A solution of Deligne's conjecture.

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ABSTRACT: Deligne asked in 1993 whether the Hochschild cochain complex of an associative ring has a natural action by the singular chains of the little 2-cubes operad. In this paper we give an affirmative answer to this question. We also show that the topological Hochschild cohomology spectrum of an associative ring spectrum has an action of an operad equivalent to the little 2-cubes.

1 Introduction.

In order to explain Deligne's question, let us first recall some facts about the Hochschild cochain complex $C^*(R)$, where R is an associative ring. An element of $C^p(R)$ is a map of abelian groups

$$x : R^{\otimes p} \to R.$$

In 1962, Gerstenhaber [9] defined a cup product in $C^*(R)$ as follows: If $x \in C^p(R)$ and $y \in C^q(R)$ then $x \smile y$ is the $(p+q)$-cochain defined by

$$(x \smile y)(r_1 \otimes \cdots \otimes r_{p+q}) = x(r_1 \otimes \cdots \otimes r_p) \cdot y(r_{p+1} \otimes \cdots \otimes r_{p+q})$$

The cup product is clearly associative and non-commutative. But Gerstenhaber showed that it is chain-homotopy commutative; the chain homotopy which demonstrates this is a combination of the operations

$$\circ_k : C^p(R) \otimes C^q(R) \to C^{p+q-1}(R)$$

which take $x \otimes y$ to the $(p+q-1)$-cochain $x \circ_k y$ with

$$(x \circ_k y)(r_1 \otimes \cdots \otimes r_{p+q-1}) = x(r_1 \otimes \cdots \otimes r_{k-1} \otimes y(r_k \otimes \cdots \otimes r_{k+q-1}) \otimes \cdots \otimes r_{p+q-1})$$

Gerstenhaber also used the $\circ_k$ operations to construct a Lie bracket

$$[,] : C^p(R) \otimes C^q(R) \to C^{p+q-1}(R)$$

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The operations $\smile$ and $[,]$ induce corresponding operations on the Hochschild cohomology $H^*(R)$, and they satisfy relations which make $H^*(R)$ into what is now known as a *Gerstenhaber algebra.*

Another example of a Gerstenhaber algebra is the homology of a 2-fold loop space: $H_*(\Omega^2 A)$ (the fact that this is a Gerstenhaber algebra is a consequence of the work of Fred Cohen [5]). In this case, the Gerstenhaber algebra structure in homology is a consequence of the fact that the little 2-cubes operad $\mathcal{C}_2$ acts on $\Omega^2 A$. In 1993, Deligne [6] asked whether there was a closer relation between these two examples: specifically, he asked whether the Gerstenhaber algebra structure of $H^*(R)$ is induced by an action on $C^*(R)$ of a chain operad quasi-isomorphic to the singular chain operad of $\mathcal{C}_2$. This is usually known as Deligne's conjecture, although in the original letter it was expressed as a desire or preference ("I would like the complex computing Hochschild cohomology to be an algebra over [the singular chain operad of the little 2-cubes] or a suitable version of it"). In this paper we give an affirmative answer to this question, except that we have to replace $C^*(R)$ by the normalized Hochschild cochains $\bar{C}^*(R)$ (a cochain x is "normalized" if $x(r_1 \otimes \cdots \otimes r_n) = 0$ whenever some $r_i = 1$; see [17, Section 10.3]), and in order to make the signs work correctly it is necessary to work with the desuspension of $\bar{C}^*$ (see [9, page 279]). Other affirmative answers to Deligne's question have been found by Tamarkin [24] and [25], Kontsevich [15], and Voronov [26]. Tamarkin also has a more recent approach which is similar to ours.

We actually show more, namely that the singular chain operad of $\mathcal{C}_2$ is quasi-isomorphic (over the integers, not just over the rationals) to a certain explicit suboperad of the endomorphism operad of the desuspended reduced Hochschild cochain functor $\Sigma^{-1}\bar{C}^*$. Specifically, for each $n \geq 0$ let $\mathcal{O}(n)$ be the graded abelian group of natural transformations

$$\nu : (\Sigma^{-1}\bar{C}^*)^{\otimes n} \to \Sigma^{-1}\bar{C}^*$$

(where the grading of ν is the amount by which it *lowers* degrees). We give $\mathcal{O}(n)$ the differential

$$\partial(\nu) = \partial \circ \nu - \nu \circ \partial.$$

Then $\mathcal{O}$ is an operad in the category of chain complexes, with the evident composition operations and symmetric group actions, and we think of $\mathcal{O}$ as the endomorphism operad of $\Sigma^{-1}\bar{C}^*$. Next we consider certain specific elements in $\mathcal{O}$. We have already mentioned the element $\smile \in \mathcal{O}(2)$. Let

$$e : \mathbb{Z} \to \bar{C}^*$$

denote the element of $\mathcal{O}(0)$ which takes 1 to the unit element of the ring R. Next recall that for each $n \geq 2$ there is a *brace operation*

$$(\bar{C}^*)^{\otimes n} \to \bar{C}^*$$

which takes $x_1 \otimes \cdots \otimes x_n$ to the cochain

$$x_1\{x_2, \ldots, x_n\}$$

defined by

$$x_1\{x_2, \ldots, x_n\} = \sum (-1)^\varepsilon x_1(\mathrm{id}, \ldots, \mathrm{id}, x_2, \mathrm{id}, \ldots, \mathrm{id}, x_n, \mathrm{id}, \ldots, \mathrm{id}),$$

where id is the identity map of R, the summation runs over all possible substitutions of $x_2,\ldots,x_n$ (in that order) into x_1 and ε is

$$\sum_{j=2}^{n} |x_j| i_j,$$

(here $|\ |$ denotes the degree in the desuspension and i_j is the total number of inputs in front of x_j). Note that if $\deg(x) < n$ then the sum is empty and the brace is zero. (The brace operation for $n=2$ was defined by Gerstenhaber in [9], and the higher braces were defined by Kadeishvili [14] and Getzler [11].)

Let $\mathcal{H}$ be the suboperad of $\mathcal{O}$ generated by e,$\smile$, and the brace operations $\{\,\}_n \in \mathcal{O}(n)$ for $n \geq 2$. Our first main result is

Theorem 1.1 *The singular chain operad of $\mathcal{C}_2$ is quasi-isomorphic as a chain operad to $\mathcal{H}$.*

We can give a more explicit description of $\mathcal{H}$ by considering the relations satisfied by e, $\smile$ and the $\{\,\}_n$ ([10, Section 1]). First we have

$$e \smile x = x \smile e = x \tag{1}$$

and

$$(x \smile y) \smile z = x \smile (y \smile z). \tag{2}$$

Because we are using normalized cochains, we have

$$x_1\{x_2, \ldots, x_n\} = 0 \quad \text{whenever some } x_i = 1. \tag{3}$$

There is a relation between $\{\}$ and $\smile$:

$$(x_1 \cdot x_2)\{y_1, \ldots, y_n\} = \sum_{k=0}^{n} (-1)^{\varepsilon} x_1\{y_1, \ldots, y_k\} \cdot x_2\{y_{k+1}, \ldots, y_n\}, \tag{4}$$

where this time $\varepsilon = |x_2| \sum_{j=1}^{k} |y_j|$. We can write this relation more compactly by using the standard notation $\nu \circ_k \nu'$ for the composition in the operad $\mathcal{O}$ which inserts ν' in the k-th input position of ν; then equation (4) becomes

$$\{\,\}_{n+1} \circ_1 \smile = \sum (\{\,\}_{k+1} \smile \{\,\}_{n-k}) \circ \tau \tag{5}$$

(where τ is the permutation that shuffles the second input to the $k+2$ position). If we compose brace operations, we have

$$\begin{aligned} x\{x_1, \ldots, x_m\}\{y_1, \ldots, y_n\} = \sum \Big[& (-1)^{\varepsilon} x\{y_1, \ldots, y_{i_1}, x_1\{y_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \\ & \ldots, y_{i_m}, x_m\{y_{i_m+1}, \ldots, y_{j_m}\}, y_{j_m+1}, \ldots, y_m\} \Big], \end{aligned} \tag{6}$$

where the sum is taken over all sequences $0 \leq i_1 \leq j_1 \leq i_2 \cdots \leq i_m \leq j_m \leq n$ and $\varepsilon = \sum_{p=1}^{m} |x_p| \sum_{q=1}^{i_p} |y_q|$; this may be written more compactly as

$$\{\,\}_{n+1} \circ_1 \{\,\}_{m+1} = \sum \{\,\}\,(\mathrm{id}, \ldots, \mathrm{id}, \{\,\}, \mathrm{id}, \ldots, \mathrm{id}, \{\,\}, \mathrm{id}, \ldots, \mathrm{id}) \tag{7}$$

where the sum is over all ways of interpreting the right-hand side. Since $\cup$ is a chain map we have

$$\partial(\smile) = 0 \tag{8}$$

and the differential on a brace operation is given by

$$\partial\{\,\}_n = -(\smile \circ_2 \{\,\}_{n-1}) \circ \tau + \left(\sum_i (\{\,\}_{n-1} \circ_i \smile)\right) - (\smile \circ_1 \{\,\}_{n-1}) \tag{9}$$

where τ is the transposition that switches the first and second entries. If we consider $\mathcal{H}$ as an operad in the category of graded abelian groups (that is, if we neglect the differential) then $\mathcal{H}$ is the quotient of the free operad generated by e, $\smile$ and the brace operations by the relations (1), (2), (3), (5) and (7), and the differential of $\mathcal{H}$ is determined by (8) and (9).

Our method for proving Theorem 1.1 is to construct a topological operad $\mathcal{C}$ whose structure is based on that of $\mathcal{H}$ and to show that the singular chain operad of $\mathcal{C}$ is quasi-isomorphic to $\mathcal{H}$ (this is Corollary 7.3) and that $\mathcal{C}$ is weakly equivalent (as an operad) to $\mathcal{C}_2$ (this is Theorem 8.1).

The operad $\mathcal{C}$ has another significant property. If $X^\bullet$ is a cosimplicial space (or spectrum) which has cup products and $\circ_k$ operations which satisfy the same relations as those in the Hochschild cochain complex, then $\mathcal{C}$ acts on $\mathrm{Tot}(X^\bullet)$. We give a careful statement of this fact in Theorem 3.3. In particular, this implies that the "topological Hochschild cohomology" of an A_∞ ring spectrum (see Example 3.4 for the definition) is a C_2-ring spectrum.

In a sequel to this paper we will construct similar models for the little n-cubes operads $\mathcal{C}_n$ when $n > 2$.

The organization of the paper is as follows. Section 2 is a warmup in which we give a sufficient condition for $\mathrm{Tot}(X^\bullet)$ to have an A_∞ structure. We also introduce our basic technical tool, prismatic subdivision. In section 3 we give the background needed to state Theorem 3.3 and some examples to which the theorem applies. In sections 4–6 we give the definition of the operad $\mathcal{C}$ and prove that it acts on $\mathrm{Tot}(X^\bullet)$ for suitable $X^\bullet$. We begin in sections 4 and 5 with a simplified model $\mathcal{C}'$ which captures most but not all of the structure of $\mathcal{C}_2$, and in section 6 we complete the definition of $\mathcal{C}$. In section 7 we prove that the singular chain operad of $\mathcal{C}$ is quasi-isomorphic to $\mathcal{H}$. In sections 8 and 9 we prove that $\mathcal{C}$ is weakly equivalent to $\mathcal{C}_2$, using a method of Fiedorowicz.

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2 Maps from $\mathrm{Tot}(X^\bullet) \times \mathrm{Tot}(Y^\bullet)$ to $\mathrm{Tot}(Z^\bullet)$.

Suppose that $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ are cosimplicial spaces. In this section we show how to construct a map

$$\mathrm{Tot}(X^\bullet) \times \mathrm{Tot}(Y^\bullet) \to \mathrm{Tot}(Z^\bullet).$$

from cosimplicial data.

Of course, the simplest way to do this is to begin with a cosimplicial map

$$X^\bullet \times Y^\bullet \to Z^\bullet$$

and then apply Tot (using the fact that Tot commutes with products) but we will give a construction more general than this.

In order to explain how a more general construction can be useful, let us consider the cobar construction $X^\bullet$ on a space A: here X^n is the Cartesian product $A^{\times n}$, the coface maps are diagonals and the codegeneracy maps are projections. In this case $\mathrm{Tot}(X^\bullet)$ is homeomorphic to ΩA and thus there is a multiplication map

$$\mathrm{Tot}(X^\bullet) \times \mathrm{Tot}(X^\bullet) \to \mathrm{Tot}(X^\bullet);$$

in fact there is one such map for each partition of the unit interval into two parts. On the other hand, there is clearly no sensible way to map $A^{\times n} \times A^{\times n}$ to $A^{\times n}$, so there is in general no cosimplicial map $X^\bullet \times X^\bullet \to X^\bullet$ that can induce the multiplication.

The following definition describes the structure at the cosimplicial level that our construction accepts as input.

Definition 2.1 **(i)** A *cup-pairing* $\phi : (X^\bullet, Y^\bullet) \to Z^\bullet$ is a family of maps

$$\phi_{p,q} : X^p \times Y^q \to Z^{p+q}$$

satisfying

(a) $d^i \phi_{p,q}(x, y) = \begin{cases} \phi_{p+1,q}(d^i x, y) & \text{if } i \le p \\ \phi_{p,q+1}(x, d^{i-p} y) & \text{if } i > p \end{cases}$

(b) $\phi_{p+1,q}(d^{p+1} x, y) = \phi_{p,q+1}(x, d^0 y)$

(c) $s^i \phi_{p,q}(x, y) = \begin{cases} \phi_{p-1,q}(s^i x, y) & \text{if } i \le p-1 \\ \phi_{p,q-1}(x, s^{i-p} y) & \text{if } i \ge p \end{cases}$

(ii) A *morphism* of cup pairings, from $\phi : (X^\bullet, Y^\bullet) \to Z^\bullet$ to $\phi' : (X'^\bullet, Y'^\bullet) \to Z'^\bullet$ is a triple of cosimplicial maps

$$\mu_1 : X \to X', \mu_2 : Y \to Y', \mu_3 : Z \to Z'$$

such that

$$\mu_3 \circ \phi'_{p,q} = \phi_{p,q} \circ (\mu_1 \times \mu_2)$$

for all p, q.

(iii) Given a cup-pairing $\phi : (X^\bullet, X^\bullet) \to X^\bullet$, a *unit* for ϕ is a sequence of points $e_n \in X^n$ such that the set $\{ e_n \}$ is closed under all cofaces and codegeneracies and $\phi_{0,p}(e_0, x) = \phi_{p,0}(x, e_0) = x$ for all x (i.e., the e_n determine a map from the trivial cosimplicial space to $X^\bullet$, and e_0 is a unit in the usual sense).

¿From now on we shall usually drop the subscripts and just write $\phi(x,y)$ for $\phi_{p,q}(x,y)$.

Remark 2.2 **(i)** The cobar construction on A has the cup-pairing

$$\phi((a_1,\dots,a_p),(b_1,\dots,b_q)) = (a_1,\dots,a_p,b_1,\dots,b_q)$$

(ii) The definition of cup-pairing is modeled on the properties of the cup product in the Hochschild cohomology complex. It is easy to check that the function $\phi(x,y) = x \smile y$ is a cup-pairing in our sense if the Hochschild complex is given the usual cosimplicial structure in which s^i inserts a unit in the $i+1$-st position and d^i is defined by

$$(d^i x)(r_1,\dots,r_{p+1}) = \begin{cases} r_1 x(r_2,\dots,r_{p+1}) & \text{if } i=0 \\ x(\dots,r_i r_{i+1},\dots) & \text{if } 0<i<p+1 \\ x(r_1,\dots,r_p)r_{p+1} & \text{if } i=p+1 \end{cases}$$

(iii) A cosimplicial map $\mu : X^\bullet \times Y^\bullet \to Z^\bullet$ induces a cup pairing by

$$\phi(x,y) = \mu(d^{p+q} d^{p+q-1} \cdots d^{p+1} x, d^0 d^0 \cdots d^0 y)$$

(that is, the last coface map is applied q times to x, and the zeroth coface map is applied p times to y, where p is the degree of x and q is the degree of y; then μ is applied to the resulting pair).

Theorem 2.3 **(i)** *A cup-pairing*

$$\phi : X^\bullet \times Y^\bullet \to Z^\bullet$$

induces a map

$$\bar\phi_u : \mathrm{Tot}(X^\bullet) \times \mathrm{Tot}(Y^\bullet) \to \mathrm{Tot}(Z^\bullet).$$

for each u with $0<u<1$.

(ii) *A morphism of cup-pairings induces a commutative diagram*

$$\begin{array}{ccc} \mathrm{Tot}(X^\bullet)\times\mathrm{Tot}(Y^\bullet) & \xrightarrow{\bar\phi_u} & \mathrm{Tot}(Z^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Tot}(X'^\bullet)\times\mathrm{Tot}(Y'^\bullet) & \xrightarrow{\bar\phi'_u} & \mathrm{Tot}(Z'^\bullet) \end{array}$$

(iii) *If the cup pairing comes from a cosimplicial map μ then each $\bar\phi_u$ is homotopic to the usual map induced by μ.*

Our next result refers to Tot', which is a construction related to Tot in the same way that the Moore loop space is related to the ordinary loop space; we will give the precise definition below.

Theorem 2.4 *A strictly associative cup-pairing on $X^\bullet$ with a unit induces a strictly associative multiplication on* $\mathrm{Tot}'(X^\bullet)$ *and an action of the little 1-cubes operad on* $\mathrm{Tot}(X^\bullet)$.

Remark 2.5 If $X^\bullet$ is the cobar construction on a space A then the action of the little 1-cubes on $\mathrm{Tot}(X^\bullet)$ is homeomorphic to the usual action of the little 1-cubes on $\Omega(X)$. Also, there is a continuous bijection from $\mathrm{Tot}'(X^\bullet)$ to the Moore loop space of A which takes the multiplication on $\mathrm{Tot}'(X^\bullet)$ to the usual multiplication on the Moore loop space.

Before giving the proof of Theorems 2.3 and 2.4, we will describe a way of subdividing a simplex which we call the "prismatic subdivision." This is a homeomorphism discovered independently and at various times by Lisica and Mardesic [16], Grayson [13], and ourselves; see also Batanin's papers [2, 3]. (There is a related "edgewise subdivision" discovered by Quillen, Segal, Bökstedt and Goodwillie; the edgewise subdivision of a simplex is a subdivision of the prismatic subdivision.)

It will be convenient from now on to let $\mathbf{s}$, $\mathbf{t}$, etc. stand for a point of a simplex and let $s_0, s_1, \ldots$ (respectively, $t_0, t_1, \ldots$, etc.) be its coordinates.

Define a space D^n for each $n \geq 0$ by

$$D^n = (\coprod_{p=0}^{n} \Delta^p \times \Delta^{n-p})/\sim,$$

where $\sim$ is defined by $(d^{p+1}\mathbf{s}, \mathbf{t}) \sim (\mathbf{s}, d^0\mathbf{t})$ if $\mathbf{s} \in \Delta^p$ and $\mathbf{t} \in \Delta^{n-p-1}$. For each u with $0 \leq u \leq 1$, there is a map

$$\sigma^n(u) : D^n \to \Delta^n$$

whose restriction to $\Delta^p \times \Delta^{n-p}$ takes

$$(s_0, \ldots, s_p), (t_0, \ldots, t_{n-p})$$

to

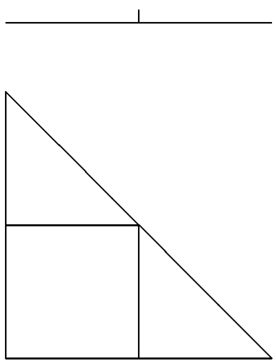
$$(us_0, \ldots, us_{p-1}, us_p + (1-u)t_0, (1-u)t_1, \ldots, (1-u)t_{n-p}).$$

It is easy to check that this map is well-defined, continuous and onto, and when $0 < u < 1$ it is also one-to-one. Thus we have:

Proposition 2.6 *$\sigma^n(u)$ is a homeomorphism if $0 < u < 1$.*

This homeomorphism is the prismatic subdivision of the simplex Δ^n.

Here are the prismatic subdivisions of the 1-simplex and the 2-simplex (with $u = \frac{1}{2}$):



We can now prove part (i) of Theorem 2.3 Recall that a point in $\mathrm{Tot}(X^\bullet)$ is a sequence

$$a_0 \in X^0, a_1 : \Delta^1 \to X^1, a_2 : \Delta^2 \to X^2, \ldots$$

which is *consistent*, i.e.,

$$d^i \circ a_n = a_{n+1} \circ d^i$$

and

$$s^i \circ a_n = a_{n-1} \circ s^i$$

Thus what is required is: given consistent sequences $a_n : \Delta^n \to X^n$ and $b_n : \Delta^n \to Y^n$, to construct a consistent sequence $c_n : \Delta^n \to Z^n$.

First observe that, by part (b) of Definition 2.1(i), the maps

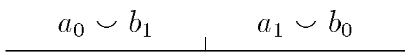
$$\phi \circ (a_p \times b_{n-p}) : \Delta^p \times \Delta^{n-p} \to Z^n$$

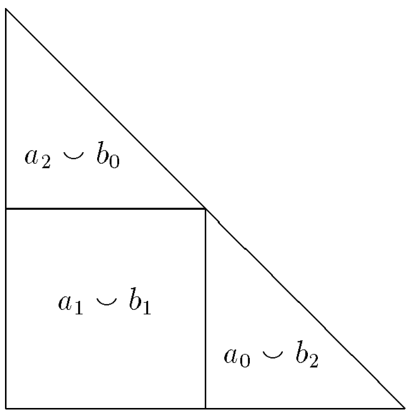
fit together to give a well-defined map

$$D^n \to Z^n$$

(where D^n is the space defined before Proposition 2.6). We define c_n for each n to be the composite of this map with $(\sigma^n(u))^{-1}$. The fact that the c_n commute with coface and codegeneracy maps is immediate from the formula for $\sigma^n(u)$ and parts (a) and (c) of Definition 2.1(i). This concludes the proof of part (i).

Here are pictures of c_1 and c_2 (with $u = \frac{1}{2}$):





Part (ii) of Theorem 2.3 is immediate from the definitions.

We now turn to the proof of part (iii). By part (ii), it suffices to consider the case where μ is the identity map of $X^\bullet \times Y^\bullet$. Let $\phi : (X^\bullet, Y^\bullet) \to X^\bullet \times Y^\bullet$ denote the pairing induced by the identity map of $X^\bullet \times Y^\bullet$ (as in Remark 2.22(iii)): thus

$$\phi(x, y) = (d^{p+q} d^{p+q-1} \cdots d^{p+1} x, d^0 d^0 \cdots d^0 y)$$

Let

$$F : \mathrm{Tot}(X^\bullet) \times \mathrm{Tot}(Y^\bullet) \to \mathrm{Tot}(X^\bullet) \times \mathrm{Tot}(Y^\bullet)$$

denote $\bar{\phi}_{1/2}$; it suffices to show that F is homotopic to the identity. Let π_1 and π_2 be the projections of $\mathrm{Tot}(X^\bullet)\times\mathrm{Tot}(Y^\bullet)$ on its first and second factors; it suffices to show that $\pi_i\circ F$ is homotopic to π_i for $i=1,2$. We consider the case $i=1$; the other case is similar. Let $\tilde{D}^n$ denote the space

$$(\coprod_p \Delta^p \times (\Delta^{n-p}\wedge[\frac{1}{2},1]))/\sim,$$

where the basepoint of $[\frac{1}{2},1]$ is taken to be 1 and $\sim$ is defined by $(d^{p+1}\mathbf{s},\mathbf{t}\wedge u)\sim(\mathbf{s},d^0\mathbf{t}\wedge u)$ if $\mathbf{s}\in\Delta^p$ and $\mathbf{t}\in\Delta^{n-p-1}$. The map $\tau:\tilde{D}^n\to\Delta^n\times[\frac{1}{2},1]$ given by

$$\tau(\mathbf{s},\mathbf{t}\wedge u)=(\sigma^n(u)(\mathbf{s},\mathbf{t}),u)$$

is a homeomorphism, since it is 1-1, onto, and continuous. For each n, the maps

$$\Delta^p\times(\Delta^{n-p}\wedge[\frac{1}{2},1])\times\mathrm{Tot}(X^\bullet)\times\mathrm{Tot}(Y^\bullet)\to X^n$$

which take $(\mathbf{s},\mathbf{t}\wedge u,\{a_n\},\{b_n\})$ to $a_n(d^nd^{n-1}\cdots d^{p+1}\mathbf{s})$ fit together to give a well-defined map

$$\tilde{D}^n\times\mathrm{Tot}(X^\bullet)\times\mathrm{Tot}(Y^\bullet)\to X^n.$$

Composing this with $(\tau_n)^{-1}$ gives a map

$$\Delta^n\times[\frac{1}{2},1]\times\mathrm{Tot}(X^\bullet)\times\mathrm{Tot}(Y^\bullet)\to X^n,$$

whose adjoint is a map

$$H_n:[\frac{1}{2},1]\times\mathrm{Tot}(X^\bullet)\times\mathrm{Tot}(Y^\bullet)\to\mathrm{Map}(\Delta^n,X^n).$$

Taken together, the H_n give a map

$$H:[\frac{1}{2},1]\times\mathrm{Tot}(X^\bullet)\times\mathrm{Tot}(Y^\bullet)\to\mathrm{Tot}(X^\bullet)$$

which is equal to $\pi_1\circ F$ when $u=\frac{1}{2}$ and to π_1 when $u=1$. Thus $\pi_1\circ F$ is homotopic to π_1 as required. $\square$

Before proving Theorem 2.4 we need to give the definition of Tot$'$. First we define Δ_p^n, for each $p\ge 0$, to be the set

$$\{(s_0,\dots,s_n)\,|\,s_i\ge 0 \text{ for all } i \text{ and } s_0+\cdots+s_n=p\}$$

(Note that $\Delta_p^\bullet$ is a cosimplicial space for each p, and that $\Delta_0^\bullet$ is is the trivial cosimplicial space consisting of a point in each degree.) Then we define $\mathrm{Tot}'(X^\bullet)$ to consist of pairs $(p,\{a_n\})$, where p is ≥ 0 and $\{a_n\}$ is a cosimplicial map $\Delta_p^\bullet\to X^\bullet$. In order to describe the topology of Tot$'$ we first observe that for $p>0$ the cosimplicial spaces $\Delta_p^\bullet$ and $\Delta^\bullet$ are isomorphic, and thus there is a bijection between $T'(X^\bullet)$ and the disjoint union

$$\mathrm{Const}(X^\bullet)\cup(\mathrm{Tot}(X)\times\mathbb{R}_{>0}),$$

where $\mathrm{Const}(X^\bullet)$ denotes the space of cosimplicial maps from $\Delta_0^\bullet$ to $X^\bullet$ and $\mathbb{R}_{>0}$ denotes the positive reals. This in turn implies that there is a bijection between $T'(X^\bullet)$ and the pushout

$$\begin{array}{ccc} \mathrm{Const}(X^\bullet) \times \mathbb{R}_{>0} & \to & \mathrm{Const}(X^\bullet) \times \mathbb{R}_{\geq 0} \\ \downarrow & & \\ \mathrm{Tot}(X^\bullet) \times \mathbb{R}_{>0} & & \end{array}$$

We give $\mathrm{Tot}'(X^\bullet)$ the topology which makes this bijection a homeomorphism. (We could have simply defined $\mathrm{Tot}'(X^\bullet)$ to be this pushout, but this would make the formulas in the rest of the paper more complicated). Note that there is a continuous projection $\rho : \mathrm{Tot}'(X^\bullet) \to \mathrm{Tot}(X^\bullet)$ which makes $T(X^\bullet)$ a deformation retract of $T'(X^\bullet)$.

Proof of Theorem 2.4. We need a slight generalization of the idea of prismatic subdivision: define a space $D^n_{p,q}$ for each $n \geq 0$ and each $p, q \geq 0$ by

$$D^n_{p,q} = (\coprod_{k=0}^{n} \Delta^k_p \times \Delta^{n-k}_q)/\sim,$$

where $\sim$ is defined by $(d^{k+1}\mathbf{s}, \mathbf{t}) \sim (\mathbf{s}, d^0\mathbf{t})$ if $\mathbf{s} \in \Delta^k_p$ and $\mathbf{t} \in \Delta^{n-k-1}_r$. Define

$$\sigma^n_{p,q} : D^n_{p,q} \to \Delta^n_{p+q}$$

by

$$\Sigma((s_0, \ldots, s_k), (t_0, \ldots, t_{n-k})) = (s_0, \ldots, s_{k-1}, s_k + t_0, t_1, \ldots, t_{n-k}).$$

Then $\sigma^n_{p,q}$ is a homeomorphism for every choice of p and q.

Now we can define the multiplication on Tot'. Suppose given a pair of points $(p, \{a_n\})$ and $(q, \{b_n\})$ in Tot'. Observe that, by part (b) of Definition 2.1(i), the maps

$$\phi \circ (a_k \times b_{n-k}) : \Delta^k_p \times \Delta^{n-k}_q \to X^n$$

fit together to give a well-defined map

$$D^n_{p,q} \to X^n$$

Composing this map with $(\sigma^n_{p,q})^{-1}$ gives a map $c_n : \Delta^n_{p+q} \to X^n$ for each n. The c_n commute with coface and codegeneracy maps by parts (a) and (c) of Definition 2.1(i), and we define the product of the points $(p, \{a_n\})$ and $(q, \{b_n\})$ to be the point $(p+q, \{c_n\})$. This multiplication is clearly associative and unital, and we leave it to the reader to check that it is continuous.

It remains to give the action of the little 1-cubes operad $\mathcal{C}_1$ on $\mathrm{Tot}(X^\bullet)$. So given a point z of $\mathcal{C}_1(k)$ and k points $a_1, \ldots, a_k$ of $\mathrm{Tot}(X^\bullet)$, we are required to define a point $\gamma(z, a_1, \ldots, a_k)$ in $\mathrm{Tot}(X^\bullet)$. The idea is to imitate the way that $\mathcal{C}_1$ acts on a loop space, that is, we scale the a's to the lengths of the corresponding little intervals and fill in the blank spaces with appropriately scaled copies of the unit element.

First observe that the point z of $\mathcal{C}_1(k)$ can be written as a sequence of $2k+1$ numbers $(p_1, p_2, \ldots, p_{2k+1})$, where the even numbered p's are the lengths of the little intervals and the odd numbered p's are the lengths of the empty spaces.

Next we introduce some notation. Given a function $f : \Delta^n \to X^n$ and a number $p > 0$, let us write $p \cdot f$ for the function $\Delta^n_p \to X^n$ which takes $\mathbf{s}$ to $f(\frac{1}{p} \cdot \mathbf{s})$. Given a point $a = \{a_n\}$ in $\mathrm{Tot}(X^\bullet)$, let us write $p \cdot a$ for the point $(p, \{p \cdot a_n\})$ in $\mathrm{Tot}'(X^\bullet)$. Recall that the cup product has a unit e, which is a map from $\Delta^\bullet_0$ to $X^\bullet$, and for each $p \geq 0$ write e_p for the composite

$$\Delta^\bullet_p \to \Delta^\bullet_0 \xrightarrow{e} X^\bullet$$

Let $*$ denote the multiplication on $\mathrm{Tot}'(X^\bullet)$. Then we define $\gamma(z, a_1, \ldots, a_k)$ to be

$$\rho(e_{p_1} * (p_2 \cdot a_1) * e_{p_3} * (p_4 \cdot a_2) * \cdots * e_{p_{2k+1}}),$$

where ρ is the projection from $\mathrm{Tot}'(X^\bullet)$ to $\mathrm{Tot}(X^\bullet)$. $\square$

3 Operads with multiplication and their associated cosimplicial objects.

In this section we give a formal statement of the fact that if $X^\bullet$ is a cosimplicial space or spectrum with $\smile$ and $\circ_k$ operations that satisfy the same relations as those in the Hochschild cochain complex then our operad $\mathcal{C}$ acts on $\mathrm{Tot}(X^\bullet)$. For this we need some background.

First recall that if $\mathcal{O}$ is an operad with structure maps

$$\gamma : \mathcal{O}(n) \times \mathcal{O}(j_1) \times \cdots \times \mathcal{O}(j_n) \to \mathcal{O}(j_1 + \cdots + j_n)$$

and identity element $\mathrm{id} \in \mathcal{O}(1)$, it is customary to write $\circ_i$ for the map

$$\mathcal{O}(n) \times \mathcal{O}(j) \to \mathcal{O}(n + j - 1)$$

which takes (o_1, o_2) to $\gamma(o_1, \mathrm{id}, \ldots, o_2, \ldots, \mathrm{id})$ (with $i-1$ id's before the o_2).

Next let R be a ring and let $\mathcal{O}_R$ denote the endomorphism operad of the underlying abelian group of R; that is, $\mathcal{O}_R(n)$ is the set of homomorphisms of abelian groups from $R^{\otimes n}$ to R, with the evident operad structure. There are special elements $\mu \in \mathcal{O}_R(2)$ (the multiplication in the ring R) and $e \in \mathcal{O}_R(0)$ (the unit element of R). Of course, $\mathcal{O}_R(n)$ is the same as the group of n-cochains in the Hochschild complex of R, and the cosimplicial structure of the Hochschild complex can be recovered from the operad structure of $\mathcal{O}_R$ and the elements μ and e:

$$(d^i x) = \begin{cases} \mu \circ_2 x & \text{if } i = 0 \\ x \circ_i \mu & \text{if } 0 < i < p+1 \\ \mu \circ_1 x & \text{if } i = p+1 \end{cases} \tag{10}$$

$$s_i(x) = x \circ_{i+1} e \tag{11}$$

When the d^i and s^i are defined in this way the cosimplicial identities follow formally from the identities

$$\mu \circ_1 \mu = \mu \circ_2 \mu \tag{12}$$

and

$$\mu \circ_1 e = \mu \circ_2 e = \mathrm{id}. \tag{13}$$

This example motivates the following definition, which is due to Gerstenhaber and Voronov [10]. Recall ([18, Definition 3.12]) that a non-Σ operad is a structure which has all the properties of an operad except those having to do with the actions of the symmetric groups. Also recall that the definition of operad makes sense in any symmetric monoidal category.

Definition 3.1 Let $\mathcal{S}$ be a symmetric monoidal category with unit object S. An *operad with multiplication* in $\mathcal{S}$ is a non-Σ operad $\mathcal{O}$ in $\mathcal{S}$ together with maps $e : S \to \mathcal{O}(0)$ and $\mu : S \to \mathcal{O}(2)$ satisfying (12) and (13). The *associated cosimplicial object* $\mathcal{O}^\bullet$ consists of the objects $\mathcal{O}(n)$ with coface and codegeneracy maps given by (10) and (11).

The reader will perhaps be relieved to know that the only symmetric monoidal categories we will be concerned with in this paper are the category of spaces, the category of S-modules [7, Chapter II], and (briefly) the category of sets.

Remark 3.2 **(i)** If we let Ass be the non-Σ operad for which $Ass(n)$ is the object S for each n (so that an algebra over Ass is an associative monoid in $\mathcal{S}$) then a more economical (but equivalent) way to define an operad with multiplication is: a non-Σ operad together with a non-Σ-operad morphism from Ass.

(ii) If $\mathcal{O}$ is an operad with multiplication then the associated cosimplicial object has a cup-pairing in the sense of Section 2: we define $\phi_{p,q}(x, y)$ to be $(\mu \circ_1 x) \circ_{p+1} y$.

We can now state our main result. Recall that a weak equivalence of operads is an operad morphism which is a weak equivalence on each object. We say that two operads are *weakly equivalent* if there is a third operad which maps to each of them by a weak equivalence.

Theorem 3.3 *There is an operad $\mathcal{C}$ in the category of spaces with the following properties:*

(i) *$\mathcal{C}$ is weakly equivalent to the little 2-cubes operad $\mathcal{C}_2$.*

(ii) *If $\mathcal{O}$ is an operad with multiplication in the category of spaces or spectra then $\mathcal{C}$ acts on* $\mathrm{Tot}(\mathcal{O}^\bullet)$.

Here are some examples to which Theorem 3.3 can be applied. In the examples that refer to the category of S-modules we will always write $\wedge$ for $\wedge_S$.

Example 3.4 (The Hochschild cohomology complex of a ring spectrum.) Let R be an S-algebra in the sense of [7, Section II.3] and let $\mathcal{O}$ be the endomorphism operad of R:

$$\mathcal{O}(n) = F_S(R^{\wedge n}, R)$$

(see [7, Section II.1] for the definition of F_S). Let $e : S \to R$ be the unit map of R and let $\mu : S \to \mathcal{O}(2)$ be adjoint to the multiplication map $R \wedge R \to R$. Then $\mathcal{O}$ is an operad with multiplication and we define the Hochschild cohomology complex of R to be the cosimplicial S-module associated to $\mathcal{O}$.

Example 3.5 (The loop space of a topological monoid.) (We would like to thank Zig Fiedorowicz for pointing out this example to us.) Let A be a topological monoid, with the unit of A chosen as basepoint. We will define an operad with multiplication $\mathcal{O}$ whose associated cosimplicial space is the cobar construction on A. Of course, we let $\mathcal{O}(n) = A^{\times n}$. In order to define the operad structure we observe (as in [19, p. 6]) that it suffices to specify the operations

$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(j) \to \mathcal{O}(n + j - 1),$$

and we define these by the equation

$$(a_1, \ldots, a_n) \circ_i (b_1, \ldots, b_j) = (a_1, \ldots, a_{i-1}, a_i b_1, \ldots, a_i b_j, a_{i+1}, \ldots a_n) \tag{14}$$

Finally, we choose $e \in \mathcal{O}(0)$ and $\mu \in \mathcal{O}(2)$ to be the basepoints. (Observe that the cup-pairing determined by μ is the same as that defined in Remark 2.2(i).)

In preparation for our next two examples it is helpful to reformulate Example 3.5 in a fancier way. Let $B_\bullet$ denote the simplicial circle $\Delta^1/\partial\Delta^1$. Recall that a non-basepoint simplex in B_n is a sequence $(b_0, b_1, \ldots, b_n)$ of $n+1$ zeroes and ones, with the zeroes coming before the ones. We can put the non-basepoint simplices of B_n in one-to-one correspondence with the numbers 1 to n by letting the number j correspond to the simplex with j zeroes, and this gives a homeomorphism α between $\mathcal{O}(n)$ and the space of based maps from B_n to A; under α the coface and codegeneracy maps of $\mathcal{O}(n)$ agree with the maps induced by the faces and degeneracies of B_n. If $x \in \mathcal{O}(n)$ and $b \in B_n$ then we write $x(b)$ for the element $\alpha(x)(b)$ of A. Now equation (14) is equivalent to the following: if $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(j)$ and $b \in B_n$ then

$$(x \circ_i y)(b) = x(b'_{ij}) \cdot y(b''_{ij}), \tag{15}$$

where $\cdot$ denotes the product in A, b'_{ij} is the simplex

$$(b_0, b_1, \ldots, b_{i-1}, b_{i+j-1}, \ldots, b_{n+j-1})$$

and b''_{ij} is the simplex

$$(b_{i-1}, \ldots, b_{i+j-1}).$$

We can generalize the definitions of b'_{ij} and b''_{ij} to any simplicial set $B_\bullet$ as follows. If $s \leq t$ we write $\partial(s,t)$ for the composite of face maps

$$\partial_s \partial_{s+1} \cdots \partial_t$$

Now if $n \geq 1$, $j \geq 0$, $1 \leq i \leq n$, and $b \in B_{n+j-1}$, we define

$$b'_{ij} = \partial(i, i+j-2)b$$

and

$$b''_{ij} = \partial(0, i-2)\partial(i+j, n+j-1)b.$$

These definitions will be familiar to many readers because they are related to Steenrod's original definition of the $\smile_1$ product [23]: if $\xi \in C^n(B_\bullet; \mathbb{Z}/2)$ and $\eta \in C^j(B_\bullet; \mathbb{Z}/2)$ are mod-2 simplicial cochains of $B_\bullet$ and $b \in B_{n+j-1}$ then

$$(\xi \smile_1 \eta)(b) = \sum_i \xi(b'_{ij})\eta(b''_{ij})$$

Example 3.6 (Ω^k of a space when $k \geq 2$.) Let A be any based space and fix $k \geq 2$. We can define a "higher" cobar construction $X^\bullet$, with $\mathrm{Tot}(X^\bullet)$ homeomorphic to $\Omega^k A$, as follows: let $B_\bullet$ be the simplicial k-sphere $\Delta^k/\partial\Delta^k$ and let X^n be the space of based maps from B_n to A, with coface and codegeneracy maps induced by the face and degeneracy maps of $B_\bullet$. Next we define an operad with multiplication $\mathcal{O}$ whose associated cosimplicial space is $X^\bullet$. Of course, we let $\mathcal{O}(n) = X^n$. We define the $\circ_i$ operations by equation (15), but note that the symbol $\cdot$ now requires interpretation, since we are not assuming that A is a monoid. We are saved by the fact that (since $k \geq 2$) either b'_{ij} or b''_{ij} is always the basepoint, and thus either $x(b'_{ij})$ or $y(b''_{ij})$ is always the basepoint $*$ of A. It therefore suffices to define $a \cdot * = * \cdot a = a$ for all $a \in A$ to make equation (15) meaningful in this context. Finally, we choose $e \in \mathcal{O}(0)$ and $\mu \in \mathcal{O}(2)$ to be the basepoints.

As background for our final example we recall that, if R is a commutative ring and $m \geq 2$, Pirashvili [20] has defined a higher Hochschild homology complex $Y_\bullet(R)$ by letting $Y_n(R)$ be the tensor product

$$\bigotimes_{B_n} R$$

(where $B_\bullet$ is the simplicial set from example 3.6), with face and degeneracy maps induced by those of $B_\bullet$; thus $Y_\bullet(R)$ is the tensor product $B_\bullet \otimes R$ in the category of simplicial commutative rings, as defined by Quillen [21].

We also recall [7, Proposition VII.1.6] that $\wedge$ is the coproduct in the category of commutative S-algebras. It follows that if R is an S-algebra, A and B any sets and $f : A \to B$ any map then f induces a map

$$f_* : \bigwedge_A R \to \bigwedge_B R$$

of S-algebras.

Example 3.7 (The Pirashvili cohomology complex of a commutative ring spectrum.) Fix $k \geq 2$. Let R be a commutative S-algebra, and define the k-th Pirashvili homology complex $Y_\bullet(R)$ by letting

$$Y_n(R) = \bigwedge_{B_n} R$$

(where $B_\bullet$ is the simplicial set from example 3.6) with face and degeneracy maps induced by those of $B_\bullet$. The inclusion of the basepoint in B_n induces a map $R \to Y_n(R)$ of commutative S-algebras which makes $Y_n(R)$ an R-module. We define the k-th Pirashvili cohomology complex of R to be the cosimplicial R-module $X^\bullet(R)$ with

$$X^n(R) = F_R(Y_n(R), R)$$

(see [7, Section III.6] for the definition of F_R), with coface and codegeneracy maps induced by the face and degeneracy maps of $Y_\bullet$. Next we want to define an operad with multiplication $\mathcal{O}$ whose associated cosimplicial object is $X^\bullet$. Let $\mathcal{O}(n) = X^n$. In order to define the operation $\circ_i$ we first observe that, since $k \geq 2$, the map

$$B_{n+j-1} \to B_n \times B_j$$

which takes b to (b'_{ij}, b''_{ij}) factors through the wedge to give a map

$$B_{n+j-1} \to B_n \vee B_j$$

and this in turn induces a map

$$f_i : Y_{n+j-1}(R) = \bigwedge_{B_{n+j-1}} R \to \bigwedge_{B_n \vee B_j} R$$

We next observe that there is a natural isomorphism

$$\bigwedge_{B_n \vee B_j} R \cong (\bigwedge_{B_n} R) \wedge_R (\bigwedge_{B_j} R) = Y_n(R) \wedge_R Y_j(R)$$

We can therefore define $\circ_i$ to be the composite

$$\begin{aligned} X^n \wedge X^j &= F_R(Y_n(R), R) \wedge F_R(Y_j(R), R) \xrightarrow{\wedge_R} F_R(Y_n(R) \wedge_R Y_j(R), R \wedge_R R) \\ &\cong F_R(\bigwedge_{B_n \vee B_j} R, R) \xrightarrow{f_i^*} F_R(Y_{n+j-1}(R), R) = X^{n+j-1} \end{aligned}$$

It remains to specify the maps $e : S \to \mathcal{O}(0)$ and $\mu : S \to \mathcal{O}(2)$. We note that $\mathcal{O}(0)$ is just R, so we can let e be the unit map of R. If $k > 2$ then $\mathcal{O}(2)$ is also R, and we can let μ be the unit map. If $k = 2$ then $\mathcal{O}(2)$ is isomorphic to $F_S(R, R)$, and we let μ be the adjoint of the identity map.

4 The spaces of the operad $\mathcal{C}'$.

Recall that $\mathcal{C}_2$ denotes the operad of little 2-cubes. The zeroth space of $\mathcal{C}_2$ is a point. We can define a suboperad $\mathcal{C}_2'$ of $\mathcal{C}_2$ by letting $\mathcal{C}_2'(n) = \mathcal{C}_2(n)$ for all $n > 0$ and letting $\mathcal{C}_2'(0)$ be the empty set. All of the information about $\mathcal{C}_2$ and its operad structure is contained in $\mathcal{C}_2'$ *except* for the so-called "degeneracy" maps

$$\circ_k : \mathcal{C}_2(n) \times \mathcal{C}_2(0) \to \mathcal{C}_2(n-1)$$

(the corresponding maps for $\mathcal{C}_2'$ have the empty set as their domain).

In this section and the next we define an operad $\mathcal{C}'$ which will turn out to be equivalent to $\mathcal{C}_2'$. In section 6 we will define the operad $\mathcal{C}$ of theorem 3.3 by adding one more ingredient to the definition of $\mathcal{C}'$.

4.1 The spaces $\mathcal{P}(n)$ and $\mathcal{F}(n)$.

The n-th space $\mathcal{C}'(n)$ is empty for $n = 0$ and for $n > 0$ it is the Cartesian product of a space $\mathcal{F}(n)$ and a contractible space $\mathcal{P}(n)$. (The idea here is that the spaces $\mathcal{F}(n)$ by themselves have a structure like that of an operad but with a composition operation γ which is only A_∞-associative. By combining the $\mathcal{F}(n)$ with the $\mathcal{P}(n)$ we get a true operad).

The space $\mathcal{P}(n)$ is easy to define: it is the empty set for $n = 0$ and for $n > 0$ it is the set of n-tuples $(p_1, \ldots, p_n)$ of positive real numbers that add up to 1.

The structure of the space $\mathcal{F}(n)$ is related to that of the algebraic operad $\mathcal{H}$ defined in the introduction. We will describe it in several stages: first we define the set that indexes the cells of $\mathcal{F}(n)$, then we define the cells themselves, and finally we define the attaching maps.

4.2 The set of "formulas of type n."

The cells of $\mathcal{F}(n)$ will be indexed by what we call "formulas of type n." Specifically, a formula of type n is a symbol constructed from the numbers $1, 2, \ldots n$ (with no repetitions) by formal cup products and formal operad compositions. (The formulas of type n correspond to certain multilinear operations in the Hochschild complex. The formal cup products correspond to cup products in the Hochschild complex, and the formal operad compositions correspond to brace operations.) For ease of notation we write the formal operad composition $\gamma(x_0; x_1, \ldots, x_n)$ as $x_0(x_1, \ldots, x_n)$.

Some examples of formulas of type 5 are: $2(1(3,5),4)$, $1(2 \smile 4, 3(5))$, $4(2,3) \smile 1(5)$.

Here is a more precise definition. Define a sequence of sets $\{S_k\}$ for $k \geq 0$ as follows: S_k is the set of positive integers when $k \neq 2$ and S_2 is the set of positive integers with the symbol μ adjoined. Take the free operad generated by the sequence of sets S_k and impose the relation (12) from Section 3. Let the resulting operad be denoted by $\mathcal{J}'$ (the prime refers to the fact that we are engaged in defining $\mathcal{C}'$). Then the set of formulas of type n is precisely the set of elements in $\mathcal{J}'(0)$ which contain each of the symbols $1, \ldots, n$ exactly once.

Remark 4.1 For clarity we should point out that, in this way of defining the formulas of type n, the cup products are denoted by μ instead of $\smile$, so that for example the formula

$1(2 \smile 4, 3(5))$ would be written $1(\mu(2,4), 3(5))$ and the formula $4(2,3) \smile 1(5)$ would be written $\mu(4(2,3), 1(5))$.

For our later definitions it will be convenient to define the *valence* of an integer in S_k to be the number k. Thus in a formula of type n the valence of a symbol $1, \ldots, n$ is the number of formal inputs for that symbol. For example, in the formula

$$3(2, 4(5,6), 1, 7) \smile 8(9)$$

the valence of the symbol 3 is 4, the valence of the symbol 4 is 2, the valence of the symbol 8 is 1 and the valences of $1, 2, 5, 6, 7, 9$ are each 0.

We will write $v(i)$ for the valence of i.

4.3 The cells of $\mathcal{F}(n)$.

To each formula f of type n we associate the product of simplices

$$\prod_{i=1}^{n} \Delta^{v(i)}$$

where $v(i)$ is the valence of the symbol i. This product of simplices is a cell of $\mathcal{F}(n)$. We will frequently denote the cell associated to a formula f by $\mathcal{F}_f$.

Here are some examples of cells of $\mathcal{F}(5)$:

$$\mathcal{F}_{2(1(3,5),4)} = \Delta^2 \times \Delta^2 \times \Delta^0 \times \Delta^0 \times \Delta^0,$$

$$\mathcal{F}_{1(2\smile 4,3(5))} = \Delta^2 \times \Delta^0 \times \Delta^1 \times \Delta^0 \times \Delta^0,$$

and

$$\mathcal{F}_{4(2,3)\smile 1(5)} = \Delta^1 \times \Delta^0 \times \Delta^0 \times \Delta^2 \times \Delta^0.$$

4.4 Identifications Along The Boundary.

The boundary of each cell $\mathcal{F}_f$ is a union of other cells which are selected by a rule modeled on the relation (9) given in the introduction. Before spelling this out we give some examples of what the rule will say.

Example. The cell $\mathcal{F}_{1(2)}$ is an interval whose initial point is identified with $\mathcal{F}_{1\smile 2}$ and whose terminal point is identified with $\mathcal{F}_{2\smile 1}$. Similarly, the cell $\mathcal{F}_{2(1)}$ is an interval whose initial point is identified with $\mathcal{F}_{2\smile 1}$ and whose terminal point is identified with $\mathcal{F}_{1\smile 2}$. Since the only cells of $\mathcal{F}(2)$ are $\mathcal{F}_{1(2)}$, $\mathcal{F}_{2(1)}$, $F_{1\smile 2}$, and $\mathcal{F}_{2\smile 1}$, we conclude that the space $\mathcal{F}(2)$ is homeomorphic to a circle.

Example. $\mathcal{F}_{1(2,3)}$ is a triangle whose three faces are identified with $\mathcal{F}_{2\smile 1(3)}$, $\mathcal{F}_{1(2\smile 3)}$ and $\mathcal{F}_{1(2)\smile 3}$.

Example. $\mathcal{F}_{1(2(3))}$ is a square whose four faces are identified with $\mathcal{F}_{2(3)\smile 1}$, $\mathcal{F}_{1\smile 2(3)}$, $\mathcal{F}_{1(3\smile 2)}$, and $\mathcal{F}_{1(2\smile 3)}$.

Next we give a formal description of the rule which underlies these examples. We will use the operad $\mathcal{J}'$ described in subsection 4.1.

Let f be a formula of type n and let i be an integer with $1 \leq i \leq n$. Let k be the valence of i in f and let j be an integer with $0 \leq j \leq k$. If $k \geq 1$ we define $\partial_{ij} f$ to be the formula obtained from f by replacing the symbol i by

$$\begin{cases} \mu \circ_2 i_{k-1} & \text{if } j = 0 \\ i_{k-1} \circ_j \mu & \text{if } 0 < j < k \\ \mu \circ_1 i_{k-1} & \text{if } j = k \end{cases}$$

Here we are writing i_k for the copy of i in S_k; also see Remark 4.1. Of course, this definition is motivated by equation (10). Here are some examples:

$$\partial_{10}\, 1(2) = 2 \smile 1 \text{ and } \partial_{11}\, 1(2) = 1 \smile 2$$

$$\partial_{10}\, 1(2,3) = 2 \smile 1(3), \quad \partial_{11}\, 1(2,3) = 1(2 \smile 3), \text{ and } \partial_{12}\, 1(2,3) = 1(2) \smile 3$$

$$\partial_{10}\, 1(2(3)) = 2(3) \smile 1, \quad \partial_{11}\, 1(2(3)) = 1 \smile 2(3)$$

$$\partial_{20}\, 1(2(3)) = 1(3 \smile 2), \quad \partial_{21}\, 1(2(3)) = 1(2 \smile 3)$$

Now let $\mathcal{I}_n$ be the set of formulas of type n. We can define a partial ordering on $\mathcal{I}_n$ as follows: given a formula f of type n, the maximal elements in the set $\{\, g \,|\, g < f \,\}$ are $\{\, \partial_{ij} f \,|\, 1 \leq i \leq n, 0 \leq j \leq v(i) \,\}$.

Let us now think of the partially ordered set $\mathcal{I}_n$ as a category in the usual way. We can make the function $\mathcal{F}$ defined in the previous subsection into a functor from $\mathcal{I}_n$ to the category of topological spaces by taking the arrow $\partial_{ij} f \to f$ to the map

$$1 \times \cdots \times d_j \times \cdots \times 1 : \prod_{k=1}^{i-1} \Delta^{v(k)} \times \Delta^{v(i)-1} \times \prod_{k=i+1}^{n} \Delta^{v(k)} \to \prod_{k=1}^{n} \Delta^{v(k)}$$

where d_j is the j-th face map

$$\Delta^{v(i)-1} \to \Delta^{v(i)}$$

Now we can give a formal definition of the space $\mathcal{F}(n)$:

Definition 4.2 $\mathcal{F}(n) = \operatorname{colim}_{\mathcal{I}_n} \mathcal{F}$

This definition implies that the boundary of a cell $\mathcal{F}_f$ is the union of the cells $\mathcal{F}_{\partial_{ij} f}$.

5 The operad structure of $\mathcal{C}'$ and the action of $\mathcal{C}'$ on $\mathrm{Tot}(X^\bullet)$.

Our next goal is to do two things: to define the operad structure of $\mathcal{C}'$ and to define the action maps

$$\theta_n : \mathcal{C}'(n) \times (\mathrm{Tot}(X^\bullet))^n \to \mathrm{Tot}(X^\bullet)$$

when $X^\bullet$ is a cosimplicial space arising from an operad with multiplication (the case where $X^\bullet$ is a cosimplicial S-module requires only routine changes and we will not discuss it separately). It turns out that the second of these tasks is easier and provides helpful background for the first, so we begin with it and defer the definition of the operad structure to the end of this section.

We will construct θ_n as a map of sets and leave it to the reader to verify continuity.

Recall that $\mathcal{C}'(n) = \mathcal{P}(n) \times \mathcal{F}(n)$. In order to construct θ_n it suffices to construct, for each choice of $\mathbf{p} = (p_1, \ldots, p_n) \in P(n)$ and of a cell $\mathcal{F}_f$ of $\mathcal{F}(n)$, a map

$$\theta_{\mathbf{p},f} : \mathcal{F}_f \times (\mathrm{Tot}(X^\bullet))^n \to \mathrm{Tot}(X^\bullet)$$

Next recall that

$$\mathrm{Tot}(X^\bullet) = \mathrm{Hom}(\Delta^\bullet, X^\bullet),$$

where Hom denotes morphisms of cosimplicial spaces, so in order to construct θ_n it suffices to construct, for each $k \geq 0$, a suitable map

$$\tilde{\theta}_{\mathbf{p},f,k} : \Delta^k \times \mathcal{F}_f \times (\mathrm{Tot}(X^\bullet))^n \to X^k.$$

We do the case $k = 0$ first, since this illustrates the general idea in a simple situation.

Example 5.1 (The case $k = 0$.) Given a formula f of type n and elements $x_i \in X^{v(i)}$ for $1 \leq i \leq n$, we can define an element $\bar{f}(x_1, \ldots, x_n) \in X^0$ by replacing each symbol i in f by x_i and then interpreting the formal compositions and cup products in f to be genuine compositions and cup products in $X^\bullet$. For example, if $f = 3(1(2 \smile 4), 6(5))$ then

$$\bar{f}(x_1, \ldots, x_6) = x_3(x_1(\mu(x_2, x_4)), x_6(x_5)).$$

This process gives a map

$$\bar{f} : \prod_{i=1}^{n} X^{v(i)} \to X^0.$$

Now let $\mathbf{s} \in \mathcal{F}_f$, and $\mathbf{a}_i \in \mathrm{Tot}(X^\bullet)$ for $1 \leq i \leq n$; thus $\mathbf{s}$ is an n-tuple $\mathbf{s}_1, \ldots, \mathbf{s}_n$ with $\mathbf{s}_i \in \Delta^{v(i)}$ and each $\mathbf{a}_i$ is a sequence $\{ a_{im} : \Delta^m \to X^m \}$. We define

$$\tilde{\theta}_{\mathbf{p},f,0}(\mathbf{s}, \mathbf{a}_1, \ldots, \mathbf{a}_n) = \bar{f}(a_{1v(1)}(\mathbf{s}_1), \ldots, a_{nv(n)}(\mathbf{s}_n)).$$

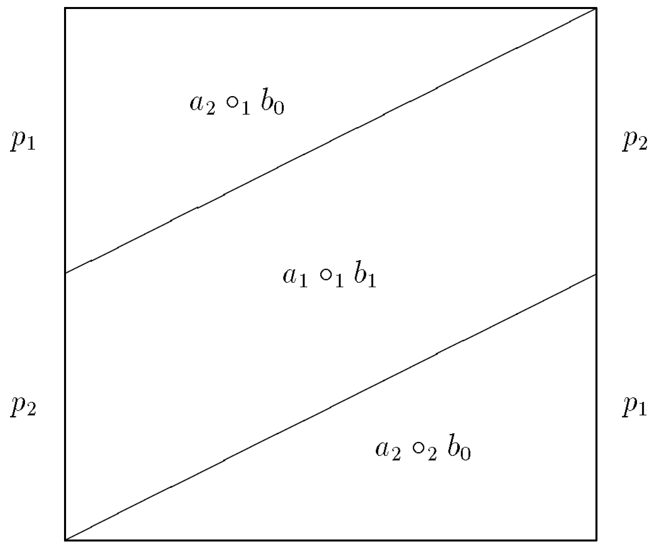
(Note that $\mathbf{p}$ plays no role in this definition, but it will have a role for $k > 0$).

In order to extend this idea to $k > 0$ we need a way of decomposing the product $\Delta^k \times \mathcal{F}_f$ as a union of products of the form $\prod_{i=1}^n \Delta^{k_i}$. Here is an example which should help the reader to follow the general description. It may also be helpful to compare this example to the relation (7) for iterated brace products given in the introduction.

Example 5.2 The following picture shows how to define the map

$$\tilde{\theta}_{\mathbf{p},f,k} : \Delta^k \times \mathcal{F}_f \times (\mathrm{Tot}(X^\bullet))^n \to X^k$$

for the case in which n is 2, f is $1(2)$, k is 1, and $\mathbf{p} = (p_1, p_2)$ is an arbitrary point of $\mathcal{P}(2)$.



Here $\mathbf{a} = \{a_m\}$ and $\mathbf{b} = \{b_m\}$ are points of $\mathrm{Tot}(X^\bullet)$, and the picture defines a map from $\Delta^1 \times \mathcal{F}_{1(2)}$ to X^1 (note: the $\mathcal{F}_{1(2)}$ coordinate is the horizontal direction). This picture is part of a homotopy from $\mathbf{b} \smile \mathbf{a}$ to $\mathbf{a} \smile \mathbf{b}$, since the left- and right-hand edges are the projections of $\mathbf{b} \smile \mathbf{a}$ and $\mathbf{a} \smile \mathbf{b}$ on $\mathrm{Hom}(\Delta^1, X^1)$.

Notice that each vertical cross-section of the picture in example 5.2 is a 3-fold prismatic subdivision of Δ^1 (except at the endpoints where it degenerates to a 2-fold prismatic subdivision). We therefore think of the picture in example 5.2 as a "fiberwise prismatic subdivision" of $\Delta^1 \times \mathcal{F}_{1(2)}$. Our next goal is to define the fiberwise prismatic subdivision of $\Delta^k \times \mathcal{F}_f$ in general, and for this we first need to define the "thickenings" of a formula f.

5.1 The thickenings of a formula.

The cells of the fiberwise prismatic subdivision of $\Delta^k \times F_f$ will correspond to a collection of formulas called the k-thickenings of f. To thicken a formula means to add extra inputs (denoted by the symbol id) to the formula (we will give a formal definition in a moment). The new formula is called a k-thickening if there are k copies of the symbol id.

Before giving the formal definition we give an example. The 1-thickenings of $1(2)$ are $1(\mathrm{id}, 2)$, $1(2, \mathrm{id})$, $1(2(\mathrm{id}))$, $1(\mathrm{id} \smile 2)$, $1(2 \smile \mathrm{id})$, $\mathrm{id} \smile 1(2))$ and $1(2) \smile \mathrm{id}$, and in Example 5.2 these correspond to the cells which are labelled by $a_2 \circ_2 b_0$, $a_2 \circ_1 b_0$, $a_1 \circ_1 b_1$, and to the two diagonal and the two horizontal edges, respectively.

For the formal definition, we use the operad $\mathcal{J}'$ defined in subsection 4.1. Let us define a formula of type (n, k) to be an element of $\mathcal{J}'(k)$ which contains each of the symbols $1, \ldots, n$ exactly once (note that such a formula is forced to contain the symbol id exactly k times). There is a reduction map

$$\rho : \mathcal{J}'(k) \to \mathcal{J}'(0)$$

which removes each copy of id, together with the typographical symbols that are immediately adjacent to it (these may be commas, parentheses, or $\smile$'s). For example, ρ takes $1(2 \smile \mathrm{id}, \mathrm{id}, 3(4(\mathrm{id})))$ to $1(2, 3(4))$.

Definition 5.3 If f is a formula of type n, the k-thickenings of f are the formulas g of type (n, k) for which $\rho(g) = f$.

5.2 Fiberwise prismatic subdivision.

Given a formula f of type n, let $\mathcal{I}_{f,k}$ be the set of k-thickenings of f. We can define a function $\mathcal{F}$ from $\mathcal{I}_{f,k}$ to the set of topological spaces exactly as in subsection 4.3, that is,

$$\mathcal{F}_g = \prod_{i=1}^{n} \Delta^{v(i)}$$

where g is in $\mathcal{I}_{f,k}$ and $v(i)$ denotes the valence of i in g.

For our present purposes we need to generalize this, using simplices of variable sizes as in Section 2 : given $\mathbf{q} = (q_1, \ldots, q_n) \in (\mathbb{R}_{\geq 0})^n$ and $g \in \mathcal{I}_{f,k}$ we define

$$\mathcal{F}_{g,\mathbf{q}} = \prod_{i=1}^{n} \Delta^{v(i)}_{q_i};$$

this gives a function $\mathcal{F}_{-,\mathbf{q}}$ from $\mathcal{I}_{f,k}$ to the set of topological spaces for each choice of $\mathbf{q}$.

Next observe that we can make $\mathcal{I}_{f,k}$ into a partially ordered set, and $\mathcal{F}_{-,\mathbf{q}}$ into a functor from $\mathcal{I}_{f,k}$ to the category of topological spaces, exactly as we did with $\mathcal{I}_n$ in subsection 4.4.

Definition 5.4 $\mathcal{F}_{f,k,\mathbf{q}}$ is the space $\operatorname{colim}_{\mathcal{I}_{f,k}} \mathcal{F}_{-,\mathbf{q}}$.

(In example 5.2, $\mathcal{F}_{1(2),(1,1),1}$ is the union of two triangles and a square, with the edge identifications indicated in the picture there.)

The fiberwise prismatic subdivision is a certain map

$$\sigma : \mathcal{F}_{f,k,\mathbf{q}} \to \Delta^k_{q_1+\cdots+q_n} \times \mathcal{F}_{f,\mathbf{q}}$$

which will turn out to be a homeomorphism. In order to describe it we need some terminology.

Let g be an element of $\mathcal{I}_{f,k}$. Then g is a character-string consisting of the integers 1,…,n together with k copies of the symbol id and some commas and parentheses. If $i \in \{1, \ldots, n\}$ and $v(i) > 0$ then there are $v(i) - 1$ commas, a left parenthesis, and a right parenthesis which *belong* to i (to be precise, the characters that belong to i are the left and right parentheses that enclose the inputs of i and the commas that separate them). If i has valence 0 then we say that i belongs to itself. Note that in either case there are $v(i) + 1$ characters which belong to i. We refer to the characters of g which belong to some i as the *eligible* characters of g; thus the eligible characters are all of the commas, all of the parentheses, and the i with valence 0.

Next let $\mathbf{s}$ be an element of $\mathcal{F}_{g,\mathbf{q}}$. Now $\mathcal{F}_{g,\mathbf{q}}$ is a product of simplices indexed by 1,...,n, and we write $\mathbf{s}_i$ for the projection of $\mathbf{s}$ on the i-th factor; thus $\mathbf{s}_i \in \Delta_{q_i}^{v(i)}$. Let us fix i temporarily. As an element of $\Delta_{q_i}^{v(i)}$, $\mathbf{s}_i$ has $v(i)+1$ coordinates, and we denote these by s_{ij}. Since there are $v(i)+1$ characters belonging to i in g, we can match the numbers s_{ij} with the characters that belong to i (going from left to right as j increases), and letting i vary we get a one-to-one correspondence between the eligible characters of g and the numbers s_{ij}, $1 \le i \le n$, $0 \le j \le v(i)$. Here is an example, for the formula $g = 3(2(1,\mathrm{id}),4,\mathrm{id},\mathrm{id},5(6))$:

$$
\begin{array}{cccccccccccccccccccccccc}
3 & (& 2 & (& 1 & , & \mathrm{id} &) & , & 4 & , & \mathrm{id} & , & \mathrm{id} & , & 5 & (& 6 &) &) \\
& s_{30} & & s_{20} & s_{10} & s_{21} & & s_{22} & s_{31} & s_{40} & s_{32} & & s_{33} & & s_{34} & & s_{50} & s_{60} & s_{51} & s_{35}
\end{array}
$$

We will refer to this diagram as the *tableau* of g. The tableau of g has two lines, with the first being the string of characters of g and the second the symbols s_{ij} in the order we have prescribed.

Now we are ready to describe the fiberwise prismatic subdivision map

$$\sigma : \mathcal{F}_{f,k,\mathbf{q}} \to \Delta^k_{q_1+\cdots+q_n} \times \mathcal{F}_{f,\mathbf{q}}$$

We write σ_2 for the projection of σ on the second factor:

$$\sigma_2 : \mathcal{F}_{f,k,\mathbf{q}} \to \mathcal{F}_{f,\mathbf{q}}$$

Since $\mathcal{F}_{f,\mathbf{q}}$ is itself a product, it suffices to define the projection of σ_2 on the i-th factor of $\mathcal{F}_{f,\mathbf{q}}$; we denote this projection by σ_{2i}. Let $g \in \mathcal{I}_{f,k}$ and let $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_n) \in \mathcal{F}_{g,\mathbf{q}}$. In order to apply σ_{2i} to $\mathbf{s}$, we select the coordinates s_{ij} from the second line of the tableau of g and then insert $+$ signs whenever the symbol id occurs in the first line. (The motivation for this is that f is obtained from g by collapsing out the symbols id that occur in g.) For example, in the tableau given above we have

$$
\begin{array}{rcl}
\sigma_{21}(\mathbf{s}) & = & s_{10} \\
\sigma_{22}(\mathbf{s}) & = & (s_{20}, s_{21} + s_{22}) \\
\sigma_{23}(\mathbf{s}) & = & (s_{30}, s_{31}, s_{32} + s_{33} + s_{34}, s_{35}) \\
\sigma_{24}(\mathbf{s}) & = & s_{40} \\
\sigma_{25}(\mathbf{s}) & = & (s_{50}, s_{51}) \\
\sigma_{26}(\mathbf{s}) & = & s_{60}
\end{array}
$$

(Note that the restriction of σ_2 to $\mathcal{F}_{g,\mathbf{q}}$ is a Cartesian product of iterated degeneracies, with the id symbols telling what degeneracies are to be used.)

Next we define the projection of σ on the first factor, which we denote by σ_1:

$$\sigma_1 : \mathcal{F}_{f,k,\mathbf{q}} \to \Delta^k_{q_1+\cdots+q_n}$$

Again let $g \in \mathcal{I}_{f,k}$ and let $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_n) \in \mathcal{F}_{g,\mathbf{q}}$. In order to apply σ_1 to $\mathbf{s}$, we insert $+$ signs in the second line of the tableau wherever the symbol id does *not* occur in the first line. For example, in the tableau given above we have $k = 3$ and

$$\sigma_1(\mathbf{s}) = (s_{30} + s_{20} + s_{10} + s_{21}, s_{22} + s_{31} + s_{40} + s_{32}, s_{33}, s_{34} + s_{50} + s_{60} + s_{51} + s_{35}).$$

When these definitions are applied to example 5.2, the formulas that result are:

- If $\mathbf{s} \in \mathcal{F}_{1(\mathrm{id},2),\mathbf{q}}$ then $\sigma_1(\mathbf{s}) = (s_{10}, s_{11} + s_{20} + s_{12})$ and $\sigma_2(\mathbf{s}) = ((s_{10} + s_{11}, s_{12}), s_{20})$.
- If $\mathbf{s} \in \mathcal{F}_{1(2,\mathrm{id}),\mathbf{q}}$ then $\sigma_1(\mathbf{s}) = (s_{10} + s_{20} + s_{11}, s_{12})$ and $\sigma_2(\mathbf{s}) = ((s_{10}, s_{11} + s_{12}), s_{20})$.
- If $\mathbf{s} \in \mathcal{F}_{1(2(\mathrm{id})),\mathbf{q}}$ then $\sigma_1(\mathbf{s}) = (s_{10} + s_{20}, s_{21} + s_{11})$ and $\sigma_2(\mathbf{s}) = ((s_{10}, s_{11}), s_{20} + s_{21})$.

Returning to the general situation, it is not difficult to see that the restriction of σ_1 to each fiber of σ_2 is a prismatic subdivision of $\Delta^k_{q_1+\cdots+q_n}$; this accounts for the name "fiberwise prismatic subdivision" and also implies

Proposition 5.5 *σ is a homeomorphism.*

Finally, it is convenient to introduce a variant of σ. Let $\mathbf{p} \in \mathcal{P}(n)$ and let f be a formula of type n. For each $g \in \mathcal{I}_{f,k}$ define

$$\cdot\mathbf{p} : \mathcal{F}_g \to \mathcal{F}_{g,\mathbf{p}}$$

to be the map which takes $(\mathbf{s}_1, \ldots, \mathbf{s}_n) \in \mathcal{F}_g$ to $(p_1\mathbf{s}_1, \ldots, p_n\mathbf{s}_n) \in \mathcal{F}_{g,\mathbf{p}}$. These maps fit together to give a map

$$\cdot\mathbf{p} : \mathcal{F}_{f,k} \to \mathcal{F}_{f,k,\mathbf{p}}$$

(where, as the reader may have guessed, $\mathcal{F}_{f,k}$ is an abbreviation for $\mathcal{F}_{f,k,(1,\ldots,1)}$). Now define

$$\sigma(\mathbf{p}) : \mathcal{F}_{f,k} \to \Delta^k \times \mathcal{F}_f$$

to be the composite

$$\mathcal{F}_{f,k} \xrightarrow{\cdot\mathbf{p}} \mathcal{F}_{f,k,\mathbf{p}} \xrightarrow{\sigma} \Delta^k \times \mathcal{F}_{f,\mathbf{p}} \xrightarrow{1\times(\cdot\mathbf{p})^{-1}} \Delta^k \times \mathcal{F}_f$$

Note that the projection of $\sigma(\mathbf{p})$ on the second factor $\mathcal{F}_f$ is the same as the projection of σ on $\mathcal{F}_f$ (that is, it is the map σ_2 defined above).

5.3 Definition of the map $\tilde{\theta}_{\mathbf{p},f,k}$

We are now ready to define the map

$$\tilde{\theta}_{\mathbf{p},f,k} : \Delta^k \times \mathcal{F}_f \times (\mathrm{Tot}(X^\bullet))^n \to X^k;$$

it is the composite

$$\Delta^k \times \mathcal{F}_f \times (\mathrm{Tot}(X^\bullet))^n \xrightarrow{\sigma(\mathbf{p})^{-1}\times 1} \mathcal{F}_{f,k} \times (\mathrm{Tot}(X^\bullet))^n \xrightarrow{\theta'_{f,k}} X^k,$$

where $\sigma(\mathbf{p})$ was defined at the end of subsection 5.2 and $\theta'_{f,k}$ is a map which we define next.

Recall that $\mathcal{F}_{f,k}$ is $\mathrm{colim}_{g\in\mathcal{I}_{f,k}} \mathcal{F}_g$. It therefore suffices to define the restriction of $\theta'_{f,k}$ to $\mathcal{F}_g$; we denote this restriction by

$$\theta'_g : \mathcal{F}_g \times (\mathrm{Tot}(X^\bullet))^n \to X^k.$$

So fix $g \in \mathcal{I}_{f,k}$. As in Example 5.1, if we are given elements $x_i \in X^{v(i)}$ for $1 \leq i \leq n$ (where $v(i)$ denotes the valence of i in g) we can define an element $\bar{g}(x_1, \ldots, x_n) \in X^k$ by replacing each symbol i in g by x_i and then interpreting the formal compositions and cup products in g to be genuine compositions and cup products in $X^\bullet$; this process gives a map

$$\bar{g} : \prod_{i=1}^{n} X^{v(i)} \to X^k.$$

Now given $\mathbf{s} \in \mathcal{F}_g$ and $\mathbf{a}_i \in \mathrm{Tot}(X^\bullet)$ for $1 \leq i \leq n$ we define

$$\theta'_g(\mathbf{s}, \mathbf{a}_1, \ldots, \mathbf{a}_n) = \bar{g}\left(a_{1v(1)}(\mathbf{s}_1), \ldots, a_{nv(n)}(\mathbf{s}_n)\right),$$

where $v(i)$ denotes the valence of i in g.

This completes the definition of the maps $\tilde{\theta}_{\mathbf{p},f,k}$. It is straightforward to check that these maps fit together to give the map $\theta_n : \mathcal{C}'(n) \times (\mathrm{Tot}(X^\bullet))^n \to \mathrm{Tot}(X^\bullet)$ that we set out to define.

5.4 The operad structure of $\mathcal{C}'$.

The operad structure of $\mathcal{C}'$ is determined by the fact that we want θ_n to be an action of $\mathcal{C}'$.

First we describe the action of the symmetric group Σ_n on $\mathcal{C}'(n)$ (we depart slightly from [18] by having the symmetric group act on the left instead of the right.) Let $\tau \in \Sigma_n$. The action of τ on the $\mathcal{P}(n)$ factor permutes the coordinates $(p_1, \ldots, p_n)$. Given a formula f of type n, we define $\tau(f)$ to be the formula obtained from f by replacing each symbol i in f by $\tau(i)$. There is an evident homeomorphism $\tau : \mathcal{F}_f \to \mathcal{F}_{\tau(f)}$ which permutes the coordinates, and passing to the colimit over f we get a homeomorphism $\tau : \mathcal{F}(n) \to \mathcal{F}(n)$.

Next we will describe the operad composition $\circ_k$. We begin by giving the collection of spaces $\mathcal{P}(n)$ an operad composition: we define

$$\circ_k : \mathcal{P}(n) \times \mathcal{P}(j) \to \mathcal{P}(n+j-1)$$

to be the map that takes the pair

$$(p_1, \ldots, p_n), (q_1, \ldots, q_j)$$

to

$$(p_1, \ldots, p_{k-1}, p_k q_1, \ldots, p_k q_j, p_{k+1}, \ldots, p_n).$$

(If we think of an element of $\mathcal{P}(n)$ as a collection of little intervals whose lengths add up to 1 then $\mathcal{P}(n)$ is a suboperad of the little intervals operad $\mathcal{C}'_1$.)

Now we want to define

$$\circ_k : (\mathcal{P}(n) \times \mathcal{F}(n)) \times (\mathcal{P}(j) \times \mathcal{F}(j)) \to (\mathcal{P}(n+j-1) \times \mathcal{F}(n+j-1))$$

for $1 \leq i \leq n$. The projection of $\circ_k$ on the $\mathcal{P}(n+j-1)$ factor is defined to be the composite

$$(\mathcal{P}(n) \times \mathcal{F}(n)) \times (\mathcal{P}(j) \times \mathcal{F}(j)) \to \mathcal{P}(n) \times \mathcal{P}(j) \to (\mathcal{P}(n+j-1),$$

where the first map is the projection and the second is the $\circ_k$ operation for the operad $\mathcal{P}$. The projection of $\circ_k$ on the $\mathcal{F}(n+j-1)$ factor is defined to be the composite

$$(\mathcal{P}(n)\times\mathcal{F}(n))\times(\mathcal{P}(j)\times\mathcal{F}(j))\to\mathcal{P}(j)\times\mathcal{F}(n)\times\mathcal{F}(j)\xrightarrow{c_{i,n,j}}\mathcal{F}(n+j-1),$$

where the first map is the projection and the map $c_{k,n,j}$ will be defined next. (This means that the operad structure of $\mathcal{C}'$ is like a semidirect product of $\mathcal{P}$ with $\mathcal{F}$, except that $\mathcal{F}$ is not an operad.)

In order to construct $c_{k,n,j}$ it suffices to construct, for each choice of $\mathbf{p}=(p_1,\ldots,p_n)\in P(n)$, each formula f of type n, and each formula f' of type j, a map

$$c_{k,\mathbf{p},f,f'}:\mathcal{F}_f\times\mathcal{F}'_f\to\mathcal{F}(n+j-1)$$

Let us write $\mathcal{F}_f^{\neq k}$ for

$$\prod_{i\neq k}\Delta^{v(i)};$$

thus $\mathcal{F}_f=\mathcal{F}_f^{\neq k}\times\Delta^{v(k)}$, where $v(k)$ denotes the valence of k in f. We define $c_{k,\mathbf{p},f,f'}$ to be the composite

$$\mathcal{F}_f\times\mathcal{F}'_f=\mathcal{F}_f^{\neq k}\times\Delta^{v(k)}\times\mathcal{F}'_f\xrightarrow{1\times\sigma(\mathbf{p})^{-1}}\mathcal{F}_f^{\neq k}\times\mathcal{F}_{f',v(k)}\xrightarrow{c'_{k,f,f'}}\mathcal{F}_{n+j-1},$$

where $c'_{k,f,f'}$ remains to be defined.

Of course, it suffices to define the restriction of $c'_{k,f,f'}$ to $\mathcal{F}_f^{\neq k}\times\mathcal{F}_{g'}$ where g' is a $v(k)$-thickening of f'; we denote this restriction by

$$c'_{k,f,g'}:\mathcal{F}_f^{\neq k}\times\mathcal{F}_{g'}\to\mathcal{F}_{n+j-1}$$

Next we define a formula $f*_k g'$ of type $n+j-1$ by "substituting g' for k," as follows: we replace the symbols $k+1,\ldots,n$ in f by $k+j,\ldots,n+j-1$ respectively, replace the symbols $1,\ldots,j$ in g' by $k,\ldots,k+j-1$ respectively, replace the symbols id in g' by the entries of k in f, and then replace k by g'. For example, if $k=1$, $f=1(2(3),4\smile 5,6(7,8)$ and $g'=1(2,\mathrm{id},3,\mathrm{id},\mathrm{id})$ then $f*_k g'=1(2,4(5),3,6\smile 7,8(9,10))$.

Observe that $\mathcal{F}_f^{\neq k}\times\mathcal{F}_{g'}=\mathcal{F}_{f*_k g'}$. Since $f*_k g'$ is a formula of type $n+j-1$, we can define $c'_{k,f,g'}$ to be the composite

$$\mathcal{F}_f^{\neq k}\times\mathcal{F}_{g'}=\mathcal{F}_{f*_k g'}\subset\mathcal{F}_{n+j-1},$$

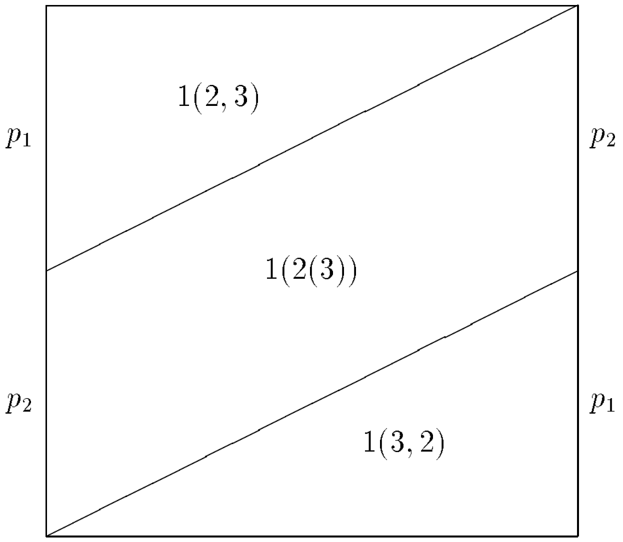
and this completes the definition of the operations $\circ_k$ for $\mathcal{C}'$.

The fact that the operations $\circ_k$ so defined make $\mathcal{C}'$ into an operad, and the fact that the maps θ_n defined earlier give an action of this operad on $\mathrm{Tot}(X^\bullet)$ when $X^\bullet$ is the cosimplicial space associated to an operad with multiplication, are both easy consequences of an associativity property of the fiberwise prismatic subdivision which will be stated and proved in the next subsection. First we pause to give two examples to illustrate the definition of $\circ_k$.

Example 5.6 Let $f=f'=1(2)$, let $\mathbf{p}=(p_1,p_2)$, and let $k=2$. Then $v(k)=0$, so there is only one g' (which is equal to f') and $c_{k,\mathbf{p},f,f'}$ is the composite

$$\mathcal{F}_f\times\mathcal{F}_{f'}=\mathcal{F}_{1(2(3))}\subset\mathcal{F}(3)$$

Example 5.7 Again let $f = f' = 1(2)$, $\mathbf{p} = (p_1, p_2)$, but now let $k = 1$. Then $v(k) = 1$, and the top-dimensional 1-thickenings g' of f' are $1(\mathrm{id}, 2)$, $1(2, \mathrm{id})$, and $1(2(\mathrm{id})$. Now $c_{k,\mathbf{p},f,f'}$ is the map of $\mathcal{F}_f \times \mathcal{F}_{f'}$ into $\mathcal{F}(3)$ indicated in the following picture.



5.5 An associativity property of the fiberwise prismatic subdivision.

Given k, n, j and l, let f be a formula of type n, f' a formula of type j, g an l-thickening of f, g' a $v_f(k)$ thickening of f' (where, of course, $v_f(k)$ denotes the valence of k in f), and h' a $v_g(k)$-thickening of f'. Note that $g *_k h'$ is an l-thickening of $f *_k g'$. Let $\mathbf{p} \in \mathcal{P}(n)$ and let $\mathbf{p}' \in \mathcal{P}(j)$.

It is straightforward to check from the definitions that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{F}_{g *_k h'} & = & \mathcal{F}_g^{\neq k} \times \mathcal{F}_{h'} \\
\sigma(\mathbf{p} \circ_k \mathbf{p}') \downarrow & & \downarrow 1 \times \sigma(\mathbf{p}') \\
\Delta^l \times \mathcal{F}_{f *_k g'} & & \mathcal{F}_g^{\neq k} \times \Delta^{v_g(k)} \times \mathcal{F}_{f'} \\
= \downarrow & & \downarrow = \\
\Delta^l \times \mathcal{F}_f^{\neq k} \times \mathcal{F}_{g'} & & \mathcal{F}_g \times \mathcal{F}_{f'} \\
1 \times 1 \times \sigma(\mathbf{p}') \downarrow & & \downarrow \sigma(\mathbf{p}) \times 1 \\
\Delta^l \times \mathcal{F}_f^{\neq k} \times \Delta^{v_f(k)} \times \mathcal{F}_{f'} & = & \Delta^l \times \mathcal{F}_f \times \mathcal{F}_{f'}
\end{array}
$$

6 The operad $\mathcal{C}$ and its action on $\mathrm{Tot}(X^\bullet)$.

In this section we modify the definition of $\mathcal{C}'$ to obtain the operad $\mathcal{C}$ of theorem 3.3.

As we have already mentioned, the reason we need to modify $\mathcal{C}'$ is that it doesn't model the degeneracy maps

$$\circ_k : \mathcal{C}_2(n) \times \mathcal{C}_2(0) \to \mathcal{C}_2(n-1)$$

of the little 2-cubes operad. Here is another way to describe the difficulty. If $X^\bullet$ is as in Theorem 3.3, the operad $\mathcal{C}'$ maps to a suboperad of the endomorphism operad of $\mathrm{Tot}(X^\bullet)$, but this suboperad is not closed under the degeneracy maps of this endomorphism operad (these are the maps that insert one or more copies of the basepoint). We will remedy this defect by adjoining new points to each $\mathcal{C}'(n-1)$ which will serve as the degeneracies of points in $\mathcal{C}'(n)$.

Before proceeding we need to specify the basepoint of $\mathrm{Tot}(X^\bullet)$. The definition of operad with multiplication provides a special element $e \in X^0$. The conditions (12) and (13) of Section 3 imply that, for each n, all n-fold iterated cofaces of e are equal, so there is a unique cosimplicial map from $\Delta^\bullet$ to $X^\bullet$ which is constant on each Δ^n and takes Δ^0 to e. This cosimplicial map is the basepoint of $\mathrm{Tot}(X^\bullet)$; we will denote it by $\bar{e}$ from now on.

Remark 6.1 For later use we describe $\bar{e}$ more explicitly. The projection of $\bar{e}$ on $\mathrm{Hom}(\Delta^m, X^m)$ will be denoted by $\bar{e}_m$. It is a constant map whose image is $e \in X^0$ for $m = 0$, $\mathrm{id} \in X^1$ for $m = 1$, $\mu \in X^2$ for $m = 2$, and for $m \geq 3$ its image is the "iterated multiplication" $\mu \circ_1 (\mu \circ_1 (\cdots \mu) \cdots)$ in X^m.

In order to define $\mathcal{C}$ we follow the general outline of sections 4 and 5. First we need to define an "indexing" operad $\mathcal{J}$ analogous to the operad $\mathcal{J}'$ defined in section 4. Define a sequence of sets $\{S_m\}$ for $m \geq 0$ as follows: S_m is the set of positive integers with the symbol ε adjoined when $m \neq 2$ and S_2 is the set of positive integers with the symbols ε and μ adjoined. Take the free operad generated by the sequence of sets S_m and impose the relation (12) from Section 3. Let the resulting operad be denoted by $\mathcal{J}$.

Let us write $\mathcal{I}_n^m$ for the set of elements of $\mathcal{J}(0)$ which contain each of the symbols $1, \ldots, n$ exactly once and the symbol ε exactly m times.

If $f \in \mathcal{I}_n^k$, we define the valence of each of the symbols $1, \ldots, n$ and of each of the copies of ε in the usual way, as the number of entries of the symbol in question. We define $\mathcal{F}_f$ to be

$$\prod_{i=1}^{n} \Delta^{v(i)} \times \prod_{\varepsilon \in f} \Delta^{v(\varepsilon)}$$

where the second product is indexed by the copies of ε in f.

We can define a partial ordering on $\mathcal{I}_n^m$ and make $\mathcal{F}$ into a functor from $\mathcal{I}_n^m$ to the category of topological spaces exactly as in subsection 4.4. We will denote the space $\mathrm{colim}_{\mathcal{I}_n^m} \mathcal{F}$ by $\mathcal{F}(n,m)$.

Next we define $\mathcal{P}(n,m)$ to be

$$\{ (p_1, \ldots, p_n, q_1, \ldots, q_m) \,|\, p_i > 0,\ q_i \geq 0, \sum p_i + \sum q_i = 1 \}$$

Next we define the spaces $\mathcal{C}(n)$. We define $\mathcal{C}(0)$ to be a point. For $n > 0$ the space $\mathcal{C}(n)$ that we are seeking to define is a quotient

$$\left(\bigcup_{m\geq 0} P(n,m)\times F(n,m)\right)/\sim$$

and our next task is to define the equivalence relation $\sim$. If $f\in\mathcal{I}_n^m$ and $1\leq k\leq m$ we write f_k for the formula obtained from f by "pruning" the k-th copy of ε in f (counting from left to right). What this means is that if the copy of ε has valence j with $j>0$ we replace that ε by $\bar{e}_j$ (see Remark 6.1). If the copy of ε has no entries then we remove it, together with the typographical symbols immediately adjacent to it (these can be commas, parentheses, or $\smile$'s). Now let $f\in\mathcal{I}_n^m$, and let

$$\mathbf{x}=(\mathbf{p},\mathbf{q}),(\mathbf{s},\mathbf{t})$$

be a point of $\mathcal{P}(n,m)\times\mathcal{F}_f$. If $q_k=0$ then $\mathbf{x}$ will be equivalent to a point of $\mathcal{P}(n,m-1)\times\mathcal{F}_{f_k}$. There are three cases. If the k-th copy of ε in f has valence 0 and if this ε is the i-th entry of the integer k' then $\mathbf{x}$ is equivalent to the point

$$(\mathbf{p},q_1,\ldots,\widehat{q_k},\ldots,q_m),(\mathbf{s}_1,\ldots,s^i\mathbf{s}_{k'},\ldots,\mathbf{s}_n,\mathbf{t}_1,\ldots,\widehat{t_k},\ldots,\mathbf{t}_m)$$

of $\mathcal{P}(n,m-1)\times\mathcal{F}_{f_k}$ (where the hats indicate that the coordinates q_k and $\mathbf{t}_k$ are deleted, and s^i is the i-th degeneracy map of the simplex $\Delta^{v_f(k')}$). If the k-th copy of ε in f has valence 0 and this copy of ε is the i-th entry of the k'-th copy of ε in f then $\mathbf{x}$ is equivalent to

$$(\mathbf{p},q_1,\ldots,\widehat{q_k},\ldots,q_m),(\mathbf{s},t_1,\ldots,s^i\mathbf{t}_{k'},\ldots,\widehat{t_k},\ldots,\mathbf{t}_m).$$

Otherwise $\mathbf{x}$ is equivalent to

$$(\mathbf{p},q_1,\ldots,\widehat{q_k},\ldots,q_m),(\mathbf{s},t_1,\ldots,\widehat{t_k},\ldots,\mathbf{t}_m).$$

This completes the definition of the equivalence relation $\sim$ and of the space $\mathcal{C}(n)$.

Remark 6.2 For later use we remark that $\mathcal{C}'(n)$ is a deformation retract of $\mathcal{C}(n)$ by the homotopy H_t with

$$H_t((\mathbf{p},\mathbf{q}),(\mathbf{s},\mathbf{t}))=\left(\frac{1-t\sum q_i}{\sum p_i}\mathbf{p},t\mathbf{q}\right),(\mathbf{s},\mathbf{t})$$

It remains to define the operad structure of $\mathcal{C}$ and the action of $\mathcal{C}$ on $\mathrm{Tot}(X^\bullet)$ when X is the cosimplicial space associated to an operad with multiplication.

First we define the degeneracy maps of $\mathcal{C}$. If $f\in\mathcal{I}_{n,m}$ and $1\leq k\leq n$ we write ${}_kf$ for the formula obtained from f by replacing the symbol k by ε, and then replacing $k+1,\ldots,n$ respectively by $k,\ldots,n-1$; thus ${}_kf$ is an element of $\mathcal{I}_{n-1}^{m+1}$. The degeneracy map

$$\circ_k:\mathcal{C}(n)\to\mathcal{C}(n-1)$$

takes a point

$$(\mathbf{p},\mathbf{q}),(\mathbf{s},\mathbf{t})$$

of $\mathcal{P}(n,m) \times \mathcal{F}_f$ to the point

$$((p_1, \ldots, \widehat{p_k}, \ldots, p_n), (q_1, \ldots, p_k, \ldots, q_n)), ((\mathbf{s}_1, \ldots, \widehat{\mathbf{s}_k}, \ldots, \mathbf{s}_n), (\mathbf{t}_1, \ldots, \mathbf{s}_k, \ldots, \mathbf{t}_n)),$$

of $\mathcal{P}(n-1, m+1) \times \mathcal{F}_{kf}$ (the hats mean that the coordinates p_k and $\mathbf{s}_k$ are deleted).

Notice in particular that every point of $\mathcal{C}(n)$ is an iterated degeneracy of some point in $\mathcal{C}'(n)$. This implies that the operad structure maps of $\mathcal{C}$ and its action on $\mathrm{Tot}(X^\bullet)$ are determined by the corresponding data for $\mathcal{C}'$, and it is straightforward to verify that the conditions for an operad structure and an operad action are satisfied. We are now finished with the definition of $\mathcal{C}$ and the proof of part (ii) of Theorem 3.3.

7 The chain operad of $\mathcal{C}$.

The (normalized) singular chains functor, applied to a topological operad, gives a chain operad since the shuffle map is strictly associative and commutative. In this section we show that the singular chain operad of our operad $\mathcal{C}$ is quasi-isomorphic to the chain operad $\mathcal{H}$ described in the introduction.

We begin with a general result, and for this we need some terminology. We will consider filtered chain complexes of abelian groups (all of our chain complexes will be non-negatively graded, and all of our filtrations will be increasing). We say that a chain complex X is *homologically filtered* if the homology of $F_iX/F_{i-1}X$ is concentrated in dimension i. (The motivating example is when X is the singular chain complex of a CW-complex A and F_iX is the singular chain complex of the i-skeleton of A.) If X is a homologically filtered chain complex we defined the *condensation* of X, denoted $\overline{X}$, to be the chain complex whose i-th group is $H_i(F_iX/F_{i+1}X)$ and whose i-th boundary operator is the boundary operator of the triple $(F_iX, F_{i-1}X, F_{i-2}X)$. (In the motivating example, the condensation of X is the usual cellular chain complex of A). Note for later use that the homology of $\overline{X}$ is naturally isomorphic to the homology of X (this follows from the usual spectral sequence argument).

By a *filtered chain operad* we mean a chain operad $\mathcal{D}$ together with an increasing filtration F_i of each $\mathcal{D}(n)$ such that for each n, j and k the operation

$$\circ_k : \mathcal{D}(n) \otimes \mathcal{D}(j) \to \mathcal{D}(n+j-1)$$

takes $F_i\mathcal{D}(n) \times F_{i'}\mathcal{D}(j)$ to $\mathcal{F}_{i+i'}\mathcal{D}(n+j-1)$. We say that $\mathcal{D}$ is *homologically filtered* if the filtration of each $\mathcal{D}(n)$ is homological. In this case we can define the *condensation* of $\mathcal{D}$ to be the chain operad $\overline{\mathcal{D}}$ whose n-th object is the condensation $\overline{\mathcal{D}(n)}$ and whose operad structure maps are determined by those of $\mathcal{D}$.

Theorem 7.1 *Let $\mathcal{D}$ be a homologically filtered chain operad. Then $\mathcal{D}$ is quasi-isomorphic in the category of chain operads to its condensation $\overline{\mathcal{D}}$.*

Before proving this we give some applications to our situation. Recall the chain operad $\mathcal{H}$ defined in the introduction. Let $\mathcal{H}'$ be the suboperad of $\mathcal{H}$ defined by $\mathcal{H}'(n) = \mathcal{H}(n)$ for $n > 0$ and $\mathcal{H}'(0) = 0$. We can filter $\mathcal{C}'$ by declaring $\mathcal{P}(n) \times \mathcal{F}_f$ to be in filtration $\sum v_f(i)$. It is easy to check that this induces a homological filtration of the singular chains operad of $\mathcal{C}'$, and that the condensation of this filtered chain operad is isomorphic to $\mathcal{H}'$. Now Theorem 7.1 implies

Corollary 7.2 *The singular chain operad of $\mathcal{C}'$ is quasi-isomorphic to $\mathcal{H}'$.*

Similarly, we can filter $\mathcal{C}$ by declaring $\mathcal{P}(n,m) \times \mathcal{F}_f$ to be in filtration $\sum v_f(i) + \sum v_f(\varepsilon)$, and this induces a homological filtration of the singular chain operad of $\mathcal{C}$. Now the deformation retraction defined in Remark 6.2 is filtration preserving, and thus the inclusion $\mathcal{C}'(n) \hookrightarrow \mathcal{C}(n)$ induces an isomorphism (not just a quasi-isomorphism) of condensations for all $n > 0$. Moreover, the definition of the degeneracy maps for $\mathcal{C}$ given in Section 6 shows that the degeneracy maps in the condensation of $\mathcal{C}$ are zero (because if we begin with a cell in $\mathcal{C}(n)$, apply a degeneracy, and then apply the retraction of Remark 6.2 we will end up in a lower filtration than that of the original cell). Together, these facts imply that the condensation of $\mathcal{C}$ is isomorphic to $\mathcal{H}$, and Theorem 7.1 implies

Corollary 7.3 *The singular chain operad of* $\mathcal{C}$ *is quasi-isomorphic to* $\mathcal{H}$.

We now turn to the proof of Theorem 7.1.

Let Op denote the category of non-negatively graded chain operads, let $\mathbb{N}$ be the non-negative integers, and let $Ch^{\mathbb{N}}$ be the category whose objects are sequences of chain complexes, indexed by $\mathbb{N}$, and whose morphisms are sequences of chain maps. The forgetful functor

$$Op \to Ch^{\mathbb{N}}$$

has a left adjoint, the free functor, which we will denote by

$$\Phi : Ch^{\mathbb{N}} \to Op$$

We will write S^n for the chain complex consisting of a copy of $\mathbb{Z}$ in dimension n, and B^n for the chain complex

$$\cdots 0 \to \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \to 0 \cdots$$

where the copies of $\mathbb{Z}$ are in dimensions n and $n-1$ (of course, the analogy is with the n-sphere and the n-disk). Given a sequence of sets $T = \{T_m\}_{m\geq 0}$, we will write $S^n(T)$ for the sequence of chain complexes

$$\{\bigoplus_{T_m} S^n\}_{m\geq 0}$$

and $B^n(T)$ for the sequence of chain complexes

$$\{\bigoplus_{T_m} B^n\}_{m\geq 0}$$

Now suppose we are given a chain operad $\mathcal{D}$, a sequence of sets T, and a sequence of chain maps

$$f = \{\, f_m : \bigoplus_{T_m} S^{n-1} \to \mathcal{D}(m)\,\}$$

Passing to the adjoint gives a map of chain operads

$$\tilde{f} : \Phi(S^{n-1}(T)) \to \mathcal{D}$$

and we can form a pushout in the category of chain operads:

$$\begin{array}{ccc} \Phi(S^{n-1}(T)) & \xrightarrow{\Phi\iota} & \Phi(B^n(T)) \\ \tilde{f}\downarrow & & \downarrow \\ \mathcal{D} & \to & \mathcal{D}' \end{array}$$

where ι is the inclusion $S^{n-1}(T) \hookrightarrow B^n(T)$. We then say that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by *attaching n-cells*. If we have a sequence of operads $\mathcal{D}_n$ such that $\mathcal{D}_0 = S^0(T)$ for some T and each $\mathcal{D}_n$ is obtained from $\mathcal{D}_{n-1}$ by attaching n-cells, we say that the colimit $\operatorname{colim}_n \mathcal{D}_n$ is a *CW chain operad*.

Lemma 7.4 *If* $\mathcal{D}$ *is any chain operad, there is a CW chain operad* $\mathcal{E}$ *and a quasi-isomorphism* $\mathcal{E} \to \mathcal{D}$.

The proof of the lemma is precisely analogous to that of the corresponding fact in the category of spaces, so we omit it.

We need to define what it means for two morphisms of chain operads to be chain-homotopic "through operad maps." First note that if C is a differential graded coalgebra and $\mathcal{D}$ is any chain operad we can define a new chain operad $\mathrm{Hom}(C, \mathcal{D})$ by letting the n-th object be $\mathrm{Hom}(C, \mathcal{D}(n))$ and letting the operad structure be determined by that of $\mathcal{D}$. Now let I be the simplicial chain complex of the standard 1-simplex. Let i_0 (respectively i_1) be the maps from S^0 to I corresponding to the endpoints of the 1-simplex. Then two chain-operad morphisms $f_1, f_2 : \mathcal{D} \to \mathcal{D}'$ are *operad-chain-homotopic* if there is a chain-operad morphism $H : \mathcal{D} \to \mathrm{Hom}(I, \mathcal{D}')$ with $i_1^* \circ H = f_1$ and $i_2^* \circ H = f_2$.

Next let us observe that if C is a chain complex, then we can give C the filtration F_i where $F_i(C)$ is equal to C in dimensions $\leq i$ and is 0 in dimensions $> i$. We call this the *filtration by degrees*. This is a homological filtration, and the associated condensation is isomorphic to the C that we started with.

Proposition 7.5 *Let $\mathcal{E}$ be a CW chain operad, $\mathcal{D}$ a homologically filtered chain operad, and $f : \mathcal{E} \to \mathcal{D}$ a morphism of chain operads. Give $\mathcal{E}$ the filtration by degrees. Then f is operad-chain-homotopic to a morphism f' of filtered chain operads.*

Before proving Proposition 7.5 we use it to prove Theorem 7.1. So let $\mathcal{D}$ be a chain operad with a homological filtration. By Lemma 7.4 there is a CW chain operad $\mathcal{E}$ and a quasi-isomorphism of chain operads $f : \mathcal{E} \to \mathcal{D}$. By Proposition 7.5, f is operad-chain-homotopic to a morphism f' of filtered chain operads (where $\mathcal{E}$ is given the filtration by degrees.) Now f' induces an operad morphism of condensations

$$\overline{f'} : \overline{\mathcal{E}} \to \overline{\mathcal{D}},$$

and $\overline{f'}$ is a quasi-isomorphism (because $\overline{f'}$ has the same effect in homology as f', and f' has the same effect in homology as f). But $\overline{\mathcal{E}}$ is isomorphic to $\mathcal{E}$ (since $\mathcal{E}$ was given the filtration by degrees), so we conclude that $\mathcal{E}$ is quasi-isomorphic to $\overline{\mathcal{D}}$. Since we also know that $\mathcal{E}$ is quasi-isomorphic to $\mathcal{D}$, we have finally that $\mathcal{D}$ is quasi-isomorphic to $\overline{\mathcal{D}}$, as required. □

It remains to prove Proposition 7.5. First we need two lemmas.

Lemma 7.6 *Let $\mathcal{D}$ be a filtered operad, let T be a sequence of sets, and for each m let $f_m : S^n(T_m) \to \mathcal{D}(m)$ be a chain map which lands in filtration n. Let f denote the sequence $\{f_m\}$. Then the chain-operad map*

$$\tilde{f} : \Phi(S^n(T)) \to \mathcal{D}$$

is filtration preserving when $\Phi(S^n(T)$ is given the filtration by degrees.

Proof. This is immediate from the explicit description of Φ, which may be found in [12]. □

Lemma 7.7 *Let*

$$\begin{array}{ccc} \mathcal{D}_0 & \to & \mathcal{D}_1 \\ \downarrow & & \downarrow \\ \mathcal{D}_2 & \to & \mathcal{D} \end{array}$$

be a pushout diagram in the category of chain operads. Suppose that $\mathcal{D}'$ is a filtered chain operad and that $f : \mathcal{D} \to \mathcal{D}'$ is a morphism of chain operads. Give $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}$ the filtration by degrees and suppose that f is filtration-preserving when restricted to $\mathcal{D}_1$ and $\mathcal{D}_2$. Then f is filtration-preserving.

Proof. Suppose first that the pushout diagram is obtained by applying Φ to a pushout diagram in the category $Ch^{\mathbb{N}}$. In this case the lemma follows from Lemma 7.6 by passage to adjoints. But the general case can be reduced to this case by using the fact that the pushout of

$$\begin{array}{ccc} \Phi\mathcal{D}_0 & \to & \Phi\mathcal{D}_1 \\ \downarrow & & \\ \Phi\mathcal{D}_2 & & \end{array}$$

surjects to $\mathcal{D}$. $\square$

Proof of Proposition 7.5. Let $\mathcal{E}_n$ be a sequence of chain operads, with $\mathcal{E}_n$ obtained from $\mathcal{E}_{n-1}$ by attaching n-cells, and let $\mathcal{E} = \operatorname{colim}_n \mathcal{E}_n$. Suppose inductively that we are given a homotopy $H_{n-1} : \mathcal{E}_{n-1} \to \mathrm{Hom}(I, \mathcal{D})$ such that $i_0^* \circ H_{n-1} = f$ and $i_1^* \circ H_{n-1}$ is filtration-preserving. It suffices to show that H_{n-1} extends to a homotopy $H_n : \mathcal{E}_n \to \mathrm{Hom}(I, \mathcal{D})$ such that $i_0^* \circ H_n = f$ and $i_1^* \circ H_n$ is filtration-preserving. Now $\mathcal{E}_n$ is defined by a pushout diagram

$$\begin{array}{ccc} \Phi(S^{n-1}(T)) & \xrightarrow{\Phi\iota} & \Phi(B^n(T)) \\ \tilde{g} \downarrow & & \downarrow \gamma \\ \mathcal{E}_{n-1} & \to & \mathcal{E}_n \end{array}$$

and it suffices (by Lemma 7.7) to show that $H_{n-1} \circ \tilde{g}$ extends to a homotopy

$$H' : \Phi(B^n(T)) \to \mathrm{Hom}(I, \mathcal{D})$$

such that $i_0^* \circ H' = f \circ \gamma$ and $i_1^* \circ H'$ is filtration-preserving.

Passing to adjoints, we need to find a suitable chain homotopy

$$s : B^n(T) \to \mathcal{D},$$

(where s raises degrees by 1) and we can treat each of the summands B^n in $B^n(T)$ separately, so we are in the following situation. Choose generators a and b for the two copies of $\mathbb{Z}$ in B^n such that $\partial b = a$. The homotopy H_{n-1} determines $s(a)$ and we have

$$\partial s(a) = f(a) - x,$$

where x is in filtration $n-1$. We need to define $s(b)$ in such a way that

$$\partial(s(b)) + s(a) = f(b) - y,$$

where y is in filtration n and $\partial y = x$. We know that

$$\partial f(b) = f(a),$$

and this implies that

$$\partial(f(b) - s(a)) = x.$$

The fact that $\mathcal{D}$ is homologically filtered now implies that there is an element z in filtration n with

$$\partial z = x.$$

Since $\partial z = \partial(f(b) - s(a)) = x$, we see that $f(b) - s(a) - z$ is a cycle, so (using again the fact that $\mathcal{D}$ is homologically filtered) there is a cycle z' in filtration n which is homologous to $f(b) - s(a) - z$; that is, there is an element w in dimension $n + 1$ with

$$\partial w = (f(b) - s(a) - z) - z'$$

We can now define $s(b) = w$ and let $y = z + z'$. $\square$

8 The equivalence between $\mathcal{C}$ and the little 2-cubes operad.

In this section and the next, we will prove

Theorem 8.1 *$\mathcal{C}$ and $\mathcal{C}_2$ are weakly equivalent operads.*

For the proof we use a method of Fiedorowicz [8]. Let $\mathcal{C}_1$ denote the operad of little intervals and let $\tau_i \in \Sigma_n$ denote the permutation that transposes i and $i+1$.

Proposition 8.2 *Suppose we are given an operad $\mathcal{D}$ with*

(a) *a morphism of (non-Σ) operads $I : \mathcal{C}_1 \to \mathcal{D}$,*

(b) *for each n, a point $c_n \in \mathcal{C}_1(n)$ and for each i a path α_i from $I(c_n)$ to $\tau_i I(C_n)$.*

Suppose moreover that

(c) *the universal cover of $\mathcal{D}(n)$ is contractible for each n, and*

(d) *for each n and i the paths*

$$\tau_i\tau_{i+1}(\alpha_i)\cdot\tau_i(\alpha_{i+1})\cdot\alpha_i$$

and

$$\tau_{i+1}\tau_i(\alpha_{i+1})\cdot\tau_{i+1}(\alpha_i)\cdot\alpha_{i+1}$$

are path homotopic (where $\cdot$ denotes concatenation of paths).

Then $\mathcal{D}$ is weakly equivalent as an operad to $\mathcal{C}_2$ (in the sense that there is a third operad which maps to each of them by a morphism which is a weak equivalence on each space.)

Proof of Proposition 8.2 (following Fiedorowicz). For each n let $\tilde{\mathcal{D}}(n)$ be the universal cover of the space $\mathcal{D}(n)$, and let $\pi : \tilde{\mathcal{D}}(n) \to \mathcal{D}(n)$ be the projection. We assume for simplicity that $I : \mathcal{C}_1(n) \to \mathcal{D}(n)$ is an inclusion. Since $\mathcal{C}_1(n)$ is contractible, $\pi^{-1}\mathcal{C}_1(n)$ is a disjoint union of copies of $\mathcal{C}_1(n)$; we arbitrarily choose one such copy for each n and declare that the basepoint of $\tilde{\mathcal{D}}(n)$ shall lie in it. Now that we have chosen basepoints, $\tilde{\mathcal{D}}$ has an induced structure of non-Σ operad. Next observe that hypothesis (b) determines a map $\tilde{\tau}_i : \tilde{\mathcal{D}}(n) \to \tilde{\mathcal{D}}(n)$ for each n and i, and hypothesis (d) implies that the braid relation

$$\tilde{\tau}_i\tilde{\tau}_{i+1}\tilde{\tau}_i = \tilde{\tau}_{i+1}\tilde{\tau}_i\tilde{\tau}_{i+1}$$

is satisfied, so each $\tilde{\mathcal{D}}(n)$ has an action of the braid group B_n. This makes $\tilde{\mathcal{D}}$ into a braided operad as defined by Fiedorowicz (this just means that the symmetric groups are replaced by the braid groups everywhere in the usual definition of operad). Next let $\tilde{\mathcal{C}}_2$ be the corresponding braided operad constructed from $\mathcal{C}_2$ (see [8, Example 3.1]). Since $\mathcal{C}_2(n)$ is a $K(B_n, 1)$ for each n, the spaces $\tilde{\mathcal{C}}_2(n)$ are contractible for each n, so by hypothesis (c) the projections

$$\tilde{\mathcal{D}} \times \tilde{\mathcal{C}}_2 \to \tilde{\mathcal{D}}$$

and

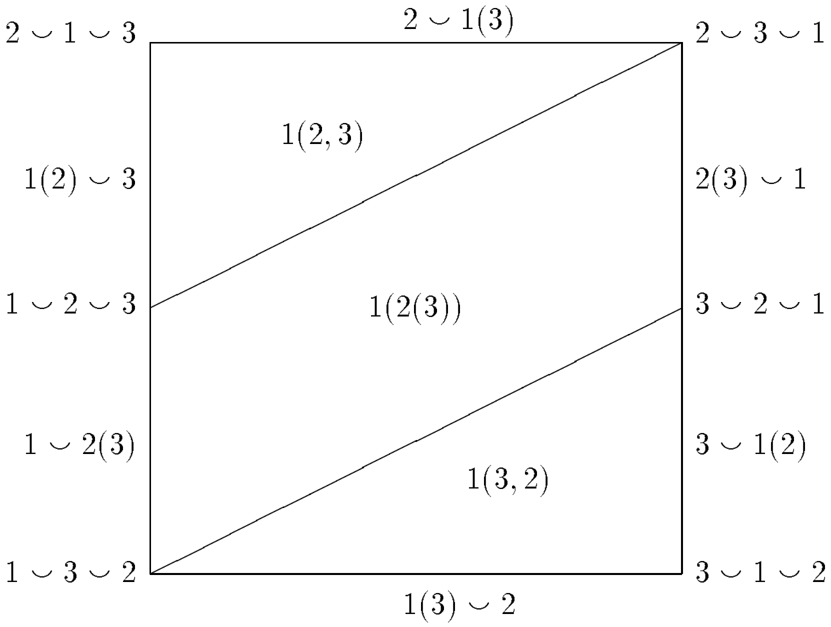
$$\tilde{\mathcal{D}} \times \tilde{\mathcal{C}}_2 \to \tilde{\mathcal{C}}_2$$

are equivalences of braided operads. If we now mod out by the action of the pure braid groups (that is, the kernels of the projections $B_n \to \Sigma_n$), we get the assertion of the proposition. $\square$.

Now we turn to the proof of Theorem 8.1. We need to show that our operad $\mathcal{C}$ satisfies the hypotheses of Proposition 8.2. For (a), let $c \in \mathcal{C}_1(n)$. Let $(p_1, \ldots, p_n$ be the lengths of the little intervals in c and let $(q_1, \ldots, q_{n+1})$ be the lengths of the gaps, and define $I(c) = (\mathbf{p}, \mathbf{q}, e \smile 1 \smile e \smile \ldots \smile n \smile e)$. For (b), we define c_n to consist of n intervals each of length $\frac{1}{n}$, and we define α_i to be the 1-simplex

$$\{c_n\} \times \mathcal{F}_{1\smile\ldots\smile(i-1)\smile i(i+1)\smile(i+2)\smile\ldots\smile n}$$

We defer the verification of (c) to the next section. For (d), assume first that $n = 3$ and $i = 1$. In this case the desired path homotopy is given by the following picture:



Here the path

$$\tau_1\tau_2(\alpha_1) \cdot \tau_1(\alpha_2) \cdot \alpha_1$$

begins at the vertex $1 \smile 2 \smile 3$ and goes clockwise to $3 \smile 2 \smile 1$ while the path

$$\tau_2\tau_1(\alpha_2) \cdot \tau_2(\alpha_1) \cdot \alpha_2$$

begins at the vertex $1 \smile 2 \smile 3$ and goes counterclockwise to $3 \smile 2 \smile 1$.

The verification of (d) for general n and i is similar: the required homotopy is again the boundary of a figure consisting of a square and two triangles, but the triangle labelled $1(2, 3)$ in the figure above is replaced by

$$1 \smile \ldots \smile (i-1) \smile i(i+1, i+2) \smile (i+3) \smile \ldots \smile n$$

and so on.

9 The homotopy type of the spaces $\mathcal{C}(n)$.

In this section we prove

Proposition 9.1 *For each $n \geq 0$ the space $\mathcal{C}(n)$ is homotopy equivalent to the space $\mathcal{C}_2(n)$.*

On passage to universal covers this will imply that hypothesis (c) of Proposition 8.2 is satisfied for the operad $\mathcal{C}$, and this will complete the proof of Theorem 8.1.

Since $\mathcal{C}(0)$ is a point we may assume $n > 0$.

The homotopy in Remark 6.2 shows that $\mathcal{C}(n)$ is homotopy equivalent to $\mathcal{F}(n)$, and it is well-known that $\mathcal{C}_2(n)$ is homotopy equivalent to the configuration space $F(n)$ of n ordered points in $\mathbb{R}^2$, so what we really need to show is that $\mathcal{F}(n)$ is weakly equivalent to $F(n)$.

The basic idea is to show that $\mathcal{F}(n)$ and $F(n)$ are both weakly equivalent to the nerve of a certain category $\mathcal{T}_n$. In order to motivate the definition of $\mathcal{T}_n$, let us observe that if f is a formula of type n then f induces a total order on the set $\{1, \ldots, n\}$ (namely the order in which these symbols appear in f as one reads from left to right) and also a partial order (generated by the relation: $i < j$ if j is an entry of i). These two orders are consistent in the sense that

(16) $\quad p \subseteq t$, and if $i < j < k$ in t and $i < k$ in p then i must be $< j$ in p.

(Here $p \subseteq t$ means that p is contained in t when both are considered as sets of ordered pairs, i.e., if $i < j$ in p then $i < j$ in t.)

Accordingly, we define $\mathcal{T}_n$ to be the following partially ordered set. The objects $\mathcal{T}_n$ are pairs (t, p), where t is a total order of the set $\{1, \ldots, n\}$ and p is a partial order of this set, subject to the consistency condition (16). In order to define the partial ordering of $\mathcal{T}_n$, let us first define t^{op} to be the ordering which is the reverse of t (that is, $i < j$ in t^{op} if and only if $j < i$ in t). Then there is a morphism in $\mathcal{T}_n$ from (t_1, p_1) to (t_2, p_2) if both of the conditions $p_1 \subseteq p_2$ and $t_2 \cap t_1^{op} \subseteq p_2$ are satisfied. (We don't know a good way to motivate this partial order on $\mathcal{T}_n$, except that it is what is needed to make the proof work. But note that if $f < f'$ in the partial order of subsection 4.4, then the pair (t, p) associated to f is $<$ the pair associated to f' in the partial order of $\mathcal{T}_n$.)

The main step in proving Proposition 9.1 is to show

Proposition 9.2 (a) *There is a functor I_n from $\mathcal{T}_n$ to spaces such that*

(i) *the spaces $I_n(t, p)$ are an open cover of $F(n)$,*

(ii) *$F(n)$ is the colimit of I_n, and*

(iii) *each $I_n(t, p)$ is contractible.*

(b) *There is a functor I'_n from $\mathcal{T}_n$ to spaces such that*

(i) *the spaces $I'_n(t, p)$ are subcomplexes of $\mathcal{F}(n)$,*

(ii) *$\mathcal{F}(n)$ is the colimit of I'_n, and*

(iii) *each $I'_n(t, p)$ is contractible.*

Proof of Proposition 9.1. Parts (a)(i) and (a)(ii), together with the proof of [22, Proposition 4.1], imply that the natural map

$$\operatorname{hocolim} I_n \to \operatorname{colim} I_n$$

is a weak equivalence. Part (a)(iii), together with the homotopy invariance of homotopy colimits ([4, XII.4.2]), implies that $\operatorname{hocolim} I_n$ is weakly equivalent to $\operatorname{hocolim} *$ (where $*$ is the constant functor that takes every object of $\mathcal{T}_n$ to a point.)

Similarly, parts (b)(i) and (b)(ii), together with [1, Proposition 6.9], imply that the natural map

$$\operatorname{hocolim} I'_n \to \operatorname{colim} I'_n$$

is a weak equivalence, and part (b)(iii) implies that $\operatorname{hocolim} I'_n$ is weakly equivalent to $\operatorname{hocolim} *$. □

We proceed to the proof of Proposition 9.2(a). Let π_1 and π_2 denote the projections of $\mathbb{R}^2$ on its two factors. Let $I_n(t,p)$ be the open subspace of $F(n)$ consisting of all configurations $\mathbf{x} = (x_1, \ldots, x_n)$ such that

if $i < j$ in t and $\pi_1(x_i) \geq \pi_1(x_j)$ then $i < j$ in p

if $i < j$ in t and $\pi_1(x_i) = \pi_1(x_j)$ then $i < j$ in p and $\pi_2(x_i) < \pi_2(x_j)$.

Part (a)(i) is evident. For part (a)(ii), since the sets $I_n(t,p)$ cover $F(n)$ it suffices to show that the map $\operatorname{colim} I_n \to F(n)$ is 1-1, and for this it suffices to show that, if $\mathbf{x}$ is in $I_n(p,t) \cap I_n(p',t')$, there is a pair (p'',t'') which is $\leq$ both (p,t) and (p',t') and is such that $\mathbf{x} \in I_n(p'',t'')$. And this in turn follows from the stronger fact that for each configuration $\mathbf{x}$ there is a unique minimal (t,p) with $\mathbf{x} \in I_n(t,p)$: define t by $i < j$ in t if and only if $x_i < x_j$ in lexicographic order, and define p by $i < j$ in p if and only if $\pi_1(x_i) = \pi_1(x_j)$ and $\pi_2(x_i) < \pi_2(x_j)$.

The proof of part (a)(iii) is by induction on n. The case $n = 1$ is obvious, so assume $n > 1$, and let $(t,p) \in \mathcal{T}_n$. To simplify the notation we treat the case where t is the standard total order $1 < 2 < \ldots < n$; the general case is precisely similar. Let t' be the standard total order of $\{1, \ldots, n-1\}$, and let p' be the restriction of p to $\{1, \ldots, n-1\}$. It suffices by induction to show that $I_n(t,p)$ is homotopy equivalent to $I_{n-1}(t',p')$. Let

$$\alpha : I_n(t,p) \to I_{n-1}(t',p')$$

be the projection map that takes $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_{n-1})$. Let

$$\beta : I_n(t,p) \to I_{n-1}(t',p')$$

take $(x_1, \ldots, x_{n-1})$ to $(x_1, \ldots, x_{n-1}, \gamma(x_1, \ldots, x_{n-1}))$, where

$$\gamma(x_1, \ldots, x_{n-1}) = (1 + \max\{\pi_1(x_i)\}, 1 + \max\{\pi_2(x_i)\}) \in \mathbb{R}^2$$

Then $\alpha \circ \beta$ is the identity map, and it suffices to show that $\beta \circ \alpha$ is homotopic to the identity map of $I_n(t,p)$. Let H be the homotopy which leaves $x_1, \ldots, x_{n-1}$ fixed and moves

x_n vertically until it has the desired second coordinate, and then horizontally until it has the desired first coordinate: that is,

$$H_t(\mathbf{x}) = \begin{cases} (x_1, \ldots, x_{n-1}, (1-2t)x_n + 2t(\pi_1(x_n), \pi_2(\gamma))) & \text{if } t \le 1/2 \\ (x_1, \ldots, x_{n-1}, (2-2t)(\pi_1(x_n), \pi_2(\gamma)) + (2t-1)\gamma) & \text{if } 1/2 \le t \end{cases}$$

where we have written γ for $\gamma(x_1, \ldots, x_{n-1})$. (To see that this homotopy stays inside $I_n(t,p)$ note that there cannot be a point in $\mathbf{x}$ directly above x_n, and if x_i is to the right of x_n we must have $i < n$ in p.) This concludes the proof of part (a) of Proposition 9.2.

For part (b), let $I'_n(t,p)$ be the following subcomplex of $\mathcal{F}(n)$:

$$I'_n(t,p) = \bigcup_{\{f \mid (t_f, p_f) \le (t,p)\}} \mathcal{F}_f$$

where (t_f, p_f) is the pair determined by the formula f and $\mathcal{F}_f$ is as in subsection 4.3.

Part (b)(i) is evident. For part (b)(ii), since the sets $I'_n(t,p)$ cover $\mathcal{F}(n)$ it suffices to show that the map $\operatorname{colim} I'_n \to \mathcal{F}(n)$ is 1-1, and for this it suffices to show that, if $\mathcal{F}_f$ is contained in $I'_n(p,t) \cap I'_n(p',t')$, there is a pair (p'',t'') which is $\le$ both (p,t) and (p',t') and is such that $\mathcal{F}_f \subseteq I'_n(p'',t'')$. And this in turn follows from the stronger fact that for each formula f the pair (t_f, p_f) is the unique minimal pair whose image under I_n contains $\mathcal{F}_f$.

We now turn to the proof of (a)(iii), so let us fix a pair (t,p). If $i \in \{1, \ldots, n\}$, we write $h(i)$ (the height of i with respect to p) for the length of the longest chain $j_1 < \ldots < i$ in p, and we write $h(p)$ (the height of p) for the length of the longest chain in p. We write $w(p)$ (the width of p) for the number of elements $i \in \{1, \ldots, n\}$ with $h(i) = h(p)$. The proof of (a)(iii) will be by double induction on $h(p)$ and $w(p)$.

Let us fix an element i with $h(i) = h(p) - 1$. There are two cases:

Case 1 There is only one j with $i < j$ in p.

Case 2 There is more than one j with $i < j$ in p.

We begin with Case 1. In this case, condition (16) implies that j is the immediate successor of i in the total order t.

We define three subcomplexes of $I_n(p,t)$. It will be convenient to write $f \le (t,p)$ to mean $(t_f, p_f) \le (t,p)$.

$A = \bigcup_{f \in S_1} \mathcal{F}_f$, where $S_1 = \{ f \le (t,p) \mid f$ contains one of the strings $i(j)$, $i \smile j$ or $j \smile i\}$.

$B = \bigcup_{f \in S_2} \mathcal{F}_f$, where $S_2 = \{ f \le (t,p) \mid i$ is to the left of j in f but j is not an entry of i in $f\}$

$C = \bigcup_{f \in S_3} \mathcal{F}_f$, where $S_3 = \{ f \le (t,p) \mid i$ is to the right of j in $f\}$.

Clearly $I_n(t,p) = A \cup B \cup C$. We will show that $I_n(t,p)$ is contractible by showing that A, B, C, $A \cap B$, and $A \cap C$ are contractible and that $B \cap C$ is empty.

It is easy to see that $B = I_n(t, p - (i,j))$ (where $p - (i,j)$ means the partial order which is the same as p except that i is no longer $< j$; this is indeed a partial order since j is the immediate successor of i in p) so B is contractible by the inductive hypothesis. Similarly, $C = I_n(\bar{t}, p - (i,j))$ (where $\bar{t}$ is the same as t except that i and j are switched; this is indeed a total order since j is the immediate successor of i in t) so C is contractible by the inductive hypothesis. It is also clear that $B \cap C$ is empty.

Next we claim that A is contractible. Let ${}_jt$ (respectively ${}_jp$) be the restriction of t (respectively, p) to $\{1, \ldots, j-1, j+1, \ldots, n\}$. If i has valence 0 in g let us write $g *_i h$ to mean "plug h in for i" (this is a little different from the use of this symbol in subsection 5.4). Then $S_1 = \{g *_i h \,|\, g \leq ({}_jt, {}_jp) \text{ and } h \leq i(j)\}$. This implies that $A = I_{n-1}({}_jt, {}_jp) \times \Delta^1$, and so A is contractible by the inductive hypothesis.

Similarly, $S_1 \cap S_2 = \{g *_i h \,|\, g \leq ({}_jt, {}_jp) \text{ and } h = i \smile j\}$. This implies that $A \cap B = I_{n-1}({}_jt, {}_jp) \times \Delta^0$ which is contractible by the inductive hypothesis. Also, $S_1 \cap S_3 = \{g *_i h \,|\, g \leq ({}_jt, {}_jp) \text{ and } h = j \smile i\}$. This implies that $A \cap B = I_{n-1}({}_jt, {}_jp) \times \Delta^0$ which is contractible by the inductive hypothesis. This concludes the proof of Case 1.

For Case 2, let $j_1, \ldots, j_m$ be the successors of i in p. We may assume that $j_1 < j_2 < \ldots < j_m$ in t. Condition (16) then implies that $i, j_1, \ldots, j_m$ is a consecutive sequence in t (that is, j_1 is the immediate successor of i, etc.).

Let t' (respectively p') denote the restriction of t (respectively p) to $\{1, \ldots, n\} - \{j_1, \ldots, j_m\}$.

Now define subcomplexes A, B, and C of $I_n(t, p)$ by

$A = \bigcup_{f \in S_1} \mathcal{F}_f$, where $S_1 = \{g *_i h \,|\, g \leq (t', p') \text{ and } h \leq i(j_1, \ldots, j_m)\ \}$.

$B = \bigcup_{f \in S_2} \mathcal{F}_f$, where $S_2 = \{\ f \,|\, j_1 \text{ is to the left of } i \text{ in } f\}$.

$C = \bigcup_{f \in S_3} \mathcal{F}_f$, where $S_3 = \{\ f \,|\, j_m \text{ is to the right of } i \text{ in } f \text{ but is not an entry of } i\}$.

Then $A \cup B \cup C = I_n(p, t)$, because if f is not in S_2 or S_3 then j_1 will be to the right of i, j_m will be an entry of i, and this will imply that f contains the string $i(j_1, \ldots, j_m)$ so that $f \in S_1$.

We will show that $I_n(t, p)$ is contractible by showing that A, B, C, $A \cap B$, $A \cap C$, $B \cap C$ and $A \cap B \cap C$ are all contractible.

First of all, B is contractible by induction because it is equal to $I_n(\bar{t}, p - (i, j_1))$ (where $\bar{t}$ is the same as t but with i and j_1 switched). Similarly, C is contractible because it is equal to $I_n(t, p - (i, j_m))$, and $B \cap C$ is contractible because it is equal to $I_n(\bar{t}, p - (i, j_1) - (i, j_s))$.

Next, A is contractible because it is homeomorphic to $I_{n-m}(t', p') \times \Delta^m$. Similarly, we have

- $S_1 \cap S_2 = \{g *_i h \,|\, g \leq (t', p') \text{ and } h \leq j_1 \smile i(j_2, \ldots, j_m)\ \}$, and so $A \cap B \approx I_{n-m}(t', p') \times \Delta^{m-1}$.

- $S_1 \cap S_3 = \{g *_i h \,|\, g \leq (t', p') \text{ and } h \leq i(j_1, \ldots, j_{m-1}) \smile j_m\ \}$, and so $A \cap C \approx I_{n-m}(t', p') \times \Delta^{m-1}$.

- $S_1 \cap S_2 \cap S_3 = \{g *_i h \,|\, g \leq (t', p') \text{ and } h \leq j_1 \smile i(j_2, \ldots, j_{m-1}) \smile j_m\ \}$, and so $A \cap B \cap C \approx I_{n-m}(t', p') \times \Delta^{m-2}$.

This concludes the proof of Case 2. $\square$

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