Krull-Remak-Schmidt fails for artinian modules over local rings

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Dedicated to Klaus W. Roggenkamp on his 60th birthday

Let R be any ring. The theorem of Krull-Remak-Schmidt is one of the basic results in representation theory [RH]. However, the usual assumption of dealing with R-modules of finite length is very special and one would be happy if one could extend its validity, for example to artinian modules over suitable rings. In 1932, Krull [K] asked whether direct decompositions of a general artinian R-module M into indecomposables are unique up to isomorphisms. In 1995, this question was answered in the negative in a joint paper of Facchini, Herbera, Levy and Vámos [FHLV]. For a comprehensive treatment, we refer to Facchini's recent book *Module Theory* [F]. In this book, Facchini recalls the positive answers in case R is noetherian or commutative, and he raises the question (Problem 11) whether one also obtains a positive answer in case R is local. The aim of this note is to show that this is not the case: we are going to present a local ring R and artinian R-modules M with essentially different direct decompositions into indecomposables. Our treatment is based on recent considerations by Pimenov and Yakovlev [PY] concerning the general Krull problem, only two slight modifications are necessary. We refrain to discuss the military importance of the result presented here; the interested reader is referred to the forthcoming Nato-meeting at Constantia, August 2000, organized by K.W.Roggenkamp, which will deal with the use of methods from ring and representation theory.

1. The Pimenov-Yakovlev setting. The Pimenov-Yakovlev approach can be phrased as follows: start with the category $\mathcal{F}$ of reduced torsionfree abelian groups F of finite rank such that $pF = F$ for almost all primes and denote by $\mathcal{A}$ the category of minimal injective resolutions of the groups in $\mathcal{F}$. Now, first, it is well-known that in $\mathcal{F}$ not all decompositions into indecomposables are unique up to isomorphism [J,Fu]. Second, it is easy to see that the categories $\mathcal{F}$ and $\mathcal{A}$ are equivalent, thus Krull-Remak-Schmidt also fails for the objects in $\mathcal{A}$. Finally, and this had been overlooked before, we may interprete $\mathcal{A}$ as a category of modules over the ring

$$T = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix},$$

and all the T-modules obtained in this way are artinian.

Let us present this approach in more detail, but for our further investigations it will be appropriate to consider $K[X]$-modules instead of abelian groups, here $K[X]$ is the polynomial ring in one variable over a field K. Of course, in the same way as one constructs torsionfree abelian groups with essentially different direct decompositions into indecomposables, one can work with modules over an arbitrary principal ideal domain provided it has sufficiently many maximal ideals. Let $\mathcal{P}$ be a finite set of elements of K, let n be its cardinality, let us assume that $n \geq 4$. We denote by $\mathcal{P}'$ the set of irreducible polynomials p in $K[X]$ which are not of the form $X - a$ with $a \in \mathcal{P}$. Let $\mathcal{F}$ be the category of all reduced torsionfree $K[X]$-modules F of finite rank such that $pF = F$ for all irreducible polynomials $p \in \mathcal{P}'$ (recall that a torsionfree $K[X]$-module F is reduced if the $K[X]$-module $K(X)$ cannot be embedded into F). This is an additive category where Krull-Remak-Schmidt fails. In order to construct a category $\mathcal{A}$ of artinian modules over some ring T such that $\mathcal{A}$ is equivalent to $\mathcal{F}$, take the matrix ring

$$T = \begin{bmatrix} K(X) & 0 \\ K(X) & K[X] \end{bmatrix}.$$

Note that the T-modules can be written as columns of the form $[N, N']^t$; actually, in order to specify such a module, we need a triple (N, N', γ) where N is a $K(X)$-space, N' a $K[X]$-module, and $\gamma\colon N \to N'$ a $K[X]$-linear map. Given a T-module $A = [N, N']^t$, we may consider its submodule $B = [0, N']^t$, this is a $K[X]$-module. Always, A/B is a semisimple T-module, thus A is artinian if and only if A/B is of finite length and B is artinian.

Let $\mathcal{A}$ be the category of all T-modules given by triples (N, N', γ), where N is a finite dimensional $K(X)$-vector space, N' is a torsion $K[X]$-module which is annihilated by all irreducible polynomials $p \in \mathcal{P}'$ and γ is surjective, such that, moreover, the kernel of γ is reduced. Note that all the T-modules in $\mathcal{A}$ are artinian.

Given a module F in $\mathcal{F}$, let $I(F)$ be its injective hull, and consider the canonical projection $\pi\colon I(F) \to I(F)/F$. Note that $I(F)$ is a finite dimensional vector space over $K(X)$, thus the triple $(I(F), I(F)/F, \pi)$ yields a T-module. Of course, $I(F)/F$ is a torsion $K[X]$-module and since we assume that $pF = F$ for all irreducible polynomials $p \in \mathcal{P}'$, it follows that the triple $(I(F), I(F)/F, \pi)$ belongs to $\mathcal{A}$.

2. First modification. Let $\mathcal{P} = \{a_1, \dots, a_n\}$ and consider the linear polynomials $p_i = X - a_i$. Let $V_i = K[X]_{(a_i)}$ be the localization of $K[X]$ at the maximal ideal $\mathbf{m}_i$ generated by p_i, for $1 \leq i \leq n$. Let us consider the $(n+1) \times (n+1)$ matrix ring

$$S = \begin{bmatrix} K(X) & 0 & 0 & \cdots & 0 \\ K(X) & V_1 & 0 & \cdots & 0 \\ K(X) & 0 & V_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ K(X) & 0 & \cdots & 0 & V_n \end{bmatrix}$$

The S-modules can be written as columns of the form $[N, N_1, \dots, N_n]^t$; in order to do so, we need the data $(N; N_1, \dots, N_n; \gamma_1, \dots, \gamma_n)$; here, N is a $K(X)$-vector space, N_i a V_i-module and $\gamma_i \colon N \to N_i$ a V_i-linear map. The map γ_i is just the multiplication with the matrix unit $e_{i+1,1}$. Given an S-module $A = [N, N_1, \dots, N_n]^t$ we may consider its submodule $B = [0, N_1, \dots, N_n]$, this is a $\prod V_i$-module. Note that A/B is a semisimple S-module, thus A is artinian if and only if A/B is of finite length and B is artinian.

The first column of S is given by the data $(K(X); K(X), \dots, K(X); 1, \dots, 1)$, we denote this S-module by $P(1)$. This is an indecomposable projective S-module, its radical is $\operatorname{rad} P(1) = (0; K(X), \dots, K(X); 1, \dots, 1)$ and the simple S-module $E(1) = P(1)/\operatorname{rad} P(1)$ is injective. Let us denote by $\mathcal{A}$ the set of S-modules M which are artinian, finitely generated by $P(1)$ and such that the simple S-module $E(1)$ cannot be embedded into M. We claim that $\mathcal{F}$ and $\mathcal{A}$ are equivalent categories, an equivalence being given by the following functor Φ_0. Given a module F in $\mathcal{F}$, let $I(F)$ be its injective hull, this is a finite dimensional vector space over $K(X)$. Of course, $I(F)/F$ is a divisible torsion $K[X]$-module, and we may write $I(F)/F$ as the direct sum of its p-torsion submodules N_p where p is an irreducible polynomial. Recall that we assume $pF = F$ for $p \in \mathcal{P}'$, thus we have $N_p \neq 0$ only in case $p = p_i$ for some i and then we write N_i instead of N_p. Of course, N_i can be considered as a V_i-module and the canonical projection $\pi_i \colon I(F) \to N_i$ is V_i-linear. Thus

$$\Phi_0(F) \;=\; (I(F); N_1, \dots, N_n; \pi_1, \dots, \pi_n)$$

is an S-module, it is finitely generated by $P(1)$ and artinian. Since F is reduced, it follows that the simple S-module $E(1)$ cannot be embedded into $\Phi_0(F)$. Thus $\Phi_0(F)$ belongs to $\mathcal{A}$.

3. Second modification. Now, let us determine the Jacobson radical $J(S) = \operatorname{rad} S$ of S. For $1 \le i \le n$, consider the elements

$$t_i = e_{i+1,1} \quad \text{and} \quad s_i = p_i e_{i+1,i+1},$$

they generate the ideal $J(S)$. Note that $S/J(S) = K(X) \times K \times \cdots \times K$.

Our next aim is to replace S by a local ring R. *Let us assume now that K and $K(X)$ are isomorphic (as fields);* for example, we may start with an arbitrary field k and take as K the field of rational functions in infinitely many variables. Using this assumption, there exists a local subring R of S which contains $J(S)$ such that all the simple S-modules remain simple when considered as R-modules.

Here is the precise construction of R. Let $a_i \in \mathcal{P}$ and recall that $\mathbf{m}_i$ denotes the maximal ideal of $K[X]$ generated by p_i. By assumption, $K[X]/\mathbf{m}_i = K$ is isomorphic as a field to $K(X)$, thus there exists a surjective ring homomorphism $\epsilon_i \colon V_i \to K(X)$ with kernel $\mathbf{m}_i$. Let R be the subring of S of all entries of the form

$$\begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ a_1 & b_1 & 0 & \cdots & 0 \\ a_2 & 0 & b_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_n & 0 & \cdots & 0 & b_n \end{bmatrix} \qquad \begin{array}{ll} \text{with} & a, a_i \in K(X),\ b_i \in V_i \\ \text{such that} & \epsilon_i(b_i) = a. \end{array}$$

Given an S-module M, we may consider M as an R-module; let us denote this functor by $\iota\colon \operatorname{Mod} S \to \operatorname{Mod} R$. In particular, we may consider the S-modules in $\mathcal{A}$ as R-modules using ι. Let $\Phi = \iota \circ \Phi_0$.

Lemma. *Let $F \in \mathcal{F}$ and $M = \Phi(F)$. Then M is a finitely generated artinian R-module and* $\operatorname{soc}_R M \subseteq \operatorname{rad}_R M$.

Proof. As we know, the S-module $\Phi_0(F)$ is finitely generated by $P(1)$, thus the R-module M is finitely generated by $\iota P(1)$. Note that $\iota P(1)$ is a factor module of ${}_RR$, namely $\iota P(1) = {}_RR/\bigoplus_{i=1}^n \mathbf{m}_i$, thus M is finitely generated.

Next, let us show that M is artinian. As we know, the S-module $\Phi_0(F)$ has a submodule which is a finite direct sum of S-modules N_i whose proper S-submodules form an $\mathbb{N}$-chain (a strictly ascending chain $0 = M_0 \subset M_1 \subset M_2 \subset \cdots$ indexed by the natural numbers) and such that $\Phi_0(F)/\bigoplus N_i$ is semisimple and of finite length. The modules ιN_i have no additional submodules when considered as R-modules, thus they are artinian also as R-modules, therefore $\bigoplus \iota N_i$ is an artinian R-module. The factor module $\Phi(F)/\bigoplus \iota N_i$ has no additional submodules, thus it is semisimple and of finite length. This shows that M is artinian.

Finally, also the property $\operatorname{soc} M \subseteq \operatorname{rad} M$ is inherited from the S-module structure. This completes the proof.

Corollary. *Let $F \in \mathcal{F}$ and $M = \Phi(F)$. Let f be an endomorphism of M with semisimple image. Then f belongs to the Jacobson radical of* $\operatorname{End}_R(M)$.

Proof: First, let us note that $\operatorname{soc} M$ is essential in M, since M is artinian, and $\operatorname{rad} M$ is small in M, since M is a finitely generated R-module and R is local.

Now, consider any endomorphism f of M with semisimple image and let us show that $1+f$ is invertible. This implies that f belongs to the Jacobson radical, since the twosided ideal generated by f contains only endomorphisms with semisimple images. Now $\operatorname{Im}(1+f) + \operatorname{Im}(f) = M$. Since $\operatorname{Im}(F) \subseteq \operatorname{soc} M \subseteq \operatorname{rad} M$ and $\operatorname{rad} M$ is small, we see that $\operatorname{Im}(1+f) = M$. Similarly, $\operatorname{Ker}(1+f) \cap \operatorname{Ker}(f) = 0$. Since $\operatorname{Ker}(f) \supseteq \operatorname{rad} M \supseteq \operatorname{soc} M$ and $\operatorname{soc} M$ is essential, it follows that $\operatorname{Ker}(1+f) = 0$. Altogether we have shown that $1+f$ is invertible.

Let us consider an R-module M. Note that $t_it_j = 0 = t_is_j$ for all i,j, thus

$$\operatorname{rad} M \subseteq \operatorname{Ker}(t_i),$$

for all i, and we may define

$$\Psi(M) = \left(\bigcap\nolimits_{i=1}^{n} \operatorname{Ker}(t_i)\right) / \operatorname{rad} M,$$

this is a subgroup of $M/\operatorname{rad} M$. An R-module homomorphisms $f\colon M \to M'$ induces an $R/J(R)$-module homomorphism $f^*\colon M/\operatorname{rad} M \to M'/\operatorname{rad} M'$. Note that $R/J(R) = K(X)$, thus f^* is $K(X)$-linear. In general, the subgroup $\Psi(M)$ will not be a $K(X)$-submodule of $M/\operatorname{rad} M$, but at least a $K[X]$-submodule, as the following lemma shows:

Lemma. *Let M, M' be R-modules, let $f\colon M \to M'$ be an R-module homomorphism. The subset $\Psi(M) \subseteq M/\operatorname{rad} M$ is a $K[X]$-submodule and the restriction $\Psi(f)$ of f^* is $K[X]$-linear.*

Proof: Consider the ring D of diagonal matrices which belong to R. This ring D is a pullback of the rings $V_1, \dots, V_n$, it is a local ring with radical $\prod_{j=1}^{n} \mathbf{m}_j$. Let $d_i(a)$ be an element of D with entry a in the position $(i+1, i+1)$, for $0 \le i \le n$.

Let M be any R-module. First of all, using the inclusion of D into R, the ring D operates on $M/\operatorname{rad} M$, and the product $\prod \mathbf{m}_j$ annihilates $M/\operatorname{rad} M$. Now $D/\prod_{j\neq i} \mathbf{m}_j = K(X)$, in this way, M/rad is a $K(X)$-module and this is the $K(X)$-structure of $M/\operatorname{rad} M$ mentioned before, we denote by $*$ this scalar multiplication $K(X) \otimes M/\operatorname{rad} M \to M/\operatorname{rad} M$; actually, we will need its restriction to $V_i \otimes M/\operatorname{rad} M$ which again will be denoted by $*$. Given $x \in M$ and $\overline{x} = x + \operatorname{rad} M$, we have $a * \overline{x} = \overline{d_0(a) \cdot x}$, for any $a \in K(X)$.

Second, let $1 \le i \le n$ and consider t_iM as a D-module (again via the inclusion of D into R). The factors $\mathbf{m}_j$ with $j \neq i$ annihilate t_iM, and $D/\prod_{j\neq i} \mathbf{m}_j = V_i$, thus t_iM is a V_i-module. We denote by $*_i$ this scalar product $V_i \otimes t_iM \to t_iM$.

We claim that the multiplication map $t_i\colon M/\operatorname{rad} M \to t_iM$ commutes with the V_i-module structures on $M/\operatorname{rad} M$ and t_iM. Let $a \in V_i$ and $x \in M$, then $a *_i t_i m = d_i(a) \cdot t_i m$. Note that for $a \in V_i$, we have $t_i \cdot d_0(a) = d_i(a) \cdot t_i$ in R (this is the matrix with entry a in the position $(i+1, 1)$ and zeroes elsewhere). Altogether we see that

$$t_i(a * x) = t_i \cdot \overline{d_0(a) \cdot x} = t_i \cdot d_0(a) \cdot x = d_i(a) \cdot t_i \cdot x = a *_i t_i m.$$

This shows that the multiplication with t_i yields a V_i-linear map $M/\operatorname{rad} M \to t_iM$ and its kernel has to be a V_i-submodule of $M/\operatorname{rad} M$. In particular, this kernel is a $K[X]$-submodule of $M/\operatorname{rad} M$, thus also $\Psi(M)$ is a $K[X]$-submodule of $M/\operatorname{rad} M$. Of course, if $f\colon M \to M'$ is R-linear, then f commutes with the multiplication by t_i. As a consequence, $\Psi(f)$ is $K[X]$-linear. This completes the proof.

In this way, we obtain a functor

$$\Psi\colon \operatorname{Mod} R \to \operatorname{Mod} K[X],$$

note that the image of any R-module under Ψ is a torsionfree $K[X]$-module.

Theorem. *The composition $\Psi \circ \Phi$ is the identity of $\mathcal{F}$. If F, F' belong to $\mathcal{F}$ and $f\colon \Phi(F) \to \Phi(F')$ is an R-linear map with $\Psi(f) = 0$, then the image of f is semisimple.*

Proof. Let $F \in \mathcal{F}$ and $M = \Phi(F) = (I(F), N_1, \dots, N_n, \pi_1, \dots, \pi_n)$. Then $I(F) = M/\operatorname{rad} M$ and $\pi_i\colon I(F) \to N_i$ is just the multiplication map $t_i\colon M/\operatorname{rad} M \to t_i M = N_i$. It follows that $F = \bigcap \operatorname{Ker} \pi_i = \Psi(M)$.

Consider $K[X]$-modules F, F' which belong to $\mathcal{F}$. Let $M = \Phi(F)$, $M' = \Phi(F')$ and take an R-linear map $f\colon M \to M'$ with $\Psi(f) = 0$. We are going to show that $f(\operatorname{rad} M) = 0$. Recall that $F \subseteq M/\operatorname{rad} M$ and $K(X)F = M/\operatorname{rad} M$. Since f^* is $K(X)$-linear, we see that $f^* = 0$. But this implies that $f(M) \subseteq \operatorname{rad} M'$.

Since M is finitely generated, the same is true for $f(M)$. Now let us consider the structure of $\operatorname{rad} M$ and $\operatorname{rad} M'$. These modules are direct sums of indecomposable S-modules whose proper submodules form an $\mathbb{N}$-chain. We observe the following: Since $\operatorname{rad} M'$ is a direct sum of modules whose proper submodules form an $\mathbb{N}$-chain, any finitely generated submodule of $\operatorname{rad} M'$ has finite length. This shows that $f(M)$ has finite length. Next, a module whose proper submodules form an $\mathbb{N}$-chain has no factor modules of finite length. This implies that $\operatorname{rad} M$ has no factor module of finite length and therefore $f(\operatorname{rad} M) = 0$.

Remark. Note that we assume that all the $K[X]$-modules in $\mathcal{F}$ are reduced. It follows that the simple R-module is not in the image of Φ. As a consequence, the embedding functor Φ is controlled by the semisimple modules in the sense of [R]: For any pair F, F' of $K[X]$-modules in $\mathcal{F}$, there is a direct decomposition of abelian groups

$$\operatorname{Hom}_R(\Phi(F), \Phi(F')) = \Phi \operatorname{Hom}_{K[X]}(F, F') \;\oplus\; \operatorname{Hom}_R(\Phi(F), \Phi(F'))_{\mathcal{E}},$$

where $\operatorname{Hom}_R(M, M')_{\mathcal{E}}$ denotes the set of R-homomorphisms $M \to M'$ with semisimple image.

Corollary. *The functor $\Phi\colon \mathcal{F} \to \operatorname{Mod} R$ preserves indecomposability and non-isomorphy.*

Proof. This follows directly from the fact that the endomorphisms of $\Phi(F)$ with semisimple image belong to the Jacobson radical of $\operatorname{End}_R(\Phi(F))$.

4. The ring R revisited. It seems to be appropriate to have another look at the local ring R exhibited above. Denote by I the ideal of all matrices of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ K(X) & 0 & \cdots & 0 \\ \cdots & & & \cdots \\ K(X) & 0 & \cdots & 0 \end{bmatrix}$$

and as before, let D be the subring of all diagonal matrices in R. Then

$$I^2 = 0, \quad I \cap D = 0, \quad I + D = R,$$

thus R is what is called the "trivial extension" of D by I, this is the idealization of the bimodule ${}_D I_D$ as propagated by Nagata.

Note that D is a local noetherian commutative ring, it is reduced, of Krull dimension 1, and Cohen-Macaulay, thus not at all a complicated ring! In our considerations above we have used the precise way D is constructed: it is a pullback ring of n discrete rank 1 valuation rings. For the construction of a ring such as D, one only needs local rings $V_1, \dots, V_n$ such that the factor rings $V_i/\operatorname{rad} V_i$ are isomorphic.

The interesting ingredient in our construction seems to be the bimodule ${}_D I_D$. Note that I considered as a right D-module is semisimple and of length n (in particular, not faithful), whereas as a left D-module, I is faithful. The construction of I relies on our assumption that $K(X)$ is isomorphic to K as a field.

5. An example. Consider the case $n = 4$. We want to present five pairwise non-isomorphic indecomposable artinian R-modules $M_1, \dots, M_5$ such that

$$M_1 \oplus M_2 \simeq M_3 \oplus M_4 \oplus M_5.$$

Actually, we will exhibit S-modules, but as we know, any S-module becomes an R-module under ι. Our S-modules will be images under Φ_0, thus we could start with $K[X]$-modules similar to finite rank abelian groups as given in [J] or [Fu] and apply Φ_0. However, one may observe that the usual procedure of writing down examples F in $\mathcal{F}$ starts with a finite dimensional $K(X)$-module N, and specifies F as a $K[X]$-submodule of N. Equally well, we may specify N/F, and this is what we will do. In this way, we construct directly examples in $\mathcal{A}$. Our example is the adaption of a special case of Theorem 90.1 of Fuchs [Fu].

We need the $K[X]$-modules $P_i = K(X)/V_i$ (these are the so called Prüfer modules) and we denote by π_i the canonical projection $K(X) \to P_i$. Note that the multiplication by p_i yields an endomorphism of P_i which is surjective and whose kernel is the socle of P_i. We will denote by π_i' the composition of π_i with the multiplication by p_i.

The S-modules $M_1 \oplus M_2$ and $M_3 \oplus M_4 \oplus M_5$ are given by data of the following form

$$(K(X)^4; P_1^{d(1)}, P_2^{d(2)}, P_3^{d(3)}, P_4^{d(4)}; \gamma_1, \gamma_2, \gamma_3, \gamma_4)$$

where $d(1) = d(2) = 2,\ d(3) = d(4) = 4$.

First, let us assume that the maps γ_i $(1 \le i \le 4)$ are given by the following matrices:

$$\left[\begin{array}{cc:cc} \pi_1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \pi_1 & 0 \end{array}\right] \quad \left[\begin{array}{cc:cc} 0 & \pi_2 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \pi_2 \end{array}\right] \quad \left[\begin{array}{cc:cc} \pi_3 & \pi_3 & 0 & 0 \\ \pi_3' & 0 & 0 & 0 \\ \hdashline 0 & 0 & \pi_3 & 0 \\ 0 & 0 & 0 & \pi_3 \end{array}\right] \quad \left[\begin{array}{cc:cc} \pi_4 & 0 & 0 & 0 \\ 0 & \pi_4 & 0 & 0 \\ \hdashline 0 & 0 & \pi_4 & \pi_4 \\ 0 & 0 & \pi_4' & 0 \end{array}\right]$$

Let us denote the corresponding S-module by $M(\gamma)$. Observe that the canonical direct decomposition $K(X)^4 = K(X)^2 \oplus K(X)^2$ yields a direct decomposition $M(\gamma) = M_1 \oplus M_2$ of S-modules. It is not difficult to check that the direct summands M_1, M_2 are indecomposable.

Here is a second choice of matrices $\gamma_1', \gamma_2', \gamma_3', \gamma_4'$, which yields an S-module which we label $M(\gamma')$:

$$\begin{bmatrix} \pi_1 & 0 & 0 & 0 \\ 0 & 0 & \pi_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \pi_2 & 0 & 0 \\ 0 & 0 & 0 & \pi_2 \end{bmatrix} \begin{bmatrix} \pi_3 & \pi_3 & 0 & 0 \\ \pi_3' & 0 & 0 & 0 \\ 0 & 0 & \pi_3 & 0 \\ 0 & 0 & 0 & \pi_3 \end{bmatrix} \begin{bmatrix} \pi_4 & \pi_4 & 0 & 0 \\ \pi_4' & 0 & 0 & 0 \\ 0 & 0 & \pi_4 & 0 \\ 0 & 0 & 0 & \pi_4 \end{bmatrix}$$

The canonical direct decomposition $K(X)^4 = K(X)^2 \oplus K(X) \oplus K(X)$ gives a direct decomposition $M(\gamma') = M_3 \oplus M_4 \oplus M_5$ of S-modules with three direct summands M_3, M_4, M_5. It is obvious that the S-modules M_4 and M_5 are indecomposable, and again it is easy to see that also M_3 is indecomposable.

Also, one checks without difficulties that the S-modules $M_1, \dots, M_5$ are pairwise non-isomorphic.

On the other hand, the two S-modules $M(\gamma)$ and $M(\gamma')$ are isomorphic. An isomorphism $M(\gamma) \to M(\gamma')$ is given by an automorphism f of $K(X)^4$ and automorphisms f_i of $P_i^{d(i)}$ for $1 \le i \le 4$ (these are endomorphisms of $K[X]$-modules) such that

$$\gamma_i' \circ f = f_i \circ \gamma_i,$$

for $1 \le i \le n.$. Let us present such automorphisms $f,\ f_1 = f_2,\ f_3,\ f_4$ via matrices:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ p_3 & 0 & p_4 & 0 \\ 0 & p_3 & 0 & p_4 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ p_3 & p_4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & p_3 & 1 \\ 0 & 1 & p_4 & 0 \\ p_3 & -1 & 0 & p_4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ p_4 & 0 & 0 & 1 \\ p_3 & 0 & 0 & 1 \\ 0 & p_3 & p_4 & -1 \end{bmatrix},$$

note that $\det f_1 = \det f_2 = p_4 - p_3 = a_3 - a_4$ and $\det f = \det f_3 = \det f_4 = (p_4 - p_3)^2 = (a_3 - a_4)^2$, thus all the matrices are invertible.

Any homomorphism of S-modules is a homomorphism of R-modules, in particular $(f; f_1, \dots, f_4)$ is an isomorphism of the R-modules $\iota M(\gamma)$ and $\iota M(\gamma')$. Similarly, the given direct decompositions of S-modules are direct decompositions of R-modules: $\iota M(\gamma) = \iota M_1 \oplus \iota M_2$ and $\iota M(\gamma') = \iota M_3 \oplus \iota M_4 \oplus \iota M_5$ (but note that the R-modules $\iota M(\gamma)$ and $\iota M(\gamma')$ have more direct decompositions than the S-modules $M(\gamma)$ and $M(\gamma')$). It remains to observe that all the modules ιM_i are indecomposable and pairwise non-isomorphic. This follows from the corresponding assertions for the S-modules M_i and the fact that ι is a representation embedding.

Let us add a further remark on the shape of the modules considered. For the modules M_1, M_2, M_3, the radical $\operatorname{rad} M_i$ is the direct sum of six modules whose proper submodules form an $\mathbb{N}$-chain, and $M/\operatorname{rad} M_i$ is semisimple of length 2.

For the modules M_4, M_5, the radical $\operatorname{rad} M_i$ is the direct sum of three modules whose proper submodules form an $\mathbb{N}$-chain, and $M/\operatorname{rad} M_i$ is simple. All these assertions remain true when we consider the R-module ιM_i instead of the S-module M_i.

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