Homotopy Algebras are Homotopy Algebras

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Abstract. We prove that strongly homotopy algebras (such as A_∞, C_∞, sh Lie, G_∞,...) are homotopy invariant concepts in the category of chain complexes. An important consequence is a rigorous proof that 'strongly homotopy structures transfer over chain homotopy equivalences.'

1. Introduction.

Historically, the first examples of homotopy algebras were strongly homotopy associative algebras, also called A_∞-algebras or sh associative algebras, discovered, as structures intrinsically present in the singular chain complex of a loop space, by Jim Stasheff [22] in 1963.

A *strongly homotopy associative algebra* is a chain complex $V = (V, \partial)$ together with a bilinear product $\mu_2 : V \otimes V \to V$ which is associative only *up to a homotopy* $\mu_3 : V^{\otimes 3} \to V$, that is, for each $a, b, c \in V$,

$$\mu_2(\mu_2(a,b)c) - \mu_2(a, \mu_2(b,c)) = [\mu_3, \partial](a,b,c),$$

where

$$[\mu_3; \partial](a,b,c) := \mu_3(\partial a, b, c) + (-1)^{|a|}\mu_3(a, \partial b, c) + (-1)^{|a|+|b|}\mu_3(a,b,\partial c) + \partial\mu_3(a,b,c)$$

is the differential induced on $Hom(V^{\otimes 3}, V)$ by ∂. Very crucially, there are some higher coherence relations involving μ_2, μ_3 and higher homotopies $\mu_n : V^{\otimes n} \to V$, $n \geq 4$. Namely, for each $n \geq 2$ and $a_1, \ldots, a_n \in V$,

$$(1) \qquad \sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{s=0}^{n-j} (-1)^{j+s(j+1)} \mu_i(a_1,\ldots, a_s, \mu_j(a_{s+1},\ldots, a_{s+j}), a_{s+j+1},\ldots, a_n) = [\mu_n, \partial](a_1,\ldots, a_n).$$

For a gentle introduction to these structures, we recommend [21].

Strongly homotopy (sh) morphisms of A_∞-algebras were introduced in 1965 by Allan Clark [3] (though strongly homotopy maps of topological spaces with additional structures were considered by M. Sugawara [23] in 1960). If $\mathbf{V} = (V, \partial, \mu_2, \mu_3, \ldots)$ and $\mathbf{W} =$

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$(W,\partial,\nu_2,\nu_3,\ldots)$ are two A_∞-algebras, then a *morphism* $\mathbf{f}:\mathbf{V}\to\mathbf{W}$ is a *sequence* $\mathbf{f}=(f_1,f_2,f_3,\ldots)$ of multilinear maps $f_n:V^{\otimes n}\to W$, $n\geq 1$, such that f_1 is a chain map which commutes with the products μ_2 and ν_2 up to a homotopy f_2,

$$f_1(\mu_2(a,b))-\nu_2(f_1(a),f_1(b))=[f_2,\partial](a,b), \tag{2}$$

and the following higher coherence relations are satisfied, for each $n\geq 2$ and $a_1,\ldots,a_n\in V$:

$$\sum_{k=2}^{n}\sum_{r_1+\ldots+r_k=n}(-1)^{\sum_{1\leq i<j\leq k}r_i(r_j+1)}\cdot\nu_k(f_{r_1}(a_1,\ldots,a_{r_1}),\ldots,f_{r_k}(a_{n-r_k+1},\ldots,a_n))-$$
$$-\sum_{\substack{i+j=n+1\\ j\geq 2}}\sum_{s=0}^{n-j}(-1)^{n+s(j+1)}\cdot f_i(a_1,\ldots,a_s,\mu_j(a_{s+1},\ldots,a_{s+j}),a_{s+j+1},\ldots,a_n)=[f_n,\partial](a_1,\ldots,a_n). \tag{3}$$

If we let `Chain` to denote the category of chain complexes over a ring R and $\mathtt{A}(\infty)$ the category of A_∞-algebras and their sh morphisms, then the correspondence

$$\mathbf{V}=(V,\partial,\mu_2,\mu_3,\ldots)\longmapsto(V,\partial),\ \mathbf{f}=(f_1,f_2,\ldots)\longmapsto f_1,$$

defines the 'forgetful' functor $\#:\mathtt{A}(\infty)\to\mathtt{Chain}$. Observe that the functor $\#$ is *not* injective on morphisms.

More examples of homotopy algebras appeared later on. Let us name at least strongly homotopy associative commutative algebras [12, 19], also called C_∞ or balanced A_∞-algebras, and strongly homotopy Lie algebras, also called sh Lie or L_∞-algebras [17, 16]. Then, in the early 90's, with the renaissance of operads, the heavens opened wide and examples of homotopy algebras became abundant.

With hindsight, one might say that by a homotopy $\mathcal{P}$-algebra, where $\mathcal{P}$ is an operad in the category of chain complexes, is today understood an algebra over a *cofibrant model* of $\mathcal{P}$. A cofibrant model of $\mathcal{P}$ is, by definition, a free operad $\mathcal{F}(E)$ equipped with a differential ∂ and a map $\alpha:(\mathcal{F}(E),\partial)\to\mathcal{P}$ that is a homology isomorphism. We also assume the existence of a suitable filtration on the space of generators E, see Definition 15 for details.

This definition is broad enough to comprise all objects which would probably be called homotopy algebras, including various types of homotopy Gerstenhaber algebras arising in connection with quantum field theory [4], the Deligne conjecture [25] and, later, in the proof of Kontsevich's formality, see [13, 14].

If $R=\mathbf{k}$, the field of characteristic zero, not only does there exist a cofibrant model for any operad $\mathcal{P}$, but there even exists, under the assumption $\mathcal{P}(1)=\mathbf{k}$, a very special type of this model, called the *minimal model* of $\mathcal{P}$. By definition, the minimal model of $\mathcal{P}$ is a cofibrant model $\mathcal{M}_{\mathcal{P}} = (\mathcal{F}(E), \partial)$ of $\mathcal{P}$ as above such that $\partial(E)$ consists of decomposable elements of the free operad $\mathcal{F}(E)$.

In case $\mathcal{P}$ is quadratic Koszul, the minimal model of $\mathcal{P}$ is isomorphic to the dual bar construction on the quadratic dual $\mathcal{P}^!$ and our conception coincides with that of [5]. In particular, if $\mathcal{P} = \mathit{Ass}$, the operad for associative algebras, homotopy $\mathcal{P}$-algebras are precisely A_∞-algebras discussed above.

In more detail, the operad Ass for associative algebras is given by

$$\mathit{Ass} := \mathcal{F}(\mu)/(\mu(\mu \otimes \mathbb{1}) - \mu(\mathbb{1} \otimes \mu)),$$

the free operad on one bilinear operation μ, modulo the associativity. The minimal model for the operad Ass is

$$\mathit{Ass} \xleftarrow{\alpha} (\mathcal{F}(\mu_2, \mu_3, \mu_4, \ldots), \partial),$$

where $\alpha(\mu_2) = \mu$ while α is trivial on the remaining generators. The differential is defined by

$$\partial(\mu_n) = \sum_{\substack{i+j=n+1 \\ i,j\geq 2}} \sum_{s=0}^{n-j} (-1)^{j+s(j+1)} \mu_i(\mathbb{1}^{\otimes s} \otimes \mu_j \otimes \mathbb{1}^{\otimes i-s-1}),$$

see [20, Example 4.8]. Comparing it with (1), we immediately see that A_∞-algebras are algebras over the operad $(\mathcal{F}(\mu_2, \mu_3, \mu_4, \ldots), \partial)$ defined above.

Let us make another brief excursion to topology. With a great simplification, we might say that key problems in algebraic topology of the sixties were related to homotopy invariant structures on topological spaces. For instance, given an associative topological monoid Y, is it true that each space Y' of the same homotopy type inherits from Y an induced associative monoid structure?

The answer is no, but to understand what of the monoidal structure remains preserved under a change of the homotopy type of the underlying space required a considerable effort. In this concrete case, the answer was that the corresponding homotopy invariant structure is that of a 'strongly homotopy associative monoid,' which is, by definition, a space acted on by the topological operad $\mathcal{K} = \{K_n\}_{n\geq 1}$ of the associahedra [22].

A conceptual solution of the problem was provided by Boardman & Vogt and their W-construction [2]. In today's language, one would say that the W-construction gives, to any topological operad $\mathcal{O}$, its specific cofibrant resolution $W\mathcal{O}$. Homotopy invariant algebras are then algebras over the operad $W\mathcal{O}$. Another, in a sense dual, solution of the problem based on 'lax' actions of operads was given by Lada [15].

Let us come back to the realm of algebra and try to formulate what homotopy invariance might mean here. The underlying category is the category `Chain` of chain complexes and their chain maps, and homotopy is provided by chain homotopy of maps. If we assume that the ground ring is a field, homotopy boils down to homology. Thus, for instance, two maps $f, g : (V, \partial) \to (W, \partial)$ are (chain) homotopic if and only if $H_*(f) = H_*(g)$.

Suppose we have a category of strongly homotopy $\mathcal{P}$-algebras $\mathtt{Alg}_{h\mathcal{P}}$, with a forgetful functor $\# : \mathtt{Alg}_{h\mathcal{P}} \to \mathtt{Chain}$. For an sh morphism $\mathbf{f}$ of strongly homotopy $\mathcal{P}$-algebras, we call the chain map $\mathbf{f}_\#$ the *underlying map* of $\mathbf{f}$. If g is a chain map and $\mathbf{f}$ such that $\mathbf{f}_\# = g$, we say that $\mathbf{f}$ is a *strongly homotopy $\mathcal{P}$-structure* on the map g. Let us remark that, generally speaking, there are certain problems with the categorial structure of strongly homotopy $\mathcal{P}$-algebras (see [2] where, for topological spaces, these subtleties required the use of weak Kan categories), but this subtlety will play no rôle in our paper.

What homotopy invariance means for topological spaces was formulated by Boardman and Vogt in the introduction to [2]. Their conditions, which we will also call *moves*, reformulated to algebra read as follows.

(M1) For each strongly homotopy $\mathcal{P}$-algebra $\mathbf{V}$, chain complex $W = (W, \partial)$ and a chain homotopy equivalence $f : (V, \partial) \to (W, \partial)$, there exist a sh $\mathcal{P}$-structure $\mathbf{W}$ on the chain complex (W, ∂) and a strongly homotopy $\mathcal{P}$-structure $\mathbf{f} : \mathbf{V} \to \mathbf{W}$ on the map f.

(M2) Suppose $\mathbf{V}$ and $\mathbf{W}$ are two strongly homotopy $\mathcal{P}$-algebras and $\mathbf{f} : \mathbf{V} \to \mathbf{W}$ a sh $\mathcal{P}$-algebra morphism. Suppose that $g : (V, \partial) \to (W, \partial)$ is a chain map that is chain homotopic to $\mathbf{f}_\#$. Then there exists a strongly homotopy $\mathcal{P}$-structure on g.

(M3) Suppose that $\mathbf{f} : \mathbf{V} \to \mathbf{W}$ is a strongly homotopy $\mathcal{P}$-algebra morphism such that $\mathbf{f}_\#$ is a chain homotopy equivalence. Suppose that $g : (W, \partial) \to (V, \partial)$ is a chain homotopy inverse of $\mathbf{f}_\#$. Then there exists a strongly homotopy $\mathcal{P}$-structure on g.

Moves **(M1)** and **(M3)** clearly imply:

(M1') For each strongly homotopy $\mathcal{P}$-algebra $\mathbf{W}$, chain complex $V = (V, \partial)$ and a chain homotopy equivalence $f : (V, \partial) \to (W, \partial)$, there exist a strongly homotopy $\mathcal{P}$-structure $\mathbf{V}$ on (V, ∂) and a strongly homotopy $\mathcal{P}$-structure $\mathbf{f} : \mathbf{V} \to \mathbf{W}$ on the map f.

Let us add, for the completeness of the exposition, also the following move which says that strongly homotopy algebras and their strongly homotopy maps form a certain weak category.

(M4) For each sequence $\mathbf{V}_1 \xrightarrow{\mathbf{f}_1} \mathbf{V}_2 \xrightarrow{\mathbf{f}_2} \cdots \xrightarrow{\mathbf{f}_{n-1}} \mathbf{V}_n \xrightarrow{\mathbf{f}_n} \mathbf{V}_{n+1}$ of sh $\mathcal{P}$-algebra morphisms and for each chain map $g : (V_1, \partial) \to (V_{n+1}, \partial)$, homotopic to the composition $\#(\mathbf{f}_n) \cdots \#(\mathbf{f}_1)$, there exist a strongly homotopy $\mathcal{P}$-structure on the map g.

The aim of this paper is to prove the above 'moves' for the case when the ground ring is a field of characteristic zero – see Proposition 24, Proposition 25 and Remark 26. We show that they are, in fact, special cases of a certain homotopy extension problem for diagrams. Our main theorem then states that, roughly speaking, the homotopy extension problem over cofibrant operads can be solved if and only if the corresponding 'classical' extension problem has a solution.

This is our understanding of the following principle that has been latent in topology since [2]:

Principle 1. *Algebras over a cofibrant operad are homotopy invariant.*

The most important seems to be move **(M1)** and its 'inverse' **(M1')** which says that 'strongly homotopy structures can be transfered over chain homotopy equivalences.' Various special cases of this statement were proved by many authors (Kadeishvili [12], Lambe, Gugenheim, Stasheff [18, 6], Huebschmann-Kadeishvili [10], Hess [8], Johannson-Lambe [11],...) and used as a folklore in deformation quantization papers of Kontsevich [13, 14]. Our theory explains the nature of 'side conditions' for strong deformation data in perturbation theory and shows why they are important for the existence of the 'transferred structure,' see Proposition 21.

Conventions. Throughout the paper, we work with algebras without units. In most situations, the results can be easily translated to the unital case, but there are important exceptions, such as move **(S)** of Example 10.

All algebraic objects are assumed to be defined over a field $\mathbf{k}$ of characteristic zero though, again, in some situations this assumption is not really necessary.

By an operad we always mean an operad in the symmetric monoidal category of chain complexes over $\mathbf{k}$.

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2. Colored operads and diagrams.

Since we will be primarily interested in algebras and their *diagrams*, we need to introduce (multi)colored operads as a tool describing these objects. These colored operads are, in fact, equivalent to the classical 'colored PROPs' of [2]. In [1] they are called 'many-sorted' operads.

Fix a (finite) *set of colors* $\mathbf{C} = \{1, \ldots, C\}$ and consider an operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ such that each $\mathcal{P}(n)$ decomposes into the direct sum

$$\mathcal{P}(n) = \bigoplus \mathcal{P}\binom{c}{c_1, \ldots, c_n}, \tag{4}$$

where the summation runs over all colors $c, c_1, \ldots, c_n \in \mathbf{C}$. We require the decomposition (4) to be, in the obvious sense, Σ_n-equivariant. More crucially, we demand the following.

Let $x \in \mathcal{P}\binom{c}{c_1, \ldots, c_n}$ and $x_i \in \mathcal{P}\binom{d_i}{d^i_1, \ldots, d^i_{k_i}}$, $1 \leq i \leq n$. Then we require that the non-triviality of the composition $x(x_1, \ldots, x_n)$ implies that

$$d_i = c_i, \text{ for } 1 \leq i \leq n, \tag{5}$$

in which case

$$x(x_1, \ldots, x_n) \in \mathcal{P}\binom{c}{d^1_1, \ldots, d^1_{k_1}, \ldots, d^n_1, \ldots, d^n_{k_n}}.$$

The intuitive meaning of (5) is that one may plug the element x_i into the i-th slot of the element x if and only if the color of the output of x_i is the same as the color of the i-th input of x, otherwise the composition is defined to be zero.

An example is provided by the colored endomorphism operad $\mathcal{E}nd_{\mathbf{U}}$ on a 'colored' chain complex $\mathbf{U} = \bigoplus_{c\in\mathbf{C}} \mathbf{U}_c$ where we put, in (4),

$$\mathcal{E}nd_{\mathbf{U}}\binom{c}{c_1, \ldots, c_n} := \mathrm{Hom}(U_{c_1} \otimes \cdots \otimes U_{c_n}, U_c).$$

Of course, any ordinary operad can be interpreted as a colored operad, with $\mathbf{C}$ = Point. From now on, by an operad we always mean a colored operad. Since most of the basic machinery for ordinary operads like free operads, ideals, presentations, etc., translate literally to the colored setting, we believe this will not lead to any confusion.

Example 2. The two-colored operad $Ass_{\bullet\to\bullet}$, with $\mathbf{C} := \{v, w\}$, describing diagrams $(V, \mu) \xrightarrow{f} (W, \nu)$, where (V, μ) and (W, ν) are associative algebras and f their homomorphism, has the presentation:

$$Ass_{\bullet\to\bullet} := \frac{\mathcal{F}(\mu, \nu, f)}{(\mu(\mu \otimes \mathbb{1}) = \mu(\mathbb{1} \otimes \mu),\ \nu(\nu \otimes \mathbb{1}) = \nu(\mathbb{1} \otimes \nu),\ f\mu = \nu(f \otimes f))}, \tag{6}$$

where

$$\mu \in Ass_{\bullet\to\bullet}\binom{v}{v, v},\ \nu \in Ass_{\bullet\to\bullet}\binom{w}{w, w} \text{ and } f \in Ass_{\bullet\to\bullet}\binom{w}{v}.$$

Here $\mathcal{F}(\mu, \nu, f)$ denotes the free operad on μ (a generator for the multiplication on V), ν (a generator for the multiplication on W) and f (a generator for the map), modulo the ideal generated by the axioms which have to be satisfied – the associativity of μ and ν together with the axiom saying that f is a homomorphism.

The diagram $(V,\mu) \xrightarrow{f} (W,\nu)$ as above is then the same as a homomorphism $A : \mathcal{A}ss_{\bullet\to\bullet} \to \mathcal{E}nd_{V,W}$ from the colored operad $\mathcal{A}ss_{\bullet\to\bullet}$ to the colored operad $\mathcal{E}nd_{V,W} := \mathcal{E}nd_{\mathbf{U}}$ where $\mathbf{U} := U_v \oplus U_w$ with $U_v := V$ and $U_w := W$. More generally, given an (ordinary) operad $\mathcal{P}$, the (colored) operad

$$\mathcal{P}_{\bullet\to\bullet} := \frac{\mathcal{P}_{\bullet_1} * \mathcal{P}_{\bullet_2} * \mathcal{F}(f)}{(fp_{\bullet_1} = p_{\bullet_2} f^{\otimes n})}, \tag{7}$$

where $*$ denotes the free product of operads, $\mathcal{P}_{\bullet_1}$ (resp. $\mathcal{P}_{\bullet_2}$) denotes a copy of $\mathcal{P}$ 'concentrated' at the first (resp. the second) space and $fp_{\bullet_1} = p_{\bullet_2} f^{\otimes n}$ means that, for each $n \geq 1$ and each operation $p \in \mathcal{P}(n)$, the arrow f is a p-homomorphism, describes homomorphisms of $\mathcal{P}$-algebras.

The constructions of Example 2 can be generalized as follows. Let S be a finite diagram, i.e. a category with finite number of objects and maps, and $\mathcal{P}$ an ordinary operad. The finiteness assumption on the diagram is not really necessary, but it will simplify the exposition and we will always assume it.

Then there exist the operad $\mathcal{P}_{\underline{S}}$ describing S-diagrams of $\mathcal{P}$-algebras. The operad $\mathcal{P}_{\underline{S}}$ can be constructed by taking generators for $\mathcal{P}$-algebra structures at each vertex of the diagram S, plus generators for the arrows of S, and then modding out the ideal generated by the axioms of the $\mathcal{P}$-structures, the axioms saying that the arrows of the diagram S are $\mathcal{P}$-homomorphisms and the axioms expressing the commutativity relations inside S.

We need also the relative version of the above construction. For a couple (S, D) of a diagram S and its subdiagram D, by an $(S, \underline{D})$-diagram of $\mathcal{P}$-algebras we mean an S-diagram of differential graded vector spaces with the property that each vector space corresponding to a vertex of the subdiagram D is equipped with a $\mathcal{P}$-algebra structure such that each arrow belonging to the subdiagram D is a homomorphism of these $\mathcal{P}$-algebras.

Thus by underlying we indicate that the corresponding part of the diagram carries a $\mathcal{P}$-structure.

Definition 3. *Let $\mathcal{P}_{(S,\underline{D})}$ denote the operad that describes $(S,\underline{D})$-diagrams of $\mathcal{P}$-algebras. The* canonical map *$\phi = \phi_{(S,\underline{D})} : \mathcal{P}_{(S,\underline{D})} \to \mathcal{P}_{\underline{S}}$ is defined by forgetting the $\mathcal{P}$-structures on the complement of D in S.*

Notice that the map ϕ in the above definition goes in the opposite direction than one would expect. Observe that $\mathcal{P}_{(D,\underline{D})} = \mathcal{P}_{\underline{D}}$, while $\mathcal{P}_{(D,\underline{\emptyset})} = \mathit{Triv}_{\underline{D}}$, the operad describing D-diagrams of graded vector spaces, i.e. algebras with no operations.

Definition 4. *We say that the couple (S, D) has the (classical) $\mathcal{P}$-*extension property *if there exist a left inverse $\psi : \mathcal{P}_{\underline{S}} \to \mathcal{P}_{(S,\underline{D})}$ of the canonical map $\phi : \mathcal{P}_{(S,\underline{D})} \to \mathcal{P}_{\underline{S}}$ of Definition 3,*

$$\psi \circ \phi = 1\!\!1. \tag{8}$$

Proposition 5. *If (S, D) has the $\mathcal{P}$-extension property, then each $(S, \underline{D})$-diagram of $\mathcal{P}$-algebras can be extended to an S-diagram of $\mathcal{P}$-algebras.*

Proof. We must show that, given a map $A_{(S,\underline{D})} : \mathcal{P}_{(S,\underline{D})} \to \mathcal{E}nd_{\mathbf{U}}$ representing an $(S, \underline{D})$-diagram of $\mathcal{P}$-algebras, there exists a map $A_{\underline{S}} : \mathcal{P}_{(S,\underline{D})} \to \mathcal{E}nd_{\mathbf{U}}$ representing an S-diagram of $\mathcal{P}$-algebras such that the diagram

$$\begin{tikzcd}
\mathcal{P}_{(S,\underline{D})} \arrow[rr, "\phi"] \arrow[dr, "A_{(S,\underline{D})}"'] & & \mathcal{P}_{\underline{S}} \arrow[dl, "A_{\underline{S}}"] \\
& \mathcal{E}nd_{\mathbf{U}} &
\end{tikzcd}$$

is commutative. The map $A_{\underline{S}} := A_{(S,\underline{D})} \circ \psi$ clearly has the desire property. ■

Let us formulate the following conjecture.

Conjecture 6. *The implication of Proposition 5 can be reversed, i.e. if any $(S, \underline{D})$ diagram of $\mathcal{P}$-algebras can be extended to an S-diagram of $\mathcal{P}$-algebras, then the couple $(S, \underline{D})$ has the $\mathcal{P}$-extension property.*

The extension property for $(S, \underline{D})$ does not depend on the operad $\mathcal{P}$ provided $\mathcal{P}$ is nontrivial, that is if it is satisfied for some nontrivial operad $\mathcal{P}$, then it is satisfied for all operads.

Example 7. (related to (M1)) Let S be the diagram

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "f"] & \circ \arrow[l, bend left, "g"]
\end{tikzcd}, \ fg = 1 \text{ and } gf = 1,$$

and let the subdiagram D consist of the 'solid vertex' and no morphisms. Then

$$\mathcal{P}_{(S,\underline{D})} := \frac{\mathcal{P}_{\bullet} * \mathcal{F}(f,g)}{(fg = 1,\ gf = 1)}, \tag{9}$$

where $\mathcal{P}_\bullet$ denotes the copy of $\mathcal{P}$ 'concentrated' at the vertex $\bullet$ and $*$ is the free product of operads. In a similar manner,

$$\mathcal{P}_{\underline{S}} := \frac{\mathcal{P}_\bullet * \mathcal{P}_\circ * \mathcal{F}(f,g)}{(fg=1,\ gf=1,\ fp_\bullet = p_\circ f^{\otimes n},\ gp_\circ = p_\bullet g^{\otimes n})},$$

where we use notation as in (7). The canonical map $\phi : \mathcal{P}_{(S,\underline{D})} \to \mathcal{P}_{\underline{S}}$ is given by $\phi(f) := f$, $\phi(g) := g$ and $\phi(p_\bullet) := p_\bullet$. Let us consider a map $\psi : \mathcal{P}_{\underline{S}} \to \mathcal{P}_{(S,\underline{D})}$ defined by

$$\psi(f) := f,\ \psi(g) := g,\ \psi(p_\bullet) := p_\bullet \text{ and } \psi(p_\circ) := fp_\bullet g^{\otimes n}. \tag{10}$$

We must verify that this definition is correct. Clearly, $\psi(fg) = fg$ and $\psi(gf) = gf$, thus ψ maps the axioms $fg = 1$ and $gf = 1$ to themselves. Next, we have $\psi(fp_\bullet) = fp_\bullet$ and

$$\psi(p_\circ f^{\otimes n}) = fp_\bullet g^{\otimes n} f^{\otimes n} = fp_\bullet (gf)^{\otimes n} = \text{ (because } gf = 1 \text{ in } \mathcal{P}_{(S,\underline{D})}) \ = fp_\bullet,$$

thus ψ respects the axiom $fp_\bullet = p_\circ f^{\otimes n}$. In a similar manner we verify that $\psi(gp_\circ) = p_\bullet g^{\otimes n} = \psi(p_\bullet g^{\otimes n})$, which finishes the verification that ψ is well-defined.

It is immediately clear by checking on the generators that $\psi \circ \phi = 1\!\!1$, so the couple (S, D) has the $\mathcal{P}$-extension property. It is very easy to verify that also $\phi \circ \psi = 1\!\!1$, thus the map ϕ is in fact an isomorphism.

Example 8. (related to (M2)) This example is rather dull. Consider the diagram S:

$$\begin{tikzcd} \bullet_1 \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \bullet_2 \end{tikzcd},\ f = g,$$

and let D be the 'thick' subdiagram. Then

$$P_{(S,\underline{D})} := \frac{\mathcal{P}_{\bullet_1} * \mathcal{P}_{\bullet_2} * \mathcal{F}(f,g)}{(f = g,\ fp_{\bullet_1} = p_{\bullet_2} f^{\otimes n})}$$

while

$$P_{\underline{S}} := \frac{\mathcal{P}_{\bullet_1} * \mathcal{P}_{\bullet_2} * \mathcal{F}(f,g)}{(f = g,\ fp_{\bullet_1} = p_{\bullet_2} f^{\otimes n},\ gp_{\bullet_1} = p_{\bullet_2} g^{\otimes n})}.$$

The canonical map $\phi : \mathcal{P}_{(S,\underline{D})} \to \mathcal{P}_{\underline{S}}$, given by $\phi(f) := f$, $\phi(g) := g$ and $\phi(p_{\bullet_1}) := p_{\bullet_1}$, is clearly an isomorphism thus the couple (S, D) has the $\mathcal{P}$-extension property.

Example 9. (related to (M3)) Let S be the diagram

$$\begin{tikzcd} \bullet_1 \arrow[r, bend left, "f"] & \bullet_2 \arrow[l, bend left, "g"] \end{tikzcd},\ fg = 1 \text{ and } gf = 1,$$

and let D be the 'thick' subdiagram. Then

$$P_{(S,\underline{D})} := \frac{\mathcal{P}_{\bullet_1} * \mathcal{P}_{\bullet_2} * \mathcal{F}(f,g)}{(fg = 1,\ gf = 1,\ fp_{\bullet_1} = p_{\bullet_2} f^{\otimes n})} \tag{11}$$

and

$$P_{\underline{S}} := \frac{\mathcal{P}_{\bullet_1} * \mathcal{P}_{\bullet_2} * \mathcal{F}(f,g)}{(fg = 1,\ gf = 1,\ fp_{\bullet_1} = p_{\bullet_2} f^{\otimes n},\ gp_{\bullet_2} = p_{\bullet_1} g^{\otimes n})}.$$

The canonical map $\phi : \mathcal{P}_{(S,\underline{D})} \to \mathcal{P}_{\underline{S}}$, given by $\phi(f) := f$, $\phi(g) := g$, $\phi(p_{\bullet_1}) := p_{\bullet_1}$ and $\phi(p_{\bullet_2}) := p_{\bullet_2}$ is again immediately seen to be an isomorphism.

Example 10. (due to P. Salvatore) In all the Examples 7, 8 and 9 above, the canonical map ϕ was an isomorphism. We give an example in which the map ϕ has a left inverse ψ, but ϕ it is not an isomorphism. Let S be the diagram in which

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] & \circ \arrow[l, bend left, "g"] \end{tikzcd}, \ gf = 1 \text{ but } \underline{\text{!not!}}\ fg = 1,$$

and let D be the solid vertex. Then

$$\mathcal{P}_{(S,\underline{D})} := \frac{\mathcal{P}_{\bullet} * \mathcal{F}(f,g)}{(gf = 1)} \quad \text{and} \quad \mathcal{P}_{\underline{S}} := \frac{\mathcal{P}_{\bullet} * \mathcal{P}_{\circ} * \mathcal{F}(f,g)}{(gf = 1,\ fp_{\bullet} = p_{\circ} f^{\otimes n},\ gp_{\circ} = p_{\bullet} g^{\otimes n})}.$$

The canonical map $\phi : \mathcal{P}_{(S,\underline{D})} \to \mathcal{P}_{\underline{S}}$ is given by $\phi(p_\bullet) := p_\bullet$, $\phi(f) = f$ and $\phi(g) = g$. Let us define the map $\psi : \mathcal{P}_{\underline{S}} \to \mathcal{P}_{(S,\underline{D})}$ by

$$\psi(f) := f,\ \psi(g) := g,\ \psi(p_\bullet) := p_\bullet \text{ and } \psi(p_\circ) := fp_\bullet g^{\otimes n}.$$

(i.e by formally the same formula as (10)). The same arguments as in Example 7 show that ψ is well-defined and that $\psi \circ \phi = \mathbb{1}$.

On the other hand, the map ϕ is not an isomorphism, because the equivalence class of $p_\circ$ in $\mathcal{P}_{\underline{S}}$ is clearly not in the image of ϕ.

The 'move' corresponding to the diagram of Example 10 is:

(S) For any strongly homotopy $\mathcal{P}$-algebra $\mathbf{V}$, chain complex $W = (W, \partial)$ and a chain map $f : (V, \partial) \to (W, \partial)$ that has a left chain-homotopy inverse, there exist a sh $\mathcal{P}$-structure $\mathbf{W}$ on the chain complex (W, ∂) and a strongly homotopy $\mathcal{P}$-structure $\mathbf{f} : \mathbf{V} \to \mathbf{W}$ on the map f.

It is crucial for the validity of the above move that we consider algebras *without units*.

Example 11. (related to (M4)) We leave it to the reader to analyze the situation when S is the diagram

$$
\begin{tikzcd}
\bullet \arrow[r, bend left, "f_1"] \arrow[rrrr, bend right, "g"'] & \bullet \arrow[r, bend left, "f_2"] & \bullet & \cdots & \bullet \arrow[r, bend left, "f_n"] & \bullet
\end{tikzcd}, \quad f_n \cdots f_2 f_1 = g,
$$

and S is the 'solid' subdiagram, that is, to describe the operads $\mathcal{P}_{(S,\underline{D})}$, $\mathcal{P}_{\underline{S}}$ and prove that the canonical map $\phi : \mathcal{P}_{(S,\underline{D})} \to \mathcal{P}_{\underline{S}}$ is an isomorphism.

Example 12. Let us give an example which *does not* have the extension property. Let S be the diagram

$$
\begin{tikzcd}
\bullet \arrow[r, "f"] & \circ
\end{tikzcd}
$$

and let D be the subdiagram consisting of the solid vertex $\bullet$. Then

$$
\mathcal{P}_{(S,\underline{D})} := \mathcal{P}_\bullet * \mathcal{F}(f),\ \mathcal{P}_{\underline{S}} := \frac{\mathcal{P}_\bullet * \mathcal{P}_\circ * \mathcal{F}(f)}{(f p_\bullet = p_\circ f^{\otimes n})},
$$

and the canonical map $\phi : \mathcal{P}_{(S,\underline{D})} \to \mathcal{P}_{\underline{S}}$ is given by $\phi(p_\bullet) := p_\bullet$ and $\phi(f) := f$.

Suppose that $\psi \circ \phi = 1\!\!1$, for some map $\psi : \mathcal{P}_{\underline{S}} \to \mathcal{P}_{(S,\underline{D})}$. Then clearly $\psi(p_\bullet) = p_\bullet$ and $\psi(f) = f$. The element $\omega := \psi(p_\circ)$ must satisfy $f p_\bullet = \omega f^{\otimes n}$ in $\mathcal{P}_{(S,\underline{D})}$, otherwise ψ will not be well-defined. But it is immediate to see that there is no such ω, thus ϕ has no left inverse.

On the algebra level this means the existence of a $\mathcal{P}$-algebra V, a vector space W and a linear map $f : V \to W$ for which there is no $\mathcal{P}$-algebra structure on W such that f is a homomorphism.

To construct an example, take any $\mathcal{P}$-algebra V with a subspace $I \subset V$ that is *not* an ideal. Put $W := V/I$ (quotient of vector spaces). Then the canonical projection $f : V \to W$ clearly has the above property.

Let us close this section with the following problem.

Problem. *Give a characterization of pairs (S, D) that have the $\mathcal{P}$-extension property. Is there a (finite) list of 'moves' such that (S, D) has the $\mathcal{P}$-extension property if and only if S is obtained from D by consecutively applying moves from the list?*

In all examples with the $\mathcal{P}$-extension property we know, the bigger diagram S is obtained from D by the moves listed above.

3. Resolutions of colored operads and homotopy diagrams.

As we mentioned in the introduction, by a homotopy $\mathcal{P}$-algebra we mean an algebra over a minimal, or at least cofibrant, resolution of the operad $\mathcal{P}$. The same approach translates to the word of colored operads. Let us illustrate this on the following example.

Example 13. Let $Ass_{\bullet\to\bullet}$ be the operad from Example 2 in presentation (6). The minimal model of $Ass_{\bullet\to\bullet}$ is given by

$$Ass_{\bullet\to\bullet} \stackrel{\alpha}{\longleftarrow} \Big(\mathcal{F}\Big(\mu_2,\mu_3,\mu_4,\ldots,f_1,f_2,f_3,\ldots,\nu_2,\nu_3,\nu_4,\ldots\Big),\partial\Big),$$

where $\alpha(\mu_2)=\mu$, $\alpha(\nu_2)=\nu$, $\alpha(f_1)=f$, while α is trivial on the remaining generators. The differential is given by the formulas

$$\begin{aligned}
\partial(\mu_n) &:= \sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{s=0}^{n-j}(-1)^{j+s(j+1)}\mu_i(\mathbb{1}^{\otimes s}\otimes\mu_j\otimes\mathbb{1}^{\otimes i-s-1}),\\
\partial(f_n) &:= \sum_{k=2}^{n}\sum_{r_1+\cdots+r_k=n}(-1)^{\sum_{1\leq i<j\leq k} r_i(r_j+1)}\nu_k(f_{r_1}\otimes\cdots\otimes f_{r_k}) -\\
&\qquad - \sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{s=0}^{n-j}(-1)^{n+s(j+1)}f_i(\mathbb{1}^{\otimes s}\otimes\mu_j\otimes\mathbb{1}^{\otimes i-s-1}),\\
\partial(\nu_n) &:= \sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{s=0}^{n-j}(-1)^{j+s(j+1)}\nu_i(\mathbb{1}^{\otimes s}\otimes\nu_j\otimes\mathbb{1}^{\otimes i-s-1}),
\end{aligned}$$

Comparing this with formulas (1) and (3), we conclude that a strongly homotopy $Ass_{\bullet\to\bullet}$-algebra on a colored chain complex $V\oplus W$ is given by an A_∞-structure on V (generators $\mu_2,\mu_3,\ldots$), an A_∞-structure on W (generators $\nu_2,\nu_3,\ldots$), and a sh morphism of these two structures (generators $f_1,f_2,\ldots$).

This motivates the following definition.

Definition 14. *A* strongly homotopy $(S,\underline{D})$-diagram of $\mathcal{P}$-algebras *is an algebra over a cofibrant resolution of the operad* $\mathcal{P}_{(S,\underline{D})}$.

Intuitively, a strongly homotopy $(S,\underline{D})$-diagram of $\mathcal{P}$-algebras is a homotopy S-diagram of chain complexes such that each complex corresponding to a vertex of the subdiagram D has a structure of a strongly homotopy $\mathcal{P}$-algebra and the maps corresponding to the arrows of the subdiagram D are sh morphisms of these strongly homotopy algebras.

At this stage, it is necessary to give a precise definition of a cofibration.

Definition 15. *By a* cofibration *we mean a map of chain operads of the form* $\iota : (\mathcal{S}, \partial) \hookrightarrow (\mathcal{S} * \mathcal{F}(E), \partial)$ *such that there exists a filtration* $\{F_i E\}_{i \in K}$ *of* E *indexed by a directed set* K, $E = \bigcup_{i \in K} F_i E$, *with the property that*

$$\partial(F_i E) \subset S * \mathcal{F}(E)_{<i}, \text{ for each } i \in K, \tag{12}$$

where $\mathcal{F}(E)_{<i}$ *denotes the suboperad of* $\mathcal{F}(E)$ *generated by* $\bigcup_{j<i} F_j E$.

We say that an operad $\mathcal{S}$ *is* cofibrant *if it is of the form* $(\mathcal{F}(E), \partial)$, *where* E *is filtered as above and the filtration satisfy*

$$\partial(F_i E) \subset \mathcal{F}(E)_{<i}, \text{ for each } i \in K. \tag{13}$$

The definition above is an 'operadic' analog of the Koszul-Sullivan algebra in rational homotopy theory, see [7]. By a slight generalization of results of [9] to colored operads, there exists a closed model category structure on the category of (colored) operads such that fibrations are epimorphisms, weak equivalences are homology equivalences and cofibrations are retractions of cofibrations in the sense of Definition 15. The factorization axiom in the form proved in [9, page 3296] then says that:

Theorem 16. *For an arbitrary map* $f : \mathcal{S} \to \mathcal{Q}$ *of differential operads, there exists a cofibration* $\iota : \mathcal{S} \to \mathcal{S} * \mathcal{F}(E)$ *and a homology isomorphism* $\alpha : \mathcal{S} * \mathcal{F}(E) \to \mathcal{Q}$ *making the diagram*

$$\begin{tikzcd} \mathcal{S} \arrow[rr, "f"] \arrow[dr, "\iota"'] & & \mathcal{Q} \\ & \mathcal{S} * \mathcal{F}(E) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commutative.

Consider a couple (S, D) and a cofibrant resolution $\alpha_{(S,\underline{D})} : \mathcal{R}_{(S,\underline{D})} \to \mathcal{P}_{(S,\underline{D})}$ of the operad $\mathcal{P}_{(S,\underline{D})}$. By Theorem 16 there exist a cofibrant resolution $\alpha_{\underline{S}} : \mathcal{R}_{\underline{S}} \to \mathcal{P}_{\underline{S}}$ and a cofibration $\iota : \mathcal{R}_{(S,\underline{D})} \to \mathcal{R}_{\underline{S}}$ such that the diagram

$$\begin{tikzcd} \mathcal{R}_{(S,\underline{D})} \arrow[r, "\iota"] \arrow[d, "\alpha_{(S,\underline{D})}"'] & \mathcal{R}_{\underline{S}} \arrow[d, "\alpha_{\underline{S}}"] \\ \mathcal{P}_{(S,\underline{D})} \arrow[r, "\phi"] & \mathcal{P}_{\underline{S}} \end{tikzcd}$$

commutes.

Definition 17. *The couple (S, D) has the* homotopy $\mathcal{P}$-extension property *if there exist a left inverse $p : \mathcal{R}_{\underline{D}} \to \mathcal{R}_{(S,\underline{D})}$ of the map ι from the diagram above.*

Exactly as in the proof of Proposition 5, we show that the $\mathcal{P}$-homotopy extension property implies that each strongly homotopy $(S, \underline{D})$-diagram of $\mathcal{P}$-algebras can be extended to a strongly homotopy S-diagram of $\mathcal{P}$-algebras. The main theorem of the paper then reads.

Theorem 18. *The couple (S, D) has the homotopy extension property if and only if it has the (classical) extension property of Definition 4.*

Proof. Suppose the couple (S, D) has the classical extension property, that is, that there exists a right inverse $\psi : \mathcal{P}_{\underline{S}} \to \mathcal{P}_{(S,\underline{D})}$ of the canonical map $\phi : \mathcal{P}_{(S,\underline{D})} \to \mathcal{P}_{\underline{S}}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{R}_{(S,\underline{D})} \arrow[r, "1\!\!1"] \arrow[d, "\iota"'] & \mathcal{R}_{(S,\underline{D})} \arrow[d, "\alpha_{(S,\underline{D})}"] \\
\mathcal{R}_{\underline{S}} \arrow[r, "\psi\alpha_{\underline{S}}"] \arrow[ur, dotted, "p"] & \mathcal{P}_{(S,\underline{D})}
\end{tikzcd}$$

Since ι is a cofibration and $\alpha_{(S,\underline{D})}$ a homology isomorphism, by [24, Lemma 3.5] there exists a map $p : \mathcal{R}_{\underline{S}} \to \mathcal{R}_{(S,\underline{D})}$ such that $p \circ \iota = 1\!\!1$, that is, p is a left inverse of the map ι.

The opposite implication is clear. If $p \circ \iota = 1\!\!1$ for some map p, then

$$H(\alpha_{(S,\underline{D})}) \circ H(p) \circ H(\alpha_{\underline{S}})^{-1} : \mathcal{P}_{\underline{S}} \to \mathcal{P}_{(S,\underline{D})}$$

(the composition of the homology of maps and their inverses) is a left inverse of the canonical map ϕ. ■

4. Back to Earth.

A little care is needed to apply the results of the previous section to concrete situations. The difficulties can be summarized by saying that the homotopy diagram is often not what we think it is, as illustrated by the following example.

Example 19. Consider the diagram S given by the following picture:

(14)
$$\begin{tikzcd} \circ \arrow[r, bend left, "f"] & \circ \arrow[l, bend left, "g"] \end{tikzcd}, \; fg = 1 \text{ and } gf = 1.$$

A 'naïve' homotopy S-diagram consist of two arrows, f and g, and homotopies $h : gf \sim 1\!\!1$ and $l : fg \sim 1$. This could be algebraically expressed as a 'resolution'

$$\mathcal{P}_{(S,\emptyset)} := \frac{\mathcal{F}(f,g)}{(fg=1,\ gf=1)} \xleftarrow{\alpha} (\mathcal{F}(f,g,h,l), \partial),$$

with $\alpha(f) := f$, $\alpha(g) := g$, $\alpha(h) = \alpha(l) = 0$, and the differential given by

$$\partial(f) = \partial(g) := 0,\ \partial(h) := gf - 1 \text{ and } \partial(l) := fg - 1.$$

The trouble is that the map α is not a homology isomorphism. For example, the element $gl - hg \in \mathcal{F}(f,g,h,l)$ is a cocycle, mapped to zero by α, that is not a coboundary. The honest cofibrant model of $\mathcal{P}_{(S,\emptyset)}$ is described follows:

(15)
$$\mathcal{P}_{(S,\emptyset)} \xleftarrow{\alpha} (\mathcal{F}(f,g,h,l,f_2,g_2,f_3,g_3,\ldots), \partial),$$

with the differential given by

$$\begin{array}{lllll} \partial f{:=}0, & \partial h{:=}gf{-}1, & \partial f_2{:=}fh{-}lf, & \partial f_3{:=}gf_2{-}hh{+}g_2f, & \partial f_4{:=}ff_3{-}lf_2{+}f_2h{-}g_3f, \ldots. \\ \partial g{:=}0, & \partial l{:=}fg{-}1, & \partial g_2{:=}gl{-}hg, & \partial g_3{:=}fg_2{-}ll{+}f_2g, & \partial g_4{:=}gg_3{-}hg_2{+}g_2l{-}f_3g, \ldots \end{array}$$

For later reference, we denote $\mathcal{P}_{(S,\emptyset)}$ by $\mathcal{P}_{\mathrm{iso}}$ and resolution (15) by $\alpha_{\mathrm{iso}} : \mathcal{R}_{\mathrm{iso}} \to \mathcal{P}_{\mathrm{iso}}$. Thus, in general, the data f, g, h and l do not lead to an algebra over $\mathcal{R}_{\mathrm{iso}}$, but it is not difficult to prove the following algebraic variant of a theorem by Boardman and Vogt.

Proposition 20. (3+1) *Given the 'naïve' data f, g, h, l as above, by changing one of f, g, h or l, we can make the 'naïve' homotopy diagram to be an honest one, i.e. an algebra over the operad $\mathcal{R}_{\mathrm{iso}}$.*

Side conditions in perturbation theory. Let us recall (see, for example, [11]) that the central object of perturbation theory, *strong deformation retract (SDR)* data, are given by the diagram

$$\begin{tikzcd} M \arrow[r, bend left, "\nabla"] & (A, \phi) \arrow[l, bend left, "f"] \end{tikzcd}$$

of chain complexes and their maps, where $f\nabla = 1\!\!1_M$ and ϕ is a chain homotopy between ∇f and $1\!\!1_A$. One then shows that one may always change the SDR-data in such a way that the following *side conditions* hold:

(16)
$$\phi\phi = 0,\ \phi\nabla = 0 \text{ and } f\phi = 0.$$

The nature and rôle of these side conditions is explained in the following proposition.

Proposition 21. *Suppose we have a 'partial' action* $a : (\mathcal{F}(f,g,h,l), \partial) \to \mathcal{E}nd_{V,W}$ *given by the SDR data as*

$$a(f) := \nabla,\ \ a(g) := f, a(h) := 0 \text{ and } a(l) := \phi$$

(we believe that the collision between f denoting the generator and f denoting the map will not cause problems). Then a can be extended to a full action

$$a : \mathcal{R}_{\rm iso} = (\mathcal{F}(f,g,h,l,f_2,g_2,f_3,g_3,\ldots), \partial) \to \mathcal{E}nd_{V,W}$$

by putting

$$a(f_i) = a(g_i) = 0 \text{ for } i \geq 2 \tag{17}$$

if and only if the side conditions (16) are satisfied.

Proof. The extension a defined by (17) clearly commutes with the differentials if and only if

$$a(\partial f_i) = 0 \text{ and } a(\partial g_i) = 0, \tag{18}$$

for each $i \geq 2$. For $i = 2$ this means that

$$a(\partial f_2) = a(fh - lf) = -\phi\nabla = 0 \text{ and } (\partial g_2) = a(gl - hg) = f\phi = 0,$$

which are the second and the third conditions of (16). For $i = 3$, equation (18) requires

$$a(\partial f_3) = a(gf_2 - h^2 + g_2 f) = 0,$$

which is satisfied automatically, because $a(f_2) = a(h) = a(g_2) = 0$, and

$$a(\partial g_3) = a(fg_2 - l^2 + f_2 g) = \phi\phi = 0,$$

which is the first condition of (16).

To finish the proof, it is enough to observe that (18) is automatically true for an arbitrary $i \geq 4$, because both $\partial(f_i)$ and $\partial(g_i)$ belong, for $i \geq 4$, to the ideal generated by $\{f_j, g_j |\ j \geq 2\}$. ■

The difference between naïve homotopy diagrams and proper ones is measured by the homology of the nerve of the category S. From this point of view, diagram (14) related to moves **(M1)** and **(M3)** was very complicated. Diagrams for other moves are much easier, as illustrated in the following example.

Example 22. The diagram

$$\circ_1 \underset{g}{\overset{f}{\rightrightarrows}} \circ_2,\ f = g,$$

related to move **(M2)** has the model

$$\mathcal{P}_{(S,\underline{\emptyset})} := \frac{\mathcal{F}(f,g)}{(f=g)} \xleftarrow{\alpha_{(S,\underline{\emptyset})}} (\mathcal{F}(f,g,h),\partial), \tag{19}$$

with $\partial(f) = \partial(g) := 0$ and $\partial(h) := f - g$.

Of course, the operad $\mathcal{P}_{(S,\underline{\emptyset})}$ is isomorphic to the free operad $\mathcal{F}(u)$ on one map $u : \circ_1 \to \circ_2$, which is cofibrant and thus it is its own cofibrant model. The (non-minimal) model (19) has two generators f and g for u and their difference killed by a 'homotopy' h. It immediately follows from this observation that the map $\alpha_{(S,\underline{\emptyset})}$ is a homology isomorphism.

We leave to the reader to verify that also the homotopy versions of diagrams related to Salvatore's move **(S)** and move **(M4)** are the naïve ones.

We need also to know that the 'initial data' of the moves induce an action of a cofibrant resolution. Let us analyze the situation.

It is immediate to see that the operad $\mathcal{P}_{(S,\underline{D})}$ of Definition 3 is the push-out of the diagram

$$\mathcal{P}_S \xleftarrow{n_1} \mathcal{P}_D \xrightarrow{n_2} \mathcal{P}_{\underline{D}} \tag{20}$$

where $\mathcal{P}_S := \mathcal{P}_{(S,\underline{\emptyset})}$, $\mathcal{P}_D := \mathcal{P}_{(D,\underline{\emptyset})}$, n_1 is the 'corestriction' map and $n_2 := \phi_{(D,\underline{\emptyset})}$. Let us assume that the operad $\mathcal{P}_D$ is free, $\mathcal{P}_D = \mathcal{F}(X_D)$, and that the maps n_1, n_2 are monomorphisms. These conditions are satisfied for all moves considered in the paper.

Suppose also that $\alpha_S : \mathcal{R}_S \to \mathcal{P}_S$ (resp. $\alpha_{\underline{D}} : \mathcal{R}_{\underline{D}} \to \mathcal{P}_{\underline{D}}$) is a cofibrant model for $\mathcal{P}_S$ (resp. $\mathcal{P}_{\underline{D}}$) of the form $\mathcal{R}_S = (\mathcal{F}(X^1_D, Y_S), d)$ (resp. $R_{\underline{D}} = (\mathcal{F}(X^2_D, Y_{\underline{D}}), d)$), where X^1_D (resp. X^2_D) is an identical copy of X_D, with the identification induced by

$$X^1_D \ni x^1_D \longleftrightarrow x_D \in X_D \text{ if and only if } \alpha_S(x^1_D) = n_1(x_D)$$

(resp.

$$X^2_D \ni x^2_D \longleftrightarrow x_D \in X_D \text{ if and only if } \alpha_{\underline{D}}(x^2_D) = n_2(x_D)).$$

The above assumptions clearly imply that $d(X^i_D) = 0$, $i = 1, 2$.

Lemma 23. *Any two actions $A_S : \mathcal{R}_S \to \mathcal{E}nd_{\mathbf{U}}$ and $A_{\underline{D}} : \mathcal{R}_{\underline{D}} \to \mathcal{E}nd_{\mathbf{U}}$ satisfying*

$$A_S(x^1_D) = A_{\underline{D}}(x^2_D) \text{ for each } x_D \in X_D, \tag{21}$$

extend to an action

$$A_{(S,\underline{D})} : \mathcal{R}_{(S,\underline{D})} \to \mathcal{E}nd_{\mathbf{U}}$$

of a cofibrant resolution $\mathcal{R}_{(S,\underline{D})}$ of the operad $\mathcal{P}_{(S,\underline{D})}$.

Remark. Intuitively, the above lemma means that a strongly homotopy $(S,\underline{\emptyset})$-diagram $A_S : \mathcal{R}_S \to \mathcal{E}nd_{\mathbf{U}}$ and a strongly homotopy $\underline{D}$-diagram $A_{\underline{D}} : \mathcal{R}_{\underline{D}} \to \mathcal{E}nd_{\mathbf{U}}$ that coincide on D induce a strongly homotopy $(S,\underline{D})$-diagram $A_{(S,\underline{D})} : \mathcal{R}_{(S,\underline{D})} \to \mathcal{E}nd_{\mathbf{U}}$ that extends A_S and $A_{\underline{D}}$.

Proof of the lemma. Let us denote by $\mathcal{P}_S *_{\mathcal{F}(X_D)} \mathcal{P}_{\underline{D}}$ the push-out of (20), with the canonical projection $\pi : \mathcal{P}_S * \mathcal{P}_{\underline{D}} \to \mathcal{P}_S *_{\mathcal{F}(X_D)} \mathcal{P}_{\underline{D}}$. Let us consider the diagram

$$\begin{tikzcd}
\mathcal{P}_S * \mathcal{P}_{\underline{D}} \arrow[d, "\pi"'] & \mathcal{R}_S * \mathcal{R}_{\underline{D}} \arrow[l, "\alpha_S * \alpha_{\underline{D}}"'] \arrow[r, "A_S * A_{\underline{D}}"] \arrow[d, dotted, "\iota"] & \mathcal{E}nd_{\mathbf{U}} \\
\mathcal{P}_S *_{\mathcal{F}(X_D)} \mathcal{P}_{\underline{D}} & \mathcal{R}_{(S,\underline{D})} \arrow[l, dotted, "\alpha_{(S,\underline{D})}"'] \arrow[ur, dotted, "A_{(S,\underline{D})}"'] &
\end{tikzcd}$$

in which $\alpha_S * \alpha_{\underline{D}}$ is a homology isomorphism, by the Künneth theorem. We will construct a cofibrant resolution $\alpha_{(S,\underline{D})} : \mathcal{R}_{(S,\underline{D})} \to \mathcal{P}_S *_{\mathcal{F}(X_D)} \mathcal{P}_{\underline{D}}$ by attaching free generators to $\mathcal{R}_S * \mathcal{R}_{\underline{D}}$, together with a cofibration $\iota : \mathcal{R}_S * \mathcal{R}_{\underline{D}} \hookrightarrow \mathcal{R}_{(S,\underline{D})}$ and an action $A_{(S,\underline{D})} : \mathcal{R}_{(S,\underline{D})} \to \mathcal{E}nd_{\mathbf{U}}$ extending $A_S * A_{\underline{S}}$ in the sense that $A_{(S,\underline{D})}(\iota) = A_S * A_{\underline{S}}$.

The resolution $\mathcal{R}_{(S,\underline{D})}$ is constructed in a standard manner by killing cycles in $\mathcal{R}_S * \mathcal{R}_{\underline{D}} = (\mathcal{F}(X_D^1, X_D^2, Y_S, Y_{\underline{D}}), d)$ that represent the homology of the kernel of $\pi(\alpha_S * \alpha_{\underline{D}})$ (see, for example, the arguments in [9]). The extension $A_{(S,\underline{D})}$ is then given by elementary obstruction theory.

At the first stage we add generators $\overline{X}_D := \uparrow X_D$ (the suspension of X_D) and define the differential by

$$d\overline{x}_D := x_D^1 - x_D^2,\ \ x_D \in X_D.$$

The maps $\alpha_{(S,\underline{D})}$ and $A_{(S,\underline{D})}$ are extended by

$$\alpha_{(S,\underline{D})}(\overline{x}_D) := 0 \text{ and } A_{(S,\underline{D})}(\overline{x}_D) := 0, \text{ for } x_D \in X_D. \tag{22}$$

Let us verify that these extensions commute with the differential. For $\alpha_{(S,\underline{D})}$ this means that $0 = \alpha_{(S,\underline{D})}(d\overline{x}_D)$, for $x_D \in X_D$. This follows from

$$\alpha_{(S,\underline{D})}(d\overline{x}_D) = \alpha_{(S,\underline{D})}(x_D^1 - x_D^2) = \alpha_{(S,\underline{D})}(x_D^1) - \alpha_{(S,\underline{D})}(x_D^2) = \pi(\alpha_S(x_S^1)) - \pi(\alpha_{\underline{D}}(x_D^2))$$

since clearly $\pi(\alpha_S(x_S^1)) = \pi(\alpha_{\underline{D}}(x_D^2))$. Similarly,

$$A_{(S,\underline{D})}(d\overline{x}_D) = A_{(S,\underline{D})}(x_D^1 - x_D^2) = A_{(S,\underline{D})}(x_D^1) - A_{(S,\underline{D})}(x_D^2) = 0$$

by (21), thus $A_{(S,\underline{D})}$ commutes with the differential as well.

Since $\alpha_S * \alpha_{\underline{D}}$ is a homology isomorphism and $\mathrm{Ker}(\pi)$ is the ideal generated in $\mathcal{P}_S * \mathcal{P}_{\underline{D}}$ by $\{n_1(x_D) - n_2(x_D);\ x_D \in X_D\}$, each cycle in the kernel of $\alpha_{(S,\underline{D})} : \mathcal{F}(X_D^1, X_D^2, Y_S, Y_{\underline{D}}) \to \mathcal{P}_S *_{\mathcal{F}(X_D)} \mathcal{P}_{\underline{D}}$ is homologous to a cycle in the ideal $(\overline{X}_D)$ generated by $\overline{X}_D$. Thus the 'higher' generators z killing the cycles in the kernel of $\alpha_{(S,\underline{D})}$ can be added in such a way that $dz \in (\overline{X}_D)$. The extension

$$A_{(S,\underline{D})}(z) := 0$$

then commutes with the differentials, because, by (22), $A_{(S,\underline{D})}|_{(\overline{X}_D)} = 0$. ■

Let us prove move **(M1)**:

Proposition 24. (M1) *For each strongly homotopy $\mathcal{P}$-algebra $\mathbf{V}$, chain complex $W = (W, \partial)$ and a chain homotopy equivalence $f : (V, \partial) \to (W, \partial)$, there exist a strongly homotopy $\mathcal{P}$-structure $\mathbf{W}$ on (W, ∂) and a strongly homotopy $\mathcal{P}$-structure $\mathbf{f} : \mathbf{V} \to \mathbf{W}$ on the map f.*

Proof. The initial data are given by an action $A_{\bullet \to \circ} : \mathcal{R}_\bullet * \mathcal{F}(f) \to \mathcal{E}nd_{V,W}$ representing a strongly homotopy $\mathcal{P}$-algebra structure on (V, ∂) and a chain map $f : (V, \partial) \to (W, \partial)$. Since f is a chain homotopy equivalence we can, by Proposition 20, extend $A_{\bullet \to \circ}$, properly choosing a homotopy inverse g and homotopies h and l, to a map $A_{(S,\underline{D})} : \mathcal{R}_\bullet * \mathcal{R}_{\mathrm{iso}} \to \mathcal{E}nd_{V,W}$.

Very crucially, $\mathcal{R}_\bullet * \mathcal{R}_{\mathrm{iso}}$ is a cofibrant resolution of the operad $\mathcal{P}_{(S,\underline{D})}$. Indeed, the operad $\mathcal{P}_{(S,\underline{D})}$ of (9) is naturally isomorphic to the free product $\mathcal{P}_\bullet * \mathcal{P}_{\mathrm{iso}}$, therefore the induced map

$$\alpha_\bullet * \alpha_{\mathrm{iso}} : \mathcal{R}_\bullet * \mathcal{R}_{\mathrm{iso}} \to \mathcal{P}_\bullet * \mathcal{P}_{\mathrm{iso}} \cong \mathcal{P}_{(S,\underline{D})} \tag{23}$$

is, by the Künneth theorem, a homology isomorphism. Of course, the above situation is that of Lemma 23 with $X_D = 0$.

By Theorem 16, there exists a cofibration $\iota_{(S,\underline{D})} : \mathcal{R}_\bullet * \mathcal{R}_{\mathrm{iso}} \to \mathcal{R}_{\underline{S}}$ and a homology isomorphism $\alpha_{\underline{S}} : \mathcal{R}_{\underline{S}} \to \mathcal{P}_{\underline{S}}$ making the diagram

$$\begin{array}{ccc} \mathcal{R}_\bullet * \mathcal{R}_{\mathrm{iso}} & \xrightarrow{\iota_{(S,\underline{D})}} & \mathcal{R}_{\underline{S}} \\ {\scriptstyle \alpha_\bullet * \alpha_{\mathrm{iso}}} \downarrow & & \downarrow {\scriptstyle \alpha_{\underline{S}}} \\ \mathcal{P}_\bullet * \mathcal{P}_{\mathrm{iso}} & \xrightarrow{\phi_{(S,\underline{D})}} & \mathcal{P}_{\underline{S}} \end{array} \tag{24}$$

commutative. Since we know, from Example 7, that the couple (D, S) has the classical $\mathcal{P}$-extension property, Theorem 18 gives an action $A_{\underline{S}} : \mathcal{R}_{\underline{S}} \to \mathcal{E}nd_{V,W}$.

We must prove the existence of strongly homotopy $\mathcal{P}$-structures on the complexes (V, ∂) and (W, ∂) and a sh structure on the map f, i.e. of an action $A_{\bullet\to\bullet} : \mathcal{R}_{\bullet\to\bullet} \to \mathcal{E}nd_{V,W}$, where $\mathcal{R}_{\bullet\to\bullet}$ is a cofibrant resolution of the operad $\mathcal{P}_{\bullet\to\bullet}$ of (7). We also require that $A_{\bullet\to\bullet}$ extends the given data, i.e. that there is a cofibration $\iota_{\bullet\to\circ} : \mathcal{P}_\bullet * \mathcal{F}(f) \to \mathcal{R}_{\bullet\to\bullet}$ such that the diagram

$$\begin{tikzcd}
\mathcal{R}_\bullet * \mathcal{F}(f) \arrow[r, "\iota_{\bullet\to\circ}"] \arrow[d, "\alpha_{\bullet\to\circ} := \alpha_\bullet * \mathbb{1}"'] & \mathcal{R}_{\bullet\to\bullet} \arrow[d, "\alpha_{\bullet\to\bullet}"] \\
\mathcal{P}_\bullet * \mathcal{F}(f) \arrow[r, "\phi_{\bullet\to\circ}"] & \mathcal{P}_{\bullet\to\bullet}
\end{tikzcd} \tag{25}$$

commutes and that $A_{\bullet\to\bullet}(\iota_{\bullet\to\circ}) = A_{\bullet\to\circ}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{R}_\bullet * \mathcal{F}(f) \arrow[rr, "\mathbb{1} * \tilde{\jmath}"] \arrow[dr, "\alpha_\bullet * \mathbb{1}"] \arrow[dd, "\iota_{\bullet\to\circ}"'] & & \mathcal{R}_\bullet * \mathcal{R}_{\mathrm{iso}} \arrow[rr, "\iota_{(S,\underline{D})}"] \arrow[dr, "\alpha_\bullet * \alpha_{\mathrm{iso}}"] \arrow[d, phantom, "\boxed{3}"] & & \mathcal{R}_{\underline{S}} \arrow[dd, "\alpha_{\underline{S}}"] \arrow[dl, phantom, "\boxed{2}"] \\
 & \mathcal{P}_\bullet * \mathcal{F}(f) \arrow[rr, "\mathbb{1} * j"] \arrow[dr, "\phi_{\bullet\to\circ}"] \arrow[dl, phantom, "\boxed{1}"] & & \mathcal{P}_\bullet * \mathcal{P}_{\mathrm{iso}} \arrow[dr, "\phi_{(S,\underline{D})}"] \arrow[dl, phantom, "\boxed{4}"] & \\
\mathcal{R}_{\bullet\to\bullet} \arrow[rr, "\alpha_{\bullet\to\bullet}"] & & \mathcal{P}_{\bullet\to\bullet} \arrow[rr, "r"] & & \mathcal{P}_{\underline{S}}
\end{tikzcd}$$

where $j : \mathcal{F}(f) \to \mathcal{P}_{\mathrm{iso}}$ maps f to the corresponding element of $\mathcal{P}_{\mathrm{iso}}$, $\tilde{\jmath} : \mathcal{F}(f) \to \mathcal{R}_{\mathrm{iso}}$ is a lift of j, i.e. a map that satisfies $\alpha_{\mathrm{iso}}(\tilde{\jmath}) = j$ (such a map clearly exists) and $r : \mathcal{P}_{\bullet\to\bullet} \to \mathcal{P}_{\underline{S}}$ is the corestriction map.

The subdiagram $\boxed{1}$ (resp. $\boxed{2}$) commutes because it is (25) (resp. (24)). The commutativity of the remaining subdiagrams $\boxed{3}$ and $\boxed{4}$ is obvious. This implies the commutativity of the 'outer diagram'

$$\begin{tikzcd}
\mathcal{R}_\bullet * \mathcal{F}(f) \arrow[r, "\iota_{(S,\underline{D})}(\mathbb{1} * \tilde{\jmath})"] \arrow[d, "\iota_{\bullet\to\circ}"'] & \mathcal{R}_{\underline{S}} \arrow[d, "\alpha_{\underline{S}}"] \\
\mathcal{R}_{\bullet\to\bullet} \arrow[r, "r(\alpha_{\bullet\to\bullet})"] & \mathcal{P}_{\underline{S}}
\end{tikzcd}$$

By [24, Lemma 3.5] there exists a map $\beta : \mathcal{R}_{\bullet\to\bullet} \to \mathcal{R}_{\underline{S}}$ such that $\beta(\iota_{\bullet\to\circ}) = \iota_{(S,\underline{D})}(\mathbb{1} * \tilde{\jmath})$. The action $A_{\bullet\to\bullet} := A_{\underline{S}}(\beta) : \mathcal{R}_{\bullet\to\bullet} \to \mathcal{E}nd_{V,W}$ then has the required property. ∎

Remark. In the proof of Proposition 24 we encountered two kinds of difficulties. First, we needed to show that the 'initial data' really do induce an action of a cofibrant resolution $\mathcal{R}_{(S,\underline{D})}$. To this end we invoked a Künneth-type argument to prove that (23) is a homology isomorphism. In more general situations, this difficulty is taken care of by Lemma 23.

The second subtlety was to verify that the action $A_{\underline{S}} : \mathcal{R}_{\underline{S}} \to \mathcal{E}nd_{V,W}$ constructed from Theorem 18 contains the data we were looking for. This we deduced from a general argument which, though seemingly very elaborate, involved nothing except general nonsense reasoning.

Let us prove **(M3)**.

Proposition 25. (M3) *Suppose* $\mathbf{f} : \mathbf{V} \to \mathbf{W}$ *is an sh* $\mathcal{P}$*-algebra map such that* $\mathbf{f}_\#$ *is a chain homotopy equivalence. Suppose that* $g : (W, \partial) \to (V, \partial)$ *is a chain homotopy inverse of* $\mathbf{f}_\#$. *Then there exists a strongly homotopy* $\mathcal{P}$*-structure on the map* g.

Proof. The 'initial data' consist of an action $A_{\bullet\to\bullet} : \mathcal{R}_{\bullet\to\bullet} \to \mathcal{E}nd_{V,W}$ representing $\mathbf{f} : \mathbf{V} \to \mathbf{W}$, and an action $A_{\mathrm{iso}} : \mathcal{R}_{\mathrm{iso}} \to \mathcal{E}nd_{V,W}$, induced by $f := \mathbf{f}_\#$ and its homotopy inverse g.

The assumptions of Lemma 23 are clearly satisfied, with S the diagram

$$\begin{tikzcd} \circ \arrow[r, bend left, "f"] & \circ \arrow[l, bend left, "g"] \end{tikzcd}, \; fg = 1 \text{ and } gf = 1,$$

$D := \circ \longrightarrow \circ$, and $X_D := \mathrm{Span}(f)$, therefore the initial data induce an action $A_{(S,\underline{D})} : \mathcal{R}_{(S,\underline{D})} \to \mathcal{E}nd_{V,W}$ of a cofibrant resolution $\mathcal{R}_{(S,\underline{D})}$ of the operad $\mathcal{P}_{(S,\underline{D})}$.

As we showed in Example 9, the couple (S, D) has the classical extension property and Theorem 18 guarantees the existence of an action $A_{\underline{S}} : \mathcal{R}_{\underline{S}} \to \mathcal{P}_{\underline{S}}$ of a cofibrant resolution of the operad $\mathcal{P}_{\underline{S}}$. The rest follows, as in the proof of Proposition 24, from a general nonsense argument. ■

Remark 26. In exactly the same manner, one can prove moves **(M2)**, **(M4)**, and also Salvatore's move **(S)**. The situation is even easier than above, because the homotopy diagrams related to these moves are the naïve ones and also Lemma 23 applies in a straightforward manner.

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