# Coherence of Operations in $A$ – theory

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## 0 Introduction

One may view $K$ - theory as an additive device for describing the infrastructure of equivalences of filtered objects in a category. This has been made precise for categories with cofibrations and weak equivalences by Waldhausen [W] by means of the $\mathcal{S}_\bullet$ construction – thereby generalizing Quillens $\mathcal{Q}$ construction for exact categories. These constructions behave like classifying spaces with respect to products, i.e. one has to take loops to adjust for internal pairings. An 'undelooped version' of $K$ - theory in interesting cases is provided by the spherical models or plus – construction approach, which allows for instance to trace a categorical product on the input to an $E_\infty$ product on the output [SV].

This type of approach becomes cumbersome, when one wants to factor operations through equivariant objects as in [W2, W3], as typically operations will not carry spherical objects to equivariant spherical objects.

Undelooped descriptions for $K$ - theory as combinatorial objects incorporating group completion avoiding spherical models have been provided by [G-G, Gu].

In this paper we generalize Grayson's set up for exterior power operations [G] to construct maps

$$\omega^k : \mathsf{sub}_k wG_\bullet \mathscr{C} \to wG^k_\bullet \mathscr{C}_{\Sigma_k} \qquad \text{for } k \geq 0.$$

Here $\mathscr{C}$ is a (nice) category with cofibrations, weak equivalences, suspension [Gu] and bi – exact pairing; $wG_\bullet\mathscr{C}$ is the suspension prespectra model for the $K$-theory of $\mathscr{C}$ of [Gu]. $wG^k_\bullet$ is the $k$-fold iteration. $\mathsf{sub}_k$ denotes subdivision. $\mathscr{C}_{\Sigma_k}$ is the $\Sigma_k$ equivariant version of the base category.

These maps assemble to a total Segal operation, which in the case of Waldhausen $A$ - theory becomes an infinite loop map

$$\omega : A(*) \to 1 \times \prod_{n\geq 1} A_{\Sigma_n}(*),$$

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where $A_{\Sigma_n}$ is the $K$ - theory of $\Sigma_n$ - spaces.

As the $A_{\Sigma_n}$ allow for a splitting – which is infinite loop also – the operations of [W2] fit into the picture by projecting to the free part.

# 1 Review of the $G_\bullet$-construction - the un-delooped model of algebraic $K$-theory

In [Gu] we described how to extend the $G_\bullet$-construction of [G-G] to the context of categories with cofibrations and weak equivalences. We will recall the definitions. Notions not explicitly recalled are taken from [W].

1.1 Definition:

1. $\mathcal{C}of$ is the category whose objects are small categories with cofibrations and whose morphisms are the exact functors [W, p. 320].
2. $w\mathcal{C}of$ is the category whose objects are small categories with cofibrations and weak equivalences and whose morphisms are the exact categories [W, p. 326].

A basic example of an object in $w\mathcal{C}of$ is the category $\mathcal{S}^f_*$ of finite pointed simplicial sets with the usual notions of cofibrations and weak equivalences.

$\mathcal{S}_\bullet : w\mathcal{C}of \to \triangle^{op} - w\mathcal{C}of$ is the functor defined in [W, p. 328]. $\triangle^{op}$ is the index category for simplicial objects.

For $\mathscr{C} \in Ob(w\mathcal{C}of)$ we let $w\mathscr{C}$ denote the category of weak equivalences in $\mathscr{C}$. $Ob(w\mathscr{C}) = Ob(\mathscr{C})$, $Mor(w\mathscr{C}) = \{f \in Mor(\mathscr{C}) | f$ is a weak equivalence in $\mathscr{C}\}$.

1.2 Definition ([Gu-definition 2.2], [G-G]): $G_\bullet : w\mathcal{C}of \to \triangle^{op} - w\mathcal{C}of$ is defined by the cocartesian square

$$\begin{array}{ccc} G_\bullet\mathscr{C} & \to & PS_\bullet\mathscr{C} \\ \downarrow & & \downarrow \\ PS_\bullet\mathscr{C} & \to & S_\bullet\mathscr{C} \end{array}$$

Here $PS_\bullet\mathscr{C}$ is the path object $PS_n\mathscr{C} = S_{n+1}\mathscr{C}$ and $PS_\bullet\mathscr{C} \to S_\bullet\mathscr{C}$ is given by the boundary map $d_0 : X_{n+1} \to X_n$. $PS_\bullet\mathscr{C}$ is simplicially homotopic to $S_0\mathscr{C} = pt$ by [W, Lemma 1.5.1, p. 341]. This gives a map

$$|wG_\bullet\mathscr{C}| \longrightarrow \Omega|wS_\bullet\mathscr{C}| \tag{1}$$

Here $\Omega|wS_\bullet\mathscr{C}|$ is the model for $K$-theory of [W]. In [Gu] we proved:

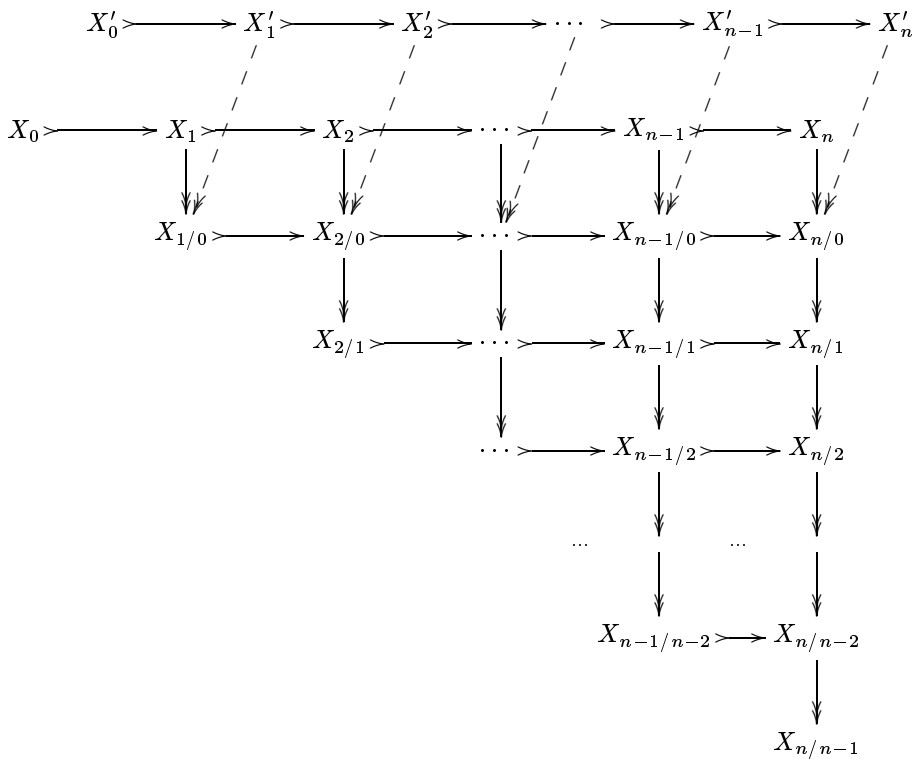
1.3 Theorem ([Gu, Theorem 2.6]):

The map $|wG_\bullet\mathscr{C}| \to \Omega|wS_\bullet\mathscr{C}|$ is a weak equivalence if $\mathscr{C}$ is pseudo-additive.

Here pseudo-additivity is an extra condition that for instance holds for exact categories (in the sense of Quillen). To get the algebraic $K$-theory in other cases, we assume that $\mathscr{C} \in w\mathscr{C}of$ has a cylinder functor satisfying $\mathcal{C}yl1$ and $\mathcal{C}yl2$ in [W, p. 343] and the cylinder axiom [W, p. 349]. Then $\mathscr{C}$ has a suspension functor $\Sigma$ and then a category of prespectra $\Sigma^\infty\mathscr{C}$ defined as

$$\mathsf{colim}(\mathscr{C} \overset{\Sigma}{\to} \mathscr{C} \overset{\Sigma}{\to} \mathscr{C} \to \cdots).$$

For $\Sigma^\infty\mathscr{C}$ the map (1) is a weak equivalence. The map $wS_\bullet\mathscr{C} \to wS_\bullet\Sigma^\infty\mathscr{C}$ is a weak equivalence by [W]. The model $wG_\bullet\mathscr{C}$ is a simplicial category whose objects in dimension $n$ can be described as diagrams




(2)

where $\rightarrowtail$ denotes cofibrations and the sequences $x'_0 \rightarrowtail x'_i \twoheadrightarrow x_{i/0}$, $x_0 \rightarrowtail x_i \twoheadrightarrow x_{i/0}$ and $x_{i/j} \rightarrowtail x_{k/j} \twoheadrightarrow x_{k/i}$ are cofibration sequences.

These diagrams can be considered as functors as follows (following [G]):

A category with exact sequences $(I, ses(I)))$ is a category $I$ together with a collection of sequences

$$ses(I) \subseteq Nerv_2(I)$$

An exact functor $F : (I_1, ses(I_1)) \to (I_2, ses(I_2))$ is a functor $F : I_1 \to I_2$ such that $Nerv_2(F)(ses(I_1)) \subseteq ses(I_2)$, i.e. $F$ maps exact sequences to exact sequences.

If $\mathscr{C} \in w\mathcal{C}of$, then the exact sequences in $\mathscr{C}$ are the cofibration sequences

$$A \rightarrowtail C \twoheadrightarrow B.$$

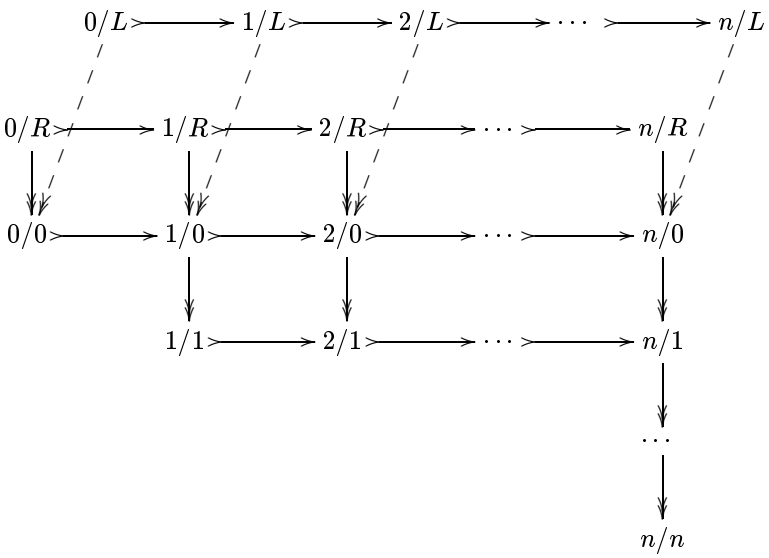
For $A \in \triangle$, we denote with $\gamma(A)$ the partially ordered set $\{L, R\} \amalg A$ with $L < a$ and $R < a$ for all $a \in A$; and for $c, d \in A$ we have $c < d$, if $c < d$ in $A$.

If $A$ is $0 \to 1 \to 2 \to \ldots \to n$, then $\gamma(A)$ is given by the diagram

$$\begin{array}{ccc} L & & \\ & \searrow & \\ & & 0 \to 1 \to 2 \to \ldots \to n \\ & \nearrow & \\ R & & \end{array}$$

$\Gamma(A)$ is the category of arrows in $\gamma(A)$, except $L \to L$ and $R \to R$.

If $A$ is $0 \to 1 \to 2 \to \ldots \to n$, as above then, $\Gamma(A)$ is given by the diagram




(3)

Here $a/b$ denotes the arrow $b \leq a$ in $\gamma(A)$ and $a/b \to c/d$ denotes the diagram

$$\begin{array}{ccc} b & \leq & d \\ \leq & & \leq \\ a & \leq & c \end{array}$$

The exact sequences in $\Gamma(A)$ are the sequences $j/k \to i/k \to i/j$ for $k \leq j \leq i$ in $\gamma(A)$, with $j/k$, $i/k$ and $i/j$ in $\Gamma(A)$. This gives an identification

$$G_A\mathscr{C} = \text{ Exact } (\Gamma(A), \mathscr{C}).$$

Observe that if $F : \Gamma(A) \to \mathscr{C}$ is exact, then $F(i/i) = pt$.

$\Gamma(A)$ is functorial in $A \in \triangle$ and this functoriality preserves exact sequences.

Hence another description [G] of $G_\bullet$ is obtained as a functor

$$w\mathcal{C}of \to (\triangle^{op} - w\mathcal{C}of).$$

# 2 Review of Grayson's construction of operations in algebraic $K$-theory of exact categories

In [G] D. Grayson used the $G_\bullet$-construction to obtain exterior power operations in algebraic $K$-theory of exact categories. As we will show, he set up things so that the techniques can be applied to the case with (nice) categories with cofibrations and weak equivalences. For the convenience of the reader, we will recall the construction in [G.].

2.1. The $k$-fold edge-wise subdivision ([G])

The concatenation $A_1, A_2, \ldots, A_k$ of a sequence $(A_1 A_2, \ldots, A_k)$ of (partially) ordered sets is the disjoint union $A_1 \coprod A_2 \coprod \ldots \coprod A_k$ ordered so that each $A_i$ is a (partially) ordered subset and with the additional relations that if $i < j$, $a_i \in A_i$ and $a_j \in A_j$ then $a_i < a_j$.

Concatenation gives rise to the standard functor

$$\begin{array}{cccc} \text{con:} & \triangle^k & \to & \triangle \\ & (A_1, \ldots, A_k) & \mapsto & A_1 A_2 \ldots A_k \end{array}$$

If $X : \triangle^{op} \to \mathscr{D}$ is a simplicial object in some category $\mathscr{D}$, then the $k$-fold edge-wise subdivision - we use - $\mathsf{sub}_k X$ of $X$, is the $k$-simplicial object in $\mathscr{D}$ given by the composite

$$\mathsf{sub}_k X : (\triangle^k)^{op} \xrightarrow{\text{con}} \triangle^{op} \xrightarrow{X} \mathscr{D}$$

For a simplicial set $X$ there is a natural homeomorphism $|\mathsf{sub}_k X| \to |X|$ – see [G] – of geometric realizations.

2.2 The functor $\Xi_K$

The combinatorics in the construction of the Segal operations are handled by [G] in the following way: There are two operations for which we use the symbols $\diamond$ (the diamond operation) and $\boxtimes$ (the boxtensor operation).

The formulas are encoded in a category $\Gamma^k(A)$ with exact sequences. $\Gamma^k(A)$ is functorial in $A \in \triangle$.

2.3 Definition ([G, §5]): For $A \in \triangle$ we define: $\Gamma^1(A) = \Gamma(A)$ and for $k > 1$, $\Gamma^k(A)$ has objects $\alpha = (i_1/l_1, *_2, i_2/l_2, *_3, \ldots, *_k, i_k/l_k)$

| | | |
|---|---|---|
| | (A1) | $i_r \in \gamma(A)$, $l_r \in \gamma(A)$ and $*_r \in \{\diamond, \boxtimes\}$ |
| such that for all $r$: | (A2) | $l_r \leq i_r$ and $i_r \in A$ |
| | (A3) | If $*_r = \diamond$ and $r > 1$, then $l_{r-1} = l_r$ and $i_{r-1} \leq i_r$ |

i.e. $Ob(\Gamma^k(A)) = Ob(\Gamma(A)) \times (\{\diamond, \boxtimes\} \times \Gamma(A))^{k-1}$. $\Gamma^k(A)$ will be a partially ordered set, considered as a category. The arrows in $\Gamma^k(A)$ are given by the requirement, that for $\alpha, \alpha' \in Ob(\Gamma^k(A))$ there is one and only one arrow $\alpha \to \alpha'$,

| | | |
|---|---|---|
| | (B1) | $i_r \leq i'_r$ |
| if for each $r$: | (B2) | $l_r \leq l'_r$ |
| | (B3) | If $*_r = \diamond$ and $*'_r = \boxtimes$, then $i_{r-1} \leq l'_r$ |

There is no map if $*_r = \boxtimes$ and $*'_r = \diamond$.

A sequence $\alpha' \to \alpha \to \alpha''$ is exact, if there exist integers $r \leq s$ such that:

(C1) for any $p < r$ or $s < p$, we have $i'_p = i_p = i''_p, l'_p = l_p = l''_p$ and $*'_p = *_p = *''_p$.

(C2) for any $p$ satisfying $r < p \leq s$, we have $*'_p = *_p = *''_p = \diamond$ and $i'_p = i_p = i''_p$.

(C3) $l_r = l'_r \leq i'_r = l''_r \leq i''_r = i_r$, $*'_r = *_r$ and $*''_r = \boxtimes$.

The Segal operations are extracted from $\Gamma^k(A)$ with the functor $\Xi_k$.

2.4 Definition ([G]): For $A_1, \ldots, A_k \in \triangle$, the functor

$$\Xi_k : \Gamma(A_1) \times \ldots \times \Gamma(A_k) \to \Gamma^k(A_1 \ldots A_k)$$

is defined by the formula $\Xi_k(i_1/j_1, \ldots, i_k/j_k) = (i_1/l_1, *_2, i_2/l_2, *_3, \ldots, *_k, i_k/l_k)$, where we define $l_1 = j_1$, and then inductively for $r > 1$ we declare:

(D1) If $j_r = L$, then $*_r = \diamond$ and $l_r = l_{r-1}$,
(D2) If $j_r \neq L$, then $*_r = \boxtimes$ and $l_r = j_r$.

2.5 Theorem ([G]):

$\Gamma^k(A)$ is a category with exact sequences, natural in the variable $A \in \triangle$, $\Xi_k$ is

a multi-exact functor (i.e. exact in each variable separately) and is natural in each of the variables $A_1, \ldots, A_k \in \triangle$.

2.6 Construction ([G])

Let $(\mathscr{M}_n)_{n\geq 0}$ be a sequence of exact categories (in the sense of Quillen [Q]) and let

$$\boxtimes_{n,p} : \mathscr{M}_n \times \mathscr{M}_p \to \mathscr{M}_{n+p}$$

be exact functors (called boxtensors) for $n, p \geq 0$. (Associative up to natural isomorphisms).

Let $F_k(\mathscr{M})$ denote the category of $k$-filtered objects in the exact category $\mathscr{M}$, i.e. sequences $v_1 \rightarrowtail \ldots \rightarrowtail v_k$ of cofibrations (admissible monomorphisms) in $\mathscr{M}$ and let

$$\diamond_{n,k} = F_k(\mathscr{M}_n) \to \mathscr{M}_{nk}$$

be functors (to be thought of as exterior powers and called diamonds.)

The notation $V_1 \diamond \cdots \diamond V_k := \diamond_{n,k}(V_1 \rightarrowtail V_2 \rightarrowtail \cdots \rightarrowtail V_k)$ is used.

The operations $(\boxtimes_{n,p})_{n\geq 0, p\geq 0}$ and $(\diamond_{n,k})_{n\geq 0, k\geq 1}$ are assumed to satisfy the following compatibility conditions (E1) - (E5), formalizing the relations between tensorproduct and exterior powers of vectorspaces:

(E1) Given $V_1 \rightarrowtail \cdots \rightarrowtail W \rightarrowtail X \rightarrowtail \cdots Y$, there is a natural map

$$(V, \diamond \cdots \diamond W) \boxtimes (X \diamond \cdots \diamond Y) \to V \diamond \cdots \diamond W \diamond X \diamond \cdots \diamond Y.$$

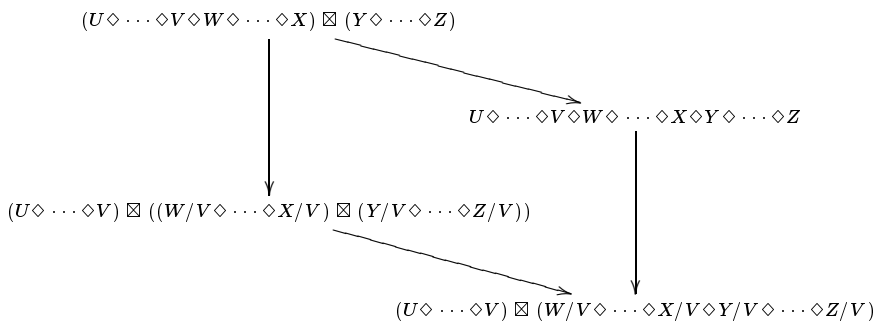
These maps are associative in the obvious sense.

(E2) Given $V \rightarrowtail \cdots \rightarrowtail W \rightarrowtail X \rightarrowtail \cdots \rightarrowtail Y$, there is a natural map

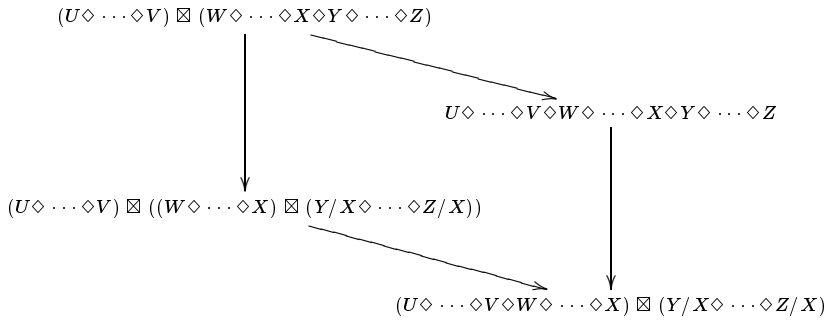
$$V \diamond \cdots \diamond W \diamond X \diamond \cdots \diamond Y \to (V \diamond \cdots \diamond W) \boxtimes (X/W \diamond \cdots \diamond Y/W).$$

These maps are associative in the obvious sense also. The condition is for any choice of quotient objects $X/W, \cdots, Y/W$.

(E3) Given $V \rightarrowtail \cdots \rightarrowtail V \rightarrowtail W \rightarrowtail \cdots \rightarrowtail X \rightarrowtail Y \rightarrowtail \cdots \rightarrowtail Z$ the following diagram commutes:




(E4) Given $V \rightarrowtail \cdots \rightarrowtail V \rightarrowtail W \rightarrowtail \cdots \rightarrowtail X \rightarrowtail Y \rightarrowtail \cdots \rightarrowtail Z$, the following diagram commutes




(E5) Given $U \rightarrowtail \cdots \rightarrowtail V \rightarrowtail W' \rightarrowtail W \rightarrowtail X \rightarrowtail \cdots \rightarrowtail Y$ the sequence

$$U\diamond\cdots\diamond V\diamond W'\diamond X\diamond\cdots\diamond Y \rightarrowtail U\diamond\cdots\diamond V\diamond W\diamond X\diamond\cdots\diamond Y \twoheadrightarrow (U\diamond\cdots\diamond V)\boxtimes(W/W'\diamond X/W'\diamond\cdots Y/W'$$

is an exact sequence.

## 2.7 The functor $\Lambda^k$

The collection of operations $\diamond$ and $\boxtimes$ define functors

$$\Lambda^k_{\diamond,\boxtimes}: \text{ Exact } (\Gamma(A), \mathscr{M}_n) \rightarrow \text{ Exact } (\Gamma^k(A), \mathscr{M}_{nk})$$

as follows ([G]): Given $M \in$ Exact $(\Gamma(A), \mathscr{M}_n)$ and given

$$(i_1/l_1, *_2, i_2/l_2, *_3, \ldots, *_k, i_k/l_k) \in Ob(\Gamma^k(A))$$

we define the functor $\Lambda^k M = \Lambda^k_{\diamond,\boxtimes} M$ on objects with the formula

$$\Lambda^k M(i_1/l_1, *_2, i_2/l_2, \ldots, *_k, i_k/l_k) = M(i_1/l_1) *_2 \ldots *_k M(i_k/l_k),$$

where $\diamond$ has precedence over $\boxtimes$, and left association is used for $\boxtimes$. $\Lambda^k M$ is defined on arrows in the natural way.

Theorem ([G]):

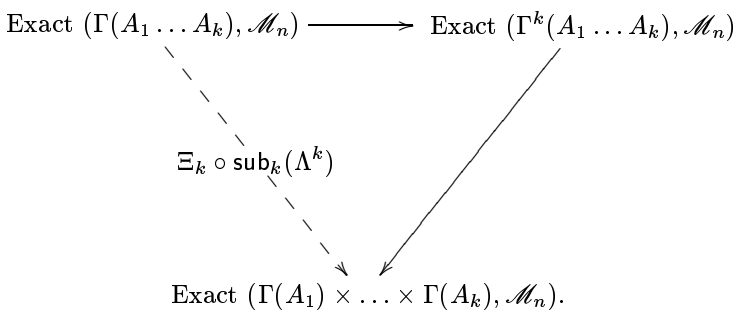
$\Lambda^k$ - defined as above - gives a well-defined functor, natural in the variable $A \in \triangle$.

The functor $\Xi_k$ above induces functors

$$\Xi_k^* : \text{ Exact } (\Gamma^k(A_1 \dots A_k), \mathscr{M}_n) \to \text{ Exact } (\Gamma(A_1) \times \dots \times \Gamma(A_k), \mathscr{M}_n),$$

where a functor $\Gamma(A_1) \times \dots \times (\Gamma(A_k) \to \mathscr{M}_n$ is exact, if it is exact in each variable separately.

The composition $\Xi_k^* \circ \mathsf{sub}_k(\Lambda^k)$ now gives the desired operations




Since there is naturality in the variables $A_1, \dots, A_k \in \triangle$, Exact $(\Gamma(A), \mathscr{M}_n) = G_A \mathscr{M}_n$ and for the the $k$-times iterated $G_\bullet$-construction, there is an isomorphism: $G_{A_1,\dots,A_k}\mathscr{M}_n \cong$ Exact $(\Gamma(A_1) \times \dots \times \Gamma(A_k), \mathscr{M}_n)$. The compositions $\omega_k := \Xi_k^* \circ \mathsf{sub}_k(\Lambda^k)$ can be viewed as functors

$$\omega^k := \mathsf{sub}_k G_\bullet \mathscr{M}_n \to G_\bullet^k \mathscr{M}_{nk},$$

whose geometric realizations give the (desired) operations

$$\omega^k : K(\mathscr{M}_n) \to K(\mathscr{M}_{nk}).$$

The last assertion comes by observing, that there is a natural inclusion $G_\bullet \mathscr{M} \to G_\bullet^k \mathscr{M}$, which is a weak equivalence for exact categories ([G-G]).

# 3 Equivariant Objects, boxtensor and diamond

For a group $G$ we let $\mathscr{C}_G$ be the category of objects in $\mathscr{C}$ with a left $G$-action and equivariant maps. A $G$-cofibration should just be a $G$-map, which is a cofibration in $\mathscr{C}$. Observe that this turns $(\mathscr{C}_G, G - cof)$ into a category with

cofibrations in the sense of [W]. Now let $\mathscr{C}$ be a category with cofibrations and a bi-exact pairing $\wedge$ such that $(\mathscr{C}, \vee, \wedge)$ is symmetric bimonoidal [M] ($\vee$ is a sum in $\mathscr{C}$). Using these data we will (following [G]) construct two sets of operations:

$$\boxtimes k, l : \mathscr{C}_{\Sigma_k} \times \mathscr{C}_{\Sigma_l} \longrightarrow \mathscr{C}_{\Sigma_{k+l}}$$

and the diamond operation

$$\diamond_k : \mathscr{C}^{[k]} \longrightarrow \mathscr{C}_{\Sigma_k}.$$

Here $\Sigma_k$ is the symmetric group. $\mathscr{C}^{[k]}$ is the category of filtered objects in $\mathscr{C}$, i.e. the category of sequences

$$p_1 \rightarrowtail p_2 \rightarrowtail \ldots \rightarrowtail p_k$$

of cofibrations and natural transformations of such sequences. The operations $\boxtimes$ and $\diamond$ corresponds to $\otimes$ and $\wedge$ in [G]. We follow an earlier notation by Grayson in order not to confuse the diamond operator with the pairing $\wedge$. Our construction is a non-linear analogue of the construction in the last example in [G]. The boxtensor operation $\boxtimes$ is defined using induction as follows. For a finite set $A$ and $C \in Ob(\mathscr{C}_H)$ we form

$$\begin{array}{rcl} A^+ \wedge C & := & C \vee \cdots \vee C \\ & & \longleftarrow |A| \longrightarrow \end{array} .$$

This construction can be made functorial using the symmetric monoidal structure by association to the left. (This convention we always will use throughout the paper for (symmmetric) monoidal structures.) $A^+$ is $A$ with a basepoint added. For a subgroup $H$ of a finite group $G$ we let

$$\mathsf{Ind}_H^G(C) := (G/H)^+ \wedge C,$$

which we give a $G$-action by the rule: Pick a set $g_1, \ldots, g_{|G/H|}$ of representatives $g_i \in G$ of the left cosets in $G/H$. If $gg_i = g_j h$, with $h \in H$, then $g$ maps the $i$-th summand of $(G/H)^+ \wedge C$ to the $j$-th summand using $h$

$$\begin{array}{lcl} (G/H)^+ \wedge C & \stackrel{g}{\longrightarrow} & (G/H)^+ \wedge C \\ \uparrow incl_i & & \uparrow incl_j \\ C & \stackrel{h}{\longrightarrow} & C \end{array}$$

Consider $\Sigma_k \times \Sigma_l$ as a subgroup of $\Sigma_{k+l}$ using the standard inclusion.

3.1 Definition: The boxtensor operation

$$\boxtimes_{k,l} : \mathscr{C}_{\Sigma_k} \times \mathscr{C}_{\Sigma_l} \longrightarrow \mathscr{C}_{\Sigma_{k+l}}$$

is given by $C_1 \boxtimes_{k,l} C_2 := \mathsf{Ind}_{\Sigma_k \times \Sigma_l}^{\Sigma_{k+l}}(C_1 \wedge C_2)$; for $k, l \geq 0$.

3.2 Lemma:

The boxtensoroperation $\boxtimes_{k,l} : \mathscr{C}_{\Sigma_k} \times \mathscr{C}_{\Sigma_l} \longrightarrow \mathscr{C}_{\Sigma_{k+l}}$ is bi-exact for all $k, l \geq 0$.

Proof: The pairing $\wedge$ is bi-exact, and $\vee$ is exact.

3.3 Remark: We write $\boxtimes$ for $\boxtimes_{k,l}$ if $k, l$ are clear from the context.

The diamond operation $(\diamond_n)$ is defined as follows: Let $\underline{P} = (P_1 \rightarrowtail \ldots \rightarrowtail P_k)$ be a sequence of cofibrations in $\mathscr{C}$. Let $\underline{\underline{k}} = \{1, \ldots, k\}$ and $I(\underline{\underline{k}}) = \{f : \underline{\underline{k}} \to \underline{\underline{k}} | (\exists \sigma \in \Sigma_k) \text{ with } (f \leq \sigma)\}$. Here $f \leq g$ means: $(\forall i \in \underline{\underline{k}}) \quad (f(i) \leq g(i))$. This makes $(I(f), \leq)$ into a partially ordered set. The sequence $\underline{P}$ defines a functor $\underline{P} : I(\underline{\underline{k}}) \to \mathscr{C}$ given on objects by

$$f \longmapsto \underline{P}_f := (P_{f(l)} \wedge \ldots \wedge P_{f(k)}).$$

If $f \leq g$, the cofibrations $P_{f(i)} \rightarrowtail P_{g(i)}$ give cofibrations $\underline{P}_f \rightarrowtail \underline{P}_g$ by bi-exactness of $\wedge$.

3.4 Definition:

i) The diamond operation $\diamond_k : \mathscr{C}^{[k]} \longrightarrow \mathscr{C}_{\Sigma_k}$ is given by $\diamond_k(\underline{P}) = \mathsf{colim}_{f \in I(k)} \underline{P}_f$, with $\Sigma_k$-action induced by permuting factors. (As $\underline{P}$ is a finite diagram of cofibrations the colimit exists.)

ii) For $n > 1$ observe, that $\diamond_k : \mathscr{C}_{\Sigma_n}^{[k]} \longrightarrow (\mathscr{C}_{\Sigma_n})_{\Sigma_k} \cong \mathscr{C}_{B_{n,k}}$,

where $B_{n,k} < \Sigma_{n \cdot k}$ denotes the group of blockpermutations. The blocks are of length $n$ and there are $k$ of them. $\diamond_{n,k} : \mathscr{C}_{\Sigma_n}^{[k]} \longrightarrow \mathscr{C}_{\Sigma_{n \cdot k}}$ then denotes the composition of $\diamond_k$ with the induction $B_{n,k}$ to $\Sigma_{n \cdot k}$.

3.5 Remark:

For $\underline{P} = (P_1 \rightarrowtail \ldots \rightarrowtail P_k)$ we use the notation $P_1 \diamond \cdots \diamond P_k := \diamond_k(\underline{P})$, in case the cofibrations are clear from the context. Note that if $P = P_1 = P_2 = \ldots = P_n$, then

$$\begin{array}{lll} P_1 \diamond P_2 \diamond \cdots \diamond P_n = & P \diamond \cdots \diamond P = & P \wedge \cdots \wedge P. \\ & \longleftarrow n \longrightarrow & \longleftarrow n \longrightarrow \end{array}$$

3.6 Example:

If $(\mathscr{C}, \vee, \wedge)$ is the usual category of finite pointed sets with inclusions as cofibrations, then $P_1 \diamond \ldots \diamond P_K$ is (isomorphic to) the union

$$\bigcup_{\sigma \in \Sigma_k} P_{\sigma(1)} \wedge \ldots \wedge P_{\sigma(k)} \subset P_k \wedge \ldots \wedge P_k,$$

i.e. the smallest $\Sigma_k$-equivariant subset of $P_k \wedge \ldots \wedge P_k$ containing $P_1 \wedge P_2 \wedge \ldots \wedge P_k$.

# 4 Grayson's conditions for categories with cofibrations

By inspection the following variant of Grayson's result holds:

4.1 Theorem:

Let $(\mathscr{M}_n)_{n\geq 0}$ be a sequence of categories with cofibrations. Let

$$\boxtimes_{m,p} : \mathscr{M}_m \times \mathscr{M}_p \longrightarrow \mathscr{M}_{m+p}$$

be exact functors associative up to natural isomorphisms. Let $F_k(\mathscr{M})$ denote the category of $k$-filtered objects in a category with cofibrations $\mathscr{M}$ and

$$\diamond_{n,k} : F_k(\mathscr{M}_n) \longrightarrow \mathscr{M}_{n\cdot k}$$

functors. Suppose the $\boxtimes_{n,p}$ and $\diamond_{n,k}$ fulfill (E1) - (E5) of section 2. Then Grayson's construction

$$w_n^k : \mathsf{sub}_k G_\bullet \mathscr{M}_n \longrightarrow G_\bullet^k \mathscr{M}_{n\cdot k}$$

is well-defined.

4.2 Remark: Observe that for a general category with cofibrations $\mathscr{M}$ the construction $G_\bullet \mathscr{M}$ may not give the $K$-theory of $\mathscr{M}$ in the sense of [W].

4.3 Definition: A category $\mathscr{C}$ with cofibrations is said to have the extension property, if for all commutative diagrams:

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\ A' & \rightarrowtail & B' & \twoheadrightarrow & C' \end{array}$$

in $\mathscr{C}$ it follows, that $i \in \mathcal{C}of(\mathscr{C})$.

4.4 Remark: The extension property implies that the natural inclusion

$$G^k_{A_1,\ldots,A_k}\mathscr{C} \longrightarrow \ \mathrm{Exact}(\Gamma(A_1) \times \ldots \times \Gamma(A_k), \mathscr{C})$$

is an isomorphism.

4.5 Proposition: Let $(\mathscr{C}, \vee, \wedge)$ be a symmetric bimonoidal category with cofibrations, extension property and bi-exact pairing $\wedge$. Then the functors

$$\boxtimes_{n,k} : \mathscr{C}_{\Sigma_n} \times \mathscr{C}_{\Sigma_k} \longrightarrow \mathscr{C}_{\Sigma_{n+k}}$$

and

$$\diamond_{n,k} : F_k(\mathscr{C}_{\Sigma_n}) \longrightarrow \mathscr{C}_{\Sigma_{n \cdot k}}$$

of section 3 fulfill (E1) - (E5).

Proof: Properties (E1) - (E4) are consequences of the assumed symmetric bimonoidal structure. It remains to verify property (E5). As $\diamond_{n,k}$ is given by the composition of $\diamond_k$ with an exact functor, it suffices to show (E5) for $\diamond_k$.

4.6 Lemma: If $\mathscr{C}$ has the extension property for cofibrations then a sequence

$$X_1 \rightarrowtail \cdots \rightarrowtail X_n \rightarrowtail A \rightarrowtail Y_1 \rightarrowtail \cdots \rightarrowtail Y_m$$

of cofibrations gives rise to cofibration sequences:

(i) $$\begin{array}{ccc} X_1 \diamond \cdots \diamond X_n \diamond A \diamond Y_2 \diamond \cdots \diamond Y_n & \rightarrowtail & X_1 \diamond \cdots \diamond X_n \diamond Y_1 \diamond \cdots \diamond Y_m \\ & & \downarrow \\ & & (X_1 \diamond \cdots \diamond X_n) \boxtimes (Y_1/A \diamond \cdots \diamond Y_m/A) \end{array}$$

and

(ii) $$A \diamond Y_2 \diamond \cdots \diamond Y_m \rightarrowtail Y_1 \diamond \cdots \diamond Y_m \twoheadrightarrow Y_1/A \diamond \cdots \diamond Y_m/A.$$

Claim (i) follows from claim (ii) by the following Lemma:

4.7 Lemma: If $X_1 \rightarrowtail \cdots \rightarrowtail X_n$ is a sequence of cofibrations then the diagrams

$$\begin{array}{ccc} (X_1 \diamond \cdots \diamond X_k) \boxtimes (X_k \diamond X_{k+2} \diamond \cdots \diamond X_n) & & \\ \downarrow & \searrow & \\ & & X_1 \diamond \cdots \diamond X_k \diamond X_k \diamond X_{k+2} \diamond \cdots \diamond X_n \\ (X_1 \diamond \cdots \diamond X_k) \boxtimes X_{k+1} \diamond X_{k+2} \diamond \cdots \diamond X_n) & & \downarrow \\ & \searrow & \\ & & X_1 \diamond \cdots \diamond X_k \diamond X_{k+1} \diamond X_{k+2} \diamond \cdots \diamond X_n \end{array}$$

are cocartesian.

**Proof:** This is a consequence of the definition of $\diamond$ and that one takes the colimit in four steps.

In the situation of Lemma 4.6 this gives

$$\begin{array}{ccc}
(X_1\diamond\cdots\diamond X_n)\boxtimes(X_n\diamond Y_2\diamond\cdots\diamond Y_m) & \longrightarrow & X_1\diamond\cdots\diamond X_n\diamond X_n\diamond Y_2\diamond\cdots\diamond Y_m \\
\downarrow & & \downarrow \\
(X_1\diamond\cdots\diamond X_n)\boxtimes(A\diamond Y_2\diamond\cdots\diamond Y_m) & \longrightarrow & X_1\diamond\cdots\diamond X_n\diamond A\diamond Y_2\diamond\cdots\diamond Y_m \\
\downarrow & & \downarrow \\
(X_1\diamond\cdots\diamond X_n)\boxtimes(Y_1\diamond Y_2\diamond\cdots\diamond Y_m) & \longrightarrow & X_1\diamond\cdots\diamond X_n\diamond Y_1\diamond Y_2\diamond\cdots Y_m
\end{array}$$

By Lemma 4.7 the large and the upper squares are cocartesian, hence the lower square is cocartesian.

By claim (ii) and the bi-exactness of the $\boxtimes$ operation the lower left vertical map is a cofibration with cofiber $(X_1\diamond\cdots\diamond X_n)\boxtimes(Y_1/A\diamond\cdots\diamond Y_m/A)$, but, as the lower square is cocartesian, this also holds for the lower right vertical map, since pushouts of cofibrations are required to be cofibrations. This concludes the proof that claim (ii) implies claim (i).

It remains to prove claim (ii). The proof is split into showing, that the first map in claim (ii) is a cofibration and showing that the cofiber is as claimed.

4.8 Lemma: The diagram

$$\begin{array}{ccc}
A\diamond Y_2\diamond\cdots\diamond Y_m & \longrightarrow & Y_1\diamond\cdots\diamond Y_m \\
\downarrow & & \downarrow \\
* & \longrightarrow & Y_1/A\diamond Y_2/A\diamond\cdots\diamond Y_m/A
\end{array}$$

is cocartesian.

Proof: We verify the universal property of a cofiber. Let

$$\underline{P}^{(i)} = (A \rightarrowtail Y_1 \rightarrowtail \hat{Y}_i \rightarrowtail \cdots \rightarrowtail Y_m)$$

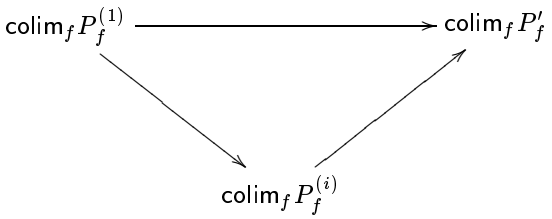
denote the cofibration sequence of length $m$ with $Y_i$ deleted for $1 \leq i \leq m$, let

$\underline{\underline{P'}} = (Y_1 \rightarrowtail Y_2 \rightarrowtail \cdots \rightarrowtail Y_m)$ and $\underline{\tilde{P}} = (Y_1/A \rightarrowtail Y_2/A \rightarrowtail \cdots \rightarrowtail Y_m/A)$.

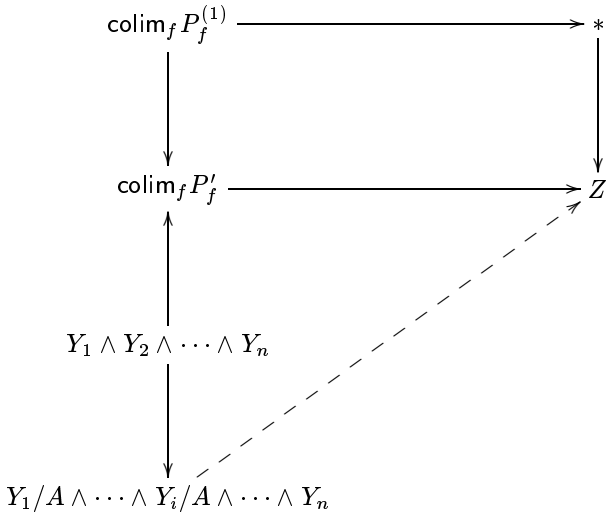
The lemma is to say that

$$\begin{array}{ccc}
\mathsf{colim}_f\, \underline{P}^{(1)}_f & \longrightarrow & \mathsf{colim}_f\, \underline{\underline{P'}}_f \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathsf{colim}_f\, \underline{\tilde{P}}_f
\end{array}$$

is cocartesian (with notation as in Section 3). For that purpose first observe that there is a factorization

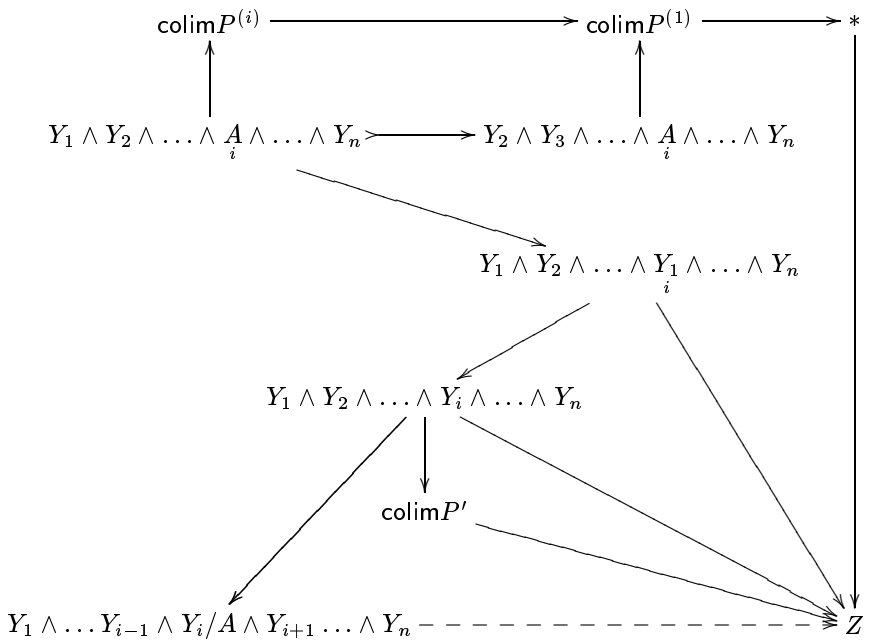



for $1 \leq i \leq m$. It suffices to check that for all commutative diagrams




the dotted arrow exists uniquely and hence the map $\mathsf{colim} P'_f \to Z$ factors uniquely over $\mathsf{colim}_f \underline{\tilde{P}}_f$. All unlabeled arrows are the natural ones.

This can be seen by the following commutative diagram in which the dotted arrow obviously exists by bi-exactness of the $\wedge$-pairing:




The claim now follows by Gertrude Stein's Lemma.

It remains to show that the left map in claim (ii) is a cofibration. This is done inductively.

$c_n$ : (i) and (ii) holds for $\diamond$-products of length $< n$.

$c^{(1)}_{n,i}$ : (i) holds for $\diamond$-products of length $< n$ and

(ii) holds for $\diamond$-products of length $n$ of the type

$$P \diamond P_1 \diamond \cdots \diamond P_1 \diamond P_{i+2} \diamond \cdots \diamond P_n \to P_1 \diamond P_1 \diamond \cdots \diamond P_1 \diamond P_{i+2} \diamond \cdots \diamond P_n.$$

4.9 Lemma: The following holds

(a) $c_2$

(b) $c_n \Rightarrow c^{(1)}_{n+1,n-1}$ for $n \geq 2$

(c) $c_n$ and $c^{(1)}_{n+1,i} \Rightarrow c^{(1)}_{n+1,i-1}$ for $2 \leq i \leq n-1$

(d) $c_n$ and $c^{(1)}_{n+1,1} \Rightarrow c_{n+1}$

Induction using Lemma 4.9 shows the remaining part of Lemma 4.6.

Proof of Lemma 4.9(a): Given a cofibration $P_1 \rightarrowtail P_2$, we obtain the following diagram

$$\begin{array}{ccccc}
P_1 \wedge P_1 & \rightarrowtail & P_1 \wedge P_2 & \twoheadrightarrow & P_1 \wedge P_2/P_1 \\
\downarrow & & \downarrow & & \downarrow = \\
P_2 \wedge P_1 & \longrightarrow & P_1 \diamond P_2 & \longrightarrow & P_1 \wedge P_2/P_1 \\
\downarrow = & & \downarrow f_2 & & \downarrow f_1 \\
P_2 \wedge P_1 & \rightarrowtail & P_2 \wedge P_1 & \longrightarrow & P_2 \wedge P_2/P_1
\end{array}$$

The top row is a cofibration sequence by bi-exactness of $\wedge$. The top left square is cocartesian, hence the middle row is a cofibration sequence. The bottom row is a cofibration sequence by bi-exactness of $\wedge$. The map of cofibers $f_1$ is a cofibration for the same reason. By the extension property for cofibrations $f_2$ is a cofibration. But $f_2$ is the map $P_1 \diamond P_2 \to P_2 \diamond P_2$.

4.10 Lemma: Given a cofibration $P \rightarrowtail P_1$, then the squares of products of length $n$,

$$\begin{array}{ccc}
(P \diamond P_1 \diamond \cdots \diamond P_1) \wedge P & \longrightarrow & (P_1 \diamond \cdots \diamond P_1) \wedge P \\
\downarrow & & \downarrow \\
(P \diamond P_1 \diamond \cdots \diamond P_1) \wedge P_1 & \longrightarrow & P \diamond P_1 \diamond \cdots \diamond P_1 \diamond P_1
\end{array}$$

is cocartesian in $\mathscr{C}$.

Proof: This follows by computing $P \diamond P_1 \diamond \cdots \diamond P_1$ in four steps.

The cofiber of the top map is by $c_n$ and the bi-exactness of $\wedge$ the object $(P_1/P \diamond \cdots \diamond P_1/P) \wedge P$ and hence this is also the cofiber of the lower map in Lemma 4.10.

Proof of Lemma 4.9(b): The map

$$(P \diamond P_1 \diamond \cdots \diamond P_1) \wedge P_1 \longrightarrow (P_1 \diamond \cdots \diamond P_1) \wedge P_1 = P_1 \diamond \cdots \diamond P_1 \diamond P_1$$

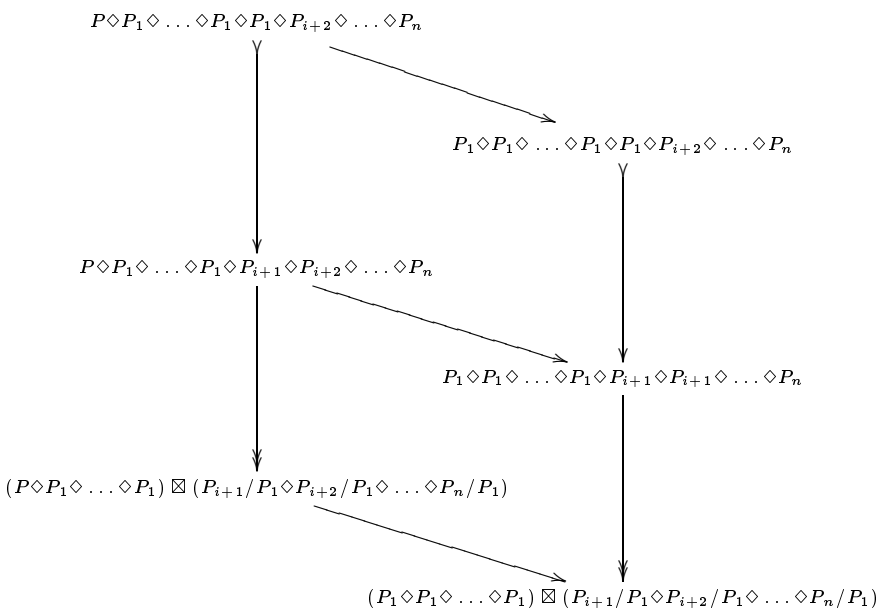
is a cofibration in $\mathscr{C}$ by $c_n$ and bi-exactness of $\wedge$, with cofiber $(P_1/P \diamond \cdots \diamond P_1/P) \wedge P_1$.

In the map of cofibration sequences (in $\mathscr{C}$)

$$\begin{array}{ccccc}
(P \diamond P_1 \diamond \cdots \diamond P_1) \wedge P_1 & \rightarrowtail & P \diamond P_1 \diamond \cdots \diamond P_1 \diamond P_1 & \twoheadrightarrow & (P_1/P \diamond \cdots \diamond P_1/P) \wedge P \\
\downarrow = & & & & \downarrow \\
(P \diamond P_1 \diamond \cdots \diamond P_1) \wedge P_1 & \rightarrowtail & P_1 \diamond P_1 \diamond \cdots \diamond P_1 \diamond P_1 & \twoheadrightarrow & (P_1/P \diamond \cdots \diamond P_1/P) \wedge P_1
\end{array}$$

the right map is a cofibration by $c_n$ and the bi-exactness of $\wedge$. Hence the middle map is a cofibration by the extension property for cofibrations.

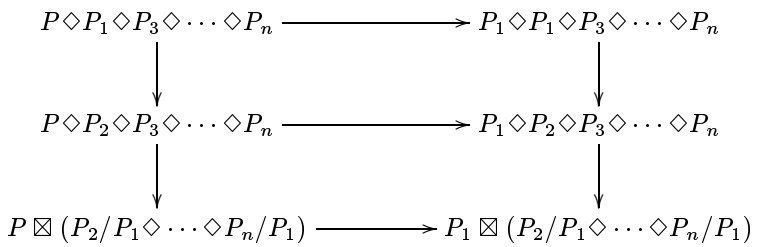
Proof of Lemma 4.9(c):




The columns are cofibration sequences by $c_n$ and Lemma 4.6. The bottom map is a cofibration by $c_n$ and the exactness of $\boxtimes$. The map on top is a cofibration by $c^{(1)}_{n+1,i}$, hence by the extension property for cofibrations the middle of the slanted maps is a cofibration. This is the claim of $c_{n+1,i-1}$.

Proof of Lemma 4.9(d):

In the commutative diagram




The columns are cofibration sequences by Lemma 4.7 and $c_n$. The top horizontal map is a cofibration by $c^{(1)}_{n+1,1}$. The bottom horizontal map is a cofibration by $c_n$ and the exactness of $\boxtimes$. By the extension property for cofibrations the middle horizontal map is a cofibration too.

# 5 Operations on retractive spaces

We first need some more notation for modeling $K$-theory of spaces with group action. As usual we will have a simplicial and a topological variant.

5.1 Definition: For a group $G$, we call a set of subgroups $\mathcal{F}$ a family of subgroups, if $\mathcal{F}$ contains at most one member of a conjugacy class of subgroups [I].

5.2 Definition: For a discrete group $G$, a $G$-simplicial set $Y$ is said (as in [W]) to have orbittypes in a family $\mathcal{F}$ relative to a $G$-simplicial set $W$, if $Y$ may be obtained from $W$ by direct limit and formation of pushouts of diagrams of the kind

$$Y \leftarrow \partial \triangle^n \times G/H \rightarrow \triangle^n \times G/H,$$

where $\triangle^n$ denotes the simplicial $n$-simplex, $\partial\triangle^n$ the simplicial subset 'boundary' and $H \in \mathcal{F}$.

5.3 Definition: For a $G$-simplicial set $W$, we denote by $\mathscr{R}(W, G, \mathcal{F})$ the category whose objects are tripels $(Y, r, s)$, with $Y$ is a $G$-simplicial set, which has orbittypes in $\mathcal{F}$ relative to a $G$-section $s : W \rightarrow Y$. The map $r : Y \rightarrow W$ is a given $G$-retraction to $sW$. The morphismus are the $G$-simplicial maps, commuting with the structure data.

5.4 Definition: With $\mathscr{R}_f(W, G, \mathcal{F})$ we denote the category with cofibrations and weak equivalences, whose objects $(Y, r, s)$ are in $\mathscr{R}(W, G, \mathcal{F})$ and finitely many pushouts

$$Y' \leftarrow \partial \triangle^n \times G/H \rightarrow \triangle^n \times G/H$$

with $H \in \mathcal{F}$ suffice to built up $Y$ from $W$. The cofibrations are the injective maps. The weak equivalences $(Y, r, s) \rightarrow (Z, t, u)$ are the maps whose underlying maps $Y \rightarrow Z$ are equivariant homotopy equivalences after realization.

5.5 Definition: For a $G$-space $|X|$, we denote by $\mathscr{R}(X, G, \mathcal{F})$ the category, whose objects a triples $(Y, r, s)$, where $Y$ is a $G - CW$-complex relative $X$ with orbittypes in $\mathcal{F}$. That is: $Y$ can be build up from $s|X|$ by direct limit and forming pushouts of the type

$$Y' \leftarrow S^{n-1} \times G/H \rightarrow D^n \times G/H,$$

with $H \in \mathcal{F}$. As in the simplicial case $s\colon |X| \rightarrow Y$ is a $G$-section with $G$-retraction
$r : Y \rightarrow s|X|$. The maps are the $G$-cellular maps commuting with the structure data.

5.6 Definition: With $\mathscr{R}_f(|W|, G, \mathcal{F})$ we denote the category of cofibrations and weak equivalences, with objects - as in the simplicial case - the finite objects of

$\mathscr{R}(|W|, G, \mathcal{F})$. The cofibrations are the maps, which are isomorphic to $G$-cellular inclusions, and the weak equivalence are the equivariant homotopy equivalences again.

There are of course - as in [W] - homotopy finite versions of these categories, which we will only formulate in the topological case.

5.7 Definition: For a $G$-space $|W|$ we denote with $\mathscr{R}_{hf}(|W|, G, \mathcal{F})$ the category with cofibrations and weak equivalences, whose objects $(Y, r, s)$ are $G$-retractive spaces over $|W|$, which are $G$-homotopy equivalent to objects in $\mathscr{R}_f(|W|, G, \mathcal{F})$. Cofibrations are now equivariant cofibrations (i.e. those maps which have the equivariant homotopy extension property) and the weak equivalences are the maps which are equivariant homotopy equivalences.

Exactly as in [W, Prop. 2.1.1] one proves:

5.8 Lemma: The approximation theorem applies to the map $\mathscr{R}_f(|W|, G, \mathcal{F}) \to \mathscr{R}_{hf}(|W|, G, \mathcal{F})$. That is: it induces an equivalence in $K$-theory.

For the necessary underlying topological assertions see again [I].

From the very definition we now get operations $w^k$ ($w_1^k$ in the notation of Theorem 4.1):

$$\omega^k : w\mathsf{sub}_k G_\bullet \mathscr{R}_f(*) \to wG_\bullet^k \mathscr{R}_f(*, \Sigma_k, \{\text{all}\}),$$

where $\{\text{all}\}$ denotes a representative for all conjugacy classes of subgroups of $\Sigma_k$. It suffices to check that the notions of weak equivalence in $\mathscr{R}_f(*)$ becomes moved to the notion of weak equivalence in $\mathscr{R}_f(*, \Sigma_K, \{\text{all}\})$ by the functors $\boxtimes$ and $\diamond$ of section 4. This follows from the fact that the fixed point sets of the realizations of the object products by $\boxtimes$ and $\diamond$ are given by (iterated pushouts of) diagonals of smash products (along cofibrations).

In the topological case observe, that $\boxtimes$ and $\diamond$ can be viewed as first taking values in categories of generalized finite pointed $\Sigma_n - CW$-complexes:

For a finite group G the category of generalized finite pointed $G-CW$-complexes has as objects spaces built up from the base point by pushouts of the kind

$$Y' \leftarrow \partial DV \times_H G \to DV \times_H G,$$

where $DV$ is the disc in an orthogonal finite dimensional representation of $H$. The maps are cellular with respect to these generalized $G$-cells [L-M-St, pg 52]. The cofibrations are then maps isomorphic to inclusions of generalized $G-CW$-sub-complexes.

Such maps become cofibrations of $CW$-complexes in the ordinary sense, when forgetting the action of $G$. So they fit into the framework of section 3. By [Mu, Prop.9.6] (see also [Hu, Thm.7.10] and [Han1, Han2]) since the spaces involved are compact G-ANR's, the class of ordinary G-cofibrations is closed under extension in the sense of definition 4.3.

As weak homotopy equivalences of $CW$-complexes are honest homotopy equivalences and $\boxtimes$ and $\diamond$ are continuous functors on finite $CW$-complexes, it's automatic, that they map weak equivalences to equivariant homotopy equivalences.

Since representations have $G-CW$ structures in the strict sense [I], we can view these target categories of $\boxtimes$ and $\diamond$ as subcategories of $\mathscr{R}_{hf}(|*|, \Sigma_k, \{\text{all}\})$.

This composition with inclusion, we will take as the definition of $(\omega^k)$ in the topological case.

To summarize:

5.9 Proposition: Let $\mathscr{R}$ denote either $\mathscr{R}_f(*)$ or $\mathscr{R}_f(|*|)$ and $\mathscr{R}_{\Sigma_k}$ denote $\mathscr{R}_f(*, \Sigma_k, \{\text{all}\})$ or $\mathscr{R}_{hf}(|*|, \Sigma_k, \{\text{all}\})$ accordingly, then the maps $\omega^k$ of section 4 induce operations, called the same way:

$$\omega^k : w\mathsf{sub}_k G_\bullet \mathscr{R}_\bullet \to wG^k_\bullet \mathscr{R}_{\Sigma_k}.$$

5.10 Remark: To obtain the complete set in the sense of section 4 of operations

$$\omega^k_n : w\mathsf{sub}_k G_\bullet \mathscr{R}_{\Sigma_n} \to wG^k_\bullet \mathscr{R}_{\Sigma_{n\cdot k}}$$

we use as the building block for the $\diamond_{n,k}$ – as in section 3.4 – the construction $\Sigma^+_{n\cdot k} \wedge_{B_{n,k}} Y^k$ for a $\Sigma_n$ object $Y$.

5.11 Variant: Starting with retractive spaces over a given space $X$ rather than pointed spaces, the preceding constructions yield operations

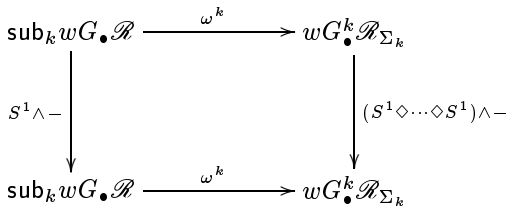
$$\omega^k_n : w\mathsf{sub}_k G_\bullet \mathscr{R}_{\Sigma_n}(X^n) \to wG^k_\bullet \mathscr{R}_{\Sigma_{n\cdot k}}(X^{n\cdot k}).$$

Here $X^n$ carries the natural $\Sigma_n$ operation. To define $\diamond_{n,k}$ for $n > 1$ in this case also, one takes the push out of the formula in 5.10 with the $\Sigma_{n\cdot k}$ equivariant map $\Sigma^+_{n\cdot k} \wedge_{B_{n,k}} (X^n)^k \to X^{n\cdot k}$.

# 6 Compatibility of $\omega^k$ with suspensions

In [Gu] we obtained a model for the $K$-theory of spaces based on the $G_\bullet$-construction using prespectra. We need to extend the operations correspondingly. For the categories $\mathscr{R}$ and $\mathscr{R}_{\Sigma_k}$ defined in section 5 we have:

6.1 Proposition: The following diagram commutes up to natural isomorphism




The presence of the natural isomorphism can be handled with the Grothendieck construction of [T], a version of homotopy colimits for categories designed for this situation. This gives operations

$$\omega^k : \mathsf{hocolim}_{S^1\wedge -} w\mathsf{sub}_k G_\bullet\mathscr{R} \longrightarrow \mathsf{hocolim}_{S^1\diamond\cdots\diamond S^1\wedge -} wG^k_\bullet\mathscr{R}_{\Sigma_k}$$

Since $S^1\wedge -$ and $(S^1\diamond\cdots\diamond S^1)\wedge -$ induce cofibrations, one obtains the same homotopy type as with ordinary colimits.

$\Sigma^\infty\mathscr{R} = \mathsf{colim}_\Sigma\mathscr{R}$ can be interpreted as a category with objects strict suspension prespectra ($S^1\wedge X_n = X_{n+1}$) for sufficiently large $n$, maps $f_n : X_n \to Y_n$ for sufficiently large $n$ such that $id_{S^1}\wedge f_n = f_{n+1}$ eventually, with the obvious equivalence relations.

$\Sigma^\infty\mathscr{R}$ is contained in the category $\mathsf{Lax}\Sigma^\infty\mathscr{R}$ of suspension prespectra with $S^1\wedge X_n \to X_{n+1}$ eventually an isomorphism and maps $f_n : X_n \to Y_n$ such that the diagrams

$$\begin{array}{ccc} S^1\wedge X_n & \xrightarrow{\sim} & X_{n+1} \\ id_{S^1}\downarrow\wedge f_n & & \downarrow f_{n+1} \\ S^1\wedge Y_n & \xrightarrow{\sim} & Y_{n+1} \end{array}$$

eventually commute.

The inclusion $\Sigma^\infty\mathscr{R} \to \mathsf{Lax}\Sigma^\infty\mathscr{R}$ is an exact equivalence of $G_\bullet$-constructions. By allowing the operations $\omega^k$ to take valves in $wG^k_\bullet\mathsf{Lax}\Sigma^\infty_{\diamond k}\mathscr{R}_{\Sigma_k}$ (where $S^1\diamond\cdots\diamond S^1$ is used instead of $S^1$ in the definition of suspension prespectrum), we can define the operations on category level (the isomorphism is incorporated into $\omega^k$).

6.2 Corollary: The operations $\omega^k$ extend to operations

$$\omega^k : w\mathsf{sub}_k G_\bullet\Sigma^\infty\mathscr{R} \longrightarrow wG^k_\bullet\mathsf{Lax}\Sigma^\infty_{\diamond k}\mathscr{R}_{\Sigma_k}$$

To see that the diagram in proposition 6.1 commutes up to a natural isomorphism we will just need the following lemmas:

6.3 Lemma: There is a natural $\Sigma_{n_1+\cdots+n_k}$-isomorphism

$$\Sigma_{n_1+\cdots+n_k}/\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}{}^{+}\wedge \underset{\leftarrow n_1\rightarrow}{(S\diamond\cdots\diamond S)}\wedge\cdots\wedge\underset{\leftarrow n_k\rightarrow}{(S\diamond\cdots\diamond S)}\wedge Y$$

$$\downarrow$$

$$\Sigma_{n_1+\cdots+n_k}/\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}{}^{+}\wedge \underset{\leftarrow n_1+\cdots+n_k\rightarrow}{(S\diamond\cdots\cdots\cdots\diamond S)\wedge Y}$$

Here $\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}$ is considered as a subgroup of $\Sigma_{n_1+\cdots+n_k}$ in the usual way and $Y$ has a $\Sigma_{n_1+\cdots+n_k}$-action. The source space has a $\Sigma_{n_1+\cdots+n_k}$-action induced by induction as defined above (section 2). (Note that this includes the case $n_1=n_2=\cdots=n_k=1$.)

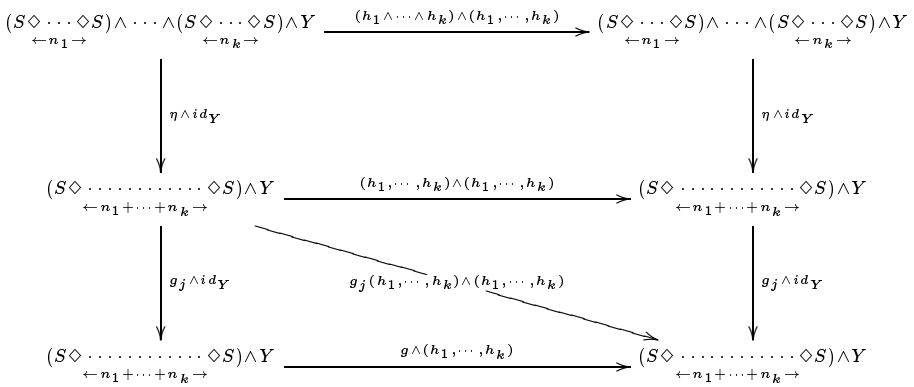
Proof: Let $g_1,\cdots,g_N$ be representatives of the left cosets of $\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}$ in $\Sigma_{n_1+\cdots+n_k}$. If $gg_i=g_j(h_1,\cdots,h_k)$ where $g\in\Sigma_{n_1+\cdots+n_k}$ and $h_i\in\Sigma_{n_i}$, we define $\Psi$ on the $i^{th}$-summand as $\Psi_i:=g_i\wedge id_Y$, mapping the $i^{th}$ summand of

$$\Sigma_{n_1+\cdots+n_k}/\Sigma_{n_1}\times\cdots\times\Sigma_{n_k}{}^{+}\wedge \underset{\leftarrow n_1\rightarrow}{(S\diamond\cdots\diamond S)}\wedge\cdots\wedge\underset{\leftarrow n_k\rightarrow}{(S\diamond\cdots\diamond S)}\wedge Y$$

to the $j^{th}$ summand of

$$\Sigma_{n_1+\cdots+n_k}/\Sigma_1\times\cdots\times\Sigma_{n_k}{}^{+}\wedge\underset{\leftarrow n_1+\cdots+n_K\rightarrow}{(S\diamond\cdots\cdots\cdots\diamond S)}\wedge Y$$

The map of the lemma is composed of coherence isomorphisms and $\Psi$. To see how this works consider the following diagram, where we will show that the vertically composed maps are $\Sigma_{n_1+\cdots+n_k}$-equivariant. The diagram describes the mapping of the $i^{th}$ component to the $j^{th}$ component induced by an element $g$ such that $gg_i=g_j(h_1,\cdots,h_k)$. $\eta$ is the natural rearrangement of the brackets.




The $g_i\wedge id_Y$ are isomorphisms, the top square commutes: This is part of the coherence data for the $\wedge$-product. The upper triangle commutes. The lower triangle commutes since

$$(g\wedge(h_1,\cdots,h_k))\circ(g_i\wedge id_Y)=gg_i\wedge(h_1,\cdots,h_k)=g_j(h_1,\cdots,h_k)\wedge(h_1,\cdots,h_k)$$

6.4 Lemma: There is a natural isomorphism

$$(S\wedge M_{i_1/l_1})*_2\cdots*_k(S\wedge M_{i_k/j_k})\to \underset{\leftarrow k\rightarrow}{(S\diamond\cdots\diamond S)}\wedge M_{i_1/l_1}*_2\cdots*_k M_{i_k/l_k}$$

Here left association and the notation from [G] is used.

Proof: This is proved using the symmetric bimonoidal structure for the $\wedge$-product to collect the $S$-factors to the left, Lemma 6.3 and induction.

6.5 Remark: In a formula $A_1\boxtimes A_2\boxtimes\cdots\boxtimes A_k$ there are natural isomorphisms between different ways of evaluating the induction, i.e. different bracketings. As a rule we have association to the left, and the brackets are suppressed from the exposition.

Proof of Proposition 6.1: This is a consequence of Lemma 6.4.

# 7 Extending $\omega^k$ to $K$-theory of spaces

7.1 Theorem:
$|\ \mathsf{hocolim}_{S\diamond\cdots\diamond S\wedge-}wG^k_\bullet\mathscr{R}_{\Sigma_k}\ |$ is homotopy equivalent to $\Omega\mid wS_\bullet\mathscr{R}_{\Sigma_k}\mid$.

Proof: We will consider homotopy colimits of

$$wG^k_\bullet\mathscr{C}_{\Sigma_k}\xrightarrow{S^V\wedge-}wG^k_\bullet\mathscr{C}_{\Sigma_k}\longrightarrow\cdots$$

If $S^V\wedge-$ is the reduced suspension $\Sigma$, then the $(\mathsf{ho})$-$\mathsf{colim}\ wG^k_\bullet\mathscr{C}_{\Sigma_k}$ gives a model for $\Omega wS_\bullet\mathscr{C}_{\Sigma_k}$ as was shown in [Gu]. We consider the three cases

i) $S^1\wedge-$ (i.e. the reduced suspension $\Sigma$).

ii) $(S^1\diamond\cdots\diamond S^1)\wedge-$ the regular representation of $\Sigma_k$.

iii) $S^V\wedge-$ all orthogonal $\Sigma_k$ - sub-representations $V$ of the infinite sum of the regular representation with itself and their inclusions (cp. [L-M-St, pg. 11]).

$$\begin{array}{ccc}\Omega\mid wS_\bullet\mathscr{R}_{\Sigma_k}\mid & \xrightarrow{\quad\simeq\quad} & \mathsf{hocolim}_\Sigma wG^k_\bullet\mathscr{R}_{\Sigma_k}\\ & & \downarrow\\ \mathsf{hocolim}_{(S\diamond\cdots\diamond S)\wedge-}wG^k_\bullet\mathscr{R}_{\Sigma_k} & \longrightarrow & \mathsf{hocolim}_{(S^V\wedge-)_{\mathrm{all}\ V}}wG^k_\bullet\mathscr{R}_{\Sigma_k}\end{array}$$

The map on top is a weak equivalence as follows from [Gu]. The lower horizontal map is a weak equivalence by cofinality since every representation (of $\Sigma_k$) is contained in a power of the regular representation ($S\diamond\cdots\diamond S$).

To see that the vertical map is a weak equivalence, we will consider the map

$$\mathsf{hocolim}_\Sigma wG_\bullet^k\mathscr{R}_{\Sigma_k} \xrightarrow{X\wedge -} \mathsf{hocolim}_\Sigma wG_\bullet^k\mathscr{R}_{\Sigma_k}$$

For this we assume that $X$ is a finite $\Sigma_k$-$CW$-complex. We are in a situation where the additivity theorem holds. Hence we can use the cofibration sequences

$$X^{(m-1)} \rightarrowtail X^{(m)} \twoheadrightarrow \bigvee_m \Sigma^k/H_m^+\wedge S^m$$

where $X^{(m)}$ is the $m$-skeleton of $X$ to obtain that $X\wedge -$ is homotopic to $\left(\bigvee_m \Sigma_k/H_m^+\wedge S^m\right)\wedge -$.

Since $S^m\wedge -$ is homotopic to $(-1)^m id$, this means that smashing with $X$ is the same as smashing with the reduced $\Sigma_k$-equivariant Eulercharacteristic. We want to show that $X\wedge -$ is a weak equivalence, when $X$ is a sphere. Since the formula

$$\chi_{red}(A\wedge B) = \chi_{red}(A)\chi_{red}(B)$$

holds for the reduced Eulercharacteristic in the Burnsidering $A_b(\Sigma_k)$ - defined using finite based $G$-sets - it suffices to show that $\chi_{red}((S^V)^L) = \pm 1$ for any $\Sigma_k$-sphere $S^V$. The reduced Eulercharakteristic of a sphere $S^k$ is $(-1)^k$. The fixpoint set in a sphere (with orthogonal action) is again a sphere. Since $[X] \longmapsto \prod_L \chi_{red}(X^k)$ is an injective ringhomomorphism $A_b(\Sigma_k) \longrightarrow \prod_L \mathbb{Z}$, it follows that $[X]\in A_b(\Sigma_k)$ is invertible, if $X$ is a sphere. The functors $X\wedge -$ define a module structure

$$A_b(\Sigma_k)\times\left[-,\mathsf{hocolim}_\Sigma wG_\bullet^k\mathcal{R}_{\Sigma_k}\right]\to\left[-,\mathsf{hocolim}_\Sigma wG_\bullet^k\mathcal{R}_{\Sigma_k}\right],$$

hence $S^V\wedge -$ induces a homotopy equivalence.

# 8 Taking the free part

The operations in [W2] are maps $\Theta^n : A(*) \longrightarrow A(B\Sigma_n)$, while the operations constructed in section 6 are maps $\omega^n : A(*) \longrightarrow A_{(\Sigma_n,\{\mathrm{all}\})}(*)$, where $A_{(\Sigma_n\{\mathrm{all}\})}(*) = \mathsf{hocolim}_\Sigma wG_\bullet^k\mathcal{R}_{\Sigma_n}$.

We will show:

8.1 Theorem: There is homotopy equivalence

$$h : A_{(\Sigma_n,\{\text{all}\})}(*) \longrightarrow \prod_{H\in\{\text{all}\}} A(B(N_{\Sigma_n}(H))/H)$$

of infinite loop spaces. (Here $N_{\Sigma_n}(H)$ denotes the normalizer of $H$ in $\Sigma_n$.). Composing $h$ with the projection $P_E$ to the factor with $H = E$ gives maps

$$\tilde{\Theta}^n : A(*) \xrightarrow{\omega^n} A_{(\Sigma_n,\{\text{all}\})}(*) \xrightarrow{P_E\circ h} A(B\Sigma_n)$$

In section 9 we will show:

8.2 Theorem: The maps $\Theta^n$ and $\tilde{\Theta}^n$ are homotopic.

By [W] there is a weak equivalence of infinite loop spaces $A_{(\Sigma_n,\text{free}_*)}(*)\tilde{\longrightarrow}A(B\Sigma_n)$, where $\text{free}_*$ denotes pointed free.

Theorem 8.1 therefore will follow from:

8.3 Theorem: There exists an infinite loop space splitting

$$|\mathsf{hocolim} wG_\bullet\mathscr{R}_G| \longrightarrow \prod_{H\in\{\text{all}\}} |\mathsf{hocolim}_\Sigma wG_\bullet\mathscr{R}_f(*,N_G(H)/H,\{e\})|$$

for finite groups $G$.

8.4 Remark: Taking $\pi_1$, we recover the splitting of the equivariant Whitehead-groups of [I].

Let $G$ be a finite group, $\mathcal{F}$ a family of subgroups. There is a partial ordering $\leq_c$ of $\mathcal{F}$. $H_1 \leq_c H_2$ means, that $H_1$ is conjugated to a subgroup of $H_2$. Extend $\leq_c$ to a linear ordering of $\mathcal{F}$ (also denoted by $\leq_c$). $H_1 <_c \bullet H_2$ will mean $H_1 <_c H_2$ and $(H_1 \leq_c K \leq_c H_2 \Rightarrow K = H_1)$. We will use this ordering to get a filtration of $id_{\mathscr{R}_f(*,G,\mathcal{F})}$. $\mathcal{F}_{>_cH}(X) := \mathsf{colim}X^K = \bigcup_{K>_cH} X^K$ defines an exact functor

$$F_{>_cH} : \mathscr{R}_f(*,G,\mathcal{F}) \longrightarrow \mathscr{R}_f(*,G,\mathcal{F})$$

Let $\mathcal{I} = \{H_i | 0 \leq i \leq n\}$, so that for $0 \leq i < n$, $H_i <_c H_{i+1}$. Then we inductively define the following exact functors:

$$S_i, Q_j : \mathscr{R}_f(*,G,\mathcal{F}) \longrightarrow \mathscr{R}_f(*,G,\mathcal{F}), \quad -1 \leq i \leq n, 0 \leq j \leq n \qquad \text{by}$$

$S_{-1}(X) = X$, i.e. $S_{-1} = id_{\mathscr{R}_f(*,G,\mathcal{F})}$ $S_i(X) = \mathcal{F}_{>_cH_i}(S_{i-1}(X))$, i.e. $S_i = \mathcal{F}_{>_cH_{i-1}} \circ S_{i-1}$ for $0 \leq i \leq n$ and the $Q_i$ are defined by the natural cofibration sequences

$$S_i(X) \rightarrowtail S_{i-1}(X) \twoheadrightarrow Q_i(X), \quad 0 \leq i \leq n$$

$Q_i$ takes values in $\mathscr{R}_f(*, G, \{H_i\})$. Let $j_i : \mathscr{R}_f(*, G, \{H_i\}) \to \mathscr{R}_f(*, G, \mathcal{F})$ be the inclusion. Then $Q_i$ factors as $j_i \circ \overline{Q}_i$, with $\overline{Q}_i$ a retraction. By the additivity theorem [Gu], there results a splitting:

$$\mathsf{hocolim} wG_\bullet \mathscr{R}_f(*, G, \{\text{all}\}) \longrightarrow \prod_{H \in \{\text{all}\}} \mathsf{hocolim} wG_\bullet \mathscr{R}_f(*, G, \{H\})$$

Now consider the exact functor:

$$\begin{array}{ccc} \mathscr{R}_f(*, G, \{H\}) & \longrightarrow & \mathscr{R}_f(*, N_G(H)/H, \{e\}) \\ X & \longmapsto & X^H \end{array}$$

The construction $Y \mapsto G^+ \wedge_{N_G(H)} Y$ gives an exact functor going the opposite way. These functors induce the disered equivalences.

# 9 Comparison with Waldhausen's operations

There is a total operation

$$\prod_k \omega^k : \prod_{k \geq 1} w\mathsf{sub}_k G_\bullet \mathscr{R}_f(*) \longrightarrow 1 \times \prod_{k \geq 1} wG^k_\bullet \mathscr{R}_f(*, \Sigma_k, \{\text{all}\})$$

and a diagonal $w\mathscr{R}_f(*) \longrightarrow \prod_k w\mathsf{sub}_k G_\bullet \mathscr{R}_f(*)$, whose $k^{th}$ component is

$$X \longmapsto \frac{X = X = \cdots = X}{\star = \star = \cdots = \star}$$

These objects are mapped via $\omega^k$ to

$$\begin{cases} X \wedge \cdots \wedge X & \text{in position } LO, L'O', \ldots, L^{(k)}O^{(k)}, \\ * & \text{otherwise.} \end{cases}$$

We now promote the composition to a map of $H$-spaces, where the $H$-space structure in $w\mathscr{R}_f(*)$ is given by categorical sum and on $1 \times \prod_{k \geq 1} wG^k_\bullet \mathscr{R}_f(*, \Sigma_k, \{\text{all}\})$ by:

$$\boxtimes_* : \quad G^k_\bullet \mathscr{R}_f(*, \Sigma_k, \{\text{all}\}) \times G^n_\bullet \mathscr{R}_f(*, \Sigma_n, \{\text{all}\}) \quad \longrightarrow \quad G^{k+n}_\bullet \mathscr{R}_f(*, \Sigma_{k+n}, \{\text{all}\}),$$

where $\boxtimes_*$ with 1 is to be interpreted as the identity.

By lemma 6.3 this map is compatible with taking colimits with respect to the regular representation. In this version the product of the $\omega^k$ is called the total Segal operation and denoted with $\omega$. (The name we will keep also for restrictions and group completions of it.) In the notation of the proof of theorem 7.1

$$\pi_0 \mathsf{hocolim}_{S^1 \diamond \ldots \diamond S^1 \wedge -} wG^k_\bullet \mathscr{R}_f(*, \Sigma_k, \{\text{all}\}) = A_b(\Sigma_k)$$

The operation $\boxtimes_*$ induces the multiplication of "power series" $1 \times \prod_{k \geq 1} A_b(\Sigma_k)$ - with leading coefficient - 1 in the "power series" ring $\mathbb{Z} \times \prod_{k \geq 1} A_b(\Sigma_k)$. We obtain a map of $H$-spaces, where the right handside is group complete.

We may now restrict to spherical objects in the domain of the total Segal operation. The group completion of this sub - $H$ - space is a model for $A(*)$ also. The comparison argument now proceeds as the construction argument of [W2]. So we have obtained

9.1 Theorem: The total Segal operation $\omega$ and the maps $\omega^k$ agree (up to homotopy) with the operations constructed by Waldhausen in [W2].

# 10 $E_\infty$-structure on the total Segal operation

10.1 Theorem: The total Segal operation:

$$\omega : A(*) \longrightarrow 1 \times \prod_{n \geq 1} A_{\Sigma_n}(*)$$

carries an infinite loop map structure.

In order to put a structure of an $E_\infty$-map on the total Segal operation, we will extend the model. The reason is that for $X \in G^n_\bullet \mathscr{C}$ and $Y \in G^m_\bullet \mathscr{C}$ the coherence data for commutativity of the $\wedge$-product

$$X(i/j_1, \ldots, i_n/j_n) \wedge Y(i'_1/j'_1, \ldots, i'_m/j'_m) \rightarrow \\ Y(i'_l/j'_l, \ldots, i'_m/j'_m) \wedge X(i_1/j_1, \ldots, i_n/j_n)$$

also have to involve a permutation of the coordinates

$$i_1/j_1, \ldots, i_n/j_n, i'_1/j'_1, \ldots, i'_m/j'_m.$$

To handle this problem we will use a Thomason homotopy colimit [Th] in a similar way as was done in [B].

Let $\mathbb{N}$ be the set of natural numbers and $F$ the category of finite subsets of $\mathbb{N}$ and injections. $F_+ \subseteq F$ is the full subcategory of nonempty sets. Let $\amalg$ denote the associative sum on $F_+$ given by

$$\{x_i | i = 1 \dots n\} \amalg \{y_i | i = 1 \dots m\} = \\ \{x_i | i = 1 \dots n\} \cup \{y_i + x_n - y_1 + 1 \,| i = 1, \dots, m\}$$

where it is assumed that $x_1 < x_2 < \dots < x_n$ and $y_1 < y_2 < \dots < y_m$.

10.2 Lemma: The category $F_+$ is contractible (i.e. its geometric realisation $|F_+|$ is contractible).

Proof: Let $t : F_+ \to F_+$ be the functor given by $x \mapsto 1 \amalg x$ where $1 = \{0\}$. Let $C_1 : F_+ \to F_+$ be the functor given by $x \mapsto 1$. There are natural transformations

$$Id_{F_+} \to t \leftarrow C_1.$$

This means that $Id_{F_+} \sim C_1$. Hence $F_+$ is contractible. $\square$

We will assume that the category $\mathscr{C}$ and the categories $\mathscr{C}_{\Sigma_n}$ satisfy the extension property for cofibrations (see 4.3). Hence the iterated $G_\bullet$-construction $G^n_\bullet \mathscr{C}$ can be identified with $Exact(\Gamma(-)^n, \mathscr{C})$. The construction

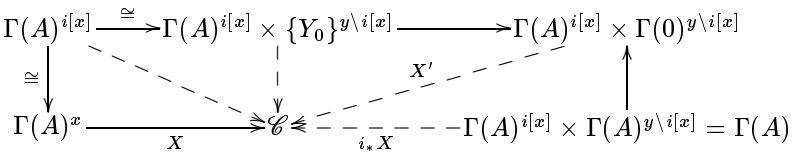
$$n \mapsto Exact(\Gamma(-)^n, \mathscr{C}) = G^n_\bullet(\mathscr{C})$$

extends to a functor

$$G^{(-)}_\bullet : F_+ \to Cat^{\triangle^{op}} \qquad G^x_A(\mathscr{C}) := Exact(\Gamma(A)^x, \mathscr{C}).$$

Here we use the diagonal, e.g. $G^3_A \mathscr{C} = G_A\,(G_A(G_A(\mathscr{C})))$.

Isomorphisms $x \to x'$ in $F$ induce natural morphisms $G^x_\bullet(\mathscr{C}) \to G^{x'}_\bullet(\mathscr{C})$ by permuting coordinates. In general an injection $i : x \to y$ induces $G^x_\bullet(\mathscr{C}) \to G^y_\bullet(\mathscr{C})$ using stabilization as follows (note that $\Gamma(0) = \{L/0, R/0\}$)




Here $X'$ is zero outside $\Gamma(A)^{i[x]} \times \{L/0\}^{y/i[x]}$. This is the stabilization by inclusion of $\mathscr{D}$ on the $L$-line in $G_0\mathscr{D}$.

We can extend $G^{()}_\bullet$ to a functor on $F$ by adding the functors $\Gamma(A)^\emptyset \to \mathscr{C}$. Observe that $\mathscr{C}$ identifies with $G^\emptyset_\bullet(\mathscr{C})$ this way.

Let $F_+/G_A\mathscr{C}$ denote Thomason's homotopy colimit [T] with objects $(x, X)$ where $X : \Gamma(A)^x \to \mathscr{C}$, and morphisms $(x, X) \to (y, Y)$ given by $i : x \to y$ in $F_+$ and a natural transformation $i_*X \to Y$ in $G_A^y\mathscr{C}$. The notation extends to $F$.

10.3 Theorem: $F/wG_\bullet\mathscr{C}$ gives a model for $K$-theory.

Proof: To simplify notation here we suppress the $w$ - variable, since it gives an independent simplicial direction. We will first show that $F_+/G_\bullet G_\bullet^\infty\mathscr{C}$ gives $K$-theory. Each $i : x \to y$ in $F$ induces a weak equivalence $G_\bullet^x G_\bullet^\infty\mathscr{C} \to G_\bullet^y C_\bullet^\infty\mathscr{C}$. This is so since $G_\bullet^\infty\mathscr{C}$ by [Gu] is a model of $K$-theory and the morphisms $G_\bullet^x G_\bullet^\infty \to G_\bullet^\infty\mathscr{C}$ induce weak equivalences. It follows that $G_\bullet G_\bullet^\infty\mathscr{C} \to F_+/G_\bullet G_\bullet^\infty\mathscr{C} \to F_+$ is a quasifibration. Since $F_+$ is contractible and $G_\bullet^\infty\mathscr{C} \to G_\bullet G_\bullet^\infty\mathscr{C}$ induces a weak equivalence it follows that $F_+/G_\bullet G_\bullet^\infty\mathscr{C}$ gives models for $K$-theory.

The functor $t : F_+ \to F_+$ above induces via $x \rightarrowtail 1 \amalg x = t(x)$ a functor $\tilde{t} : F_+/G_\bullet\mathscr{C} \to F_+/G_\bullet\mathscr{C}$ which on the classifying space level is homotopic to the identity. $\tilde{t}$ is also injective, i.e. a cofibration.

It follows that

$$F_+/G_\bullet\mathscr{C} \to colim(F_+/G_\bullet\mathscr{C} \xrightarrow{\tilde{t}} F_+/G_\bullet\mathscr{C} \xrightarrow{\tilde{t}} \dots)$$

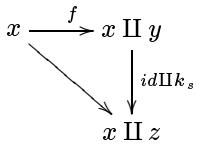
is a weak equivalence. The colimit can be identified with $F_+/G_\bullet G_\bullet^\infty\mathscr{C}$.

Since $F_+$ is cofinal in $F$, we conclude that $|F/wG.\mathscr{C}|$ is naturally homotopic to $\Omega|wS_\bullet\mathscr{C}|$. $\square$

We note the following variation of a folklore statement:

10.4 Lemma: If $T : F \to Top_*$ is such that $T(i)$ is a homotopy equivalence for all $i \in Mor(F_)$ then for all $i, j : x \to y$ in $F$, $T(i)$ is homotopic to $T(j)$.

Proof: Consider the commutative diagram for $k_s : y \to z \quad s = 1, 2$




in $F$. $(id \amalg k_1) \circ f = (id \amalg k_2) \circ f$.

Since $T(f)$ is a homotopy equivalence by assumption it follows that

$$T(id \amalg k_1) \sim T(id \amalg k_2).$$

Next consider the commutative diagrams

$$\begin{array}{ccc} x \amalg y & \xleftarrow{g} & y \\ {\scriptstyle id \amalg k_i}\downarrow & & \downarrow{\scriptstyle k_i} \\ x \amalg z & \xleftarrow{h} & z \end{array}$$

$T(h \circ k_1) = T((id \amalg k_1) \circ g) \sim$ [by the above] $\sim T((id \amalg k_2) \circ g) = T(h \circ k_2)$. Since $T(h)$ is a homotopy equivalence it follows $T(k_1) \sim T(k_2)$. $\square$

10.5 Remark: The lemma also holds for full subcategories which are closed under the sum.

In particular it follows that each $\sigma \in \Sigma_n$ induces a map $G_\bullet^n G_\bullet^\infty \mathscr{C} \to G_\bullet^n G_\bullet^\infty \mathscr{C}$ that is homotopic to the identity, and that all stabilization maps $G_\bullet^n \mathscr{C} \to G_\bullet^\infty \mathscr{C}$ are homotopic. In the pseudo-additive case [Gu] this holds for maps $G_\bullet^n \mathscr{C} \to G_\bullet^m \mathscr{C}$ induced by injections.

The $E_\infty$-structure on the total Segal operation will be described using the following diagram.

$$\begin{array}{ccc} \mathscr{C} & & \\ \downarrow{\scriptstyle \alpha_1} & & \\ 1 \times \prod_{n\geq 1} \mathscr{C}_{\Sigma_n} & \xrightarrow{\beta_1} & 1 \times \prod_{n\geq 1} G_\bullet^n \mathscr{C}_{\Sigma_n} \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_3} \\ 1 \times \prod_{n\geq 1} F/G_\bullet \mathscr{C}_{\Sigma_n} & \xrightarrow{\beta_2} & 1 \times \prod_{n\geq 1} F/G_\bullet G_\bullet^n \mathscr{C}_{\Sigma_n} \end{array}$$

Observe that the $w$ - direction is still suppressed from notation.

Binary operations will be given on the categories so that the following theorem holds.

10.6 Theorem: Let $\mathscr{C}$ be a category with cofibrations and weak equivalences, let $\vee$ be a categorical sum on $\mathscr{C}$, and let $\wedge$ be a symmetric monoidal biexact product on $\mathscr{C}$. Then $\alpha_1$ and $\alpha_2$ are maps of symmetric monoidal categories, $\beta_1, \beta_2$ and $\beta_3$ are maps of monoidal categories. $\beta_2$ is a homotopy equivalence and in the pseudo-additive case $\beta_3$ is also a homotopy equivalence.

Proof: The maps $\alpha_2$ and $\beta_3$ are given by the identification $\mathscr{C} \cong \{\emptyset\}/G_\bullet \mathscr{C} \cong G_\bullet^\emptyset \mathscr{C}$. On $\mathscr{C}$ the operation $\vee$ is used. $\alpha_1(c) = (c \overset{\leftarrow n \rightarrow}{\wedge} \ldots \wedge c)$ where $\Sigma_n$ acts by permuting factors via the coherence data for $\wedge$. $\beta_1$ and $\beta_2$ are given by stabilization maps $\mathscr{C}_{\Sigma_n} \to G_\bullet^n \mathscr{C}_{\Sigma_n}$. On $1 \times \prod_{n\geq 1} \mathscr{C}_{\Sigma_n}$ the product $(c_n) \tilde{\boxtimes} (d_n) = (\bigvee_{s+t=n} Ind_{\Sigma_s \times \Sigma_t}^{\Sigma_n} c_s \wedge d_t)$ is used, where the sum is by left association with $s$ from 0 to $n$ and the induction is as described prior to definition 3.1.

A composition rule $\widetilde{\boxtimes}$ on $1 \times \prod_{n\geq 1} G^n_\bullet \mathscr{C}_{\Sigma_n}$ is given by

$$((X_n),(Y_n)) \longmapsto (\bigvee_{s+t=n} \Gamma(A)^{s+t} \overset{X_s \wedge Y_t}{\longrightarrow} \mathscr{C}_{\Sigma_s} \times \mathscr{C}_{\Sigma_t} \overset{\wedge}{\longrightarrow} \mathscr{C}_{\Sigma_s \times \Sigma_t} \overset{Ind^{\Sigma_n}_{\Sigma_s \times \Sigma_t}}{\longrightarrow} \mathscr{C}_{\Sigma_n})$$

i.e. $((X_n) \widetilde{\boxtimes} (Y_n))_n = \bigvee_{s+t=n} Ind^{\Sigma_n}_{\Sigma_s \wedge \Sigma_t} X_s \wedge Y_t$

where the sum is given by left association and induction as above.

The binary operation $\hat{\boxtimes}$ on $1 \times \prod_{n\geq 1} F/G_\bullet \mathscr{C}_{\Sigma_n}$ is given by

$$((x_n, X_n))\hat{\boxtimes}((y_n, Y_n)) =$$
$$(\coprod_{u+v=n}(x_u \amalg y_v), \bigvee_{s+t=n} (l^{\coprod_{u+v=n}(x_u \amalg y_v)}_{x_s \amalg y_t})_* Ind^{\Sigma_n}_{\Sigma_s \times \Sigma_t}(X_s \wedge Y_t))$$

Here $l^{\coprod_{u+v=n}(x_u \amalg y_v)}_{x_s \amalg y_t} : x_s \amalg y_t \to \coprod_{u+v=n}(x_u \amalg y_v)$ is the inclusion. The composition on $1 \times \prod_{n\geq 1} F/G_\bullet G^n_\bullet \mathscr{C}_{\Sigma_n}$ is defined the same way.

To check for coherence we consider the $k$-fold iterated operation on $1 \times \prod_{n\geq 1} F/G_\bullet \mathscr{C}_{\Sigma_n}$

$$(x^1_n, X^1_n)\hat{\boxtimes} \cdots \hat{\boxtimes}(x^k_n, X^k_n) \cong$$

$$\left( \coprod_{t_l + \cdots t_k = n} (x^1_{t_1} \amalg \ldots \amalg x^k_{t_k}) \right.,$$

$$\left. \bigvee_{s_1 + \cdots + s_k = n} \left( l^{\coprod_{t_1+\cdots+t_k=n}(x^1_{t_1} \amalg \ldots \amalg x^k_{t_k})}_{x^1_{s_1} \amalg \ldots \amalg x^k_{s_k}} \right)_* Ind^{\Sigma_n}_{\Sigma_{s_1} \times \cdots \times \Sigma_{s_k}} (X^1_{s_1} \wedge \ldots \wedge X^k_{s_k}) \right)$$

Here $X^i_{s_i} \in G^{x^i_{s_i}}_\bullet(\mathscr{C}_{\Sigma_{s_i}})$ and $X^1_{s_1} \wedge \ldots \wedge X^k_{s_k} \in G^{x^1_{s_1} \amalg \ldots \amalg x^k_{s_k}}_\bullet \mathscr{C}_{\Sigma_{s_1} \times \cdots \times \Sigma_{s_k}}$.

The latter is induced up to

$$Ind^{\Sigma_n}_{\Sigma_{s_1} \times \cdots \times \Sigma_{s_k}} (X^1_{s_1} \wedge \ldots \wedge X^k_{s_k}) \in G^{x^1_{s_1} \amalg \ldots \amalg x^k_{s_k}}_\bullet \mathscr{C}_{\Sigma_n}$$

using a representation of $Ind$ with sums in $G^{x^1_{s_1} \amalg \ldots \amalg x^k_{s_k}}_\bullet \mathscr{C}$.

The various $Ind^{\Sigma_n}_{\Sigma_{s_1} \times \cdots \times \Sigma_{s_k}} (X^1_{s_1} \wedge \ldots \wedge X^k_{s_k})$ are then stabilized to $G^{\coprod_{t_1+\cdots t_k=n}(x^1_{t_1} \amalg \ldots \amalg x^k_{t_k})}_\bullet \mathscr{C}_{\Sigma_n}$.

Note that the construction makes each part have its own coordinate part. The formula for the $k$-fold operations comes from the following. Let $l^n_{s_1, s_2, \cdots, s_k}$ be

the inclusion $x^1_{s_1} \amalg x^2_{s_2} \amalg \ldots \amalg x^k_{s_k} \to \coprod_{t_1+t_2+\cdots+t_k=n} (x^1_{t_1} \amalg x^2_{t_2} \amalg \ldots \amalg x^k_{t_k})$.

Let $X^1_{s_1} \boxtimes X^2_{s_2} = Ind^{\Sigma_{s_1+s_2}}_{\Sigma_{s_1}\times\Sigma_{s_1}} X^1_{s_1} \wedge X^2_{s_2}$

$$\bigvee_{s_1+t=n} (l^n_{s_1,t})_* \left( X^1_{s_1} \boxtimes \left( \bigvee_{s_2+s_3=t} (l^t_{s_2,s_3})_* (X^2_{s_2} \boxtimes X^3_{s_3}) \right) \right) =$$

$$\bigvee_{s_1+t=n} \left( \bigvee_{s_2+s_3=t} (l^n_{s_1,t})_* \left( X^1_{s_1} \boxtimes \left( (l^t_{s_2,s_3})_* (X^2_{s_2} \boxtimes X^3_{s_3}) \right) \right) \right) =$$

$$\bigvee_{s_1+s_2+s_3=n} (l^n_{s_1,s_2,s_3})_* \left( X^1_{s_1} \boxtimes (X^2_{s_2} \boxtimes X^3_{s_3}) \right)$$

$$\bigvee_{t+s_3=n} (l^n_{t,s_3})_* \left( \left( \bigvee_{s_1+s_2=t} (l^t_{s_1,s_2})_* (X^1_{s_1} \boxtimes X^2_{s_2}) \right) \boxtimes X^3_{s_3} \right) =$$

$$\bigvee_{t+s_3=n} \left( \bigvee_{s_1+s_2=t} (l^n_{t,s_3})_* \left( l^t_{s_1,s_2} \right)_* (X^1_{s_1} \boxtimes X^2_{s_2}) \boxtimes X^3_{s_3} \right) \right) =$$

$$\bigvee_{s_1+s_2+s_3=n} (l^n_{s_1,s_2,s_3})_* \left( (X^1_{s_1} \boxtimes X^2_{s_2}) \boxtimes X^3_{s_3} \right)$$

The desired coherence now follows from the coherence of associativity of $\boxtimes$.

The symmetric part is then by permuting summands, permuting factors and changing from one subgroup $\Sigma_{s_1} \times \ldots \times \Sigma_{s_k} \subseteq \Sigma_n$ to another by a block permutation of $n$. The indexing makes this well defined, as it does with the identification of the $k$-fold operation.

It remains to check that there is no coherency obstruction coming from the rearrangement in the base category, but this works by Laplaza's coherence theorem [L, Prop.10] for distributivity.

Since $\vee$ is a sum Laplaza's 30 diagrams (including the diagrams for $\vee$ and $\wedge$) reduce to the coherence for $\wedge$.

Hence by the above the Cartan formula

$$\bigwedge_n \left( \bigvee_{i=1}^k c_i \right) \cong \bigvee_{s_1+\cdots+s_k=n} Ind^{\Sigma_n}_{\Sigma_{s_1}\times\ldots\times\Sigma_{s_k}} \bigwedge_{i=1}^k \left( \bigwedge_{s_i} c_i \right)$$

coherently holds in $\mathscr{C}$.
($\bigwedge_n d$ denotes the n-fold $\wedge$ of $d$ with itself given by left association as always).

Since the action of $\mathcal{F}$ commutes with sums and $\wedge$- products the result follows.

□

10.7 Remark: When $\mathscr{C}$ has an internal suspension as in [Gu], then using $X \mapsto S^1 \wedge X$ in $\mathscr{C}$, $X \mapsto \left( \bigwedge_n S^1 \right) \wedge X$ in $\mathscr{C}_{\Sigma_n}$ for internal suspension, we obtain lax

symmetric monoidal (resp. monoidal) functors and can pass to prespectra as in [Gu]. Hence Theorem (-) carries over to $K$ - theory in the situation of [Gu].

Finally we use that we can reduce the calculation of $K$-theory of spaces to spherical objects $w\mathcal{S}^n$. That is $K$-theory of spaces is also given by the group completion of $(colim_n w\mathcal{S}^n, \vee)$ as an $E_\infty$-space and $E_\infty$-maps survive group completion.

10.8 Remark:

1. The sum - operation

$$(x, X), (y, Y) \mapsto (x \amalg y, (l_x^{x \amalg y})_* X \vee (l_y^{x \amalg y})_* Y)$$

gives $F/G_\bullet \mathscr{C}$ an $E_\infty$ - structure which is compatible with the one coming from $\vee$ on $G_\bullet^\infty \mathscr{C}$. The construction of the sum - operation only uses that $\vee$ is exact. Hence it generalizes and gives pairings $F/G_\bullet \mathscr{C} \times F/G_\bullet \mathscr{D} \to F/G_\bullet \mathscr{E}$ for exact functors $\mathscr{C} \times \mathscr{D} \to \mathscr{E}$ [cf. G-G].

2. The multiplication

$$(x, X), (y, Y) \mapsto (x \amalg y, X \wedge Y)$$

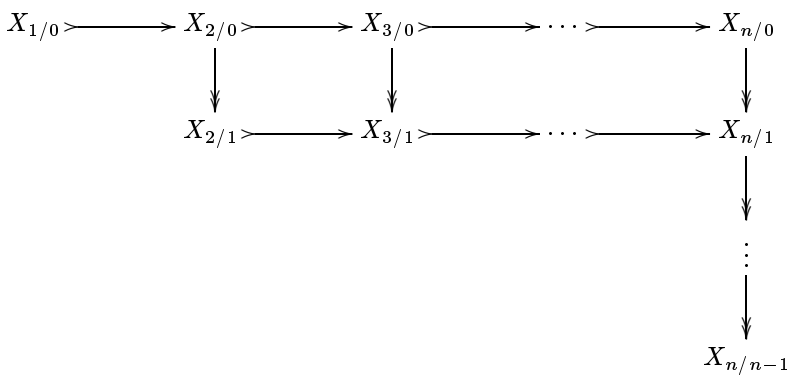
also gives an $E_\infty$ - structure on $F/G_\bullet \mathscr{C}$. This construction works for biexact pairings on category level.

3. These two types of operations do not satisfy the distributivity conditions for $E_\infty$ - rings on the nose. In general we only know that distributivity holds up to non - coherent homotopy. A similar remark applies to the Cartan formula. In the $A$ - theory case the $E_\infty$ - ring property has been derived by other methods in [SV].

# Appendix

The construction $\Gamma(A) \mapsto \Gamma^k(A)$ by Grayson (reviewed in section 2 above) can be extended to categories with exact sequences. This results in a factorization of the operation $\omega^n$ (A5 below).

A1 1. The $S_n$-construction of [W] can be extended to the context of categories with exact sequences (and weak equivalences). The requirement on a diagram



is that all sequences $X_{j/i} \rightarrowtail X_{k/i} \twoheadrightarrow X_{k/j}$ are exact sequences. $S_n\mathscr{C}$ can also be interpreted as $\mathrm{Exact}(Ar([n]), \mathscr{C})$, where $Ar([n])$ is the category of arrows $i/j = (j \to i)$ in $[n] = (0 \to 1 \to \ldots \to n)$, except the identity arrows, and where the exact sequences are the sequences $j/i \to k/i \to k/j$ for $i < j < k$.

2. The $\Gamma^k(A)$-construction can be expressed using the $S_\bullet$-construction. An object $(i_1/j_1, *_2, i_2/j_2, \ldots, *_k, i_k/j_k)$ has the form

$$\Big(i_1/j_1, \diamond, \ldots, \diamond, i_{s_1}/j_1, \boxtimes, i_{s_1+1}/j_{s_1+1}, \diamond, \ldots, \diamond, i_{s_1+s_2}/j_{s_1+1}, \boxtimes, \ldots, \boxtimes, \ldots,$$

$$i_{s_1+\ldots+s_{l-1}+1}/j_{s_1+\ldots+s_{l-1}+1}, \diamond, \ldots, i_{s_1+\ldots+s_{l-1}+s_l}/j_{s_1+\ldots+s_{l-1}+1}\Big)$$

The parts

$$\Big(i_{s_1+\ldots+s_{r-1}+1}/j_{s_1+\ldots+s_{r-1}+1}, \diamond, \ldots, \diamond, i_{s_1+\ldots+s_{r-1}+s_r}/j_{s_1+\ldots+s_{r-1}+1}\Big)$$

can be interpreted as an object in $S_r(\Gamma(A))$. The interpretation is that $(i_1/j_1, \diamond, \ldots, \diamond, i_m/j_1)$ is viewed as

$$\begin{array}{ccccccc} i_1/j_1 & \rightarrowtail & i_2/j_1 & \rightarrowtail \ldots \rightarrowtail & i_m/j_1 \\ & & \downarrow & & \downarrow \\ & & i_2/i_1 & \rightarrowtail \ldots \rightarrowtail & i_m/i_1 \\ & & & & \downarrow \\ & & & & \vdots \\ & & & & \downarrow \\ & & & & i_m/i_{m-1} \end{array}$$

in $S_m(\Gamma(A))$. Since $\Gamma(A)$ is a partially ordered set, the inserted "quotient objects" are uniquely defined.

A2

To get the maps in the category $\Gamma^k(A)$ we need to consider the functors

$$S_{n_1+\ldots+n_l}\mathscr{C} \to S_{n_1}\mathscr{C} \times \ldots \times S_{n_l}\mathscr{C}$$

for an exact category $\mathscr{C}$ (here $\Gamma(A)$).

$$\begin{array}{lllll}
X_{1/0} \rightarrowtail & X_{2/0} \rightarrowtail \cdots \rightarrowtail & X_{n_1/0} \rightarrowtail & X_{n_1+1/0} \rightarrowtail \cdots \rightarrowtail & X_{n_1+\cdots+n_l/0} \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
& X_{2/1} \rightarrowtail \cdots \rightarrowtail & X_{n_1/1} \rightarrowtail & X_{n_1+1/1} \rightarrowtail \cdots \rightarrowtail & X_{n_1+\cdots+n_l/1} \\
& & & \downarrow & \downarrow \\
& & & \vdots & \vdots \\
& & & \downarrow & \downarrow \\
& & & X_{n_1+1/n_1} \rightarrowtail \cdots \rightarrowtail & X_{n_1+\cdots+n_l/n_1} \\
& & & & \vdots \\
& & & & \vdots \\
& & & & \downarrow \\
& & & & X_{n_1+\cdots+n_l/n_1+\cdots+n_l-1}
\end{array}$$

The $S_{n_1}C$-component is $X_{1/0} \rightarrowtail \cdots \rightarrowtail X_{n_{1/0}}$ (with subquotients) and the $S_{n_i}C$-component is

$$X_{n_1+\ldots+n_{i-1}+1}/_{n_1+\ldots+n_{i-1}} \rightarrowtail \cdots \rightarrowtail X_{n_1+\ldots+n_{i-1}+n_i}/_{n_1+\ldots+n_{i-1}}$$

(with subquotients). This can be described using a Grothendieck construction ([T]), $I_k^{op} \int F_{k,\mathscr{C}}$ as follows.

Let $I_k$ be the category of partitions of the integer $R$. This is the partially ordered set with objects $Ob(I_k) = \{(n_1, \ldots, n_l) | n_i \geq 1 \text{ and } \sum_{s=1}^{l} n_i = k\}$. The morphisms are

$$\begin{array}{c}
\left(n_1, \ldots, n_{s_1}, n_{s_1+1}, \ldots, n_{s_1+s_2}, \ldots n_{s_1+\ldots+s_{i-1}+1}, \ldots, n_{s_1+\ldots+s_r}\right) \\
\downarrow \\
\left(\sum_{i=1}^{s_1} n_i, \sum_{i=s_1+1}^{s_1+s_2} n_i, \ldots, \sum_{i=s_1+\ldots+s_{r-1}+1}^{s_1+\ldots+s_r} n_i\right)
\end{array}$$

The maps A2 give the morphism part of a functor $F_{k,\mathscr{C}} : I_k^{op} \to Cat$ given on objects by

A3 $$F_{k,\mathscr{C}}((n_1, \ldots, n_l)) = S_{n_1}\mathscr{C} \times \ldots \times S_{n_l}\mathscr{C}.$$

A4

Lemma: Let $(\omega) - ExCat$ be the category of categories with exact sequences, exact functors (and weak equivalences). Then the construction

$$\mathscr{C} \longmapsto \mathscr{C}^{<k>} := I_R^{op} \int F_{k,\mathscr{C}}$$

is a functor $(\omega) - ExCat \to (\omega) - ExCat$. The exact sequences in $C^{<k>}$ are the sequences (where the subobjects are omitted from the notation and $\underline{X}_{n_i}$ denotes objects in $S_{n_i}$):

$$\begin{array}{c} \left(\underline{X}_{n_1}, \ldots, \underline{X}_{n_s}, \underline{X}_{n_s+n_{s+1}} \underline{X}_{n_{s+2}}, \ldots, \underline{X}_{n_l}\right) \\ \downarrow \\ \left(\underline{X}_{n_1}, \ldots, \underline{X}_{n_s}, \underline{X}'_{n_s+n_{s+1}}, \underline{X}_{n_{s+2}}, \ldots, \underline{X}_{n_l}\right) \\ \downarrow \\ \left(\underline{X}_{n_1}, \ldots, \underline{X}_{n_s}, \underline{Y}_{n_s}, \underline{Y}_{n_{s+1}}, \underline{X}_{n_{s+2}}, \ldots, \underline{X}_{n_l}\right) \end{array}$$

where the sequence

$$\left(\underline{X}_{n_s+n_{s+1}}\right) \to \left(\underline{X}'_{n_s+n_{s+1}}\right) \to \left(\underline{Y}_{n_s}, \underline{Y}_{n_{s+1}}\right)$$

is exact, and this is so iff it has the form

$$\begin{array}{c} \left(X_1 \rightarrowtail \ldots \rightarrowtail X_n \rightarrowtail A \rightarrowtail Y_2 \rightarrowtail \ldots \rightarrowtail Y_m\right) \\ \downarrow \\ \left(X_1 \rightarrowtail \ldots \rightarrowtail X_n \rightarrowtail Y_1 \rightarrowtail Y_2 \rightarrowtail \ldots \rightarrowtail Y_m\right) \\ \downarrow \\ \left(X_1 \rightarrowtail \ldots \rightarrowtail X_n, Y_1/A \rightarrowtail \ldots \rightarrowtail Y_m/A\right) \end{array}$$

for $(X_1 \rightarrowtail \ldots \rightarrowtail X_n \rightarrowtail A \rightarrowtail Y_1 \rightarrowtail Y_2 \rightarrowtail \ldots \rightarrowtail Y_m)$ in $S_{n+m+1}C$.

Here we allow $n = 0$ (i.e. that the part $X_1 \rightarrowtail \ldots \rightarrowtail X_n$ is missing).

Lemma: $\Gamma^k(A) = \Gamma(A)^{<k>}$

Proof: The construction gives a functor Exact $(\Gamma(A), \mathscr{C}) \to$ Exact $(\Gamma^k(A), \mathscr{C}^{<k>})$

composing with $\Xi_k^*$ gives a functor

$$\text{Exact } (\Gamma(A_1 \ldots A_k), \mathscr{C}) \to \text{ Exact } (\Gamma(A_1) \times \ldots \times \Gamma(A_k), \mathscr{C}^{<k>}).$$

Now if we have $\boxtimes$ and $\diamond$ operations for $\mathscr{C}$ satisfying E1 - E5, then we can consider $\diamond$ as a functor $\diamond : S_n\mathscr{C} \to \mathscr{C}_{\Sigma_n}$. The maps

$$S_{n_1}\mathscr{C} \times \ldots \times S_{n_l}\mathscr{C} \to \mathscr{C}_{\Sigma_{n_1}} \times \ldots \times \mathscr{C}_{\Sigma_{n_l}} \overset{\boxtimes}{\to} \cdots \to \mathscr{C}_{\Sigma_{n_1+\ldots+n_l}}$$

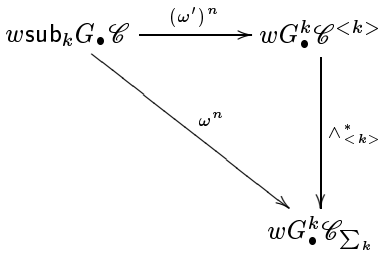
(with left association for $\boxtimes$), then give a functor $\wedge_{<k>} : \mathscr{C}^{<k>} \to \mathscr{C}_{\Sigma_k}$ , which is exact by the condition E5.

If we let $G_\bullet^{<k>}(\mathscr{C}) =$ Exact $(\Gamma^k(-), \mathscr{C})$ , we obtain for the operation $\omega^k$

$$w\mathsf{sub}_k G_\bullet \mathscr{C} \longrightarrow w\mathsf{sub}_k G_\bullet^{<k>}(\mathscr{C}^{<k>}) \longrightarrow wG_\bullet^k \mathscr{C}^{<k>} \longrightarrow wG_\bullet^k \mathscr{C}_{\Sigma_k},$$

i.e. a factorization

A5




where only $\wedge^*_{<k>}$ depends on the choice of operations $\boxtimes$ and $\diamond$ satisfying E1 - E5.

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