AUSLANDER-REITEN THEORY VIA BROWN REPRESENTABILITY

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Abstract. We develop an Auslander-Reiten theory for triangulated categories which is based on Brown's representability theorem.

In a fundamental article [3], Auslander and Reiten introduced almost split sequences for the category of finitely generated modules over an artin algebra. These are short exact sequences which look almost like split exact sequences, but many authors prefer to call them Auslander-Reiten sequences. This concept is one of the most successful in modern algebra representation theory (cf. [4] for a good introduction). In fact, the existence theorem for almost split sequences has been generalized in various directions and became the starting point of what is now called Auslander-Reiten theory. Let us mention some of the main ingredients of classical Auslander-Reiten theory:

- the Auslander-Reiten formula,
- almost split sequences,
- morphisms determined by objects,
- Auslander's defect formula.

In this paper we discuss analogous concepts and results for compactly generated triangulated categories. This includes, for example, the stable homotopy category of CW-spectra, or the derived category of modules over some fixed ring.

For each of the above concepts from classical Auslander-Reiten theory there is a corresponding section in this paper. In addition, we have included an appendix which provides a brief introduction into classical Auslander-Reiten theory and sketches the parallel between Auslander-Reiten theory for module categories and the new Auslander-Reiten theory for triangulated categories.

In this paper, no attempt has been made to present a unified approach towards a general Auslander-Reiten theory which covers module categories and triangulated categories at the same time. For this we refer to recent work of Beligiannis [5].

1. Brown representability

Throughout this paper we fix a triangulated category $\mathcal{S}$ and make the following additional assumptions:

- $\mathcal{S}$ has arbitrary coproducts;
- the isomorphism classes of compact objects in $\mathcal{S}$ form a set;
- $\operatorname{Hom}(C, X) = 0$ for all compact C implies $X = 0$ for every object X in $\mathcal{S}$.

Recall that an object X in $\mathcal{S}$ is *compact* if the representable functor $\operatorname{Hom}(X, -)$ preserves arbitrary coproducts. A functor $\mathcal{S} \to \mathrm{Ab}$ into the category of abelian groups is *exact* if it sends every triangle to an exact sequence. The following characterization of representable functors is our main tool for proving the existence of maps and triangles.

Theorem (Brown). *A contravariant functor $F\colon \mathcal{S} \to \mathrm{Ab}$ is isomorphic to a representable functor $\operatorname{Hom}(-, X)$ for some $X \in \mathcal{S}$ if and only if F is exact and sends arbitrary coproducts to products.*

Proof. See [7, 11]. □

Let us give an example. We fix a compact object C in $\mathcal{S}$ and a ring homomorphism $\Gamma \to \operatorname{End}(C)$. Given any injective Γ-module I, Brown's theorem provides an object $T_C I$ in $\mathcal{S}$ such that

$$\operatorname{Hom}_\Gamma(\operatorname{Hom}(C,-),I) \cong \operatorname{Hom}(-,T_C I).$$

It follows from Yoneda's lemma that the representing object $T_C I$ is unique up to a canonical isomorphism. The defining isomorphism for $T_C I$ is functorial in I; it is also functorial in C if Γ acts centrally on $\mathcal{S}$. For example, if $\mathcal{S}$ is the appropriate derived category of sheaves on some space $\mathbf{X}$ over a field k, and $\Gamma = k$, then the functoriality in C gives the corresponding Serre duality functor for $\mathbf{X}$ (cf. [6]). However, in this paper we shall concentrate on the functoriality in I.

2. Auslander-Reiten triangles

The analogue of an almost split sequence for a triangulated category was introduced by Happel (cf. [9]).

Definition 2.1. A triangle $X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z \overset{\gamma}{\to} \Sigma X$ is called *Auslander-Reiten triangle* if the following conditions hold:

(1) every map $X \to Y'$ which is not a section factors through α;
(2) every map $Y' \to Z$ which is not a retraction factors through β;
(3) $\gamma \neq 0$.

Note that the *end terms* X and Z of an Auslander-Reiten triangle are indecomposable objects with local endomorphism rings. Moreover, each end term determines an Auslander-Reiten triangle up to isomorphism.

Theorem 2.2. *Let Z be a compact object and suppose that the endomorphism ring $\Gamma = \operatorname{End}(Z)$ is local. Denote by $\mu\colon \Gamma/\operatorname{rad}\Gamma \to I$ an injective envelope in the category of Γ-modules and let $T_Z I$ be the object in $\mathcal{S}$ such that*

$$(*) \qquad \operatorname{Hom}_\Gamma(\operatorname{Hom}(Z,-),I) \cong \operatorname{Hom}(-,T_Z I).$$

Then there exists an Auslander-Reiten triangle

$$\Sigma^{-1}(T_Z I) \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} T_Z I$$

where γ denotes the map which corresponds under $(*)$ to the canonical map

$$\operatorname{Hom}(Z,Z) \xrightarrow{\pi} \Gamma/\operatorname{rad}\Gamma \xrightarrow{\mu} I.$$

The existence of the object $T_Z I$ is guaranteed by Brown's representability theorem. In fact, $\Sigma^{-1}(T_Z I)$ is the analogue of the *dual of transpose* $D\operatorname{Tr} C$ for a finitely presented module C which leads to an almost split sequence $0 \to D\operatorname{Tr} C \to B \to C \to 0$.

Before giving the proof of the theorem, let us recall some generalities from Auslander-Reiten theory. A map $\alpha\colon X \to Y$ is called *left almost split* if α is not a section and any map $X \to Y'$ which is not a section factors through α. Dually, $\beta\colon Y \to Z$ is *right almost split* if β is not a retraction and any map $Y' \to Z$ which is not a retraction factors through β. We note the following easy observation.

Lemma 2.3. *Let $X \to Y$ be left almost split. Then X has a local endomorphism ring.*

A map $\alpha\colon X \to Y$ is called *left minimal* if every endomorphism $\phi\colon Y \to Y$ satisfying $\phi \circ \alpha = \alpha$ is an isomorphism. Dually, $\beta\colon Y \to Z$ is *right minimal* if every endomorphism $\phi\colon Y \to Y$ satisfying $\beta \circ \phi = \beta$ is an isomorphism.

Lemma 2.4. *Let $\alpha\colon X \to Y$ be a non-zero map and suppose that* $\mathrm{End}(Y)$ *is local. Then α is left minimal.*

Proof. Let $\phi\colon Y \to Y$ be a map such that $\phi \circ \alpha = \alpha$. Applying Nakayama's lemma to the $\mathrm{End}(Y)$-submodule of $\mathrm{Hom}(X,Y)$ generated by α, one shows that ϕ does not belong to the radical of $\mathrm{End}(Y)$. Thus ϕ is invertible since $\mathrm{End}(Y)$ is local. □

Lemma 2.5. *Let $X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z \overset{\gamma}{\to} \Sigma X$ be a triangle. Then β is right minimal if and only if γ is left minimal.*

Proof. Straightforward. □

The following lemma is due to Assem, Beligiannis, and Marmaridis [1]; it is the analogue of a characterization of almost split sequences due to Auslander and Reiten. We include a short proof; it is different from the one in [1] which also covers right triangulated catories.

Lemma 2.6. *Let $\varepsilon\colon X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z \overset{\gamma}{\to} \Sigma X$ be a triangle and suppose that β is right almost split. Then the following are equivalent:*

(1) $\mathrm{End}(X)$ *is local.*
(2) *β is right minimal.*
(3) *α is left almost split.*
(4) *ε is an Auslander-Reiten triangle.*

Proof. (1) ⇒ (2) Suppose that $\mathrm{End}(X) \cong \mathrm{End}(\Sigma X)$ is local. Using Lemma 2.4, it follows that γ is left minimal. An application of Lemma 2.5 shows that β is right minimal.

(2) ⇒ (3) The map α is not a section since β is not a retraction. Now suppose that $\phi\colon X \to X'$ is a map which is not a section. Completing the composition $\phi \circ (-\Sigma^{-1}\gamma)$ to a triangle, we get the following map between triangles:

$$\begin{array}{ccccccc}
X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \| & & \downarrow{\scriptstyle\Sigma\phi} \\
X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z & \xrightarrow{\gamma'} & \Sigma X'
\end{array}$$

We claim that α' is a section. If not, the map β' factors through β, say $\beta' = \beta \circ \psi'$, and we get another commutative diagram:

$$\begin{array}{ccccccc}
X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z & \xrightarrow{\gamma'} & \Sigma X' \\
\downarrow{\scriptstyle\phi'} & & \downarrow{\scriptstyle\psi'} & & \| & & \downarrow{\scriptstyle\Sigma\phi'} \\
X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X
\end{array}$$

(2) implies that $\psi' \circ \psi$ is an isomorphism. Therefore $\phi' \circ \phi$ is an isomorphism and ϕ is a section. This contradicts our assumption on ϕ, and therefore α' is a section. The map ϕ factors through α via $(\alpha')^{-1} \circ \psi$ and this shows that α is left almost split.

(3) ⇔ (4) This is just a reformulation of the definitions.

(3) ⇒ (1) Use Lemma 2.3. □

Proof of Theorem 2.2. We check the conditions (1) – (3) for the triangle

$$\Sigma^{-1}(T_Z I) \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} T_Z I.$$

(3) The map γ corresponds, by definition, under $(*)$ to a non-zero map. Thus $\gamma \neq 0$.

(2) Let $\phi\colon Y' \to Z$ be a map in $\mathcal{S}$ which is not a retraction. It follows that the image of the induced map $\operatorname{Hom}(Z, Y') \to \operatorname{Hom}(Z, Z)$ is contained in the radical of $\operatorname{End}(Z)$. Therefore the composition with $\mu \circ \pi\colon \operatorname{Hom}(Z, Z) \to I$ is zero. However, $\operatorname{Hom}(Z, \phi) \circ \mu \circ \pi$ corresponds under $(*)$ to the map $\gamma \circ \phi$, and this implies $\gamma \circ \phi = 0$. Thus ϕ factors through β.

(1) Applying Lemma 2.6, it is sufficient to show that the endomorphism ring of $\Sigma^{-1}(T_Z I)$ is local. This ring is isomorphic to $\operatorname{End}(T_Z I)$, and applying the isomorphism $(*)$ twice we obtain

$$\operatorname{End}(T_Z I) \cong \operatorname{Hom}_\Gamma(\operatorname{Hom}(Z, T_Z I), I) \cong \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Gamma(\operatorname{Hom}(Z, Z), I), I) \cong \operatorname{End}_\Gamma(I).$$

The injective Γ-module I is indecomposable since $\Gamma/\operatorname{rad}\Gamma$ is simple, and therefore $\operatorname{End}_\Gamma(I)$ is local. □

Given a compact object Z with local endomorphism ring, it seems to be an interesting project to compute the other endterm of an Auslander-Reiten triangle

$$\Sigma^{-1}(T_Z I) \longrightarrow Y \longrightarrow Z \longrightarrow T_Z I.$$

We shall discuss this problem in three examples.

(1) Let Λ be a finite dimensional algebra over a field k and consider the derived category $D(\Lambda)$ of unbounded complexes of Λ-modules. An object in $D(\Lambda)$ is compact if and only if it is isomorphic to a bounded complex $(P^i)_{i\in\mathbb{Z}}$ of finitely generated projective Λ-modules. The Auslander-Reiten triangle corresponding to an indecomposable compact object $Z = (P^i)_{i\in\mathbb{Z}}$ has been computed by Happel in [9]. One gets

$$T_Z I \cong (P^i \otimes_\Lambda D\Lambda)_{i\in\mathbb{Z}}$$

where $D\Lambda = \operatorname{Hom}_k(\Lambda, k)$. Note that $T_Z I$ is compact whenever the Λ-module $D\Lambda$ has finite projective dimension.

(2) Let Λ be a symmetric finite dimensional algebra, for example the group algebra of a finite group. We consider the stable category $\underline{\operatorname{Mod}}\,\Lambda$ of Λ-modules where for two modules X and Y one defines $\operatorname{Hom}(X, Y)$ to be the group of Λ-module homomorphisms modulo the subgroup of all maps which factor through a projective module. An object Z in $\underline{\operatorname{Mod}}\,\Lambda$ is compact if and only if it is isomorphic to a finitely generated Λ-module. Moreover, if $\operatorname{End}(Z)$ is local then we can assume that Z is a finitely generated indecomposable and non-projective Λ-module. There exists an almost split sequence

$$0 \longrightarrow \Omega^2 Z \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

in the category of Λ-modules (cf. [4]), and this induces an Auslander-Reiten triangle

$$\Omega^2 Z \longrightarrow Y \longrightarrow Z \longrightarrow \Omega Z$$

in $\underline{\operatorname{Mod}}\,\Lambda$ where ΩX denotes the kernel of a projective cover for a Λ-module X. Note that ΩX is finitely generated if X is finitely generated.

(3) Consider the stable homotopy category of CW-spectra. A spectrum Z is a compact object if and only if it is finite. Suppose therefore that Z is finite with local endomorphism ring and let $Z \to T_Z I$ be the third map in the Auslander-Reiten triangle corresponding to Z. This map induces the zero map $\pi_*(Z) \to \pi_*(T_Z I)$ between the stable homotopy groups since no map $S^n \to Z$ is a retraction. Assuming Freyd's Generating Hypothesis (cf. [8]) it follows that $T_Z I$ is not a finite spectrum. Hence the other end term $\Sigma^{-1}(T_Z I)$ of the Auslander-Reiten triangle for Z is not compact.

3. Pure-injective objects

The concept of purity for triangulated categories has been introduced in [10]. For the purpose of this paper it is important to have relative versions of some results in [10]. In fact, we shall always work with a fixed set of compact objects instead of taking a representative set of all compact objects.

Let $\mathbf{C}$ be a set of objects in $\mathcal{S}$. We shall view $\mathbf{C}$ as a full subcategory of $\mathcal{S}$. A $\mathbf{C}$-*module* is by definition an additive functor $\mathbf{C}^{\mathrm{op}} \to \mathrm{Ab}$ into the category Ab of abelian groups, and a map between two $\mathbf{C}$-modules is a natural transformation. The $\mathbf{C}$-modules form an abelian category which we denote by $\operatorname{Mod}\mathbf{C}$. For example, if $\mathbf{C}$ consists of one object C, then $\operatorname{Mod}\mathbf{C}$ is the category of modules over the endomorphism ring of C. Note that (co)kernels and (co)products in $\operatorname{Mod}\mathbf{C}$ are computed *pointwise*: for instance, a sequence $X \to Y \to Z$ of maps between $\mathbf{C}$-modules is exact if and only if the sequence $X(C) \to Y(C) \to Z(C)$ is exact in Ab for all $C \in \mathbf{C}$. Every object X in $\mathcal{S}$ gives rise to a $\mathbf{C}$-module

$$\operatorname{Hom}(\mathbf{C}, X) = \operatorname{Hom}(-, X)|_{\mathbf{C}} \colon \mathbf{C}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$$

and we get a functor

$$\operatorname{Hom}(\mathbf{C}, -) \colon \mathcal{S} \longrightarrow \operatorname{Mod}\mathbf{C}, \quad X \mapsto \operatorname{Hom}(\mathbf{C}, X).$$

Suppose now that $\mathbf{C}$ is a set of compact objects, and let I be an injective $\mathbf{C}$-module. Then we denote by $T_{\mathbf{C}}I$ the *dual* of $\mathbf{C}$ with respect to I which is defined by the isomorphism

$$\operatorname{Hom}(\operatorname{Hom}(\mathbf{C}, -), I) \cong \operatorname{Hom}(-, T_{\mathbf{C}}I).$$

In fact, the contravariant functor $\operatorname{Hom}(\operatorname{Hom}(\mathbf{C}, -), I) \colon \mathcal{S} \to \mathrm{Ab}$ is exact and sends coproducts to products, since I is injective and every object in $\mathbf{C}$ is compact; it is therefore representable by Brown's theorem. Next we discuss some basic properties of $T_{\mathbf{C}}I$.

Lemma 3.1. $\operatorname{Hom}(\mathbf{C}, T_{\mathbf{C}}I) \cong I$.

Proof. Combine the defining isomorphism of $T_{\mathbf{C}}I$ with Yoneda's lemma. □

Proposition 3.2. *Let $\mathbf{C}$ be a set of compact objects and Y be an arbitrary object in $\mathcal{S}$. Then the following conditions are equivalent:*

1. *$Y \cong T_{\mathbf{C}}I$ for some injective $\mathbf{C}$-module I.*
2. *The map $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(\operatorname{Hom}(\mathbf{C}, X), \operatorname{Hom}(\mathbf{C}, Y))$, $\phi \mapsto \operatorname{Hom}(\mathbf{C}, \phi)$, is an isomorphism for every X in $\mathcal{S}$.*

Proof. (1) ⇒ (2) We get an inverse for the map

$$\operatorname{Hom}(X, T_{\mathbf{C}}I) \longrightarrow \operatorname{Hom}(\operatorname{Hom}(\mathbf{C}, X), \operatorname{Hom}(\mathbf{C}, T_{\mathbf{C}}I)), \quad \phi \mapsto \operatorname{Hom}(\mathbf{C}, \phi)$$

if we take the composition

$$\operatorname{Hom}(\operatorname{Hom}(\mathbf{C}, X), \operatorname{Hom}(\mathbf{C}, T_{\mathbf{C}}I)) \cong \operatorname{Hom}(\operatorname{Hom}(\mathbf{C}, X), I) \cong \operatorname{Hom}(X, T_{\mathbf{C}}I)$$

where the first map is induced by the isomorphism in Lemma 3.1 and the second is the defining isomorphism for $T_{\mathbf{C}}I$.

(2) ⇒ (1) Let $\mu \colon \operatorname{Hom}(\mathbf{C}, Y) \to I$ be an injective envelope, and put $Z = T_{\mathbf{C}}I$. The map μ corresponds to a map $Y \to Z$ which we complete to a triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X.$$

Note that μ is isomorphic to $\operatorname{Hom}(\mathbf{C}, \beta)$ by Lemma 3.1. Therefore the induced map $\operatorname{Hom}(\mathbf{C}, X) \to \operatorname{Hom}(\mathbf{C}, Y)$ is zero, and (2) implies that $\alpha = 0$. Thus β is a section, and it follows that μ is a section. In fact, the minimality of an injective envelope implies that μ is an isomorphism. Now consider a map $\beta'\colon Z \to Y$ such that $\beta' \circ \beta = \mathrm{id}_Y$. It follows that $\beta \circ \beta'$ induces the identity

$$\mathrm{Hom}(\mathbf{C}, Z) \longrightarrow \mathrm{Hom}(\mathbf{C}, Y) \longrightarrow \mathrm{Hom}(\mathbf{C}, Z).$$

Therefore $\beta \circ \beta' = \mathrm{id}_Z$ by the first part of the proof and we conclude that $Y \cong T_{\mathbf{C}} I$. □

Corollary 3.3. *Let $\mathbf{C}$ be a set of compact objects in $\mathcal{S}$ and denote by $\operatorname{Inj}\mathbf{C}$ the full subcategory of injective $\mathbf{C}$-modules. Then the assignment $I \mapsto T_{\mathbf{C}} I$ induces a fully faithful functor $\operatorname{Inj}\mathbf{C} \to \mathcal{S}$.*

Let us include a characterization of those objects in $\mathcal{S}$ which are of the form $T_{\mathbf{C}} I$ for some set $\mathbf{C}$ of compact objects and some injective $\mathbf{C}$-module I. To this end recall the following definition from [10].

Definition 3.4. An object X in $\mathcal{S}$ is *pure-injective* if any map $X \to Y$ is a section whenever it induces a monomorphism $\mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y)$ for every compact C in $\mathcal{S}$.

Proposition 3.5. *The following are equivalent for an object X in $\mathcal{S}$:*
1. *X is pure-injective.*
2. *$X \cong T_{\mathbf{C}} I$ for some set $\mathbf{C}$ of compact objects and some injective $\mathbf{C}$-module I.*
3. *The summation map $\coprod_\Omega X \to X$ factors through the canonical map $\coprod_\Omega X \to \prod_\Omega X$ for every set Ω.*

Proof. Adapt the argument of Theorem 1.8 in [10]. □

4. Morphisms determined by objects

Given an object Y in $\mathcal{S}$, we may ask for a classification of all maps $X \to Y$ ending in Y, where two such maps $\alpha_i\colon X_i \to Y$ $(i = 1, 2)$ are *isomorphic* if there exists an isomorphism $\phi\colon X_1 \to X_2$ such that $\alpha_1 = \alpha_2 \circ \phi$.

In this section we give an answer to that question which is based on the concept of a map which is determined by an object. Originally, this concept was introduced by Auslander in order to give a conceptual explanation for the existence of left or right almost split maps.

Definition 4.1. A map $\alpha\colon X \to Y$ is said to be *right determined* by a set $\mathbf{C}$ of objects if for every map $\alpha'\colon X' \to Y$ the following conditions are equivalent:
1. α' factors through α;
2. for every $C \in \mathbf{C}$ and every map $\phi\colon C \to X'$ the map $\alpha' \circ \phi$ factors through α.

Given a map $\alpha\colon X \to Y$ in $\mathcal{S}$, we denote by $\operatorname{Im}\mathrm{Hom}(\mathbf{C}, \alpha)$ the image of the induced map $\mathrm{Hom}(\mathbf{C}, X) \to \mathrm{Hom}(\mathbf{C}, Y)$. Note that an equivalent formulation of condition (2) is

(2') $\operatorname{Im}\mathrm{Hom}(\mathbf{C}, \alpha') \subseteq \operatorname{Im}\mathrm{Hom}(\mathbf{C}, \alpha)$.

For example, a map $\alpha\colon X \to Y$ is right almost split if and only if $\Gamma = \mathrm{End}(Y)$ is a local ring, α is right determined by Y, and $\operatorname{Im}\mathrm{Hom}(Y, \alpha) = \operatorname{rad}\Gamma$.

Our main existence result for maps determined by objects is the following.

Theorem 4.2. *Let $\mathbf{C}$ be a set of compact objects and Y be an arbitrary object in $\mathcal{S}$. Suppose that H is a $\mathbf{C}$-submodule of $\mathrm{Hom}(\mathbf{C}, Y)$. Then there exists, up to isomorphism, a unique right minimal map $\alpha\colon X \to Y$ which is right determined by $\mathbf{C}$ and satisfies $\operatorname{Im}\mathrm{Hom}(\mathbf{C}, \alpha) = H$.*

Proof. Let $\mu\colon \operatorname{Hom}(\mathbf{C},Y)/H \to I$ be an injective envelope and define $Z = T_{\mathbf{C}}I$ which gives an isomorphism
$$\operatorname{Hom}(\operatorname{Hom}(\mathbf{C},-),I) \cong \operatorname{Hom}(-,Z).$$
Note that $I \cong \operatorname{Hom}(\mathbf{C},Z)$ by Yoneda's lemma. Now let $\beta\colon Y \to Z$ be the map corresponding to the composition
$$\operatorname{Hom}(\mathbf{C},Y) \xrightarrow{\pi} \operatorname{Hom}(\mathbf{C},Y)/H \xrightarrow{\mu} I.$$
We complete β to a triangle $X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z \overset{\gamma}{\to} \Sigma X$ and claim that α has the desired properties. Clearly, $\operatorname{Im}\operatorname{Hom}(\mathbf{C},\alpha) = H$ since there exists an exact sequence
$$\operatorname{Hom}(\mathbf{C},X) \longrightarrow \operatorname{Hom}(\mathbf{C},Y) \longrightarrow \operatorname{Hom}(\mathbf{C},Z) \cong I$$
where the composition $\operatorname{Hom}(\mathbf{C},Y) \to I$ is just $\mu\circ\pi$. The right minimality of α is equivalent to the left minimality of β by Lemma 2.5. Now let $\phi\colon Z \to Z$ be a map such that $\phi\circ\beta = \beta$. Applying $\operatorname{Hom}(\mathbf{C},-)$, we get a map $\psi\colon I \to I$ such that $\psi\circ\mu = \mu$. The map μ is an injective envelope and therefore left minimal. Thus ψ is an isomorphism and it follows from Proposition 3.2 that ϕ is an isomorphism. It remains to show that α is right determined by $\mathbf{C}$. To this end let $\alpha'\colon X' \to Y$ be a map such that for every $C \in \mathbf{C}$ and every map $\phi\colon C \to X'$ the map $\alpha'\circ\phi$ factors through α. Thus $\operatorname{Im}\operatorname{Hom}(\mathbf{C},\alpha') \subseteq H$ and therefore $\operatorname{Hom}(\mathbf{C},\beta\circ\alpha') = 0$. Another application of the defining isomorphism for Z gives $\beta\circ\alpha' = 0$ and this implies that α' factors through α. □

The next result characterizes the maps which are determined by compact objects. Recall that the *cofibre* of a map $\alpha\colon X \to Y$ is the object which occurs as the third term in a triangle
$$X \xrightarrow{\alpha} Y \longrightarrow Z \longrightarrow \Sigma X.$$

Theorem 4.3. *Let $\mathbf{C}$ be a set of compact objects. Then the following conditions are equivalent for a map $\alpha\colon X \to Y$.*

1. *The map α is right determined by $\mathbf{C}$.*
2. *There exists a decomposition $X = X'\coprod X''$ such that $\alpha|_{X'}$ is right minimal and right determined by $\mathbf{C}$, and $\alpha|_{X''} = 0$.*
3. *There exists a decomposition $X = X'\coprod X''$ such that the cofibre of $\alpha|_{X'}$ is isomorphic to $T_{\mathbf{C}}I$ for some injective $\mathbf{C}$-module I, and $\alpha|_{X''} = 0$.*

Moreover, the restriction $\alpha|_{X'}$ of α as in (2) is unique up to isomorphism.

Proof. (1) $\Rightarrow$ (2) Let $H = \operatorname{Im}\operatorname{Hom}(\mathbf{C},\alpha)$ and denote by $\alpha'\colon X' \to Y$ the right minimal and right $\mathbf{C}$-determined map satisfying $\operatorname{Im}\operatorname{Hom}(\mathbf{C},\alpha') = H$ which exists by Theorem 4.2. Assuming (1), there are maps $\phi\colon X' \to X$ and $\phi'\colon X \to X'$ such that $\alpha' = \alpha\circ\phi$ and $\alpha = \alpha'\circ\phi'$. Thus $\alpha' = \alpha'\circ(\phi'\circ\phi)$ and $\phi'\circ\phi$ is an isomorphism since α' is right minimal. It follows that ϕ is a section, and completing it to a triangle $X' \to X \to X'' \to \Sigma X'$ gives a decomposition $X = X'\coprod X''$ as in (2).

(2) $\Rightarrow$ (3) Let $X = X'\coprod X''$ be a decomposition with $\alpha|_{X''} = 0$. We complete $\alpha|_{X'}$ to a triangle $X' \to Y \to Z \to \Sigma X'$. Now suppose that the map $\alpha|_{X'}$ is right minimal and right determined by $\mathbf{C}$. Then we deduce from the construction given in the proof of Theorem 4.2 that $Z \cong T_{\mathbf{C}}I$ for some injective $\mathbf{C}$-module I.

(3) $\Rightarrow$ (1) Let $X = X'\coprod X''$ be a decomposition as in (3). Clearly, α is right determined by $\mathbf{C}$ if and only $\alpha|_{X'}$ is right determined by $\mathbf{C}$. Therefore we may assume $X'' = 0$. Now complete α to a triangle
$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} T_{\mathbf{C}}I \xrightarrow{\gamma} \Sigma X.$$

In order to show that α is right determined by $\mathbf{C}$ let $\alpha'\colon W \to Y$ be a map such that for every $C \in \mathbf{C}$ and every map $\phi\colon C \to W$ the map $\alpha' \circ \phi$ factors through α. We need to show that α' factors through α. Clearly, this equivalent to showing that $\beta \circ \alpha' = 0$. However, it follows from Proposition 3.2 that $\beta \circ \alpha' = 0$ since $\operatorname{Hom}(\mathbf{C}, \beta \circ \alpha') = 0$ by the assumption on α'. Thus α' factors through α. □

Corollary 4.4. *The following conditions are equivalent for a map $\alpha\colon X \to Y$.*

(1) *The map α is right determined by a set of compact objects.*
(2) *There exists a decomposition $X = X' \coprod X''$ such that the cofibre of $\alpha|_{X'}$ is pure-injective and $\alpha|_{x''} = 0$.*

5. A defect formula

The defect of a triangle is the analogue of the defect of a short exact sequence.

Definition 5.1. Given a triangle $\varepsilon\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ and a set $\mathbf{C}$ of objects in $\mathcal{S}$, the *defect* of ε with respect to $\mathbf{C}$ is defined as follows:

$$\begin{aligned} \Delta_\varepsilon(\mathbf{C}) &= \operatorname{Coker}(\operatorname{Hom}(\mathbf{C}, Y) \to \operatorname{Hom}(\mathbf{C}, Z)), \\ \nabla_\varepsilon(\mathbf{C}) &= \operatorname{Coker}(\operatorname{Hom}(Y, \mathbf{C}) \to \operatorname{Hom}(X, \mathbf{C})). \end{aligned}$$

Note that Δ_ε and ∇_ε are functorial in ε. More precisely, any map $\varepsilon \to \varepsilon'$ between triangles induces natural transformations $\Delta_\varepsilon \to \Delta_{\varepsilon'}$ and $\nabla_{\varepsilon'} \to \nabla_\varepsilon$.

The *covariant defect* $\nabla_\varepsilon(\mathbf{C})$ and the *contravariant defect* $\Delta_\varepsilon(\mathbf{C})$ of a triangle ε are related by a formula which is an analogue of a formula of Auslander for the defect of a short exact sequence.

Theorem 5.2. *Let ε be a triangle and $\mathbf{C}$ be a set of compact object in $\mathcal{S}$. Also let I be an injective $\mathbf{C}$-module. Then there exists an isomorphism*

$$\operatorname{Hom}(\Delta_\varepsilon(\mathbf{C}), I) \cong \nabla_\varepsilon(\Sigma^{-1}(T_{\mathbf{C}} I))$$

which is functorial in ε and I.

Proof. Fix a triangle $\varepsilon\colon X \to Y \to Z \to \Sigma X$. Applying the defining isomorphism of $T_{\mathbf{C}}I$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} ((\mathbf{C}, \Sigma Y), I) & \to & ((\mathbf{C}, \Sigma X), I) & \xrightarrow{\phi} & ((\mathbf{C}, Z), I) & \to & ((\mathbf{C}, Y), I) \\ \| \wr & & \| \wr & & \| \wr & & \| \wr \\ (\Sigma Y, T_{\mathbf{C}} I) & \to & (\Sigma X, T_{\mathbf{C}} I) & \to & (Z, T_{\mathbf{C}} I) & \to & (Y, T_{\mathbf{C}} I) \\ \| \wr & & \| \wr & & \| \wr & & \| \wr \\ (Y, \Sigma^{-1}(T_{\mathbf{C}} I)) & \to & (X, \Sigma^{-1}(T_{\mathbf{C}} I)) & \xrightarrow{\psi} & (\Sigma^{-1} Z, \Sigma^{-1}(T_{\mathbf{C}} I)) & \to & (\Sigma^{-1} Y, \Sigma^{-1}(T_{\mathbf{C}} I)) \end{array}$$

The assertion now follows since

$$\operatorname{Hom}(\Delta_\varepsilon(\mathbf{C}), I) \cong \operatorname{Im} \phi \cong \operatorname{Im} \psi \cong \nabla_\varepsilon(\Sigma^{-1}(T_{\mathbf{C}} I)).$$

□

Note that the isomorphism of the defect formula specializes to the isomorphism defining $T_{\mathbf{C}}I$ if one takes a triangle $X \to Y \to Z \to \Sigma X$ with $Y = 0$. The following application illustrates the defect formula.

Corollary 5.3. *Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ be a triangle in $\mathcal{S}$. Let $\mathbf{C}$ be a set of compact objects in $\mathcal{S}$ and suppose that I is an injective cogenerator for the category of $\mathbf{C}$-modules. Then the following are equivalent:*

(1) *every map $C \to Z$ with $C \in \mathbf{C}$ factors through β;*
(2) *every map $X \to \Sigma^{-1} T_{\mathbf{C}} I$ factors through α.*

Appendix: Classical Auslander-Reiten theory

In this appendix we sketch the parallel between the classical Auslander-Reiten theory and the material of this paper. We follow Auslander's exposition in his Philadelphia notes [2]. Throughout we fix an associative ring Λ with identity. The category of Λ-modules is denoted by $\operatorname{Mod}\Lambda$, and $\operatorname{mod}\Lambda$ denotes the full subcategory of finitely presented modules. The *stable category* of $\operatorname{mod}\Lambda$ modulo projectives is denoted by $\underline{\operatorname{mod}}\,\Lambda$. The *transpose*

$$\operatorname{Tr}\colon (\underline{\operatorname{mod}}\,\Lambda)^{\mathrm{op}} \longrightarrow \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$$

is an equivalence which relates the stable categories of Λ and its opposite ring Λ^{op} (cf. [2, p.27]).

5.1. **The Auslander-Reiten formula.** The main ingredient for constructing maps and triangles in the previous sections of this paper is the fact that for every compact object C with $\Gamma = \operatorname{End}(C)$ and for every injective Γ-module I, there exists an object $T_C I$ such that

$$\operatorname{Hom}_\Gamma(\operatorname{Hom}(C,-), I) \cong \operatorname{Hom}(-, T_C I).$$

The following Auslander-Reiten formula plays an analogous role for the category of Λ-modules (cf. Proposition I.3.4 in [2]).

Theorem (Auslander/Reiten). *Let C be a finitely presented Λ-module and suppose that $\operatorname{Tr} C$ is a Λ^{op}-Γ-bimodule. Let I be an injective Γ-module. Then there exists an isomorphism*

$$\operatorname{Hom}_\Gamma(\underline{\operatorname{Hom}}_\Lambda(C,-), I) \cong \operatorname{Ext}^1_\Lambda(-, \operatorname{Hom}_\Gamma(\operatorname{Tr} C, I))$$

which is functorial in I.

5.2. **Almost split sequences.** A short exact sequence $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ is called *almost split* if α is left almost split and β is right almost split. The following existence result is taken from Proposition II.5.1 in [2]; it generalizes the corresponding result for modules over artin algebra from [3].

Theorem (Auslander). *Let Z be a finitely presented non-projective Λ-module with local endomorphism ring. Let $\Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} Z)$ and denote by I an injective envelope of $\Gamma/\operatorname{rad}\Gamma$. Then there exists an almost split sequence*

$$0 \longrightarrow \operatorname{Hom}_\Gamma(\operatorname{Tr} X, I) \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

5.3. **Pure-injective modules.** Auslander noticed the relevance of pure-injective modules (cf. Section I.10 in [2]) even though he did not exploit this fact systematically. In any case, the following result (cf. Proposition I.3.8 in [2]) deals implicitly with pure-injectives and plays a key role; it is the analogue of our Corollary 3.3.

Proposition (Auslander). *Let C be a finitely presented Λ^{op}-module. Let $\Gamma = \operatorname{End}_{\Lambda^{\mathrm{op}}}(C)$ and denote by $\operatorname{Inj}\Gamma$ the full subcategory of injective Γ-modules. Then the functor*

$$\operatorname{Inj}\Gamma \longrightarrow \operatorname{Mod}\Lambda, \quad I \mapsto \operatorname{Hom}_\Gamma(C, I)$$

is fully faithful.

Note that $\operatorname{Hom}_\Gamma(C,I)$ is a pure-injective Λ-module whenever I is an injective Γ-module. The inverse for the functor $I \mapsto \operatorname{Hom}_\Gamma(C,I)$ is given by $X \mapsto X \otimes_\Lambda C$. Therefore $\operatorname{Hom}_\Gamma(C,-)$ is the analogue of our functor $T_{\mathbf{C}}$, whereas $- \otimes_\Lambda C$ is the analogue of our functor $\operatorname{Hom}(\mathbf{C},-)$.

5.4. **Morphisms determined by modules.** The concept of a map which is determined by an object was introduced by Auslander in order to give a conceptual explanation for the existence of almost split sequences. The main existence result is the following theorem (cf. Theorem I.3.9 in [2]).

Theorem (Auslander). *Let C be a finitely presented Λ-module. Let Y be an arbitrary Λ-module and suppose that H is an $\operatorname{End}_\Lambda(C)$-submodule of $\operatorname{Hom}_\Lambda(C,Y)$. Then there exists, up to isomorphism, a unique right minimal map $\alpha\colon X \to Y$ which is right determined by C and satisfies $\operatorname{Im}\operatorname{Hom}_\Lambda(C,\alpha) = H$.*

5.5. **Auslander's defect formula.** Given a short exact sequence $\varepsilon\colon 0 \to X \to Y \to Z \to 0$ and a module C, the definition of the *defect* $\Delta_\varepsilon(C)$ and $\nabla_\varepsilon(C)$ is completely analogous to that of Definition 5.1. The following formula (cf. Theorem III.4.1 in [2]) relates the contravariant and the covariant defect of a short exact sequence.

Theorem (Auslander). *Let ε be a short exact sequence and C be a finitely presented module in $\operatorname{Mod}\Lambda$. Suppose that $\operatorname{Tr} C$ is a Λ^{op}-Γ-bimodule and let I be an injective Γ-module. Then there exists an isomorphism*

$$\operatorname{Hom}_\Gamma(\Delta_\varepsilon(C), I) \cong \nabla_\varepsilon(\operatorname{Hom}_\Gamma(\operatorname{Tr} C, I))$$

which is functorial in ε and I.

Note that this isomorphism specializes to the Auslander-Reiten formula mentioned above if one takes a short exact sequence $\varepsilon\colon 0 \to X \to Y \to Z \to 0$ with Y projective.

Acknowledgement. I would like to thank Apostolos Beligiannis and Idun Reiten for a number of helpful conversations about the subject of this paper.

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