{ik}(\xi_\mu) = \begin{cases} \xi_\mu, & \mu < i \text{ or } \mu > k \\ \xi_i\,[\xi_i, \xi_k], & \mu = i \\ [\xi_i, \xi_k]^{-1}\,\xi_k, & \mu = k \\ [\xi_i, \xi_k]^{-1}\,\xi_\mu\,[\xi_i, \xi_k], & i < \mu < k. \end{cases}$$

A set of defining relations for P_n in terms of the generators A_{ik} can be found in [11, Sect. 3.7, Theorem N8]. Note that the automorphisms $A_{ik}(\xi_\mu)$ induce the identity on $F_n/[F_n, F_n]$, which is important for the definition of the representation of P_n given in [8].

A natural linear representation $\beta_n : B_n \to GL_n\big(\mathbb{Z}[t, t^{-1}]\big)$, which generalizes the standard representation of S_n by permutation matrices, was introduced by Burau in [5]. It is defined on the generators of B_n by

$$\beta_n(\sigma_\nu) = I_{\nu-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-\nu-1},$$

where I_k denotes the k–dimensional identity matrix. As it turns out, the Burau representation is faithful for $n \leq 3$, but not for $n \geq 6$; cf. [12] and [9]. The cases $n = 4, 5$ are still open. Despite the fact that the Burau representation fails to be faithful for almost all n, braid groups share important structural properties with linear groups. For example, it follows from more general results concerning mapping class groups derived in [7] that all subgroups of braid groups satisfy the Tits alternative, i.e., they are either virtually solvable or they contain non–abelian free subgroups. In 1961 Gassner [6] [5] defined a more complicated homomorphism $\gamma_n : P_n \to GL_n\big(\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]\big)$, whose definition is based on the fact that P_n can be identified with the subgroup $\langle A_{ik}(\xi_\mu)\rangle$ of $\mathrm{Aut}(F_n)$ and which coincides with the map β_n restricted to P_n after specializing $t_1 = \ldots = t_n =: t$. However, in view of the greater freedom due to additional variables, γ_n might indeed be a better candidate for a faithful representation of P_n, which would prove that P_n (and hence B_n) is linear. Building on results of [10], Lipschutz [8, Sect. 5] showed that the faithfulness of the Gassner representation γ_n follows from the injectivity of γ_{n-1} in conjunction with the injectivity of γ_n restricted to R_{n-1}. Hence, once we knew that all $R's$ were faithfully represented by the corresponding $\gamma' s$, the faithfulness of γ_n would

[5] Cf. also [8].

follow inductively for all $n \geq 2$, starting from the trivial case $n = 2$. Since R_{n-1} is free of rank $n-1$ and finitely generated free groups are hopfian, a proof of the faithfulness of γ_n reduces to showing that $\{\gamma_n(\tau_1), \ldots, \gamma_n(\tau_{n-1})\}$, a set of $n-1$ explicitly given matrices with entries in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, generates a free group of rank $n-1$.

3. A counter–example to Bachmuth's Theorem C

The representation of R_{n-1} considered by Bachmuth results from Lipschutz' variant of the Gassner representation restricted to R_{n-1}[6] by introducing the variables $x_\ell := t_{\ell+1}$ ($\ell = 1, \ldots, n-1$) and $t := t_1^{-1}$, reducing the representation by cutting off the first row and column, and transposing the resulting matrices. Applying the latter representation to the generators $\tau_1, \ldots, \tau_{n-1}$ of R_{n-1} yields Bachmuth's $((n-1) \times (n-1))$–matrices $\mathcal{M}_1, \ldots, \mathcal{M}_{n-1}$. One easily checks that $\gamma_n(\tau_1), \ldots, \gamma_n(\tau_{n-1})$ generate a free group of rank $n-1$ if and only if the same holds for $\mathcal{M}_1, \ldots, \mathcal{M}_{n-1}$. Hence, as explained in Section 2, it would suffice to show that the group $G(n,t)$ generated by the matrices $\mathcal{M}_1, \ldots, \mathcal{M}_n \in GL_n\big(\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, t^{\pm 1}]\big)$ is free of rank n to ensure faithfulness of the Gassner representation γ_{n+1} for all $n \geq 1$. This is what Bachmuth claims to prove. Setting $t = 1$ in the matrices $\mathcal{M}_i$, we obtain the matrices

$$M_i = \begin{array}{cc} \begin{pmatrix} x_i & & & & & & & \\ & \ddots & & & & & & \\ & & x_i & & & & & \\ 1-x_1 & \cdots & 1-x_{i-1} & 1 & 1-x_{i+1} & \cdots & 1-x_n \\ & & & & x_i & & \\ & & & & & \ddots & \\ & & & & & & x_i \end{pmatrix} & \begin{array}{c} \\ \\ \\ i^{\text{th}} \text{ row} \\ \\ \\ \\ \end{array} \end{array}$$

which generate a group $G(n,1)$. Similarly, letting $x_1 = x_2 = \ldots = x_n = 1$, the matrices $\mathcal{M}_1, \ldots, \mathcal{M}_n$ reduce to $T_1 = I$ (the identity matrix) and

$$T_i = \begin{array}{cc} \begin{pmatrix} t & & & & & & \\ & \ddots & & & & & \\ & & t & & & & \\ 1-t & \cdots & 1-t & 1 & 0 & \cdots & 0 \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} & \begin{array}{c} \\ \\ \\ i^{\text{th}} \text{ row} \quad (i = 2, \ldots, n), \\ \\ \\ \end{array} \end{array}$$

which generate a free abelian group $\mathcal{T}(n)$ of rank $n-1$ with basis $T_2, \ldots, T_n$. We have

$$\mathcal{M}_i = M_i T_i \quad (i = 1, \ldots, n). \tag{1}$$

Define a vector

$$\underline{v} = (1-x_1, 1-x_2, \ldots, 1-x_n).$$

[6]Cf. [8, Sects. 2 and 3]. Note that the second formula in [8, (3.3)] is used in order to define the representation on R_{n-1}.

Obviously, we have $\underline{v}\, m = \underline{v}$ for every $m \in G(n,1)$. Moreover, as Bachmuth observes (correctly), each such element $m \in G(n,1)$ has a (unique) decomposition

$$m = D(u) + \begin{pmatrix} \lambda_1 \underline{v} \\ \lambda_2 \underline{v} \\ \vdots \\ \lambda_n \underline{v} \end{pmatrix}, \tag{2}$$

where u is a unit in $\mathbb{Z}[x_1^{\pm 1}, \dots , x_n^{\pm 1}]$, $D(u) = u\, I$, and $\sum \lambda_i\, (1 - x_i) = 1 - u$. The assignment $m \mapsto u$ defines a homomorphism $u : G(n,1) \to \mathbb{Z}[x_1^{\pm 1}, \dots , x_n^{\pm 1}]^*$.

By (1) each element in $G(n,t)$ can be written as an alternating product

$$m_1\, t_1\, m_2\, t_2 \dots m_k\, t_k \tag{3}$$

with $m_j \in G(n,1)$ and $t_j \in \mathcal{T}(n)$. Bachmuth attempts to prove that every non–trivial freely reduced word in the $\mathcal{M}_i$ is not the identity by trying to show that such a word cannot leave the vector $\underline{v}$ invariant. Even more, Bachmuth claims that any non–empty reduced alternating product (3) does not leave $\underline{v}$ invariant. Clearly, it is of crucial importance how the notion “reduced” is defined in this context. Bachmuth never offers a precise definition; however, every restriction on these products explicitly stated (and used) by Bachmuth is contained in the following.

Definition 1. *A non–empty alternating product $m_1 t_1 \dots m_k t_k$ with $m_j \in G(n,1)$ and $t_j \in \mathcal{T}(n)$ is called reduced, if $u(m_j) \neq 1$ and $t_j \neq I$ for all $j = 1, \dots , k$.*

Bachmuth's main theorem is as follows.

Theorem C. *If an alternating product $m_1 t_1 \dots m_k t_k$ with $m_j \in G(n,1)$ and $t_j \in \mathcal{T}(n)$ satisfies $\underline{v}\, m_1 t_1 \dots m_k t_k = \underline{v}$, then $m_1 t_1 \dots m_k t_k$ is not reduced.*

As the following example shows, this assertion is false, even when adopting the above definition of reducedness (which may be stronger than what Bachmuth might have had in mind).

Example 1. *We have*

$$M_n^{-2}\, T_n^{-1}\, M_n\, T_n\, M_n\, T_n^{-1}\, M_n^{-2}\, T_n\, M_n\, T_n^{-1}\, M_n\, T_n = I. \tag{4}$$

Proof. Denote the left–hand side of (4) by W. We first show that $\underline{v}\, W = \underline{v}$. Given integers $a_1, \dots , a_k$ and $\ell_1, \dots , \ell_k$, we obtain

$$\underline{v}\, M_n^{\ell_1}\, T_n^{a_1} \dots M_n^{\ell_j}\, T_n^{a_j} = \underline{v} + \gamma_j\, \underline{x} \quad \text{for} \quad 1 \leq j \leq k,$$

where

$$\underline{x} := (x_n - x_1, x_n - x_2, \dots , x_n - x_{n-1}, 0),$$

and

$$\gamma_j := (t^{a_j} - 1) + t^{a_j}\, x_n^{\ell_j}\, (t^{a_{j-1}} - 1) + \dots + t^{a_j + \dots + a_2}\, x_n^{\ell_j + \dots + \ell_2}\, (t^{a_1} - 1).$$

This is easily verified by induction on j, using the formulas $\underline{v}\, M_n^\ell = \underline{v}$, $\underline{x}\, M_n^\ell = x_n^\ell\, \underline{x}$, $\underline{v}\, T_n^a = \underline{v} + (t^a - 1)\, \underline{x}$, and $\underline{x}\, T_n^a = t^a\, \underline{x}$. In our example we have $k = 6$,

$$(a_1, a_2, \dots , a_6) = (-1, 1, -1, 1, -1, 1),$$

and

$$(\ell_1, \ell_2, \dots, \ell_6) = (-2, 1, 1, -2, 1, 1),$$

implying $\gamma_6 = 0$.

Denote by V the group of matrices of the form

$$\begin{pmatrix} * & & & & \\ & \ddots & & 0 & \\ 0 & & \ddots & & \\ & & & * & \\ * & \cdots & \cdots & * & 1 \end{pmatrix}$$

with entries in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, t^{\pm 1}]$ and units on the diagonal. We have $W \in V$ since $M_n, T_n \in V$. More specifically, because of $\ell_1 + \dots + \ell_6 = 0 = a_1 + \dots + a_6$,

$$W = \begin{pmatrix} 1 & & & \\ & \ddots & 0 & \\ 0 & & \ddots & \\ \omega_1 & \cdots & \omega_{n-1} & 1 \end{pmatrix}.$$

The fact that $\underline{v}\, W = \underline{v}$ now gives $\omega_1 = \ldots = \omega_{n-1} = 0$, and hence $W = I$. □

Remark 1. *Since M_n, T_2, T_3, ... , T_n are lower triangular matrices, the group $\langle M_n, \mathcal{T}(n)\rangle$ is solvable; in particular there have to exist non–trivial relations between M_n and T_n, of which* (4) *is just one example. In fact, there exists a multitude of further relations between elements of $G(n,1)$ and $\mathcal{T}(n)$; for example, the group $\langle M_i, T_i\rangle$ is solvable for every $2 \le i \le n$. Bachmuth's assertion* [3, pp. 60–61] *that such relations can only exist if the m_j are allowed to be in the commutator subgroup of $G(n,1)$ is false. In* [3, pp. 59–60, Remark (2)] *Bachmuth sketches an inductive "proof" that any reduced alternating product $m_1t_1 \dots m_kt_k$ is different from λI for every $\lambda \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The absurdity of this assertion is sufficiently illuminated by our Example* 1 *and the above observations.*

4. Analysis of Bachmuth's arguments

The fact that Bachmuth's Theorem C is false would still leave open the possibility that some of his arguments might contribute to resolving the original question as to whether or not the Gassner representation is faithful. To demonstrate that this is not the case is the aim of this concluding section.

Bachmuth's starting point is the equation $\underline{v}\, W(\mathcal{M}_i) = \underline{v}$, which according to his claim leads to a contradiction for every non–empty freely reduced word W in the $\mathcal{M}_i$. Decomposing $W(\mathcal{M}_i)$ as an alternating product (3) gives an equation

$$\underline{v}\, m_1\, t_1\, m_2\, t_2 \,\dots\, m_k\, t_k = \underline{v}. \tag{5}$$

Bachmuth claims that even every such equation leads to a contradiction as long as the product $m_1t_1 \dots m_kt_k$ is reduced. That this is not the case was already demonstrated in Example 1. The question remains whether he can derive such a contradiction at least in the (crucial) case when $m_1 t_1 \dots m_k t_k = W(\mathcal{M}_i)$ in the group $\langle G(n,1), \mathcal{T}(n)\rangle$ with $W(\mathcal{M}_i)$ as above. Again, this is not the case. Rewriting $\underline{v}\, m_1\, t_1 \,\dots\, m_k\, t_k$ by means of the decomposition (2) as

$$\alpha_1\, \underline{v}\, t_1 \,\dots\, t_k \,+\, \alpha_2\, \underline{v}\, t_2 \,\dots\, t_k \,+\, \dots \,+\, \alpha_k\, \underline{v}\, t_k$$

with coefficients $\alpha_j \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, t^{\pm 1}]$, Bachmuth transforms (5) into the equation[7]

$$\alpha_1\, \underline{v}\, t_1 \dots t_k \,+\, \dots \,+\, \alpha_k\, \underline{v}\, t_k = \underline{v}. \tag{6}$$

In the rest of his paper Bachmuth derives from (6) various sets of conditions on the elements

$$\overline{\alpha}_i := \alpha_i\,|_{t=1} = \begin{cases} u_2 \,\dots\, u_k, & i = 1 \\ (1 - u_i)\, u_{i+1} \,\dots\, u_k, & i \geq 2, \end{cases}$$

where $u_j = u(m_j)$. In each case he then completes his argument by claiming that these respective conditions cannot occur. However, this is not true. An important point to note here is that one would at least have to bring the full complexity of the coefficients α_j in (6) into the picture instead of specializing them by setting $t = 1$ (as Bachmuth always does).

Example 2. *Consider the word*

$$\begin{aligned} W_0(\mathcal{M}_i) &= \mathcal{M}_1^{-2}\, \mathcal{M}_n^{-1}\, \mathcal{M}_1\, \mathcal{M}_n\, \mathcal{M}_1\, \mathcal{M}_n^{-1}\, \mathcal{M}_1^{-2}\, \mathcal{M}_n\, \mathcal{M}_1\, \mathcal{M}_n^{-1}\, \mathcal{M}_1\, \mathcal{M}_n \\ &= M_1^{-2}\, T_n^{-1}\, (M_n^{-1}\, M_1\, M_n)\, T_n\, M_1\, T_n^{-1}\, (M_n^{-1}\, M_1^{-2}\, M_n)\, T_n\, M_1\, T_n^{-1}\, (M_n^{-1}\, M_1\, M_n)\, T_n \\ &=: m_1\, t_1\, m_2\, t_2 \,\dots\, m_6\, t_6. \end{aligned}$$

Put $t_j = T_n^{a_j}$ *and* $u_j = u(m_j)$ *for* $j = 1, 2, \dots, 6$. *Then*

$$(a_1, a_2, \dots, a_6) = (-1, 1, -1, 1, -1, 1),$$

$$(u_1, u_2, \dots, u_6) = (x_1^{-2}, x_1, x_1, x_1^{-2}, x_1, x_1),$$

and

$$(\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_6) = (x_1^2, x_1 - x_1^2, 1 - x_1, x_1^2 - 1, x_1 - x_1^2, 1 - x_1).$$

W_0 is reduced in any natural sense; in particular it satisfies the restrictions of Definition 1. In what follows we shall contrast Example 2 with those passages of [3] in which Bachmuth claims to have reached a contradiction.

(i) In the situation of Example 2, Bachmuth's equation (9) (which actually concerns our $\overline{\alpha}$'s) is only relevant for the generator T_n, as $T_2, \dots, T_{n-1}$ do not occur in our t_j's. Bachmuth's argument then amounts to claiming that the two equations

$$\overline{\alpha}_1 \,+\, \overline{\alpha}_2 \,+\, \dots \,+\, \overline{\alpha}_k = 1 \tag{7}$$

$$(a_1 \,+\, \dots \,+\, a_k)\,\overline{\alpha}_1 \,+\, (a_2 \,+\, \dots \,+\, a_k)\,\overline{\alpha}_2 \,+\, \dots \,+\, a_k\,\overline{\alpha}_k = 0 \tag{8}$$

imply that $\overline{\alpha}_1 = 1$ and $\overline{\alpha}_2 = \dots = \overline{\alpha}_k = 0$. However, our example satisfies (7) and (8) with $\overline{\alpha}_1 - 1, \overline{\alpha}_2, \dots, \overline{\alpha}_6 \neq 0$. Bachmuth in the case $t_1 t_2 \dots t_k = I$ also creates $k-1$ new

[7] This is Bachmuth's equation (C1); cf. [3, pp. 53–54].

equations (which however are linearly dependent on the original equations (7) and (8)), from which he builds a $(k \times k)$–matrix T. He claims that T has rank k and bases his argument on the assumption that T contains a $((k-1) \times (k-1))$–submatrix of rank $k-1$. But in our example T becomes

$$\begin{pmatrix} a_1 & 0 & a_1+a_3+\cdots+a_k & \cdots & a_1+a_k \\ a_1+a_2 & a_2 & 0 & \cdots & a_1+a_2+a_k \\ \vdots & \vdots & \vdots & & \vdots \\ a_1+\cdots+a_{k-1} & a_2+\cdots+a_{k-1} & a_3+\cdots+a_{k-1} & \cdots & 0 \\ 0 & a_2+\cdots+a_k & a_3+\cdots+a_k & \cdots & a_k \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

which has rank 2. By the way, one can easily see that in general Bachmuth's matrix T has rank at most n.

(ii) Bachmuth's Remark (1) in [3, Sect. 3] boils down to another "proof" that the matrix equation

$$T \begin{pmatrix} \overline{\alpha}_1 - 1 \\ \overline{\alpha}_2 \\ \vdots \\ \overline{\alpha}_k \end{pmatrix} = \underline{0}$$

forces $\overline{\alpha}_1 - 1 = \overline{\alpha}_2 = \ldots = \overline{\alpha}_k = 0$. That this is not the case has already been observed in (i).

(iii) Bachmuth's Remark (2) in [3, pp. 59–60] concerning the inequality $m_1 t_1 \ldots m_k t_k \neq \lambda I$ has already bee discussed (and rejected) in the concluding remark of the previous section.

Remark 2. *By analyzing the situation described in Example* 2 *more carefully, we can in fact show that* $\underline{v} m_1 t_1 \ldots m_6 t_6 \neq \underline{v}$. *But we have demonstrated above that none of Bachmuth's "proofs" establishes this fact.*

References

[1] E. Artin, Theorie der Zöpfe, *Abh. Math. Sem. Univ. Hamburg* **4** (1926), 47–72.

[2] E. Artin, Theory of braids, *Ann. of Math.* (2) **48** (1947), 101–126.

[3] S. Bachmuth, Braid groups are linear groups, *Adv. in Math.* **121** (1996), 50–61.

[4] J. Birman, Review of Bachmuth's paper, *Math. Reviews* 98 h: 20061 (1998), 4897–4898.

[5] W. Burau, Über Zopfgruppen und gleichsinnig verdrillte Verkettungen, *Abh. Math. Sem. Univ. Hamburg* **11** (1937), 179–186.

[6] B. Gassner, On braid groups, *Abh. Math. Sem. Univ. Hamburg* **25** (1961), 10–22.

[7] N. Ivanov, *Subgroups of Teichmüller modular groups*, Translations of Mathematical Monographs Vol. 115, American Mathematical Society, Providence, Rhode Island, 1992.

[8] S. Lipschutz, On a finite matrix representation of the braid group, *Arch. Math.* **12** (1961), 7–12.

[9] D. Long and M. Paton, The Burau representation is not faithful for $n \geq 6$, *Topology* **32** (1993), 439–447.

[10] W. Magnus, Über Automorphismen von Fundamentalgruppen berandeter Flächen, *Math. Ann.* **109** (1934), 617–646.

[11] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory*, John Wiley & Sons, New York, 1966.

[12] J. Moody, The faithfulness question for the Burau representation, *Proc. Amer. Math. Soc.* **119** (1993), 671–679.

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