# Representations of Infinitesimal Groups of Characteristic 2

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## 1. Introduction

This short note is concerned with infinitesimal groups of finite representation type that are defined over an algebraically closed field $k$ of characteristic $p = 2$. In [3, (2.7)] it was shown that the principal block $\mathcal{B}_0(\mathcal{G})$ of the distribution algebra $H(\mathcal{G})$ of an infinitesimal group $\mathcal{G}$ is representation-finite if and only if its second Frobenius kernel $\mathcal{G}_2$ enjoys this property. More precisely, the following result was established:

**Theorem 1.1 (cf. (2.7) of [3])** *Let $\mathcal{G}$ be an infinitesimal $k$-group. Then the following statements are equivalent:*

(1) $\mathcal{B}_0(\mathcal{G})$ *has finite representation type.*
(2) $\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \times \mu_{p^n}$ *is a semidirect product with a $V$-uniserial normal subgroup $\mathcal{U}$.*
(3) $H(\mathcal{G})$ *is a Nakayama algebra.*
(4) $\dim \mathcal{V}_{\mathcal{G}_2}(k) \leq 1$.
(5) $H(\mathcal{G}_2)$ *is a Nakayama algebra.*

Theorem 1.1 depends on the following Lemma (cf. [3, (2.5)]) that was verified in [3] for groups of characteristic $p \geq 3$:

**Lemma 1.2** *Let $\mathcal{U}$ be a unipotent, infinitesimal $k$-group, $\mathcal{U}' \subset \mathcal{U}$ a central subgroup such that*

(a) $\mathcal{U}' \cong \alpha_p$, *and*
(b) $\mathcal{U}/\mathcal{U}'$ *is commutative of length $\ell(\mathcal{U}/\mathcal{U}') = 2$, and*
(c) $V_{\mathcal{U}/\mathcal{U}'} = 0$, *and*
(d) $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{U})} \leq 1$.

*Then $\mathcal{U}$ contains a subgroup isomorphic to $\alpha_{p^2}$.*

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Here condition (c) refers to the *Verschiebung*, a homomorphism $V_{\mathcal{G}} : \mathcal{G}^{(p)} \longrightarrow \mathcal{G}$ that is defined for any commutative infinitesimal $k$-group (cf. [2, (IV.§3(4.3)]). Given any infinitesimal $k$-group $\mathcal{G}$ with Lie algebra $\mathrm{Lie}(\mathcal{G})$, we let $\mathcal{V}_{\mathrm{Lie}(\mathcal{G})} := \{x \in \mathrm{Lie}(\mathcal{G}) \ ; \ x^p = 0\}$ denote the *rank variety* of $\mathrm{Lie}(\mathcal{G})$. If $R$ is a commutative $k$-algebra, $M_R$ is the category of all $R$-algebras. The reader is referred to [1, 3, 4] for all other undefined notions.

The arguments of [3] yielding a proof of (1.2) break down in even characteristic. We shall show in §3 that (1.2), and thereby (1.1) also holds for $p = 2$.

## 2. Bilinear Forms

Throughout this section we assume that $k$ is an algebraically closed field of characteristic $p = 2$. We shall compute bilinear forms between certain commutative unipotent infinitesimal $k$-groups.

**Lemma 2.1** *Let $\mathcal{W}$ be an commutative infinitesimal group, $\varphi : \mathcal{W} \times \mathcal{W} \longrightarrow \alpha_2$ a bilinear form. Then the following statements hold:*

(1) *If $\mathcal{W} = \alpha_2 \times \alpha_2$, then there exist $\lambda_{ij} \in k$ for $1 \leq i, j \leq 2$ such that*

$$\varphi((x_1, x_2), (y_1, y_2)) = \sum_{i,j=1}^{2} \lambda_{ij} x_i y_j \quad \forall \ x_i, \ y_i \in \alpha_2.$$

(2) *If $\mathcal{W} = \alpha_{2^2}$, then there exist $\lambda_i \in k$ $1 \leq i \leq 3$ such that*

$$\varphi(x, y) = \lambda_1 x^2 y + \lambda_2 x y^2 + \lambda_3 x^2 y^2 \qquad \forall \ x, \ y \in \alpha_{2^2}.$$

*Proof.* In view of [2, (I.§2(7.1))] there is a canonical isomorphism

$$\mathrm{Bil}(\mathcal{W} \times \mathcal{W}, \alpha_2) \cong \mathrm{Hom}(\mathcal{W}, \mathcal{HOM}(\mathcal{W}, \alpha_2)).$$

Here the first and second groups are those of the bilinear forms and homomorphisms, while $\mathcal{HOM}$ denotes the scheme of homomorphisms. We recall the canonical identifications

$$\mathcal{HOM}(\alpha_2, \alpha_2) \cong \alpha_k \ \text{ and } \ \mathrm{Hom}(\alpha_2, \alpha_k) = \mathrm{Hom}(\alpha_2, \alpha_2) \cong k.$$

Consequently, every bilinear form $\varphi \in \mathrm{Bil}(\alpha_2 \times \alpha_2, \alpha_2)$ is of the form $\varphi(x, y) = \lambda xy$ for a suitable $\lambda \in k$.

(1). Suppose that $\mathcal{W} \cong \alpha_2 \times \alpha_2$. By the above, every bilinear form $\varphi \in \mathrm{Bil}(\mathcal{W} \times \mathcal{W}, \alpha_2)$ has the from

$$\varphi((x_1, x_2), (y_1, y_2)) = \sum_{i,j=1}^{2} \lambda_{ij} x_i y_j,$$

where the $\lambda_{ij}$ are suitably chosen elements of $k$.

(2). Suppose that $\mathcal{W} = \alpha_{2^2}$. Since

$$\mathcal{HOM}(\alpha_{2^2}, \alpha_2)(R) \cong \mathrm{Hom}_{\mathrm{Hopf}}(R[Y]/(Y^2), R[X]/(X^4)) \cong \alpha_2(R) \times \alpha_k(R)$$

for every commutative $k$-algebra $R$, the map

$$\Gamma_R : \alpha_2 \times \alpha_k \longrightarrow \mathcal{HOM}(\alpha_{2^2}, \alpha_2) \quad ; \quad \Gamma(\zeta, \eta)(y) = \zeta y + \eta y^2$$

for every $\zeta \in \alpha_2(R)$, $\eta \in \alpha_k(R)$, $y \in \alpha_{2^2}(S)$, $R \in M_k$, $S \in M_R$ is an isomorphism.

We continue by computing $\mathrm{Hom}(\alpha_{2^2}, \alpha_2 \times \alpha_k) \cong \mathrm{Hom}(\alpha_{2^2}, \alpha_2) \times \mathrm{Hom}(\alpha_{2^2}, \alpha_k)$. Since $\Gamma_k$ is an isomorphism $k \cong \mathcal{HOM}(\alpha_{2^2}, \alpha_2)(k) = \mathrm{Hom}(\alpha_{2^2}, \alpha_k)$, every homomorphism $h : \alpha_{2^2} \longrightarrow \alpha_2$ is of the form $h(x) = \lambda_1 x$ for some $\lambda_1 \in k$. Given an endomorphism $g : \alpha_{2^2} \longrightarrow \alpha_{2^2}$ there exist $\lambda_2$, $\lambda_3 \in k$ such that $g(x) = \lambda_2 x + \lambda_3 x^2$ for every $x \in \alpha_{2^2}$. Combination of the above steps implies that every bilinear form $\varphi \in \mathrm{Bil}(\alpha_{2^2} \times \alpha_{2^2}, \alpha_2)$ can be written as

$$\varphi(x, y) = \lambda_1 x^2 y + \lambda_2 x y^2 + \lambda_3 x^2 y^2 \quad \forall\, x, y \in \alpha_{2^2},$$

for suitably chosen elements $\lambda_i \in k$. $\square$

# 3. Commutator Forms for Groups of Characteristic 2

In this final section we shall prove Lemma 1.2. In view of [3, (2.5)] we may assume that $p = 2$. Given a commutative infinitesimal group $\mathcal{W}$ we observe that the *symmetrization*

$$s : \mathrm{Bil}(\mathcal{W} \times \mathcal{W}, \alpha_2) \longrightarrow \mathrm{Bil}(\mathcal{W} \times \mathcal{W}, \alpha_2) \quad ; \quad s(f)(x, y) := f(x, y) + f(y, x)$$

maps $\mathrm{Bil}(\mathcal{W} \times \mathcal{W}, \alpha_2)$ into the subspace $\mathrm{Alt}(\mathcal{W} \times \mathcal{W}, \alpha_2)$ of the alternating forms. For $p \geq 3$ every alternating form is the "symmetrization" of some bilinear form. Lemma 2.1 illustrates that is is no longer the case for groups of even characteristic. However, as we show in Lemma 3.1 below, the so-called commutator form $\chi_{\mathcal{U}}$ still has the requisite property.

Consider $\mathcal{U}$ as in (1.2), and let $\pi : \mathcal{U} \longrightarrow \mathcal{U}/\mathcal{U}'$ be the quotient map. The commutator product induces a bilinear form

$$\chi_{\mathcal{U}} : \mathcal{U}/\mathcal{U}' \times \mathcal{U}/\mathcal{U}' \longrightarrow \alpha_2$$

satisfying $\chi_{\mathcal{U}}(\pi(a), \pi(b)) = aba^{-1}b^{-1}$ for all $a, b \in \mathcal{U}$.

**Lemma 3.1** *Let $\mathcal{U}$ satisfy the properties of (1.2). Then there exists $\varphi \in \mathrm{Bil}(\mathcal{U}/\mathcal{U}' \times \mathcal{U}/\mathcal{U}', \alpha_2)$ such that $\chi_{\mathcal{U}} \in \mathrm{Alt}(\mathcal{U}/\mathcal{U}' \times \mathcal{U}/\mathcal{U}', \alpha_2)$ is of the form $\chi_{\mathcal{U}} = s(\varphi)$.*

*Proof.* We put $\mathcal{W} := \mathcal{U}/\mathcal{U}'$. Owing to (c) and [2, (IV.§3(6.9))] we have $\mathcal{W} \cong \alpha_2 \times \alpha_2$ or $\mathcal{W} \cong \alpha_{2^2}$.

We first consider the case $\mathcal{W} = \alpha_2 \times \alpha_2$. According to (2.1(1)) there exist $\lambda_{ij} \in k$ such that $\chi_{\mathcal{U}}((x_1, x_2), (y_1, y_2)) = \sum_{i,j=1}^{2} \lambda_{ij} x_i y_j$. Since $\chi_{\mathcal{U}}$ is alternating, we have $\lambda_{12} = \lambda_{21}$. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be the pre-images of the two copies of $\alpha_2$ under $\mathcal{U} \longrightarrow \mathcal{W}$. Thanks to [2, (II.6(7.1))] each $\mathcal{U}_i$ is commutative. This implies that $0 = \chi_{\mathcal{U}}((x, 0), (y, 0)) = \lambda_{11} xy$ for every $x, y \in \alpha_2$. Thus, $\lambda_{11} = 0$, and the same reasoning shows $\lambda_{22} = 0$. We obtain $\chi_{\mathcal{U}} = s(\varphi)$, where $\varphi((x_1, x_2), (y_1, y_2)) = \lambda_{12} x_1 y_2$.

Now assume that $\mathcal{W} = \alpha_{2^2}$. By (2.1(2)) we have $\chi_{\mathcal{U}}(x,y) = \lambda_1 x^2 y = \lambda_2 xy^2 + \lambda_3 x^2y^2$ for some $\lambda_i \in k$. As the form $\chi_{\mathcal{U}}$ is alternating, we may write

$$\chi_{\mathcal{U}}(x,y) = \gamma_1 x^2 y + \gamma_1 xy^2 + \gamma_2 x^2 y^2.$$

We propose to show that $\gamma_2 = 0$. To that end, we first consider the exact sequence

$$e_k \longrightarrow \alpha_2 \longrightarrow \mathcal{U} \longrightarrow \alpha_{2^2} \longrightarrow e_k. \tag{1}$$

Application of the morphism $j : \alpha_2 \hookrightarrow \alpha_k$ of abelian groups with trivial $\alpha_{2^2}$-operation to (1) gives a central extension

$$e_k \longrightarrow \alpha_k \longrightarrow \tilde{\mathcal{U}} \longrightarrow \alpha_{2^2} \longrightarrow e_k \tag{2}$$

(cf. [2, (III.§6(1.7))]). The commutator form $\chi_{\tilde{\mathcal{U}}}$ takes values in $\alpha_2 \hookrightarrow \alpha_k$, and we have $\chi_{\tilde{\mathcal{U}}} = \chi_{\mathcal{U}}$. Since $E\tilde{x}^1(\alpha_{2^2}, \alpha_k) = H_0^2(\alpha_{2^2}, \alpha_k)$ [2, (III.§4 (1.8),(6.6))], the exact sequence (2) is a Hochschild extension. Thus, if $f \in Z^2(\alpha_{2^2}, \alpha_k)$ is a cocycle describing the above extension of groups, then there is a section $s : \alpha_{2^2} \longrightarrow \tilde{U}$ of schemes such that

$$s(a)s(b) = s(a+b)f(a,b) \qquad \forall\ a, b \in \alpha_{2^2}.$$

Consequently, observing the fact that $f(a,b)$ lies centrally in $\tilde{U}$, we obtain

$$\begin{aligned}
\chi_{\mathcal{U}}(a,b) &= \chi_{\tilde{\mathcal{U}}}(a,b) = \chi_{\tilde{\mathcal{U}}}(s(a), s(b)) = s(a+b)f(a,b)f(b,a)^{-1}s(a+b)^{-1} \\
&= f(a,b)f(b,a)^{-1} = f(a,b) + f(b,a).
\end{aligned}$$

Thanks to [2, (II.§3 (4.5),(4.7))] we have

$$H_0^2(\alpha_{2^2}, \alpha_k) = k\,\bar{X}\bar{Y} \oplus k\,\bar{X}\bar{Y}^2,$$

where $\bar{X}, \bar{Y}$ are the residue classes of $X, Y$ in $k[X,Y]/(X^4, Y^4) \cong \mathcal{O}(\alpha_{2^2} \times \alpha_{2^2})$. Since $\delta(g)(a,b) + \delta(g)(b,a) = 0$ for every 1-cochain $g$, we obtain

$$\chi_{\mathcal{U}}(x,y) = \gamma_1(xy^2 + yx^2),$$

so that $\chi_{\mathcal{U}} = s(\varphi)$ with $\varphi(x,y) = \gamma_1\, xy^2$. $\square$

*Proof of Lemma 1.2.* Assuming $p = 2$, we put $\mathcal{W} := \mathcal{U}/\mathcal{U}'$. If $\mathcal{U}$ is commutative, then we consider the commutative diagram

$$\begin{tikzcd}
\mathcal{U}^{(2)} \arrow[r, "V_{\mathcal{U}}"] \arrow[d, "\pi^{(2)}"] & \mathcal{U} \arrow[d, "\pi"] \\
\mathcal{W}^{(2)} \arrow[r, "V_{\mathcal{W}}"] & \mathcal{W},
\end{tikzcd}$$

where the maps $\pi$ and $\pi^{(2)}$ are the canonical epimorphisms. Since $V_{\mathcal{W}} = 0$, the Verschiebung $V_{\mathcal{U}}$ sends $\mathcal{U}^{(2)}$ into $\mathcal{U}'$. It now follows from (b) that $\ker V_{\mathcal{U}}$ has length $\geq 2$. As $k$ is perfect, there exists a subgroup $\mathcal{K} \subset \mathcal{U}$ such that $\ker V_{\mathcal{U}} = \mathcal{K}^{(2)}$. Owing to [2, (IV.§3(6.11))] we thus have an isomorphism

$$\mathcal{K} \cong \prod_{i=1}^{s} \alpha_{2^{r_i}}.$$

From condition (d) we obtain

$$s = dim \mathcal{V}_{\mathrm{Lie}(\mathcal{K})}(k) \leq dim \mathcal{V}_{\mathrm{Lie}(\mathcal{U})}(k) \leq 1,$$

so that $r_1 = \ell(\mathcal{K}) \geq 2$.

In the general case we consider the commutator form $\chi_{\mathcal{U}} : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{U}'$. According to (3.1) we have $\chi_{\mathcal{U}} = s(\varphi)$ for a suitably chosen $\varphi \in \mathrm{Bil}(\mathcal{W} \times \mathcal{W}, \alpha_2)$. We define a new composition on $\mathcal{U}$ via

$$a \circ b := ab\, \varphi(\pi(a), \pi(b)) \qquad \forall\ a, b \in \mathcal{U}.$$

Direct computation reveals that this product endows $\mathcal{U}$ with the structure of a commutative group (with the inverse $\hat{a}$ of an element $a$ being given by $\hat{a} := a^{-1}\varphi(\pi(a), \pi(a))$). Since the induced structures on $\mathcal{U}/\mathcal{U}'$ and $\mathcal{U}'$ coincide with the original ones, $(\mathcal{U}, \circ)$ is, as an extension of two unipotent groups, unipotent (cf. [2, (IV.§2 (2.3.(c)))]). Every subgroup of $(\mathcal{U}, \circ)$ is obviously also a subgroup of $\mathcal{U}$. According to (d) $\mathcal{U}'$ is the only subgroup of $\mathcal{U}$ that is isomorphic to $\alpha_2$. Consequently, $(\mathcal{U}, \circ)$ has the same property, so that $(\mathcal{U}, \circ)$ also satisfies (d). By the first part of the proof the commutative group $(\mathcal{U}, \circ)$ contains a subgroup $(\mathcal{K}, \circ) \cong \alpha_{2^2}$. Since the induced group structures on $\mathcal{U}/\mathcal{U}'$ and $\mathcal{U}'$ coincide, it follows that $(\mathcal{K}, \cdot)$ is a central extension of $\alpha_2$ by $\alpha_2$. Owing to [2, (III.§6(7.7))] $\mathcal{K}$ is a commutative subgroup of $\mathcal{U}$, and the new structure coincides on $\mathcal{K}$ with the old one. $\square$

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