Double centralizer properties, dominant dimension and tilting modules

by

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Abstract

Double centralizer properties play a central role in many parts of algebraic Lie theory. Soergel's double centralizer theorem relates the principal block of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of a semisimple complex Lie algebra with the coinvariant algebra (i.e. the cohomology algebra of the corresponding flag manifold). Schur–Weyl duality relates the representation theories of general linear and symmetric groups in defining characteristic, or (via the quantized version) in non–defining characteristic. In this paper we exhibit algebraic structures behind these double centralizer properties. We show that the finite dimensional algebras relevant in this context have dominant dimension at least two with respect to some projective–injective or tilting modules. General arguments which combine methods from ring theory (QF–3 rings and dominant dimension) with tools from representation theory (approximations, tilting modules) then imply the validity of these double centralizer properties as well as new ones. In contrast to the traditional proofs (e.g. by the fundamental theorems of invariant theory) no computations are necessary.

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1 Introduction

The aim of this paper is to give unified and computation–free proofs for the following two theorems:

Theorem 1.1 (Soergel [21], 'Struktursatz 9') *Let $\mathfrak{g}$ be a semisimple complex Lie algebra and A the finite dimensional algebra of some block of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$. Then there is an indecomposable projective–injective A–module Ae with local endomorphism ring eAe such that there is the following double centralizer property:*

$$A \simeq End(Ae_{eAe})$$

The local algebra eAe has a combinatorial description (see [21], 'Endomorphismensatz 7') in terms of invariant theory. In the most important case, that of the principal block, eAe actually equals the coinvariant algebra which is isomorphic to the cohomology algebra of the associated flag manifold. The structure of Ae as an eAe–module is combinatorially described in [21]. Therefore, the theorem states that the whole algebra A can be recovered from combinatorial data.

Theorem 1.2 (Schur, Green, Jimbo, Dipper–James, Du–Parshall–Scott, e.a.) ('Schur–Weyl duality') *Let n and r be two natural numbers and k an infinite field of any characteristic. Let the general linear group $GL_n(k)$ act diagonally from the left on $E = (k^n)^{\otimes r}$ (with natural action on k^n) and let the symmetric group Σ_r act from the right by place permutations. Denote by $S_k(n,r)$ the algebra generated by the image of the GL_n–action (the 'Schur algebra').*
(a) Suppose $n \geq r$. Then there is a double centralizer property

$$S_k(n,r) \simeq End_{k\Sigma_r}(E)$$

$$k\Sigma_r \simeq End_{S_k(n,r)}(E)$$

(b) Suppose $n < r$. Denote by B the quotient of $k\Sigma_r$ modulo the kernel of the action of $k\Sigma_r$ on E. Then there is a double centralizer property

$$S_k(n,r) \simeq End_B(E)$$

$$B \simeq End_{S_k(n,r)}(E)$$

(c) Parts (a) and (b) remain true if one replaces the Schur algebra $S_k(n,r)$ by the quantized Schur algebra $S_q(n,r)$ and the group algebra $k\Sigma_r$ of the symmetric group by the Hecke algebra $\mathcal{H}_q(r)$ $(q \neq 0)$ of type A.

Schur proved this result in characteristic zero in order to relate the representation theories of the general linear and the symmetric groups. There are various proofs available for the general assertion; in particular, it can be seen as a reformulation of the first and the second fundamental theorem of invariant theory for general linear groups (see [5] for a characteristic free approach). The quantized version is due in the 'generic case' independently to Jimbo [19] and Dipper and James [7] and in full generality to Du, Parshall and Scott [11] (and at roots of unity to Du [10]).

Our main observation is that both the blocks of category $\mathcal{O}$ appearing in Soergel's result and the (classical or quantized) Schur algebras with $n \geq r$ have dominant dimension at least two. This means that such an algebra A is the first term in an exact sequence $0 \to A \to T_1 \to T_2$ where both T_1 and T_2 are projective–injective A–modules. In the case of category $\mathcal{O}$ this follows from well–known elementary properties; in the other case, this observation is almost trivial.

For rings of dominant dimension at least two, Tachikawa and others developped about 25 years ago a general theory focussing on examples rather far away from algebraic Lie theory. However, the technology of their theory fits quite well to the examples we are interested in, and we derive in a short and elementary way a criterion for double centralizer properties which implies the first theorem and the $n \geq r$–part of the second one.

In the case $n < r$ this approach cannot work; in fact, $S_k(n,r)$ need not have a faithful projective–injective module. Therefore, we have to generalize our technology appropriately to handle this case as well. It turns out that the right way to do that is as follows. Instead of projective–injective modules we will use (partial) tilting modules, that is, self–dual modules which are filtered by Weyl modules. Again we use an exact sequence $0 \to A \xrightarrow{\delta} T_1 \to T_2$, but now T_1 and T_2 are tilting modules. We have to require that the map δ is an $add(T_1)$–approximation in the sense of Auslander and Reiten [1]. This condition is automatic in the above case of projective–injective modules, whereas it is a (necessary) restriction in the more general situation.

Altogether, we have the following result:

Theorem 1.3 *(a) A block of category $\mathcal{O}$ has dominant dimension at least two.*
(b) A (classical or quantized) Schur algebra $S(n,r)$ with $n \geq r$ has dominant dimension at least two.
(c) For a (classical or quantized) Schur algebra $S(n,r)$ with $n < r$ there is a tilting module T such that $S(n,r)$ has T–dominant dimension at least two.

In each of the situations of the theorem, our criterion 2.8 then implies the validity of a double centralizer property. This criterion relates a resolution from the definition of dominant dimension with an approximation property and a description of the double centralizer as a left module which is given in Theorem 2.7.

In the case of Schur–Weyl duality we get a stronger result than just the existence of a double centralizer property (with a Schur algebra on the left hand side and an endomorphism ring on the right hand side). In fact, the right hand side can be identified with a quotient of the group algebra of a symmetric group. This algebra obviously is contained in the endomorphism ring, and the problem is to prove that the endomorphism ring cannot be larger. Our results give an 'upper bound' for the endomorphism ring in question. In the case of category $\mathcal{O}$, the dimension of the endomorphism ring is clear, and unfortunately our methods do not yield any additional information about the endomorphism ring appearing on the right hand side of the double centralizer property.

Our proof of the double centralizer property for category $\mathcal{O}$ immediately carries over to another situation studied in [13, 14, 15]. There a generalized category $\mathcal{O}$ has been investigated where generalized Verma modules are produced by inducing up infinite dimensional modules over a parabolic subalgebra of the given Lie algebra. Our method reproves Theorem B in [14] and one assertion in the more general Main Theorem in [15] (both stating the validity of a double centralizer property). The finite dimensional associative algebras arising in this situation are not quasi–hereditary any more, but our method still works.

Here is an outline of this paper. Section two collects definitions and facts from ring theory, in particular on QF–3 rings, dominant dimension and approximations. Theorem 2.7 describes the double centralizer algebra as a certain submodule. From this we deduce Theorem 2.8 which is our tool for proving double centralizer properties. In the case of projective–injective modules this specializes to Corollary 2.10; in the case of tilting modules we get Theorem 2.15. In order to pass information from the case $n \geq r$ in

Schur–Weyl duality to the case $n < r$ we also need to compare endomorphisms of certain tilting modules. This is done by Theorem 2.16.

Section three discusses the examples. First, in 3.1 we show that blocks of category $\mathcal{O}$ have dominant dimension at least two, which implies Soergel's result and also Theorem B in [14] and part of Theorem 1 in [15]. Then, in 3.2 we explain how (classical or quantized) Schur–Weyl duality for $n \geq r$ can be demonstrated in the same way. In section 3.3 we settle the case $n < r$. Section 3.4 explains how by the same method we get in fact a large supply of new double centralizer properties from the classical situations. Schur–Weyl duality as stated above then becomes just an example of a whole series of double centralizer properties.

2 Techniques

In this section we collect the necessary definitions and results from ring theory and from representation theory, and we derive our tools for verifying double centralizer properties.

Throughout, our algebras are finitely generated and free over some commutative noetherian domain k and they are associative with unit. Modules are finitely generated, and left modules unless stated otherwise. A k–space is a finitely generated free k–module of finite rank. (In the applications, k is a field.) By $add(M)$ of a module M we mean the full subcategory of direct summands of finite direct sums of M.

The first and the second subsection follow quite closely the classical theory of dominant dimension and QF–3 rings, as developped by Tachikawa and others (see [23, 22, 16]).

2.1 Dominant dimension and QF–3 rings

Definition 2.1 *Let A be an algebra and T an A–module. Let X be an A–module. Then the* **dominant dimension of X relative to T**, *$T - domdim(X)$, is the supremum of all $n \in \mathbb{N}$ such that there exists an exact sequence*

$$0 \to X \to T_1 \to T_2 \to \ldots \to T_n$$

with all T_i in $add(T)$.
If T is a faithful projective–injective module, then the symbol T is dropped in the notation. The left dominant dimension of the algebra itself is defined to be $domdim({}_AA)$.

Definition 2.2 *The algebra A is* **left QF–3** *if and only if there exists a faithful projective–injective A–module.*

QF–3 rings have been introduced by R.M.Thrall as one of several generalizations of quasi–Frobenius rings.

Lemma 2.3 *Let X be a faithful projective–injective A–module. Then every indecomposable projective–injective A–module P is isomorphic to a direct summand of X. In particular, all endomorphism rings of faithful projective–injective modules are Morita equivalent.*

Proof. Since X is faithful, there is an inclusion $0 \to A \to X^n$ for some $n \in \mathbb{N}$ which implies the statement. ∎

For a QF–3 algebra we do not have to distinguish between the various kinds of dominant dimensions relative to projective–injective modules:

Proposition 2.4 ([23], 7.7) *Let A be left QF–3 with minimal faithful left ideal Ae. Then the left dominant dimension of A coincides with its dominant dimension relative to Ae. Moreover, the left dominant dimension coincides with the right dominant dimension.*

2.2 Approximations and double centralizers

Definition 2.5 *Let A be an algebra and let $\mathcal{C}$ be a subcategory of A–mod. Let M be an A–module. Then a homomorphism $f : M \to C$ is called a* **left $\mathcal{C}$–approximation** *of M if and only if C is an object of $\mathcal{C}$ and the induced morphism $Hom(f,-) : Hom_A(C,D) \to Hom_A(M,D)$ is an epimorphism for all objects D in $\mathcal{C}$.*

We will concentrate on the special case of $\mathcal{C}$ being the category $add(T)$ for a faithful module T.

If A is a finite dimensional algebra, then the existence of $add(T)$–approximations is easy to establish in the following way: Let M be the module to be approximated. Choose a basis $f_1, \dots, f_n$ of the k–space $Hom_A(M,T)$. Then any map $M \to T$ factors through $M \stackrel{(f_1,\dots,f_n)}{\to} T^n$, which thus is a left $add(T)$–approximation of M.

Note that such an argument forces us to replace T by some T^n. This does not cause a problem for the double centralizer properties since their validity is in fact Morita invariant:

Proposition 2.6 (Tachikawa, [23] 10.1) *Let A and B be algebras and suppose there is a Morita equivalence $F : A-mod \to B-mod$ sending a module ${}_AM$ to ${}_BN$. If there is a double centralizer property on ${}_AM$, i.e. the canonical map $A \to End(M_{End_A(M)})$ is surjective, then there is also a double centralizer property on ${}_BN$.*

The following result is a direct generalization of a known fact for the case of projective–injective modules (see [16], proof of Theorem 2.29). We will need only a part of it, namely the claim that the double centralizer C is contained in the module Q. That is, there is a chain of A–modules $A \subset Q \subset T$.

Theorem 2.7 *Let A be an algebra and T an A–module. Suppose there exists an injective left $add(T)$–approximation $0 \to A \stackrel{\delta}{\to} T$. Denote by B the centralizer algebra $B := End({}_AT)$ and by C the double centralizer $C := End(T_B)$. Then C can be identified (as an A–module) with a subspace of T as follows:*

$$C \simeq Q := \bigcap_{f \in B, f(A)=0} ker(f)$$

Proof. We regard A as a subset of T via δ (which we notationally treat as an inclusion). By definition, Q is an A–submodule of T. Denote by B_0 the subspace of B consisting of maps which are zero on A.

By assumption, T is a faithful A–module. Thus A is contained in the double centralizer C. There is a left B– and right C– (hence also right A–) bimodule structure on T. We write maps on appropriate sides to distinguish these two structures.

We define two k–linear maps $\beta : B \to T$ with $b \mapsto b(1)$ and $\gamma : C \to T$ with $c \mapsto (1)c$.

Claim: β is surjective. In fact, choose an element $t \in T$. There is a homomorphism $\varphi : A \to T$ sending 1 to $\varphi(1) = t$. Since $\delta : A \to T$ is an approximation, φ factors through T, thus defining an element $b \in B$ which sends 1 to t.

Claim: γ is injective. In fact, pick an element $c \in C$ with $c \neq 0$. Its γ–image is $(1)c$. Since c is not zero, there exists an element $t \in T$ such that $(t)c \neq 0$. By the previous claim we can write $t = \beta(b)$ for some $b \in B$. Using the bimodule structure of T we have: $0 \neq (t)c = (\beta(b))c = (b(1))c = b((1)c)$, hence $(1)c$ cannot be zero.

Claim: The image of γ is contained in Q. In fact, let $f \in B$ be an endomorphism which is zero on A. Choose $c \in C$. We have to show $\gamma(c) = (1)c$ is in the kernel of f. Again using the bimodule structure on T we have $f((1)c) = (f(1))c$ which is zero because of $1 \in A$.

Thus γ identifies C with a subspace of Q. Clearly, γ preserves the left A–module structure.

It remains to show that the image of γ is all of Q. Pick an element $q \in Q$. Define a linear map $\psi : B \to T$ by sending b to $(b)\psi = b(q)$. By construction, ψ sends B_0 to zero. But B_0 is the kernel of the epimorphism β. Hence there is a factorization $\mu : T \to T$ with $\psi = \beta\mu$, that is, $(b(1))\mu = b(q)$ for all $b \in B$. Clearly, μ is an element of C. Denote by id the identity map on T. Then $\gamma(\mu) = (1)\mu = (id(1))\mu = id(q) = q$. Therefore, q is in the image of γ. ∎

Now the T–dominant dimension of A comes into play, in a way familiar from Tachikawa's theory [23].

Theorem 2.8 *Let A be an algebra and T an A–module. Suppose there exists an injective left $add(T)$–approximation $0 \to A \xrightarrow{\delta} T$ which can be continued to an exact sequence $0 \to A \xrightarrow{\delta} T \xrightarrow{\epsilon} T^n$ for some $n \in \mathbb{N}$ (this implies that A has T–dominant dimension at least two). Denote by B the centralizer algebra $B := End({}_AT)$ and by C the double centralizer $C := End(T_B)$. Then C equals A.*

Proof. This is an application of Theorem 2.7. Denote by p_i for $i = 1, \ldots, n$ the projection of T^n onto the i–th component. Then ϵp_i is in B and A can be written as intersection of the kernels of all the ϵp_i. ∎

The two conditions in the theorem (left $add(T)$–approximation and T–dominant dimension at least two) are also necessary for the natural morphism from A to C to be an isomorphism. This follows from results of Auslander and Solberg (see [2], 2.1) which also provide an alternative proof of Theorem 2.8.

To check the property of a given δ to be an $add(T)$–approximation might be hard. In all our applications we check a stronger property, namely that of being an approximation with respect to a much larger category, which is either the category of all modules (and then the approximation is just an injective hull) or that of modules having a certain filtration.

Our previous discussion (at the beginning of this subsection) implies the following: Suppose we are given a faithful T. Then we can always modify the situation (without loosing anything with respect to the double centralizer property aimed at) in such a way that we have an approximation, i.e. the first condition in Theorem 2.8 is satisfied. The real difficulty lies in checking the second condition, that is, in embedding the cokernel of δ into some object from $add(T)$. If T is projective–injective, then the obstruction for doing this can be formulated in terms of cohomology as follows.

Proposition 2.9 *Let M be an A–module and T a projective–injective A–module. Call a simple A–module of the first kind if it occurs (up to isomorphism) in the socle of T, and of the second kind otherwise. Then there exists an exact sequence $0 \to A \xrightarrow{\delta} T \xrightarrow{\epsilon} T^n$ for some $n \in \mathbb{N}$ if and only if the following two conditions are satisfied:*
(1) All simple modules in the socle of A are of the first kind.
(2) For all simple modules L of the second kind, $Ext^1_A(L, A)$ vanishes.

If T is not projective–injective, the obstruction still can be formulated in a similar way, using some relative cohomology. However, for the applications we have in mind, this does not help.

2.3 Double centralizer properties on projective–injective modules

At first, we discuss the easiest case of T being a faithful projective–injective module. Combining Theorem 2.8 with Lemma 2.3 and Proposition 2.4 we get back the following result equivalent to a theorem of Tachikawa.

Theorem 2.10 (see [23], (7.1)) *Let A be left QF–3 with minimal faithful left ideal Ae. Then the following two assertions are equivalent:*
(1) $Ae - domdim(A) \geq 2$
(2) There holds a double centralizer property $A = End(Ae_{eAe})$.

An easier (and known) special case is the following:

Corollary 2.11 *Let A be a self–injective algebra, M a faithful A–module and $B := End({}_AM)$. Then there is a double centralizer property $A = End(M_B)$.*

Proof. Since A is self–injective and M is faithful we get $A \in add(M)$. If I is an injective A–module, then $Hom_A(M, I)$ is projective and injective as left B–module.

Choose an injective resolution of M, say $0 \to {}_AM \to I_0 \to I_1 \to \dots$. Applying $Hom_A(M, -)$ yields an exact sequence of B–modules

$$0 \to Hom_A(M, M) \to Hom_A(M, I_0) \to Hom_A(M, I_1)$$

with $Hom_A(M, I_0)$ and $Hom_A(M, I_1)$ both being projective and injective over B. Thus B has dominant dimension at least two, and Theorem 2.10 applies. ■

Another proof of this corollary can be found in [6], 59.6. We remark that some of the examples considered below fit into this context. However, in the case of category $\mathcal{O}$ an application of Corollary 2.11 would not yield the desired result, since it does not tell us anything about the algebra B. Therefore we need the more sophisticated criterion 2.10.

2.4 Double centralizer properties on tilting modules

Now we proceed to a more general criterion. This is based on the notion of tilting modules for quasi–hereditary algebras as developped by Ringel [20] (based on earlier work of Auslander and Reiten [1]).

Definition 2.12 (Cline, Parshall and Scott, [4]) *Let A be a finite dimensional algebra over a field, and Λ the set of isomorphism classes of simple A–modules. Choose representatives $L(\lambda)$ of the elements of Λ. Let $\leq$ be a partial order on I. Then $(A, \leq)$ is called* **quasi–hereditary** *if and only if the following assertions are true:*

(a) For each $\lambda \in \Lambda$, there exists a finite dimensional A–module $\Delta(\lambda)$ with an epimorphism $\Delta(\lambda) \to L(\lambda)$ such that the composition factors $L(\mu)$ of the kernel satisfy $\mu < \lambda$.

(b) For each $\lambda \in \Lambda$, a projective cover $P(\lambda)$ of $L(\lambda)$ maps onto $\Delta(\lambda)$ such that the kernel has a finite filtration with factors $\Delta(\mu)$ satisfying $\mu > \lambda$.

The module $\Delta(i)$ is called **standard module** of index i. Injective A–modules are filtered by (costandard) modules $\nabla(i)$ (which are dual to the standard modules of the quasi–hereditary algebra $(A^{op}, \leq)$).

By $\mathcal{F}(\Delta)$ we denote the full subcategory of A–mod which has objects filtered by standard modules. By $\mathcal{F}(\nabla)$ we denote the full subcategory of A–mod which has objects filtered by costandard modules.

The two categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are orthogonal to each other, in the following sense. Let X be in $\mathcal{F}(\Delta)$ and Y in $\mathcal{F}(\nabla)$. Then $Ext^i_A(X, Y)$ vanishes for all $i > 0$.

Theorem 2.13 (Ringel, [20]) *Let $(A, \leq)$ be a quasi–hereditary algebra as above. Then, for each $\lambda \in \Lambda$, there is a unique (up to isomorphism) indecomposable module $T(\lambda)$ which has both a filtration with subquotients of the form $\Delta(\mu)$ (for $\mu \leq \lambda$ and $\Delta(\lambda)$ itself occuring with multiplicity one) and another filtration with subquotients of the form $\nabla(\lambda)$ (for $\mu \leq \lambda$ and $\nabla(\lambda)$ itself occuring with multiplicity one).*

The module $T(\lambda)$ is characterized by its 'highest weight' λ; that is, it is unique among the $T(\mu)$ with the property that $[T(\lambda) : L(\lambda)] = 1$ and $[T(\lambda) : L(\nu)] = 0$ for $\nu \not\leq \lambda$.

In the typical applications, $A - mod$ has always a duality fixing simple modules. Then $T(\lambda)$ is characterized by the following three properties; it is indecomposable, it is self–dual, and it has a filtration by standard modules where λ is the largest occuring index.

In [20] the direct sum $T := \oplus_{\lambda \in \Lambda} T(\lambda)$ is called the **characteristic tilting module** of $(A, \leq)$. Slightly abusing language, the indecomposable modules $T(\lambda)$ or any sum of those, nowadays are just called 'tilting modules'. A **full tilting module** is one which contains at least one direct summand from each isomorphism class of indecomposable tilting modules.

Projective–injective modules are examples of tilting modules in this sense.

A full tilting module is a 'generalized tilting module' in the sense of representation theory of finite dimensional algebras: it has finite projective dimension, it admits no self–extensions, and it cogenerates the algebra.

The tilting module $T(\lambda)$ comes with two important exact sequences which relate it to the theory of left and right approximations (developped by Auslander and Reiten, see [1]):

$$0 \to \Delta(\lambda) \to T(\lambda) \to C \to 0,$$

where C has a filtration by standard modules, and

$$0 \to K \to T(\lambda) \to \nabla(\lambda) \to 0,$$

where K has a filtration by costandard modules. Each sequence can be used as a starting point for constructing $T(\lambda)$ by iterated universal extensions (see [20]).

The module $T(\lambda)$ is the left $\mathcal{F}(\nabla)$–approximation of $\Delta(\lambda)$ and the right $\mathcal{F}(\Delta)$–approximation of $\nabla(\lambda)$. This follows from the exact sequences above together with the following well–known criterion:

Lemma 2.14 *Let $(A, \leq)$ be quasi–hereditary. Let M be in $\mathcal{F}(\Delta)$ and let X be both in $\mathcal{F}(\Delta)$ and in $\mathcal{F}(\nabla)$. Then $0 \to M \to X$ is a left $\mathcal{F}(\nabla)$–approximation of M if and only if the cokernel is in $\mathcal{F}(\Delta)$.*

Proof. Denote the cokernel of $M \to X$ by Y. Pick a module Z having a costandard filtration. Applying $Hom_A(-, Z)$ to the exact sequence $0 \to M \to X \to Y \to 0$ yields a long exact cohomology $0 \to Hom_A(Y,Z) \to Hom_A(X,Z) \to Hom_A(M,Z) \to Ext^1_A(Y,Z) \to Ext^1_A(X,Z) \to \ldots$. By orthogonality of $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$, the last term always is zero. Moreover, the space $Ext^1_A(Y,Z)$ is zero for all Z if and only if Y is in $\mathcal{F}(\Delta)$. Vanishing of $Ext^1_A(Y,Z)$ is equivalent to surjectivity of $Hom_A(X,Z) \to Hom_A(M,Z)$ which is equivalent to the lifting property required in the definition of approximation. ∎

Combining the Lemma 2.14 with Theorem 2.8 we get a general criterion:

Theorem 2.15 *Let $(A, \leq)$ be quasi–hereditary and T a (not necessarily full) tilting module. Suppose there is an exact sequence $0 \to A \xrightarrow{\delta} T \to T'$ with $T' \in add(T)$ and $coker(\delta) \in \mathcal{F}(\Delta)$. Then there is a double centralizer property*

$$A = End(T_{End_A(T)})$$

2.5 Endomorphisms of tilting modules

The main result of this subsection relates the endomorphism ring of an A–tilting module T with the endomorphism ring of the eAe–tilting module eT for a suitable idempotent e. Applying this to Schur–Weyl duality will allow us to compare the cases $n \geq r$ and $n < r$.

For simplicity, we assume now that k is a splitting field for A.

Let T be a tilting module, that is, T has a filtration by standard modules and another filtration by costandard modules. Then there are two short exact sequences

$$0 \to \Delta(\lambda)^{n_1} \to T \to C(\lambda) \to 0$$

$$0 \to K(\mu) \to T \to \nabla(\mu)^{m_1} \to 0$$

where the first cokernel $C(\lambda)$ has a Δ–filtration with all occuring $\Delta(\nu)$ satisfying $\nu < \lambda$ and the second kernel $K(\lambda)$ has a ∇–filtration with all occuring $\nabla(\nu)$ satisfying $\nu < \mu$. Considering indecomposable direct summands of T we get $\lambda = \mu$. We denote the algebra $End(_AT)$ by E.

We denote by $\lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_n$ the set of indices of all Δ's or ∇'s occuring in the above filtrations of T. Note that we do not suppose the existence of a duality, thus the index set for the Δ's need not be the same as that for the ∇'s. We are using here the union of the two index sets. These filtrations can be split into series of short exact sequences:

$$0 \to \Delta(\lambda_1)^{n_1} \to T(\lambda) \to C(\lambda_1) \to 0$$

$$0 \to \Delta(\lambda_2)^{n_2} \to C(\lambda_1) \to C(\lambda_2) \to 0$$

$$0 \to \Delta(\lambda_3)^{n_3} \to C(\lambda_2) \to C(\lambda_3) \to 0$$

and so on (which also serve as definitions of the terms occuring therein). (Note that some of these sequences may be trivial, since $\Delta(\lambda_l)$ need not occur for some l, i.e. n_l can be zero.) Similar sequences exist for the ∇'s.

Now we apply Hom–functors to produce a filtration of E. Here we use that $Ext^i(\mathcal{F}(\Delta), \mathcal{F}(\nabla))$ is zero for all $i \geq 1$, thus various extension groups vanish. We also use that $Hom(\Delta(\mu), \nabla(\nu))$ is non–zero only if $\mu = \nu$, which by the Ext–vanishing can be generalized to a similar statement about homomorphisms from certain objects in $\mathcal{F}(\Delta)$ to objects in $\mathcal{F}(\nabla)$.

Applying $Hom(T, -)$ to the first sequence induced by the ∇–filtration gives:

$$0 \to Hom(T, K(\lambda_1)) \to E \to Hom(T, \nabla(\lambda_1)^{m_1}) \to 0.$$

Applying $Hom(-, K(\lambda_1))$ to the first sequence induced by the Δ–filtration gives:

$$0 \to Hom(C(\lambda_1), K(\lambda_1)) \to Hom(T, K(\lambda_1)) \to Hom(\Delta(\lambda_1)^{n_1}, K(\lambda_1)) \to 0,$$

where the last term is zero, since $\nabla(\lambda_1)$ does not appear in the filtration of $K(\lambda_1)$. Hence the first map must be an isomorphism. Similarly, we can identify $Hom(T, \nabla(\lambda_1)^{m_1})$ with $Hom(\Delta(\lambda_1)^{n_1}, \nabla(\lambda_1)^{m_1})$ which equals $k^{n_1 \times m_1}$. Hence we have

$$0 \to Hom(C(\lambda_1), K(\lambda_1)) \to E \to k^{n_1 \times m_1} \to 0$$

and the first term actually is an E–bimodule because it is isomorphic to $Hom(T, K(\lambda_1))$ and also to $Hom(C(\lambda_1), T)$. Thus we have found a maximal ideal of E. It consists of those endomorphisms of T which can be factored $T \to C(\lambda_1) \to K(\lambda_1) \to T$. Now we continue inductively to filter E. The next step (by analogous arguments) is

$$0 \to Hom(C(\lambda_2), K(\lambda_2)) \to Hom(C(\lambda_1), K(\lambda_1)) \to k^{n_2 \times m_2} \to 0$$

where $n_2 = [T : \Delta(\lambda_2)]$ and $m_2 = [T : \nabla(\lambda_2)]$ (these multiplicities may be zero). As before this is a sequence of E–bimodules.

Inductively, we get a filtration

$$Hom_A(C(\lambda_n), K(\lambda_n)) \subset \ldots \subset Hom_A(C(\lambda_1), K(\lambda_1)) \subset E$$

of E which we use for constructing a special k–**basis** of E as follows. A basis element is obtained by choosing a (unique up to scalar multiple) map $\Delta(\nu) \to \nabla(\nu)$ (this depends on the choice of these modules as direct summands of $\Delta(\nu)^{n_\nu}$ and $\nabla(\nu)^{m_\nu}$ and lifting that map along the above filtrations. Letting ν vary (and choosing a full set of direct summands) yields a k–basis of E which is compatible with the filtration constructed before. By construction, the basis element just constructed is distinguished by ν being the largest index contributing a non–zero term to the associated map. 'Lower order' entries in the basis element can be modified by adding suitable linear combinations of other basis elements which are associated with strictly bigger indices.

This description has two applications. A well–known fact is that endomorphism rings of tilting modules do not change dimension under change of scalars. That is, given another commutative ring k' which is a k–module we get an isomorphism $End_A(T) \simeq End_{k' \otimes_k A}(k' \otimes_k T)$.

Moreover, this description allows comparing endomorphisms of tilting modules over A and eAe as follows. Let $(A, \leq)$ be quasi–hereditary and eAe a quasi–hereditary centralizer algebra belonging to an ideal I in the index set Λ. Thus, $eL(\lambda)$ is zero for $\lambda \notin I$ and simple otherwise. Similarly, $e\Delta(\lambda)$ for $\lambda \in I$ is a standard module and $e\nabla(\lambda)$ is a costandard module over eAe (cf. [4]). Thus eT is a tilting module over eAe and its Δ– and ∇–filtrations are obtained by multiplying the filtrations of T by e.

Obviously, multiplication by e is an algebra homomorphism from $End_A(T)$ to $End_{eAe}(eT)$.

Now we can state the result.

Theorem 2.16 *Multiplication by e induces a surjective k–algebra homomorphism $End_A(T) \to End_{eAe}(eT)$.*

Proof. First we observe that $Hom_A(\Delta(\lambda), \nabla(\mu))$ does not vanish if and only if $Hom_{eAe}(e\Delta(\lambda), e\nabla(\mu))$ does not vanish and then the elements in the second space are just restrictions of the elements in the first space. Suppose λ equals μ. Fix a subquotient $\Delta(\lambda)$ (respectively, $e\Delta(\lambda)$) in a Δ–filtration of T (respectively, eT) and a subquotient $\nabla(\lambda)$ (respectively, $e\nabla(\lambda)$) in a ∇–filtration. Pick a non–zero element φ

in $Hom_{eAe}(e\Delta(\lambda), e\nabla(\mu))$ and construct from it an element $\varphi' \in End_{eAe}(eT)$ as above. There exists an element $\psi \in Hom_A(\Delta(\lambda), \nabla(\mu)$ such that φ equals $e\psi$. We can perform the same construction with ψ as input to get an element $\psi' \in End_A(T)$. By construction, the difference $\varphi' - e\psi'$ vanishes on the fixed copy of $e\Delta(\lambda)$ and is then a linear combination of basis elements of $End_{eAe}(eT)$ which are associated with indices strictly bigger than λ. Proceeding inductively we may assume that these basis elements are in the image of multiplication by e. Hence φ' is so as well. ■

3 Applications

Subsections one and two contain applications of the easier criterion Theorem 2.10 (dealing with projective–injective modules) or even Corollary 2.11. The third and fourth subsection are using the more involved Theorem 2.15 based on tilting modules.

3.1 Category $\mathcal{O}$

Let $\mathfrak{g}$ be a finite dimensional semisimple complex Lie algebra. Fix a triangular decomposition of $\mathfrak{g}$ and decompose the corresponding category $\mathcal{O}$ into blocks. Let $\mathcal{O}_\chi$ be some block. Let A be a quasi–hereditary algebra whose module category is equivalent to $\mathcal{O}_\chi$. (This algebra has been constructed in [3]. The Verma modules play the role of standard modules in this quasi–hereditary structure.)

We recall some well–known elementary facts from Lie theory:

Lemma 3.1 *Let A be as before. Then:*
(1) There is a simple A–module L such that the socle of each standard module is isomorphic to L.
(2) The injective envelope of L is projective as well, say of the form Ae for some primitive idempotent $e \in A$. Every indecomposable projective–injective A–module is isomorphic to Ae.
(3) Tensoring with a finite dimensional module and then projecting onto the block $\mathcal{O}_\chi$ is an exact functor which has an exact left adjoint and an exact right adjoint. Hence it sends projective objects to projective objects and injective objects to injective objects.
(4) There is a unique projective Verma module, say $\Delta(max)$. Every indecomposable projective module in $\mathcal{O}_\chi$ is a direct summand of a module of the form $\Delta(max) \otimes E$ for some finite dimensional $\mathfrak{g}$–module E.

Proof. Statement (1) is part (II) of Proposition 7.6.3. in [8]. The module Ae is the unique direct summand in this block of a module of the form $\Delta(\mu) \otimes E$, where $\Delta(\mu)$ is a simple and projective Verma module and E is finite dimensional. Thus, $\Delta(\mu) \otimes E$ is projective and self–dual, hence injective as well. Tensoring with a finite dimensional $\mathfrak{g}$–module is an exact functor. Thus it sends projective objects to projective objects. This proves (3). The module $\Delta(max)$ occuring in (4) has as highest weight the unique largest weight occuring in this block. Its projectivity thus follows from the universal property of highest weight modules. Weight considerations together with (3) yield the second statement in (4) (see e.g. chapter 4 in [18] for details). ■

Theorem 3.2 *Let A be as before. Then:*
(1) The algebra A is left QF–3 and has dominant dimension at least two.
(2) (Soergel [21]) There is a double centralizer property

$$A = End(Ae_{eAe})$$

Proof. The projective A–modules are filtered by Verma modules. Hence, by (1) in Lemma 3.1, the socle of A is a direct sum of copies of L. Thus the injective envelope of A is a direct sum of copies of Ae. Consequently, Ae is faithful and A is left QF–3.

The module Ae is the injective envelope of $\Delta(max)$. Therefore, there exists a short exact sequence

$$0 \to \Delta(max) \to Ae \to C \to 0,$$

whose cokernel C is filtered by Verma modules. In fact, Ae is filtered by Verma modules. The composition factor $L = top(\Delta(max))$ belongs to the unique highest weight occuring in this block. Thus any Verma factor $\Delta(max)$ in a filtration of Ae must occur as a submodule by the universal property of the highest weight module $\Delta(max)$.

Since C has a filtration by Verma modules, its socle (by 3.1 (1)) is a direct sum of copies of L and we can embed C into a direct sum of copies of Ae, thus getting an exact sequence

$$0 \to \Delta(max) \to (Ae) \to (Ae)^a$$

for some natural number a. Tensoring with finite dimensional $\mathfrak{g}$–modules (and transporting the result back to $A - mod$) produces short exact sequences of the form

$$0 \to P_0 \to P_1 \to P_2,$$

where (by (4) in Lemma 3.1) the first term P_0 is projective and the other two terms P_1 and P_2 are projective–injective, thus direct sums of copies of Ae. It follows from assertion (3) in Lemma 3.1 that any indecomposable projective A–module is isomorphic to a direct summand of some P_0. Hence A has dominant dimension at least two.

Part (b) is now a consequence of Theorem 2.10. ■

The proof just given carries over without any changes to the situation studied in [13, 14, 15]. Thus we also get a new proof of Theorem 1 in [14] and of one assertion in the main result (Theorem 1) of [15] (which is more general than Theorem 1 of [14]). In particular we get (see [15] for notation):

Theorem 3.3 *The algebra associated with the category $\mathcal{O}(\mathcal{P}, \Lambda(st\mathcal{F}_{int}))$ is left QF–3 and has dominant dimension at least two.*

3.2 Schur–Weyl duality for $\mathbf{n \geq r}$

Let $n \geq r$ be two natural numbers. Let B be either the group algebra of the symmetric group Σ_r over a field k or the corresponding Hecke algebra $\mathcal{H}_q(r)$ (where the parameter q has to be different from zero). Then B is well–known to be self–injective. Let M be the tensor space $(k^n)^{\otimes r}$. This is a faithful B–module (where the B–action is by 'place permutation'). Let A be the endomorphism algebra $A = End(M_B)$. Then A is the classical or the q–Schur algebra (see [17] or [9]). Thus Schur–Weyl duality follows immediately from Corollary 2.11. Moreover:

Proposition 3.4 *Let $n \geq r$ and let A be the classical Schur algebra $S_k(n, r)$ (over some field k) or the quantized Schur algebra $S_q(n, r)$. Then A is left QF–3 and has dominant dimension at least two.*

3.3 Schur–Weyl duality for $\mathbf{n < r}$

In the case $n < r$, the Schur algebra does not always have a faithful projective–injective module, hence it is in general not left QF–3. For example, the Schur algebra $S_k(2, 4)$ for k a field of characteristic two is Morita equivalent to the basic algebra A given by quiver and relations as follows (see [12], 5.6):

A has quiver

$$\begin{tikzcd} \underset{3}{\bullet} \arrow[r, shift left, "\alpha"] & \underset{1}{\bullet} \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\gamma"] & \underset{2}{\bullet} \arrow[l, shift left, "\delta"] \end{tikzcd}$$

and relations:

$\delta \cdot \gamma \cdot \alpha = 0$, $\beta \cdot \delta \cdot \gamma = 0$,

$\beta \cdot \alpha = 0$, $\gamma \cdot \delta = 0$.

The projective modules are as follows:

projective module:

$$\begin{tikzcd}[row sep=small, column sep=small] & & & & 2 \arrow[d] \\ & & 1 \arrow[dl] \arrow[dr] & & 1 \arrow[d] \\ 3 \arrow[d] & 3 \arrow[d] & & 2 \arrow[d] & 3 \arrow[d] \\ 1 \arrow[d] & 1 \arrow[d] & & 1 & 1 \arrow[d] \\ 2 & 2 & & & 2 \end{tikzcd}$$

Loewy length:	3	4	5

Clearly, there is no faithful projective–injective module.

Now we prepare for applying Theorem 2.15 to this situation.

We collect several auxiliary facts making use of the situation dealt with before. Again, the classical Schur algebra and the quantized one can be handled simultaneously. In the following a notation like $S(n,r)$ always means that both cases are covered. Schur algebras are known to be quasi–hereditary with respect to the dominance order $\leq$ on partitions (which index isomorphism classes of simple modules). We are following Green's method (see chapter six of [17]) to relate two different Schur algebras, one with $N \geq r$ and another one with $n < r$.

Lemma 3.5 *Let $N \geq r > n$. Let F be a vector space of dimension N and E one of dimension n. Then $S(N,r)$ contains an idempotent such that $eF^{\otimes r} \simeq E^{\otimes r}$ and $eS(N,r)e \simeq S(n,r)$. The module $E^{\otimes r}$ is a tilting module over $S(n,r)$. Multiplying an exact sequence $0 \to S(N,r) \xrightarrow{\delta} (F^{\otimes r})^l \to (F^{\otimes r})^m$ (with δ a left $\mathcal{F}(\nabla)$–approximation of $S(N,r)$) by e yields an exact sequence $0 \to S(n,r) \oplus X \xrightarrow{\delta} (E^{\otimes r})^l \to (E^{\otimes r})^m$ (with δ a left $\mathcal{F}(\nabla)$–approximation of $S(n,r) \oplus X$) for some module X.*

Proof. The first part is taken from chapter six of [17]. The idempotent e can be written as a sum of pairwise orthogonal idempotents e_λ where λ runs through the partitions of r into at most n parts. These partitions form an ideal in the dominance order, that is, if $\mu \leq \lambda$ and λ has at most n parts, then μ cannot have more than n parts. Thus the general theory of quasi–hereditary algebras [4] implies that multiplication by e sends an $S(N,r)$–standard module $\Delta(\lambda)$ to an $S(n,r)$–standard module if λ has at most n parts. A similar statement is true for costandard modules. Consequently, multiplication by e sends tilting modules to tilting modules. The remaining assertions follow from exactness of multiplication by e. ∎

Since forming approximations is additive, we can cancel the summand X. Because of the Morita invariance of double centralizer properties, we also may forget about the exponent l.

The last part of 3.5 is enough to verify the assumptions of Theorem 2.15, hence we get the following statement which is valid for any n and r:

Theorem 3.6 *The Schur algebra $S(n,r)$ has $E^{\otimes r}$–dominant dimension at least two.*

Let B be the endomorphism ring of $E^{\otimes r}$ over $S(n,r)$. Then there is a double centralizer property

$$S(n,r) = End_B(E^{\otimes r})$$

It remains to determine B. Fortunately, Proposition 2.16 tells us that $End_{S(n,r)}(E^{\otimes r})$ must be a quotient of $End_{S(N,r)}(F^{\otimes r})$ which we know already to coincide with $k\Sigma_r$. Moreover, the dimension of B does not depend on the choice of k.

3.4 From A to eAe

The method applied previously relies on several abstract properties only. We summarize the general result as follows:

Corollary 3.7 *Let $(A, \leq)$ be quasi–hereditary and T a (not necessarily full) tilting module. Suppose there is an exact sequence*

$$0 \to A \xrightarrow{\delta} T \to T^n$$

(for some $n \in \mathbb{N}$) with δ a left $add(T)$–approximation of A. Choose an idempotent e in A which is associated with an ideal in the set of weights of A. Then there is a double centralizer property

$$eAe = End(eT_{End_{eAe}(eT)})$$

Moreover, there is a surjective ring homomorphism

$$End_A(T) \to End_{eAe}(eT) \to 0$$

where eAe acts faithfully, but $End_A(T)$ does not necessarily act faithfully.

For example, this establishes many double centralizer properties between endomorphism algebras of projective modules over Schur algebras and quotients of group algebras of symmetric groups or quotients of Hecke algebras. The two cases $n \geq r$ and $n < r$ in Schur–Weyl duality become two members in a series of double centralizer properties (indexed by ideals in the poset of partitions of r).

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