Uniqueness of invariant measures and maximal dissipativity of diffusion operators on L^1

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Abstract

It is proved that there exists at most one probability measure μ on $\mathbb{R}^d$, so that $L^*\mu = 0$, where $L = a^{ij}\partial_i\partial_j + b^i\partial_i$, provided $\big(L, C_0^\infty(\mathbb{R})\big)$ is maximally dissipative on $L^1(\mathbb{R}^d, \nu)$ for at least one ν, so that $L^*\nu = 0$. Here it is assumed that (a^{ij}) is non-degenerate, $a^{ij} \in H^{p,1}_{\text{loc}}$, and $b^i \in L^p_{\text{loc}}$. We also present a whole class of examples (even for $a^{ij} = \delta^{ij}$), where $L^*\mu = 0$ has more than one solution. Furthermore, recent related results are reviewed.

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Contents

1 Introduction and framework

Let Ω be a connected open set in $\mathbb{R}^d$, $d \geq 2$ (see the appendix for the case $d = 1$), $A = (a^{ij})$ a Borel–measurable mapping on Ω with values in the non-negative matrices on $\mathbb{R}^d$, and let $b = (b^i) : \Omega \longrightarrow \mathbb{R}^d$ be a Borel–measurable vector field. Let us set

$$L_{A,b}\varphi = a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi, \quad \varphi \in C_0^\infty(\Omega), \tag{1.1}$$

where we use the standard summation rule for repeated indices. Suppose that μ is a locally finite (not necessarily non-negative) Borel measure on Ω, i.e. a measure on the Borel σ–algebra $\mathcal{B}(\Omega)$ of Ω, such that

$$L^*_{A,b}\mu = 0 \tag{1.2}$$

in the following sense: $a^{ij}, b^i \in L^1_{\text{loc}}(\mu)$ and

$$\int_\Omega L_{A,b}\varphi \, d\mu = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{1.3}$$

Measures μ satisfying (1.2) are called *infinitesimally invariant* or simply *invariant* if there is no confusion possible (cf. Subsect. 2.5 below for the motivation of this terminology).

Define

$$\mathcal{M}_{\text{ell}}^{A,b} := \big\{\mu | \mu \text{ probability measure on } \Omega \text{ satisfying (1.2)}\big\}. \tag{1.4}$$

The purpose of this paper is to survey and, in particular, to complete recent results (cf. [ABR99, Sta99]) on the question whether or not $\mathcal{M}_{\text{ell}}^{A,b}$ contains at most one element in the case $\Omega = \mathbb{R}^d$.

It turns out that this is related to the question whether the operator $\big(L_{A,b}, C_0^\infty(\mathbb{R}^d)\big)$ is maximally dissipative on $L^1(\mathbb{R}^d, \mu)$ for $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$, and whether μ is invariant for the C_0–semigroup generated by the closure of $\big(L_{A,b}, C_0^\infty(\mathbb{R}^d)\big)$ on $L^1(\mathbb{R}^d, \mu)$. In fact, we shall prove that if there exists one $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$ so that $\big(L_{A,b}, C_0^\infty(\mathbb{R}^d)\big)$ is maximally dissipative on $L^1(\mathbb{R}^d, \mu)$, then $\#\mathcal{M}_{\text{ell}}^{A,b} = 1$ (cf. Theorem 3.1 below).

We also give a whole class of examples for which $\#\mathcal{M}_{\text{ell}}^{A,b} > 1$ (cf. Proposition 3.2). Hence $\big(L_{A,b}, C_0^\infty(\mathbb{R}^d)\big)$ is not maximally dissipative on $L^1(\mathbb{R}^d, \mu)$ for all $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$ in these cases. Both results have important consequences for the relation between infinitesimal invariance as defined above and invariance w.r.t. a semigroup (cf. Subsect. 2.5 below), which we will discuss in detail.

The organization of this paper is as follows:

In Section 2 we survey known results related to the above problem or needed subsequently. We also prove some generalizations.

In Section 3 we state the main results and discuss their consequences.

Finally, we give proofs for the main results in Section 4.

In the appendix we give an elementary approach to the case when $d = 1$, where (1.2) can be solved explicitly and there always is at most one solution, which is a probability measure.

Let us introduce some notation. If μ is a (not necessarily non-negative) locally finite Borel measure on Ω, we denote by $L^1_{\mathrm{loc}}(\Omega, \mu)$ the class of all funtions f such that $\chi f \in L^1(\Omega, \mu)$ for every $\chi \in C_0^\infty(\Omega)$. Here $L^p(\Omega, \mu) = L^p(\Omega, |\mu|)$, where $|\mu|$ is the variation of μ. The Lebesgue measure is denoted by dx, and as usual $L^p_{\mathrm{loc}}(\Omega) := L^p_{\mathrm{loc}}(\Omega, dx)$. The same notation is used for spaces of vector-valued mappings on Ω. We also occasionally identify μ with its Radon–Nikodym density w.r.t. the Lebesgue measure if this density exists.

Let $C_0^\infty(\Omega)$ be the class of all infinitely differentiable functions with compact support in Ω. We set $\partial_i \varphi = \frac{\partial \varphi}{\partial x_i}$.

Let $H^{p,r}(\mathbb{R}^d)$, $p \geq 1$, $r \geq 0$, be the standard Sobolev space of functions on $\mathbb{R}^d$ whose derivatives up to order r are in $L^p(\mathbb{R}^d)$, equipped with its natural norm, defined as in e.g. [Ada75]. By $H^{p,r}_{\mathrm{loc}}(\Omega)$ we denote the class of all functions f on Ω such that $\chi f \in H^{p,r}(\mathbb{R}^d)$ for every $\chi \in C_0^\infty(\Omega)$.

All measures on Ω considered in this paper are defined on the Borel σ–algebra $\mathcal{B}(\Omega)$ of Ω.

2 Survey of known results and some generalizations

2.1 Regularity

The following result on the regularity of measures satisfying (1.2) has been proved recently in [BKR], generalizing earlier results in [BKR97].

Theorem 2.1. *Let μ be a locally finite (not necessarily non-negative)* Borel *measure satisfying* (1.2). *Assume that for $p > d$*

(A1) $a^{ij} \in H^{p,1}_{loc}(\Omega)$, (a^{ij}) *non-degenerate in* Ω,

(A2) $b^i \in L^p_{loc}(\Omega, dx)$.

Then $\mu \ll dx$ with $\frac{d\mu}{dx} \in H^{p,1}_{loc}$ $\big(\subset C^{1-d/p}(\Omega)\big)$. If ρ denotes the continuous version of $\frac{d\mu}{dx}$, then for all compact $K \subset \Omega$ there exists $c_K \in]0, \infty[$ such that

$$\sup_K \rho \leq c_K \inf_K \rho. \tag{2.1}$$

In particular, either $\rho \equiv 0$ or $\rho(x) > 0$ $\forall x \in \Omega$.

Proof. See [BKR, Corollaries 2.8 and 2.9]. □

2.2 Existence

Let $\Omega = \mathbb{R}^d$. Then obviously $\mathcal{M}^{A,b}_{\mathrm{ell}} = \emptyset$, if $A = \mathrm{Id}$, $b = 0$. The question which (minimal) conditions imply $\mathcal{M}^{A,b}_{\mathrm{ell}} \neq \emptyset$ has been studied in [BR95, Sect. 5] and for the finite dimensional case in particular in [BR]. The latter contains the following, so far most general result on $\mathbb{R}^d$.

Theorem 2.2. *Suppose $\Omega = \mathbb{R}^d$ and assume that conditions* (A1) *and* (A2) *of Theorem 2.1 hold. Assume that there exists $V \in C^2(\mathbb{R}^d)$ ("*Lyapunov *function") such that*

$$\lim_{|x|\to\infty} V(x) = +\infty \quad \text{and} \quad \lim_{|x|\to\infty} LV(x) = -\infty. \tag{2.2}$$

Then $\mathcal{M}_{\mathrm{ell}}^{A,b}$ (as defined in (1.4)*) is non-empty.*

Proof. See [BR, Theorem 1.6]. □

Lemma 2.3. *If, in the situation of Theorem 2.2, the functions a^{ij} are locally* Lipschitz *and globally bounded and if, in addition,*

$$\lim_{|x|\to\infty} b^i(x)x_i = -\infty, \tag{2.3}$$

then $V(x) := |x|^2$, $x \in \mathbb{R}^d$, fulfills (2.2)*. In particular, $\mathcal{M}_{\mathrm{ell}}^{A,b} \neq \emptyset$.*

Proof. See [BR, Corollary 1.7]. □

2.3 Preliminaries on maximal dissipativity

Throughout this section we assume that conditions (A1) and (A2) of Theorem 2.1 hold. Fix $\mu \in \mathcal{M}_{\mathrm{ell}}^{A,b}$. Since by Theorem 2.1, μ is equivalent to Lebesgue measure, and therefore is strictly positive on all non-empty open subsets of Ω, $C_0^\infty(\Omega)$ can be identified with a subset of $L^1(\Omega,\mu)$, since each corresponding μ–class has a unique continuous μ–version. Hence the operator $\big(L_{A,b}, C_0^\infty(\Omega)\big)$ is well-defined on $L^1(\Omega,\mu)$.

Remark 2.4. For μ as above we define $\beta_\mu := (\beta_\mu^i)$ by

$$\beta_\mu^i := \partial_j a^{ij} + a^{ij}\frac{\partial_j \rho}{\rho}, \tag{2.4}$$

where ρ is the continuous version of $\frac{d\mu}{dx}$. Then $\beta_\mu^i \in L^p_{\mathrm{loc}}(\Omega)$ for $p > d$. Then on $C_0^\infty(\Omega)$

$$L_{A,b} = L_{A,\beta_\mu} + (b^i - \beta_\mu^i)\partial_i, \tag{2.5}$$

L_{A,β_μ} is μ–symmetric, i.e.

$$\int L_{A,\beta^\mu}\varphi\,\psi\,d\mu = \int \varphi\, L_{A,\beta_\mu}\psi\,d\mu \quad \forall \varphi,\psi \in C_0^\infty(\Omega), \tag{2.6}$$

and

$$\int (b^i - \beta_\mu^i)\partial_i\varphi\,d\mu = 0 \quad \forall \varphi \in C_0^\infty(\Omega) \tag{2.7}$$

or shortly

$$\operatorname{div}_\mu(b-\beta_\mu)=0. \tag{2.8}$$

Then, clearly,

$$L_{A,2\beta_\mu-b}=L_{A,\beta_\mu}-(b^i-\beta^i_\mu)\partial_i \qquad \text{on } C_0^\infty(\Omega) \tag{2.9}$$

and obviously $L_{A,2\beta_\mu-b}$ is "formally adjoint"[1] to $L_{A,b}$, i.e.

$$\int L_{A,b}\varphi\,\psi\,d\mu=\int \varphi\,L_{A,2\beta_\mu-b}\psi\,d\mu \quad \forall\varphi,\psi\in C_0^\infty(\Omega). \tag{2.10}$$

We note that by (2.8)

$$\mu\in\mathcal{M}_{\text{ell}}^{A,\beta_\mu}\cap\mathcal{M}_{\text{ell}}^{A,2\beta_\mu-b}, \tag{2.11}$$

and that (A2) holds for $2\beta^\mu-b$. We recall that by [BR95, Theorem 6.2] $\mathcal{M}_{\text{ell}}^{A,\beta_\mu}=\{\mu\}$.

Lemma 2.5. *$\left(L_{A,b},C_0^\infty(\Omega)\right)$ is dissipative on $L^1(\Omega,\mu)$ and therefore, in particular, closable.*

Proof. This is completely standard, since μ satisfies (1.2) (cf. e.g. [Ebe97, Lemma 1.8], resp. [RS75, Sect. X.8] for the last statement). □

As a consequence, we have for the closure $\left(\bar{L}^\mu_{A,b},D(\bar{L}^\mu_{A,b})\right)$ of $\left(L_{A,b},C_0^\infty(\Omega)\right)$ on $L^1(\Omega,\mu)$ the following result (cf. [Ebe97, Appendix A]).

Proposition 2.6. *The following assertions are equivalent:*

1. *$\left(\bar{L}^\mu_{A,b},D(\bar{L}^\mu_{A,b})\right)$ generates a* C_0–semigroup $(T_t)_{t\geq 0}$ *(i.e. a strongly continuous semigroup of bounded operators T_t, $t\geq 0$) on $L^1(\Omega,\mu)$.*
2. *For one (hence all) $\lambda\in]0,\infty[$ the set $\left(\lambda-L_{A,b}\right)\left(C_0^\infty(\Omega)\right)$ is dense in $L^1(\Omega,\mu)$, i.e. $\left(L_{A,b},C_0^\infty(\Omega)\right)$ is* maximally dissipative *in $L^1(\Omega,\mu)$.*
3. *$(T_t)_{t\geq 0}$ is the only C_0–semigroup on $L^1(\Omega,\mu)$ which has a generator extending $\left(L_{A,b},C_0^\infty(\Omega)\right)$.*

In case any (hence all) of the assertions (1)–(3) hold, $(T_t)_{t\geq 0}$ is a contraction *semigroup (i.e. each T_t has norm less than one) and is* sub–Markovian *(i.e. $f\in L^1(\Omega,\mu)$, $0\leq f\leq 1$ implies $0\leq T_tf\leq 1$ for all $t\geq 0$).*

Proof. The equivalence of (1) and (2) is a consequence of Lemma 2.5 and the well-known Lumer–Phillips Theorem (cf. e.g. [Paz85, chap. I, Theorem 4.3]). The implication "(1)⇒(3)" is trivial, and "(3)⇒(1)" is due to W. Arendt [Are86, A–II, Theorem 1.33].

For the last part, we note that $(T_t)_{t\geq 0}$ must consist of contractions by the dissipativity of $\left(L_{A,b},C_0^\infty(\Omega)\right)$, and the sub–Markov property was proved in [Ebe97, Lemma 1.9]. □

[1] Note that we are not working on the dual $L^\infty(\mathbb{R}^d,\mu)$ of $L^1(\mathbb{R}^d,\mu)$.

Remark 2.7. For bounded Ω (with smooth boundary), assertion (3) in Proposition 2.6 does not hold, even if $A = \mathrm{Id}$, $b \equiv 0$. So below, we shall mainly consider the case $\Omega = \mathbb{R}^d$.

For $\Omega = \mathbb{R}^d$ we introduce the following subset of $\mathcal{M}_{\text{ell}}^{A,b}$:

$$\mathcal{M}_{\text{ell,md}}^{A,b} := \left\{ \mu \in \mathcal{M}_{\text{ell}}^{A,b} \middle| \left(L_{A,b}, C_0^\infty(\mathbb{R}^d)\right) \text{ is maximally dissipative on } L^1(\mathbb{R}^d,\mu) \right\}. \tag{2.12}$$

2.4 Analytic characterization of maximal dissipativity

Let $\Omega = \mathbb{R}^d$. We recall the following analytic characterization of maximal dissipativity of $\left(L_{A,b}, C_0^\infty(\mathbb{R}^d)\right)$ on $L^1(\mathbb{R}^d,\mu)$ for a given $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$, proved in [Sta99]. In particular, we shall see that $\left(L_{A,b}, C_0^\infty(\mathbb{R}^d)\right)$ is maximally dissipative on $L^1(\mathbb{R}^d,\mu)$ if and only if its "formally adjoint" $\left(L_{A,2\beta_\mu - b}, C_0^\infty(\mathbb{R}^d)\right)$ (cf. Remark 2.4) is so.

For $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$ we define $H^{2,1}(\mathbb{R}^d,\mu)$ analogously to $H^{2,1}(\mathbb{R}^d, dx)$, i.e. as the set of all $f \in L^2(\mathbb{R}^d,\mu)$ so that there exist $\varphi_n \in C_0^\infty(\mathbb{R}^d)$ with the property that $\varphi_n \longrightarrow f$ in $L^2(\mathbb{R}^d,\mu)$ as $n \to \infty$ and $(\partial_i \varphi_n)_{n\in\mathbb{N}}$ is a Cauchy–sequence in $L^2(\mathbb{R}^d,\mu)$ for all $1 \le i \le d$. We then set

$$\partial_i^\mu f := \lim_{n\to\infty} \partial_i \varphi_n \quad \text{in } L^2(\mathbb{R}^d,\mu),\ 1 \le i \le d. \tag{2.13}$$

Correspondingly, $H_{\text{loc}}^{2,1}(\mathbb{R}^d,\mu)$ denotes the set of all $f \in L^2(\mathbb{R}^d,\mu)$ such that $\chi f \in H^{2,1}(\mathbb{R}^d,\mu)$ for all $\chi \in C_0^\infty(\mathbb{R}^d)$. Then $\partial_i^\mu f$, $1 \le i \le d$, is defined for all $f \in H_{\text{loc}}^{2,1}(\mathbb{R}^d,\mu)$.

Theorem 2.8. *Let $\Omega = \mathbb{R}^d$ and assume that conditions* (A1) *and* (A2) *of Theorem 2.1 hold. Let $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$. Then the following assertions are equivalent:*

1. $\mu \in \mathcal{M}_{\text{ell,md}}^{A,b}$.
2. *There exist $\chi_n \in H_{loc}^{2,1}(\mathbb{R}^d,\mu)$ and $\alpha \in]0,\infty[$ such that $(1-\chi_n)^+ \in L^\infty(\mathbb{R}^d,\mu)$, $\{(1-\chi_n)^+ > 0\}$ is bounded, $\lim_{n\to\infty} \chi_n = 0$ μ–a.e. and for $\eta = 1$ or -1*

$$\int a^{ij}\, \partial_i^\mu \chi_n\, \partial_j \varphi\, d\mu + \alpha \int \chi_n \varphi\, d\mu + \eta \int \left(b^i - \beta_\mu^i\right) \partial_i^\mu \chi_n\, \varphi\, d\mu \ge 0 \tag{2.14}$$

   for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi \ge 0$, and all $n \in \mathbb{N}$.
3. $\mu \in \mathcal{M}_{\text{ell,md}}^{A,2\beta_\mu - b}$.

Proof. For "(1) $\iff$ (2)" see [Sta99, Prop. 1.9 and Cor. 2.2]. "(1) $\iff$ (3)" follows by the same results and [Sta99, Remark 1.11]. $\square$

There are explicit sufficient conditions implying that the equivalent statements in Theorem 2.8 hold, which are easy to check in applications:

Proposition 2.9. *In the situation of Theorem 2.8, sufficient conditions for the equivalent assertions (1)–(3) to hold are each of the following:*

1. $a^{ij}, b^i - \beta_i^\mu \in L^1(\mathbb{R}^d, \mu)$ *for all* $1 \leq i, j < d$.
2. *There exist* $u \in C^2(\mathbb{R}^d)$ *and* $\alpha \in]0, \infty[$ *such that* $\lim_{|x|\to\infty} u(x) = +\infty$ *and*
$$a^{ij}\partial_i\partial_j u + b^i\partial_i u - \alpha u \leq 0. \tag{2.15}$$
3. *There exists* $V \in C^2(\mathbb{R}^d)$ *("*Lyapunov *function") such that* (2.2) *holds.*
4. *There exists* $M \in]0, \infty[$ *such that*
$$\begin{aligned} &-2\big(|x|^2+1\big)^{-1} a^{ij}(x)x_i x_j + a_{ii}(x) + b^i(x)x_i \\ &\qquad \leq M\big(|x|^2 \ln\big(|x|^2+1\big)+1\big) \quad \textit{for all } x \in \mathbb{R}^d. \end{aligned} \tag{2.16}$$

Proof. (1), (2) and (4) have been proved in [Sta99, Prop. 1.10 and Cor. 2.2]. For (3), we note that by adding a constant we may assume that $V \geq 1$. Let $R > 0$ such that $LV(x) \leq V(x)$ for all $x \in \mathbb{R}^d \setminus B_R$, where $B_R := \big\{x \in \mathbb{R}^d \,\big|\, |x| < R\big\}$. Let V_0 denote the solution to the Dirichlet problem

$$(L-1)h = 0 \text{ on } B_R, \quad h = V \text{ on } \partial B_R, \tag{2.17}$$

and define

$$u := \begin{cases} V_0 & \text{on } B_R, \\ V & \text{on } \mathbb{R}^d \setminus B_R. \end{cases} \tag{2.18}$$

Then u is continuous, superharmonic in the weak sense on all of $\mathbb{R}^d$, strictly positive, and $\lim_{|x|\to\infty} u(x) = +\infty$. Letting $\chi_n := \frac{u}{n}$, assertion (2) in Theorem 2.8 is easily verified. □

2.5 Infinitesimal invariance and semigroup invariance

As announced in the introduction and proved below (cf. Proposition 3.2) it can happen that $\#\mathcal{M}_{\text{ell}}^{A,b} > 1$. In this case, by the main result of this paper (Theorem 3.1), the closure of $\big(L_{A,b}, C_0^\infty(\mathbb{R}^d, \mu)\big)$ does not generate a C_0–semigroup on $L^1(\mathbb{R}^d, \mu)$ for *any* $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$. Nevertheless, as proved in [Sta99], it always has an extension that does. Let us recall the corresponding result in our (w.r.t. [Sta99], however, more special) situation:

Theorem 2.10. *Let* $\Omega = \mathbb{R}^d$ *and assume that conditions* (A1) *and* (A2) *of Theorem 2.1 hold. Let* $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$. *Then there exists a closed extension* $\big(L_{A,b}^\mu, D(L_{A,b}^\mu)\big)$ *of* $\big(L_{A,b}, C_0^\infty(\mathbb{R}^d)\big)$ *that generates a sub–Markovian* C_0*–semigroup* $(T_t^\mu)_{t\geq 0}$ *on* $L^1(\mathbb{R}^d, \mu)$. *Furthermore,* μ *is* $(T_t^\mu)_{t\geq 0}$–sub-invariant*, i.e.*

$$\int T_t^\mu f\, d\mu \leq \int f\, d\mu \quad \textit{for all } f \in L^\infty(\mathbb{R}^d, \mu),\ f \geq 0, \textit{ and all } t \geq 0. \tag{2.19}$$

If $b=\beta_\mu$, then $\big(L^\mu_{A,\beta_\mu}, D(L^\mu_{A,\beta_\mu})\big)$ *can be identified with the* Friedrichs *extension of* $\big(L_{A,\beta_\mu}, C_0^\infty(\mathbb{R}^d)\big)$ *on* $L^2(\mathbb{R}^d,\mu)$.

Proof. [Sta99, Theorem 1.5]. □

Remark 2.11. In the situation of Theorem 2.10, since $\mu \in \mathcal{M}_{\rm ell}^{A,2\beta_\mu - b}$ by Remark 2.4, also the "formally adjoint" $\big(L_{A,2\beta_\mu-b}, C_0^\infty(\mathbb{R}^d)\big)$ of $\big(L_{A,b}, C_0^\infty(\mathbb{R}^d)\big)$ thus has a closed extension $\big(L^\mu_{A,2\beta_\mu-b}, D(L^\mu_{A,2\beta_\mu-b})\big)$ generating a sub–Markovian C_0–semigroup $(T_t'^\mu)_{t\geq 0}$ on $L^1(\mathbb{R}^d,\mu)$. Then by [Sta99, Remark 1.7(ii)]

$$\int T_t^\mu f\, g\, d\mu = \int f\, T_t'^\mu g\, d\mu \quad \forall f,g \in L^1(\mathbb{R}^d,\mu)\cap L^\infty(\mathbb{R}^d,\mu). \tag{2.20}$$

The same relation holds for the corresponding resolvents $(G^\mu_\alpha)_{\alpha>0}$ and $(G'^\mu_\alpha)_{\alpha>0}$.

(2.20), in particular, immediately implies that μ is $(T_t^\mu)_{t\geq 0}$–invariant (cf. the following theorem) if and only if μ is $(T_t'^\mu)_{t\geq0}$–invariant. Hence, since both semigroups are sub–Markovian, this is the case if and only if $T_t 1 = 1$ for all $t\geq 0$, which in turn is equivalent to $T_t'^\mu 1 = 1$ for all $t \geq 0$.

The connection to maximal dissipativity is now given by:

Theorem 2.12. *Consider the situation of Theorem 2.10. Then $\mu \in \mathcal{M}^{A,b}_{\rm ell,md}$ if and only if μ is $(T_t^\mu)_{t\geq0}$–invariant, i.e.*

$$\int T_t^\mu f\, d\mu = \int f\, d\mu \quad \textit{for all } f \in L^\infty(\mathbb{R}^d,\mu) \textit{ and all } t\geq 0. \tag{2.21}$$

Proof. [Sta99, Cor. 2.2]. □

So, by Theorem 2.12 and our main result (Theorem 3.1 below), if $\Omega = \mathbb{R}^d$ and $\#\mathcal{M}^{A,b}_{\rm ell} > 1$, no $\mu \in \mathcal{M}^{A,b}_{\rm ell}$ is $(T_t^\mu)_{t\geq0}$–invariant.

The following generalizes [ABR99, Prop. 2.6(i)]:

Proposition 2.13. *Let Ω be a connected open set in $\mathbb{R}^d$. Assume that conditions* (A1), (A2) *of Theorem 2.1 hold. Let $\mu \in \mathcal{M}^{A,b}_{\rm ell}$ and let $(T_t^\mu)_{t\geq0}$ be the corresponding semigroup specified in Theorem 2.10. Let ν be a probability measure on Ω such that*

1. $\nu \ll dx$
2. $b^i \in L^q_{loc}(\Omega,\nu)$ *for some* $q \in]1,\infty[$.
3. $\int T_t^\mu f\, d\nu = \int f\, d\nu$ *for all* $f\in L^\infty(\Omega,\mu)$ *and all* $t>0$.

Then $\nu \in \mathcal{M}^{A,b}_{\rm ell}$.

Proof. Let $f \in L^\infty(\Omega, \mu)$. Then, since $(T_t^\mu)_{t\geq 0}$ is sub–Markovian, for all $t > 0$

$$\int \left|T_t^\mu f\right|^q d\nu \leq \int T_t^\mu |f|^q \, d\nu = \int |f|^q \, d\nu \tag{2.22}$$

and, likewise, for all $\varphi \in C_0^\infty(\Omega)$

$$\int \left|T_t^\mu L_{A,b}\varphi\right|^q d\nu \leq \int \left|L_{A,b}\varphi\right|^q d\nu < \infty, \tag{2.23}$$

where we used (2) and we note that (3) extends to all positive Borel measurable functions f by monotone convergence and hence to all ν–integrable functions f. In particular, we have uniform ν–integrability of $\{T_t^\mu f, T_t^\mu L_{A,b}\varphi \big| t \geq 0\}$. Therefore, $t \longmapsto T_t^\mu L_{A,b}\varphi$ is continuous as a map into $L^1(\Omega,\nu)$ by Lebesgue's (generalized) Dominated Convergence Theorem, since this is so as a map into $L^1(\Omega,\mu)$. Hence, the $L^1(\Omega,\nu)$–valued Riemann integrals

$$\int_0^t T_s^\mu L_{A,b}\varphi \, ds, \quad t > 0, \tag{2.24}$$

exist and by the Main Theorem of Calculus it follows for the $L^1(\Omega,\nu)$–derivative that

$$\frac{d}{dt}\int_0^t T_s^\mu L_{A,b}\varphi \, ds = T_t^\mu L_{A,b}\varphi, \quad t > 0. \tag{2.25}$$

On the other hand, since $\nu \ll dx \sim \mu$, the integral in (2.24) is equal to the corresponding $L^1(\Omega,\mu)$ integral, therefore it coincides with $T_t^\mu\varphi - \varphi$. Hence

$$\frac{d}{dt} T_t^\mu \varphi = T_t^\mu L_{A,b}\varphi \text{ in } L^1(\Omega,\nu), \tag{2.26}$$

and therefore,

$$\begin{aligned}\int L_{A,b}\varphi \, d\nu = \int \frac{d}{dt}\left(T_t^\mu L_{A,b}\varphi\right)_{t=0} d\nu &= \frac{d}{dt}\left(\int T_t^\mu L_{A,b}\varphi \, d\nu\right)_{t=0} \\ &= \frac{d}{dt}\left(\int L_{A,b}\varphi \, d\nu\right)_{t=0} = 0, \end{aligned} \tag{2.27}$$

by (3), extended to all ν–integrable f. □

Remark 2.14. In the situation of Proposition 2.13, the semigroup $(T_t^\mu)_{t\geq 0}$ extends to a C_0–semigroup $(T_t^{\mu,\nu})_{t\geq 0}$ on $L^1(\Omega,\nu)$, whose generator extends $\big(L_{A,b}, C_0^\infty(\Omega)\big)$. However, $(T_t^{\mu,\nu})_{t\geq 0}$ may not coincide with the semigroup $(T_t^\nu)_{t\geq 0}$, specified in Theorem 2.10, with ν replacing μ. If

$$\left(T_t^{\mu,\nu}\right)_{t\geq 0} = \left(T_t^\nu\right)_{t\geq 0} \tag{2.28}$$

it would follow by Theorem 2.12 that $\nu \in \mathcal{M}_{\text{ell,md}}^{A,b}$, so $\#\mathcal{M}_{\text{ell}}^{A,b} = 1$ by our main result Theorem 3.1 below. At present, however, we cannot prove (2.28), neither have we a counter-example.

We close this section with the following result, which was proved in [BKR], generalizing [ABR99, Thm. 1.4(iv)] (with, however, almost identical proofs). It will be used in the proof of Theorem 3.1 below.

Proposition 2.15. *Let Ω be a connected open set in $\mathbb{R}^d$ and assume that conditions* (A1) *and* (A2) *of Theorem 2.1 hold. Let $\mu \in \mathcal{M}_{\text{ell}}^{A,b}$ and let $(T_t^\mu)_{t\geq 0}$ be the corresponding semigroup of Theorem 2.10.*

1. *For all $\varphi \in C_0^\infty(\Omega)$ and all $t \geq 0$, $T_t^\mu\varphi$ has a continuous μ–version $\widetilde{T_t^\mu\varphi}$.*
2. *Suppose μ is $(T_t^\mu)_{t\geq 0}$–invariant (i.e. $\mu \in \mathcal{M}_{\text{ell,md}}^{A,b}$ if $\Omega = \mathbb{R}^d$). Let ν be a probability measure on Ω such that*

$$\int \widetilde{T_t^\mu\varphi}\, d\nu = \int \varphi\, d\nu \qquad \text{for all } \varphi \in C_0^\infty(\Omega),\ t > 0. \tag{2.29}$$

   Then $\mu = \nu$.

Proof. [BKR, Cor. 4.2]. □

3 Main results and consequences

The following is the main result of this paper. As announced in the introduction, it establishes a quite surprising link between maximal dissipativity and uniqueness of (infinitesimally) invariant measures.

Theorem 3.1. *Let $\Omega = \mathbb{R}^d$ and assume that conditions* (A1) *and* (A2) *of Theorem 2.1 hold. Then*

$$\mathcal{M}_{\text{ell,md}}^{A,b} \neq \emptyset \implies \#\mathcal{M}_{\text{ell}}^{A,b} = 1. \tag{3.1}$$

We note, that the converse is false at least for $d = 1$ (cf. the appendix, Remark A.3).

As mentioned in Subsection 2.2, it may occur that $\mathcal{M}_{\text{ell}}^{A,b} = \emptyset$ (e.g. if $\Omega = \mathbb{R}^d$, $A = \text{Id}$, $b \equiv 0$). But it can also happen that $\#\mathcal{M}_{\text{ell}}^{A,b} > 1$, hence $\mathcal{M}_{\text{ell,md}}^{A,b} = \emptyset$. The following proposition provides a whole class of examples.

Proposition 3.2.

1. *Let $\Omega = \mathbb{R}^d$, $A = \text{Id}$. Let $f \in C^2(\mathbb{R})$, bounded, such that $f, f' > 0$ and $f', f'' \in L^1(\mathbb{R}, dx)$ and let $\sigma : \{1, \ldots, d\} \longrightarrow \{1, \ldots, d\}$ be one-to-one, such that $\sigma(i) \neq i$ for all $i \in \{1, \ldots, d\}$. Define $b = (b^i) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ by*

$$b^i(x) := \frac{f''(x_i)}{f'(x_i)} + 2\frac{f''(x_{\sigma(i)})}{f'(x_i)}, \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \tag{3.2}$$

   and

$$\mu(dx) := c_1 \prod_{i=1}^d f'(x_i)\, dx, \qquad \nu(dx) := c_2 \sum_{i=1}^d f(x_i)\mu(dx), \tag{3.3}$$

where $c_1, c_2 \in]0,\infty[$ *are normalization constants.*

Then $\mu \neq \nu$, *but* $\mu, \nu \in \mathcal{M}_{\mathrm{ell}}^{\mathrm{Id},b}$.

2. $f(t) := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{t} e^{-s^2/2}\, ds$, $t \in \mathbb{R}$, *fulfills all assumptions in (1). In this case*

$$b^i(x) = -x_i - 2x_{\sigma(i)} e^{(x_i^2 - x_{\sigma(i)}^2)/2}, \quad x \in \mathbb{R}^d, \tag{3.4}$$

and μ *is the standard Gaussian measure on* $\mathbb{R}^d$.

The proofs of both Theorem 3.1 and Proposition 3.2 will be given in Section 4 below. Now we formulate two immediate consequences.

Corollary 3.3. *In the existence theorem 2.2 above, we also have uniqueness, i.e.* $\#\mathcal{M}_{\mathrm{ell}}^{A,b} = 1$.

Proof. By Theorem 2.2 we have $\#\mathcal{M}_{\mathrm{ell}}^{A,b} \geq 1$ and, by assumption, Proposition 2.9(3) applies. So, Theorem 2.8 yields $\#\mathcal{M}_{\mathrm{ell,md}}^{A,b} \geq 1$, hence $\#\mathcal{M}_{\mathrm{ell}}^{A,b} = 1$ by Theorem 3.1. □

Corollary 3.4. *Consider the situation of Theorem 2.10. If there exists one* $\mu \in \mathcal{M}_{\mathrm{ell}}^{A,b}$ *which is invariant with respect to its corresponding semigroup* $(T_t^\mu)_{t\geq 0}$, *then* $\#\mathcal{M}_{\mathrm{ell}}^{A,b} = 1$.

Proof. Theorems 2.12 and 3.1. □

However, we would like to emphasize that in contrast to Theorem 2.8 (and Proposition 2.9) Corollary 3.4 is of limited practical use, since $(T_t^\mu)_{t\geq 0}$ is not given in an explicit way.

We close this section with a few simple results on the extreme points of the convex set $\mathcal{M}_{\mathrm{ell}}^{A,b}$. Define $\operatorname{ext}\mathcal{M}_{\mathrm{ell}}^{A,b}$ to be the set of all $\mu \in \mathcal{M}_{\mathrm{ell}}^{A,b}$ which cannot be written as the non-trivial convex combination of two others. The following is standard. We include a proof for completeness.

Lemma 3.5. *Let* $\mu \in \mathcal{M}_{\mathrm{ell}}^{A,b}$. *Then the following are equivalent:*

1. $\mu \in \operatorname{ext}\mathcal{M}_{\mathrm{ell}}^{A,b}$.
2. $\rho \in L^\infty(\mathbb{R}^d, \mu)$, $\rho\mu \in \mathcal{M}_{\mathrm{ell}}^{A,b}$, *implies* $\rho = 1$ μ*–a.e.*

Proof. Assume (1) holds and let $\rho \in L^\infty(\mathbb{R}^d,\mu)$ such that $\rho\cdot\mu \in \mathcal{M}_{\mathrm{ell}}^{A,b}$. Then for $M := \sup\{\rho(x) \,|\, x \in \mathbb{R}^d\}$

$$\mu_1 := \frac{M-\rho}{M-1}\cdot\mu \in \mathcal{M}_{\mathrm{ell}}^{A,b} \tag{3.5}$$

and $\mu = \frac{M-1}{M}\mu_1 + \frac{1}{M}(\rho\cdot\mu)$. Hence by (1)

$$\frac{M-\rho}{M-1} = \rho, \tag{3.6}$$

so $\rho = 1$ μ–a.e.

Assume (2) holds and let $\mu_1, \mu_2 \in \mathcal{M}^{A,b}_{\text{ell}}$, $\alpha \in]0,1[$ such that

$$\mu = \alpha\mu_1 + (1-\alpha)\mu_2. \tag{3.7}$$

Then both μ_1 and μ_2 are absolutely continuous w.r.t. μ with bounded Radon–Nikodym densities. Hence by (2) $\mu_1 = \mu = \mu_2$. □

Corollary 3.6. *Let $\Omega = \mathbb{R}^d$ and assume that conditions* (A1) *and* (A2) *of Theorem 2.1 hold. Let $\mu = \rho_\mu\, dx \in \operatorname{ext}\mathcal{M}^{A,b}_{\text{ell}}$. Then for all $\nu = \rho_\nu\, dx \in \mathcal{M}^{A,b}_{\text{ell}} \backslash \{\mu\}$ the function ρ_ν/ρ_μ is unbounded. Here, ρ_μ and ρ_ν are the continuous versions of $\frac{d\mu}{dx}$ and $\frac{d\nu}{dx}$, resp. (cf. Theorem 2.1).*

Proof. Since $\mu = (\rho_\mu/\rho_\nu)\cdot\nu$, this is a direct consequence of Lemma 3.5. □

4 Proofs of Theorem 3.1 and Proposition 3.2

In all of this section we assume $\Omega = \mathbb{R}^d$ and conditions (A1) and (A2) of Theorem 2.1 holding. We need some preparations:

Lemma 4.1. *Let $\mu \in \mathcal{M}^{A,b}_{\text{ell,md}}$ and let $(T^\mu_t)_{t\geq 0}$ be the corresponding semigroup specified in Theorem 2.10. Let $\nu \in \mathcal{M}^{A,b}_{\text{ell}}$ be such that $\rho_1 := \frac{d\nu}{d\mu}$ is bounded. Then ν is $(T^\mu_t)_{t\geq 0}$–invariant.*

Proof. (cf. [ABR99, Prop. 2.6(ii)])

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then, since ρ_1 is bounded and since by assumption $\big(L^\mu_{A,b}, D(L^\mu_{A,b})\big) = \big(\bar{L}^\mu_{A,b}, D(\bar{L}^\mu_{A,b})\big)$, we have

$$\int \big(T^\mu_t\varphi - \varphi\big)\, d\nu = \int_0^t \int \bar{L}^\mu_{A,b}(T^\mu_s\varphi)\, d\nu\, ds \quad \forall t > 0. \tag{4.1}$$

But for $s > 0$ there exist $\varphi_k \in C_0^\infty(\mathbb{R}^d)$, $k \in \mathbb{N}$, such that, as $k \to \infty$, $L\varphi_k \longrightarrow \bar{L}^\mu T^\mu_s\varphi$ in $L^1(\mathbb{R}^d,\mu)$, hence in $L^1(\mathbb{R}^d,\nu)$. Therefore, since $\nu \in \mathcal{M}^{A,b}_{\text{ell}}$, it follows that

$$\int \bar{L}^\mu_{A,b}(T^\mu_s\varphi)\, d\nu = 0 \quad \text{for all } s > 0, \tag{4.2}$$

and, consequently, ν is $(T^\mu_t)_{t\geq 0}$–invariant by (4.1). □

Lemma 4.2. $\mathcal{M}^{A,b}_{\text{ell,md}} \subset \operatorname{ext}\mathcal{M}^{A,b}_{\text{ell}}$.

Proof. Let $\mu \in \mathcal{M}^{A,b}_{\text{ell,md}}$. Suppose $\mu_i = \rho_i\, dx \in \mathcal{M}^{A,b}_{\text{ell}}$, $i = 1,2$, such that for some $\alpha \in]0,1[$

$$\mu := \alpha\mu_1 + (1-\alpha)\mu_2 \tag{4.3}$$

Let $(T_t^\mu)_{t\geq 0}$ be the semigroup corresponding to μ, as specified in Theorem 2.10. Since

$$\frac{d\mu_i}{d\mu} = 2\frac{\rho_i}{\rho_1+\rho_2}, \quad i=1,2, \tag{4.4}$$

are bounded functions, we can apply Lemma 4.1 to conclude that μ_1 and μ_2 are $(T_t^\mu)_{t\geq 0}$–invariant. But by Theorem 2.12, also μ is $(T_t^\mu)_{t\geq 0}$–invariant, so Proposition 2.15 applies. Hence $\mu = \mu_1 = \mu_2$. □

Lemma 4.3. *$\mathcal{M}_{\mathrm{ell,md}}^{A,b} \neq \emptyset$ implies $\mathcal{M}_{\mathrm{ell,md}}^{A,b} = \mathcal{M}_{\mathrm{ell}}^{A,b}$.*

Proof. Let $\mu = \rho_\mu\, dx \in \mathcal{M}_{\mathrm{ell,md}}^{A,b}$ and $\nu = \rho_\nu\, dx \in \mathcal{M}_{\mathrm{ell}}^{A,b}$, where ρ_μ and ρ_ν are the strictly positive, continuous versions of the respective Radon–Nikodym derivatives of $\frac{d\mu}{dx}, \frac{d\nu}{dx} \in H_{\mathrm{loc}}^{p,1}(\mathbb{R}^d)$ (cf. Theorem 2.1). Let χ_n, $n\in\mathbb{N}$, be as in Theorem 2.8(2) (applied to μ). Now we shall show that the assertion of Theorem 2.8(2) also holds for ν with the same $(\chi_n)_{n\in\mathbb{N}}$.

Obviously, $H_{\mathrm{loc}}^{2,1}(\mathbb{R}^d,\mu) = H_{\mathrm{loc}}^{2,1}(\mathbb{R}^d,\nu)$ and $\partial_i^\mu = \partial_i^\nu$. Furthermore, for $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi \geq 0$ by (2.4)

$$\begin{aligned}
&\int a^{ij}\,\partial_i^\nu \chi_n\,\partial_j\varphi\, d\nu + \int \chi_n\varphi\, d\nu + \int (\beta_\nu^i - b^i)\,\partial_i^\nu\chi_n\,\varphi\, d\nu \\
&\quad= \int a^{ij}\partial_i^\mu\chi_n\,\partial_j\Big(\varphi\frac{\rho_\nu}{\rho_\mu}\Big)\, d\mu + \int \chi_n\varphi\, d\nu - \int a^{ij}\partial_i^\mu\chi_n\,\partial_j\Big(\frac{\rho_\nu}{\rho_\mu}\Big)\,\varphi\, d\mu \\
&\quad+ \int (\beta_\mu^i - b^i)\,\partial_i^\mu\chi_n\,\Big(\varphi\frac{\rho_\nu}{\rho_\mu}\Big)\, d\mu + \int a^{ij}\Big(\frac{\partial_j\rho_\nu}{\rho_\nu} - \frac{\partial_j\rho_\mu}{\rho_\mu}\Big)\frac{\rho_\nu}{\rho_\mu}\,\partial_i^\mu\chi_n\,\varphi\, d\mu \geq 0,
\end{aligned} \tag{4.5}$$

since the last summand cancels the third and since $\varphi\frac{\rho_\nu}{\rho_\mu} \in H_{\mathrm{loc}}^{2,1}(\mathbb{R}^d,\mu)$. □

Proof of Theorem 3.1. By Lemmas 4.2 and 4.3

$$\mathcal{M}_{\mathrm{ell,md}}^{A,b} \neq \emptyset \implies \mathcal{M}_{\mathrm{ell}}^{A,b} = \mathrm{ext}\,\mathcal{M}_{\mathrm{ell}}^{A,b}, \tag{4.6}$$

hence $\#\mathcal{M}_{\mathrm{ell}}^{A,b} = 1$. □

Proof of Proposition 3.2. Assertion (2) is trivial, so we only prove (1). Obviously,

$$\beta_\mu^i(x) = \frac{f''(x_i)}{f'(x_i)}. \tag{4.7}$$

Hence, to prove that $\mu \in \mathcal{M}_{\mathrm{ell}}^{A,b}$ by Remark 2.4 we have to show that

$$\mathrm{div}_\mu\Big(2\frac{f''(x_{\sigma(i)})}{f'(x_i)}\Big)_{1\leq i\leq d} = 0. \tag{4.8}$$

But integrating by parts for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ (not using the summation convention) we have, since $\sigma(i) \neq i$,

$$\sum_{i=1}^d \int \frac{f''(x_{\sigma(i)})}{f'(x_i)} \partial_i\varphi(x)\,\mu(dx) = c_1 \sum_{i=1}^d \int \cdots \int \left(\int \partial_i \varphi(x)\,dx_i \right) f(x_{\sigma(i)}) \times$$

$$\times f'(x_1) \cdots \widehat{f'(x_i)} \cdots f'(x_d)\; dx_i \cdots \widehat{dx_i} \cdots dx_d = 0, \quad (4.9)$$

with the hat (" $\hat{}$ ") indicating the term below it being absent. Now (4.8) follows.

To show that also $\nu \in \mathcal{M}_{\text{ell}}^{A,b}$, we first note that, obviously,

$$\beta_\nu^i(x) = \frac{f'(x_i)}{\sum_{j=1}^d f(x_j)} + \frac{f''(x_i)}{f'(x_i)}. \quad (4.10)$$

Hence, by Remark 2.4 we have to show

$$\operatorname{div}_\nu \left(2\frac{f''(x_{\sigma(i)})}{f'(x_i)} - \frac{f'(x_i)}{\sum_{j=1}^d f(x_j)} \right)_{1 \le i \le d} = 0. \quad (4.11)$$

But for all $\varphi \in C_0^\infty(\mathbb{R}^d)$ integrating by parts we have

$$\begin{aligned}
&\sum_{i=1}^d \int \left(2\frac{f''(x_{\sigma(i)})}{f'(x_i)} - \frac{f'(x_i)}{\sum_{j=1}^d f(x_j)} \right) \partial_i \varphi(x)\,\nu(dx) \\
&= c_2 \sum_{i=1}^d \int \left(2 \sum_{j=1}^d f(x_j)\ \frac{f''(x_{\sigma(i)})}{f'(x_i)} - f'(x_i) \right) \partial_i\varphi(x)\,\mu(dx) \\
&= c_1 c_2 \sum_{i=1}^d \int \cdots \int \left(\int 2 f'(x_i) f''(x_{\sigma(i)}) \varphi(x) - f'(x_i)^2 \partial_i \varphi(x)\,dx \right) \times \\
&\qquad\qquad \times f'(x_1) \cdots \widehat{f'(x_i)} \cdots f'(x_d)\; dx_1 \cdots \widehat{dx_i} \cdots dx_d \\
&= 2c_1c_2 \sum_{i=1}^d \left(\int f''(x_{\sigma(i)}) \varphi(x)\,d\mu - \int f''(x_i)\varphi(x)\,d\mu \right) = 0,
\end{aligned} \quad (4.12)$$

and (4.11) follows. □

A Appendix: The one–dimensional case

For completion, we give a brief account of the case $d = 1$ in this section.

So, let $\Omega = I$ be an open interval in $\mathbb{R}$. We adopt the notation of the main part of this paper.

Proposition A.1. *Suppose μ is a locally finite* Borel *measure on Ω such that $L_{A,b}^*\mu = 0$.*[2] *Then $A\mu = g\,dx$, where g is of locally bounded variation. Its distributional derivative is in $L^1_{\text{loc}}(I, dx)$, provided $A > 0$ μ–a.e.*

[2] So, in particular, we assume $A, b \in L^1_{\text{loc}}(I, \mu)$

Proof. Since $A\mu$, $b\mu$ are by assumption locally bounded measures, $L^*_{A,b}\mu = 0$ implies that

$$(A\mu)'' - (b\mu)' = 0 \tag{A.1}$$

in the sense of distributions. Hence

$$(A\mu)' - b\mu = c \in \mathbb{R}, \tag{A.2}$$

and, consequently,

$$A\mu(t) = \int_{\{s\in I\,|\,s\geq t\}} \big(b\,d\mu + c\,dx\big), \tag{A.3}$$

and the first part of the assertion follows. If $A > 0$ μ–a.e. then

$$\mu = 1_{\{A>0\}}\frac{g}{A}\,dx. \tag{A.4}$$

Since $g' = (A\mu)'$ is a distribution, it follows by (A.2) that g' $\big(= b1_{\{A>0\}}\frac{g}{A}\big)$ is in $L^1_{\mathrm{loc}}(I, dx)$. □

Proposition A.2. *Let μ, A and b be as in Proposition A.1, and assume, in addition, that A is locally absolutely continuous and $A > 0$ everywhere. Then:*

1. *$\frac{d\mu}{dx}$ (which exists by Proposition A.1) has a locally absolutely continuous version ρ.*

2. *If, in addition, $b \in L^1_{\mathrm{loc}}(I, dx)$, then for all $t_0 \in I$*

$$\rho(t) = c\exp\left\{\int_{t_0}^{t}\frac{b}{A}(s)\,ds\right\}\left(\int_{t_0}^{t}\exp\left\{-\int_{t_0}^{s}\frac{b}{A}(r)\,dr\right\}ds + \tilde{c}\right), \tag{A.5}$$

for some $c, \tilde{c} \in \mathbb{R}$ and any such ρ is a solution to $L^*_{A,b}\mu = 0$. In particular, $\#\mathcal{M}^{A,b}_{\mathrm{ell}} \leq 1$.

Proof.

1. By (the proof of) Proposition A.1, it follows that $\rho = \frac{g}{A}$, with distributional derivative in $L^1_{\mathrm{loc}}(I, dx)$.

2. Dividing by A, we may assume that $A \equiv 1$. By the proof of Proposition A.1, we know that

$$\rho' - b\rho = -c \tag{A.6}$$

which by elementary calculus is equivalent to (A.5), if $b \in L^1_{\mathrm{loc}}(I, dx)$. Now assume that $\int_I \rho\,dx = 1$ and that $\nu = \rho_\nu\,dx \in \mathcal{M}^{A,b}_{\mathrm{ell}}$. Then $\rho - \rho_\nu$ solves (A.6) with $c = 0$ and must be zero at some point, since its integral vanishes and I is connected. Hence $\rho = \rho_\nu$ by (A.5). □

Remark A.3. By [Sta99, Example 1.12] we know that for $A = 1$ and $b(x) := -2x - 6e^{x^2}$, $x \in \mathbb{R}$, we have $\mathcal{M}^{A,b}_{\mathrm{ell}} = \{\mu\}$ with $\mu(dx) = e^{-x^2}\,dx$, but $\big(L_{A,b}, C_0^\infty(\mathbb{R})\big)$ is not maximally dissipative on $L^1(\mathbb{R}, \mu)$. So, the converse of Theorem 3.1 does not hold in the case $d = 1$.

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