On Infinitesimal Groups of Tame Representation Type

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0. Introduction

The modular representation theory of cocommutative Hopf algebras can naturally be divided into two disciplines that differ in regard to their methods and results. The classical modular representation theory of finite groups largely depends on Mackey's decomposition theorem as well as Green's work on vertices and sources. The absence of analogous techniques for infinitesimal groups has led to the introduction of geometric methods involving module varieties and schemes of tori.

In continuation of [13] we will investigate in this paper infinitesimal groups whose distribution algebras admit a principal block of tame representation type. This problem has two interrelated aspects concerning the isomorphism types of the underlying groups and the Morita equivalence classes of their Hopf algebras. The corresponding program for representation-finite infinitesimal groups was carried out in [11, 12], and we will occasionally invoke the results of these papers in order to deal with tame infinitesimal groups.

Our approach combines basic results on dimensions of rank varieties [34, 27, 16] with recent work on schemes of tori of restricted Lie algebras [13]. Tame infinitesimal groups are known to possess two-dimensional rank varieties, a property that is inherited by all subgroups. On the other hand, factor groups of tame groups are tame or representation-finite, so that the dimensions of their varieties are also bounded by 2. Schemes of tori of maximal dimension enable us, via the construction of certain characteristic ideals, to link properties of rank varieties to the structure of the Lie algebras of infinitesimal groups. Roughly speaking, the geometric methods usually afford a reduction to the consideration of relatively small groups, whose Gabriel quivers are well enough understood to determine their representation type.

Our paper is organized as follows. In section 1 we collect basic results on the solvable radical and the center of infinitesimal groups with a tame principal block. We then turn

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to solvable groups and prove in Theorem 2.4 that their principal blocks are not tame. The complementary case of tame, semisimple infinitesimal groups, which is addressed in §4 and §5, requires basic facts on algebraic families of Lie algebras These are collected in section 3, where we introduce the generic α_p-rank of such a family. Algebraic families defined by schemes of maximal tori are crucial in our determination of the characteristic semisimple restricted Lie algebras with two-dimensional rank varieties. As these prove to be isomorphic to $s\ell(2)$, all tame semisimple infinitesimal groups live inside SL(2). Our main result of section 5, Theorem 5.5, combines this fact with homological methods to identify these groups as products $\mathrm{SL}(2)_1 T_n$ of the first Frobenius kernel of SL(2) with the n-th Frobenius kernel of its standard torus. Their distribution algebras are shown to be special biserial of domestic representation-type. In addition, we have a complete description of their basic algebras and their Auslander-Reiten quivers. The final section addresses arbitrary tame infinitsimal groups by providing results on the structure of their solvable radicals. These groups turn out to be largely determined by their first Frobenius kernels. Moreover, for tame groups of height ≤ 1 the solvable radical coincides with the center, so that such groups are non-split extensions of $s\ell(2)$ by a representation-finite center.

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1. Preliminaries

Throughout this paper $\mathcal{G}$ is assumed to be an infinitesimal group, defined over an algebraically closed field k of characteristic $p \geq 3$. The solvable radical of $\mathcal{G}$ will be denoted $\mathcal{R}(\mathcal{G})$. We let $H(\mathcal{G})$ be the *distribution algebra* of $\mathcal{G}$. By definition, $H(\mathcal{G}) = \mathcal{O}(\mathcal{G})^*$ is the dual of the *function algebra* $\mathcal{O}(\mathcal{G})$ of the affine group scheme $\mathcal{G}$. We refer the reader to [4] and [35] concerning undefined notions and the general theory of affine group schemes. With regard to quotients we will follow the notation of [35].

We will be interested in the representation theory of the principal block $\mathcal{B}_0(\mathcal{G})$ of the cocommutative Hopf algebra $H(\mathcal{G})$. By general theory this finite-dimensional associative algebra has either finite, tame, or wild representation type. We will employ [1] and [5] as general references on these matters. For our immediate purposes it suffices to note that tameness of $\mathcal{B}_0(\mathcal{G})$ entails that the *rank variety* $\mathcal{V}_{\mathrm{Lie}(\mathcal{G})}$ has dimension ≤ 2 (cf. [27, Thm.2] ,[34]). Given any restricted Lie algebra $(L, [p])$, the affine, conical variety $\mathcal{V}_L$ is defined by

$$\mathcal{V}_L := \{x \in L \; ; \; x^{[p]} = 0\}.$$

Recall that an infinitesimal k-group $\mathcal{G}$ is referred to as *supersolvable* if it admits a composition series with each composition factor being isomorphic to α_p or μ_p.

Lemma 1.1 *Let $\mathcal{N} \subset \mathcal{G}$ be a normal subgroup.*

(1) *If $\mathcal{N}$ is multiplicative, then $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{N})$.*

(2) *If $\mathcal{B}_0(\mathcal{G})$ is tame, then $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ is tame or representation-finite.*

(3) *If $\mathcal{B}_0(\mathcal{G})$ is tame, and $\mathcal{G}/\mathcal{N}$ is not supersolvable, then $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ is tame.*

Proof. Let $I := H(\mathcal{G})H(\mathcal{N})^\dagger$ be the (two-sided) ideal of $H(\mathcal{G})$ that is generated by the augmentation ideal $H(\mathcal{N})^\dagger$ of $H(\mathcal{N})$. The canonical projection $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$ induces an isomorphism $H(\mathcal{G})/I \cong H(\mathcal{G}/\mathcal{N})$ of Hopf algebras (cf. [32, p.45]). In particular, the central primitive idempotent e_0 defining $\mathcal{B}_0(\mathcal{G})$ is sent via the surjection $\pi : H(\mathcal{G}) \longrightarrow H(\mathcal{G}/\mathcal{N})$ to a central idempotent f_0 that is not annihilated by the augmentation of $H(\mathcal{G}/\mathcal{N})$. Consequently, $\pi(\mathcal{B}_0(\mathcal{G}))$ is a sum of blocks containing the principal block $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$. Part (2) is now a direct consequence of [5, (I.4.7)].

(1). Since $H(\mathcal{N})$ is, as a commutative semisimple algebra, a product of copies of the ground field k, the augmentation ideal $H(\mathcal{N})^\dagger$ is a sum of blocks. In other words, there exist orthogonal idempotents $e_1, \ldots, e_r \in H(\mathcal{N})$ such that $H(\mathcal{N})^\dagger = \bigoplus_{i=1}^r H(\mathcal{N})e_i$. By the rigidity of multiplicative groups [35, (7.7)] the normal subgroup $\mathcal{N}$ is contained in the center of $\mathcal{G}$. Hence each e_i is a central idempotent of $H(\mathcal{G})$, and $I = H(\mathcal{G})H(\mathcal{N})^\dagger = \bigoplus_{i=1}^r H(\mathcal{G})e_i$ is a sum of blocks of $H(\mathcal{G})$. Consequently, there exist blocks $\mathcal{B}_1, \ldots, \mathcal{B}_s$ of $H(\mathcal{G})$ such that the projection $H(\mathcal{G}) \longrightarrow H(\mathcal{G}/\mathcal{N})$ induces an isomorphism $\bigoplus_{i=1}^s \mathcal{B}_i \cong H(\mathcal{G}/\mathcal{N})$. Since the projection is a map of augmented algebras, it follows that $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{N})$.

(3). Thanks to (2) the principal block $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ is of tame or finite representation type. In the latter case [11, (2.1)] shows that $\mathcal{G}/\mathcal{N}$ is supersolvable, a contradiction. $\square$

If $(L, [p])$ is a restricted Lie algebra over k, then $T(L)$ will denote its *toral radical*, i.e., the largest p-ideal that is also a torus. For this and other basic concepts from the theory of restricted Lie algebras the reader is referred to [29].

The following subsidiary result, which we require in §2, will be sharpened in §5 after we have a better understanding of semisimple groups of tame representation type.

Lemma 1.2 *If $\mathcal{B}_0(\mathcal{G})$ is tame, then $\mathcal{R}(\mathcal{G})$ is supersolvable.*

Proof. We put $\mathcal{R} := \mathcal{R}(\mathcal{G})$ and proceed by induction on $\mathrm{ord}(\mathcal{R}) := \dim_k H(\mathcal{R})$, the case $\mathcal{R} = e_k$ being trivial.

Let $\mathcal{M}(\mathcal{G})$ be the multiplicative center of $\mathcal{G}$, so that $\mathcal{M}(\mathcal{G}) \subset \mathcal{R}$. We put $\mathcal{G}' := \mathcal{G}/\mathcal{M}(\mathcal{G})$ and apply (1.1) to obtain $\mathcal{B}_0(\mathcal{G}') \cong \mathcal{B}_0(\mathcal{G})$. Since $\mathcal{R}(\mathcal{G}')$ is isomorphic to $\mathcal{R}/\mathcal{M}(\mathcal{G})$, the inductive hypothesis shows that $\mathcal{R}(\mathcal{G}')$ is supersolvable whenever $\mathcal{M}(\mathcal{G}) \neq e_k$. Thus, $\mathcal{R}$ is supersolvable in this case as well, and we will henceforth assume that $\mathcal{M}(\mathcal{G}) = e_k$.

Let $\mathcal{N} \subset \mathcal{R}$ be a minimal normal subgroup of $\mathcal{G}$. Then $\mathcal{N}$ is an abelian subgroup of the first Frobenius kernel $\mathcal{G}_1$ of $\mathcal{G}$. We consider $L := \mathrm{Lie}(\mathcal{G})$ as well as $N := \mathrm{Lie}(\mathcal{N})$. Then N is an abelian p-ideal of L, so that $N^{[p]}$ is contained in the center $C(L)$ of L. Since N is a simple $\mathcal{G}$-module (with respect to the adjoint representation), $N^{[p]} \subset N$ is a submodule, and $T(L) = \mathrm{Lie}(\mathcal{M}(\mathcal{G})) = (0)$, it follows that $N^{[p]} = (0)$. Consequently,

$$\dim_k N \leq \dim \mathcal{V}_L \leq 2.$$

Since $p \geq 3$, and every simple $\mathcal{R}$-module has dimension a p-power (cf. [33, (1.1)] or [12, (2.1)]), we see that N has an $H(\mathcal{R})$-composition series with one-dimensional factors. Accordingly, $\mathcal{R}$ is supersolvable whenever $\mathcal{R}/\mathcal{N}$ is. By (1.1) the block $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ is tame or representation-finite. In the latter case, [11, (2.1)] ensures that the group $\mathcal{G}/\mathcal{N}$ is supersolvable, so its solvable radical $\mathcal{R}/\mathcal{N}$ also enjoys this property. If $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ is tame, then the inductive hypothesis yields the same conclusion. Consequently, $\mathcal{R}$ is supersolvable. □

Following [4, (IV.§3no.4,II.§7no.1)] we denote the Verschiebung of a commutative infinitesimal k-group $\mathcal{U}$ by $V_{\mathcal{U}}$. Recall that a commutative, unipotent infinitesimal k-group $\mathcal{U}$ is referred to as *V-uniserial* if the Verschiebung induces an exact sequence

$$\mathcal{U}^{(p)} \xrightarrow{V_{\mathcal{U}}} \mathcal{U} \longrightarrow \alpha_p \longrightarrow e_k.$$

Our next result gives the structure of the center of a tame infinitesimal group by showing that the corresponding Hopf algebra is of finite representation type.

Proposition 1.3 *Suppose that $\mathcal{B}_0(\mathcal{G})$ is tame. Then the unipotent radical $\mathcal{U}$ of the center of $\mathcal{G}$ is V-uniserial.*

Proof. We consider the functor

$$F : \mathrm{mod}_{H(\mathcal{U})} \longrightarrow \mathrm{mod}_{\mathcal{B}_0(\mathcal{G})} \ \ ; \ \ M \mapsto \mathcal{B}_0(\mathcal{G}) \otimes_{H(\mathcal{U})} M$$

between the categories of finite-dimensional modules for $H(\mathcal{U})$ and $\mathcal{B}_0(\mathcal{G})$. Thanks to [23, (2.6)] the algebra $H(\mathcal{G})$ is a free (right) $H(\mathcal{U})$-module. Consequently, $\mathcal{B}_0(\mathcal{G})$ is, as a projective module of the local algebra $H(\mathcal{U})$, also free. There thus exists a subset $\{x_1, \ldots, x_r\}$ of $\mathcal{B}_0(\mathcal{G})$ such that

$$\mathcal{B}_0(\mathcal{G}) \otimes_{H(\mathcal{U})} M = \bigoplus_{i=1}^{r} x_i \otimes M$$

for every $H(\mathcal{U})$-module M. Since $H(\mathcal{U})$ lies in the center of $H(\mathcal{G})$, the summands in the above decomposition are $H(\mathcal{U})$-modules. Consequently, $F(M)|_{H(\mathcal{U})} \cong M^r$ for every $M \in \mathrm{mod}\, H(\mathcal{U})$.

For a positive integer $d > 0$ we let $\mathrm{Ind}_{H(\mathcal{U})}(d)$ be the set of isoclasses of indecomposable $H(\mathcal{U})$-modules of dimension d. We define $Q(d)$ to be the set of those isoclasses of indecomposable $\mathcal{B}_0(\mathcal{G})$-modules that are direct summands of $F(M)$ for some $[M] \in \mathrm{Ind}_{H(\mathcal{U})}(d)$. By construction, the dimensions of the elements of $Q(d)$ are bounded. Since the algebra $\mathcal{B}_0(\mathcal{G})$ is tame, there exist $(\mathcal{B}_0(\mathcal{G}), k[T])$-bimodules $X_1, \ldots, X_q$ such that

(a) each X_i is a finitely generated, free right $k[T]$-module, and

(b) for each $[N] \in Q(d)$ there exists a simple $k[T]$-module k_λ and $i \in \{1, \ldots, q\}$ such that $N \cong X_i \otimes_{k[T]} k_\lambda$.

Our above observation in conjunction with the theorem of Krull-Remak-Schmidt entails that each $[M] \in \mathrm{Ind}_{H(\mathcal{U})}(d)$ is a direct summand of a suitable $(X_i \otimes_{k[T]} k_\lambda)|_{H(\mathcal{U})}$. Thanks to [24, (I.2.3)] this implies that $H(\mathcal{U})$ is of tame or finite representation type. According to [12, (2.2)] the latter alternative obtains, and [11, (2.6)] ensures that the group $\mathcal{U}$ is V-uniserial. □

Remark. The arguments of (1.3) rely on an "analogue" of the Mackey decomposition theorem. For infinitesimal groups we do in general not have an analogue of Mackey's result, even if the inducing subgroup is normal and abelian. Let L be the Heisenberg algebra with toral center, $S := u(L) \otimes_{u(K)} k_\lambda$ a p-dimensional simple module. Then $K \subset L$ is an abelian ideal, and $\mathrm{Soc}_{u(K)}(S) \cong k_\lambda$ (see [8, (2.9)]). Hence k_λ is not a direct summand of $S|_{u(K)}$.

2. Solvable Groups

The main result of this section shows that principal blocks of solvable infinitesimal groups are not tame. This fact will be used repeatedly in our study of arbitrary tame infinitesimal groups. We begin by determining the Gabriel quiver and the relations of a certain class of trigonalizable infinitesimal groups.

Let $\mathcal{G} = \mathcal{U} \rtimes \mathcal{M}$ be a semidirect product with a unipotent normal subgroup $\mathcal{U}$ and a multiplicative group $\mathcal{M}$. In the sequel, we will frequently employ the *adjoint representation* of $H(\mathcal{M})$ on $H(\mathcal{U})$. Letting η denote the antipode of the Hopf algebra $H(\mathcal{M})$, this action is given by

$$m \cdot u := \sum_{(m)} m_{(1)} u\, \eta(m_{(2)}) \quad \forall\ m \in H(\mathcal{M}),\ u \in H(\mathcal{U}).$$

The adjoint representation endows $H(\mathcal{U})$ with the structure of an $H(\mathcal{M})$-module algebra, and the multiplication $H(\mathcal{U}) \otimes_k H(\mathcal{M}) \longrightarrow H(\mathcal{G})$ induces an isomorphism between the smash product $H(\mathcal{U})\#H(\mathcal{M})$ and $H(\mathcal{G})$ (cf. [22, p.40f]).

The character group of $H(\mathcal{M})$ will be denoted $\mathcal{X}(\mathcal{M})$. We will write the convolution product on $\mathcal{X}(\mathcal{M})$ additively, i.e.,

$$(\lambda + \mu)(m) = \sum_{(m)} \lambda(m_{(1)})\mu(m_{(2)}) \qquad \forall\ m \in H(\mathcal{M}),\ \lambda, \mu \in \mathcal{X}(\mathcal{M}).$$

Observe that the augmentation is the zero element, and that $-\lambda = \lambda \circ \eta$ for all $\lambda \in \mathcal{X}(\mathcal{M})$. Since $\mathcal{M}$ is infinitesimal, the group $\mathcal{X}(\mathcal{M})$ is an abelian p-group.

The adjoint representation induces an action of $H(\mathcal{M})$ on $J := H(\mathcal{U})^\dagger/(H(\mathcal{U})^\dagger)^2$, and we let $J = \bigoplus_{\alpha \in \Phi} J_\alpha$ be the corresponding weight space decomposition. Observe that $\Phi \subset \mathcal{X}(\mathcal{M})$. In the following result we identify $\mathcal{X}(\mathcal{M})$ with the character group $\mathcal{X}(\mathcal{G})$ via the restriction map.

Lemma 2.1 *Let $\mathcal{G} = \mathcal{U} \rtimes \mathcal{M}$ be a semidirect product with a unipotent group $\mathcal{U}$ and a multiplicative group $\mathcal{M}$. Then we have $\mathrm{Ext}^1_{H(\mathcal{G})}(k_\lambda, k_\gamma) \cong (H(\mathcal{U})^\dagger/(H(\mathcal{U})^\dagger)^2)_{\gamma-\lambda}$.*

Proof. Note that $R := H(\mathcal{G})H(\mathcal{U})^\dagger$ is the Jacobson radical of $H(\mathcal{G})$. According to [12, (1.1)] the module $P(\lambda) := H(\mathcal{G}) \otimes_{H(\mathcal{M})} k_\lambda$ is the projective cover of k_λ. General theory (cf. [2, (2.4.3)]) then provides an isomorphism

$$\mathrm{Ext}^1_{H(\mathcal{G})}(k_\lambda, k_\gamma) \cong \mathrm{Hom}_{H(\mathcal{G})}(RP(\lambda)/R^2P(\lambda), k_\gamma).$$

The latter space is readily seen to be isomorphic to $\mathrm{Hom}_{H(\mathcal{M})}(J \otimes_k k_\lambda, k_\gamma) \cong J_{\gamma-\lambda}$. $\square$

Let G be a finite abelian group, $\Phi \subset G$ a subset. Given $\mathbf{n} := (n_\alpha)_{\alpha\in\Phi} \in \mathbb{N}^\Phi$, we consider the quiver $Q(G,\mathbf{n}) := (G, A(\mathbf{n}))$, where for each $\lambda \in G$, and $\alpha \in \Phi$ there are n_α arrows $\lambda \mapsto \lambda + \alpha$ belonging to $A(\mathbf{n})$. These will be denoted (λ, α, i) for $1 \le i \le n_\alpha$. We shall consider the following relations in the path algebra $k[Q(G,\mathbf{n})]$:

(a) $c(\lambda,\alpha,\beta,i,j) := (\lambda+\alpha,\beta,j)(\lambda,\alpha,i) - (\lambda+\beta,\alpha,i)(\lambda,\beta,j)$ for $\lambda \in G$, $\alpha,\beta \in \Phi$, $1 \le i \le n_\alpha$, $1 \le j \le n_\beta$, and

(b) $p(\lambda,\alpha,i) := (\lambda+(p-1)\alpha,\alpha,i)(\lambda+(p-2)\alpha,\alpha,i)\cdots(\lambda+\alpha,\alpha,i)(\lambda,\alpha,i)$ for $\lambda \in G$, $\alpha \in \Phi$, $1 \le i \le n_\alpha$.

According to (2.1) the quiver $Q(\mathcal{X}(\mathcal{M}),\mathbf{n})$ with $\mathbf{n} := (\dim_k J_\alpha)_{\alpha\in\Phi}$ is the Gabriel quiver of $H(\mathcal{U} \times \mathcal{M})$. Our next result shows that for those commutative unipotent groups $\mathcal{U}$ that are annihilated by the Verschiebung $V_\mathcal{U}$, the relations are precisely the ones given in (a) and (b).

Proposition 2.2 *Let $\mathcal{G} = \mathcal{U} \times \mathcal{M}$ be a semidirect product of a multiplicative group $\mathcal{M}$ and a unipotent group $\mathcal{U} \cong \prod_{i=1}^r \alpha_{p^{n_i}}$. Let $J = \bigoplus_{\alpha\in\Phi} J_\alpha$ be the weight space decomposition of $J := H(\mathcal{U})^\dagger/(H(\mathcal{U})^\dagger)^2$ relative to $\mathcal{M}$, and set $\mathbf{n} := (\dim_k J_\alpha)_{\alpha\in\Phi}$. Then $H(\mathcal{G}) \cong k[Q(\mathcal{X}(\mathcal{M}),\mathbf{n})]/I$, where*

$I = \langle\{c(\lambda,\alpha,\beta,i,j)\,,\, p(\lambda,\alpha,i)\ ;\ \lambda \in \mathcal{X}(\mathcal{M}),\ \alpha,\beta \in \Phi,\ 1 \le i \le n_\alpha,\ 1 \le j \le n_\beta\}\rangle.$

Proof. Since $H(\mathcal{M})$ is semisimple, there exists an $H(\mathcal{M})$-invariant subspace $V \subset H(\mathcal{U})^\dagger$ such that the canonical projection $H(\mathcal{U})^\dagger \longrightarrow J$ induces an isomorphism $V \cong J$. Our assumption on $\mathcal{U}$ implies $x^p = 0$ for every $x \in H(\mathcal{U})^\dagger$. Consequently, writing $V = \bigoplus_{\alpha\in\Phi} V_\alpha$ and choosing a basis $(v_{\alpha,i})_{1\le i\le n_\alpha}$ of V_α for every $\alpha \in \Phi$, we have $H(\mathcal{U}) \cong k[v_{\alpha,i}]/(v_{\alpha,i}^p)$. Since $H(\mathcal{U})$ is an $H(\mathcal{M})$-module algebra, the weight space decomposition of the adjoint representation satisfies $H(\mathcal{U})_\lambda H(\mathcal{U})_\mu \subset H(\mathcal{U})_{\lambda+\mu}$ for every $\lambda,\mu \in \mathcal{X}(\mathcal{M})$. The commutative, semisimple k-algebra $H(\mathcal{M})$ decomposes into a direct sum $H(\mathcal{M}) = \bigoplus_{\lambda\in\mathcal{X}(\mathcal{M})} ku_\lambda$, with primitive orthogonal idempotents u_λ. Accordingly, we have $au_\lambda = \lambda(a)u_\lambda$ for every element $a \in H(\mathcal{M})$, so that $\lambda(u_\mu) = \delta_{\mu,\lambda}\ \ \forall\ \lambda,\mu \in \mathcal{X}(\mathcal{M})$.

We decompose the adjoint $H(\mathcal{M})$-module $H(\mathcal{U})$ into its eigenspaces and write $H(\mathcal{U}) = \bigoplus_{\alpha\in\Psi} H(\mathcal{U})_\alpha$, where $\Phi \subset \Psi \subset \mathcal{X}(\mathcal{M})$.

(*) *Let x be an element of $H(\mathcal{U})_\alpha$. Then we have $u_\mu x u_\lambda = \delta_{\mu,\lambda+\alpha} x u_\lambda$ for $\lambda,\mu \in \mathcal{X}(\mathcal{M})$.*
Directly from the definitions we obtain

$$\begin{aligned} u_\mu x u_\lambda &= \sum_{(u_\mu)} (u_{\mu,(1)} \cdot x) u_{\mu,(2)} u_\lambda = \sum_{(u_\mu)} \alpha(u_{\mu,(1)}) x \lambda(u_{\mu,(2)}) u_\lambda = (\lambda+\alpha)(u_\mu) x u_\lambda \\ &= \delta_{\mu,\lambda+\alpha} x u_\lambda. \end{aligned}$$

It follows directly from (*) that there exists a unique homomorphism $\Gamma : k[Q(\mathcal{X}(\mathcal{M}),\mathbf{n})] \longrightarrow H(\mathcal{G})$ of k-algebras such that

$$\Gamma(e_\lambda) = u_\lambda \quad ; \quad \Gamma((\lambda,\alpha,i)) = v_{\alpha,i} u_\lambda.$$

Since $u_\lambda \in \operatorname{im}\Gamma$ and

$$v_{\alpha,i} = \sum_{\lambda\in\mathcal{X}(\mathcal{M})} v_{\alpha,i} u_\lambda \in \operatorname{im}\Gamma,$$

the image $\Gamma(k[Q(\mathcal{X}(\mathcal{M}),\mathbf{n})])$ contains a set of generators of $H(\mathcal{G})$. Consequently, the homomorphism Γ is surjective.

By virtue of (*) we have

$$\begin{aligned}\Gamma(c(\alpha,i,\beta,j)) &= v_{\beta,j}u_{\lambda+\alpha}v_{\alpha,i}u_\lambda - v_{\alpha,i}u_{\lambda+\beta}v_{\beta,j}u_\lambda\\ &= v_{\beta,j}v_{\alpha,i}u_\lambda - v_{\alpha,i}v_{\beta,j}u_\lambda = 0,\end{aligned}$$

as well as

$$\Gamma(p(\lambda,\alpha,i)) = v_{\alpha,i}^p u_\lambda = 0.$$

Consequently, Γ factors through to a surjective homomorphism $\hat{\Gamma} : k[Q(\mathcal{X}(\mathcal{M}),\mathbf{n})]/I \longrightarrow H(\mathcal{G})$.

Let $\pi : k[Q(\mathcal{X}(\mathcal{M}),\mathbf{n})] \longrightarrow k[Q(\mathcal{X}(\mathcal{M}),\mathbf{n})]/I$ be the canonical projection. Since

$$\begin{aligned}[\sum_{\lambda\in\mathcal{X}(\mathcal{M})}(\lambda,\alpha,i),\sum_{\mu\in\mathcal{X}(\mathcal{M})}(\mu,\beta,j)] &= \sum_{\lambda,\mu}(\lambda,\alpha,i)(\mu,\beta,j)-(\mu,\beta,j)(\lambda,\alpha,i)\\ &= \sum_{\mu}(\mu+\beta,\alpha,i)(\mu,\beta,j)-\sum_{\lambda}(\lambda+\alpha,\beta,j)(\lambda,\alpha,i)\\ &= -\sum_{\lambda}c(\lambda,\alpha,\beta,i,j)\in I,\end{aligned}$$

and

$$(\sum_{\lambda\in\mathcal{X}(\mathcal{M})}(\lambda,\alpha,i))^p = (\sum_{\lambda\in\mathcal{X}(\mathcal{M})}p(\lambda,\alpha,i))\in I,$$

there exist a unique algebra homomorphism $\Omega : H(\mathcal{U}) \longrightarrow k[Q(\mathcal{X}(\mathcal{M}),\mathbf{n})]/I$ such that $\Omega(v_{\alpha,i}) = \pi(\sum_{\lambda\in\mathcal{X}(\mathcal{M})}(\lambda,\alpha,i))$. We extend Ω to a linear map $\hat{\Omega}$ on $H(\mathcal{G})\cong H(\mathcal{U})\otimes_k H(\mathcal{M})$ via

$$\hat{\Omega}(a\otimes u_\lambda) := \Omega(a)\pi(e_\lambda)\quad \forall\ a\in H(\mathcal{U}),\ \lambda\in\mathcal{X}(\mathcal{M}).$$

Direct computation shows that

$$\hat{\Omega}((v_{\alpha,i}^m u_\mu)(v_{\beta,j}^n u_\lambda)) = \hat{\Omega}(v_{\alpha,i}^m u_\mu)\hat{\Omega}(v_{\beta,j}^n u_\lambda),$$

proving that $\hat{\Omega}$ is a homomorphism of k-algebras.

Let R be the Jacobson radical of $k[Q(\mathcal{X}(\mathcal{M}),\mathbf{n})]/I$. The identity

$$(\lambda,\alpha,i) = e_{\lambda+\alpha}(\sum_{\mu\in\mathcal{X}(\mathcal{M})}(\mu,\alpha,i))$$

yields $k[Q(\mathcal{X}(\mathcal{M}),\mathbf{n})]/I = \hat{\Omega}(H(\mathcal{G}))+R^2$. We may now apply [2, (1.2.8)] to see that $\hat{\Omega}$ is surjective.

As a result, the algebras $H(\mathcal{G})$ and $k[Q(\mathcal{X}(\mathcal{M}),\mathbf{n})]/I$ have the same dimension, so that $\hat{\Gamma}$ and $\hat{\Omega}$ are isomorphisms. □

Lemma 2.3 *Let $\mathcal{U}$ be V-uniserial. Then every extension*

$$e_k \longrightarrow \alpha_p \longrightarrow \mathcal{V} \longrightarrow \mathcal{U} \longrightarrow e_k$$

is commutative.

Proof. The group $\mathcal{V}$ is, as an extension of two unipotent groups, unipotent [4, (IV.§2 (2.3))]. Conjugation induces a homomorphism $\mathcal{V} \longrightarrow \mathcal{AUT}(\alpha_p)$ from $\mathcal{V}$ into the automorphism scheme of α_p. As the latter coincides with μ_k, the above map is trivial, and α_p lies in the center of $\mathcal{V}$.

Since $\mathcal{U}$ is abelian, the commutator $[a,b] := aba^{-1}b^{-1}$ belongs to the central subgroup α_p for every $a, b \in \mathcal{V}$. There thus results an alternating bilinear map

$$\chi : \mathcal{U} \times \mathcal{U} \longrightarrow \alpha_p \quad ; \quad (\bar{a}, \bar{b}) \mapsto [a,b].$$

Letting $\mathcal{F} : \mathcal{U} \longrightarrow \mathcal{U}^{(p)}$ be the Frobenius homomorphism, we obtain from [4, (IV.§3(4.7))] the identity

$$V_{\alpha_p}(\chi^{(p)}(\mathcal{F}(\bar{a}), \bar{b})) = \chi(\bar{a}, V_{\mathcal{U}}(\bar{b})).$$

Since $V_{\alpha_p} : {\alpha_p}^{(p)} \longrightarrow \alpha_p$ is trivial, the inverse image $\mathcal{Z}$ of $V_{\mathcal{U}}(\mathcal{U}^{(p)})$ under the canonical projection $\mathcal{V} \longrightarrow \mathcal{U}$ is a central subgroup of length $\ell(\mathcal{V}) - 1$. It follows that

$$H(\mathcal{V}/\mathcal{Z}) \cong H(\alpha_p) \cong k[x]/(x^p).$$

Consequently, $H(\mathcal{Z})$ is central in $H(\mathcal{V})$, and letting $X \in H(\mathcal{V})$ be a pre-image of x under $H(\mathcal{V}) \longrightarrow H(\mathcal{V}/\mathcal{Z})$, we obtain

$$H(\mathcal{V}) = H(\mathcal{Z})[X] + H(\mathcal{Z})^{\dagger} H(\mathcal{V}).$$

Since the augmentation ideal $H(\mathcal{Z})^{\dagger}$ is the Jacobson radical of $H(\mathcal{Z})$, this implies that $H(\mathcal{V}) = H(\mathcal{Z})[X]$ is abelian. As a result, the group $\mathcal{V}$ is commutative. $\square$

Recall that the *complexity* $c_{H(\mathcal{G})}(M)$ of a $H(\mathcal{G})$-module M is the rate of growth of a minimal projective resolution of M. If $\mathcal{G}' \subset \mathcal{G}$ is a subgroup, then, observing that $H(\mathcal{G})$ is a free $H(\mathcal{G}')$-module [23, (2.6)], we have $c_{H(\mathcal{G}')}(M) \leq c_{H(\mathcal{G})}(M)$. As $H(\alpha_{p^n}) \cong k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$ Künneth's theorem implies $c_{H(\alpha_{p^n})}(k) = n$.

Theorem 2.4 *Let $\mathcal{G}$ be a solvable infinitesimal group. Then $\mathcal{B}_0(\mathcal{G})$ is not tame.*

Proof. Assume the theorem to be false, and let $\mathcal{G}$ be a counterexample of minimal order. Thanks to (1.2) the group $\mathcal{G}$ is supersolvable. Moreover, a consecutive application of (1.1(1)) and [11, (2.3)] ensures that $\mathcal{G} = \mathcal{U} \times \mathcal{M}$, where $\mathcal{U}$ is the unipotent radical of $\mathcal{G}$, and $\mathcal{M} \subset \mathcal{G}$ denotes a multiplicative subgroup.

Since $\mathcal{G}$ is supersolvable with trivial multiplicative center, rigidity (cf. [35, (7.7)]) implies the existence of a normal subgroup $\mathcal{N} \subset \mathcal{G}$ such that $\mathcal{N} \cong \alpha_p$. The unipotent normal subgroup $\mathcal{N}$ is contained in $\mathcal{U}$, so that $\mathcal{G}/\mathcal{N} \cong [\mathcal{U}/\mathcal{N}] \times \mathcal{M}$. According to (1.1) the principal block $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ has finite representation type. Thus, [11, (2.7)] yields that $\mathcal{U}/\mathcal{N}$ is V-uniserial. Owing to (2.3) the group $\mathcal{U}$ is commutative.

According to [4, (IV.§3(4.6))] the image $\mathcal{U}' := V_{\mathcal{U}}(\mathcal{U}^{(p)}) \subset \mathcal{U}$ of the Verschiebung is a normal subgroup of $\mathcal{G}$. If $\mathcal{U}' \neq e_k$, then the above arguments show that the factor group $\mathcal{U}/\mathcal{U}'$ is V-uniserial. Since the Verschiebung annihilates this group [4, (IV.§3(4.6))], it readily follows from [4, (IV.§3(6.11))] that $\mathcal{U}/\mathcal{U}' \cong \alpha_p$. Thus, $\mathcal{U}$ is V-uniserial, and [11, (2.7)] implies that $\mathcal{B}_0(\mathcal{G})$ has finite representation type, a contradiction. Consequently, $V_{\mathcal{U}}(\mathcal{U}^{(p)}) = e_k$, so that [4, (IV.§3(6.11))] yields $\mathcal{U} \cong \prod_{i=1}^{r} \alpha_{p^{n_i}}$. Since $\mathcal{B}_0(\mathcal{G})$ is tame, Rickard's Theorem [27, Thm.2] shows that $c_{H(\mathcal{G})}(k) \leq 2$. Künneth's theorem then gives rise to

$$\sum_{i=1}^{r} n_i = c_{H(\prod_{i=1}^{r} \alpha_{p^{n_i}})}(k) = c_{H(\mathcal{U})}(k) \leq c_{H(\mathcal{G})}(k) \leq 2.$$

Consequently, $\mathcal{U} \cong \alpha_p \times \alpha_p$ or $\mathcal{U} \cong \alpha_{p^2}$, and we have $\dim_k H(\mathcal{U})^\dagger/(H(\mathcal{U})^\dagger)^2 = 2$ in either case. Thanks to [11, (2.4)] the algebra $H(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G})$ has exactly one block.

We let $H(\mathcal{M})$ operate on $H(\mathcal{U}) = k[T_1, T_2]/(T_1^p, T_2^p)$ via the adjoint representation. As before, we can choose the generators T_1, T_2 to belong to $H(\mathcal{M})$-eigenspaces with characters λ_1 and λ_2, respectively. Let G be the subgroup generated by $\{\lambda_1, \lambda_2\}$. Then $\mathcal{M}' := \mathrm{Spec}_k(k[\mathcal{X}(\mathcal{M})/G])$ is a closed subgroup of $\mathcal{M}$ that operates trivially on $H(\mathcal{U})$. Thus, $\mathcal{M}' \subset \mathcal{M}(\mathcal{G}) = e_k$, so that $\mathcal{X}(\mathcal{M})$ is generated by $\{\lambda_1, \lambda_2\}$. In view of [12, (2.2)] we have $\mathcal{M} \neq e_k$, whence

$$1 \leq \dim_{\mathrm{GF}(p)} \mathcal{X}(\mathcal{M})/p\mathcal{X}(\mathcal{M}) \leq 2.$$

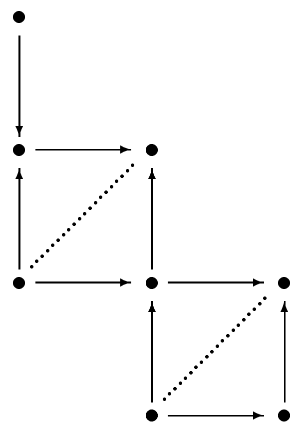
If the dimension equals 2, then λ_1 and λ_2 generate $\mathcal{X}(\mathcal{M}) = \mathbb{Z}/(p^n) \times \mathbb{Z}/(p^m)$, and $\mathcal{M} = \mu_{p^n} \times \mu_{p^m}$. We may now apply (2.2) to obtain the Gabriel quiver $Q_{n,m}$ and the relations of $H(\mathcal{G})$. The set of vertices is $\mathbb{Z}/(p^n) \times \mathbb{Z}/(p^m)$, and the arrows and relations are given by

$$\alpha_{(i,j)} : (i,j) \to (i+1,j), \quad \beta_{(i,j)} : (i,j) \to (i,j+1) \qquad (i,j) \in \mathbb{Z}/(p^n) \times \mathbb{Z}/(p^m),$$

and

$$\begin{aligned} c_{(i,j)} := \beta_{(i+1,j)}\alpha_{(i,j)} - \alpha_{(i,j+1)}\beta_{(i,j)}, \qquad p_{(i,j)} &:= \alpha_{(i+p-1,j)} \cdots \alpha_{(i,j)}, \\ q_{(i,j)} &:= \beta_{(i,j+p-1)} \cdots \beta_{(i,j)} \quad (i,j) \in \mathbb{Z}/(p^n) \times \mathbb{Z}/(p^m), \end{aligned}$$

respectively. Let $I_{n,m}$ be the set of relations and denote by $\mathrm{mod}_{(Q_{n,m},I_{n,m})}$ the module category of the bound quiver $(Q_{n,m}, I_{n,m})$. As $p \geq 3$, the category $\mathrm{mod}_{(Q_{n,m},I_{n,m})}$ contains the module category of the following bound quiver, in which the relations are marked by dotted lines, as a full subcategory



Here the downward arrow corresponds to $(p^n-1,2) \mapsto (0,2)$. According to [31, p.4] this module category is wild. Consequently, $H(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G})$ is wild, a contradiction.

Alternatively, $\mathcal{X}(\mathcal{M}) \cong \mathbb{Z}/(p^n)$ is cyclic with generator λ_1, say. Hence there is $0 \le r \le p^n-1$ such that $\lambda_2 = r\lambda_1$. By the same token, we have $\mathcal{M} = \mu_{p^n}$, and (2.2) shows that the Gabriel quiver of $H(\mathcal{G})$ has set of vertices $\mathbb{Z}/(p^n)$, arrows

$$\alpha_i : i \to i+1, \quad \beta_i : i \to i+r \qquad i \in \mathbb{Z}/(p^n),$$

and relations

$$c_i := \beta_{i+1}\alpha_i - \alpha_{i+r}\beta_i, \quad p_i := \alpha_{i+(p-1)} \cdots \alpha_i, \quad q_i := \beta_{i+(p-1)r} \cdots \beta_i \qquad i \in \mathbb{Z}/(p^n).$$

We denote this quiver by $Q_{n,r}$, and let $I_{n,r} \subset k[Q_{n,r}]$ be the ideal generated by the set $\{c_i, p_i, q_i \ ; \ i \in \mathbb{Z}/(p^n)\}$.

In order to determine the representation type of the algebra $k[Q_{n,r}]/I_{n,r}$ we construct a Galois covering of $\tilde{Q}_{n,r}$ of $Q_{n,r}$. We define the set of vertices of $\tilde{Q}_{n,r}$ to be $\mathbb{Z}/(p^n) \times \mathbb{Z}$. The arrows are

$$\tilde{\alpha}_{i,j} : (i,j) \to (i+1,j) \ \ ; \ \ \tilde{\beta}_{i,j} : (i,j) \to (i+r, j+1).$$

Given $s \in \mathbb{Z}$, we let $\lambda_s : \tilde{Q}_{n,r} \longrightarrow \tilde{Q}_{n,r}$ be the automorphism that is defined by

$$\lambda_s(i,j) = (i, j+s).$$

Observe that $G := \{\lambda_s \ ; \ s \in \mathbb{Z}\} \cong \mathbb{Z}$ is a subgroup of the automorphism group of $\tilde{Q}_{n,r}$.

We define

$$\pi : \tilde{Q}_{n,r} \longrightarrow Q_{n,r} \ \ ; \ \ (i,j) \mapsto i.$$

Direct computation now shows that $\tilde{Q}_{n,r}$ is a Galois covering of $Q_{n,r}$ with free Galois group G.

Next, we put

$$\begin{aligned} \tilde{c}_{i,j} &:= \tilde{\beta}_{i+1,j} \cdot \tilde{\alpha}_{i,j} - \tilde{\alpha}_{i+r,j+1} \cdot \tilde{\beta}_{i,j} \ \ ; \ \ \tilde{p}_{i,j} := \tilde{\alpha}_{i+p-1,j} \cdot \tilde{\alpha}_{i+p-2,j} \cdot \ldots \cdot \tilde{\alpha}_{i,j} \\ \tilde{q}_{i,j} &:= \tilde{\beta}_{i+(p-1)r,j+p-1} \cdot \tilde{\beta}_{i+(p-2)r,j+p-2} \cdot \ldots \cdot \tilde{\beta}_{i,j}. \end{aligned}$$

Let $\tilde{I}_{n,r} := \{\tilde{c}_{i,j}, \ \tilde{p}_{i,j}, \ \tilde{q}_{i,j} \ ; \ (i,j) \in \mathbb{Z}/(p^n) \times \mathbb{Z}\}$. Since G is free [17, (3.5)] shows that the push-down functor $\mathrm{mod}_{(\tilde{Q}_{n,r}, \tilde{I}_{n,r})} \longrightarrow \mathrm{mod}_{k[Q_{n,r}]/I_{n,r}}$ sends indecomposables to indecomposables and reflects isomorphisms. Accordingly, the algebra $k[Q_{n,r}]/I_{n,r}$ is wild whenever the category $\mathrm{mod}_{(\tilde{Q}_{n,r}, \tilde{I}_{n,r})}$ enjoys this property. As $p \geq 3$ the latter follows again by identifying a subquiver of the type given in the above picture. $\square$

For $r \geq 1$, we let $\mathcal{G}_r$ be the r-th Frobenius kernel of $\mathcal{G}$. Recall that $H(\mathcal{G}_1) \cong u(\mathrm{Lie}(\mathcal{G}))$ is the restricted enveloping algebra of the Lie algebra of $\mathcal{G}$.

Corollary 2.5 *Let $\mathcal{G}$ be an infinitesimal group with tame principal block $\mathcal{B}_0(\mathcal{G})$.*

(1) $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G})} = 2$

(2) *If $L := \mathrm{Lie}(\mathcal{G})$ admits a self-centralizing torus, then $L/T(L) \cong s\ell(2)$. In particular, $\mathcal{B}_0(\mathcal{G}_1)$ is tame.*

Proof. (1). If $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G})} \leq 1$, then [11, (2.1)] implies that $\mathcal{G}$ is supersolvable. As this contradicts (2.4), we have $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G})} \geq 2$. The reverse inequality has already been noted.

(2). From (1) and [13, (4.3)] we conclude that L is either supersolvable, or $L/T(L) \cong s\ell(2)$. The former alternative entails the solvability of $\mathcal{G}$. Since this contradicts (2.4), our result follows from (1.1(1)). $\square$

Corollary 2.6 *Let $\mathcal{G}$ be an infinitesimal group of characteristic $p \geq 3$. Then the following statements hold:*

(1) *$H(\mathcal{G})$ does not possess any tame, basic blocks.*

(2) *If $\mathcal{G}$ is trigonalizable, then $H(\mathcal{G})$ does not possess any tame blocks.*

Proof. (1). Let $\mathcal{B} \subset H(\mathcal{G})$ be a tame basic block. Then there exists a homomorphism $\lambda : H(\mathcal{G}) \longrightarrow k$ that sends the idempotent of $\mathcal{B}$ to 1. We consider the convolution $\lambda * id_{H(\mathcal{G})}$ that is given by

$$(\lambda * id_{H(\mathcal{G})})(h) = \sum_{(h)} \lambda(h_{(1)})h_{(2)} \quad \forall\ h \in H(\mathcal{G}).$$

Letting $\varepsilon : H(\mathcal{G}) \longrightarrow k$ be the counit, we observe that $\varepsilon \circ (\lambda * id_{H(\mathcal{G})}) = \lambda$. Consequently, $(\lambda * id_{H(\mathcal{G})})(\mathcal{B}) = \mathcal{B}_0(\mathcal{G})$, implying that the principal block of $H(\mathcal{G})$ is also basic and tame. Thanks to [32, (I.2.40)] the former property entails the supersolvability of $\mathcal{G}$. In view of (2.4) we have thus arrived at a contradiction.

(2). Since $\mathcal{G}$ is trigonalizable, every block of $H(\mathcal{G})$ is basic, and (1) yields the desired conclusion. $\square$

3. The Generic α_p-Rank

In this section we collect a few subsidiary results involving the scheme of tori that we will require in our study of the structure of Lie algebras of tame infinitesimal groups. Throughout $(L, [p])$ is assumed to be a restricted Lie algebra, defined over the algebraically closed field k of characteristic $p \geq 3$. If $K \subset L$ is a p-subalgebra, then $\dim \mathcal{V}_K$ will occasionally be referred to as the *α_p-rank* of K. In what follows, $N(L)$ will be the nilpotent radical of L. Given a subspace $V \subset L$, we denote by V_p the smallest $[p]$-invariant subspace of L containing V. We let $C_L(V)$ be the centralizer of V in L, and begin by recording a technical result which will be frequently referred to in the sequel.

Lemma 3.1 *Let $T \subset L$ be a maximal torus. If $\dim \mathcal{V}_L \leq 1$, then L is either abelian, or the root space decomposition relative to T consists of a single root space of dimension* 1.

Proof. Suppose that L is not abelian. Thanks to [7, (3.2)] and [7, (4.3)] the abelian ideal $N(L)$ has codimension 1. If $T \subset N(L)$, then $T = T(L)$, and [29, (II.4.1)] shows that L is nilpotent, a contradiction. Hence there exists a toral element $t \in L$, such that $L = kt \oplus N(L)$. By [7, (4.3)] there is a p-nilpotent element $z \in N(L)$ such that $N(L) = T(L) \oplus (kz)_p$. Consequently,

$$T(L) \oplus (kz^{[p]})_p = N(L)^{[p]} \subset C(L),$$

so that $N(L) = C(L) + kz$. Hence $L = kt + N(L) = kt + C(L) + kz = C_L(T) + kz$. As L is not nilpotent, [29, (II.4.1)] implies $L \neq C_L(T)$, proving that L admits a single root space of dimension 1. $\square$

Given a torus $T \subset L$, we recall from [13, §1] the affine algebraic scheme $\tilde{\mathcal{T}}_L : M_k \longrightarrow \mathrm{Ens}$ that associates to every commutative k-algebra R the set of split injective homomorphisms $T \otimes_k R \longrightarrow L \otimes_k R$ of restricted R-Lie algebras. Observe that this set can be identified with the set of those homomorphisms $\varphi : T \longrightarrow L \otimes_k R$ of restricted k-Lie algebras for which $R\varphi(T)$ is an R-direct summand of $L \otimes_k R$. According to [13, (1.3)] the scheme $\tilde{\mathcal{T}}_L$ is smooth. Consequently, its irreducible components are its connected components, and there exists exactly one irreducible component $\mathcal{X}_T \subset \tilde{\mathcal{T}}_L$ such that the given inclusion $T \hookrightarrow L$ corresponds to a rational point $x_0 \in \mathcal{X}_T(k)$. By [13, (1.3)] $\mathcal{X}_T$ is an affine algebraic scheme, and we write $\mathcal{X}_T \cong \mathrm{Spec}_k(A)$ for some finitely generated integral domain A over k. We will henceforth identify these schemes and suppress the isomorphism.

We define $\tilde{L} := L \otimes_k A$, and consider the *universal embedding* $j : T \hookrightarrow \tilde{L}$ of restricted k-Lie algebras that corresponds to $id_A \in \mathrm{Spec}_k(A)(A)$. Any element $\varphi \in \tilde{\mathcal{T}}_L(R)$ can be obtained via specialization along an element $x \in \mathrm{Spec}_k(A)(R)$:

$$\varphi = (id_L \otimes x) \circ j.$$

For any A-direct summand $P \subset \tilde{L}$ and any $x \in \mathrm{Spec}_k(A)(k)$ we put $P(x) := (id_L \otimes x)(P) \subset L$. By general principles there exists a weight space decomposition

$$\tilde{L} = \tilde{L}_0 \oplus \bigoplus_{\alpha \in \Phi} \tilde{L}_\alpha$$

of $\tilde{L}$ relative to T. Here $\Phi \subset T^* \setminus \{0\}$ is the finite set of weights, and for $\alpha \in \Phi \cup \{0\}$ the weight space $\tilde{L}_\alpha = \{v \in \tilde{L} \ ; \ [j(t), v] = \alpha(t)v \ \forall \ t \in T\}$ is an A-direct summand of $\tilde{L}$. Given an arbitrary element $x \in \mathcal{X}_T(k)$, we have

$$L = \tilde{L}_0(x) \oplus \bigoplus_{\alpha \in \Phi} \tilde{L}_\alpha(x),$$

where $\tilde{L}_\alpha(x) = \{v \in L \ ; \ [(id_L \otimes x)(j(t)), v] = \alpha(t)v \quad \forall \ t \in T\}$ for $\alpha \in \Phi \cup \{0\}$. In other words, if $\varphi := (id_L \otimes x) \circ j$ is the embedding corresponding to $x \in \mathcal{X}_T(k)$, then $\tilde{L}_\alpha(x)$ is the root space with root $\alpha \circ \varphi^{-1}$ relative to the torus $\varphi(T) \subset L$. In particular, specialization along x_0 yields the root space decomposition

$$L = C_L(T) \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

of L relative to T. According to general theory, the centralizer $C_L(T)$ of the torus T is a Cartan subalgebra of L if and only if T is a maximal torus (cf. [29, (II.4.1)]).

Let M be a *restricted* L-module, i.e., an L-module for which $x^{[p]} \in L$ acts via the p-th power of $x \in L$. The restricted L-modules are precisely those L-modules whose structure can be lifted to the restricted enveloping algebra $u(L)$. Given such a module M, the A-module $\tilde{M} := M \otimes_k A$ carries the structure of an $\tilde{L}$-module, and we have a weight space decomposition

$$\tilde{M} = \bigoplus_{\lambda \in \Lambda} \tilde{M}_\lambda$$

relative to T with weight spaces $\tilde{M}_\lambda = \{m \in \tilde{M} \ ; \ j(t)m = \lambda(t)m \ \forall \ t \in T\}$. As before, we set $\tilde{M}(x) := (id_M \otimes x)(\tilde{M})$ for every $x \in \mathcal{X}_T(k)$.

Let $P \subset \tilde{L}$ be an A-direct summand of $\tilde{L}$. Thanks to [13, (2.5)] there exists a non-empty open subset $U_P \subset \mathcal{X}_T(k)$, and a natural number $c_P(L,T) \in \mathbb{N}_0$ such that

$$\dim P(x) \cap \mathcal{V}_L = c_P(L,T) \qquad \forall \ x \in U_P.$$

The number $c_P(L,T)$ is the *generic α_p-rank* of the algebraic family $(P(x))_{x \in \mathcal{X}_T(k)}$ of subspaces of L. Given a subset $\Psi \subset T^*$, the A-submodule $\tilde{L}^\Psi := \bigoplus_{\alpha \in \Psi} \tilde{L}_\alpha$ satisfies this condition, and we write $c_\Psi(L,T) := c_{\tilde{L}^\Psi}(L,T)$ as well as $U_\Psi := U_{\tilde{L}^\Psi}$. For $\alpha \in \Phi \cup \{0\}$ we put $(\alpha) := \mathrm{GF}(p)\alpha$ and define $c_\alpha(L,T) := c_{(\alpha)}(L,T)$ and $U_\alpha := U_{(\alpha)}$. Note that $c_\alpha(L,T)$ is the generic α_p-rank associated to the p-subalgebra $\tilde{L}^{(\alpha)} := \bigoplus_{i=0}^{p-1} \tilde{L}_{i\alpha}$ of $\tilde{L}$.

Lemma 3.2 *Let $T \subset L$ be a torus.*

(1) *$c_0(L,T) = 0$ if and only if $C_L(T) = T$.*

(2) *If T is a torus of maximal dimension, and $c_0(L,T) = 1$, then $\tilde{L}_0$ is abelian.*

Proof. (1). Suppose that $c_0(L,T) = 0$. Let $x \in U_0$, and put $\varphi := (id_L \otimes x) \circ j$. Then $\mathcal{V}_{\tilde{L}_0(x)} = \{0\}$ and [29, (II.3.10)] shows that $\tilde{L}_0(x) = C_L(\varphi(T))$ is a torus. Since $C_L(\varphi(T))$ is self-normalizing, it is a Cartan subalgebra of L. Thus, $(Aj(T))(x) = \varphi(T)$ is a maximal torus of L, and we conclude that $(Aj(T))(x) = \tilde{L}_0(x)$ for every $x \in U_0$. Since $Aj(T)$ and $\tilde{L}_0$ are A-direct summands of $\tilde{L}$, we have

$$\dim_k C_L(T) = \dim_k \tilde{L}_0(x_0) = \dim_k \tilde{L}_0(x) = \dim_k (Aj(T))(x) = \dim_k (Aj(T))(x_0) = \dim_k T,$$

so that $T = C_L(T)$.

Conversely, assume that $T = C_L(T)$. Then we have $\dim_k (Aj(T))(x_0) = \dim_k \tilde{L}_0(x_0)$. As $Aj(T)$ and $\tilde{L}_0$ are A-direct summands of $\tilde{L}$, we conclude that $(Aj(T))(x) = \tilde{L}_0(x)$ for every $x \in \mathcal{X}_T(k)$. This shows that $c_0(L,T) = 0$.

(2). By assumption we have $\dim \mathcal{V}_{\tilde{L}_0(x)} = 1$ for every $x \in U_0$. Since T has maximal dimension, $(Aj(T))(x)$ is a maximal torus of L, and $\tilde{L}_0(x)$ is nilpotent (cf. [29, (II.4.1)]). It now follows from (3.1) that $\tilde{L}_0(x)$ is abelian $\forall \ x \in U_0$. Let $\{v_1, \ldots, v_n\}$ be a basis of L. Given $r, s \in \tilde{L}_0$ we write $[r,s] = \sum_{i=1}^n v_i \otimes a_i$, where $a_i \in A$. For $x \in U_0$ we have

$$\sum_{i=1}^n x(a_i)v_i = (id_L \otimes x)([r,s]) = [(id_L \otimes x)(r), (id_L \otimes x)(s)] = 0,$$

so that U_0 is contained in the zero locus $Z(\sum_{i=1}^n Aa_i)$ of the ideal $\sum_{i=1}^n Aa_i$ of A. Since $U_0 \subset \mathcal{X}_T(k)$ is dense, Hilbert's Nullstellensatz implies $a_i = 0$ for all i, whence $[r,s] = 0$. $\square$

We continue by recording basic facts concerning tori. The maximum dimension of all tori of L will be denoted $\mu(L)$.

Lemma 3.3 *Let $I \subset L$ be a p-ideal. Then the following statements hold:*

(1) *If $T \subset L$ is a torus of maximal dimension, then $(T+I)/I \subset L/I$ is a torus of maximal dimension.*

(2) $\mu(L) = \mu(I) + \mu(L/I)$.

(3) *Let $T \subset L$ be a maximal torus. Then I is p-nilpotent if and only if $T \cap I = (0)$.*

Proof. (1). We put $L' := L/I$ and consider the canonical projection $\pi : L \longrightarrow L'$. Note that $T' := \pi(T) = (T+I)/I$ is a torus of L'. Let $S \subset L'$ be a torus of maximal dimension. Pick a maximal torus T_1 of $\pi^{-1}(S)$ such that $T_1 \cap I$ is a torus of I of maximal dimension. By [29, (II.4.5)] we have $\pi(T_1) = S$, so that

$$\dim_k S = \dim_k T_1 - \dim_k T_1 \cap I \leq \dim_k T - \dim_k T_1 \cap I \leq \dim_k T - \dim_k T \cap I = \dim_k T'.$$

Consequently, we have equality throughout, and T' has maximal dimension.

(2). The foregoing calculation shows $\mu(L') = \mu(L) - \mu(L \cap I)$.

(3). Suppose that $T \cap I = (0)$. We decompose I into its weight spaces relative to T, and write $I = I_0 \oplus \bigoplus_{\alpha\in\Phi} I_\alpha$. Since T commutes with I_0, our current assumption implies that I_0 does not contain any semisimple elements. Let $x \in I_\alpha$. According to [29, (II.3.5)] the element $x = x_s + x_n$ decomposes into its semisimple and p-nilpotent parts. Since $I_\alpha^{[p]} \subset I_0$, we see that $x_s \in I_0$ is semisimple. Thus, $x = x_n$ is p-nilpotent. By the Engel-Jacobson Theorem (cf. [29, (I.3.1)]) the ideal I is nilpotent. Since $T(I) = T(L) \cap I \subset T \cap I = (0)$, our assertion follows. □

We let $\mathcal{G}_L := \mathcal{AUT}(L)^\circ$ be the connected component of the automorphism scheme of the restricted Lie algebra $(L,[p])$. Given $\Psi \subset T^*$, we put $L^\Psi := \tilde{L}^\Psi(x_0) = \bigoplus_{\alpha\in\Psi} L_\alpha$ and let Q^Ψ be the largest $\mathcal{G}_L$-invariant ideal of L contained in L^Ψ. For $\alpha \in \Phi \cup \{0\}$, let $V^{(\alpha)} := \bigoplus_{\beta\in\Phi\setminus \mathrm{GF}(p)\alpha} L_\beta$. Then $V^{(\alpha)}$ is an $L^{(\alpha)}$-submodule of L, and $C_{L^{(\alpha)}}(V^{(\alpha)})$ is the kernel of the representation $L^{(\alpha)} \longrightarrow \mathrm{gl}(V^{(\alpha)})$. In particular, $Q^{(\alpha)} \subset C_L(V^{(\alpha)})$.

Lemma 3.4 *Let $T \subset L$ be a torus, M a restricted L-module, $\Psi \subset T^*$.*

(1) *If $N \subset M$ is an L-submodule such that there is $\Gamma \subset \Lambda$ with $N \subset \bigoplus_{\lambda\in\Gamma} \tilde{M}_\lambda(y)$ for some $y \in \mathcal{X}_T(k)$, then $N \subset \bigoplus_{\lambda\in\Gamma} \tilde{M}_\lambda(x)$ for all $x \in \mathcal{X}_T(k)$.*

(2) $Q^\Psi = \bigcap_{x\in\mathcal{X}_T(k)} \tilde{L}^\Psi(x)$

(3) *If $c_\Psi(L,T) = \dim \mathcal{V}_L$, then $\dim Q^\Psi \cap \mathcal{V}_L = \dim \mathcal{V}_L$.*

Proof. (1). We consider the restricted $\tilde{L}$-module $\tilde{N} := N \otimes_k A \subset \tilde{M}$. Let $\tilde{N} = \bigoplus_{\gamma\in\Omega} \tilde{N}_\gamma$ be its weight space decomposition. Then we have

$$N = \tilde{N}(x) = \bigoplus_{\gamma\in\Omega} \tilde{N}_\gamma(x) \quad \forall\ x \in \mathcal{X}_T(k).$$

Since $\tilde{N}_\gamma$ is an A-direct summand of $\tilde{N}$, the function $x \mapsto \dim_k \tilde{N}_\gamma(x)$ is constant on $\mathcal{X}_T(k)$. Hilbert's Nullstellensatz now yields $\tilde{N}_\gamma(y) \neq (0)$ for every $\gamma \in \Omega$. Consequently, $\Omega \subset \Gamma$, as desired.

(2). Thanks to [13, (3.4)] the A-module $\tilde{L}^\Psi$ is invariant under the diagonal action of $\mathcal{G}_L$ on $\tilde{L}$. We may now apply [13, (2.2)] to see that $I_{\mathcal{X}_T(k)}(\tilde{L}^\Psi) := \bigcap_{x\in\mathcal{X}_T(k)} \tilde{L}^\Psi(x) \subset L^\Psi$ is a $\mathcal{G}_L$-invariant ideal of L. Consequently, $I_{\mathcal{X}_T(k)}(\tilde{L}^\Psi) \subset Q^\Psi$. Since $Q^\Psi \subset L^\Psi = \tilde{L}^\Psi(x_0)$ is a submodule of L, the reverse inclusion follows directly from (1).

(3). As $c_\Psi(L,T) = \dim \mathcal{V}_L$, [13, (2.1)] and (2) imply that

$$\dim Q^\Psi \cap \mathcal{V}_L = \dim I_{\mathcal{X}_T(k)}(\tilde{L}^\Psi) \cap \mathcal{V}_L = \dim \mathcal{V}_L. \qquad \square$$

In the following $\mathrm{rad}_p(L)$ will denote the p-nilpotent radical of the restricted Lie algebra $(L,[p])$. Our next result extends [13, (4.3)] by providing information on the root systems of certain Lie algebras of α_p-rank 2.

Lemma 3.5 *Let L be centerless and of α_p-rank 2. Suppose that $T \subset L$ is a torus of maximal dimension such that $c_0(L,T) = 1$. Then there exists a dense open subset $U \subset \mathcal{X}_T(k)$ and a root $\alpha_0 \in \Phi$ such that $\dim \mathcal{V}_{\tilde{L}^{(\alpha_0)}}(x) = 2$ for every $x \in U$.*

Proof. Since $c_0(L,T) = 1$ we have $1 \leq c_\alpha(L,T) \leq 2$ for every $\alpha \in \Phi$. Suppose $c_\alpha(L,T) = 1$ for every $\alpha \in \Phi$. Then there exists a nonempty open subset $O_1 \subset \mathcal{X}_T(k)$ such that $\dim \mathcal{V}_{\tilde{L}^{(\alpha)}(x)} = 1$ for every $x \in O_1$ and $\alpha \in \Phi$. Since T is a torus of maximal dimension, $\tilde{L}_0(x)$ is a Cartan subalgebra of L, and (3.1) implies

$$\tilde{L}^{(\alpha)}(x) = \tilde{L}_0(x) \oplus \tilde{L}_\alpha(x) \qquad \forall\, x \in O_1,$$

as well as

$$\dim_k \tilde{L}_\alpha(x) = 1 \qquad \forall\, x \in O_1.$$

As the function $x \mapsto \dim_k \tilde{L}_\alpha(x)$ is constant on $\mathcal{X}_T(k)$, we conclude that $\dim_k L_\alpha = 1$ for every $\alpha \in \Phi$.

Owing to (3.2(2)) the algebra $H := C_L(T) = \tilde{L}_0(x_0)$ is abelian, so we have a decomposition $H = T \oplus \mathrm{rad}_p(H)$. In view of the above $[\mathrm{rad}_p(H), L_\alpha] = (0)$, and we conclude that $\mathrm{rad}_p(H)$ is contained in $C(L) = (0)$. Thus $H = T$ is a torus, and (3.2(1)) yields $c_0(L,T) = 0$, a contradiction. Accordingly, there exists a root $\alpha_0 \in \Phi$, and a dense open subset $U \subset \mathcal{X}_T(k)$ such that $\dim \mathcal{V}_{\tilde{L}^{(\alpha_0)}(x)} = 2$ for every $x \in U$. $\square$

4. Characteristic Semisimple Lie Algebras

In this section we will employ the geometric methods of §3 to classify certain Lie algebras of α_p-rank 2. Since the Lie algebras of tame, semisimple infinitesimal groups belong to this class, our results are also a key ingredient in our determination of these group schemes.

Throughout, we let $(L,[p])$ be a restricted Lie algebra with automorphism scheme $\mathcal{G}_L = \mathcal{AUT}(L)^\circ$. Recall our general convention $p \geq 3$.

We begin by recording a few basic properties of Lie algebras whose roots generate a $\mathrm{GF}(p)$-vector space of dimension 1.

Lemma 4.1 *Let $T \subset L$ be a maximal torus. Suppose that $T(L) = (0)$ and that there is $\alpha \in \Phi$ such that $\Phi \subset \mathrm{GF}(p)\alpha$.*

(1) *Let $\beta \in \Phi$ be a root. Every element $x \in L_\beta$ is p-nilpotent.*

(2) *If $I \subset L$ is a p-ideal, then either I is p-nilpotent, or $\sum_{\beta\in\Phi}[L_\beta, L_{-\beta}] \oplus \bigoplus_{\beta\in\Phi} L_\beta \subset I$.*

(3) *If $[L_\beta, L_{-\beta}] = (0)$ for every $\beta \in \Phi$, then L is solvable.*

Proof. (1). Given $\beta \in \Phi$, we decompose an element $x \in L_\beta$ into its semisimple and nilpotent parts, and write $x = x_s + x_n$. Then $x_s \in (kx^{[p]})_p \subset C_L(T) = H$ is semisimple, so that $x_s \in T$. Since $0 = [x_s, x] = \beta(x_s)x$, it follows that $x_s \in \ker\beta$. Thus, $x_s \in \bigcap_{\gamma\in\Phi}\ker\gamma = T(L) = (0)$. As a result, $x = x_n$ is p-nilpotent.

(2). Let $I \subset L$ be a p-ideal. Since $T(L) = (0)$, it follows that $\dim_k T \leq 1$. If $I \cap T = (0)$, then, (3.3(3)) yields the p-nilpotence of I. Alternatively, $T \subset I$, and we have $L_\beta \subset I$ for every $\beta \in \Phi$, so that $\bigoplus_{\beta\in\Phi} L_\beta \subset I$. This readily implies our assertion.

(3). In view of (1), our assumption implies that $\bigcup_{\beta\in\Phi} L_\beta$ is a nil Lie set of L. By the Engel-Jacobson theorem, the ideal $\bigoplus_{\beta\in\Phi} L_\beta$ is nilpotent. Consequently, the Lie algebra L is solvable. □

Definition. A Lie algebra $L \neq (0)$ is called *characteristic semisimple* if it does not possess any non-zero solvable $\mathcal{G}_L$-invariant p-ideals.

Theorem 4.2 *If L is characteristic semisimple and of α_p-rank 2, then $L \cong s\ell(2)$.*

Proof. Let $T \subset L$ be a torus of maximal dimension. Since L is characteristic semisimple, the Cartan subalgebra $H = C_L(T)$ does not contain any non-zero $\mathcal{G}_L$-invariant p-ideals of L. Thus, (3.4(3)) yields $0 \leq c_0(L,T) \leq 1$. Assume that $c_0(L,T) = 1$.

(i) *There exists $\alpha_0 \in \Phi$ such that $\Phi \subset \mathrm{GF}(p)\alpha_0$.*

Since $c_0(L,T) = 1$, Lemma 3.5 applies and there exists a root $\alpha_0 \in \Phi$ such that $c_{\alpha_0}(L,T) = 2$. Owing to (3.4(3)) the p-ideal $Q^{(\alpha_0)}$ has α_p-rank 2. Since $T(Q^{(\alpha_0)}) \subset T(L) = (0)$, we have $\ker\alpha_0|_{T\cap Q^{(\alpha_0)}} = (0)$, so that $\dim_k T \cap Q^{(\alpha_0)} \leq 1$.

If $Q^{(\alpha_0)} \cap T = (0)$ we apply (3.3(3)) to see that $Q^{(\alpha_0)}$ is a p-nilpotent $\mathcal{G}_L$-invariant p-ideal of L. This, however, contradicts the characteristic semisimplicity of L.

Thus, $Q^{(\alpha_0)} \cap T$ is one-dimensional, and $C_L(Q^{(\alpha_0)}) \cap Q^{(\alpha_0)} \subset C(Q^{(\alpha_0)}) \subset Q_0^{(\alpha_0)} \subset H$ is a $\mathcal{G}_L$-invariant p-ideal of L. This implies $C_L(Q^{(\alpha_0)}) \cap Q^{(\alpha_0)} = (0)$, so that

$$2 = \dim\mathcal{V}_{Q^{(\alpha_0)}\oplus C_L(Q^{(\alpha_0)})} = \dim\mathcal{V}_{Q^{(\alpha_0)}} + \dim\mathcal{V}_{C_L(Q^{(\alpha_0)})} = 2 + \dim\mathcal{V}_{C_L(Q^{(\alpha_0)})}.$$

Hence $C_L(Q^{(\alpha_0)})$ is a toral p-ideal and $C_L(Q^{(\alpha_0)}) = (0)$. Since $L_\beta \subset C_L(Q^{(\alpha_0)})$ for all $\beta \in \Phi \setminus \mathrm{GF}(p)\alpha_0$, we obtain $\Phi \subset \mathrm{GF}(p)\alpha_0$.

(ii) $\Phi = \{\alpha_0, -\alpha_0\}$.

Since L is not solvable, (4.1(3)) provides a root $\beta_0 \in \Phi$ such that $L_{-\beta_0} \neq (0)$. We consider $\Psi := \{0, \beta_0, -\beta_0\}$ as well as the A-direct summand $\tilde{L}^\Psi = \tilde{L}_0 \oplus \tilde{L}_{\beta_0} \oplus \tilde{L}_{-\beta_0}$ of $\tilde{L}$. Note that $1 \leq c_\Psi(L,T) \leq 2$.

Suppose that $c_\Psi(L,T) = 1$. By switching to another element of $\mathcal{X}_T(k)$ if necessary we may assume without loss of generality that $\dim L^\Psi \cap \mathcal{V}_L \leq 1$. By (3.2(2)) H is abelian, so there are one-dimensional H-submodules $kx \subset L_{\beta_0}$, and $ky \subset L_{-\beta_0}$. Hence $S := H \oplus kx \oplus ky$ is a p-subalgebra of L that is contained in L^{Ψ}. This implies $\dim \mathcal{V}_S \leq 1$, which contradicts (3.1).

Consequently, $c_{\Psi}(L,T) = 2$, and (3.4(3)) implies $\dim Q^{\Psi} \cap \mathcal{V}_L = 2$. Since $Q^{\Psi} \subset L^{\Psi}$, we have $Q^{\Psi} = Q^{\Psi}_0 \oplus Q^{\Psi}_{\beta_0} \oplus Q^{\Psi}_{-\beta_0}$. Thus, $Q^{\Psi}_p \subset L^{\Psi}$ is a $\mathcal{G}_L$-invariant p-ideal, which, by characteristic semisimplicity, is not p-nilpotent. Hence (4.1(2)) yields $\bigoplus_{\beta\in\Phi} L_\beta \subset Q^{\Psi}$, and $\Phi \subset \{\beta_0, -\beta_0\}$. Since $L_{\alpha_0} \neq (0)$, our assertion follows.

For $i \in \{1,-1\}$ we put $\Psi_i := \{0, i\alpha_0\}$, and consider $\tilde{L}^{\Psi_i} = \tilde{L}_0 \oplus \tilde{L}_{i\alpha_0}$. If $c_{\Psi_i}(L,T) = 2$, then the arguments employed in (ii) provide a nonzero $\mathcal{G}_L$-invariant p-ideal contained in L^{Ψ_i}. By (4.1(2)) we thus have $L_{-i\alpha_0} \subset L^{\Psi_i}$. As $p \geq 3$, this contradicts (ii).

Consequently, $c_{\Psi_i}(L,T) = 1$ for $i = 1,-1$, and by switching to another torus of $\mathcal{X}_T(k)$ if necessary, we may assume that

(a) $H = kt \oplus (kz_0)_p$ is abelian, $z_0^{[p]^n} = 0 \neq z_0^{[p]^{n-1}}$ for some $n \geq 1$, $\alpha_0(t) = 1$, and

(b) $\dim \mathcal{V}_L \cap (H \oplus L_{\alpha_0}) = 1 = \dim \mathcal{V}_L \cap (H \oplus L_{-\alpha_0})$.

We consider the p-subalgebra $Q = C_L(z_0) = C_L(\mathrm{rad}_p(H))$. Since the maximal torus $T = kt$ is contained in Q, we have $Q = Q_{\alpha_0} \oplus H \oplus Q_{-\alpha_0}$.

(iii) $L_\gamma = Q_\gamma$ for $\gamma \in \{\alpha_0, -\alpha_0\}$.

Suppose that $Q_{\alpha_0} \neq L_{\alpha_0}$. From the decomposition of L_{α_0} into its $(ad z_0)$-cyclic spaces we obtain the existence of $x \in Q_{\alpha_0}$ and $y \in L_{\alpha_0}$ such that $x = [z_0, y]$. Let $q := [x,y]$. We propose to show that $q = 0$. Since $q \in L_{2\alpha_0}$ this is clear for $p \geq 5$. We thus assume that $p = 3$. As $[z_0, q] = [[z_0,x],y] + [x,[z_0,y]] = [x,x] = 0$, the element q belongs to $Q_{-\alpha_0}$. In view of (a), (4.1(1)) entails $L_\gamma^{[3]} \subset \mathrm{rad}_3(H)$ for $\gamma \in \{\alpha_0, -\alpha_0\}$. Observing $[q,y] \in H$, we thus obtain $\lambda, \xi \in k$, $v \in \mathrm{rad}_3(H)^{[3]}$, and $w \in \mathrm{rad}_3(H)$ such that

$$q^{[3]} = \lambda z_0 + v \quad ; \quad [q,y] = \xi t + w.$$

As $q \in Q_{-\alpha_0}$ we have $[q,[q,y]] = \xi[q,t] = \xi q$, whence $[q^{[3]}, y] = (ad\, q)^3(y) = 0$. By virtue of $[\mathrm{rad}_3(H)^{[3]}, y] = (0)$, this yields

$$0 = [q^{[3]}, y] = [\lambda z_0 + v, y] = \lambda[z_0, y] = \lambda x,$$

so that $\lambda = 0$ and $q^{[3]} = z^{[3]}$ for some $z \in \mathrm{rad}_3(H)$. Since q and z commute, we have $q - z \in \mathcal{V}_L$. By the same token

$$k(q-z) + kz_0^{[3]^{n-1}} \subset \mathcal{V}_L \cap (H \oplus L_{-\alpha_0}),$$

and condition (b) now implies $q = 0$.

Returning to the general condition $p \geq 3$, we conclude that $S := H \oplus kx \oplus ky$ is a p-subalgebra of L, which, owing to (b), has α_p-rank 1. This, however, contradicts (3.1). We obtain $Q_{\alpha_0} = L_{\alpha_0}$, and the equality $Q_{-\alpha_0} = L_{-\alpha_0}$ follows analogously.

In view of (iii), the radical $\mathrm{rad}_p(H)$ operates trivially on the root spaces. Since H is abelian, we obtain $\mathrm{rad}_p(H) \subset C(L) = (0)$. Consequently, $H = T$, and (3.2(1)) yields $c_0(L,T) = 0$, a contradiction.

Hence we have $c_0(L,T)=0$. Since $T(L)=(0)$ a consecutive application of (3.2(1)) and [13, (4.3)] now implies $L\cong s\ell(2)$. $\square$

In the sequel, we let $R(L)$ be the solvable radical of the Lie algebra L. Note that $\mathrm{Lie}(\mathcal{R}(\mathcal{G}))\subset R(\mathrm{Lie}(\mathcal{G}))$ for every infinitesimal group $\mathcal{G}$.

Corollary 4.3 *Let $\mathcal{G}$ be an infinitesimal group such that $\mathcal{B}_0(\mathcal{G})$ is tame.*

(1) *If $\mathcal{G}$ is semisimple, then* $\mathrm{Lie}(\mathcal{G})\cong s\ell(2)$.

(2) *We have* $\mathrm{Lie}(\mathcal{G})/R(\mathrm{Lie}(\mathcal{G}))\cong s\ell(2)$ *as well as* $R(\mathrm{Lie}(\mathcal{G}))=\mathrm{Lie}(\mathcal{R}(\mathcal{G}))$.

Proof. (1). Since $\mathcal{B}_0(\mathcal{G})$ is tame, (2.5) shows that the characteristic semisimple Lie algebra $\mathrm{Lie}(\mathcal{G})$ has α_p-rank 2. Our assertion thus follows directly from (4.2).

(2). Let $\bar{\mathcal{G}}:=\mathcal{G}/\mathcal{R}(\mathcal{G})$. According to (2.4) $\bar{\mathcal{G}}\neq e_k$ is not supersolvable, and (1.1(3)) implies that $\mathcal{B}_0(\bar{\mathcal{G}})$ is tame. Thanks to (1) we have $\mathrm{Lie}(\bar{\mathcal{G}})\cong s\ell(2)$. Consider the exact sequence

$$(0)\longrightarrow \mathrm{Lie}(\mathcal{R}(\mathcal{G}))\longrightarrow \mathrm{Lie}(\mathcal{G})\longrightarrow \mathrm{Lie}(\bar{\mathcal{G}}).$$

If $\mathrm{Lie}(\mathcal{G})/\mathrm{Lie}(\mathcal{R}(\mathcal{G}))$ has α_p-rank ≤ 1, then [11, (2.1)] implies that $\mathrm{Lie}(\mathcal{G})$ is solvable. Consequently, $\mathcal{G}$ is solvable, which contradicts (2.4). As a result, $\mathrm{Lie}(\mathcal{G})/\mathrm{Lie}(\mathcal{R}(\mathcal{G}))$ is, as a p-subalgebra of $\mathrm{Lie}(\bar{\mathcal{G}})\cong s\ell(2)$ of α_p-rank 2, isomorphic to the latter. This readily yields the desired result. $\square$

5. Semisimple Infinitesimal Groups of Tame Type

In this section we turn to the classification of semisimple infinitesimal groups with tame principal blocks. We begin by considering two classess of such groups. For a natural number $n\ge 1$ we let $\mathcal{A}_{[n]}$ and $\mathcal{Q}_{[n]}$ be the closed subgroups of SL(2) that are given by

$$\mathcal{A}_{[n]}(R):=\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2)(R)\ ;\ a^{p^n}=1=d^{p^n},\ b^{p^2}=0=c^p\}$$

and

$$\mathcal{Q}_{[n]}(R):=\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2)(R)\ ;\ a^{p^n}=1=d^{p^n},\ b^{p}=0=c^p\}$$

for every commutative k-algebra R, respectively. Note that $\mathcal{Q}_{[n]}$ is the n-th Frobenius kernel of the product $\mathrm{SL}(2)_1\mathrm{T}$ of the first Frobenius kernel of SL(2) with the standard torus $\mathrm{T}\subset\mathrm{SL}(2)$.

Let $\mathcal{G}=\mathcal{A}_{[n]},\ \mathcal{Q}_{[n]}$. Recall that η and ε denote the antipode and the augmentation of $H(\mathcal{G})$, respectively. As before we will write the composition of the character group $\mathcal{X}(\mathcal{G})$ additively. Note that $\mathcal{X}(\mathcal{G})\cong\mathcal{X}(\mathcal{G}/\mathcal{G}_1)\cong\mathcal{X}(\mu_{p^{n-1}})\cong\mathbb{Z}/(p^{n-1})$. In the sequel we will occasionally identify elements of $\mathcal{G}$ with homomorphisms $\mathcal{G}\longrightarrow\mu_k$.

Throughout this section the notions "$\mathcal{G}$-module" and "$H(\mathcal{G})$-module" will be used interchangeably. Given a $\mathcal{G}$-module M and $\lambda\in\mathcal{X}(\mathcal{G})$, we put $M_\lambda:=M\otimes_k k_\lambda$. Note that $M\mapsto M_\lambda$ is an auto-equivalence of the module category $\mathrm{mod}_{H(\mathcal{G})}$. The invariants of M relative to a subgroup $\mathcal{H}\subset\mathcal{G}$ will be denoted $M^{\mathcal{H}}$. We will occasionally write $M|_{\mathcal{H}}$ for the restriction of M to the Hopf algebra $H(\mathcal{H})\subset H(\mathcal{G})$.

Lemma 5.1 *Suppose that $\mathcal{G} = \mathcal{A}_{[n]},\ \mathcal{Q}_{[n]}$, and let M, N be simple $\mathcal{G}$-modules.*

(1) *If $\mathrm{Hom}_{H(\mathcal{G}_1)}(M,N) \neq (0)$, then there is $\lambda \in \mathcal{X}(\mathcal{G})$ such that $N \cong M_\lambda$.*

(2) *$M|_{\mathcal{G}_1}$ is simple.*

(3) *$M \cong M_\lambda$ if and only if $\lambda = \varepsilon$.*

Proof. (1). We endow $\mathrm{Hom}_k(M,N)$ with the canonical structure of a $H(\mathcal{G})$-module via

$$(x\cdot f)(m) := \sum_{(x)} x_{(1)} f(\eta(x_{(2)})m) \quad \forall\ f \in \mathrm{Hom}_k(M,N),\ m \in M.$$

Since $\mathcal{G}_1$ is a normal subgroup of $\mathcal{G}$, the subspace $\mathrm{Hom}_{H(\mathcal{G}_1)}(M,N)$ obtains the structure of a $H(\mathcal{G}/\mathcal{G}_1)$-module. As $\mathcal{G}/\mathcal{G}_1 \hookrightarrow \alpha_p \times \mu_{p^{n-1}}$ is trigonalizable, there exists a character $\lambda \in \mathcal{X}(\mathcal{G})$ with $\lambda|_{H(\mathcal{G}_1)} = \varepsilon$, and a non-zero $\varphi \in \mathrm{Hom}_k(M,N)$ such that

$$x\cdot\varphi = \lambda(x)\varphi \quad \forall\ x \in H(\mathcal{G}).$$

Consequently, observing the cocommutativity of $H(\mathcal{G})$ along with $\eta^2 = id_{H(\mathcal{G})}$, we obtain,

$$\varphi(xm) = \sum_{(x)} \varepsilon(x_{(1)})\varphi(x_{(2)}m) = \sum_{(x)} x_{(1)}\eta(x_{(2)})\varphi(x_{(3)}m) = \sum_{(x)} x_{(1)}\lambda(\eta(x_{(2)}))\varphi(m).$$

As a result, $\varphi : M \longrightarrow N_{-\lambda}$ is, as a non-zero homomorphism between two simple $\mathcal{G}$-modules, an isomorphism. Hence $M \cong N_{-\lambda}$, so that $N \cong M_\lambda$.

(2). Let $V \subset M|_{\mathcal{G}_1}$ be a simple $\mathcal{G}_1$-module. According to [20, (II.3.15)] there is an SL(2)-module structure on V that restricts to the given $\mathcal{G}_1$-structure. Consequently, V is a simple $\mathcal{G}$-module such that $V|_{\mathcal{G}_1} \hookrightarrow M|_{\mathcal{G}_1}$. By (1) we have $V \cong M_\lambda$ for some $\lambda \in \mathcal{X}(\mathcal{G})$, so that $V|_{\mathcal{G}_1} \cong (M_\lambda)|_{\mathcal{G}_1} \cong M|_{\mathcal{G}_1}$. Consequently, $M|_{\mathcal{G}_1}$ is simple.

(3). Suppose that $M \cong M_\lambda$. According to (2) and Schur's Lemma the $H(\mathcal{G}/\mathcal{G}_1)$-module $\mathrm{Hom}_{H(\mathcal{G}_1)}(M,M_\lambda)$ is one-dimensional. By the same token, $\mathrm{Hom}_{H(\mathcal{G}_1)}(M,M) = \mathrm{Hom}_{H(\mathcal{G})}(M,M)$ is the trivial $\mathcal{G}$-module, and we have

$$\mathrm{Hom}_{H(\mathcal{G}_1)}(M,M_\lambda) \cong (M^*\otimes_k M\otimes_k k_\lambda)^{\mathcal{G}_1} \cong (M^*\otimes_k M)^{\mathcal{G}_1}\otimes_k k_\lambda \cong \mathrm{Hom}_{H(\mathcal{G}_1)}(M,M)\otimes_k k_\lambda \cong k_\lambda.$$

Let φ be the given isomorphism. Then φ induces an isomorphism

$$\mathrm{Hom}_{H(\mathcal{G}_1)}(M,M) \longrightarrow \mathrm{Hom}_{H(\mathcal{G}_1)}(M,M_\lambda) \ \ ; \ \ f \mapsto \varphi\circ f$$

of $\mathcal{G}$-modules. Consequently, $k \cong k_\lambda$, and $\lambda = \varepsilon$. $\square$

Let $\mathcal{G} = \mathcal{A}_{[n]},\ \mathcal{Q}_{[n]}$ and consider the characters $\nu,\ \alpha : \mathcal{G} \longrightarrow \mu_k$ that are given by

$$\nu(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a^{2p} \quad \text{and} \quad \alpha(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = d^p,$$

respectively. As $p \geq 3$, both characters generate $\mathcal{X}(\mathcal{G})$, and we have $\nu = -2\alpha$.

In order to determine the Gabriel quiver of the Hopf algebra $H(\mathcal{G})$, we require the following two subsidiary results.

Proposition 5.2 *Suppose that $\mathcal{G} = \mathcal{A}_{[n]}$, $\mathcal{Q}_{[n]}$, and let M, N be simple $\mathcal{G}$-modules.*

(1) *If $\mathcal{G} = \mathcal{Q}_{[n]}$, then we have $\mathrm{Ext}^1_{H(\mathcal{G})}(M, M_\lambda) = (0)$ for every $\lambda \in \mathcal{X}(\mathcal{G})$.*

(2) *If $\mathcal{G} = \mathcal{A}_{[n]}$, then we have $\dim_k \mathrm{Ext}^1_{H(\mathcal{G})}(M, M_\lambda) = \delta_{\nu,\lambda}$ for every $\lambda \in \mathcal{X}(\mathcal{G})$.*

(3) *If $M|_{\mathcal{G}_1} \not\cong N|_{\mathcal{G}_1}$, then we have an isomorphism $\mathrm{Ext}^1_{H(\mathcal{G})}(M, N) \cong \mathrm{Ext}^1_{H(\mathcal{G}_1)}(M, N)^{\mathcal{G}}$.*

Proof. We put $S := \mathrm{Hom}_k(M, N)$ and consider the following terms of cohomology five-term sequence (cf. [28, (11.2)]):

$$(0) \longrightarrow H^1(\mathcal{G}/\mathcal{G}_1, S^{\mathcal{G}_1}) \longrightarrow H^1(\mathcal{G}, S) \longrightarrow H^1(\mathcal{G}_1, S)^{\mathcal{G}} \longrightarrow H^2(\mathcal{G}/\mathcal{G}_1, S^{\mathcal{G}_1}).$$

Turning to (1) and (2) we set $N := M_\lambda$. According to (5.1(2)) and ([26, Thm.1] or [14, p.15]) we have $H^1(\mathcal{G}_1, S) \cong \mathrm{Ext}^1_{H(\mathrm{SL}(2)_1)}(M, M) = (0)$. There thus results an isomorphism

$$H^1(\mathcal{G}, S) \cong H^1(\mathcal{G}/\mathcal{G}_1, S^{\mathcal{G}_1}).$$

From the proof of (5.1(3)) we recall that $S^{\mathcal{G}_1} \cong \mathrm{Hom}_{H(\mathcal{G}_1)}(M, M_\lambda) \cong k_\lambda$. Accordingly, we obtain

$$\mathrm{Ext}^1_{H(\mathcal{G})}(M, M_\lambda) \cong H^1(\mathcal{G}/\mathcal{G}_1, k_\lambda).$$

Here we have identified λ with the corresponding character $\mathcal{G}/\mathcal{G}_1 \longrightarrow \mu_k$. Part (1) now follows from the fact that the Hopf algebra $H(\mathcal{G}/\mathcal{G}_1)$ is semisimple. Now suppose that $\mathcal{G} = \mathcal{A}_{[n]}$. Since $\mathcal{M}(\mathcal{G}/\mathcal{G}_1) = e_k$ we may apply [11, (2.4)] to see that the algebra $H(\mathcal{G}/\mathcal{G}_1)$ has exactly one block. Let $\omega : \mathcal{G}/\mathcal{G}_1 \longrightarrow \mu_k$ be the character induced by ν. By the proof [12, (4.1)] we have $\dim_k H^1(\mathcal{G}/\mathcal{G}_1, k_\lambda) = \delta_{\lambda,\omega}$, whence $\dim_k \mathrm{Ext}^1_{H(\mathcal{G})}(M, M_\lambda) = \delta_{\lambda,\nu}$.

(3). By virtue of (5.1(2)) the restrictions of M and N to $\mathcal{G}_1$ are simple, so that $S^{\mathcal{G}_1} = (0)$. The desired isomorphism now follows directly from the above exact sequence. □

We recall a few basic facts concerning simple SL(2)-modules. For each natural number $n \geq 0$ there exists a unique simple SL(2)-module $L(n)$ whose highest weight is the character of T that raises the $(1,1)$-entry of a diagonal matrix to its n-th power. For $0 \leq n \leq p-1$, the module $L(n)|_{\mathrm{SL}(2)_1}$ is simple of dimension $n+1$, and every simple $\mathrm{SL}(2)_1$-module arises in this fashion. Since all simple SL(2)-modules are self-dual (cf. [20, (II.3.13)]), a $\mathcal{G}_1$-module $V = L(n)|_{\mathrm{SL}(2)_1}$ gives rise to a self-dual $\mathcal{G}$-module $\tilde{V} = L(n)|_{\mathcal{G}}$. Thanks to (5.1) $\tilde{V}$ is the unique self-dual $\mathcal{G}$-module whose restriction coincides with V. Let W be another simple $\mathcal{G}_1$-module. Using standard properties of duality, and the fact that $X \mapsto X_{\lambda+\gamma}$ is an auto-equivalence of $\mathrm{mod}_{H(\mathcal{G})}$, we obtain

$$\mathrm{Ext}^1_{H(\mathcal{G})}(\tilde{V}_\lambda, \tilde{W}_\gamma) \cong \mathrm{Ext}^1_{H(\mathcal{G})}((\tilde{W}_\gamma)^*, (\tilde{V}_\lambda)^*) \cong \mathrm{Ext}^1_{H(\mathcal{G})}(\tilde{W}_{-\gamma}, \tilde{V}_{-\lambda}) \cong \mathrm{Ext}^1_{H(\mathcal{G})}(\tilde{W}_\lambda, \tilde{V}_\gamma).$$

Proposition 5.3 *Let $\lambda \in \mathcal{X}(\mathcal{G})$ be a character. Suppose that $V \not\cong W$ are two simple $\mathcal{G}_1$-modules such that $\dim_k \mathrm{Ext}^1_{H(\mathcal{G}_1)}(V, W) = 2$.*

(1) *If $\mathcal{G} \cong \mathcal{A}_{[n]}$, then $\dim_k \mathrm{Ext}^1_{H(\mathcal{G})}(\tilde{V}, \tilde{W}_\lambda) = \delta_{\lambda,\alpha} = \dim_k \mathrm{Ext}^1_{H(\mathcal{G})}(\tilde{W}, \tilde{V}_\lambda)$.*

(2) *If $\mathcal{G} \cong \mathcal{Q}_{[n]}$, then $\dim_k \mathrm{Ext}^1_{H(\mathcal{G})}(\tilde{V}, \tilde{W}_\lambda) = \delta_{\lambda,\alpha} + \delta_{\lambda,-\alpha} = \dim_k \mathrm{Ext}^1_{H(\mathcal{G})}(\tilde{W}, \tilde{V}_\lambda)$.*

Proof. We write $V = L(n)|_{\mathrm{SL}(2)_1}$, $W = L(m)|_{\mathrm{SL}(2)_1}$, and consider the $\mathcal{G}$-modules $\tilde{V}$ and $\tilde{W}$. Since $n+m = p-2$ (cf. [26, Thm.1]) the main theorem of [25, p.171] provides an isomorphism

$$\mathrm{Ext}^1_{H(\mathcal{G}_1)}(V, W) \cong L(1)^{(p)}$$

of SL(2)-modules. Here the $L(1)^{(p)}$ is the Frobenius twist of the natural SL(2)-module $L(1)$.

Given an arbitrary $H(\mathcal{G})$-module M, we consider the functor $\mathrm{Hom}_{H(\mathcal{G}_1)}(M, \cdot) : \mathrm{mod}_{H(\mathcal{G})} \longrightarrow \mathrm{mod}_{H(\mathcal{G})}$. For $\lambda \in \mathcal{X}(\mathcal{G})$ the twist $T_\lambda(X) = X_\lambda$ is an auto-equivalence of $\mathrm{mod}_{H(\mathcal{G})}$ such that

$$\mathrm{Hom}_{H(\mathcal{G}_1)}(M, T_\lambda(N)) \cong (M^* \otimes_k N \otimes_k k_\lambda)^{\mathcal{G}_1} \cong (M^* \otimes_k N)^{\mathcal{G}_1} \otimes_k k_\lambda \cong T_\lambda(\mathrm{Hom}_{H(\mathcal{G}_1)}(M, N)).$$

Thus, the functors $T_\lambda \circ \mathrm{Hom}_{H(\mathcal{G}_1)}(M, \cdot)$ and $\mathrm{Hom}_{H(\mathcal{G}_1)}(M, \cdot) \circ T_\lambda$ are naturally equivalent, and the same holds for the derived functors $\mathrm{Ext}^n_{H(\mathcal{G}_1)}(M, \cdot)$. There results an isomorphism $\mathrm{Ext}^1_{H(\mathcal{G}_1)}(M, N_\lambda) \cong \mathrm{Ext}^1_{H(\mathcal{G}_1)}(M, N) \otimes_k k_\lambda$ of $H(\mathcal{G})$-modules, so that in particular

$$\mathrm{Ext}^1_{H(\mathcal{G}_1)}(\tilde{V}, \tilde{W}_\lambda) \cong L(1)^{(p)} \otimes_k k_\lambda$$

as $H(\mathcal{G})$-modules. Since $\mathrm{Ext}^1_{H(\mathcal{G}_1)}(\tilde{V}, \tilde{W}_\lambda)^{\mathcal{G}} \cong \mathrm{Hom}_{H(\mathcal{G})}(k_{-\lambda}, L(1)^{(p)})$ the left-hand identities of (1) and (2) now follow from (5.2(3)) and the fact that $\mathrm{Soc}_{H(\mathcal{G})}(L(1)^{(p)}) \cong k_{-\alpha}$ for $\mathcal{G} = \mathcal{A}_{[n]}$ and $\mathrm{Soc}_{H(\mathcal{G})}(L(1)^{(p)}) \cong k_\alpha \oplus k_{-\alpha}$ for $\mathcal{G} = \mathcal{Q}_{[n]}$. The right-hand identities are a direct consequence of the observation preceding (5.3). □

Our last subsidiary result characterizes the groups studied above in terms of their representation theory.

Lemma 5.4 *Suppose that $\mathcal{G} \subset \mathrm{SL}(2)$ is an infinitesimal group of height $n \geq 1$ such that*
(a) *$\mathcal{G}_1 = \mathrm{SL}(2)_1$, and*
(b) *$\mathcal{B}_0(\mathcal{G}/\mathcal{G}_1)$ has finite representation type.*
Then the following statements hold:
(1) *If $\mathcal{G}/\mathcal{G}_1$ is not multiplicative, then $\mathcal{G} \cong \mathcal{A}_{[n]}$ for $n \geq 3$, and $\mathcal{G} \cong \mathcal{A}_{[1]},\ \mathcal{A}_{[2]}$ for $n = 2$.*
(2) *If $\mathcal{G}/\mathcal{G}_1$ is multiplicative, then $\mathcal{G} \cong \mathcal{Q}_{[n]}$.*

Proof. (1). A consecutive application of [35, (8.3)] and [4, (IV.§2(1.1))] shows that every infinitesimal, unipotent subgroup of SL(2) of height r is isomorphic to α_{p^r}. In view of [4, (IV.§3(6.11))] such a group is V-uniserial if and only if $r = 1$. Since the toral radical of $\mathrm{Lie}(\mathcal{G}/\mathcal{G}_1) \subset s\ell(2)$ is trivial, we have $\mathcal{M}(\mathcal{G}/\mathcal{G}_1) = e_k$. In view of (b), [11, (2.7)] provides a natural number $\ell \geq 0$ such that $\mathcal{G}/\mathcal{G}_1 \cong \alpha_p \times \mu_{p^\ell}$. Note that the height of $\mathcal{G}$ is $n = \ell + 1$ for $\ell \geq 2$, and 2 for $\ell = 0, 1$. Let $\mathcal{F} : \mathrm{SL}(2) \longrightarrow \mathrm{SL}(2)^{(p)}$ be the Frobenius homomorphism. By the above, there exists $g \in \mathrm{GL}(2)(k)$ such that

$$g\mathcal{F}(\mathcal{G})g^{-1}(R) = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}(2)^{(p)}(R)\ ;\ a^{p^\ell} = 1 = d^{p^\ell},\ b^p = 0\}$$

for every commutative k-algebra R. Since k is perfect, we can find an element $h \in \mathrm{GL}(2)(k)$ such that $\mathcal{F}(h) = g$. This implies $h\mathcal{G}h^{-1}(R) \subset \mathcal{A}_{[\ell+1]}(R)$. By (a) both groups have order $p^{\ell+4}$, and we obtain $\mathcal{G} \cong \mathcal{A}_{[\ell+1]}$, as desired.

(2). This can be shown analogously. $\square$

We are now in a position to establish the main result of this section. When dealing with path algebras and relations we will follow the conventions of [1, Chapter III]. Recall that an algebra Λ with basic algebra $k[Q]/I$ is called *special biserial* if

(a) any vertex of Q is the starting point and end point of at most two arrows, and

(b) for any arrow α there exists at most one arrow β ending at the starting point of α such that $\alpha \circ \beta \notin I$, and

(c) for any arrow α there exists at most one arrow γ starting at the endpoint of α such that $\gamma \circ \alpha \notin I$.

Theorem 5.5 *Let $\mathcal{G}$ be a semisimple infinitesimal group of characteristic $p \geq 3$. Then the following statements are equivalent:*

(1) *$\mathcal{B}_0(\mathcal{G})$ is tame.*

(2) *There exists $n \geq 1$ such that $\mathcal{G} \cong \mathcal{Q}_{[n]}$.*

(3) *$H(\mathcal{G})$ is special biserial.*

(4) *$H(\mathcal{G})$ is tame.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathcal{B}_0(\mathcal{G})$ is tame. According to (4.3) we have $\mathrm{Lie}(\mathcal{G}) \cong s\ell(2)$. Consequently, the adjoint representation of $\mathcal{G}$ gives rise to a homomorphism

$$\varrho : \mathcal{G} \longrightarrow \mathcal{AUT}(s\ell(2)) \cong \mathrm{PSL}(2)$$

of group schemes whose differential $\mathrm{Lie}(\varrho) : s\ell(2) \longrightarrow s\ell(2)$ is bijective. Hence ϱ is a closed embedding, and in view of [4, (II.§7 (4.1))] the image $\varrho(\mathcal{G})$ contains the first Frobenius kernel of PSL(2). As $\mathcal{G}$ is infinitesimal, the homomorphism ϱ factors through a suitable Frobenius kernel of PSL(2). But PSL(2) is a factor group of SL(2) by an étale subgroup, so $\mathrm{PSL}(2)_r \cong \mathrm{SL}(2)_r$ for every $r \geq 1$. Consequently, we may consider $\mathcal{G}$ a closed subgroup of SL(2) such that $\mathcal{G}_1 = \mathrm{SL}(2)_1$.

By (1.1(2)) the principal block of $\mathcal{G}/\mathcal{G}_1 \hookrightarrow \mathrm{SL}(2)$ is representation-finite or tame. In the latter case its Lie algebra, being a subalgebra of $s\ell(2)$ of α_p-rank 2, coincides with $s\ell(2)$. Thus, we have $\mathrm{ord}(\mathcal{G}_2) = \mathrm{ord}(\mathrm{SL}(2)_2)$, whence $\mathcal{G}_2 = \mathrm{SL}(2)_2$. From [30, (7.8),(6.8)] we obtain

$$c_{H(\mathcal{G})}(k) \geq c_{H(\mathcal{G}_2)}(k) = c_{H(\mathrm{SL}(2)_2)}(k) = 3,$$

which contradicts Rickard's Theorem [27, Thm.2]. Consequently, $\mathcal{B}_0(\mathcal{G}/\mathcal{G}_1)$ is representation-finite, and $\mathcal{G}$ satisfies the conditions of Lemma 5.4. Hence $\mathcal{G}$ is isomorphic to $\mathcal{A}_{[n]}$ or $\mathcal{Q}_{[n]}$.

Suppose that $\mathcal{G} \cong \mathcal{A}_{[n]}$, and note that $\mathcal{X}(\mathcal{G}) \cong \mathcal{X}(\mathcal{G}/\mathcal{G}_1) \cong \mathcal{X}(\mu_{p^{n-1}}) \cong \mathbb{Z}/(p^{n-1})$. Recall that ν is a generator of $\mathcal{X}(\mathcal{G})$. Owing to (5.2) two simple $\mathcal{G}$-modules M, N belong to the same block if and only if $N \cong M_\lambda$ for some $\lambda \in \mathcal{X}(\mathcal{G})$, and $N|_{\mathcal{G}_1}$ belongs to the block of $M|_{\mathcal{G}_1}$. In view of [26, 14] the simple modules of the principal block $\mathcal{B}_0(\mathcal{G})$ are given by the twists of $\tilde{V} := L(p-2)|_{\mathcal{G}}$ and k, respectively. Now (5.1(3)) in conjunction with (5.2), (5.3) and the identity $\nu = -2\alpha$ shows that the Gabriel quiver Q of $\mathcal{B}_0(\mathcal{G})$ has vertex set $G = \mathbb{Z}/(p^{n-1}) \times \mathbb{Z}/(2)$ and arrows

$$\alpha_{(i,j)} : (i,j) \to (i+1,j) \quad ; \quad \beta_{(i,j)} : (i,j) \to (i+r, j+1),$$

where $r := \frac{p^{n-1}-1}{2}$. In particular, two arrows start and end at each vertex.

We continue by showing that the paths $\alpha_{(1,0)} \circ \alpha_{(0.0)}$ and $\beta_{(1,0)} \circ \alpha_{(0,0)}$ are not summands of any relation in $k[Q]$. Let I denote the ideal of $k[Q]$ that is generated by the relations of $\mathcal{B}_0(\mathcal{G})$. Thus, $k[Q]/I$ and $\mathcal{B}_0(\mathcal{G})$ are Morita equivalent. In the above presentation of Q the vertices are labelled in such a way that the idempotents $e_{(i,0)}$ and $e_{(i,1)}$ correspond to the $\mathcal{B}_0(\mathcal{G})$-modules $k_{i\nu}$, and $\tilde{V}_{i\nu}$, respectively. According to [11, (2.7)] the algebra $H(\mathcal{G}/\mathcal{G}_1)$ is a (basic) Nakayama algebra. As $p \geq 3$, $H(\mathcal{G}/\mathcal{G}_1) \cong H(\alpha_p \times \mu_{p^{n-1}})$ thus possess a uniserial module of length 3 with top k. Its pull-back N to $H(\mathcal{G})$ is uniserial, has composition factors (listed from top to bottom) k, k_ν, and $k_{2\nu}$. In particular, the $H(\mathcal{G})$-module N belongs to $\mathcal{B}_0(\mathcal{G})$. Consequently, $k[Q]/I$ possesses a uniserial module $M = (M_q)_{q \in Q}$ of length 3 with composition factors corresponding to the idempotents $e_{(0,0)}$, $e_{(1,0)}$, and $e_{(2,0)}$. Thus, for $n \geq 2$, we have $M_{(i,j)} = k$ for $(i,j) = (0,0)$, $(1,0)$, $(2,0)$ and $M_{(i,j)} = (0)$ otherwise. The non-zero maps are $\alpha^M_{(0,0)} = id_k = \alpha^M_{(1,0)}$. For $n = 1$, $M_{(0,0)} = kv_1 \oplus kv_2 \oplus kv_3$ is the only non-zero vertex, and the only non-zero map $\alpha^M_{(0,0)}$ satisfies $\alpha^M_{(0,0)}(v_1) = v_2$, $\alpha^M_{(0,0)}(v_2) = v_3$, and $\alpha^M_{(0,0)}(v_3) = 0$. In either case $\alpha_{(1,0)} \circ \alpha_{(0,0)}$ is the only path of length ≥ 2 starting at $(0,0)$ and ending at $(2,0)$ that does not annihilate M. As a result, this path is not a summand of a relation of $\mathcal{B}_0(\mathcal{G})$.

In order to deal with the path $\beta_{(1,0)} \circ \alpha_{(0,0)}$, we consider the simple $\mathcal{G}$-module $\tilde{V} = L(p-2)|_{\mathcal{G}}$ as well as the projective cover $P_1(\tilde{V})$ of $\tilde{V}|_{\mathcal{G}_1}$. According to [21, (4.7)], this module possesses an SL(2)-structure that extends the given $\mathcal{G}_1$-structure. Thanks to [19, Thm.3] the SL(2)-module $P_1(\tilde{V})$ is uniserial with composition factors $L(p-2)$, $L(p)$, and $L(p-2)$. By Steinberg's tensor product theorem (cf. [20, (II.3.17)]), we have $L(p) \cong L(0) \otimes_k L(1)^{(p)} \cong L(1)^{(p)}$. Consequently, $P_1(\tilde{V})$ is a uniserial $H(\mathcal{G})$-module with composition factors $\tilde{V}$, k_α, $k_{-\alpha}$, and $\tilde{V}$. Thus, $\mathrm{Rad}(P_1(\tilde{V}))_{-\alpha}$ is a uniserial module of length 3 belonging to $\mathcal{B}_0(\mathcal{G})$ with composition factors k, k_ν, and $\tilde{V}_{-\alpha} = \tilde{V}_{\nu+\alpha}$. As these factors correspond to the idempotents $e_{(0,0)}$, $e_{(1,0)}$, and $e_{(r+1,1)}$, we may proceed as above to see that the path $\beta_{(1,0)} \circ \alpha_{(0,0)}$ is not a summand of a relation.

In summary, the quiver Q of $\mathcal{B}_0(\mathcal{G})$ has two arrows starting and ending at each vertex. Moreover, there exist arrows α, β, γ such that neither $\gamma \circ \alpha$ nor $\beta \circ \alpha$ are involved in a relation. According to [5, (I.10.9)] this implies that $\mathcal{B}_0(\mathcal{G})$ is wild, a contradiction. Consequently, $\mathcal{G} \cong \mathcal{Q}_{[n]}$.

(2) $\Rightarrow$ (3). Assume that $\mathcal{G} \cong \mathcal{Q}_{[n]}$. Let $V = L(r)|_{\mathcal{G}_1}$ be a simple $\mathcal{G}_1$-module. As before, we endow the projective cover $P_1(V)$ of V with the structure of a $\mathcal{G}$-module and call the resulting module $P_1(\tilde{V})$. Since $\mathcal{G}/\mathcal{G}_1$ is multiplicative, the functor

$$\mathrm{Hom}_{H(\mathcal{G})}(P_1(\tilde{V}), \cdot) = \mathrm{Hom}_{H(\mathcal{G}/\mathcal{G}_1)}(k, \cdot) \circ \mathrm{Hom}_{H(\mathcal{G}_1)}(P_1(\tilde{V}), \cdot)$$

is, as the composite of exact functors, exact. Consequently $P_1(\tilde{V})$ is $H(\mathcal{G})$-projective, and by (5.1) the principal indecomposable $\mathcal{G}$-modules are of the from $P_1(\tilde{V})_\lambda$, where V is a simple $\mathcal{G}_1$-module and $\lambda \in \mathcal{X}(\mathcal{G})$.

Let $V = L(r)|_{\mathcal{G}_1}$ be a non-projective simple $H(\mathcal{G}_1)$-module, and put $W := L(p-2-r)|_{\mathcal{G}_1}$. Owing to [19, Thm.3] the SL(2)-module $P_1(V)$ has Loewy series $L(r)$, $L(2p-2-r)$, $L(r)$. By Steinberg's tensor product theorem, we have $L(2p-2-r) \cong L(p-2-r) \otimes_k L(1)^{(p)}$, so that $L(2p-2-r)|_{\mathcal{G}} \cong \tilde{W}_\alpha \oplus \tilde{W}_{-\alpha}$. Accordingly, $P_1(\tilde{V})$ has Loewy factors $\tilde{V}$, $\tilde{W}_\alpha \oplus \tilde{W}_{-\alpha}$, and $\tilde{V}$.

Let $\mathcal{B} \subset H(\mathcal{G})$ be a non-simple block. Since the character α generates the group $\mathcal{X}(\mathcal{G})$, (5.2) and (5.3) imply that the Gabriel quiver Q of $\mathcal{B}$ has vertex set $G = \mathbb{Z}/(p^{n-1}) \times \mathbb{Z}/(2)$ and arrows

$$\alpha_{(i,j)} : (i,j) \to (i+1, j+1) \quad ; \quad \beta_{(i,j)} : (i,j) \to (i-1, j+1).$$

From the Loewy structure of the principal indecomposables we obtain the following relations (see [14, p.21f] for the case $n = 1$):

$$\alpha_{(i+1,j+1)} \circ \alpha_{(i,j)} = 0 = \beta_{(i-1,j+1)} \circ \beta_{(i,j)} \ ; \ \beta_{(i+1,j+1)} \circ \alpha_{(i,j)} = \alpha_{(i-1,j+1)} \circ \beta_{(i,j)}.$$

As a result, the block $\mathcal{B}$ is special biserial, and $H(\mathcal{G})$ enjoys the same property.

(3) $\Rightarrow$ (4). According to [5, (II.3.1)] or [3, Thm.A] the algebra $H(\mathcal{G})$ is tame or of finite representation type. Since $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G})} = \dim \mathcal{V}_{s\ell(2)} = 2$, [11, (2.7)] implies that the latter alternative does not obtain.

(4) $\Rightarrow$ (1). This follows directly from [5, (I.4.7)] and [11, (2.7)]. $\square$

Remarks. (1) According to (5.5) there exists for each $r \geq 1$ exactly one tame, semisimple infinitesimal group of height r. Moreover, in analogy with representation-finite infinitesimal groups [11], semisimple groups are tame precisely when their second Frobenius kernels enjoy this property.

(2) Consider the group $\mathcal{A}_{[n]}$ as well as the subgroup $\mathcal{U}$ that is given by

$$\mathcal{U}(R) := \{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2)(R) \ ; \ b^{p^2} = 0 \}$$

for every commutative k-algebra R. We claim that $\mathcal{U}$ is the unique unipotent subgroup of $\mathcal{A}_{[n]}$ of height 2. Given another such subgroup $\mathcal{H} \subset \mathcal{A}_{[n]}$ we observe that $\mathrm{ord}(\mathcal{H}) = p^2 = \mathrm{ord}(\mathcal{U})$. By general theory (cf. [35, (8.3)]), there exists an element $g \in \mathrm{GL}(2)(k)$ such that $g\mathcal{H}g^{-1} = \mathcal{U}$. On the other hand, $\mathcal{F}(\mathcal{U})$ is the unique unipotent subgroup of $\mathcal{F}(\mathcal{A}_{[n]}) \cong \alpha_p \times \mu_{p^{n-1}}$, so that $\mathcal{F}(\mathcal{H}) = \mathcal{F}(\mathcal{U})$. Consequently, $\mathcal{F}(g)$ normalizes $\mathcal{F}(\mathcal{U})$, and direct computation shows that g is an upper triangular matrix. As a result, $\mathcal{H} = g^{-1}\mathcal{U}g = \mathcal{U}$, as desired.

Since the automorphism scheme $\mathcal{AUT}(\alpha_{p^2}) \cong \mu_k \times \alpha_k$ has dimension 2, our foregoing observations imply that the rank variety $\hat{\mathcal{V}}_{(\mathcal{A}_{[n]})_2}(k) := \mathrm{Hom}(\alpha_{p^2}, \mathcal{A}_{[n]})$ has dimension 2. Thus, in contrast to representation-finite groups (cf. [11, (2.7)]), the condition $\dim \hat{\mathcal{V}}_{\mathcal{G}_2}(k) = 2$ does in general not entail the tameness of $\mathcal{B}_0(\mathcal{G})$.

We conclude this section by determining the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{Q}_{[n]})$ of the algebra $H(\mathcal{Q}_{[n]})$. For $n = 1$, the structure of this directed graph is well understood (see comments below).

Let Λ be a Frobenius algebra over k. The *stable Auslander-Reiten quiver* $\Gamma_s(\Lambda)$ is the directed graph whose vertices are the isoclasses $[M]$ of the non-projective indecomposable Λ-modules, and whose arrows arise from the so-called irreducible maps. The AR-quiver $\Gamma_s(\Lambda)$ is fitted with a certain quiver automorphism τ that, by general theory, satisfies

$$\tau([M]) = \Omega^2_\Lambda([M^{(\mu)}]).$$

Here Ω_Λ denotes the *Heller operator*, and $M^{(\mu)}$ is the twist of M by the inverse of a Nakayama automorphism of Λ. We refer the reader to [1] and [2] for the details.

The algebra Λ is called *domestic* if it does not have finite representation type, and if there exist $(\Lambda, k[T])$-bimodules $Q_1, \ldots, Q_n$ such that

(a) Q_i is a finitely generated free right $k[T]$-module, and

(b) for each natural number $d > 0$, all but a finite number of isoclasses of indecomposable Λ-modules of dimension d are of the form $[Q_i \otimes_{k[T]} V]$ for some indecomposable $k[T]$-module V.

Let $\mathcal{G}$ be an infinitesimal group, $\mathcal{N} \subset \mathcal{G}$ a subgroup such that $\mathcal{G}/\mathcal{N}$ is multiplicative. From the identity $\mathrm{Hom}_{H(\mathcal{G})}(\cdot,\cdot) \cong \mathrm{Hom}_{H(\mathcal{G}/\mathcal{N})}(k,\cdot) \circ \mathrm{Hom}_{H(\mathcal{N})}(\cdot,\cdot)$, we see that every short exact sequence

$$(0) \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow (0)$$

of $\mathcal{G}$-modules that splits upon restriction to $\mathcal{N}$, is split exact. In other words, every $\mathcal{G}$-module M is $(H(\mathcal{G}), H(\mathcal{N}))$-projective, and is thus a direct summand of $H(\mathcal{G}) \otimes_{H(\mathcal{N})} M$. This implies $c_{H(\mathcal{G})}(M) \leq c_{H(\mathcal{N})}(M)$. Since the reverse inequality follows from the fact that $H(\mathcal{G})$ is a projective $H(\mathcal{N})$-module, we obtain

$$c_{H(\mathcal{G})}(M) = c_{H(\mathcal{N})}(M)$$

for every $\mathcal{G}$-module M. (This identity can also be seen by considering rank varieties.)

The proof our our next result necessitates a few details concerning the structure of the stable AR-quiver $\Gamma_s(\mathrm{SL}(2)_1)$ of the symmetric algebra $H(\mathrm{SL}(2)_1)$ (see [9, p.176f]). Each non-simple block $\mathcal{B} \subset H(\mathrm{SL}(2)_1)$ has two simple modules V and W, and the complexity of an arbitrary $\mathrm{SL}(2)_1$-module is bounded by 2. If $P(V)$ and $P(W)$ are the projective covers of V and W, then the set of isoclasses of indecomposable $\mathcal{B}$-modules of complexity 2 decomposes into 4 τ-orbits whose representatives are $[V]$, $[W]$, $[P(V)/\mathrm{Soc}(P(V))]$, and $[P(W)/\mathrm{Soc}(P(W))]$, respectively. The first and fourth, and the second and third orbit constitute 2 components of type $\mathbb{Z}[\tilde{A}_{1,2}]$.

Theorem 5.6 *Let $n \geq 2$. The stable Auslander-Reiten quiver $\Gamma_s(\mathcal{Q}_{[n]})$ is the disjoint union of $p-1$ components of type $\mathbb{Z}[\tilde{A}_{2p^{n-1}-1}]$, $2(p-1)$ components of type $\mathbb{Z}[A_\infty]/(\tau^{p^{n-1}})$, and infinitely many components of type $\mathbb{Z}[A_\infty]/(\tau)$. Moreover, the algebra $H(\mathcal{Q}_{[n]})$ is domestic.*

Proof. We put $\mathcal{G} := \mathcal{Q}_{[n]}$ for notational convenience. Recall that $\mathcal{G}/\mathcal{G}_1$ is multiplicative, $\mathcal{G}_1 = \mathrm{SL}(2)_1$, and $\mathcal{X}(\mathcal{G}) \cong \mathbb{Z}/(p^{n-1})$. Let $\zeta : H(\mathcal{G}) \longrightarrow k$ be the modular function of the Hopf algebra $H(\mathcal{G})$. According to [15, (1.5)] the convolution $\zeta * id_{H(\mathcal{G})}$ is a Nakayama automorphism of the Frobenius algebra $H(\mathcal{G})$. In view of [7, (3.1)] the module k_ζ is the socle of the projective cover of k. From the structure of the principal indecomposable $H(\mathcal{G})$-modules (5.5) we have $k \cong k_\zeta$, so that $H(\mathcal{G})$ is symmetric. Accordingly, $\tau = \Omega^2_{H(\mathcal{G})}$.

(i) *Let M be a $\mathcal{G}$-module, $V \subset M|_{\mathcal{G}_1}$ a simple $\mathcal{G}_1$-direct summand. Then there exists a character $\lambda \in \mathcal{X}(\mathcal{G})$ such that $\tilde{V}_\lambda$ is a direct summand of M.*

By assumption, there exist homomorphisms $\varphi \in \mathrm{Hom}_{H(\mathcal{G}_1)}(\tilde{V}, M)$ and $\gamma \in \mathrm{Hom}_{H(\mathcal{G}_1)}(M, \tilde{V})$ such that $\gamma \circ \varphi = id_{\tilde{V}}$. Since $\mathcal{G}/\mathcal{G}_1$ is multiplicative, the $\mathcal{G}/\mathcal{G}_1$-module $\mathrm{Hom}_{H(\mathcal{G}_1)}(\tilde{V}, M)$

decomposes into its spaces of semi-invariants. Accordingly, we may write $\varphi = \sum_{\lambda\in\mathcal{X}(\mathcal{G})}\varphi_\lambda$. By the above there exists a character $\lambda_0 \in \mathcal{X}(\mathcal{G})$ such that $\gamma\circ\varphi_{\lambda_0} \neq 0$. Thus, $\gamma\circ\varphi_{\lambda_0}$ is an isomorphism of $\mathcal{G}_1$-modules.

As noted earlier, the injective map $\varphi_{\lambda_0} : \tilde{V}_{\lambda_0} \longrightarrow M$ is a homomorphism of $H(\mathcal{G})$-modules. By choice of φ_{λ_0} the exact sequence

$$(0) \longrightarrow \tilde{V}_{\lambda_0} \xrightarrow{\varphi_{\lambda_0}} M \longrightarrow \operatorname{coker}\varphi_{\lambda_0} \longrightarrow (0)$$

splits upon restriction to $H(\mathcal{G}_1)$. Since $\mathcal{G}/\mathcal{G}_1$ is multiplicative, the sequence also splits over $H(\mathcal{G})$, as desired.

(ii) *Let M be a non-projective indecomposable $H(\mathcal{G})$-module such that $c_{H(\mathcal{G})}(M) = 2$. Then there exists an integer $\ell \in \mathbb{Z}$ such that $\Omega^\ell_{H(\mathcal{G})}(M)$ is simple.*

According to the remarks preceding the theorem, statement (ii) does hold for $H(\mathcal{G}_1)$-modules. Given an indecomposable $H(\mathcal{G})$-module M of complexity 2, we have $c_{H(\mathcal{G}_1)}(M) = 2$. Hence there exist $\mathcal{G}_1$-modules X, Y such that $M|_{\mathcal{G}_1} = X \oplus Y$, with X being indecomposable of complexity 2. Let $\ell \in \mathbb{Z}$ be given such that $\Omega^\ell_{H(\mathcal{G}_1)}(X) \cong V$ is simple. Basic properties of the Heller operator (cf. [2, p.29]) then yield

$$\begin{aligned}\Omega^\ell_{H(\mathcal{G})}(M)|_{\mathcal{G}_1} &\cong \Omega^\ell_{H(\mathcal{G}_1)}(M|_{\mathcal{G}_1}) \oplus (\mathrm{proj}) \cong \Omega^\ell_{H(\mathcal{G}_1)}(X) \oplus \Omega^\ell_{H(\mathcal{G}_1)}(Y) \oplus (\mathrm{proj})\\ &\cong V \oplus \Omega^\ell_{H(\mathcal{G}_1)}(Y) \oplus (\mathrm{proj}).\end{aligned}$$

Thanks, to (i) this implies that $\tilde{V}_\lambda$ is a direct summand of $\Omega^\ell_{H(\mathcal{G})}(M)$ for a suitably chosen $\lambda \in \mathcal{X}(\mathcal{G})$. Since $\Omega^\ell_{H(\mathcal{G})}(M)$ is indecomposable, our assertion follows.

(iii) *Let $M \not\cong N$ be simple $H(\mathcal{G})$-modules, $P \not\cong Q$ principal indecomposable $H(\mathcal{G})$-modules. Then M, N, $P/\mathrm{Soc}(P)$ and $Q/\mathrm{Soc}(Q)$ belong to different τ-orbits.*

Let X be an $H(\mathcal{G})$-module. If $\ell \in \mathbb{Z}$ is an integer such that $\Omega^\ell_{H(\mathcal{G})}(X)|_{\mathcal{G}_1}$ is indecomposable, then general properties of the Heller operator imply that

$$\Omega^\ell_{H(\mathcal{G})}(X)|_{\mathcal{G}_1} \cong \Omega^\ell_{H(\mathcal{G}_1)}(X). \tag{1}$$

Suppose that M and N belong to the same τ-orbit. Then there exists $\ell \in \mathbb{Z}\setminus\{0\}$ such that $N \cong \Omega^{2\ell}_{H(\mathcal{G})}(M)$. By (1) we have $N|_{\mathcal{G}_1} \cong \Omega^{2\ell}_{H(\mathcal{G}_1)}(M)$, so that $M|_{\mathcal{G}_1}$ and $N|_{\mathcal{G}_1}$ lie in the same τ-orbit. This contradicts our remarks concerning $\Gamma_s(\mathrm{SL}(2)_1)$.

Assume that $\Omega^{2\ell}_{H(\mathcal{G})}(P/\mathrm{Soc}(P)) \cong M$ for some $\ell \in \mathbb{Z}\setminus\{0\}$. Since $M|_{\mathcal{G}_1}$ is indecomposable, (1) yields $\Omega^{2\ell}_{H(\mathcal{G}_1)}(P/\mathrm{Soc}(P)) \cong M|_{\mathcal{G}_1}$. This, however, conflicts with the structure of the quiver $\Gamma_s(\mathrm{SL}(2)_1)$.

Finally, if $P/\mathrm{Soc}(P)$ and $Q/\mathrm{Soc}(Q)$ belong to the same τ-orbit, so do $\mathrm{Soc}(P)$ and $\mathrm{Soc}(Q)$. Since $P \not\cong Q$, the first part of the proof renders this impossible.

(iv) *Let $\mathcal{B} \subset H(\mathcal{G})$ be a non-simple block. Then there are $4p^{n-1}$ τ-orbits containing isoclasses of $\mathcal{B}$-modules of complexity* 2.

Since the character α is a generator of $\mathcal{X}(\mathcal{G})$, a consecutive application of (5.2) and (5.3) shows that $\mathcal{B}$ possesses exactly $2p^{n-1}$ simple modules $\tilde{V}_\lambda$, $\tilde{W}_\lambda$ $(\lambda \in \mathcal{X}(\mathcal{G}))$. Owing to (ii), the number of $\Omega_{H(\mathcal{G})}$-orbits of $\mathcal{B}$-modules of complexity 2 is therefore bounded by $2p^{n-1}$. Hence there are at most $4p^{n-1}$ such τ-orbits. Thanks to (iii) the $2p^{n-1}$ simple $\mathcal{B}$-modules and the $2p^{n-1}$ modules $P/\mathrm{Soc}(P)$ associated to the principal indecomposable $\mathcal{B}$-modules all belong to different τ-orbits. Consequently, (iv) holds.

Let $\mathcal{B} \subset H(\mathcal{G})$ be a non-simple block. According to (5.5), $\mathcal{B}$ is a self-injective special biserial algebra. Let $\tilde{V} := L(r)|_{\mathcal{G}}$ be a simple $\mathcal{B}$-module, and put $\tilde{W} := L(p-2-r)|_{\mathcal{G}}$ $(0 \le r \le p-2)$. We denote by $P(r)$ the projective cover of $\tilde{V}$ and consider the standard almost split sequence

$$(0) \longrightarrow \mathrm{Rad}(P(r)) \longrightarrow P(r) \oplus \mathrm{Rad}(P(r))/\mathrm{Soc}(P(r)) \longrightarrow P(r)/\mathrm{Soc}(P(n)) \longrightarrow (0).$$

Since $n \ge 2$, the module $\mathrm{Rad}(P(r))/\mathrm{Soc}(P(r)) \cong \tilde{W}_\alpha \oplus \tilde{W}_{-\alpha}$ has two non-isomorphic summands (5.1(3)), implying that $P(r)/\mathrm{Soc}(P(r))$ possesses two predecessors in $\Gamma_s(\mathcal{B}) \subset \Gamma_s(\mathcal{G})$. We may now apply [6, (2.1),(2.2)] to see that the component $\Theta(r) \subset \Gamma_s(\mathcal{G})$ containing the isoclass of the non-periodic module $P(r)/\mathrm{Soc}(P(r))$ is isomorphic to $\mathbb{Z}[A_\infty^\infty]$ or $\mathbb{Z}[\tilde{A}_\ell]$. In view of [10, (2.1)] a component of type $\mathbb{Z}[A_\infty^\infty]$ gives rise to infinitely many τ-orbits containing modules of complexity 2. As this contradicts (ii), we obtain $\Theta(r) \not\cong \mathbb{Z}[A_\infty^\infty]$. Thus, [6, (2.1)] applies, and there exist natural numbers $m, a, b > 0,\ a+b+1 = \ell$ such that $\Gamma_s(\mathcal{B})$ is the disjoint union of m components of type $\mathbb{Z}[\tilde{A}_\ell]$, m components of type $\mathbb{Z}[A_\infty]/(\tau^a)$, m components of type $\mathbb{Z}[A_\infty]/(\tau^b)$, and infinitely many components of type $\mathbb{Z}[A_\infty]/(\tau)$.

The auto-equivalence $T_\nu : \mathrm{mod}_{H(\mathcal{G})} \longrightarrow \mathrm{mod}_{H(\mathcal{G})},\ M \mapsto M_\nu$ maps $\mathrm{mod}_{\mathcal{B}}$ onto itself and thus defines an automorphism on $\Gamma_s(\mathcal{B})$. Since the isoclasses of $\tilde{W}_\alpha$ and $\tilde{W}_{-\alpha} = T_\nu(\tilde{W}_\alpha)$ belong to the component $\Theta(r)$, we obtain $T_\nu(\Theta(r)) = \Theta(r)$. Consequently, $T_\lambda(\Theta(r)) = \Theta(r)$ for every $\lambda \in \mathcal{X}(\mathcal{G})$. Thanks to (iii), this shows that $\Theta(r)$ contains at least $2p^{n-1}$ τ-orbits. As this also holds for the component $\Theta(p-2-r)$ containing $[P(p-2-r)/\mathrm{Soc}(P(p-2-r))]$, we have $1 \le m \le 2$.

To prove $m = 2$, we consider SL(2) along with its standard torus T and standard Borel subgroup B $\supset$ T. Consider the character $\zeta \in \mathcal{X}(\mathrm{B})$ that is given by raising the $(1,1)$-entry of an upper triangular matrix to its r-th power. We let $Z(r) := H(\mathrm{SL}(2)_1) \otimes_{H(\mathrm{B}_1)} k_\zeta$ be the *baby Verma module* with highest weight ζ. Then $Z(r)$ has the structure of an $\mathrm{SL}(2)_1\mathrm{T}$-module. Thus, $Z(r)$ is an indecomposable $\mathcal{G}$-module of dimension p with simple top $\tilde{V} = L(r)|_{\mathcal{G}}$ and simple socle $\tilde{W}_\alpha$ (cf. [20, (II.9.5)]). Since $P(r)$ possesses a Z-filtration of length 2 (cf. [20, (II.11.2)]), we obtain $\Omega_{H(\mathcal{G})}(Z(r)) \cong Z(p-2-r)_{-\alpha}$, whence $\tau([Z(r)]) = [Z(r)_{-2\alpha}]$. Consequently, $M := \bigoplus_{\lambda \in \mathcal{X}(\mathcal{G})} Z(r)_\lambda$ is a periodic $\mathcal{G}$-module with $\tau([M]) = [M]$.

Let Υ be the set of isoclasses of indecomposable $\mathcal{B}$-modules of complexity 2. Consider the function

$$d_M : \Upsilon \longrightarrow \mathbb{N}_0 \quad ; \quad [X] \mapsto \dim_k \underline{\mathrm{Hom}}_{H(\mathcal{G})}(M, X),$$

where $\underline{\mathrm{Hom}}_{H(\mathcal{G})}$ denotes the homomorphisms of the stable module category of $H(\mathcal{G})$. According to [6, (3.2)] the function $d_M = d_M \circ \tau$ is additive on every AR-component $\Theta \subset \Upsilon$. Since $d_M([\tilde{V}]) = 1$ and $d_M(\tilde{W}) = 0$, general theory now shows that Υ contains at least two components. By the above observations we thus have $\Upsilon = \Theta(r) \cup \Theta(p-2-r)$, so that $m = 2$ and $\ell = 2p^{n-1} - 1$. Moreover, since $Z(r)$ has τ-period p^{n-1}, it is contained in a component of type $\mathbb{Z}[A_\infty]/(\tau^{sp^{n-1}})$ for some $s \ge 1$. As $a + b = 2p^{n-1}$, we see that $a = p^{n-1} = b$. Hence $\Gamma_s(\mathcal{B})$ possesses exactly 4 components of type $\mathbb{Z}[A_\infty]/(\tau^{p^{n-1}})$.

Since the algebra $H(\mathcal{G})$ has $\frac{p-1}{2}$ non-simple blocks, we obtain the desired structure of the quiver $\Gamma_s(\mathcal{G})$. Thanks to [6, (2.1)], $H(\mathcal{G})$ is domestic. $\square$

6. The Structure of the Solvable Radical

Throughout, we will be considering an infinitesimal k-group $\mathcal{G}$ of characteristic $p \geq 3$. In this section we will determine the structure of the solvable radical $\mathcal{R}(\mathcal{G})$ for tame groups. In case $\mathcal{G}$ has height ≤ 1 this leads to a complete description of Gabriel quiver of the tame principal blocks. We begin with the following sharpening of Lemma 1.2.

Theorem 6.1 *Suppose that $\mathcal{B}_0(\mathcal{G})$ is tame. Then $\mathcal{R}(\mathcal{G})$ is nilpotent.*

Proof. As in (1.2) we use induction on the order of $\mathcal{R} := \mathcal{R}(\mathcal{G})$. Let $\mathcal{M}(\mathcal{G}) \subset \mathcal{R}$ be the multiplicative center of $\mathcal{G}$. If $\mathcal{M}(\mathcal{G}) \neq e_k$, then $\mathcal{R}(\mathcal{G}/\mathcal{M}(\mathcal{G})) \cong \mathcal{R}/\mathcal{M}(\mathcal{G})$ is nilpotent, so that $\mathcal{R}$ also enjoys this property.

We thus continue by assuming that $\mathcal{M}(\mathcal{G}) = e_k$, and let $\mathcal{N} \subset \mathcal{R}$ be a minimal normal subgroup of $\mathcal{G}$. Since $\mathcal{N}$ is solvable, it is abelian. By [4, (IV.§3 (1.1))] the subgroup $\mathcal{M}(\mathcal{N}) \subset \mathcal{N}$ is normal in $\mathcal{G}$. We may now apply [35, (7.7)] to see that $\mathcal{M}(\mathcal{N}) \subset \mathcal{M}(\mathcal{G}) = e_k$. Consequently, $\mathcal{N}$ is unipotent (cf. [35, (9.5)]). Moreover, $\mathcal{N}$ is contained in the first Frobenius kernel $\mathcal{G}_1$ of $\mathcal{G}$. Since $\mathcal{V}_{\mathcal{N}}(\mathcal{N}^{(p)}) \subset \mathcal{N}$ is a proper normal subgroup of $\mathcal{N}$ (cf. [4, (IV.§3 (4.6))]), we see that $\mathcal{N}$ is annihilated by the Verschiebung. Moreover, $N := \mathrm{Lie}(\mathcal{N})$ has α_p-rank ≤ 2, so that [4, (IV.§3 (6.11))] yields $\mathcal{N} \cong \alpha_p$ or $\mathcal{N} \cong \alpha_p \times \alpha_p$.

According to (2.4) the group $\mathcal{G}$ is not solvable. Hence $\mathcal{G}/\mathcal{N}$ is not supersolvable, and (1.1(3)) ensures the tameness of $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$. The inductive hypothesis thus implies that $\mathcal{R}(\mathcal{G}/\mathcal{N}) \cong \mathcal{R}/\mathcal{N}$ is nilpotent. We set $L := \mathrm{Lie}(\mathcal{G})$ and $R := \mathrm{Lie}(\mathcal{R})$.

By choice of $\mathcal{N}$, the p-ideal N is a simple $\mathcal{G}$-module on which $\mathcal{N}$ operates trivially. Thus, it also has the structure of a simple $\mathcal{G}/\mathcal{N}$-module, and the multiplicative normal subgroup $\mathcal{M}(\mathcal{R}/\mathcal{N}) \subset \mathcal{G}/\mathcal{N}$ operates on N via a single character. Since $R/N \subset \mathrm{Lie}(\mathcal{R}/\mathcal{N})$ is a p-ideal, it follows that $T(R/N) \subset T(\mathrm{Lie}(\mathcal{R}/\mathcal{N}))$ operates on N via a linear form. We let $N' \subset L$ be the pre-image of $T(R/N)$ under the canonical projection $L \longrightarrow L/N$. As $T(R/N) \subset T(L/N)$ is a p-ideal in L/N, the subspace N' is a p-ideal of L. In view of [29, (II.4.4)] there exists a torus $T \subset N'$ such that $N' = T \oplus N$. By the above, T acts on N via a linear form $\alpha \in T^*$. Since $[T, L] \subset N$, we readily obtain $\ker \alpha \subset T(L) = (0)$. Hence, α is injective, and $\dim_k T \leq 1$.

Assuming $T = kt \neq (0)$, we have $(\ker ad\, t) \cap N = (0)$ as well as $(ad\, t)(L) = [T, L] = N$. Consequently, $L = C_L(T) \oplus N$ is a semidirect product of restricted Lie algebras. Thanks to (2.5) $C_L(T) \cong L/N \hookrightarrow \mathrm{Lie}(\mathcal{G}/\mathcal{N})$ has α_p-rank ≤ 2. In virtue of [11, (2.1)] the assumption $\dim \mathcal{V}_{C_L(T)} \leq 1$ entails that L/N is solvable. Hence L and $\mathcal{G}$ are solvable. As this contradicts (2.4), we obtain $\dim \mathcal{V}_{C_L(T)} = 2$. Since $p \geq 3$ and $\dim_k N \leq 2$, one readily verifies $\dim \mathcal{V}_L \geq 3$, a contradiction.

Hence $T = (0)$, and we conclude that R/N is p-nilpotent. Consequently, R is also p-nilpotent Thanks to [4, (IV.§2 (2.13)] the first Frobenius kernel $\mathcal{R}_1$ is unipotent. By [4, (II.§4 (1.5))] we have an embedding $\mathrm{Lie}(\mathcal{R}/\mathcal{R}_1) \hookrightarrow R^{(p)}$, so that the former Lie algebra is also p-nilpotent. In view of [4, (IV.§2 (2.3))] we may now proceed by induction on the height of $\mathcal{R}$ to see that $\mathcal{R}$ is unipotent. $\square$

Lemma 6.2 *Suppose that $\mathcal{B}_0(\mathcal{G})$ is tame.*

(1) *If* $\mathrm{Lie}(\mathcal{R}(\mathcal{G}))$ *has α_p-rank ≤ 1, then* $\mathrm{Lie}(\mathcal{R}(\mathcal{G})) = C(\mathrm{Lie}(\mathcal{G}))$.

(2) *$\mathcal{R}(\mathcal{G})/\mathcal{R}(\mathcal{G})_1$ is multiplicative, and the unipotent radical $\mathcal{U}(\mathcal{G})$ has height ≤ 1.*

Proof. (1). We put $L := \mathrm{Lie}(\mathcal{G})$ as well as $R := \mathrm{Lie}(\mathcal{R}(\mathcal{G}))$. Thanks to (4.3(2)) $R = R(L)$ is the solvable radical of L. If $\dim \mathcal{V}_R = 0$, then R is a torus and $R = T(L) = C(L)$, as desired.

Now assume that R has α_p-rank 1. By (6.1) the p-ideal R is nilpotent, and (3.1) thus implies that R is abelian. Moreover, there exists a p-nilpotent element $z \in R\setminus\{0\}$ such that $R = T(R) \oplus (kz)_p = T(L) \oplus (kz)_p$. In particular, $R^{[p]} \subset C(L)$ and $R/R^{[p]}$ has dimension 1. By virtue of (4.3(2)) the Lie algebra $L/R \cong s\ell(2)$ operates trivially on $R/R^{[p]}$, and thus nilpotently on R. Since L/R is simple, Engel's Theorem implies that L/R operates trivially on R, so that $R \subset C(L)$.

(2). In view of (4.3(2)) the ideal R coincides with $R(L)$ and $L/R \cong s\ell(2)$. Consider the exact sequence

$$(0) \longrightarrow \mathrm{Lie}(\mathcal{R}(\mathcal{G})_1) \longrightarrow L \longrightarrow \mathrm{Lie}(\mathcal{G}/\mathcal{R}(\mathcal{G})_1) =: \hat{L}.$$

Since $R = \mathrm{Lie}(\mathcal{R}(\mathcal{G})_1)$, it follows that $L/\mathrm{Lie}(\mathcal{R}(\mathcal{G})_1) \cong s\ell(2)$ is a p-ideal of $\hat{L}$. Since the Lie algebra $s\ell(2)$ is complete, we obtain

$$\hat{L} = s\ell(2) \oplus C_{\hat{L}}(s\ell(2)).$$

As $\hat{L}$ has α_p-rank 2, it follows that the second summand is a toral ideal. There results a decomposition

$$\hat{L} = L/\mathrm{Lie}(\mathcal{R}(\mathcal{G})_1) \oplus T(\hat{L}).$$

Next, we consider the exact sequence

$$(0) \longrightarrow \mathrm{Lie}(\mathcal{R}(\mathcal{G})/\mathcal{R}(\mathcal{G})_1) \longrightarrow \mathrm{Lie}(\mathcal{G}/\mathcal{R}(\mathcal{G})_1) \longrightarrow \mathrm{Lie}(\mathcal{G}/\mathcal{R}(\mathcal{G})).$$

Consequently, $\mathrm{Lie}(\mathcal{R}(\mathcal{G})/\mathcal{R}(\mathcal{G})_1) \subset \mathrm{Lie}(\mathcal{G}/\mathcal{R}(\mathcal{G})_1)$ is a solvable ideal, and therefore contained in $T(\hat{L})$. As a result, the factor group $\mathcal{R}(\mathcal{G})/\mathcal{R}(\mathcal{G})_1$ is multiplicative. This implies that $\mathcal{U}(\mathcal{G})$ is contained in the kernel of the natural projection $\mathcal{R}(\mathcal{G}) \longrightarrow \mathcal{R}(\mathcal{G})/\mathcal{R}(\mathcal{G})_1$. Accordingly, $\mathcal{U}(\mathcal{G}) = \mathcal{U}(\mathcal{G}) \cap \mathcal{G}_1$ has height ≤ 1. $\square$

Throughout the remainder of this section we will be concerned with infinitesimal groups of height ≤ 1. By general theory, these may be identified with their Lie algebras, so we will only consider the latter. If M is a module for the restricted enveloping algebra $u(L)$, then the restricted cohomology groups of L with coefficients in M are defined by means of $H^n_*(L, M) := \mathrm{Ext}^n_{u(L)}(k, M)$.

Lemma 6.3 *Suppose that $\mathcal{B}_0(L)$ is tame. Then $R(L) = C(L)$ has α_p-rank ≤ 1.*

Proof. We set $R := R(L)$ for notational convenience and consider the Lie algebra $L' := L/R$. Thanks to (4.3(2)) we have $L' \cong s\ell(2)$.

Consider the p-ideal $I := [R, R] + \langle R^{[p]} \rangle$. If $\dim \mathcal{V}_R = 2$, then the principal block $\mathcal{B}_0(R)$ of the enveloping algebra $u(R)$ is not representation-finite. By (6.1) the p-ideal R is nilpotent, and [32, (2.41)]) now shows that $\mathcal{B}_0(R)$ is local. From [18, (2.1)] we thus obtain

$$2 \leq \dim_k \operatorname{Ext}^1_{u(R)}(k, k) = \dim_k H^1_*(R, k) = \dim_k R/I.$$

Since $\mathcal{B}_0(L)$ is tame and L/I is not supersolvable, (1.1(3)) implies that $\mathcal{B}_0(L/I)$ is tame. Hence $R/I \subset \mathcal{V}_{L/I}$ has dimension 2.

Note that $\bar{L} := L/I$ is an abelian extension of $s\ell(2)$ by the two-dimensional $s\ell(2)$-module $\bar{R} := R/I$. If $\bar{R}$ is the standard $s\ell(2)$-module, then [26, Thm.1] (see also [14] for $p = 3$) yields $H^2_*(s\ell(2), \bar{R}) = (0)$, and thanks to [18, (3.3)] the extension splits. As $p \geq 3$ and $\dim \mathcal{V}_{\bar{R}} = 2$, this readily implies $\dim \mathcal{V}_{\bar{L}} \geq 3$, a contradiction. Alternatively, $\bar{R}$ is a self-extension of the trivial $s\ell(2)$-module. Since $H^1_*(s\ell(2), k) = (0)$, such an extension splits, and we conclude that $\bar{R} \subset C(\bar{L})$. This, in turn, contradicts (1.3).

As a result the p-ideal R has α_p-rank ≤ 1, and our assertion now is a direct consequence of (6.2(1)). $\square$

Remark. If $L = \operatorname{Lie}(\mathcal{G})$ is the Lie algebra of an infinitesimal group $\mathcal{G}$ with tame principal block, then the arguments of (6.3) can be used to show that $R(L) = C_n(L)$ coincides with a term of the ascending central series of L. In particular, $L \cong s\ell(2)$ whenever $C(L) = (0)$.

Theorem 6.4 *Let $(L, [p])$ be a restricted Lie algebra such that $\mathcal{B}_0(L)$ is tame.*

(1) *The Lie algebra L is a non-split central extension of $s\ell(2)$ by a center $C(L) = (kz)_p \oplus T(L)$, $z^{[p]^n} = 0$. Moreover, if $C(L) \neq (0)$, then there exist linearly independent elements $\{h, e, f\} \subset L$ and $\gamma \in k \setminus \{0\}$ such that $[h, e] = 2e$; $[h, f] = 2f$; $[e, f] = h$ and $h^{[p]} \equiv h \bmod(C(L))$; $f^{[p]} \equiv \gamma e^{[p]} \not\equiv 0 \bmod(C(L)^{[p]})$.*

(2) *The block $\mathcal{B}_0(L)$ possesses exactly two simple modules S_1 and S_{p-1} of dimensions 1 and $p-1$, respectively. Moreover, the Gabriel quivers of $\mathcal{B}_0(L)$ and $\mathcal{B}_0(s\ell(2))$ are isomorphic.*

Proof. (1). The first assertion follows directly from a consecutive application of (4.3(2)), (6.3), and [7, (4.3)].

Now assume that $T(L) = (0)$, and let $T \subset L$ be the standard (one-dimensional) maximal torus of L, $\mathcal{X}_T \cong \operatorname{Spec}_k(A)$ the corresponding irreducible algebraic scheme of tori. Relative to T the Lie algebra posseses a root space decomposition

$$L = L_\alpha \oplus L_0 \oplus L_\alpha$$

with one-dimensional root spaces. Our introductory remarks of §3 now show that

$$\tilde{L} = \tilde{L}_{-\alpha} \oplus \tilde{L}_0 \oplus \tilde{L}_\alpha.$$

In particular, $P := \tilde{L}_0 \oplus \tilde{L}_\alpha$ is a p-subalgebra of $\tilde{L}$. If the generic α_p-rank of the algebraic family $(P(x))_{x \in \mathcal{X}_T(k)}$ is 2, then (3.4(3)) provides a p-ideal $I \subset L_0 \oplus L_\alpha$ of L of α_p-rank 2. Since $L/C(L)$ is simple, we have $I \subset C(L)$, a contradiction. Thus the generic α_p-rank is

1, and by applying the same procedure to $-\alpha$ we obtain from the irreducibility of $\mathcal{X}_T(k)$ the existence of an element $x \in \mathcal{X}_T(k)$ such that $\tilde{L}_0(x) \oplus \tilde{L}_\alpha(x)$ and $\tilde{L}_0(x) \oplus \tilde{L}_{-\alpha}(x)$ both have α_p-rank 1. Their factor algebras by $C(L)^{[p]}$ have the same property, and we thus have $\tilde{L}_\eta(x)^{[p]} \not\subset C(L)^{[p]}$ for $\eta = \alpha, -\alpha$.

Let T_x be the torus corresponding to $x \in \mathcal{X}_T(k)$, and consider the subspace $\tilde{L}_{-\alpha}(x) \oplus T_x \oplus \tilde{L}_\alpha(x)$. Since $L/C(L) \cong s\ell(2)$ there exist elements $e \in \tilde{L}_\alpha(x)$, $f \in \tilde{L}_{-\alpha}(x)$, and a toral element $t \in T_x$ with $\alpha(t) = 2$ such that $[e, f] = t + q$ for some $q \in C(L)$. Setting $h := t + q$ we immediately obtain the asserted commutator relations as well as $h^{[p]} \equiv h \bmod(C(L))$. Since $C(L) = kz \oplus C(L)^{[p]}$ the remaining congruences follow from the above observations.

(2). By (1.1(1)) we have $\mathcal{B}_0(L) \cong \mathcal{B}_0(L/T(L))$; so we may assume that $T(L) = (0)$. In particular, the center $C(L)$ is p-nilpotent. Thus, the algebras $u(L)$ and $u(s\ell(2))$ have the same simple modules. Let V, W be two simple $u(L)$-modules, and set $M := \mathrm{Hom}_k(V, W)$. From the cohomology five term sequence we obtain, observing $H^n_*(L, M) \cong \mathrm{Ext}^n_{u(L)}(V, W)$, $C(L) \cdot M = (0)$, as well as $H^1_*(C(L), M) \cong M$, the exact sequence

$$(0) \longrightarrow \mathrm{Ext}^1_{u(s\ell(2))}(V, W) \longrightarrow \mathrm{Ext}^1_{u(L)}(V, W) \longrightarrow \mathrm{Hom}_{u(L)}(V, W)$$

Consequently, $\mathrm{Ext}^1_{u(s\ell(2))}(V, W) \cong \mathrm{Ext}^1_{u(L)}(V, W)$ whenever $V \not\cong W$. From Pollack's work [26] (cf. also [14]) we now obtain that $\mathcal{B}_0(L)$ has precisely two simple modules S_1 and S_{p-1} of the asserted dimensions. By the same token, we have

$$\dim_k \mathrm{Ext}^1_{u(L)}(S_1,\, S_{p-1}) = 2 = \dim_k \mathrm{Ext}^1_{u(L)}(S_{p-1},\, S_1).$$

Since $\mathcal{B}_0(L)$ is tame, (ii) of [5, (I.10.8)] ensures that the modules S_1 and S_{p-1} admit no non-trivial self-extensions. In view of [26, 14] this shows that the Gabriel quivers of $\mathcal{B}_0(L)$ and $\mathcal{B}_0(s\ell(2))$ coincide. $\square$

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