THE GROUPS $H^3(F(X)/F)$ AND $\mathrm{CH}^2(X)$ FOR GENERIC SPLITTING VARIETIES OF QUADRATIC FORMS

OLEG T. IZHBOLDIN

Abstract. Let F be a field of characteristic different from 2 and ϕ be a quadratic form over F. Let X be an arbitrary projective homogeneous generic splitting variety of ϕ. For example, we can take X to be equal to the variety $X_{\phi,m}$ of totally isotropic m-dimensional subspaces of V, where V is the quadratic space corresponding to ϕ and $m < \frac{1}{2}\dim V$. In this paper, we study the groups $\mathrm{CH}^2(X)$ and $H^3(F(X)/F) = \ker(H^3(F) \to H^3(F(X)))$. One of the main results of this paper claims that the group $\operatorname{Tors}\mathrm{CH}^2(X)$ is always zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In many cases we prove that $\operatorname{Tors}\mathrm{CH}^2(X) = 0$ and compute the group $H^3(F(X)/F)$ completely. As an application of main results, we give a criterion of motivic equivalence of 8-dimensional forms except for the case where the Shur indeces of their Clifford algebras equal 4.

Contents

1991 *Mathematics Subject Classification.* Primary 12G05, 11E04; Secondary 14C25, 19E15.

Key words and phrases. Quadratic form, Galois cohomology, Chow groups.

This research was supported by the Alexander von Humboldt–Stiftung.

0. Introduction

Let F be a field of characteristic $\neq 2$ and let $H^n(F)$ be the Galois cohomology group of F with $\mathbb{Z}/2\mathbb{Z}$-coefficients. For a given field extension L/F, we denote by $H^n(L/F)$ the kernel of the natural homomorphism $H^n(F) \to H^n(L)$. Now, let X be a projective homogeneous variety over F. The group $H^n(F(X)/F)$ is obviously zero in the cases where $n = 0$ or 1. If $n = 2$, the group $H^2(F(X)/F) = \mathrm{Br}_2(F(X)/F)$ is always finite and can be computed effectively for any projective homogeneous variety X.

Computation of the group $H^3(F(X)/F)$ is much more complicated. The first nontrivial result in this direction due to J. K. Arason. In [Ara, 1975] he computed the group $H^3(F(X_\phi)/F)$ where X_ϕ is the projective quadric given by the homogeneous equation $\phi = 0$. Arason's theorem (in particular) claims that: *(a) if* $\dim \phi > 4$, *the group* $H^3(F(X_\phi)/F)$ *is zero except for the case where* ϕ *is an anisotropic 3-fold Pfister neighbor; (b) if* ϕ *is an anisotropic 3-fold neighbor then* $H^3(F(X_\phi)/F) \simeq \mathbb{Z}/2\mathbb{Z}$.

This result is closely related to a result of N. Karpenko concerning the second Chow group of quadrics. In [K1, 1990], Karpenko has proved that: *(a) the group* $\mathrm{Tors}\,\mathrm{CH}^2(X_\phi)$ *is zero except for the case where* ϕ *is an anisotropic 3-fold Pfister neighbor; (b) if* ϕ *is an anisotropic 3-fold neighbor then* $\mathrm{Tors}\,\mathrm{CH}^2(X) \simeq \mathbb{Z}/2\mathbb{Z}$.

An evident comparison of Arason's and Karpenko's results shows that for all forms ϕ of dimension > 4, the groups $H^3(F(X_\phi)/F)$ and $\mathrm{Tors}\,\mathrm{CH}^2(X_\phi)$ are isomorphic. Later on, it was shown that there exists a natural explanation of this isomorphism. In the papers of A. Merkurev [M3, 1995] and E. Peyre [P2, 1998], the homomorphism (constructed in [Su, §24])

$$\tau_X : H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) \to \mathrm{Tors}\,\mathrm{CH}^2(X)$$

was studied for any homogeneous projective variety X. In particular, the kernel of this homomorphism was computed completely. In [P2], E. Peyre proved that the homomorphism τ_X is surjective. The results of Merkurev and Peyre show that for a projective homogeneous variety X, the computation of the group $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$ is actually equivalent to that of the group $\mathrm{Tors}\,\mathrm{CH}^2(X)$. We also note that the computation of $\mathrm{CH}^2(X)$ is equivalent to the computation of its torsion subgroup $\mathrm{Tors}\,\mathrm{CH}^2(X)$ because $\mathrm{CH}^2(X)$ is a finitely generated group and the group $\mathrm{CH}^2(X)_{\mathbb{Q}}$ can be easily computed for all geometrically cellular varieties.

The structure of the group $\mathrm{CH}^2(X)$ was studied for quadrics ([K1]), Severi-Brauer varieties ([K2]) and their products ([IK3], [IK1]). The complete answer is known only in the case of quadrics. In the case of Severi-Brauer varieties and their products, the group $\mathrm{CH}^2(X)$ was computed completely in many specific situations [K2].

The main purpose of this paper is to study one more class of projective homogeneous varieties. Namely, we consider "generic splitting varieties of quadratic forms". Let us give a precise definition. Let ϕ be a quadratic form over F

and let $F = F_0, F_1, F_2, \dots, F_h$ be Knebusch's generic splitting tower of ϕ. We say that X is a *generic splitting variety of* ϕ if X is a projective homogeneous variety and there exists n such that the extension $F(X)/F$ is (stable) equivalent to F_n/F. In fact, this means that X is a projective homogeneous variety corresponding to the orthogonal group $\mathbf{O}(\phi)$. The quadric X_ϕ is a simplest example of a generic splitting variety (in this case $F(X_\phi) = F_1$). An obvious generalization of this example is the following. Let V be the quadratic space corresponding to quadratic form ϕ. We define $X_{\phi,m}$ as the variety of totally isotropic subspaces of V of dimension $m < \frac{1}{2}\dim\phi$. Clearly, $X_{\phi,m}$ is a projective homogeneous variety (with a natural transitive action of $\mathbf{O}(\phi)$).

One of the principal results of this paper is the following theorem.

Theorem 0.1. *Let ϕ be a quadratic form over F and X be an arbitrary generic splitting variety of ϕ (for example, $X = X_{\phi,m}$ with $m < \frac{1}{2}\dim\phi$). Then the group* $\operatorname{Tors}\operatorname{CH}^2(X)$ *is zero or isomorphic to* $\mathbb{Z}/2\mathbb{Z}$.

Our computation of $\operatorname{Tors}\operatorname{CH}^2(X)$ is based on the computation of the group $H^3(F(X)/F)$ and applying of Merkurev-Peyre results mentioned above. In many cases we compute the group $\operatorname{Tors}\operatorname{CH}^2(X)$ explicitely (see, e.g., Theorem 10.1 and Proposition 10.8). However, we cannot present the answer in a short form because it depends essentially on many invariants of our quadratic form ϕ. Really, it is necessary to consider separately at least six different subcases depending on integer m, dimension of ϕ, determinant of ϕ, and the index of the even Clifford algebra $C_0(\phi)$. All results concerning the groups $H^3(F(X)/F)$ and $\operatorname{CH}^2(X)$ can be found in the main body of the paper (see Sections 7–10). To give an impression on our theorems, we formulate of one of them.

Theorem 0.2. *Let ϕ be an even-dimensional quadratic form with trivial discriminant and* $\operatorname{ind} C(\phi) \geq 8$. *Let X be an arbitrary generic splitting variety of ϕ (we always assume that X is projective homogeneous). Then*

- *if $\phi_{F(X)}$ is nonhyperbolic, then* $H^3(F(X)/F) = \operatorname{Tors}\operatorname{CH}^2(X) = 0$;
- *if $\phi_{F(X)}$ is hyperbolic, then*

$$\operatorname{Tors}\operatorname{CH}^2(X) \simeq \operatorname{Tors}\operatorname{CH}^2(\operatorname{SB}(D)) \simeq \frac{H^3(F(X)/F)}{c(\phi)\cdot H^1(F)}$$

where D is a division algebra corresponding to $c(\phi) \in \operatorname{Br}_2(F)$. *Moreover, if D is decomposable, then* $\operatorname{Tors}\operatorname{CH}^2(X) = 0$ *and*

$$H^3(F(X)/F) = c(\phi)H^1(F) = \{e^3(\phi\,\langle\langle a\rangle\rangle) \mid a \in F^*\}.$$

This theorem shows that the calculation of the groups $\operatorname{Tors}\operatorname{CH}^2(X)$ and $H^3(F(X)/F)$ is closely related to the computation of the second Chow group of Severi-Brauer varieties. The situation with even-dimensional forms with nontrivial discriminant d is much more difficult (an essential part of this paper is devoted to this case). Let us note that in this case, the algebra $C_0(\phi)$ is cental simple over the quadratic extension $L = F(\sqrt{d})$. It turns out that instead of the Severi-Brauer variety, we need to work with the Weil restriction

$R_{L/F}(\mathrm{SB}(C_0(\phi)))$ (which we will denote by $X_{C_0(\phi)}$). Also, in this case we will need to compute the group $H^3(F(X_{C_0(\phi)})/F)$. For this, we develop some technique concerning generic splitting field of separable (not necessary cental simple) algebras A (see Sections 3–6).

Some words about the structure of the paper and notation. In §§1–4, we present machinery and terminology concerning "generic splitting fields". In §5, we give some new version of Merkurev–Peyre theorem (see Theorem 5.9). In §6, we compute the groups $H^3(F(X)/F)$ for the generic splitting varieties of even Clifford algebras. In §7, we prove that $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X)/F)$ for all generic splitting varieties of quadratic forms and compute the group $H^n(F_n/F)$ in the case where $n < h'(\phi)$. Sections 8–10 contain our main results concerning $H^3(F(X)/F)$ and $\mathrm{Tors}\,\mathrm{CH}^2(X)$. In §11, we apply our results for study of motivic equivalence of forms.

Mainly we use the same terminology and notation as in the books of T. Y. Lam [Lam], W. Scharlau [Sch], and fundamental papers of M. Knebusch concerning generic splitting fields of quadratic forms [Kn1, Kn2]. We use the notation $\langle\langle a_1, \dots, a_n\rangle\rangle$ for the Pfister form $\langle 1, -a_1\rangle \otimes \dots \otimes \langle 1, -a_n\rangle$. We denote by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to n-fold Pfister forms is denoted by $GP_n(F)$. We recall that the Arason–Pfister Hauptsatz (APH in what follows) states that: *every quadratic form over F of dimension $< 2^n$ which lies in $I^n(F)$ is necessarily hyperbolic; if $\phi \in I^n(F)$ and $\dim \phi = 2^n$ then the form ϕ necessarily belongs to $GP_n(F)$.* Let A be a central simple algebra over F. By $\deg(A)$, $\mathrm{ind}(A)$, and $[A]$ we denote respectively the degree of A, the Shur index of A, and the class of A in the Brauer group $\mathrm{Br}(F)$. By $\mathrm{SB}(A)$ we denote the Severi–Brauer variety of A. Suppose an algebra B has the form $B = A \times A$ where A is a central simple algebra (for example, $B = C_0(\phi)$ with $\phi \in I^2(F)$). We set $\mathrm{ind}\, B = \mathrm{ind}\, A$. For a quadratic form ϕ we set $\mathrm{ind}\,\phi = \mathrm{ind}(C_0(\phi))$. All needed properties of Galois cohomology groups $H^n(F)$ and $H^n(F, \mathbb{Q}/\mathbb{Z}(i))$ can be found in [Ka] or [P1]. For $n = 1, 2, 3$, the group $H^n(F)$ is naturally identified with the 2-torsion subgroup of $H^n(F, \mathbb{Q}/\mathbb{Z}(n-1))$. For any quadratic form ϕ we have $H^3(F(\phi)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\phi)/F)$. All needed results concerning Galois cohomology in degree three and homogeneous varieties can be found in [P2] and [M3]. For $n \le 3$, we use the notation e^n for Arason's cohomological invariants $e^n : I^n(F) \to H^n(F)$. The homomorphism e^n gives rise to the isomorphism $I^n(F)/I^{n+1}(F) \simeq H^n(F)$ for $n = 0, 1, 2, 3$ (see [M1, MS, R]). When we refer the reader to Arason's theorem, we always means the following statement ([Ara]): *if ϕ is a form of dimension > 4 such that $H^3(F(\phi)/F) \neq 0$ then ϕ is a Pfister neighbor of some 3-fold Pfister form π and $H^3(F(\phi)/F)$ is generated by $e^3(\pi)$.* For a finite set A, we denote by $|A|$ the order of A. In the last section, we use terminology concerning motivic equivalence and half-neighbors. All needed definitions can be found in [Ho1] and [Izh1].

1. Generic splitting fields

We start from the following definition.

Definition 1.1. Let F_1/k and F_2/k be field extensions.

- We will write $F_1/k \prec F_2/k$ if there exists a purely transcendental extension $\tilde{F}_2/F_2$ and a k-homomorphism $F_1 \to \tilde{F}_2$.
- We will write $F_1/k \sim F_2/k$ if $F_1/k \prec F_2/k$ and $F_2/k \prec F_1/k$.

Clearly, "$\prec$" is an order relation on the category $Fields/k$. Hence, "$\sim$" is an equivalence relation on $Fileds/k$.

Lemma 1.2. *If $F_1/k \prec F_2/k$ then $H^n(F_1/k) \subset H^n(F_2/k)$ for all n.*

Proof. Let $u \in H^n(F_1/F)$. This means that $u \in H^n(F)$ and $u_{F_1} = 0$. We need to prove that $u_{F_2} = 0$. By Definition 1.1, there exists a purely transcendental extension $\tilde{F}_2/F_2$ and a k-homomorphism $F_1 \to \tilde{F}_2$. Since $u_{F_1} = 0$, we obviously get $u_{\tilde{F}_2} = 0$. Since $\tilde{F}_2/F_2$ is purely transcendental, we have $u_{F_2} = 0$. □

Corollary 1.3. *If $F_1/k \sim F_2/k$ then $H^n(F_1/k) = H^n(F_2/k)$.*

Definition 1.4. Let $\mathcal{K}/k$ be a set of extensions of the field k. We say that $\mathcal{K}/k$ *contains a generic extension* if there exists an extension $K_{gen}/k \in \mathcal{K}/k$ such that $K_{gen}/k \prec K/k$ for all $K/k \in \mathcal{K}/k$.

In this case, we say that K_{gen}/k is a generic extension of the set $\mathcal{K}/k$.

It is clear that if a generic extension exists then it is uniquely defined up to $\sim$-equivalence.

Example 1.5. (Kersten–Rehmann, [KR]). Let X be a projective homogeneous variety over F. Let

$$\mathcal{K}_X/F = \{K/F \mid \text{the variety } X_K \text{ has a rational } K\text{-point}\}.$$

Then $F(X)/F$ is a generic extension of the set $\mathcal{K}_X/F$. Moreover,

$$\mathcal{K}_X/F = \{K/F \mid X_K \text{ a rational } K\text{-variety}\} = \{K/F \mid F(X)/F \prec K/F\}.$$

Proof. Items (i), (ii) and (ii)' of [KR, Th. 3.10] claim that for a projective homogeneous F-variety X and for any extension K/F the following conditions are equivalent:

- the set $X(K)$ is nonempty (i.e., X_K has a rational K-point),
- the extension $K(X)/K$ is purely transcendental (i.e, X_K is a rational variety),
- $F(X)/F \prec K/F$.

By Definition 1.4, $F(X)/F$ is obviously a generic extension of $\mathcal{K}_X/F$. □

Proposition 1.6. *Let ϕ be a quadratic form over F. Let s be an integer such that $s \geq 1$. Then the set of extensions*

$$\mathcal{K}_{\phi,s}/F = \{K/F \mid \dim(\phi_K)_{an} \leq s\}.$$

contains a generic extension.

In fact, this proposition is well-known. But our main purpose here, not only to prove this statement, but give two alternative constructions of generic extensions. The role of the both constructions will be clear in the main sections of this paper.

Example 1.7. (`Knebusch's construction`). It is a construction based on generic splitting tower of Knebusch ([Kn1]). Let $F = F_1, F_2, \dots, F_h$ be the Knebusch's generic splitting tower of ϕ. Let n be the minimal integer such that $\dim(\phi_{F_n})_{an} \le s$. Existence of n follows from the condition $s \ge 1$.

It follows from [Kn1, Cor. 3.9 and Prop. 5.13] that for any extension K/F, the following conditions are equivalent:

(i) $\dim(\phi_K)_{an} \le s$,

(ii) the extension $(K \cdot F_n)/K$ is purely transcendental, where $K \cdot F_n$ denotes the free composite of K and F_n over F.

Condition (i) means that $K/F \in \mathcal{K}_{\phi,s}$. Since $F_n \subset K \cdot F_n$, condition (ii) implies that $F_n/F \prec K/F$. All this means that the extension F_n/F is a generic extension of the set $\mathcal{K}_{\phi,s}$. We put $K^1_{\phi,s} \overset{\text{Def}}{=} F_n$. □

Example 1.8. (`A geometric construction`). Here we assume that $s \ge 2$. The condition $\dim(\phi_K)_{an} \le s$ is equivalent to the condition $i_W(\phi_K) = \frac{1}{2}(\dim\phi - \dim(\phi_K)_{an}) \ge \frac{1}{2}(\dim\phi - s)$. Hence,

$$\mathcal{K}_{\phi,s}/F = \{K/F \mid i_W(\phi_K) \ge m\}$$

where m is the minimal integer such that $m \ge \frac{1}{2}(\dim\phi - s)$. Since $s \ge 2$, we have $m < \frac{1}{2}\dim\phi$. Let V be the quadratic F-space corresponding to the form ϕ. The condition $i_W(\phi_K) \ge m$ means that the quadratic space $V_K = V \otimes_F K$ contains an m-dimensional totally isotropic linear subspace. Let us define F-variety $X = X_{\phi,m}$ as follows: *for any extension K/F, the set of K-points of X coincides with the set of m-dimensional totally isotropic subspaces of V_K*. It is well known that $X = X_{\phi,m}$ is a geometrically integral homogeneous projective variety (here we need the condition $m < \frac{1}{2}\dim\phi$). By the definition, the variety X_K has a rational point if and only if $i_W(\phi_K) \ge m$. Taking into account Example 1.5, we see that $F(X_{\phi,m})/F$ is a generic field extension of the set $\mathcal{K}_{\phi,s}/F$. Moreover, Example 1.5 shows that

$$\mathcal{K}_{\phi,s}/F = \{K/F \mid F(X_{\phi,m})/F \prec K/F\}.$$

We put $K^2_{\phi,s} \overset{\text{Def}}{=} F(X_{\phi,m})$.

Corollary 1.9. *Let $s \ge 2$ and $K_{\phi,s}/F$ be an arbitrary generic extension of the set $\mathcal{K}_{\phi,s} = \{K/F \mid \dim(\phi_K)_{an} \le s\}$. Then*

- $K_{\phi,s}/F \sim K^1_{\phi,s}/F \sim K^2_{\phi,s}/F$,
- $\mathcal{K}_{\phi,s}/F = \{K/F \mid K_{\phi,s}/F \prec K/F\}$.

Proof. Equivalence of extensions is obvious because all three extensions are generic for the set $\mathcal{K}_{\phi,s}/F$. The second statement follows from the equivalence $K_{\phi,s}/F \sim K^2_{\phi,s}/F$ (see the property of $K^2_{\phi,s}/F$). □

2. The generic splitting field F_ϕ

Let ϕ be a quadratic form over F. The main purpose of this section is to define the extension F_ϕ/F and to study properties of this extension. To define F_ϕ, we consider three cases.

Case 1. ϕ is an even-dimensional form with $d_\pm\phi \in F^{*2}$ (in other words, $\phi \in I^2(F)$). In this case, we define F_ϕ/F as a generic extension of the set

$$\mathcal{K}_\phi/F = \{K/F \mid \ \phi_K \text{ is hyperbolic}\}.$$

By Knebusch's construction, we can take F_ϕ to be equal F_h.

Case 2. ϕ is an odd-dimensional form. In this case, we define F_ϕ/F as a generic extension of the set

$$\mathcal{K}_\phi/F = \{K/F \mid \phi \text{ splits over } K \text{ (i.e., } \dim\phi_K = 1) \ \}.$$

By Knebusch's construction, we can take F_ϕ to be equal F_h.

Case 3. ϕ is an even-dimensional form with $d_\pm\phi \notin F^{*2}$ (in other words, $\deg\phi = 1$). In this case, we define F_ϕ/F as a generic extension of the set

$$\mathcal{K}_\phi/F = \{K/F \mid \ \dim\phi_K \leq 2 \ \}.$$

Knebusch's construction shows that we can put $F_\phi = F_{h-1}$. Indeed, since $\deg\phi = 1$, we have $\dim(\phi_{F_{h-1}})_{an} = 2$. Hence $\min\{i \mid \dim(\phi_{F_i})_{an} \leq 2\} = h-1$.

All three cases considered just above make natural the following notation.

Definition 2.1. Let ϕ be a form over F. We define the integer $h'(\phi)$ as follows:

- if $\dim\phi$ is odd or $\phi \in I^2(F)$, we set $h'(\phi) = h(\phi)$,
- if $\dim\phi$ is even and $\phi \notin I^2(F)$, we set $h'(\phi) = h(\phi) - 1$.

Our previous discussion shows that F_ϕ can be defined as $F_{h'(\phi)}$.

Remark 2.2. It is not difficult to suggest a common definition of the set $\mathcal{K}_\phi$ for all three cases considered above: $\mathcal{K}_\phi/F = \{K/F \mid \dim(\phi_K)_{an} \leq 2\}$. Indeed, if $\phi \in I^2(F)$ then the condition $\dim(\phi_K)_{an} \leq 2$ implies that ϕ_K is hyperbolic. If $\dim\phi$ is odd, then the conditions $\dim(\phi_K)_{an} \leq 2$ and $\dim(\phi_K)_{an} = 1$ are equivalent.

The following lemma is obvious

Lemma 2.3. *F_ϕ/F is a generic extension of the set $\{K/F \mid \dim(\phi_K)_{an} \leq 2\}$ and $h'(\phi) = \min\{n \mid \dim(\phi_{F_n})_{an} \leq 2\}$.*

The following lemma gives a geometric interpretation of $h'(\phi)$.

Lemma 2.4. *Let ϕ be an anisotropic quadratic form of dimension ≥ 3 over F and F_n/F be the n-th generic splitting field of ϕ. Then the following conditions are equivalent:*

(i) $0 \leq n \leq h'(\phi)$.

(ii) *there exists a projective homogeneous F-variety X such that $F(X)/F \sim F_n/F$,*

(iii) *there exists a geometrically irreducible F-variety X such that $F(X)/F \sim F_n/F$,*

Proof. (i)$\Rightarrow$(ii). By Example 1.7, F_n/F is a generic extension of the set $\mathcal{K}_{\phi,s} = \{K/F \mid \dim(\phi_K)_{an} \leq s\}$ where $s = \dim(\phi_{F_{n-1}})_{an} - 1$. Since $n \leq h'(\phi)$, Lemma 2.3 implies that $\dim(\phi_{F_{n-1}})_{an} \geq 3$. Hence, $s \geq 2$. Then Example 1.8 shows that there exists a projective homogeneous variety X such that $F(X)/F \sim F_n/F$.

Implication (ii)$\Rightarrow$(iii) is obvious. To prove (iii)$\Rightarrow$(i), we note that the conditions $n \leq h'(\phi)$ and $n \leq h = h(\phi)$ are equivalent except for the case where:

$n = h$ and ϕ is an even-dimensional form with nontrivial discriminant.

Hence, we can consider only this exceptional case. Then we obviously have $L \subset F_n$ where $L = F(\sqrt{d_{\pm}\phi})$ is the *discriminant* extension. If we assume that $F(X)/F \sim F_n/F$, we necessary get $L \subset F(X)$. But this is impossible because X is a geometrically irreducible variety. $\square$

Two following lemmas reduce a construction of F_ϕ to the case where $\phi \in I^2(F)$.

Lemma 2.5. *Let ϕ be an odd-dimensional form and $\tilde{\phi} = \phi \perp \langle d_{\pm}\phi \rangle$. Then $F_\phi/F \sim F_{\tilde{\phi}}/F$.*

Proof. Since $\tilde{\phi} \in I^2(F)$, it follows that for any extension K/F the conditions $\dim(\phi_K)_{an} = 1$ and $\dim(\tilde{\phi}_K)_{an} = 0$ are equivalent. Hence, $\mathcal{K}_\phi/F = \mathcal{K}_{\tilde{\phi}}/F$. Therefore, $F_\phi/F \sim F_{\tilde{\phi}}/F$. $\square$

Lemma 2.6. *Let ϕ be an even-dimensional form with $d = d_{\pm}\phi \notin F^{*2}$. Let $\tilde{F} = F(t)$ and $\tilde{\phi} = \phi \perp t\langle\langle d \rangle\rangle$. Then $F_\phi/F \sim \tilde{F}_{\tilde{\phi}}/F$.*

Proof. It suffices to prove that $\tilde{F}_{\tilde{\phi}}/F$ is a generic extension of the set $\mathcal{K}_\phi/F = \{K/F \mid \dim(\phi_K)_{an} \leq 2\}$. In other words, we need to prove that $\dim(\phi_{\tilde{F}_{\tilde{\phi}}}) \leq 2$ and for any extension K/F satisfying the condition $\dim(\phi_K)_{an} \leq 2$, we necessarily have $\tilde{F}_{\tilde{\phi}}/F \prec K/F$.

Let us verify inequality $\dim(\phi_{\tilde{F}_{\tilde{\phi}}}) \leq 2$. Since $\tilde{\phi} \in I^2(\tilde{F})$ it follows that $\tilde{\phi}_{\tilde{F}_{\tilde{\phi}}}$ is hyperbolic. Since $\phi_{\tilde{F}} = \tilde{\phi} - t\langle\langle d \rangle\rangle$, it follows that $\phi_{\tilde{F}_{\tilde{\phi}}}$ is Witt equivalent to the 2-dimensional form $-t\langle\langle d \rangle\rangle$. Hence $\dim(\phi_{\tilde{F}_{\tilde{\phi}}})_{an} \leq 2$.

Now, let K/F be an extension such that $\dim(\phi_K)_{an} \leq 2$. Since $\dim\phi$ is even and $d_{\pm}\phi = d$, it follows that there exists $x \in K^*$ such that $(\phi_K)_{an} = (x\langle\langle d \rangle\rangle)_{an}$. Let $E = K(t)(\sqrt{-t/x})$. Clearly, $(\phi_E)_{an} = (x\langle\langle d \rangle\rangle_E)_{an} = (-t\langle\langle d \rangle\rangle_E)_{an}$. Therefore, $\tilde{\phi}_E = \phi_E + t\langle\langle d \rangle\rangle_E = 0$. By the generic property of $\tilde{F}_{\tilde{\phi}}$, we have $\tilde{F}_{\tilde{\phi}}/\tilde{F} \prec E/\tilde{F}$. Therefore, $\tilde{F}_{\tilde{\phi}}/F \prec E/F$. Obviously, E/K is purely transcendental. Hence $E/F \prec K/F$. Therefore, $\tilde{F}_{\tilde{\phi}}/F \prec E/F \prec K/F$. $\square$

3. Generic splitting fields of semisimple separable algebras

Let A be a central simple F-algebra. We say that A splits over K if there exists an isomorphism $A \otimes_F K \simeq M_n(K)$. In other words this means that $\operatorname{ind} A_K = 1$.

Proposition 3.1. *Let A be a central simple algebra over F and $F(A) = F(\mathrm{SB}(A))$ be the function field of the Severi–Brauer variety of A. Then $F(A)/F$ is a generic extension of the set*

$$\mathcal{K}_A/F = \{K/F \mid A \text{ splits over } K\}.$$

Moreover, $\mathcal{K}_A/F = \{K/F \mid F(A)/F \prec K/F\}$.

Proof. Basic property of Severi–Brauer varieties claims that the algebra A splits over K if and only if $\mathrm{SB}(A)$ has a rational point over K. Since $\mathrm{SB}(A)$ is a projective homogeneous variety, the required result follows immediately form Example 1.5. □

Now, let us recall some definitions. We say that A is a *separable simple F-algebra* if there exists a finite separable extension L/F such that A is a central simple L-algebra. We say that A splits, if $A \simeq M_n(L)$ for suitable n. Clearly, this equivalent to the condition $\operatorname{ind} A = 1$. We say that A is a *separable semisimple F-algebra* if there exists a separable simple algebras $A_1, \dots, A_s$ such that $A \simeq A_1 \times \cdots \times A_s$. Let L_i denotes the center of A_i. We say that A has exponent 2, if $[A_i] \in \mathrm{Br}_2(L_i)$ for all $i = 1, \dots, s$. We say that the (semisimple) algebra A splits if all (simple) algebras $A_1, \dots, A_s$ split. It is well-known that if A is a semisimple separable F-algebra and K/F is an arbitrary extension then $A_K \overset{\text{Def}}{=} A \otimes_F K$ is a semisimple separable K-algebra. We say that A splits over K if A_K splits.

Example 3.2. Let ϕ be a quadratic form over F. Then $C_0(\phi)$ is a separable semisimple F-algebra. Clearly, $C_0(\phi)$ splits if and only if $\operatorname{ind} \phi = 1$. For any extension K/F we have $C_0(\phi) \otimes_F K \simeq C_0(\phi_K)$. Hence, $C_0(\phi)$ splits over K if and only if $\operatorname{ind} \phi_K = 1$.

The main task of this section is to study generic extensions of the set

$$\mathcal{K}_A/F = \{K/F \mid A \text{ splits over } K\}$$

for any separable semisimple algebra A. We start from the following definition.

Definition 3.3. Let $A = A_1 \times \cdots \times A_s$ be a semisimple separable F-algebra, where $A_1, \dots, A_s$ are simple F-algebras with the centers $L_1, \dots, L_s$.

- We define the F-variety X_A by the following formula
$$X_A = R_{L_1/F}(\mathrm{SB}(A_1)) \times \cdots \times R_{L_s/F}(\mathrm{SB}(A_s)).$$
Since $\mathrm{SB}(A_i)$ are projective homogeneous L_i-varieties and the extensions L_i/F are separable, it follows that X_A is a projective homogeneous F-variety.
- We denote by $F(A)$ the function field of F-variety X_A.

Proposition 3.4. *Let A be a semisimple separable F-algebra. Then for any extension K/F the following conditions are equivalent:*

- *X_A has a rational point over K,*
- *A splits over K.*

Proof. It suffices to consider the case where $s = 1$. In other words, we can assume that A is a central simple L-algebra where L/F is a finite separable extension. Then $X_A = R_{L/F}(\mathrm{SB}(A))$. Since L/F is separable, $L \otimes_F K$ has the form $K_1 \times \cdots \times K_r$ where K_i/K are finite separable extensions. Moreover, for all $i = 1, \ldots, r$ there exist natural F-homomorphisms of the fields, $L \to K_i$. Hence we can regard the fields K_i as extensions of L. Then

$$A_K = A \otimes_F K \simeq A \otimes_L L \otimes_F K \simeq (A \otimes_L K_1) \times \cdots \times (A \otimes_L K_r).$$

Therefore, A_K splits if and only if all simple algebras $A \otimes_L K_i$ split. The algebra $A \otimes_L K_i$ splits if and only if the L-variety $\mathrm{SB}(A)$ has a rational K_i-point (see the proof of Proposition 3.1). Therefore the algebra A_K splits if and only if the set $\mathrm{Mor}_{\mathrm{Spec}\, L}(\mathrm{Spec}(K_i), \mathrm{SB}(A))$ is nonempty for all $i = 1, \ldots, r$. On the other hand, the variety X_A has a rational point over K if and only if the set $\mathrm{Mor}_{\mathrm{Spec}\, F}(\mathrm{Spec}(K), X_A)$ is not imply. Using the key property of Weil restriction, we get

$$\begin{aligned}\mathrm{Mor}_{\mathrm{Spec}\, F}(\mathrm{Spec}(K), X_A) &= \mathrm{Mor}_{\mathrm{Spec}\, F}(\mathrm{Spec}(K), R_{L/F}(\mathrm{SB}(A)))\\ &= \mathrm{Mor}_{\mathrm{Spec}\, L}(\mathrm{Spec}(L \otimes_F K), \mathrm{SB}(A))\\ &= \mathrm{Mor}_{\mathrm{Spec}\, L}(\coprod \mathrm{Spec}(K_i), \mathrm{SB}(A)).\end{aligned}$$

This identity (together with the previous discussion) shows that X_A has a rational point over K if and only if the algebra A splits over K. □

Proposition 3.5. *Let A be a semisimple separable algebra over F. Then $F(A)/F$ is a generic extension of the set $\mathcal{K}_A/F = \{K/F \mid A \text{ splits over } K\}$. Moreover,*

$$\mathcal{K}_A/F = \{K/F \mid F(A)/F \prec K/F\} = \{K/F \mid X_A \text{ is rational over } K\}.$$

Proof. Follows from Proposition 3.4 and Example 1.5. □

Proposition 3.6. *Let A be a central simple F-algebra, $L = F(\sqrt{d})$ be a quadratic extension, and $k \in F^*$. Let $\tilde{F} = F(t)$ be the field of rational functions over F and $\tilde{A} = A_{F(t)} \otimes (d, kt)$ be an $\tilde{F}$-algebra. Then $F(A_L)/F \sim \tilde{F}(\tilde{A})/F$.*

Proof. It suffices to prove that $\tilde{F}(\tilde{A})/F$ is a generic extension of the set

$$\begin{aligned}\mathcal{K}_{A_L}/F =&\{K/F \mid F\text{-algebra } A_L \text{ splits over } K/F\}\\ =&\{K/F \mid \operatorname{ind} A_{K(\sqrt{d})} = 1\}.\end{aligned}$$

In other words, we need to prove that

(i) $\operatorname{ind} A_{\tilde{F}(\tilde{A})(\sqrt{d})} = 1$,

(ii) If K/F is such that $\operatorname{ind} A_{K(\sqrt{d})} = 1$, then $\tilde{F}(\tilde{A})/F \prec K/F$.

First property (i) is obvious because $A_{\tilde F} \simeq \tilde A \otimes (d,kt)$ and $\tilde A_{\tilde F(\tilde A)}$ splits. To prove property (ii), let us consider an extension K/F such that $\operatorname{ind} A_{K(\sqrt{d})} = 1$. Then $[A_K] \in \operatorname{Br}(K(\sqrt{d})/K) = \{(d,x) \mid x \in K^*\}$. Let $x \in K^*$ be such that $[A_K] = (d,x)$ in the Brauer group $\operatorname{Br}_2(K)$. Set $E = K(t)(\sqrt{tk/x})$. Clearly, $(d,tk)_E = (d,x)_E$. Since $[A_K] = (d,x)$ and $K \subset E$, we have $[A_E] = (d,x)_E$. Therefore, $[\tilde A_E] = [A_E] + (d,tk)_E = (d,x)_E + (d,x)_E = 0$. The "generic property" of the extension $\tilde F(\tilde A)/\tilde F$ shows that $\tilde F(\tilde A)/\tilde F \prec E/\tilde F$. Therefore, $\tilde F(\tilde A)/F \prec E/F$. Obviously, E/K is a purely transcendental extension. Hence $E/F \prec K/F$. Therefore, $\tilde F(\tilde A)/F \prec E/F \prec K/F$. □

4. Generic splitting fields of the Clifford algebra $C_0(\phi)$

Let ϕ be a quadratic form over F and $C_0(\phi)$ be the even Clifford algebra of F. We recall that $\operatorname{ind}\phi = \operatorname{ind} C_0(\phi)$. Hence, for any extension K/F the condition $\operatorname{ind}\phi_K = 1$ holds if and only if the F-algebra $C_0(\phi)$ splits over K/F. The main task of this section is to study generic extensions of the set

$$\mathcal{K}_{C_0(\phi)}/F = \{K/F \mid \operatorname{ind}\phi_K = 1\} = \{K/F \mid C_0(\phi) \text{ splits over } K/F\}.$$

An arbitrary generic extension of this set will be called as *a generic splitting field of the even Clifford algebra of* ϕ. The results of the previous section show that a generic extension exists and can be defined as $F(C_0(\phi))/F$ (see Definition 3.3). The extension $F(C_0(\phi))/F$ is of special interest in our paper in view of the following easy lemma.

Lemma 4.1. *For an F-form ϕ we have $F(C_0(\phi))/F \prec F_\phi/F$.*

Proof. It suffices to prove that $F_\phi/F \in \mathcal{K}_{C_0(\phi)}/F$. In other words, we need to prove that $\operatorname{ind}\phi_{F_\phi} = 1$. It is obvious because $\dim(\phi_{F_\phi})_{an} \le 2$. □

Corollary 4.2. *For an F-form ϕ we have $H^3(F(C_0(\phi))/F) \subset H^3(F_\phi/F)$.*

Remark 4.3. Taking into account the results of the previous section and properties of Clifford algebras, we can suggest the following constructions of generic splitting fields of the even Clifford algebra.

- Let ϕ be either an odd-dimensional form or $\phi \in I^2(F)$. Let A be a central simple F-algebra such that $[A] = c(\phi)$ in the Brauer group $\operatorname{Br}_2(F)$. Then $F(\operatorname{SB}(A))/F$ is a generic splitting field for $C_0(\phi)$.
- Let ϕ be an even-dimensional form with $d = d_\pm\phi \notin F^{*2}$. Let A be a central simple F-algebra such that $[C_0(\phi)] = [A_L]$ in the Brauer group of $L = F(\sqrt{d})$. For example, we can take A to be equal to $C(\phi)$. Then $F(A_L)/F = F(R_{L/F}(\operatorname{SB}(A_L)))/F$ is a generic splitting field of $C_0(\phi)$.

Lemma 4.4. *Let ϕ be an odd-dimensional form and $\tilde\phi = \phi \perp \langle d_\pm\phi\rangle$. Then $F(C_0(\phi))/F \sim F(C_0(\tilde\phi))/F$.*

Proof. It suffices to prove that $\mathcal{K}_{C_0(\phi)}/F = \mathcal{K}_{C_0(\tilde\phi)}/F$. In other words, we need to prove that for any extension K/F the conditions $\operatorname{ind}(\phi_K)_{an} = 1$ and $\operatorname{ind}(\tilde\phi_K)_{an} = 1$ are equivalent. It is obvious because $c(\phi) = c(\tilde\phi)$. □

Lemma 4.5. *Let Let ϕ be an even-dimensional form such that $d = d_\pm\phi \notin F^{*2}$. Let $\tilde{F} = F(t)$ and $\tilde{\phi} = \phi \perp t\langle\langle d\rangle\rangle$. Then $F(C_0(\phi))/F \sim \tilde{F}(C_0(\tilde{\phi}))/F$ and $\operatorname{ind}\tilde{\phi} = 2\operatorname{ind}\phi$.*

Proof. We have $F(C_0(\phi))/F \sim F(A_L)/F$ with $A = C(\phi)$ and $L = F(\sqrt{d})$. Let $\psi = \phi \perp \langle\langle d\rangle\rangle$. Since $\psi \in I^2(F)$, it follows that $C(\psi)$ is a central simple F-algebra. Since ϕ_L and ψ_L are Witt equivalent, the L-algebras A_L and $C(\psi)_L$ are Brauer equivalent. Therefore, $[A] - c(\psi) \in \mathrm{Br}(L/F)$. Since $\mathrm{Br}(L/F) = \{(d,k) \mid k \in F^*\}$, it follows that $[A] - c(\psi) = (d,k)$ for suitable $k \in F^*$. In the Witt ring $W(F(t))$, we have $\psi - \tilde{\phi} = \langle\langle d\rangle\rangle - t\langle\langle d\rangle\rangle = \langle\langle d,t\rangle\rangle$. Hence $c(\psi_{\tilde{F}}) - c(\tilde{\phi}) = (d,t)$. Therefore, $c(\tilde{\phi}) = c(\psi_{\tilde{F}}) + (d,t) = [A_{\tilde{F}}] + (d,k) + (d,t) = [A_{\tilde{F}}] + (d,tk)$. Letting $\tilde{A} = A_{\tilde{F}} \otimes (d,tk)$, we have $\tilde{F}(C_0(\tilde{\phi}))/\tilde{F} \sim \tilde{F}(\tilde{A})/\tilde{F}$. By Proposition 3.6, we have $\tilde{F}(C_0(\tilde{\phi}))/F \sim \tilde{F}(\tilde{A})/F \sim F(A_L)/F \sim F(C_0(\phi))/F$.

Now, it suffices to prove that $\operatorname{ind}\tilde{\phi} = 2\operatorname{ind}\phi$. By Tignol's theorem [T], we have $\operatorname{ind}\tilde{\phi} = \operatorname{ind}(A \otimes (d,tk)) = 2\operatorname{ind} A_{F(\sqrt{d})} = 2\operatorname{ind} C_0(\phi) = 2\operatorname{ind}\phi$. □

5. A version of Peyre–Merkurev theorem

Let X be a smooth variety. Sometimes, it is very easy to construct some elements of the group $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$. If we know that a central simple algebra A splits over $F(X)$, we get a homomorphism $F^* \to H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$, $x \to [A] \cup (x)$ where $[A]$ is the class of the algebra A in the Brauer group $H^2(F, \mathbb{Q}/\mathbb{Z}(1))$ and (x) is the element corresponding to $x \in F^*$ via the natural homomorphism $F^* \to H^1(F, \hat{\mathbb{Z}}(1))$. Thus, for any $[A] \in \mathrm{Br}(F(X)/F)$ and $x \in F^*$ we get the element $[A] \cup (x)$ of the group $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$. There exists an evident generalization of this construction. Let L/F be a finite field extension and A be a central simple algebra over L such that A splits over $L(X)$. Then we get the homomorphism $L^* \to H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$, $x \to N_{L/F}([A] \cup (x))$, where $[A]$ is the class of the algebra A in the Brauer group $H^2(L, \mathbb{Q}/\mathbb{Z}(1))$ and (x) is the element corresponding to $x \in L^*$ via the natural homomorphism $L^* \to H^1(L, \hat{\mathbb{Z}}(1))$.

Definition 5.1. Let $A = A_1 \times \cdots \times A_s$ be a semisimple separable F-algebra where $A_1, \ldots, A_s$ are simple algebras with centers $L_1, \ldots, L_s$.

- We say that A splits over X if the algebra $A_{F(X)}$ splits. This means that A_i splits over $L_i(X)$ for all $i = 1, \ldots, s$.
- We denote by $H^3_A(F, \mathbb{Q}/\mathbb{Z}(2))$ the subgroup of $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ defined as follows:
$$H^3_A(F, \mathbb{Q}/\mathbb{Z}(2)) = \sum_{i=0\ldots s} \{N_{L_i/F}([A_i] \cup (x_i)) \mid x_i \in L_i^*\}$$
- If A has exponent 2, we obviously have $H^3_A(F, \mathbb{Q}/\mathbb{Z}(2)) \subset H^3(F)$. In this case, we use the notation $H^3_A(F)$ instead of $H^3_A(F, \mathbb{Q}/\mathbb{Z}(2))$.

Example 5.2. Let A be a separable F-algebra and X_A be the generic splitting variety of A constructed in Definition 3.3. Let $F(A) = F(X_A)$. Since A splits over $F(A)$, we have $H^3_A(F,\mathbb{Q}/\mathbb{Z}(2)) \subset H^3(F(A)/F,\mathbb{Q}/\mathbb{Z}(2))$. If A has exponent 2 then $H^3_A(F) \subset H^3(F(A)/F)$. In particular, $H^3_{C_0(\phi)}(F) \subset H^3(F(C_0(\phi))/F)$.

Remark 5.3. If A splits over X, and E/F is a field extension then A_E splits over X_E and hence $H^3_{A_E}(E,\mathbb{Q}/\mathbb{Z}(2)) \subset H^3(E(X)/E,\mathbb{Q}/\mathbb{Z}(2))$. To simplify notation, we will write $H^3_A(E,\mathbb{Q}/\mathbb{Z}(2))$ instead of $H^3_{A_E}(E,\mathbb{Q}/\mathbb{Z}(2))$.

Definition 5.4. Let X be a smooth variety over F and A be a separable F-algebra. We say that A is *maximal with respect to* X if A splits over X and for any purely transcendental extension $\tilde{F}/F$ (not necessarily finite), the factor group

$$\frac{H^3(\tilde{F}(X)/\tilde{F},\mathbb{Q}/\mathbb{Z}(2))}{H^3_A(\tilde{F},\mathbb{Q}/\mathbb{Z}(2))}$$

is finite.

Example 5.5. (E. Peyre, A. S. Merkurev). Let X be a projective homogeneous variety. By [Su, §24], there exists a canonical (with respect to F) homomorphism

$$\tau_X : H^3(F(X)/F,\mathbb{Q}/\mathbb{Z}(2)) \to \operatorname{Tors}\operatorname{CH}^2(X)$$

In [P2], it was proved that the homomorphism τ_X is surjective. Moreover, it was shown that $\ker\tau_X = H^3_A(F,\mathbb{Q}/\mathbb{Z}(2))$ where A is the separable F-algebra associated with X (see [M3, §2]). Merkurev-Peyre theorem claims that the algebra A splits over X and there exists a natural isomorphism

$$\frac{H^3(F(X)/F,\mathbb{Q}/\mathbb{Z}(2))}{H^3_A(F,\mathbb{Q}/\mathbb{Z}(2))} \simeq \operatorname{Tors}\operatorname{CH}^2(X).$$

Since X is a projective homogeneous variety, the group $K(X_E)$ is finitely generated for all extensions E/F ([Pa]). Since $\operatorname{CH}^2(X_E) \simeq K(X_K)^{(2/3)}$, it follows that the group $\operatorname{Tors}\operatorname{CH}^2(X_E)$ is finite for all extensions E/F. This implies that A is maximal with respect to X. □

Proposition 5.6. *Let X be a smooth F-variety and let A and B be separable F-algebras. Suppose that A splits over X and B is maximal with respect to X. Then*

$$H^3_A(F,\mathbb{Q}/\mathbb{Z}(2)) \subset H^3_B(F,\mathbb{Q}/\mathbb{Z}(2)).$$

Proof. Clearly, we can assume that A as a central simple algebra over some separable extension L/F. Since L/F is separable, we can write L in the form $L = F(\theta)$. For any extension E/F, we define the group $V(E)$ as the factor group

$$\frac{H^3_A(E,\mathbb{Q}/\mathbb{Z}(2)) + H^3_B(E,\mathbb{Q}/\mathbb{Z}(2))}{H^3_B(E,\mathbb{Q}/\mathbb{Z}(2))}$$

Since B is maximal with respect to X, the group $V(E)$ is finite for all purely transcendental extensions E/F. For any extensions $F \subset E_1 \subset E_2$ we have the natural homomorphism $V(E_1) \to V(E_2)$ induced by the homomorphism $H^3(E_1,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(E_2,\mathbb{Q}/\mathbb{Z}(2))$.

Lemma 5.7. *Let E/F be an extension and let $\tilde{E}/E$ be a purely transcendental extension of degree $n = [L : F]$. Then the homomorphism $V(E) \to V(\tilde{E})$ is injective. If $V(E) \neq 0$ then the homomorphism $V(E) \to V(\tilde{E})$ is not surjective.*

Before we prove this lemma, we complete the proof of Proposition 5.6. Since the condition $V(F) = 0$ and $H^3_A(F, \mathbb{Q}/\mathbb{Z}(2)) \subset H^3_B(F, \mathbb{Q}/\mathbb{Z}(2))$ are obviously equivalent, we can assume that $V(F) \neq 0$. Let us construct the fields $E_0, E_1, \dots$ as follows: we set $E_0 = F$ and for $i > 0$, we define E_i as the purely transcendental extension of E_{i-1} of degree n. Let $E = \cup_i E_i$. Lemma 5.7 (together with our assumption $V(F) \neq 0$) implies that $0 < |V(E_0)| < |V(E_1)| < \dots < |V(E_i)| \leq |V(E)|$. Therefore $|V(E)| > i$ for all $i \geq 0$. Hence the group $V(E)$ is infinite. We get contradiction. The proposition is proved.

Now, we prove Lemma 5.7. Injectivity of $V(E) \to V(\tilde{E})$ is obvious (it suffices to apply standard "specialization" arguments). Let us prove that $V(E) \to V(\tilde{E})$ is not surjective. Changing notation, we can assume that $E = F$ and $\tilde{E} = \tilde{F} = F(t_0, \dots, t_{n-1})$. Since $V(F) \neq 0$, there exists $x \in L^*$ such that $N_{L/F}([A] \cup (x)) \notin H^3_B(F, \mathbb{Q}/\mathbb{Z}(2))$. We denote by u the element of $V(F)$ given by $N_{L/F}([A] \cup (x))$. Let us write x in the form $x = \sum_{i=0}^{n-1} a_i \theta^i$ with $a_0, \dots, a_i \in F$. Set $\tilde{L} = \tilde{F}(\theta)$ and $\tilde{x} = \sum_{i=0}^{n-1} t_i \theta^i \in \tilde{L}^*$. Let $\tilde{u}$ be the element of $V(\tilde{F})$ given by $N_{\tilde{L}/\tilde{F}}([A] \cup (\tilde{x})) \in H^3_A(\tilde{F}, \mathbb{Q}/\mathbb{Z}(2))$. To complete the proof of the lemma it suffices to show that $\tilde{u} \notin im(V(F) \to V(\tilde{F}))$. Suppose the contrary, $\tilde{u} \in im(V(F) \to V(\tilde{F}))$. Then for any places $p_1, p_2 : \tilde{F} \to F$, we obviously have $(p_1)_*(\tilde{u}) = (p_2)_*(\tilde{u})$. Let us define p_1 and p_2 as follows

$$p_1 : (t_0, t_1, \dots, t_{n-1}) \mapsto (a_0, a_1, \dots, a_{n-1}) \qquad p_1 : (t_0, t_1, \dots, t_{n-1}) \mapsto (1, 0, \dots, 0).$$

Clearly, $(p_1)_*(\tilde{u}) = u$ and $(p_2)_*(\tilde{u}) = 0$. Since $u \neq 0 \in V(F)$, we get a contradiction. The proof is complete. □

Corollary 5.8. *Let X be a smooth F-variety and let A and B be separable F-algebras. Suppose that A and B are maximal with respect to X. Then*

$$H^3_A(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3_B(F, \mathbb{Q}/\mathbb{Z}(2)).$$

The following theorem should be considered as a version of Peyre–Merkurev theorem ([P2], [M3]).

Theorem 5.9. *For any projective homogeneous F-variety X there exists at least one separable algebra which is maximal with respect to X. If a separable algebra A is maximal with respect to X, then there exists a canonical isomorphism*

$$\frac{H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))}{H^3_A(F, \mathbb{Q}/\mathbb{Z}(2))} \simeq \operatorname{Tors} \operatorname{CH}^2(X).$$

Proof. Obvious consequence of Example 5.5 and Corollary 5.8. □

Corollary 5.10. *Let X be a projective homogeneous variety over F such that the group $H^3(\tilde{F}(X)/\tilde{F}, \mathbb{Q}/\mathbb{Z}(2))$ is finite for any purely transcendental extension $\tilde{F}/F$. Then* $\operatorname{Tors}\operatorname{CH}^2(X) \simeq H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2))$.

Proof. Clearly, the trivial algebra $A = F$ is maximal with respect to X. By Theorem 5.9, we are done. □

6. Computing the group $H^3(F(C_0(\phi))/F)$

In this section, we compute the group $H^3(F(X)/F)$ in the case where $X = X_{C_0(\phi)}$ is a generic splitting variety of the Clifford algebra $C_0(\phi)$. For this, we need the following result of E. Peyre [P1] (see also [IK3, Appendix A]).

Theorem 6.1. (E. Peyre). *Let A be a central simple F-algebra of exponent 2. The factor group $H^3(F(A)/F)/[A]H^1(F)$ is canonically isomorphic to 2-torsion subgroup of the second Chow group* $\operatorname{CH}^2(\operatorname{SB}(A))$.

This theorem shows that the computation of the group $H^3(F(A)/F)$ is equivalent to the computation of the second Chow group of the Severi–Brauer variety of A. The group $\operatorname{CH}^2(\operatorname{SB}(A))$ was studied by N. Karpenko in [K2]. To state one the main results of Karpenko's paper we recall a definition.

Definition 6.2. Let D be a central division algebra over F. We say that D is *decomposable* if there exist algebras $D_1 \neq F$ and $D_2 \neq F$ such that $D \simeq D_1 \otimes_F D_2$. Now, let A be an arbitrary central simple algebra over F. By the main Wederbarn theorem, A can be written in the form $M_n(D)$ where D is a division F-algebra. We say that D is *the division part of A*.

Theorem 6.3. (N. Karpenko, [K2]). *Let A be a central simple F-algebra of exponent 2. Then the group* $\operatorname{Tors}\operatorname{CH}^2(\operatorname{SB}(A))$ *is either zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. If the division part of A is decomposable or* $\operatorname{ind} A \leq 4$ *then* $\operatorname{Tors}\operatorname{CH}^2(\operatorname{SB}(A)) = 0$.

Corollary 6.4. *Let A be a central simple algebra over F. Then*
- *the factor group $H^3(F(A)/F)/[A]H^1(F)$ is zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$,*
- *if the division part of A is decomposable or* $\operatorname{ind} A \leq 4$ *then*
$$H^3(F(A)/F) = [A]H^1(F).$$

Definition 6.5. Let E/F be an extension and B be a central simple algebra over E. We say that *the division part of B is defined over F* if there exists a central simple F-algebra D such that D_E is isomorphic to the division part of A. In other word, this means that D_E is a division algebra and $[A] = [D_E]$ in the Brauer group $\operatorname{Br}(E)$.

Proposition 6.6. *Let A be a central simple F-algebra of exponent 2 and let $L = F(\sqrt{d})$ be a quadratic extension. Then*
1. *$H^3_{A_L}(F) = \{[A] \cup (a) \mid a \in F^* \text{ is such that } (a,d) = 0\}$,*
2. *The factor group $H^3(F(A_L)/F)/H^3_{A_L}(F)$ is zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$,*
3. *if the division part of A_L is defined over F then $H^3(F(A_L)/F) = H^3_{A_L}(F)$.*

Proof. (1) By the definition, $H^3_{A_L}(F) = \{N_{L/F}([A_L] \cup (x)) \mid x \in L^*\}$. The projection formula shows that $N_{L/F}([A_L] \cup (x)) = [A] \cup (N_{L/F}(x))$. Since x runs over all elements of L^*, it follows that $N_{L/F}(x)$ runs over all elements $a \in F^*$ satisfying the condition $(a, d) = 0$.

(2) Let $\tilde{F} = F(t)$ and $\tilde{A} = A_{\tilde{F}} \otimes (t, d)$. By Proposition 3.6, we have $F(A_L)/F \sim \tilde{F}(\tilde{A})/F$. Therefore, $H^3(F(A_L)/F) = H^3(\tilde{F}(\tilde{A})/F)$. Let us consider $\tilde{F} = F(t)$ as a subfield of $\hat{F} = F((t))$. Set $\hat{A} = \tilde{A}_{\hat{F}}$ Since $\tilde{F} \subset \hat{F}$, we have $H^3(\tilde{F}(\tilde{A})/F) \subset H^3(\hat{F}(\hat{A})/F)$ and hence $H^3(F(A_L)/F) \subset H^3(\hat{F}(\hat{A})/F)$. Therefore, the image of the group $H^3(F(A_L)/F)$ under the homomorphism $H^3(F) \to H^3(\hat{F})$ belongs to the group

$$U \overset{\text{Def}}{=} H^3(\hat{F}(\hat{A})/\hat{F}) \cap im(H^3(F) \to H^3(\hat{F})).$$

Set

$$U_0 = [\hat{A}]H^1(\hat{F}) \cap im(H^3(F) \to H^3(\hat{F})).$$

By Corollary 6.4, the group $H^3(\hat{F}(\hat{A})/\hat{F})/[\hat{A}]H^1(\hat{F})$ is zero or $\mathbb{Z}/2\mathbb{Z}$. Therefore, the factor group U/U_0 is zero or $\mathbb{Z}/2\mathbb{Z}$. To compute the group U_0, we use well-known isomorphism $H^m(\hat{F}) \simeq H^m(F) \oplus (t) \cup H^{m-1}(F)$. In what follows, we will identify the group $H^m(\hat{F})$ with $H^m(F) \oplus (t) \cup H^{m-1}(F)$. In particular, we will regard the group $H^3(F)$ as a subgroup of $H^3(\hat{F})$. Using these agreements, we can write U and U_0 in the forms

$$U = H^3(\hat{F}(\hat{A})/\hat{F}) \cap H^3(F) \qquad \text{and} \qquad U_0 = [\hat{A}]H^1(\hat{F}) \cap H^3(F).$$

In particular, one can consider U_0 and U as subgroups of $H^3(F)$.

Lemma 6.7. *$U_0 \subset H^3_{A_L}(F) \subset H^3(F(A_L)/F) \subset U$.*

Proof. The condition $H^3_{A_L}(F) \subset H^3(F(A_L)/F) \subset U$ was already proved. It suffices to prove that $U_0 \subset H^3_{A_L}(F)$. Let $u \in U_0 = [\hat{A}]H^1(\hat{F}) \cap H^3(F)$. It is well-known that

$$H^1(\hat{F}) \simeq \hat{F}^*/\hat{F}^{*2} = \{at^i \mid\ a \in F^* \pmod{F^{*2}} \text{ and } i \in \mathbb{Z} \pmod{2\mathbb{Z}}\}.$$

Hence, u can be written in the form $u = [\hat{A}] \cup (at^i)$ with $a \in F^*$ and $i \in \mathbb{Z}$. We have

$$\begin{aligned} u =& [\hat{A}] \cup (at^i) = ([A] + (d, t)) \cup (at^i) \\ =& [A] \cup (a) + i[A] \cup (t) + (d, t, a) + i(d, t, t) \\ =& [A] \cup (a) + i[A] \cup (t) + (d, t, a) + i(d, -1, t) \\ =& [A] \cup (a) + (t) \cup \big(i[A] + (d, (-1)^i a)\big) \end{aligned}$$

Since $u \in H^3(F)$, it follows that $u = [A] \cup (a)$ and $i[A] + (d, (-1)^i a) = 0$. Let us consider two cases $i = 0$ and $i = 1 \pmod{2\mathbb{Z}}$.

If $i = 0 \pmod{2\mathbb{Z}}$ then the equation $i[A] + (d, (-1)^i a) = 0$ simply means that $(d, a) = 0$. This implies that $u = [A] \cup (a) \subset H^3_{A_L}(F)$.

If $i = 1 \pmod{2\mathbb{Z}}$ then the equation $i[A] + (d, (-1)^i a) = 0$ means that $[A] = (d, -a)$. Then $u = [A] \cup (a) = (d, -a) \cup (a) = (d, -a, a) = 0$.

We have proved that $u \in H^3_{A_L}(F)$. The proof of the lemma is complete. $\square$

Since $|U/U_0| \leq 2$, Lemma 6.7 shows that $|H^3(F(A_L)/F)/H^3_{A_L}(F)| \leq 2$. This completes the proof of statement (2) of the proposition.

(3). Taking into account Lemma 6.7, it suffices to show that $U_0 = U$. For this, it suffices to prove that $H^3(\hat{F}(\hat{A})/\hat{F}) = [\hat{A}]H^1(\hat{F})$. Corollary 6.4 shows that it suffices to verify that either $\operatorname{ind}\hat{A} \leq 4$ or the division part of $\hat{A}$ is decomposable.

By the hypotheses of the proposition, the division part of algebra A_L is defined over F. This means that there exists an F-algebra D such that $[A_L] = [D_L]$ and D_L is a division algebra. Since $[A_L] = [D_L]$, we have $[A] - [D] \in \operatorname{Br}(L/F)$. Hence, there exists $k \in F^*$ such that $[A] - [D] = (k, d)$. Therefore, $[\hat{A}] = [A] + (d,t) = [D] + (k,d) + (d,t) = [D] + (d,kt)$. We claim that $D_{\hat{F}} \otimes (d,kt)$ is the division part of $\hat{A}$. Since $[\hat{A}] = [D_{\hat{F}} \otimes (d,kt)]$, it suffices to prove that $\operatorname{ind}(D_{\hat{F}} \otimes (d,kt)) = \deg(D_{\hat{F}} \otimes (d,kt))$. By Tignol's theorem [T], we have $\operatorname{ind}(D_{\hat{F}} \otimes (d,kt)) = 2\operatorname{ind} D_L$. Since D_L is a division algebra, it follows that $\operatorname{ind} D_L = \deg D$. Therefore,

$$\operatorname{ind}(D_{\hat{F}} \otimes (d,kt)) = 2\operatorname{ind} D_L = 2\deg D = \deg(D_{\hat{F}} \otimes (d,kt)).$$

Hence, $D_{\hat{F}} \otimes (d,kt)$ is the division part of $\hat{A}$. Clearly, $D_{\hat{F}} \otimes (d,kt)$ is either decomposable (if $[D] \neq 0$), or has index 2 (if $[D] = 0$). This completes the proof. $\square$

To apply our results to the Clifford algebra $C_0(\phi)$, we need the following

Lemma 6.8. *Let ϕ be a quadratic form over F. Then*

(1) *if $\dim\phi$ is odd then $H^3_{C_0(\phi)}(F) = c(\phi)H^1(F)$,*

(2) *if $\dim\phi$ is even and ϕ belongs to $I^2(F)$ then*

$$H^3_{C_0(\phi)}(F) = c(\phi)H^1(F) = \{e^3(\langle\langle a\rangle\rangle\,\phi) \mid a \in F^*\},$$

(3) *if $\dim\phi$ is even and $\phi \notin I^2(F)$ then*

$$H^3_{C_0(\phi)}(F) = \{e^3(\langle\langle a\rangle\rangle\,\phi) \mid a \in F^* \text{ is such that } \langle\langle a\rangle\rangle\,\phi \in I^3(F)\}.$$

Proof. The case where $\dim\phi$ is odd is obvious because $C_0(\phi)$ is a central simple algebra with $[C_0(\phi)] = c(\phi)$. If $\phi \in I^2(F)$ then $C_0(\phi)$ is of the form $A \times A$ where A is a central simple algebra satisfying the condition $[A] = c(\phi)$. Hence,

$$H^3_{C_0(\phi)}(F) = c(\phi)H^1(F) = e^2(\phi)\cdot\{e^1(\langle\langle a\rangle\rangle) \mid a \in F^*\} = \{e^3(\phi\,\langle\langle a\rangle\rangle) \mid a \in F^*\}.$$

Now, let ϕ be an even-dimensional form with nontrivial discriminant d. Let $L = F(\sqrt{d})$, $\phi = \psi \perp \langle\langle d\rangle\rangle$ and $A = C(\psi)$. Since $[C_0(\phi)] = [A_L]$, Proposition 6.6(1) shows that it suffices to prove that:

(i) for any $a \in F^*$, the conditions $(d,a) = 0$ and $\phi\,\langle\langle a\rangle\rangle \in I^3(F)$ are equivalent,

(ii) if $a \in F^*$ is such that $(d,a) = 0$ then $[A] \cup (a) = e^3(\phi\,\langle\langle a\rangle\rangle)$.

By APH, the conditions $\langle\langle d,a\rangle\rangle \in I^3(F)$ and $(d,a)=0$ are equivalent. We have $\phi\,\langle\langle a\rangle\rangle + \langle\langle d,a\rangle\rangle = \psi\,\langle\langle a\rangle\rangle \in I^2(F)\cdot I(F) \subset I^3(F)$. Hence the conditions $\phi\,\langle\langle a\rangle\rangle \in I^3(F)$ and $(d,a)=0$ are equivalent. Statement (i) is proved. Now, we assume that $(d,a)=0$. Then $\phi\,\langle\langle a\rangle\rangle = \psi\,\langle\langle a\rangle\rangle$ and hence $[A]\cup(a) = e^2(\psi)\cup e^1(\langle\langle a\rangle\rangle) = e^3(\psi\,\langle\langle a\rangle\rangle) = e^3(\phi\,\langle\langle a\rangle\rangle)$. □

Theorem 6.9. *For any quadratic form ϕ, the factor group*

$$\frac{H^3(F(C_0(\phi))/F)}{H^3_{C_0(\phi)}(F)}$$

is zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Moreover, the identity $H^3(F(C_0(\phi))/F) = H^3_{C_0(\phi)}(F)$ is necessarily holds in the following cases:

- $\dim\phi$ *is odd (or $\phi \in I^2(F)$) and* $\operatorname{ind}\phi \le 4$,
- $\dim\phi$ *is odd (or $\phi \in I^2(F)$) and the division algebra corresponding to $c(\phi)$ is decomposable,*
- $\dim\phi$ *is even, $\phi \notin I^2(F)$ and the division part of $C_0(\phi)$ is defined over F (for example,* $\operatorname{ind}\phi \le 2$*),*

Proof. First, we suppose that $\dim\phi$ is odd or $\phi \in I^2(F)$. By Lemma 6.8, we have $H^3_{C_0(\phi)}(F) = c(\phi)H^1(F)$. After this, the required result follows immediately from Corollary 6.4. Now, we assume that $\dim\phi$ is even and $\phi \notin I^2(F)$. Then $C_0(\phi)$ has the form A_L where A is a central simple F-algebra and L is the discriminant extension. In this case, the required result follows from Proposition 6.6. □

7. The group $H^3(F_n/F)$ in the case $n < h'(\phi)$

During this section we work with an anisotropic form ϕ over F. By $F_0 = F, F_1, \dots, F_h$, we denote the generic splitting tower of ϕ. For $n = 0,\dots,h$, we set $\phi_n = (\phi_{F_n})_{an}$.

The main task of this section is to prove the following theorem.

Theorem 7.1. *Let ϕ be an F-form and n be an integer such that $n < h'(\phi)$.*

(i) *the group $H^3(F_n/F)$ is always zero except (possibly) the case where:*

$$\dim\phi \text{ is odd}, \quad \operatorname{ind}\phi = 2 \text{ and } n = h'(\phi)-1.$$

(ii) *If $\dim\phi$ is odd, $\operatorname{ind}\phi = 2$ and $n = h'(\phi)-1$ then the group $H^3(F_n/F)$ is generated by $e^3(\phi \perp -\mu)$ where μ is a 3-dimensional form satisfying the conditions $d_\pm\mu = d_\pm\phi$ and $c(\mu) = c(\phi)$.*

We start the proof from the following lemma.

Lemma 7.2. *Let ϕ be a form such that either $\dim\phi \le 4$ or ϕ is a 3-fold neighbor. Then*

(i) *if L/F is a field extension such that ϕ_L is isotropic then $\dim(\phi_L)_{an} \le 2$ except (possibly) the case where ϕ is a 5-dimensional Pfister neighbor.*

(ii) *there exists an extension L/F of degree ≤ 2 such that $\dim(\phi_L)_{an} \le 2$.*

Proof. In the case $\dim\phi \leq 4$, the both statements (i) and (ii) are obvious. Now, we assume that ϕ is a 3-fold neighbor. Then there exist π and μ such that $\pi = \phi \perp \mu \in GP_3(F)$. If L/F is an extension such that ϕ_L is isotropic, then π_L is also isotropic and hence hyperbolic. Then the forms ϕ_L and $(-\mu)_L$ are Witt equivalent. Therefore, $\dim(\phi_L)_{an} \leq \dim\mu = 6 - \dim\phi$. If $\dim\phi \geq 6$ then $\dim(\phi_L)_{an} \leq 8 - 6 \leq 2$. This completes the proof of statement (i). To prove (ii), we can assume that ϕ is a 5-dimensional Pfister neighbor. Then ϕ is similar to a form of the type $\langle\langle a, b\rangle\rangle \perp \langle c\rangle$. Setting $L = F(\sqrt{a})$, we get $(\phi_L)_{an} = \langle c\rangle$ and hence $\dim(\phi_L)_{an} \leq 2$. ☐

Lemma 7.3. *Let $k \leq h'(\phi)$ be the maximal integer such that $H^3(F_k/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$. Then*

(i) *ether $\dim\phi_k \leq 4$ or ϕ_k is an anisotropic 3-fold Neighbor,*
(ii) *$k \geq h'(\phi) - 1$ except for the case where:*

$$\phi_k \text{ is a 5-dimensional Pfister neighbor and } k = h'(\phi) - 2.$$

(iii) *there exists an extension L/F_k of degree ≤ 2 such that $\dim(\phi_L)_{an} \leq 2$,*

Proof. (i) If $k = h'(\phi)$ then $\dim\phi_k \leq 2$ and the proof is complete. Now, let $k < h'(\phi)$. By the definition of k, we obviously have

$$H^3(F_k/F, \mathbb{Q}/\mathbb{Z}(2)) = 0 \quad \text{and} \quad H^3(F_{k+1}/F, \mathbb{Q}/\mathbb{Z}(2)) \neq 0.$$

Since $F_{k+1} = F_k(\phi_k)$, we have $H^3(F_k(\phi_k)/F_k, \mathbb{Q}/\mathbb{Z}(2)) \neq 0$. Finally, Arason's theorem shows that ether $\dim\phi_k \leq 4$ or ϕ_k is an anisotropic 3-fold Neighbor.

(ii) If ϕ_k is a 5-dimensional form then $h'(\phi_k) = 2$ and hence $k = h'(\phi) - 2$. Now, we can assume that ϕ_k is not a 5-dimensional Pfister neighbor and $k < h'(\phi) - 1$. Since $\phi_{k+1} = ((\phi_k)_{F_{k+1}})_{an}$, Lemma 7.2(i) shows that $\dim\phi_{k+1} \leq 2$. By the definition of $h'(\phi)$, we get $k + 1 \geq h'(\phi)$. Hence, $k \geq h'(\phi) - 1$.

(iii) Obvious in view of Item (i) and Lemma 7.2(ii). ☐

Proposition 7.4. *For any form ϕ, we have $2H^3(F_\phi/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$. In other words, $H^3(F_\phi/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F_\phi/F)$.*

Proof. Let k and L/F_k be as in Lemma 7.3. Since $[L : F_k] \leq 2$, standard transfer arguments show that $2H^3(L/F_k, \mathbb{Q}/\mathbb{Z}(2)) = 0$. Since $H^3(F_k/F, \mathbb{Q}/Z(2)) = 0$, we have $2H^3(L/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$. Since $\dim(\phi_L)_{an} \leq 2$, the generic property of F_ϕ shows that $F_\phi/F \prec L/F$. Hence $H^3(F_\phi/F, \mathbb{Q}/\mathbb{Z}(2)) \subset H^3(L/F, \mathbb{Q}/\mathbb{Z}(2))$. Therefore, $2H^3(F_\phi/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$. ☐

Corollary 7.5. *$H^3(F_n/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F_n/F)$ for all $n \leq h'(\phi)$. Moreover, $H^3(F(C_0(\phi))/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(C_0(\phi))/F)$*

Proof. Follows form Proposition 7.4 and the relations $F_n/F \prec F_\phi/F$ and $F(C_0(\phi))/F \prec F_\phi/F$. ☐

Proposition 7.6. *Let ϕ be a quadratic form over F. Let X be either a generic splitting variety of X or a generic splitting variety of the even Clifford algebra $C_0(\phi)$. Then $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X)/F)$.*

Proof. Follows from the previous corollary and Lemma 2.4. □

The following lemma is an easy consequence of Merkurev's index reduction formula [M2].

Lemma 7.7. (see [HR, Th. 1.6] or [Ho2, Prop.2.1]). *If* $\operatorname{ind}\phi \geq 2^m > 1$ *then there exists* $n \leq h(\phi)$ *such that* $\operatorname{ind}(\phi_{F_n})_{an} = 2^m$ *and*

$$\dim(\phi_{F_n})_{an} = \begin{cases} 2m+1, & \textit{if } \dim\phi \textit{ is odd,} \\ 2m+2, & \textit{if } \dim\phi \textit{ is even.} \end{cases}$$

Proof of Theorem 7.1. (i) Let k be as in Lemma 7.3. If $n \leq k$ then the group $H^3(F_n/F, \mathbb{Q}/\mathbb{Z}(2))$ is zero and the proof is complete. Hence, we can assume that $n > k$. Then $k < n \leq h'(\phi) - 1$. Then Item (ii) of Lemma 7.3 shows that ϕ_k is an anisotropic 5-dimensional Pfister neighbor and $k = h'(\phi) - 2$. Since $h'(\phi) - 2 = k < n \leq h'(\phi) - 1$, it follows that $n = h'(\phi) - 1$. Since $\dim\phi_k = 5$, it follows that ϕ is an odd-dimensional form. Now, it suffices to prove that $\operatorname{ind}\phi = 2$. Since ϕ_k is a 5-dimensional Pfister neighbor, we have $\operatorname{ind}\phi_k = 2$. Then Lemma 7.7 shows that $\operatorname{ind}\phi = 2$.

(ii) First, we prove existence of μ. Since $\operatorname{ind}\phi = 2$, we have $c(\phi) = (a, b)$ for suitable $a, b \in F^*$. Set $\mu = d\langle -a, -b, ab\rangle$ with $d = d_\pm\phi$. Then $d_\pm\phi = d = d_\pm\mu$ and $c(\phi) = (a, b) = c(\mu)$.

We obviously have $\phi \equiv \mu \pmod{I^3(F)}$. Letting $\pi = \phi \perp -\mu$, we have $\pi \in I^3(F)$. Since $n = h'(\phi) - 1$ and $\operatorname{ind}\phi = 2$, Lemma 7.7 shows that $\dim\phi_n = 3$. Hence $\dim(\pi_{F_n})_{an} \leq \dim\phi_n + \dim\mu = 3 + 3 < 8$. Then APH shows that π_{F_n} is hyperbolic. Hence, $e^3(\pi) \in H^3(F_n/F)$. Now, we can assume that $H^3(F_n/F) \neq 0$. Let k be as in Lemma 7.3 (in particular, $H^3(F_k/F) = 0$). Since $H^3(F_n/F) \neq 0$, we get $k < n = h'(\phi) - 1$. Lemma 7.3(ii) shows that $k = h'(\phi) - 2$ and $\dim\phi_k = 5$. Let $\pi_k = \phi_k \perp -\mu$ be an F_k-form. Obviously, $\dim\pi_k = 8$ and $\pi_k = \pi_{F_k} \in I^3(F_k)$. Hence $\pi_k \in GP_3(F)$. Therefore, ϕ_k is a Pfister neighbor of π_k. By Arason's theorem the group $H^3(F_{k+1}/F_k) = H^3(F_k(\phi_k)/F_k)$ is generated by $e^3(\pi_k) = e^3(\pi)_{F_k}$. Since the group $H^3(F_k/F)$ is zero, it follows that the group $H^3(F_{k+1}/F)$ is generated by $e^3(\pi)$. Since $k + 1 = h'(\pi) - 2 + 1 = n$, the proof is complete. □

Corollary 7.8. *Let ϕ be an F-form and n be an integer such that $n < h'(\phi)$. Then $H^3(F_n/F)$ is zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

8. The group $H^3(F_\phi/F)$ in the case of heigh index of ϕ

In the previous section, we have studied the group $H^3(F_n/F)$ in the case $n < h'(\phi)$. Now, it suffices to consider the case where $n = h'(\phi)$. In other words, we need to compute the group $H^3(F_\phi/F)$. The main task of this section is to prove the following theorem.

Theorem 8.1. *Let ϕ be a quadratic form over F such that $\operatorname{ind}\phi \geq 8$. Then $H^3(F_\phi/F) = H^3(F(C_0(\phi))/F)$.*

To prove this theorem, we need several lemmas. We also use the following notation: if L/F is an extension, ϕ is a form over F and A is separable semisimple algebra over F, we set $L(\phi) = L(\phi_L)$ and $L(A) = L(A_L)$. Besides, we set $L(\phi, A) = L(\phi)(A)$.

Lemma 8.2. *Let $\phi \in I^2(F)$ be a quadratic form and A be an F-algebra such that $c(\phi) = [A]$ in the Brauer group. In the case where $\dim \phi = 8$ we suppose additionally that $\operatorname{ind} A = 8$. Then $H^3(F(\phi, A)/F) = H^3(F(A)/F)$.*

Proof. In the (most difficult) case $\dim \phi = 8$ the statement is proved in [IK2, Cor. 6.10]. Now we assume that $\dim \phi \neq 8$. It suffices to show that $H^3(E(\phi)/E) = 0$ where $E = F(A)$. The case where $\dim \phi > 8$ is obvious in view of Arason's theorem. Now, let $\dim \phi \leq 6$. Since $c(\phi_E) = [A_E] = 0$, we have $\phi_E \in I^3(E)$. Then APH implies that ϕ_E is hyperbolic and hence $H^3(E(\phi)/E) = 0$. □

Corollary 8.3. *Let $\phi \in I^2(F)$ be a quadratic form such that $\operatorname{ind} \phi \geq 8$. Let A be an algebra such that $c(\phi) = [A]$. Let $F_0 = F, F_1, \dots, F_h$ be the generic splitting tower of ϕ. Then $H^3(F_n(A)/F) = H^3(F(A)/F)$ for all $n = 0, \dots, h$.*

Proof. It suffices to prove that $H^3(F_{n+1}(A)/F) = H^3(F_n(A)/F)$ for all $n = 0, \dots, h-1$. Obviously, $F_{n+1}(A) = F_n(\phi_n, A)$. If $\dim \phi_n = 8$, then Lemma 7.7 shows that $\operatorname{ind} A_{F_n} = 8$. Applying Lemma 8.2, we have $H^3(F_n(\phi_n, A)/F_n) = H^3(F_n(A)/F_n)$. Hence

$$H^3(F_{n+1}(A)/F) = H^3(F_n(\phi_n, A)/F) = H^3(F_n(A)/F).$$

□

Proof of Theorem 8.1. We consider three cases.

Case 1. ϕ is an even-dimensional form from $I^2(F)$. Let $A = C(\phi)$. Applying Corollary 8.3 to the case $n = h(\phi)$, we get $H^3(F_\phi(A)/F) = H^3(F(A)/F)$. Hence, $H^3(F_\phi/F) \subset H^3(F(A)/F)$. By Corollary 4.2, we have $H^3(F_\phi/F) \supset H^3(F(A)/F)$. Hence, $H^3(F_\phi/F) = H^3(F(A)/F)$ and the proof is complete.

Case 2. ϕ is an odd-dimensional form. Let $\tilde\phi = \phi \perp \langle d_\pm \phi\rangle$. Since $\tilde\phi \in I^2(F)$ and $\operatorname{ind}\tilde\phi = \operatorname{ind}\phi \geq 8$, we can apply the result of Case 1 to the form $\tilde\phi$. We get $H^3(F_{\tilde\phi}/F) = H^3(F(C_0(\tilde\phi))/F)$. Now, applying Lemmas 2.5 and 4.4, we get

$$H^3(F_\phi/F) = H^3(F_{\tilde\phi}/F) = H^3(F(C_0(\tilde\phi))/F) = H^3(F(C_0(\phi))/F).$$

Case 3. ϕ is an even-dimensional form such that $d_\pm\phi \notin F^{*2}$. In this case, the theorem, follows readily from the following statement.

Proposition 8.4. *Let ϕ be an even-dimensional form with nontrivial discriminant such that $\operatorname{ind}\phi \geq 4$. Then $H^3(F_\phi/F) = H^3(F(C_0(\phi))/F)$.*

To prove this proposition, we set $\tilde F = F(t)$ and $\tilde\phi = \phi \perp t\langle\langle d\rangle\rangle$. By Lemma 4.5, we have $\operatorname{ind}\tilde\phi = 2\operatorname{ind}\phi \geq 8$. Since $\tilde\phi \in I^2(\tilde F)$, we can apply the result of Case 1 of the proof of Theorem 8.1 to the $\tilde F$-form $\tilde\phi$. We get $H^3(\tilde F_{\tilde\phi}/\tilde F) =$

$H^3(\tilde{F}(C_0(\tilde{\phi}))/\tilde{F})$ and hence $H^3(\tilde{F}_{\tilde{\phi}}/F) = H^3(\tilde{F}(C_0(\tilde{\phi}))/F)$. Then Lemmas 2.6 and 4.5(3) show that

$$H^3(F_\phi/F) = H^3(\tilde{F}_{\tilde{\phi}}/F) = H^3(\tilde{F}(C_0(\tilde{\phi}))/F) = H^3(F(C_0(\phi))/F).$$

Proof of Proposition 8.4 and Theorem 8.1 is complete. □

The following two corollaries are obvious consequences of the results of this section, Lemma 6.8 and Theorem 6.9.

Corollary 8.5. *Let $\phi \in I^2(F)$ be a form with $\operatorname{ind}\phi \geq 8$. Suppose that the division part of $C(\phi)$ is decomposable. Then*

$$H^3(F_\phi/F) = c(\phi)H^1(F) = \{e^3(\phi\,\langle\langle a\rangle\rangle) \mid a \in F^*\}.$$

Corollary 8.6. *Let ϕ be even dimensional form with $d = d_\pm\phi \notin F^{*2}$ and $\operatorname{ind}\phi \geq 4$. Suppose that the division part of $C_0(\phi)$ is defined over F. Then*

$$H^3(F_\phi/F) = \{e^3(\phi\,\langle\langle a\rangle\rangle) \mid a \in F^* \text{ is such that } \phi\,\langle\langle a\rangle\rangle \in I^3(F)\}.$$

9. The group $H^3(F_\phi/F)$ in the case of low index of ϕ

In the previous section we have considered the case where $\operatorname{ind}\phi \geq 8$. To complete the computation of the group $H^3(F_\phi/F)$, it suffices to consider only the cases where $\operatorname{ind}\phi \leq 4$. The case where ϕ is an even-dimensional form with $d_\pm\phi \notin F^{*2}$ and $\operatorname{ind}\phi = 4$ was also studied in the previous section (see Proposition 8.4). Hence, we can exclude this case from our future considerations.

Theorem 9.1. *Let ϕ be a quadratic form such that $\operatorname{ind}\phi \leq 4$. If ϕ is an even-dimensional form with nontrivial discriminant, we suppose additionally that $\operatorname{ind}\phi \neq 4$. Then*

(1) *there exits a quadratic form μ such that*
   (i) *either $\dim\mu \leq 5$ or μ is a 6-dimensional Albert form [1];*
   (ii) *$\phi \equiv \mu \pmod{I^3(F)}$ (in particular, $C_0(\phi) \sim C_0(\mu)$).*

(2) *For any form μ satisfying Condition (1), the factor group*

$$\frac{H^3(F_\phi/F)}{H^3_{C_0(\phi)}(F)}$$

is generated by the element $e^3(\phi \perp -\mu)$.

To prove this theorem, we need several lemmas.

Lemma 9.2. *Let ϕ be either a form of dimension ≤ 5 or a 6-dimensional Albert form. Then $H^3(F_\phi/F) = H^3(F(C_0(\phi))/F) = H^3_{C_0(\phi)}(F)$.*

Proof. The identity $H^3(F(C_0(\phi))/F) = H^3_{C_0(\phi)}(F)$ follows from Theorem 6.9. To complete the proof, it suffices to show that $F(C_0(\phi))/F \sim F_\phi/F$. By Lemmas 2.5 and 4.4, it suffices to consider only even-dimensional forms. Thus, we can consider only two cases: (a) ϕ is a 4-dimensional form; (b) ϕ is a 6-dimensional Albert form. In Case (a), Wadsworth's theorem [Wa] claims that

[1] We say that μ is an *Albert form* if $\dim\mu = 6$ and $\mu \in I^2(F)$

the form ϕ_K is isotropic (i.e. $\dim(\phi_K)_{an} \le \dim\phi - 2 = 2$) if and only if the algebra $C_0(\phi_K)$ splits. Hence $F(C_0(\phi))/F \sim F_\phi/F$. In Case (b), we can apply Albert's theorem: *an Albert form splits if and only if its Clifford algebra splits.* This completes the proof. □

Lemma 9.3. *If $\pi \in I^3F$ then the group $H^3(F_\pi/F)$ is generated by $e^3(\pi)$.*

Proof. Since $\pi \in I^3(F)$, we have $F_\pi/F \sim F_h/F$ with $h = h(\pi)$. Clearly, $e^3(\pi) \in H^3(F_h/F)$. Hence, we can assume that $H^3(F_h/F) \neq 0$. By Theorem 7.1, $H^3(F_{h-1}/F) = 0$. Hence $H^3(F_h/F_{h-1}) \neq 0$. Since $F_h = F_{h-1}(\pi_{h-1})$, Arason's theorem shows that $\dim\pi_{h-1} \le 8$. Since $\pi \in I^3(F)$, it follows that $\pi_{h-1} \in GP_3(F_{h-1})$. Hence the group $H^3(F_h/F_{h-1})$ is generated by $e^3(\pi_{n-1}) = e^3(\pi)_{F_{h-1}}$. Since $H^3(F_{h-1}/F) = 0$, the group $H^3(F_h/F)$ is generated by $e^3(\pi)$. □

Lemma 9.4. *Let $\pi \in I^3F$ and K/F be a field extension. Then $H^3(K_\pi/F) = H^3(K/F) + H^3(F_\pi/F)$.*

Proof. It suffices to prove that $H^3(K_\pi/F) \subset H^3(K/F) + H^3(F_\pi/F)$. Let $u \in H^3(K_\pi/F)$. Then $u_K \in H^3(K_\pi/K)$. By Lemma 9.3, there exists an integer m such that $u_K = m \cdot e^3(\pi_K)$. Then $u - m \cdot e^3(\pi) \in H^3(K/F)$. Therefore, $u \in H^3(K/F) + m \cdot e^3(\pi) \subset H^3(K/F) + H^3(F_\pi/F)$. □

Proof of Theorem 9.1. First item (1) of this theorem is well-known (see, e.g, the similar statement in [Izh3]).

(2) Let $\pi = \phi \perp -\mu$. Clearly, $\pi \in I^3(F)$. We claim that $\dim(\pi_{F_\phi})_{an} < 8$. To prove this, we consider two cases: $\dim\mu \le 5$ and $\dim\mu = 6$. If $\dim\mu \le 5$ then $\dim(\pi_{F_\phi})_{an} \le \dim(\phi_{F_\phi})_{an} + \dim\mu \le 2 + 5 < 8$. If $\dim\mu = 6$ then (by the construction) $\mu \in I^2(F)$ and hence $\phi \in I^2(F)$. Hence, $\dim(\phi_{F_\phi})_{an} = 0$. Therefore, $\dim(\pi_{F_\phi})_{an} \le \dim(\phi_{F_\phi})_{an} + \dim\mu \le 0 + 6 < 8$.

Since $\dim(\pi_{F_\phi})_{an} < 8$ and $\pi \in I^3(F)$, APH shows that π_{F_ϕ} is hyperbolic. Hence $e^3(\pi) \in H^3(F_\phi/F)$. Taking into account Lemma 4.2, we have

$$H^3(F_\phi/F) \supset H^3(F(C_0(\phi))/F) + e^3(\pi)H^0(F) \supset H^3_{C_0(\phi)}(F) + e^3(\pi)H^0(F).$$

To complete the proof of the theorem, it suffices to show that

$$H^3(F_\phi/F) \subset H^3_{C_0(\phi)}(F) + e^3(\pi)H^0(F).$$

Since $\mu \equiv \phi \pmod{I^3(F)}$, the algebra $C_0(\phi)$ is Brauer-equivalent to $C_0(\mu)$. Then Lemma 9.2 shows that $H^3_{F(C_0(\phi))}(F) = H^3_{F(C_0(\mu))}(F) = H^3(F_\mu/F)$. Letting $K = F_\mu$, we have $H^3(K/F) = H^3_{F(C_0(\phi))}(F)$. Obviously, $\dim(\mu_K)_{an} = \dim(\mu_{F_\mu})_{an} \le 2$. Since $\phi = \mu + \pi$ and π_{K_π} is hyperbolic, it follows that $(\phi_{K_\pi})_{an} \simeq (\mu_{K_\pi})_{an}$. Hence, $\dim(\phi_{K_\pi})_{an} \le \dim(\mu_K)_{an} \le 2$. Therefore, $F_\phi/F \prec K_\pi/F$. Hence $H^3(F_\phi/F) \subset H^3(K_\pi/F)$. Applying Lemmas 9.3 and 9.4, we get

$$H^3(F_\phi/F) \subset H^3(K_\pi/F) = H^3(K/F) + H^3(F_\pi/F) = H^3_{C_0(\phi)}(F) + e^3(\pi)H^0(F).$$

□

Proposition 9.5. *For any F-form ϕ the factor group $H^3(F_\phi/F)/H^3_{C_0(\phi)}(F)$ is zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Proof. In view of Theorem 9.1, we can assume that either $\operatorname{ind}\phi \geq 8$ or ϕ is an even-dimensional form with nontrivial discriminant and $\operatorname{ind}\phi = 4$. In these cases Theorem 8.1 and Proposition 8.4 show that $H^3(F_\phi/F) = H^3(F(C_0(\phi))/F)$. Then Theorem 6.9 completes the proof. □

10. Second Chow group of $X_{\phi,m}$

Let ϕ be a quadratic form and $n \leq h'(\phi)$ be a positive integer. The field extension F_n/F can be constructed as the function field of some projective homogeneous variety X (see Lemma 2.4). For example, we can take X to be equal to the variety $X_{\phi,m}$ of totally isotropic m-dimensional subspaces where $m = i_W(\phi_{F_n})$ (see Example 1.8). In this section, we compute the torsion subgroup of the second Chow group of X.

In the case $n < h'(\phi)$, we can compute the group $\operatorname{Tors}\operatorname{CH}^2(X)$ completely.

Theorem 10.1. *Let ϕ be an F-form, n be a positive integer such that $n < h'(\phi)$, and X be a projective homogeneous variety such that $F(X)/F \sim F_n/F$. Then*

(1) $\operatorname{Tors}\operatorname{CH}^2(X) \simeq H^3(F(X)/F) \simeq H^3(F_n/F)$,
(2) $\operatorname{Tors}\operatorname{CH}^2(X)$ *is zero or* $\mathbb{Z}/2\mathbb{Z}$,
(3) $\operatorname{Tors}\operatorname{CH}^2(X) = 0$ *except (possibly) the case where:*

$$\dim\phi \text{ is odd, } \operatorname{ind}\phi = 2 \text{ and } n = h'(\phi) - 1.$$

(4) *If* $\dim\phi$ *is odd,* $\operatorname{ind}\phi = 2$ *and* $n = h'(\phi) - 1$ *then the group* $\operatorname{Tors}\operatorname{CH}^2(X)$ *is zero if and only if* $\phi \equiv \mu \pmod{I^4(F)}$ *for some 3-dimensional form* μ.

Proof. Since $F(X)/F \sim F_n/F$, we have $H^3(F(X)/F) \simeq H^3(F_n/F)$.

(1). By Proposition 7.6, $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X)/F) = H^3(F_n/F)$. Corollary 7.8 shows that the variety X satisfies hypotheses of Corollary 5.10. Hence $\operatorname{Tors}\operatorname{CH}^2(X) \simeq H^3(F(X)/F)$.

(2)–(4). Obvious in view of Item (1) and Theorem 7.1. □

Now it suffices to consider the case $n = h'(\phi)$. In this case we have $F_n = F_\phi$.

Proposition 10.2. (cf. [M3, §10, Example 4]). *Let ϕ be an F-form and X be a projective homogeneous variety such that $F(X)/F \sim F_\phi/F$. Then*

$$\operatorname{Tors}\operatorname{CH}^2(X) \simeq \frac{H^3(F_\phi/F)}{H^3_{C_0(\phi)}(F)}.$$

Moreover, $\operatorname{Tors}\operatorname{CH}^2(X) = 0$ *or* $\mathbb{Z}/2\mathbb{Z}$.

Proof. By Proposition 7.6, we have $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X)/F)$. Proposition 9.5 shows that the algebra $C_0(\phi)$ is maximal with respect to X (see Definition 5.4). After this, Theorem 5.9 completes the proof. □

Lemma 10.3. *Let ϕ be a quadratic form over F and $X = X_{C_0(\phi)}$ be an arbitrary generic splitting variety of $C_0(\phi)$. Then there exists a natural isomorphism*
$$\operatorname{Tors}\operatorname{CH}^2(X_{C_0(\phi)}) \simeq \frac{H^3(F(C_0(\phi))/F)}{H^3_{C_0(\phi)}(F)}.$$
Moreover, the group $\operatorname{Tors}\operatorname{CH}^2(X_{C_0(\phi)})$ *is zero or* $\mathbb{Z}/2\mathbb{Z}$.

Proof. By Proposition 7.6, we have $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) \simeq H^3(F(X)/F)$. By Theorem 6.9, the algebra $C_0(\phi)$ is maximal with respect to X. After this, Theorem 5.9 completes the proof. $\square$

Proposition 10.4. *Let ϕ be a quadratic form over F and X be a projective homogeneous variety such that $F(X)/F \sim F_\phi/F$. Then*

- *if* $\dim\phi$ *is odd and* $\operatorname{ind} C_0(\phi) \geq 8$ *then*
$$\operatorname{Tors}\operatorname{CH}^2(X) = \operatorname{Tors}\operatorname{CH}^2(\operatorname{SB}(C_0(\phi))).$$
- *if* $\phi \in I^2(F)$ *and* $\operatorname{ind} C(\phi) \geq 8$ *then*
$$\operatorname{Tors}\operatorname{CH}^2(X) = \operatorname{Tors}\operatorname{CH}^2(\operatorname{SB}(C(\phi))).$$
- *if* $\dim\phi$ *is even,* $\phi \notin I^2(F)$ *and* $\operatorname{ind} C_0(\phi) \geq 4$ *then*
$$\operatorname{Tors}\operatorname{CH}^2(X) = \operatorname{Tors}\operatorname{CH}^2(R_{L/F}(\operatorname{SB}(C_0(\phi))))$$
where $L = F(\sqrt{d_\pm\phi})$.

Proof. By Propositions 10.2 and 8.4, Theorem 8.1 and Lemma 10.3, we have
$$\operatorname{Tors}\operatorname{CH}^2(X) \simeq \frac{H^3(F_\phi/F)}{H^3_{C_0(\phi)}(F)} \simeq \frac{H^3(F(C_0(\phi))/F)}{H^3_{C_0(\phi)}(F)} \simeq \operatorname{Tors}\operatorname{CH}^2(X_{C_0(\phi)}).$$
$\square$

Now, we can prove Theorems 0.1 and 0.2 from Introduction.

Proof of Theorem 0.1. Let $n \leq h'(\phi)$ be such that $F(X)/F \sim F_n/F$. If $n < h'(\phi)$ then the required results follows from Theorem 10.1. If $n = h'(\phi)$ then $F_n = F_\phi$ and the required result follows from Proposition 10.2. $\square$

Proof of Theorem 0.2. Let $n \leq h'(\phi)$ be such that $F(X)/F \sim F_n/F$. If $\phi_{F(X)}$ is nonhyperbolic then $n < h'(\phi)$. In this case Theorem 10.1 shows that $\operatorname{Tors}\operatorname{CH}^2(X) = 0$. Now, we assume that $\phi_{F(X)}$ is hyperbolic. Then $n = h'(\phi)$ and $F(X)/F \sim F_\phi/F$. Th this case, the required result follows readily from Proposition 10.4, Corollary 8.5, and Theorem 6.1. $\square$

We complete this section by consideration of "*forms with maximal Shur index*". Let us recall some properties of Clifford algebras.

(i) Let ϕ be an odd-dimensional form with $\dim\phi = 2n+1$. Then $C_0(\phi)$ is isomorphic to a tensor product of n quaternion algebras. In particular, $\operatorname{ind}\phi \leq 2^n$. If $\operatorname{ind}\phi = 2^n$ and $n > 1$, then the algebra $C_0(\phi)$ is obviously division and decomposable.

(ii) Let ϕ be a $2n$-dimensional form from $I^2(F)$. Then $C_0(\phi)$ has the form $A \times A$ where A is a tensor product of $n-1$ quaternion algebras. In particular, $\operatorname{ind}\phi \leq 2^{n-1}$. If $\operatorname{ind}\phi = 2^{n-1}$ and $n > 2$, then the algebra A is obviously division and decomposable.

(iii) Let ϕ be a $2n$-dimensional form with nontrivial discriminant d. Then $C_0(\phi)$ has the form A_L where A is a tensor product of $n-1$ quaternion F-algebras and $L = F(\sqrt{d})$. In particular, $\operatorname{ind}\phi \leq 2^{n-1}$. If $\operatorname{ind}\phi = 2^{n-1}$ then the algebra A_L is division. Then the division part of $C_0(\phi)$ is defined over F.

Definition 10.5. Let ϕ be a quadratic form. We say that ϕ *has maximal Shur index* if $C_0(\phi)$ is a division algebra or has the form $A \times A$ where A is a division algebra. In other words, this means that

- if $\dim\phi = 2n+1$ then $\operatorname{ind}\phi = 2^n$,
- if $\dim\phi = 2n$ then $\operatorname{ind}\phi = 2^{n-1}$.

Proposition 10.6. *Let ϕ be an F-form with maximal Shur index. Then $H^3(F_\phi/F) = H^3(F(C_0(\phi))/F) = H^3_{C_0(\phi)}(F)$.*

Proof. By Lemma 9.2, we can consider only the cases where either $\dim\phi \geq 7$ or $\dim\phi = 6$ and ϕ has nontrivial discriminant. Since ϕ has maximal Shur index, the condition $\dim\phi \geq 7$ implies that $\operatorname{ind}\phi \geq 8$ and the condition $\dim\phi = 6$ implies that $\operatorname{ind}\phi = 4$. Applying Theorem 8.1 and Proposition 8.4 to our form ϕ, we get $H^3(F_\phi/F) = H^3(F(C_0(\phi))/F)$. By Theorem 6.9, we have $H^3(F(C_0(\phi))/F) = H^3_{C_0(\phi)}(F)$. □

Corollary 10.7. *If ϕ is a form such that $\dim\phi = 2n$ and $\operatorname{ind}\phi = 2^{n-1}$ then*

$$H^3(F_\phi/F) = \{e^3(\phi\,\langle\langle a\rangle\rangle) \mid a \in F^* \text{ is such that } \phi\,\langle\langle a\rangle\rangle \in I^3(F)\}.$$

Proof. Obvious in view of Proposition 10.6 and Lemma 6.8. □

Proposition 10.8. *Let ϕ be an F-form with maximal Shur index and let X be a generic splitting variety of ϕ. Then* $\operatorname{Tors}\operatorname{CH}^2(X) = 0$.

Proof. Let $n \leq h'(\phi)$ be such that $F(X)/F \sim F_n/F$. If $n = h'(\phi)$, then Proposition 10.6 shows that $H^3(F_\phi/F) = H^3_{C_0(\phi)}(F)$. Then Proposition 10.2 show that $\operatorname{Tors}\operatorname{CH}^2(X) = 0$. Now let $n < h'(\phi)$. If we assume that $\operatorname{Tors}\operatorname{CH}^2(X) \neq 0$ then Theorem 10.1 shows that $\dim\phi$ is odd and $\operatorname{ind}\phi = 2$. Since ϕ has maximal Shur index, we have $\dim\phi = 3$. Then $h'(\phi) = 1$. Since $n < h'(\phi) = 1$, we have $n = 0$. Therefore $F_n = F$ and $H^3(F_n/F) = 0$. By Theorem 10.1, $\operatorname{Tors}\operatorname{CH}^2(X) = 0$. □

11. Motivic equivalence of 8-dimensional forms

Let ϕ and ψ be forms over a field F of characteristic $\neq 2$, and let X_ϕ and X_ψ be their associated quadrics. In [Vi], A. Vishik proved [2] that the motives

[2] A. Vishik proved his theorem under assumption $char(F) = 0$. Recently, Karpenko has given a simplified proof of Vishik's theorem for all characteristics [K3].

of X_ϕ and X_ψ are equivalent in the category of Chow motives if and only if $\dim\phi = \dim\psi$ and for all field extensions L/F, the Witt indices of ϕ and ψ coincide. In this case, we say that ϕ and ψ are *motivically equivalent.*

There are some evident examples of motivic equivalent forms:

- if ϕ is similar to ψ, then ϕ and ψ are motivic equivalent,
- if ϕ and ψ are half-neighbors, then ϕ and ψ are also motivic equivalent.

In [Izh1], it was shown that if $\dim\phi \le 7$ then ϕ and ψ are motivically equivalent if and only if ϕ and ψ are similar. There are examples of motivically equivalent nonsimilar forms in dimension 8 provided by nonsimilar half-neighbors [Ho1]. In this section, we discuss the following question: *Let ϕ and ψ be motivically equivalent forms of dimension 8. Is it necessary true that ϕ and ψ are half-neighbors?* At the time, we do not know complete answer to this question. But we can prove, that the answer is positive except (possibly) the case where $\operatorname{ind} C_0(\phi) = 4$.

Theorem 11.1. *Let ϕ and ψ be 8-dimensional forms. If $\operatorname{ind}\phi = 4$, we suppose additionally that the division part of $C_0(\phi)$ is defined over F. Then the following conditions are equivalent.*

(i) *ϕ and ψ are motivically equivalent,*
(ii) *ϕ and ψ are half-neighbors.*

Proof. It suffices to prove the implication (i)$\Rightarrow$(ii). First, we suppose that at least one of the forms ϕ or ψ is isotropic. In this case, $\dim\phi_{an} = \dim\psi_{an} < 8$ and the forms ϕ_{an} and ψ_{an} are motivically equivalent. Now, we recall that forms of dimension < 8 are motivically equivalent only if they are similar. Hence, $\phi_{an} \overset{sim}{\sim} \psi_{an}$. Therefore, $\phi \overset{sim}{\sim} \psi$ and hence, ϕ and ψ are half-neighbors.

Now, we assume that ϕ and ψ are anisotropic. In the case where $\phi \in I^2(F)$, the theorem is proved in [Izh1]. Hence, we can assume that $d = d_\pm\phi \notin F^{*2}$. By [K3], we have $d_\pm\psi = d$ and $C_0(\phi) \simeq C_0(\psi)$. Hence there exists $s \in F^*$ such that $\phi \equiv s\psi \pmod{I^3(F)}$ (see, e.g, [Izh2]). Let $\pi = \phi \perp -s\psi$ and $E = F_\phi$.

Since $\dim(\phi_E)_{an} = \dim(\phi_{F_\phi})_{an} \le 2$ and ψ is motivically equivalent to ϕ, we have $\dim(\psi_E)_{an} \le 2$. Therefore, $\dim(\pi_E)_{an} \le 2+2 < 8$. Since $\pi \in I^3(F)$, APH shows that π_E is hyperbolic. Hence $e^3(\pi) \in H^3(E/F) = H^3(F_\phi/F)$. Since ϕ is an even-dimensional form with nontrivial discriminant, Lemma 6.8 shows that

$$H^3_{C_0(\phi)}(F) = \{e^3(\phi\langle\langle a\rangle\rangle) \mid a \in F^* \text{ is such that } \phi\langle\langle a\rangle\rangle \in I^3(F)\}.$$

Lemma 11.2. *If $\operatorname{ind}\phi = 8$ or 4 then $H^3(F_\phi/F) = H^3_{C_0(\phi)}(F)$.*

Proof. The case $\operatorname{ind}\phi = 8$ is obvious in view of Proposition 10.6. If $\operatorname{ind}\phi = 4$ then (by the hypotheses of the theorem) the division part of $C_0(\phi)$ is defined over F. Then the required result is proved in Theorem 6.9. $\square$

Now, we return to the proof of Theorem 11.1. First, we consider the case where $e^3(\pi) \in H^3_{C_0(\phi)}(F)$. Then there exists $a \in F^*$ is such that $\phi\langle\langle a\rangle\rangle \in I^3(F)$ and $e^3(\pi) = e^3(\phi\langle\langle a\rangle\rangle)$. Since the homomorphism $e^3 : I^3(F)/I^4(F) \to H^3(F)$

is injective, we get $\pi \equiv \phi \langle\langle a\rangle\rangle \pmod{I^4(F)}$. Since $\pi = \phi \perp -s\psi$ and $\phi\langle\langle a\rangle\rangle = \phi - a\phi$, we have $s\psi \equiv a\phi \pmod{I^4(F)}$. Therefore ϕ and ψ are half-neighbors.

Now, we can assume that $e^3(\pi) \notin H^3_{C_0(\phi)}(F)$. Since $e^3(\pi) \in H^3(F_\phi/F)$, it follows that $H^3(F_\phi/F) \neq H^3_{C_0(\phi)}(F)$. By Lemma 11.2, we have $\operatorname{ind}\phi \leq 2$.

To study the case $\operatorname{ind}\phi \leq 2$, we use Theorem 9.1. By this theorem, there exists a form μ of dimension ≤ 6 such that $\phi \equiv \mu \pmod{I^3(F)}$ and $H^3(F_\phi/F) = H^3_{C_0(\phi)}(F) + e^3(\phi \perp -\mu)H^0(F)$. Since $e^3(\pi) \in H^3(F_\phi/F)$ and $e^3(\pi) \notin H^3_{C_0(\phi)}(F)$, it follows that there exists $a \in F^*$ is such that $\phi\langle\langle a\rangle\rangle \in I^3(F)$ and $e^3(\pi) = e^3(\phi\langle\langle a\rangle\rangle) + e^3(\phi \perp -\mu)$. Hence,

$$e^3(\phi - s\psi) = e^3(\pi) = e^3(\phi\langle\langle a\rangle\rangle) + e^3(\phi \perp -\mu) = e^3(\phi\langle\langle a\rangle\rangle) + e^3(a(\phi \perp -\mu)).$$

Since the homomorphism $e^3 : I^3(F)/I^4(F) \to H^3(F)$ is injective, we have $\phi - s\psi \equiv \phi - a\phi + a(\phi - \mu) \pmod{I^4(F)}$. Hence $s\psi \perp -a\mu \in I^4(F)$. Since $\dim(s\psi \perp -a\mu) \leq 8 + 6 < 16$, APH shows that $s\psi \perp -a\mu$ is hyperbolic. Therefore $s\psi_{an} = a\mu_{an}$ and hence $\dim\psi_{an} \leq \dim\mu = 6$. On the other hand, ψ is an anisotropic 8-dimensional form, a contradiction. $\square$

References

[Ara] Arason, J. Kr. *Cohomologische Invarianten quadratischer Formen.* J. Algebra **36** (1975), 448–491.

[Art] Artin, M. *Brauer-Severi varieties.* Lect. Notes Math. **917** (1982), 194–210.

[Ho1] Hoffmann, D. W. *Similarity of quadratic forms and half-neighbors.* J. Algebra **204** (1998), no. 1, 255–280.

[Ho2] Hoffmann, D. W. *Splitting patterns and invariants of quadratic forms.* Math. Nachr. **190** (1998), 149–168.

[HR] Hurrelbrink, J.; Rehmann, U. *Splitting patterns of quadratic forms.* Math. Nachr. **176** (1995), 111–127.

[Izh1] Izhboldin O. T. *Motivic Equivalence of Quadratic Forms.* Doc.Math.J.DMV **3** (1998) 341-351

[Izh2] Izhboldin O. T. *On the isotropy of quadratic forms over the function field of a quadric.* (Russian) Algebra i Analiz **10** (1998), no. 1, 32–57; Engl. transl. in: St. Petersburg Math. J. **10** (1999), no. 1, 25–43

[Izh3] Izhboldin, O. T. *Field with u-invariant 9.* Preprint 1999. Bielefeld.

[IK1] Izhboldin O. T.; Karpenko N. A. *Isotropy of six-dimensional quadratic forms over function fields of quadrics.* J. Algebra **209** (1998), no. 1, 65–93.

[IK2] Izhboldin, O. T.; Karpenko, N. A. *On the group $H^3(F(\psi, D)/F)$.* Doc. Math. **2** (1997), 297–311

[IK3] Izhboldin, O. T.; Karpenko, N. A. *Generic Splitting Fields of Central Simple Algebras: Galois Cohomology and Nonexcellence.* Algebras and Representation Theory **2** (1999), 19–59.

[Ka] Kahn, B., *Descente galoisienne et K_2 des corps de nombres.* K-Theory **7**, No.1, (1993) 55-100.

[K1] Karpenko, N. A. *Algebro-geometric invariants of quadratic forms.* (Russian) Algebra i Analiz **2** (1990), no. 1, 141–162; Engl. transl. in: Leningrad (St. Petersburg) Math. J. **2** (1991), no. 1, 119–138.

[K2] Karpenko, N. A. *Codimension 2 cycles on Severi-Brauer varieties.* K-Theory **13**, No.4, 305-330 (1998).

[K3] Karpenko, N. A. *Criteria for motivic equivalence for quadratic forms and central simple algebras*, Preprint 38, SFB 478, Univ. Munster, Munster, 1999.

[KR] Kersten, I.; Rehmann, U., *Generic splitting of reductive groups.* Tôhoku Math. J. (2) **46** (1994), no. 1, 35–70.

[Kn1] Knebusch, M.: *Generic splitting of quadratic forms, I.* Proc. London Math. Soc **33** (1976), 65–93.

[Kn2] Knebusch, M.: *Generic splitting of quadratic forms, II.* Proc. London Math. Soc **34** (1977), 1–31.

[Lam] Lam, T. Y. *The algebraic Theory of Quadratic Forms.* Massachusetts: Benjamin 1973 (revised printing 1980).

[M1] Merkurjev, A. S. *On the norm residue symbol of degree 2.* Dokl. Akad. Nauk SSSR **261** (1981), 542–547. Engl. transl.: Soviet Math. Dokl. **24** (1981), 546–551.

[M2] Merkurjev, A. S. *Simple algebras and quadratic forms.* Izv. Akad. Nauk SSSR, Ser. Mat. **55** (1991), 218–224. Engl. transl.: Math. USSR Izv. **38** (1992), no. 1, 215–221.

[M3] Merkurjev, A. S. *The group $H^1(X, K_2)$ for projective homogeneous varieties.* (Russian) Algebra i Analiz **7** (1995), no. 3, 136–164; Engl. transl. in: St. Petersburg Math. J. **7** (1996), no. 3, 421–444.

[MS] Merkurjev, A. S.; Suslin, A. A. *The group K_3 for a field.* Izv. Akad. Nauk SSSR Ser. Mat. (Russian) **54** (1990), no. 3, 522–545. Engl. transl.: Math. USSR, Izv. **36** (1991), no.3, 541–565.

[Pa] Panin, I. A. *On the algebraic K-theory of twisted flag varieties.* K-Theory **8** (1994), no. 6, 541–585.

[P1] Peyre, E. *Products of Severi-Brauer varieties and Galois cohomology.* Proc. Symp. Pure Math. **58.2** (1995), 369–401.

[P2] Peyre, E. *Galois cohomology in degree three and homogeneous varieties.* K-Theory **15** (1998), no. 2, 99–145.

[R] Rost, M. *Hilbert 90 for K_3 for degree-two extensions.* Preprint (1986).

[Sch] Scharlau, W. *Quadratic and Hermitian Forms* Springer, Berlin, Heidelberg, New York, Tokyo (1985).

[Su] Suslin, A. A. *Algebraic K-theory and the norm residue homomorphism.* J. Soviet Math. **30** (1985), 2556–2611.

[T] Tignol, J.-P. *Algèbres indécomposables d'exposant premier.* Adv. Math **65** (1987), no 3, 205-228.

[Vi] Vishik, A. *Integral motives of quadrics.* Max Planck Institut Für Mathematik, Bonn, preprint MPI-1998-13, 1–82.

[Wa] Wadsworth, A. R. *Similarity of quadratic forms and isomorphism of their function fields.* Trans. Amer. Math. Soc. **208** (1975), 352–358.

Oleg Izhboldin, Universität Bielefeld, Fakultät für Mathematik, Postfach 100131, 33501 Bielefeld, Germany

E-mail address: `oleg@mathematik.uni-bielefeld.de`