ALMOST SPLIT SEQUENCES OF VERMA MODULES

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0. Introduction

The Auslander-Reiten theory of restricted Lie algebras is not nearly as well understood as its precursor for finite groups. In particular, a comprehensive block theory for restricted Lie algebras has remained elusive, and techniques of descent, like those set forth in [17], usually lose their utility in this context. In recent work [11, 12] methods related to the notion of support variety have proven to be most effective, particularly when the underlying Lie algebra is associated to a reductive algebraic group.

As blocks of enveloping algebras of reductive Lie algebras are rarely tame or representation-finite [11, 12], there is little hope of obtaining a comprehensive description of their indecomposable modules. In default of a general classification one is led to study the location of certain classes of indecomposables within the stable Auslander-Reiten quiver. Simple modules either belong to components of type $\mathbb{Z}[\tilde{A}_{12}]$, or are located at ends of $\mathbb{Z}[A_\infty]$, with the former occurring precisely when the corresponding block is tame [11, 12].

In this paper we use the algebro-geometric approach to study the location of *Verma modules* within the stable Auslander-Reiten quiver. Our main result, Theorem 6.1, shows that these are located at ends of Auslander-Reiten components.

Throughout, we are working over an algebraically closed field k of characteristic $p > 2$. All k-vector spaces are assumed to be finite dimensional. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra with universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Given a linear form $\chi \in \mathfrak{g}^*$, we consider the *reduced enveloping algebra*

$$u(\mathfrak{g}, \chi) := \mathcal{U}(\mathfrak{g})/(\{x^p - x^{[p]} - \chi(x)^p 1 \mid x \in \mathfrak{g}\}).$$

We refer the reader to [30] concerning basic properties of $u(\mathfrak{g}, \chi)$, and only recall a few facts that are relevant for our purposes. The algebra $u(\mathfrak{g}, \chi)$ is a finite dimensional *Frobenius algebra* whose Nakayama automorphism μ is given by (cf. [13, (1.2)])

$$\mu(x) = x - \operatorname{tr}(\operatorname{ad} x) \quad \text{for all } x \in \mathfrak{g}.$$

The *restricted enveloping algebra* $u(\mathfrak{g}) := u(\mathfrak{g}, 0)$ of $\mathfrak{g}$ is a cocommutative Hopf algebra.

If $\mathfrak{h} \subset \mathfrak{g}$ is a p-subalgebra, and M is a $u(\mathfrak{g}, \chi)$-module, then we write $M|_{u(\mathfrak{h}, \chi|_{\mathfrak{h}})}$ for the restriction of M to the subalgebra $u(\mathfrak{h}, \chi|_{\mathfrak{h}}) \subset u(\mathfrak{g}, \chi)$.

The paper is organized as follows. In §§1 and 2 we collect basic facts on Verma modules of reductive groups and their varieties. In view of the results of [11, 12], we are primarily interested in the case, where the variety has dimension 2. As is shown in Section 3, whose computational part is relegated to an appendix §7, this naturally leads to the discussion of certain semisimple groups of rank 2. After a technical section on Auslander-Reiten components, we discuss in Section 5 Verma modules for groups of type $A_1 \times A_1$ and A_2. For the former type Verma modules are shown to be *quasi-simple* (i.e., located at ends of components of tree class A_∞). For the Lie algebra $\mathfrak{sl}(3)$ this also holds, with the possible exception of those blocks whose three highest weights lie on walls

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of the upper alcove of the fundamental Weyl chamber. Here we prove that Verma modules are situated at ends of components of type $\mathbb{Z}[A_\infty]$ or $\mathbb{Z}[D_\infty]$. The occurrence of the latter case would imply that K. Erdmann's remarkable result [6, Thm.1] on the structure of the AR-components of wild blocks of group algebras would fail in our context.

As general references for algebraic groups we cite the books by A. Borel [4] and T. Springer [29].

1. Verma Modules for Reductive Lie Algebras

Throughout, G denotes a (connected) reductive algebraic group defined over k. Let T be a fixed maximal torus of G, and consider a Borel subgroup $B = TU$ of G with unipotent radical U. We denote the Lie algebra of G by $\operatorname{Lie} G$ or $\mathfrak{g}$; likewise for subgroups of G. Let $\Psi := \Psi(G,T)$ be the set of roots of G with respect to T. For a root β of G we let U_β be the corresponding one-parameter unipotent subgroup of G normalized by T. The root subspace $\operatorname{Lie} U_\beta$ of $\mathfrak{g}$ is denoted $\mathfrak{g}_\beta$. There results the root space decomposition of $\mathfrak{g}$ relative to T:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha.$$

By Ψ^+ and Ψ^- we denote the positive and negative roots of G, respectively, and we define

$$\mathfrak{u} := \bigoplus_{\alpha\in\Psi^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{u}^- := \bigoplus_{\alpha\in\Psi^-} \mathfrak{g}_\alpha.$$

Let F_G be the *Frobenius endomorphism* of G induced by taking p-th powers on the coordinate ring $k[G]$ of G. The infinitesimal algebraic k-groups $G_t := \ker F_G^t$ are called the *Frobenius kernels* of G, [22].

By $X(T)$ we denote the character group of T. Given $\lambda \in X(T)$, we let

$$Z(\lambda) := u(\mathfrak{g}) \otimes_{u(\mathfrak{b})} k_\lambda$$

be the *Verma module* with highest weight λ. We view k_λ as a $\operatorname{Lie} B$-module by letting $\operatorname{Lie} T$ and $\operatorname{Lie} U$ act via the differential $d(\lambda)$ and 0, respectively. The module $Z(\lambda)$ is indecomposable with simple top and simple socle. Accordingly, $Z(\lambda)$ and $Z(\lambda')$ are isomorphic if and only if $d(\lambda) = d(\lambda')$. Moreover, $Z(\lambda)$ is a G_1T-module, i.e., it is a G_1-module (i.e., a $\mathfrak{g}$-module), and a T-module via $t \cdot (u \otimes s) := \operatorname{Ad}(t)(u) \otimes \lambda(t)s$. Given $t \in T$ and $u \in u(\mathfrak{g})$ we have $t \cdot (u \cdot v) = \operatorname{Ad}(t)(u)(t \cdot v)$ for every $v \in Z(\lambda)$, and the differential of the T-action coincides with the action of $\operatorname{Lie} T$ on $Z(\lambda)$.

For a p-subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ we consider the induced module

$$Z_{\mathfrak{h}}(\lambda) := u(\mathfrak{h}) \otimes_{u(\mathfrak{h}\cap\mathfrak{b})} k_{\lambda|_{\mathfrak{h}\cap\mathfrak{b}}}.$$

Note that $Z_{\mathfrak{h}}(\lambda)$ is a submodule of $Z(\lambda)|_{u(\mathfrak{h})}$.

For every root α there is an associated coroot $\alpha^\vee$, i.e., a certain one-parameter subgroup $\alpha^\vee : k \setminus \{0\} \to T$ defined by the pairing $\langle \zeta, \eta\rangle = \zeta \circ \eta$ with values in $\mathbb{Z}$ such that the coroot $\alpha^\vee$ is dual to the root α relative to this pairing.

We begin with a few simple observations concerning the structure of Verma modules. Let $\mathfrak{g}' = \operatorname{Lie} G^{(1)}$ be the Lie algebra of the derived group of G. It follows from [4, §14] that there exists a torus $\mathfrak{t}_0 \subset \mathfrak{t}$ such that $\mathfrak{g} = \mathfrak{t}_0 \oplus \mathfrak{g}'$.

Lemma 1.1. *Let G be reductive.*

(i) *Suppose that $\mathfrak{g} = \mathfrak{t}_0 \oplus \mathfrak{h}$ is a sum of p-subalgebras with $\mathfrak{t}_0 \subseteq \mathfrak{t}$. Then there is an isomorphism $Z_{\mathfrak{h}}(\lambda) \cong Z(\lambda)|_{u(\mathfrak{h})}$.*

(ii) *Let H and K be normal subgroups of G such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$. Then there is an isomorphism of $u(\mathfrak{g})$-modules $Z(\lambda) \cong Z_{\mathfrak{h}}(\lambda) \otimes_k Z_{\mathfrak{k}}(\lambda)$.*

Proof. (i). Since both $u(\mathfrak{h})$-modules have the same dimension, the canonical inclusion $Z_{\mathfrak{h}}(\lambda) \hookrightarrow Z(\lambda)|_{u(\mathfrak{h})}$ is an isomorphism.

(ii). We begin by showing that $\mathfrak{b} = (\mathfrak{b} \cap \mathfrak{h}) \oplus (\mathfrak{b} \cap \mathfrak{k})$. Thanks to [4, (13.18)] all root spaces of $\mathfrak{g}$ are one-dimensional. By assumption, both summands of $\mathfrak{g}$ are invariant under the adjoint representation of G. In particular, we have $\mathfrak{q} \cap \mathfrak{b} = \mathfrak{q} \cap \mathfrak{t} \oplus \bigoplus_{\alpha \in \Psi^+} \mathfrak{q}_\alpha$ for $\mathfrak{q} \in \{\mathfrak{h}, \mathfrak{k}\}$. Since $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Psi^+} \mathfrak{g}_\alpha$ (cf. [4, §13]), the assertion follows.

Direct computation shows that the map

$$u(\mathfrak{h}) \times k_\lambda \times u(\mathfrak{k}) \times k_\lambda \longrightarrow Z(\lambda) \;\; ; \;\; (u, r, v, s) \mapsto uv \otimes rs$$

induces a linear map

$$\varphi : Z_{\mathfrak{h}}(\lambda) \otimes_k Z_{\mathfrak{k}}(\lambda) \longrightarrow Z(\lambda).$$

Since $u(\mathfrak{g}) \cong u(\mathfrak{h}) \otimes_k u(\mathfrak{k})$, this map is $u(\mathfrak{g})$-linear and surjective. By the first part of the proof, the left-hand space has dimension $p^{\dim \mathfrak{g} - \dim \mathfrak{b}}$, so that φ is an isomorphism. □

Let Π be a subset of the set of simple roots Σ of G. We denote the (standard) parabolic subgroup of G associated to Π by P_Π, so that in particular $P_\emptyset = B$. Moreover, for $P = P_\Pi$ we consider the Levi decomposition $P = LP_u$, where L and P_u denote the standard Levi subgroup and the unipotent radical of P, respectively. Accordingly, we have $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where $\operatorname{Lie} P_u = \mathfrak{n}$. We consider the associated triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n} \tag{1}$$

of $\mathfrak{g}$, where $\mathfrak{n}^-$ is the Lie algebra of the unipotent radical of the parabolic subgroup of G which is *opposite* to P. Note that each of the direct summands in (1) is a p-subalgebra of $\mathfrak{g}$, and $\mathfrak{n}$ and $\mathfrak{n}^-$ are L-invariant subspaces. Consequently, the adjoint action of $\mathfrak{l}$ on $\mathfrak{n}^-$ induces an action of $u(\mathfrak{l})$ on $u(\mathfrak{n}^-)$, and we denote the corresponding $u(\mathfrak{l})$-module by $u(\mathfrak{n}^-)_{ad}$. We maintain this notation for the remainder of this section.

For a $u(\mathfrak{l})$-module M viewed as a $u(\mathfrak{p})$-module with $\mathfrak{n}$ acting trivially, the induced $u(\mathfrak{g})$-module $u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} M$ is frequently called the *Harish-Chandra induced module of M* (with respect to $\mathfrak{p}$).

Lemma 1.2. *Let $\lambda \in X(T)$.*

(i) *There is an isomorphism $Z(\lambda)|_{u(\mathfrak{l})} \cong u(\mathfrak{n}^-)_{ad} \otimes_k Z_{\mathfrak{l}}(\lambda)$.*
(ii) *$Z_{\mathfrak{l}}(\lambda)$ is a direct summand of $Z(\lambda)|_{u(\mathfrak{l})}$.*
(iii) *There is an isomorphism $Z(\lambda) \cong u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda)$.*

Proof. (i). We denote by ι the natural embedding $Z_{\mathfrak{l}}(\lambda) \hookrightarrow Z(\lambda)$, and consider the linear map $\varphi : u(\mathfrak{n}^-)_{ad} \otimes_k Z_{\mathfrak{l}}(\lambda) \longrightarrow Z(\lambda)$ given by $\varphi(u \otimes z) := u \cdot z := u \cdot \iota(z)$. Since $\mathfrak{u}^- = \mathfrak{n}^- \oplus (\mathfrak{l} \cap \mathfrak{u}^-)$, the map φ is surjective. By the same token, both spaces have the same dimension, so that φ is an isomorphism.

Let x be in $\mathfrak{l}$, $u \in u(\mathfrak{n}^-)_{ad}$, and $z \in Z_{\mathfrak{l}}(\lambda)$. Then we have

$$\begin{aligned} \varphi(x \cdot (u \otimes z)) &= \varphi([x, u] \otimes z + u \otimes x \cdot z) = [x, u] \cdot z + u \cdot (x \cdot z) \\ &= (xu) \cdot z - (ux) \cdot z + (ux) \cdot z = (xu) \cdot z = x \cdot \varphi(u \otimes z). \end{aligned}$$

Thus φ is $\mathfrak{l}$-linear, as claimed.

(ii). Letting $u(\mathfrak{n}^-)^\dagger$ be the augmentation ideal of $u(\mathfrak{n}^-)$, we observe that $u(\mathfrak{n}^-)_{ad} = u(\mathfrak{n}^-)^\dagger \oplus k \cdot 1$ is a decomposition of $\mathfrak{l}$-submodules, with $\mathfrak{l}$ acting trivially on the second summand. Consequently, $Z(\lambda)|_{u(\mathfrak{l})} \cong u(\mathfrak{n}^-)_{ad} \otimes_k Z_{\mathfrak{l}}(\lambda) \cong (u(\mathfrak{n}^-)^\dagger \otimes_k Z_{\mathfrak{l}}(\lambda)) \oplus Z_{\mathfrak{l}}(\lambda)$.

(iii). Note that $\dim u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda) = p^{\dim \mathfrak{n}} \cdot \dim Z_{\mathfrak{l}}(\lambda) = p^{\dim \mathfrak{n}} \cdot p^{\dim(\mathfrak{u} \cap \mathfrak{l})} = p^{\dim \mathfrak{u}} = \dim Z(\lambda)$. We consider the map $f : u(\mathfrak{g}) \times Z_{\mathfrak{l}}(\lambda) \longrightarrow Z(\lambda)$ given by $(u, v) \mapsto uv$. In order to show that f is $u(\mathfrak{p})$-balanced, it suffices to check this for elements from $\mathfrak{n}$. Let $x \in \mathfrak{n}$, $a \in u(\mathfrak{l})$, and $u \in u(\mathfrak{g})$. Then $f(ux, a \otimes 1) = uxa \otimes 1 = 0$, since $xa \in u(\mathfrak{n})^\dagger u(\mathfrak{l}) = u(\mathfrak{l}) u(\mathfrak{n})^\dagger$. Hence we have a surjective $u(\mathfrak{g})$-linear map $\psi : u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda) \longrightarrow Z(\lambda)$ such that $\psi(u \otimes (1 \otimes 1)) = u \otimes 1$. For dimension reasons, ψ is bijective. $\square$

Remark. For $\lambda \in X(T)$ let $L(\lambda)$ and $L_{\mathfrak{l}}(\lambda)$ denote the simple $u(\mathfrak{g})$-module and simple $u(\mathfrak{l})$-module of highest weight λ, respectively. Note that $\mathrm{Top}(Z_{\mathfrak{l}}(\lambda)) = L_{\mathfrak{l}}(\lambda)$, cf. [22, (II.3.10(2))]. The $u(\mathfrak{p})$-linear surjection $Z_{\mathfrak{l}}(\lambda) \longrightarrow L_{\mathfrak{l}}(\lambda)$ induces a surjective map $u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda) \longrightarrow u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} L_{\mathfrak{l}}(\lambda)$. By Lemma 1.2(iii) the left-hand module has top $L(\lambda)$. Consequently, $\mathrm{Top}(u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} L_{\mathfrak{l}}(\lambda)) \cong L(\lambda)$.

For $\lambda \in X(T)$ we define the coinduced module

$$Z'(\lambda) := \mathrm{Hom}_{u(\mathfrak{b}^-)}(u(\mathfrak{g}), k_\lambda),$$

cf. [22, (II.9.1)]. If $\mathfrak{h} \subseteq \mathfrak{g}$ is a p-subalgebra, we put

$$Z'_{\mathfrak{h}}(\lambda) := \mathrm{Hom}_{u(\mathfrak{b}^- \cap \mathfrak{h})}(u(\mathfrak{h}), k_{\lambda|_{\mathfrak{b}^- \cap \mathfrak{h}}}).$$

Given a $u(\mathfrak{l})$-module M, we consider the coinduced $u(\mathfrak{g})$-module $\mathrm{Hom}_{u(\mathfrak{p})}(u(\mathfrak{g}), M)$ where $\mathfrak{n}$ is assumed to act trivially.

Let $\mathfrak{p}^- := \mathfrak{n}^- \oplus \mathfrak{l}$ be the parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{p}$. For future reference we record the following counterpart to Lemma 1.2(iii).

Lemma 1.3. *Let $\lambda \in X(T)$. There is an isomorphism $Z'(\lambda) \cong \mathrm{Hom}_{u(\mathfrak{p}^-)}(u(\mathfrak{g}), Z'_{\mathfrak{l}}(\lambda))$.*

Proof. Since $u(\mathfrak{g})$ is a free $u(\mathfrak{p}^-)$-module of rank $p^{\dim \mathfrak{n}}$, we have $\dim \mathrm{Hom}_{u(\mathfrak{p}^-)}(u(\mathfrak{g}), Z'_{\mathfrak{l}}(\lambda)) = p^{\dim \mathfrak{n}} \dim Z'_{\mathfrak{l}}(\lambda) = p^{\dim \mathfrak{n} + \dim(\mathfrak{u} \cap \mathfrak{l})} = \dim Z'(\lambda)$. Direct computation shows that the map

$$\Phi : \mathrm{Hom}_{u(\mathfrak{p}^-)}(u(\mathfrak{g}), Z'_{\mathfrak{l}}(\lambda)) \longrightarrow Z'(\lambda), \quad \text{given by} \quad \Phi(f)(u) := f(u)(1)$$

is well-defined and $u(\mathfrak{g})$-linear. Let $f \in \ker \Phi$. Given $u \in u(\mathfrak{g})$ and $a \in u(\mathfrak{l})$, we have

$$f(u)(a) = [a \cdot f(u)](1) = f(au)(1) = \Phi(f)(au) = 0.$$

Thus $f(u) = 0$ for all $u \in u(\mathfrak{g})$, so that $f = 0$. Hence Φ is injective and therefore bijective. $\square$

We close this section with an application of Lemma 1.3 (cf. [22, (II.2.11)]). For a $u(\mathfrak{g})$-module M and a p-subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ we let $M^{\mathfrak{h}}$ be the set of $\mathfrak{h}$-invariants of M.

Corollary 1.4. *Let $\lambda \in X(T)$. Then we have an isomorphism of $u(\mathfrak{l})$-modules*

$$L(\lambda)^{\mathfrak{n}} \cong L_{\mathfrak{l}}(\lambda).$$

Proof. Let $X \subseteq L(\lambda)^{\mathfrak{n}}$ be a simple $u(\mathfrak{l})$-module and $0 \neq x \in X$ a highest weight vector. Since $\mathfrak{u} = \mathfrak{n} \oplus (\mathfrak{l} \cap \mathfrak{u})$, we see that x in fact is a highest weight vector for $\mathfrak{g}$. If the weight space $L(\lambda)^{\mathfrak{n}}_\eta$ contains a highest weight vector, then $0 \neq \dim \mathrm{Hom}_{u(\mathfrak{l})}(k_\eta, L(\lambda)) = \dim \mathrm{Hom}_{u(\mathfrak{g})}(Z(\eta), L(\lambda)) = \delta_{\eta\lambda}$. Thus, $x = \xi v_\lambda$, where v_λ is a highest weight vector in $L(\lambda)$ and $\xi \in k$. Consequently, $X = u(\mathfrak{l})x = u(\mathfrak{l})v_\lambda$, and thus we have

$$\mathrm{Soc}_{u(\mathfrak{l})}(L(\lambda)^{\mathfrak{n}}) \cong L_{\mathfrak{l}}(\lambda). \tag{2}$$

We write $V(\lambda) := \mathrm{Hom}_{u(\mathfrak{p}^-)}(u(\mathfrak{g}), L_{\mathfrak{l}}(\lambda))$, and note that the canonical restriction map induces an isomorphism

$$V(\lambda) \cong \mathrm{Hom}_k(u(\mathfrak{n}), L_{\mathfrak{l}}(\lambda))$$

of $u(\mathfrak{n})$-modules. Since $u(\mathfrak{n})$ is symmetric, we have $u(\mathfrak{n})$-module isomorphisms

$$V(\lambda) \cong u(\mathfrak{n})^* \otimes_k L_{\mathfrak{l}}(\lambda) \cong u(\mathfrak{n}) \otimes_k L_{\mathfrak{l}}(\lambda)$$

with $u(\mathfrak{n})$ acting trivially on the second factors. As the trivial $u(\mathfrak{n})$-module occurs in $u(\mathfrak{n})$ with multiplicity one, we obtain k-vector space isomorphisms

$$V(\lambda)^{\mathfrak{n}} \cong u(\mathfrak{n})^{\mathfrak{n}} \otimes_k L_{\mathfrak{l}}(\lambda) \cong L_{\mathfrak{l}}(\lambda). \tag{3}$$

Since $\mathrm{Soc}_{u(\mathfrak{g})}(Z'(\lambda)) \cong L(\lambda)$ and $\mathrm{Soc}_{u(\mathfrak{l})}(Z'_{\mathfrak{l}}(\lambda)) \cong L_{\mathfrak{l}}(\lambda)$ [22, (II.3.9(4))], it follows from Lemma 1.3 that there is an embedding $L(\lambda) \hookrightarrow V(\lambda)$, so that $L(\lambda)^{\mathfrak{n}} \hookrightarrow V(\lambda)^{\mathfrak{n}}$. By (2) we have injections

$$L_{\mathfrak{l}}(\lambda) \hookrightarrow L(\lambda)^{\mathfrak{n}} \hookrightarrow V(\lambda)^{\mathfrak{n}}.$$

Finally, by (3), each of these inclusions is an isomorphism of $u(\mathfrak{l})$-modules, and the desired result follows. □

2. Rank Varieties

Let $(\mathfrak{g}, [p])$ be a reductive Lie algebra. Following Friedlander-Parshall [15] we associate to a $u(\mathfrak{g})$-module M its *rank variety*

$$\mathcal{V}_{\mathfrak{g}}(M) := \{x \in \mathfrak{g} \mid x^{[p]} = 0 \text{ and } M|_{u(kx)} \text{ is not projective}\} \cup \{0\}.$$

The reader is referred to [14, 15, 16] concerning basic properties of rank varieties. We let $\varrho := \frac{1}{2}\sum_{\alpha\in\Psi^+}\alpha$ be the half-sum of the positive roots and recall that $\varrho(\sigma^\vee) = 1$ for every simple root σ. The Verma module $Z((p-1)\varrho)$ is called the *Steinberg module*. By the *linkage principle* [19, (3.1)] and [24, Thm.1] the Steinberg module is the only projective Verma module in case G semisimple. We employ this fact repeatedly in the sequel.

Let $\lambda \in X(T)$ and $\mathfrak{h} \subset \mathfrak{g}$ a p-subalgebra. Since the module $Z_{\mathfrak{h}}(\lambda)$ is $(u(\mathfrak{h}), u(\mathfrak{b}\cap\mathfrak{h}))$-projective, we have $\mathcal{V}_{\mathfrak{h}}(Z_{\mathfrak{h}}(\lambda)) = \mathcal{V}_{\mathfrak{h}\cap\mathfrak{b}}(Z_{\mathfrak{h}}(\lambda))$ (cf. [10, (3.4)]). The latter set is readily seen to be contained in $\mathcal{V}_{\mathfrak{u}\cap\mathfrak{h}}(k)$. As twisting the module structure of $Z(\lambda)$ by an automorphism $\mathrm{Ad}(b)$, for b in B, yields an isomorphic module, we have

Lemma 2.1. *Let $\lambda \in X(T)$. Then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda))$ is a conical B-invariant subvariety of $\mathfrak{u}$.* □

Let $\{h_\sigma, e_\alpha \mid \sigma \in \Sigma, \alpha \in \Psi\}$ be a *Chevalley basis* of $\mathfrak{g}$ such that $\mathfrak{g}_\beta = ke_\beta$ for each $\beta \in \Psi$.

Lemma 2.2. *Let G be reductive and $\sigma \in \Sigma$. Then e_σ is an element of $\mathcal{V}_{\mathfrak{g}}(Z(\lambda))$ precisely when $\langle\lambda, \sigma^\vee\rangle \not\equiv -1 \mod p$.*

Proof. Letting $\Pi = \{\sigma\}$, we observe that $\mathfrak{l} \cong \mathfrak{sl}(2) \oplus \mathfrak{z}(\mathfrak{l})$ is the direct sum of a copy of $\mathfrak{sl}(2)$ with the center of $\mathfrak{l}$. Note that the latter is a torus. From Lemma 1.2(i) we obtain

$$Z(\lambda)|_{u(\mathfrak{sl}(2))} \cong (u(\mathfrak{n}^-)_{ad})|_{u(\mathfrak{sl}(2))} \otimes_k Z_{\mathfrak{l}}(\lambda)|_{u(\mathfrak{sl}(2))}.$$

Thanks to Lemma 1.1(i) and Lemma 1.2(ii) the second factor is a Verma module for $u(\mathfrak{sl}(2))$ as well as a direct summand of $Z(\lambda)|_{u(\mathfrak{sl}(2))}$. In view of [14, (2.1)] it follows that $\mathcal{V}_{\mathfrak{sl}(2)}(Z_{\mathfrak{l}}(\lambda)) = \mathcal{V}_{\mathfrak{sl}(2)}(Z(\lambda))$. Consequently, e_σ lies in $\mathcal{V}_{\mathfrak{g}}(Z(\lambda))$ if and only if $\langle\lambda, \sigma^\vee\rangle \not\equiv -1 \mod p$. □

Let W_p be the affine Weyl group associated to $\mathfrak{g}$. The "dot" action of $w \in W_p$ on $X(T)$ is defined as follows:

$$w.\lambda := w(\lambda + \varrho) - \varrho.$$

For $\lambda \in X(T)$ we put

$$\Psi_\lambda := \{\alpha \in \Psi \mid \langle\lambda + \varrho, \alpha^\vee\rangle \in p\mathbb{Z}\}.$$

We say that λ is *p-regular* provided $\Psi_\lambda = \varnothing$. Else Ψ_λ is a semisimple subsystem of Ψ [21, 2.7].

Denote by $\mathcal{C}$ the category of finite dimensional G_1T-modules. For $\mu \in X(T)$ let $L(\mu)$ be the simple G_1T-module of highest weight μ. For each $\lambda \in X(T)$ let $\mathcal{C}_\lambda$ be the full subcategory of $\mathcal{C}$ consisting of those objects whose composition factors $L(\mu)$ satisfy $\mu \in W_p.\lambda$.

Given two elements $\lambda, \mu \in X(T)$ belonging to the standard alcove, we let T_μ^λ be the *translation functor* from $\mathcal{C}_\mu$ to $\mathcal{C}_\lambda$ [22, (II.9.19)]. It follows from the definition of T_μ^λ and [14, (2.1)] that

$$\mathcal{V}_{\mathfrak{g}}(T_\mu^\lambda M) \subseteq \mathcal{V}_{\mathfrak{g}}(M)$$

for any module M in $\mathcal{C}_\mu$.

Throughout the remainder of this section we let h be the Coxeter number of G.

Lemma 2.3. *Let $\lambda \in X(T)$ and suppose that $p \geq h$. Then λ is p-regular if and only if*

$$\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{u}.$$

Proof. If λ is not p-regular, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda))$ does not contain regular nilpotent elements [21, Lemma 2.3] and, by Lemma 2.1, is thus properly contained in $\mathfrak{u}$.

In case $\lambda = 0$ we observe that the counit $\varepsilon : u(\mathfrak{g}) \longrightarrow k$ induces a $u(\mathfrak{g})$-linear map

$$\zeta : Z(0) \longrightarrow k \quad ; \quad u \otimes \xi \mapsto \varepsilon(u)\xi.$$

Since $\xi \mapsto 1 \otimes \xi$ is a $u(\mathfrak{b})$-linear splitting of ζ, it follows that k is a direct summand of $Z(0)|_{u(\mathfrak{u})}$. As $p \geq h$, an application of [14, (2.1)] yields $\mathfrak{u} = \mathcal{V}_{\mathfrak{u}}(k) \subseteq \mathcal{V}_{\mathfrak{u}}(Z(0)) = \mathcal{V}_{\mathfrak{g}}(Z(0))$, as desired.

Since any weight λ in the standard alcove C_0 is p-regular and belongs to the same facet as 0, [23, (11.21)] shows that T_0^λ is an equivalence of categories between $\mathcal{C}_0$ and $\mathcal{C}_\lambda$ sending $Z(w.0)$ to $Z(w.\lambda)$ for any $w \in W_p$. Consequently, the associated rank varieties are equal, and $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{u}$ for every $\lambda \in C_0$. Since W_p permutes the alcoves transitively (cf. [22, (II.6.2)]) we also have $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathcal{V}_{\mathfrak{g}}(Z(\gamma))$, whenever λ and γ are p-regular elements belonging to the same alcove.

Let C be an alcove such that $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{u}$ for some (and hence every) p-regular element $\lambda \in C$. We choose an alcove D adjacent to C and let s be the unique affine reflection relative to the adjoining wall $\overline{F} = \overline{C} \cap \overline{D}$ (facet of codimension 1). Note that $\eta := s.\lambda \in D$ is a p-regular element. Next, we choose an element $w \in W_p$ such that $w.C = C_0$. Then $w.D$ is adjacent to C_0 with adjoining wall $w.F$ and associated reflection $s' = wsw^{-1}$. Observe that $w^{-1}s'(w.\lambda) = s.\lambda = \eta$. Now let $\mu \in w.F$ be a weight on the wall $w.F$. Suppose that $\lambda < s.\lambda$. An application of [22, (II.9.19(3))] provides an exact sequence

$$(0) \longrightarrow Z(\lambda) \longrightarrow T_\mu^{w.\lambda}(Z(w^{-1}.\mu)) \longrightarrow Z(s.\lambda) \longrightarrow (0),$$

so that

$$\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \subseteq \mathcal{V}_{\mathfrak{g}}(T_\mu^{w.\lambda}(Z(w^{-1}.\mu))) \cup \mathcal{V}_{\mathfrak{g}}(Z(s.\lambda)) \subseteq \mathcal{V}_{\mathfrak{g}}(Z(w^{-1}.\mu)) \cup \mathcal{V}_{\mathfrak{g}}(Z(s.\lambda)).$$

Let x in $\mathfrak{u}$ be a regular element. Then $x \in \mathcal{V}_{\mathfrak{g}}(Z(\lambda))$ and by [21, Lemma 2.3] we see that x is not in $\mathcal{V}_{\mathfrak{g}}(Z(w^{-1}.\mu))$. Consequently, $x \in \mathcal{V}_{\mathfrak{g}}(Z(s.\lambda))$, and $\mathfrak{u} = \overline{B \cdot x} \subseteq \mathcal{V}_{\mathfrak{g}}(Z(s.\lambda)) = \mathcal{V}_{\mathfrak{g}}(Z(\eta))$. Therefore, $\mathcal{V}_{\mathfrak{g}}(Z(\gamma)) = \mathfrak{u}$ for any p-regular element $\gamma \in D$. The case $\lambda > s.\lambda$ can be treated similarly.

Since W_p acts transitively on the set of alcoves, we are done. □

For $i = 1, \ldots, r := \operatorname{rank} G$ we consider the parabolic subgroups $P_i := P_{\{\sigma_i\}}$ of G of semisimple rank 1. The Lie algebra of the unipotent radical of P_i is denoted $\mathfrak{n}_i$.

Corollary 2.4. *If $\lambda \in X(T)$ is not p-regular and $p \geq h$, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \subseteq \bigcup_{i=1}^r \mathfrak{n}_i$.*

Proof. For $x \in \mathfrak{g}$ let $C_G(x)$ denote the centralizer of x in G; likewise for subgroups of G. If x is an element of $\mathfrak{n}_i$, then $\dim C_{P_i}(x) \geq r+2$; whence $\dim C_G(x) \geq r+2$, i.e., x is not regular. If $x \in \mathfrak{u}$ is not in $\cup_i \mathfrak{n}_i$, then x is of the form $x = \sum c_\alpha x_\alpha$ where $c_\alpha \neq 0$ for each simple root α. But then x is regular nilpotent by [28, (5.8)]. Thus, $\cup_i \mathfrak{n}_i$ is precisely the set of all non-regular elements of $\mathfrak{u}$. By Lemma 2.3 the variety $\mathcal{V}_{\mathfrak{g}}(Z(\lambda))$ is contained in this set. □

For $\lambda \in X(T)$ we define $\Sigma_\lambda := \Sigma \cap \Psi_\lambda$. Let ϕ be an automorphism of $u(\mathfrak{g})$ and M a $u(\mathfrak{g})$-module. We define the *twisted module structure on M by ϕ* by $u \cdot m := \phi^{-1}(u)m$ for $m \in M$ and $u \in u(\mathfrak{g})$, and denote this module by $M^{(\phi)}$. For $g \in G$ we write $M^{(g)}$ instead of $M^{(\mathrm{Ad}(g))}$. We close this section with a refinement of Lemma 2.1.

Lemma 2.5. *Let $\lambda \in X(T)$ and set $P := P_{\Sigma_\lambda}$. Then we have*

(i) *$\mathcal{V}_{\mathfrak{g}}(Z(\lambda))$ is P-invariant;*

(ii) $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \subseteq \operatorname{Lie} P_u =: \mathfrak{n}$.

Proof. Let $P = LP_u$ be the standard Levi decomposition of P.

(i). By construction, $Z_{\mathfrak{l}}(\lambda)$ is the Steinberg module for $u(\mathfrak{l})$. Thus it is, as a simple $u(\mathfrak{l})$-module, also a simple $u(\mathfrak{p})$-module. Since G is connected, so is P ([4, (11.16)], [29, (7.3.8)]). It follows that $Z_{\mathfrak{l}}(\lambda)^{(g)} \cong Z_{\mathfrak{l}}(\lambda)$ for any $g \in P$, cf. [22, (II.3.11)].

Observe that the canonical map $u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda) \longrightarrow u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda)^{(g)}$ given by $u \otimes m \mapsto \operatorname{Ad}(g)(u) \otimes m$ induces an isomorphism $(u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda))^{(g)} \cong u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda)^{(g)}$. According to Lemma 1.2(iii) we have

$$Z(\lambda)^{(g)} \cong (u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda))^{(g)} \cong u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda)^{(g)} \cong u(\mathfrak{g}) \otimes_{u(\mathfrak{p})} Z_{\mathfrak{l}}(\lambda) \cong Z(\lambda)$$

for $g \in P$. This implies the P-invariance of $\mathcal{V}_{\mathfrak{g}}(Z(\lambda))$.

(ii). By Lemma 2.1 we have $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \subseteq \mathfrak{u} = (\mathfrak{l} \cap \mathfrak{u}) \oplus \mathfrak{n}$. Suppose $x + n \in \mathcal{V}_{\mathfrak{g}}(Z(\lambda))$, where $x \in \mathfrak{l} \cap \mathfrak{u}$ and $n \in \mathfrak{n}$. Let w_0^L be the longest element of the Weyl group W_L of L. By part (i) we have $\mathfrak{u} \supseteq \mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \ni y := w_0^L x + w_0^L n$. Since $x \in \mathfrak{l} \cap \mathfrak{u}$, we get $w_0^L x \in \mathfrak{l} \cap \mathfrak{u}^-$. Thus, since $\mathfrak{n}$ is a P-invariant subspace of $\mathfrak{u}$, we conclude $w_0^L x = y - w_0^L n \in \mathfrak{u}^- \cap \mathfrak{u} = \{0\}$. Hence, $x = 0$, and $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \subseteq \mathfrak{n}$. □

3. Rank Varieties of Dimension two

We start with the computation of the rank varieties of Verma modules for certain simple Lie algebras of rank 2.

Let λ be in $X(T)$ and $\{\sigma_1, \sigma_2\} = \Sigma$; in case of two root lengths, σ_1 is assumed to be long. The unique highest root is denoted $\tilde{\alpha}$. We put $\Psi_\lambda^+ := \Psi_\lambda \cap \Psi^+$, and let $e_{\sigma_1}, e_{\sigma_2}$ be the standard generators of the corresponding root spaces.

Lemma 3.1. *Suppose that $p \geq 3$. Let G be of type A_2 and $\lambda \in X(T)$.*

(i) *If λ is p-regular, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{u}$.*
(ii) *If $\Psi_\lambda^+ = \{\sigma_1\}$, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{n}_1$.*
(iii) *If $\Psi_\lambda^+ = \{\sigma_2\}$, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{n}_2$.*
(iv) *If $\Psi_\lambda^+ = \{\sigma_1 + \sigma_2\}$, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{n}_1 \cup \mathfrak{n}_2$.*
(v) *If $\Psi_\lambda^+ = \Psi^+$, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \{0\}$.*

Proof. The statement of part (i) follows from Lemma 2.3. If $\Psi_\lambda^+ = \Psi^+$, then the $u(\mathfrak{g})$-module $Z(\lambda)$ is isomorphic to the Steinberg module, so that (v) holds. In each of the other instances Corollary 2.4 yields $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \subseteq \mathfrak{n}_1 \cup \mathfrak{n}_2$.

Comparing dimensions, we obtain $\mathfrak{n}_1 = \overline{B \cdot e_{\sigma_2}}$ and likewise $\mathfrak{n}_2 = \overline{B \cdot e_{\sigma_1}}$. Consequently, in case (iv) we immediately derive the desired equality from Lemmas 2.1 and 2.2. By the same token, in case (ii) we have $\mathfrak{n}_1 \subseteq \mathcal{V}_{\mathfrak{g}}(Z(\lambda))$ and Lemma 2.5(ii) implies $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \subseteq \mathfrak{n}_1$. Thus, we have $\mathfrak{n}_1 = \mathcal{V}_{\mathfrak{g}}(Z(\lambda))$. By symmetry, the equality in (iii) also follows. □

Lemma 3.1(iv) illustrates that the rank variety of a Verma module, although projectively connected, need not be irreducible.

Remark. Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(3)$ with simple roots $\{\sigma_1, \sigma_2\}$. Suppose $\lambda \in X(T)$ satisfies $0 \not\equiv \langle \lambda, \sigma_1^\vee \rangle \not\equiv \langle \lambda, \sigma_2{}^\vee \rangle \equiv -1 \mod p$. Then, by Lemma 3.1(iii), we have $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{g}_{\sigma_1} \oplus \mathfrak{g}_{\tilde{\alpha}}$; thus $e := e_{\sigma_1} + e_{\sigma_2} \notin \mathcal{V}_{\mathfrak{g}}(Z(\lambda))$. We consider the subalgebra $\mathfrak{s} \cong \mathfrak{sl}(2)$ of $\mathfrak{g}$ associated to e. Since $\langle \lambda, (\sigma_1 + \sigma_2)^\vee \rangle \not\equiv -1 \mod p$ implies that $e \in \mathcal{V}_{\mathfrak{s}}(Z_{\mathfrak{s}}(\lambda))$, it follows that $Z_{\mathfrak{s}}(\lambda) \subseteq Z(\lambda)$ is not a direct summand of $Z(\lambda)|_{u(\mathfrak{s})}$ (cf. Lemma 1.2).

Next we consider B_2; recall that σ_1 is the long simple root.

Lemma 3.2. *Suppose that $p \geq 5$. Let G be of type B_2 and $\lambda \in X(T)$.*

(i) *If λ is p-regular, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{u}$.*
(ii) *If $\Psi^+_\lambda = \{\sigma_1\}$, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{n}_1$.*
(iii) *If $\Psi^+_\lambda = \{\sigma_2\}$, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \mathfrak{n}_2$.*
(iv) *If $\emptyset \neq \Psi^+_\lambda \subset \Psi^+ \setminus \Sigma$, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \supset \mathfrak{n}_1$.*
(v) *If $\Psi^+_\lambda = \Psi^+$, then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = \{0\}$.*

Proof. The proofs of parts (i) and (v) are analogous to those of the corresponding statements in Lemma 3.1. In each of the other instances Corollary 2.4 yields $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \subseteq \mathfrak{n}_1 \cup \mathfrak{n}_2$.

We prove (ii) and (iii) simultaneously. So we have $\Psi^+_\lambda = \{\sigma_i\}$ for $i = 1, 2$. Since $\dim B \cdot e_{\sigma_2} = 3$ (cf. [5, p. 31]), we have $\mathfrak{n}_1 = \overline{B \cdot e_{\sigma_2}}$, and by Lemmas 2.1 and 2.2 this variety is contained in the rank variety of $Z(\lambda)$ for $i = 1$. If $i = 2$, then we have $\mathfrak{n}_2 \subseteq \mathcal{V}_{\mathfrak{g}}(Z(\lambda))$ by Proposition 7.3 below. The containment $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \subseteq \mathfrak{n}_i$ follows in each case directly from Lemma 2.5(ii). This yields the desired equalities.

Finally, for λ as in part (iv) we get $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \supseteq \mathfrak{n}_1 = \overline{B \cdot e_{\sigma_2}}$ by Lemma 2.2. □

Lemma 3.3. *Suppose that $p \geq 5$. Let G be reductive and σ a simple root of G so that $\dim B \cdot e_\sigma \leq 2$. Then G admits a factorization as an almost direct product $G = SH$, such that*

(i) *S is an almost simple normal subgroup of G of type A_1, A_2, or B_2, and*
(ii) *$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$, and $e_\sigma \in \mathfrak{s}$.*

Proof. First we suppose that G is almost simple. Consider the orbit map $B \to B \cdot e_\sigma$ given by $b \mapsto \mathrm{Ad}(b)(e_\sigma)$; its differential $\mathfrak{b} \to \mathrm{T}_{e_\sigma}(B \cdot e_\sigma)$ is given by $x \mapsto [x, e_\sigma]$. Thus, the image of this map,

$$\mathrm{ad}(e_\sigma)\mathfrak{b} = ke_\sigma \oplus \mathrm{ad}(e_\sigma)\mathfrak{u}$$

is at most two-dimensional. If $\dim B \cdot e_\sigma = 1$, then $\mathrm{ad}(e_\sigma)\mathfrak{u} = \{0\}$, and there is no other simple root in Ψ (since the Dynkin diagram of G is connected). In this instance G is of type A_1. Next suppose that $\dim B \cdot e_\sigma = 2$; thus $\dim \mathrm{ad}(e_\sigma)\mathfrak{u} = 1$. Therefore, there is precisely one other simple root, say τ, which is not orthogonal to σ, and $\pm e_{\sigma+\tau} = [e_\sigma, e_\tau]$ lies in $\mathrm{ad}(e_\sigma)\mathfrak{u}$. Suppose that $\mathrm{rank}\, G \geq 3$, and say $\epsilon \neq \sigma$ is another simple root linked to τ in the Dynkin diagram of G. But then $[e_\sigma, e_{\tau+\epsilon}] = \pm e_{\sigma+\tau+\epsilon}$. Consequently, $\dim \mathrm{ad}(e_\sigma)\mathfrak{u} \geq 2$, a contradiction. This implies that $\mathrm{rank}\, G = 2$. For G_2 we have $\dim B \cdot e_\sigma = 3$ or 4 from [5, pp. 31], so that this group does not occur.

In the general case we note that σ belongs to a simple component S of G, and e_σ belongs to the Lie algebra $\mathfrak{s}$ of S. By general theory, $B \cap S$ is a Borel subgroup of S. By the first part we know that S is of type A_1, A_2, or B_2. Since $p \geq 5$, the Killing form of $\mathfrak{s}$ is non-degenerate in all cases. Thanks to a classical result due to Zassenhaus [30, (I.7.9)] the p-ideal $\mathfrak{s}$ is *complete* (i.e., without center and outer derivations). Hence $\mathfrak{g}$ decomposes into a direct sum

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}),$$

where the second summand denotes the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$. Let H be the almost direct product of the center of G and the remaining simple components. Since $H \subseteq C_G(S)$, we have $\mathfrak{h} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$. Note that $\dim G = \dim S + \dim H$ and $\dim \mathfrak{g} = \dim \mathfrak{s} + \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$. Therefore, $\dim \mathfrak{h} = \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$; consequently, these two subalgebras coincide, and the desired result follows. □

We denote the *stable Auslander-Reiten quiver* of $u(\mathfrak{g})$ by $\Gamma_s(\mathfrak{g})$. The general theory of these quivers may be found in [2]. An AR-component is referred to as *regular* if it is not attached to a principal indecomposable $u(\mathfrak{g})$-module. Let $\Theta \subseteq \Gamma_s(\mathfrak{g})$ be an Auslander-Reiten component. Then we have

$$\mathcal{V}_{\mathfrak{g}}(M) = \mathcal{V}_{\mathfrak{g}}(N) \quad \text{for all isoclasses } [M] \text{ and } [N] \text{ in } \Theta,$$

cf. [9, (5.2)], so that we can speak of the variety $\mathcal{V}_{\mathfrak{g}}(\Theta)$ of the component Θ. Since Θ only contains isoclasses of non-projective modules, we have $\mathcal{V}_{\mathfrak{g}}(\Theta) \neq \{0\}$. Given $\lambda \in X(T)$, we let $\Theta(\lambda) \subseteq \Gamma_s(\mathfrak{g})$ be the Auslander-Reiten component containing the Verma module $Z(\lambda)$.

Proposition 3.4. *Let $\lambda \in X(T)$. Then the following statements hold:*

(i) *The component $\Theta(\lambda)$ is regular.*
(ii) *The isoclass $[Z(\lambda)]$ is a vertex of $\Theta(\lambda)$ of minimal dimension.*

Proof. (1). Suppose that $\Theta(\lambda)$ is not regular. Then, by definition, there exists a principal indecomposable $u(\mathfrak{g})$-module P such that the extreme terms of its standard AR-sequence

$$(0) \longrightarrow \operatorname{Rad}(P) \longrightarrow P \oplus H(P) \longrightarrow P/\operatorname{Soc}(P) \longrightarrow (0)$$

define vertices of $\Theta(\lambda)$.

As G is connected, the module P, and thereby $\operatorname{Rad}(P)$, is isomorphic to all its twists by elements of G. Consequently, $\mathcal{V}_{\mathfrak{g}}(\Theta(\lambda))$ is invariant under the adjoint representation of G. On the other hand, $\mathcal{V}_{\mathfrak{g}}(\Theta(\lambda)) = \mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \subseteq \mathfrak{u}$. By the foregoing observations we have $\mathcal{V}_{\mathfrak{g}}(\Theta(\lambda)) \subseteq \mathfrak{u} \cap \mathfrak{u}^- = \{0\}$, a contradiction. Thus, $\Theta(\lambda)$ is regular.

(2). We write $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{b}$. As $\mathcal{V}_{\mathfrak{g}}(\Theta(\lambda)) \subseteq \mathfrak{u}$, every module M belonging to $\Theta(\lambda)$ is $u(\mathfrak{u}^-)$-projective. Since $u(\mathfrak{u}^-)$ is local, there exists $q_M > 0$ so that $\dim M = q_M \dim u(\mathfrak{u}^-) = q_M \dim Z(\lambda)$. Consequently, $[Z(\lambda)]$ is a vertex of minimal dimension. □

Since the structure of those AR-components of $\mathfrak{g}$ whose support variety is not of dimension 2 is well-understood [7, 9, 11], we provide in our next result a reduction for the remaining case.

Theorem 3.5. *Suppose $p \geq 5$. Let $\lambda \in X(T)$ so that $\dim \mathcal{V}_{\mathfrak{g}}(\Theta(\lambda)) = 2$. Then $\mathfrak{g}$ is the direct sum of p-ideals $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ such that*

(i) *$\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ or $\mathfrak{sl}(3)$, and*
(ii) *there is a quiver isomorphism $\Theta_{\mathfrak{s}}(\lambda) \cong \Theta(\lambda)$, originating in the component $\Theta_{\mathfrak{s}}(\lambda)$ of $\Gamma_s(\mathfrak{s})$ containing the Verma module $Z_{\mathfrak{s}}(\lambda)$. Moreover, this isomorphism maps $Z_{\mathfrak{s}}(\lambda)$ to $Z(\lambda)$.*

Proof. Since $\mathcal{V}_{\mathfrak{g}}(\Theta(\lambda)) \neq \{0\}$, it follows from Lemmas 2.1 and 2.2 that there is a simple root σ of G such that $B \cdot e_\sigma$ lies in this rank variety and thus $\dim B \cdot e_\sigma \leq 2$. Therefore, by Lemma 3.3 the algebra $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{s}' \oplus \mathfrak{k}$$

into p-ideals corresponding to almost direct factors S' and K of G, with S' being of type A_1, A_2, or B_2. Note that S' and K are reductive subgroups of G. Thanks to Lemma 1.1 we may write $Z(\lambda)$ as an outer tensor product $Z(\lambda) = Z_{\mathfrak{s}'}(\lambda) \otimes_k Z_{\mathfrak{k}}(\lambda)$ of Verma modules. From [14, (2.1)] we obtain

$$\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \cong \mathcal{V}_{\mathfrak{s}'}(Z_{\mathfrak{s}'}(\lambda)) \times \mathcal{V}_{\mathfrak{k}}(Z_{\mathfrak{k}}(\lambda)).$$

Since e_σ is in $\mathcal{V}_{\mathfrak{s}'}(Z_{\mathfrak{s}'}(\lambda))$ and $\dim \mathcal{V}_{\mathfrak{g}}(Z(\lambda)) = 2$, either the dimensions of the two factors equal 1 or else $\dim \mathcal{V}_{\mathfrak{s}'}(Z(\lambda)) = 2$ and $\dim \mathcal{V}_{\mathfrak{k}}(Z(\lambda)) = 0$. In the latter case $Z_{\mathfrak{k}}(\lambda)$ is projective, and Lemma 3.2 ensures that S' is not of type B_2.

In the former instance we have $\mathfrak{s}' \cong \mathfrak{sl}(2)$. We now apply Lemma 3.3 to the reductive group K and the Verma module $Z_{\mathfrak{k}}(\lambda)$. Accordingly, we may decompose $\mathfrak{k}$ as

$$\mathfrak{k} = \mathfrak{s}'' \oplus \mathfrak{k}',$$

where $\mathfrak{s}''$ is the Lie algebra of an almost simple subgroup S'' of G of type A_1. As before, we have $Z_{\mathfrak{k}}(\lambda) \cong Z_{\mathfrak{s}''}(\lambda) \otimes_k Z_{\mathfrak{k}'}(\lambda)$, where the second factor is again projective. By applying Lemma 1.1 to $\mathfrak{s}' \oplus \mathfrak{s}''$, we have

$$Z(\lambda) \cong Z_{\mathfrak{s}' \oplus \mathfrak{s}''}(\lambda) \otimes_k Z_{\mathfrak{k}'}(\lambda).$$

Now setting $\mathfrak{s} := \mathfrak{s}' \cong \mathfrak{sl}(3)$ in the first instance and $\mathfrak{s} := \mathfrak{s}' \oplus \mathfrak{s}'' \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ in the second, and $\mathfrak{q} = \mathfrak{k}$ in the first instance and $\mathfrak{q} = \mathfrak{k}'$ in the second, we obtain a decomposition

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{q},$$

with both summands being reductive Lie algebras, such that the second factor of the tensor product

$$Z(\lambda) \cong Z_{\mathfrak{s}}(\lambda) \otimes_k Z_{\mathfrak{q}}(\lambda)$$

is projective. Thus, by Lemma 1.1(i), the restriction of $Z_{\mathfrak{q}}(\lambda)$ to a suitable p-subalgebra of $\mathfrak{q}$ is a Steinberg module. Hence $Z_{\mathfrak{q}}(\lambda)$ is simple and projective.

Now let $\mathcal{B}$ be the block of $u(\mathfrak{g})$ containing $Z(\lambda)$. Since $Z(\lambda) \cong Z_{\mathfrak{s}}(\lambda) \otimes_k Z_{\mathfrak{q}}(\lambda)$, it follows that $\mathcal{B} \cong \mathcal{B}_{\mathfrak{s}} \otimes_k \mathcal{B}_{\mathfrak{q}}$, where $\mathcal{B}_X \subset u(X)$ is the block containing $Z_X(\lambda)$, for $X \in \{\mathfrak{s}, \mathfrak{q}\}$. In particular, the block $\mathcal{B}_{\mathfrak{q}}$ is simple, and the functor

$$\mathrm{mod}_{\mathcal{B}_{\mathfrak{s}}} \longrightarrow \mathrm{mod}_{\mathcal{B}} \quad ; \quad M \mapsto M \otimes_k Z_{\mathfrak{q}}(\lambda)$$

sends $Z_{\mathfrak{s}}(\lambda)$ to $Z(\lambda)$. To see that this functor is an equivalence, we consider

$$\mathrm{mod}_{\mathcal{B}} \longrightarrow \mathrm{mod}_{\mathcal{B}_{\mathfrak{s}}} \;\; ; \;\; N \mapsto \mathrm{Hom}_{\mathcal{B}_{\mathfrak{q}}}(Z_{\mathfrak{q}}(\lambda), N).$$

Let N be a $\mathcal{B}$-module. Since $\mathcal{B}_{\mathfrak{q}}$ is simple, $N|_{\mathcal{B}_{\mathfrak{q}}}$ is a direct sum of copies of $Z_{\mathfrak{q}}(\lambda)$. Using this fact direct computation shows that

$$\varrho_N : \mathrm{Hom}_{\mathcal{B}_{\mathfrak{q}}}(Z_{\mathfrak{q}}(\lambda), N) \otimes_k Z_{\mathfrak{q}}(\lambda) \longrightarrow N \quad \text{given by} \quad \varrho_N(f \otimes z) = f(z)$$

is an isomorphism. Thus, the ϱ_N define a natural equivalence. Since the maps

$$\nu_M : M \longrightarrow \mathrm{Hom}_{\mathcal{B}_{\mathfrak{q}}}(Z_{\mathfrak{q}}(\lambda), M \otimes_k Z_{\mathfrak{q}}(\lambda)) \quad \text{given by} \quad m \mapsto (z \mapsto m \otimes z)$$

are also bijective, it follows that both functors are equivalences. $\square$

4. Subsidiary results on Auslander-Reiten Components

4.1. **A Remark on Subadditive Functions.** Let $(\mathfrak{g}, [p])$ be an arbitrary finite-dimensional restricted Lie algebra over k. We fix a p-subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and decompose the restriction of a $u(\mathfrak{g})$-module M into its projective and non-projective parts:

$$M|_{u(\mathfrak{h})} = M_{\mathfrak{h}} \oplus (\mathrm{proj}),$$

where $M_{\mathfrak{h}}$ does not have any projective direct summands. According to the theorem of Krull-Remak-Schmidt $M_{\mathfrak{h}}$ is unique up to isomorphism.

In the sequel, we let $\Gamma_s(\mathfrak{g})$ and $\Omega := \Omega_{u(\mathfrak{g})}$ be the stable AR-quiver and the *Heller operator* of $u(\mathfrak{g})$, respectively. We refer the reader to [3] concerning the definition and basic properties of Ω. The Heller operator is linked to the *Auslander-Reiten translation* τ of $\Gamma_s(\mathfrak{g})$ via $\tau(M) = \Omega^2(M^{(1)})$, where the superscript indicates the twisting of the module M by the inverse of the Nakayama automorphism of $u(\mathfrak{g})$, i.e.,

$$u \cdot m := \mu^{-1}(u)m \quad \text{for all } m \in M^{(1)}, u \in u(\mathfrak{g}).$$

Lemma 4.1. *Let* $\Theta \subset \Gamma_s(\mathfrak{g})$ *be a component such that* $\dim(\mathcal{V}_{\mathfrak{g}}(\Theta) \cap \mathfrak{h}) = 1$. *Then the map*

$$d : \Theta \longrightarrow \mathbb{N}; \quad [X] \mapsto \dim X_{\mathfrak{h}}$$

is subadditive and $d \circ \tau = d$.

Proof. By assumption $X|_{u(\mathfrak{h})}$ is not projective, whenever $[X] \in \Theta$. Consequently, $\dim X_{\mathfrak{h}} \neq 0$ for all $[X] \in \Theta$. For $[X] \in \Theta$ we consider the almost split sequence

$$(0) \longrightarrow Y \xrightarrow{\iota} E \xrightarrow{\pi} X \longrightarrow (0)$$

terminating in X. To verify subadditivity, we have to show that

$$\dim E_{\mathfrak{h}} \leq \dim Y_{\mathfrak{h}} + \dim X_{\mathfrak{h}}.$$

We write $Y = Y_{\mathfrak{h}} \oplus P_Y$. Since P_Y is injective, $\iota(P_Y) \cong P_Y$ is a direct summand of the $u(\mathfrak{h})$-module E. Hence there exists a $u(\mathfrak{h})$-module E_1 such that $E = E_1 \oplus \iota(P_Y)$. It follows that $Y = \iota^{-1}(E_1 \cap \iota(Y)) \oplus P_Y$. Consequently, $Y_{\mathfrak{h}} \cong \iota^{-1}(E_1 \cap \iota(Y))$, so that $\dim Y_{\mathfrak{h}} = \dim(E_1 \cap \iota(Y))$. Since $X = \pi(E) = \pi(E_1 \oplus \iota(P_Y)) = \pi(E_1)$ we have an exact sequence

$$(0) \longrightarrow E_1 \cap \iota(Y) \longrightarrow E_1 \xrightarrow{\pi|_{E_1}} X \longrightarrow (0)$$

of $u(\mathfrak{h})$-modules. As E_1 surjects onto the projective summand P_X of X, there exists a submodule $E_2 \subset E_1$ such that $E_1 \cong E_2 \oplus P_X$. It follows that

$$E \cong E_1 \oplus P_Y \cong E_2 \oplus P_X \oplus P_Y,$$

so that $\dim E_2 \geq \dim E_{\mathfrak{h}}$. Consequently,

$$\begin{aligned} \dim E_{\mathfrak{h}} &\leq \dim E_2 = \dim E - \dim P_X - \dim P_Y \\ &= \dim X - \dim P_X + \dim Y - \dim P_Y = \dim X_{\mathfrak{h}} + \dim Y_{\mathfrak{h}}. \end{aligned}$$

As a result, the function d is subadditive.

Let $[X] \in \Theta$ be a vertex. By assumption we have $\dim \mathcal{V}_{\mathfrak{h}}(X_{\mathfrak{h}}) = 1$. Owing to [9, (2.5)], this implies that $X_{\mathfrak{h}} \cong \Omega^2_{u(\mathfrak{h})}(X_{\mathfrak{h}})$. Using standard properties of the Heller operator, we thus obtain

$$\Omega^2_{u(\mathfrak{g})}(X)|_{u(\mathfrak{h})} \cong \Omega^2_{u(\mathfrak{h})}(X) \oplus (\text{proj}) \cong \Omega^2_{u(\mathfrak{h})}(X_{\mathfrak{h}}) \oplus (\text{proj}) \cong X_{\mathfrak{h}} \oplus (\text{proj}).$$

As a result, we have $(\Omega^2_{u(\mathfrak{g})}(X))_{\mathfrak{h}} \cong X_{\mathfrak{h}}$. This readily implies $(\tau(X))_{\mathfrak{h}} \cong (X^{(1)})_{\mathfrak{h}}$, so that

$$d(\tau([X])) = \dim(X^{(1)})_{\mathfrak{h}} = \dim X_{\mathfrak{h}} = d([X]),$$

as desired. $\square$

Corollary 4.2. *Let $\Theta \subset \Gamma_s(\mathfrak{g})$ be a component such that $\dim(\mathcal{V}_{\mathfrak{g}}(\Theta) \cap \mathfrak{h}) = 1$.*

(i) *If $\Theta \cong \mathbb{Z}[A_\infty^\infty]$, then $\dim X_{\mathfrak{h}} = \dim Y_{\mathfrak{h}}$ for all $[X], [Y] \in \Theta$.*
(ii) *If $\Theta \cong \mathbb{Z}[D_\infty]$, then there exists $\ell > 0$ such that $\dim X_{\mathfrak{h}} \in \{\ell, 2\ell\}$ for every $[X] \in \Theta$. Moreover, $[X]$ is located at an end of Θ if and only if $\dim X_{\mathfrak{h}} = \ell$.*

Proof. Let d be the subadditive function defined in Lemma 4.1. Since $d([X]) > 0$ we may apply Lemma 4.1 and [2, (VII.3.4),(VII.3.5)] consecutively to obtain the assertion. $\square$

Remark. Suppose that $\Theta \cong \mathbb{Z}[A_\infty^\infty], \mathbb{Z}[D_\infty]$, and let $g : X \longrightarrow Y$ be a *right properly irreducible morphism* with kernel A, where $[X]$ is in Θ. Then A is *periodic* (cf. [12, (2.1)]), i.e., $\Omega^n(A) \cong A$ for some $n \geq 1$. Consider the exact sequence $(0) \longrightarrow A \longrightarrow X \xrightarrow{g} Y \longrightarrow (0)$. In the proof of Lemma 4.1 the inequality $\dim X_{\mathfrak{h}} \leq \dim Y_{\mathfrak{h}} + \dim A_{\mathfrak{h}}$ was verified. We have $\mathcal{V}_{\mathfrak{g}}(A) \subset \mathcal{V}_{\mathfrak{g}}(\Theta)$. If $\mathcal{V}_{\mathfrak{h}}(A) = \{0\}$, then the above sequence splits over $u(\mathfrak{h})$, and we have $X|_{u(\mathfrak{h})} \neq X_{\mathfrak{h}}$.

4.2. **Ends of Components.** Let $\mathfrak{g}$ be a restricted Lie algebra.

Lemma 4.3. *Let M be a non-projective, indecomposable $u(\mathfrak{g})$-module so that*

(i) *M is of minimal dimension in its AR-component,*
(ii) *M is not simple,*
(iii) *$\mathrm{Soc}(M)$ is simple, and*
(iv) *$\mathrm{Hom}_{u(\mathfrak{g})}(\mathrm{Soc}(M), \Omega^2(M^{(1)})) = (0)$.*

If one of the following two conditions is satisfied, then M has exactly one predecessor in $\Gamma_s(\mathfrak{g})$.

(a) *For every simple submodule X of $\Omega^2(M^{(1)})$ there is a non-zero submodule Y of M such that $\mathrm{Ext}^1_{u(\mathfrak{g})}(Y, X) = (0)$.*
(b) *$\mathrm{Ext}^1_{u(\mathfrak{g})}(\mathrm{Soc}(M), \mathrm{Soc}(M)) = (0)$ and $\mathrm{Soc}(\Omega^2(M^{(1)})$ is simple.*

Proof. Let $(0) \longrightarrow \Omega^2(M^{(1)}) \longrightarrow E \longrightarrow M \longrightarrow (0)$ be the AR-sequence terminating in M. By minimality of $\dim M$ the middle term E has no projective summands.

Suppose E is decomposable, i.e., $E = E_1 \oplus E_2$ with $E_i \neq (0)$. Owing to (ii), [2, (V.3.2)] applies, and we have

$$\mathrm{Soc}(E) = \mathrm{Soc}(E_1) \oplus \mathrm{Soc}(E_2) = \mathrm{Soc}(\Omega^2(M^{(1)})) \oplus \mathrm{Soc}(M).$$

Since M has minimal dimension in its component, every irreducible morphism $f : E_1 \longrightarrow M$ is surjective [2, (V.5.1(a))] with non-trivial kernel.

(a). Since $\mathrm{Soc}(M)$ is simple, we may assume without loss that $\mathrm{Soc}(E_1) \subseteq \mathrm{Soc}(\Omega^2(M^{(1)}))$. Let $X \subseteq \ker f$ be a simple submodule. According to [2, (V.5.1(c))] the map f induces an isomorphism $E_1/X \cong \ker f/X \oplus M$. By assumption there is a submodule $(0) \neq Y \subseteq M$ so that $\mathrm{Ext}^1_{u(\mathfrak{g})}(Y, X) = (0)$. Let Z be the inverse image of Y under the canonical projection $E_1 \to E_1/X$. Since $\mathrm{Ext}^1_{u(\mathfrak{g})}(Y, X) = (0)$, the exact sequence $(0) \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow (0)$ splits, so that $Z \cong X \oplus Y$. Hence $\mathrm{Soc}(M) = \mathrm{Soc}(Y) \subset \mathrm{Soc}(Z) \subseteq E_1$, which, by choice of E_1, contradicts (iv).

(b). Now assume that $\mathrm{Soc}(M) \subseteq E_1$. By assumption on $\Omega^2(M^{(1)})$, we see that $\mathrm{Soc}(E_1)$ is simple. Hence $\mathrm{Soc}(M) \subseteq \ker f$. By choosing $X = \mathrm{Soc}(M) = Y$ in the proof of (a), we see that $Z = \mathrm{Soc}(M) \oplus \mathrm{Soc}(M) \subseteq E_1$, a contradiction.

Consequently, E is indecomposable in both instances. $\square$

5. Modules for Groups of Rank 2

5.1. **The group $A_1 \times A_1$.** In this section we first consider the case where $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the direct sum of two copies of $\mathfrak{sl}(2)$. By Lemma 1.1, the Verma modules of $u(\mathfrak{g})$ are tensor products of Verma modules over the summands. The maximal toral subalgebra of $\mathfrak{g}$ is of the form $\mathfrak{t} = kh_1 \oplus kh_2 = \mathfrak{t}_1 \oplus \mathfrak{t}_2$. Let σ_i be the simple root of $\mathfrak{g}_i$ for $i = 1, 2$. If $\Psi^+_\lambda = \{\sigma_1, \sigma_2\}$, then $Z(\lambda)$ is projective. In case $|\Psi^+_\lambda| = 1$, a consecutive application of Lemmas 1.1 and 2.2, and [14, (2.1)] implies that $\dim \mathcal{V}(Z(\lambda)) = 1$. This shows that $Z(\lambda)$ is a periodic module. By work of Happel-Preiser-Ringel [18] the isoclass $[Z(\lambda)]$ belongs to a component that is either finite or of tree class A_∞. Since $u(\mathfrak{sl}(2))$ does not possess non-simple blocks of finite representation type, Auslander's theorem (cf. [2, (VI.1.4)]) rules out the former alternative. It remains to consider the instance when $\Psi^+_\lambda = \varnothing$. Accordingly, $\Theta(\lambda)$ has a two-dimensional support variety. Since $\Theta(\lambda)$ is regular (cf. Proposition 3.4) and not periodic, it follows from the classification of AR-components (cf. [7, Thm.1], [9, (5.7)]) that $\Theta(\lambda) \cong \mathbb{Z}[A_\infty], \mathbb{Z}[A_\infty^\infty]$, or $\mathbb{Z}[D_\infty]$.

According to Lemma 1.1 we write

$$Z(\lambda) \cong Z_1(\lambda) \otimes_k Z_2(\lambda),$$

where $Z_i(\lambda) := Z_{\mathfrak{g}_i}(\lambda)$. Let $\mathcal{B}_i \subset u(\mathfrak{g}_i)$ be the block containing $Z_i(\lambda)$. Then $\mathcal{B} = \mathcal{B}_1 \otimes_k \mathcal{B}_2$ is the block associated to $Z(\lambda)$. Thus, if $e_i \in \mathcal{B}_i$ are the primitive central idempotents defining $\mathcal{B}_i$, then $e := e_1 \otimes e_2$ is the central primitive idempotent defining $\mathcal{B}$. The identity $e = (e_1 \otimes 1)(1 \otimes e_2)$ implies that each of the two factors operates invertibly on a $\mathcal{B}$-module M. As a consequence we have

Lemma 5.1. *Let M be a $\mathcal{B}$-module. Then $M|_{u(\mathfrak{g}_i)}$ belongs to $\mathcal{B}_i$.* □

By Pollack's work [27, Thm.1] the block $\mathcal{B}_i$ has exactly two simple modules S_i and T_i of dimensions $\lambda(h_i) + 1$ and $p - 1 - \lambda(h_i)$, respectively.

Proposition 5.2. *Suppose that $\Psi^+_\lambda = \varnothing$. Then $\Theta(\lambda) \cong \mathbb{Z}[A_\infty]$.*

Proof. We denote the projective cover of a simple module S by $P(S)$. By [27, Thm.1] we have

$$\dim P(X_i) = 2p \quad X \in \{S, T\}, \quad 1 \le i \le 2.$$

Moreover, there are two short exact sequences

$$(0) \longrightarrow Z_i(\lambda) \longrightarrow P(T_i) \longrightarrow P(S_i) \longrightarrow Z_i(\lambda) \longrightarrow (0),$$

and these may be combined to the initial terms of a minimal projective resolution of $Z(\lambda)$:

$$\begin{aligned}(0) \longrightarrow \Omega^2(Z(\lambda)) &\longrightarrow (P(T_1) \otimes_k P(S_2)) \oplus (P(S_1) \otimes_k P(T_2)) \\ &\longrightarrow P(S_1) \otimes_k P(S_2) \longrightarrow Z(\lambda) \longrightarrow (0).\end{aligned}$$

We record two immediate consequences:

(a) *We have* $\dim \Omega^2(Z(\lambda)) = 5p^2$ *and* $\mathrm{Soc}(\Omega^2(Z(\lambda))) = (T_1 \otimes_k S_2) \oplus (S_1 \otimes_k T_2)$.

We assume from now on that $\Theta(\lambda) \cong \mathbb{Z}[A_\infty^\infty]$, or $\mathbb{Z}[D_\infty]$.

By $\underline{\mathrm{Hom}}$ we denote the homomorphisms in the *stable category*, i.e., $\underline{\mathrm{Hom}}$ is the quotient space of the homomorphisms modulo those that factor through a projective. Since $Z|_{u(\mathfrak{g}_1)}$ belongs to $\mathcal{B}_1$ for every $[Z] \in \Theta(\lambda)$ we may apply [8, (3.2)] to see that the functions

$$d_i^V : \Theta(\lambda) \longrightarrow \mathbb{N}_0; \quad [Z] \mapsto \dim \underline{\mathrm{Hom}}_{u(\mathfrak{g}_i)}(V_i, Z), \quad \text{and}$$
$$\delta_i^V : \Theta(\lambda) \longrightarrow \mathbb{N}_0; \quad [Z] \mapsto \dim \underline{\mathrm{Hom}}_{u(\mathfrak{g}_i)}(Z, V_i),$$

where $V \in \{S, T\}$ and $1 \le i \le 2$ are additive with $d_i^V \circ \tau = d_i^V$ and $\delta_i^V \circ \tau = \delta_i^V$. Observe that $d_i^V([Z])$ and $\delta_i^V([Z])$ are the multiplicities of V_i in the socle and top of $Z_{\mathfrak{g}_i}$, respectively. Since

$$d_i^S([Z(\lambda)]) = 0, \quad d_i^T([Z(\lambda)]) = p, \quad \delta_i^S([Z(\lambda)]) = p, \quad \delta_i^T([Z(\lambda)]) = 0,$$

it follows from [2, (VII.3.5)] that there exists a function $r : \Theta(\lambda) \longrightarrow \{1, 2\}$ such that

(i) $\mathrm{Soc}(Z_{\mathfrak{g}_i}) \cong r([Z])pT_i$, $\quad \mathrm{Top}(Z_{\mathfrak{g}_i}) \cong r([Z])pS_i$ $\quad$ for all $[Z] \in \Theta(\lambda)$, and
(ii) if $\Theta(\lambda) \cong \mathbb{Z}[D_\infty]$, then $r([Z]) = 1$ if and only if $[Z]$ is located at an end of $\Theta(\lambda)$.

We continue by considering the Auslander-Reiten sequence

$$(0) \longrightarrow \Omega^2(Z(\lambda)) \longrightarrow E \longrightarrow Z(\lambda) \longrightarrow (0)$$

terminating in $Z(\lambda)$. From (a) above and [2, (V.3.2)] we obtain, letting $\ell(M)$ denote the length of a module M,

(b) $\dim E = 6p^2$, *and* $\ell(\mathrm{Soc}(E)) = 3$.

Next we show
(c) $\Theta(\lambda) \not\cong \mathbb{Z}[A_\infty^\infty]$.

Observing (a) we let X_i be the i-th summand of $\mathrm{Soc}(\Omega^2(Z(\lambda))) = (T_1 \otimes_k S_2) \oplus (S_1 \otimes_k T_2)$ for $i = 1, 2$. Consider the submodules $Y_1 := Z_1(\lambda) \otimes_k T_2$ and $Y_2 := T_1 \otimes_k Z_2(\lambda)$ of $Z(\lambda)$. The Künneth formula shows that $\operatorname{Ext}^1_{u(\mathfrak{g})}(Y_i, X_i) = (0)$ for $i = 1, 2$. Since $\operatorname{Soc}(Z(\lambda)) \cong T_1 \otimes_k T_2$, the assertion now follows from a consecutive application of Proposition 3.4 and Lemma 4.3(a).

Owing to (c) and Lemma 4.3 the vertex $[Z(\lambda)]$ is located at an end of $\Theta(\lambda) \cong \mathbb{Z}[D_\infty]$. This implies the existence of another almost split sequence

$$(0) \longrightarrow X \longrightarrow E \longrightarrow Y \longrightarrow (0).$$

Since the socle of E has length 3, either X or Y has a simple socle. As $[X]$ and $[Y]$ are located at an end of $\Theta(\lambda)$, (ii) implies $r([X]) = 1 = r([Y])$. Suppose that X has a simple socle. Then there is an embedding $X \hookrightarrow P_1 \otimes_k P_2$, where P_i is a principal indecomposable $\mathcal{B}_i$-module. There results an embedding

$$\operatorname{Soc}(X|_{u(\mathfrak{g}_1)}) \hookrightarrow \operatorname{Soc}(P_1) \otimes_k P_2$$

of $u(\mathfrak{g})$-modules. It follows from (i) above that $\operatorname{Soc}(P_1) = T_1$, so that $P_1 = P(T_1)$. By the same token, we have $P_2 = P(T_2)$. Hence we have embeddings

$$T_1 \otimes_k T_2 \hookrightarrow \operatorname{Soc}(X|_{u(\mathfrak{g}_1)}) \overset{\iota}{\hookrightarrow} T_1 \otimes_k P(T_2)$$

of $u(\mathfrak{g})$-modules. We write $X|_{u(\mathfrak{g}_1)} = X_{\mathfrak{g}_1} \oplus P$, with a projective $\mathcal{B}_1$-module P. Since $r([X]) = 1$, we have

$$\operatorname{Soc}(X|_{u(\mathfrak{g}_1)}) \cong \operatorname{Soc}(X_{\mathfrak{g}_1}) \oplus \operatorname{Soc}(P) \cong pT_1 \oplus rT_1,$$

where $r \leq p$. Now Corollary 4.2 implies

$$\dim X = \dim X_{\mathfrak{g}_1} + \dim P = p^2 + 2rp.$$

Since $p^2 | \dim X$ and $p \neq 2$, it follows that $r = 0$, or p. Consequently, $\dim X$ equals p^2 or $3p^2$.

Assume that $r = p$, so that

$$\dim \operatorname{Soc}(X|_{u(\mathfrak{g}_1)}) = 2p \dim T_1 = \dim(T_1 \otimes_k P(T_2))$$

Accordingly, the above embedding ι is an isomorphism

$$\operatorname{Soc}(X|_{u(\mathfrak{g}_1)}) \cong T_1 \otimes_k P(T_2)$$

of $u(\mathfrak{g})$-modules. Consequently, the $u(\mathfrak{g}_2)$-module $\operatorname{Soc}(X|_{u(\mathfrak{g}_1)})$ is $u(\mathfrak{g}_2)$-projective. Since $\mathcal{B}_1$ is not simple, the assumption that the $\mathcal{B}_1$-module $Y|_{u(\mathfrak{g}_1)}$ is semisimple, implies that $Y = Y_{\mathfrak{g}_1}$ (cf. Lemma 5.1). As $\dim Y = \dim E - \dim X = 6p^2 - 3p^2 = 3p^2$, this contradicts Corollary 4.2. An application of the functor $\operatorname{Hom}_{u(\mathfrak{g})}(u(\mathfrak{g})/J_1 u(\mathfrak{g}), \cdot)$ to the above almost split sequence therefore yields a split exact sequence

$$(0) \longrightarrow \operatorname{Soc}_{u(\mathfrak{g}_1)}(X) \longrightarrow \operatorname{Soc}_{u(\mathfrak{g}_1)}(E) \longrightarrow \operatorname{Soc}_{u(\mathfrak{g}_1)}(Y) \longrightarrow (0)$$

of $u(\mathfrak{g})$-modules. It follows that $\operatorname{Soc}_{u(\mathfrak{g}_1)}(E)$ has a $u(\mathfrak{g}_2)$-projective summand. By the same token, there is a decomposition

$$\operatorname{Soc}_{u(\mathfrak{g}_1)}(E) \cong \operatorname{Soc}_{u(\mathfrak{g}_1)}(Z(\lambda)) \oplus \operatorname{Soc}_{u(\mathfrak{g}_1)}(\Omega^2(Z(\lambda))) \cong (T_1 \otimes_k Z_2(\lambda)) \oplus \operatorname{Soc}_{u(\mathfrak{g}_1)}(\Omega^2(Z(\lambda)))$$

of $u(\mathfrak{g})$-modules. As the first summand is indecomposable and not $u(\mathfrak{g}_2)$-projective, it follows from the theorem of Krull-Remak-Schmidt that $T_1 \otimes_k P(T_2)$ is a direct summand of $\operatorname{Soc}_{u(\mathfrak{g}_1)}(\Omega^2(Z(\lambda)))$. But then we have $T_1 \otimes_k T_2 \subset \operatorname{Soc}(\Omega^2(Z(\lambda)))$ which contradicts (a). As a result, $r = 0$ and $\dim X = p^2$.

Since $r = 0$, we have $X_{\mathfrak{g}_1} = X$, whence

$$\operatorname{Soc}_{u(\mathfrak{g}_1)}(X) = \operatorname{Soc}_{u(\mathfrak{g}_1)}(X_{\mathfrak{g}_1}) \cong pT_1.$$

(d) $\operatorname{Soc}_{u(\mathfrak{g}_1)}(X) = J_1 X$, *where* J_1 *denotes the radical of* $u(\mathfrak{g}_1)$.

Since $X = X_{\mathfrak{g}_1}$, property (i) above implies that $\dim \mathrm{Soc}_{u(\mathfrak{g}_1)}(X) = \dim J_1X$. Hence it suffices to show that $\mathrm{Soc}_{u(\mathfrak{g}_1)}(X) \subseteq J_1X$. If this is not the case, then there exists a maximal $u(\mathfrak{g}_1)$-submodule $M \subset X$ with $\mathrm{Soc}_{u(\mathfrak{g}_1)}(X) \not\subset M$. Hence there exists an isomorphic copy of $T_1 \subset X$ with $T_1 \not\subset M$. This readily yields $X = M \oplus T_1$. According to (i) we have $X/M \cong S_1$, a contradiction.

Recall that the socle of X is isomorphic to $T_1 \otimes_k T_2$, so that there is an embedding

$$\varepsilon : X \hookrightarrow P(T_1) \otimes_k P(T_2).$$

This induces an embedding

$$\iota : \mathrm{Soc}_{u(\mathfrak{g}_1)}(X) \hookrightarrow T_1 \otimes_k P(T_2).$$

(e) *There exists a p-dimensional indecomposable $u(\mathfrak{g}_2)$-submodule X_2 of $P(T_2)$ such that* $\operatorname{im}\iota = T_1 \otimes_k X_2$.

We define $X_2 := \{q \in P(T_2) \mid T_1 \otimes_k kq \subseteq \operatorname{im}\iota\}$. Then X_2 is a $u(\mathfrak{g}_2)$-submodule of $P(T_2)$ satisfying $T_1 \otimes_k X_2 \subseteq \mathrm{Soc}_{u(\mathfrak{g}_1)}(\operatorname{im}\iota)$. This implies in particular that $\dim X_2 \leq p$. Since $\mathrm{Soc}_{u(\mathfrak{g})}(X) \subseteq \mathrm{Soc}_{u(\mathfrak{g}_1)}(X)$, we have $T_2 \subseteq X_2$ (i.e., $X_2 \neq (0)$).

For $q \in P(T_2)$ we consider the $u(\mathfrak{g}_1)$-linear map

$$f_q : T_1 \longrightarrow P(T_1) \otimes_k P(T_2); \quad x \mapsto x \otimes q.$$

As $\operatorname{im} f_q \subset T_1 \otimes_k P(T_2)$, the map $q \mapsto f_q$ induces a k-linear map $\gamma : P(T_2) \longrightarrow \mathrm{Hom}_{u(\mathfrak{g}_1)}(T_1, T_1 \otimes_k P(T_2)/\operatorname{im}\iota)$. By definition, we have $X_2 = \ker\gamma$. Since $(\operatorname{im}\iota)|_{u(\mathfrak{g}_1)} \cong pT_1$, it follows from Schur's Lemma that $\dim \operatorname{im}\gamma \leq p$. Consequently, $\dim X_2 \geq p$. Thus, X_2 is p-dimensional, and $T_1 \otimes_k X_2 = \operatorname{im}\iota$. As $X_2 \subset P(T_2)$, it is an indecomposable $u(\mathfrak{g}_2)$-module.

(f) *There is a p-dimensional indecomposable $u(\mathfrak{g}_1)$-submodule X_1 of $P(T_1)$ such that $X \cong X_1 \otimes_k X_2$.*

According to (e) we have an embedding $\mathrm{Soc}_{u(\mathfrak{g}_1)}(X) \hookrightarrow P(T_1) \otimes_k X_2$ of $u(\mathfrak{g})$-modules. We begin by showing that $\operatorname{im}\varepsilon \subset P(T_1) \otimes_k X_2$. Consider the canonical map

$$\bar{\varepsilon} : X \longrightarrow P(T_1) \otimes_k (P(T_2)/X_2).$$

It follows from (e) that $\mathrm{Soc}_{u(\mathfrak{g}_1)}(X) \subseteq \ker\bar{\varepsilon}$. From (d) and (i) we conclude $(X/\mathrm{Soc}_{u(\mathfrak{g}_1)}(X))|_{u(\mathfrak{g}_1)} \cong pS_1$. Since $\mathrm{Soc}_{u(\mathfrak{g}_1)}(P(T_1) \otimes_k (P(T_2)/X_2)) \cong pT_1$, we have $\bar{\varepsilon} = 0$. Consequently, $\operatorname{im}\varepsilon \subseteq P(T_1) \otimes_k X_2$.

Repetition of steps (d) and (e) yields the existence of a p-dimensional $u(\mathfrak{g}_1)$-module X_1 such that $\operatorname{im}\varepsilon \subseteq X_1 \otimes_k P(T_2)$. Hence,

$$\operatorname{im}\varepsilon \subseteq (P(T_1) \otimes_k X_2) \cap (X_1 \otimes_k P(T_2)) = X_1 \otimes_k X_2.$$

According to (f) and [14, (2.1)], we have

$$\mathcal{V}_{\mathfrak{g}_1}(X_1) \times \mathcal{V}_{\mathfrak{g}_2}(X_2) = \mathcal{V}_{\mathfrak{g}}(X) \subset ke_1 \times ke_2,$$

so that $\mathcal{V}_{\mathfrak{g}_i}(X_i) = ke_i$. Since X_i is not $u(ke_i)$-projective, the $\mathfrak{t}_i$-invariant space $\mathrm{Soc}_{u(ke_i)}(X_i)$ has dimension at least 2. As $\dim \mathrm{Soc}_{u(ke_i)}(T_i) = 1$, it follows that $\mathrm{Soc}_{u(ke_i)}(X_i) \not\subset T_i$. Consequently, there exists a highest weight vector $x_i \in X_i$ that generates X_i. Note that $\lambda|_{\mathfrak{t}_i}$ is the corresponding highest weight. Thus, there are surjections $Z_i(\lambda) \longrightarrow X_i$, which, for dimension reasons, are isomorphisms. It follows that $X \cong Z_1(\lambda) \otimes_k Z_2(\lambda) \cong Z(\lambda)$, a contradiction. $\square$

5.2. **The group A_2.** Finally, we study the case $\mathfrak{g} = \mathfrak{sl}(3)$. Let σ_1 and σ_2 be the simple roots of $\mathfrak{g}$ and $\mu = (\mu_1, \mu_2)$ be in $X(T)$, where μ_i is the coefficient of the fundamental weight ω_i. According to Lemma 3.1 the only relevant cases are those where $\lambda \neq (p-1)\varrho$ lies on one of the walls of the "upper" alcove of A_2. The dot action of W partitions the set of these elements into $p-1$ orbits each of cardinality 3. As orbit representatives we may take $(p-1, n)$, where $n \in \{0, \dots, p-2\}$; the corresponding orbit is $\{(p-1,n), (p-n-2, p-1), (n, p-n-2)\}$. Given ν in $X(T)$ we write ν' for $w_0.\nu$, where w_0 is the longest word in W. We consider the block $\mathcal{B}_n$ with highest weights $\lambda_n := (p-1, n)$, $\lambda'_n := (p-n-2, p-1)$, and $\gamma_n := (n, p-n-2)$, where $n \in \{0, \dots, p-2\}$. By [26, p. 29] the dimensions of the corresponding simple modules are given by Weyl's character formula. Thus, we have $\dim L(\lambda_n) = \frac{1}{2}p(n+1)(p+n+1)$, $\dim L(\lambda'_n) = \frac{1}{2}p(p-n-1)(2p-n-1)$, and $\dim L(\gamma_n) = \frac{1}{2}p(n+1)(p-n-1)$. According to [19, Thm.2.2] each Verma module is *sincere*, i.e., each simple module of our block $\mathcal{B}_n$ occurs in each $Z(\mu)$ with multiplicity at least 1 for every weight μ belonging to $\mathcal{B}_n$. Following [19, p. 68] we write $d_{\nu\eta}$ for the multiplicity of $L(\eta)$ in $Z(\nu)$. Since, for fixed n, the dimensions of these simple modules are pairwise distinct, $\dim Z(\nu) = p^3$ for each ν, and $\dim L(\lambda_n) + \dim L(\lambda'_n) + \dim L(\gamma_n) = p^3 - \dim L(\gamma_n)$, we obtain

$$d_{\mu\lambda_n} = 1 = d_{\mu\lambda'_n}; \quad d_{\mu\gamma_n} = 2, \quad \text{where } \mu \in \{\lambda_n, \lambda'_n, \gamma_n\}. \tag{4}$$

As before, let $P(\mu)$ be the projective cover of $L(\mu)$. Then we have $\dim P(\lambda_n) = 3p^3 = \dim P(\lambda'_n)$, while $\dim P(\gamma_n) = 6p^3$ [19, (4.5)].

By [19, §4] we know that every projective indecomposable $u(\mathfrak{g})$-module has a *Z-filtration*, i.e., a filtration by Verma modules. Observe that whenever $Z(\mu) \hookrightarrow P(\nu)$, then $\mathrm{Soc}(Z(\mu)) = \mathrm{Soc}(P(\nu)) = L(\nu)$; thus $\mu = \nu'$, [22, (II.3.12)]. We continue by recalling further general facts of Verma modules and their coinduced counterparts.

Since, for $\lambda, \mu \in X(T)$, the module $Z(\lambda)^* \otimes_k Z'(\mu)$ has trivial support, we have

$$\mathrm{Ext}^1_{u(\mathfrak{g})}(Z(\lambda), Z'(\mu)) = \mathrm{H}^1(u(\mathfrak{g}), Z(\lambda)^* \otimes_k Z'(\mu)) = (0) \quad \text{for all } \lambda, \mu \in X(T), \tag{5}$$

cf. [22, (II.11.1)]. According to [22, (II.3.9(4))] we have an exact sequence

$$(0) \longrightarrow L(\mu) \longrightarrow Z'(\mu) \longrightarrow X(\mu) \longrightarrow (0). \tag{6}$$

Frobenius reciprocity yields

$$\mathrm{Hom}_{u(\mathfrak{g})}(Z(\lambda), Z'(\mu)) \cong \mathrm{Hom}_{u(\mathfrak{b}^-)}(Z(\lambda), k_\mu).$$

As $Z(\mu)$ is the projective cover of the $u(\mathfrak{b}^-)$-module k_μ, cf. [22, (II.9.4(a))], we obtain

$$\dim \mathrm{Hom}_{u(\mathfrak{g})}(Z(\lambda), Z'(\mu)) = \delta_{\lambda\mu},$$

cf. [22, (II.11.1)]. By (5), the exact sequence (6) induces an exact sequence

$$\begin{aligned}(0) \longrightarrow\ & \mathrm{Hom}_{u(\mathfrak{g})}(Z(\lambda), L(\mu)) \longrightarrow \mathrm{Hom}_{u(\mathfrak{g})}(Z(\lambda), Z'(\mu)) \\ \longrightarrow\ & \mathrm{Hom}_{u(\mathfrak{g})}(Z(\lambda), X(\mu)) \longrightarrow \mathrm{Ext}^1_{u(\mathfrak{g})}(Z(\lambda), L(\mu)) \longrightarrow (0).\end{aligned}$$

By the above, the second arrow is an isomorphism, whence

$$\mathrm{Ext}^1_{u(\mathfrak{g})}(Z(\lambda), L(\mu)) \cong \mathrm{Hom}_{u(\mathfrak{g})}(Z(\lambda), X(\mu)). \tag{7}$$

We write

$$\mathrm{Top}(\Omega(Z(\lambda))) = \bigoplus_{\mu \in X_1(T)} m_{\lambda\mu} L(\mu),$$

where $X_1(T) := \{\mu \mid 0 \leq \langle \mu, \sigma^\vee \rangle < p \text{ for all } \sigma \in \Sigma\}$ is the set of *p-restricted weights*. According to [3, (2.5.4)], we have

$$m_{\lambda\mu} = \dim \mathrm{Hom}_{u(\mathfrak{g})}(\Omega(Z(\lambda)), L(\mu)) = \dim \mathrm{Ext}^1_{u(\mathfrak{g})}(Z(\lambda), L(\mu)) \\ = \dim \mathrm{Hom}_{u(\mathfrak{g})}(Z(\lambda), X(\mu)). \tag{8}$$

We now return to our A_2 setting. Observe that once we have computed $\mathrm{Ext}^1_{u(\mathfrak{g})}(Z(\lambda_n), L(\mu))$ for *all* $n \in \{0, \ldots, p-2\}$ and $\mu \in \{\lambda_n, \lambda'_n, \gamma_n\}$, the results for λ'_n in place of λ_n follow by symmetry.

Using (4), the fact that $L(\mu)$ does not admit any self-extensions [1, (4.5)], the fact that the extension groups between $L(\lambda_n)$ and $L(\lambda'_n)$ are trivial [32, (3.3.2)], that $\gamma'_n = \gamma_n$, and the fact that $Z(\mu)$ and $Z'(\mu)$ define the same element in the Grothendieck group [22, (II.9.7)], we can determine the Loewy series of the various $X(\mu)$'s:

$$X(\lambda_n) : L(\lambda'_n)/L(\gamma_n) \oplus L(\gamma_n);$$
$$X(\lambda'_n) : L(\lambda_n)/L(\gamma_n) \oplus L(\gamma_n);$$
$$X(\gamma_n) : L(\gamma_n)/L(\lambda'_n) \oplus L(\lambda_n).$$

From these we now obtain

$$\begin{aligned} &\dim \mathrm{Hom}_{u(\mathfrak{g})}(Z(\lambda_n), X(\lambda_n)) = 0 && \dim \mathrm{Hom}_{u(\mathfrak{g})}(Z(\gamma_n), X(\lambda_n)) = 2 \\ &\dim \mathrm{Hom}_{u(\mathfrak{g})}(Z(\lambda_n), X(\lambda'_n)) \leq 1 && \dim \mathrm{Hom}_{u(\mathfrak{g})}(Z(\gamma_n), X(\lambda'_n)) = 2 \\ &\dim \mathrm{Hom}_{u(\mathfrak{g})}(Z(\lambda_n), X(\gamma_n)) = 1 && \dim \mathrm{Hom}_{u(\mathfrak{g})}(Z(\gamma_n), X(\gamma_n)) \leq 1. \end{aligned} \tag{9}$$

Next we address the inequalities in (9); as it turns out, these groups are trivial. The proof of the following lemma was communicated to us by J.C. Jantzen.

Lemma 5.3. *Let* $\mu = \lambda_n$*, or* γ_n*. Then* $\mathrm{Hom}_{u(\mathfrak{g})}(Z(\mu), X(\mu')) = (0)$.

Proof. Let $\mu = \lambda_n$ or γ_n and observe that $\gamma_n = \gamma'_n$. Suppose that $\mathrm{Hom}_{u(\mathfrak{g})}(Z(\mu), X(\mu')) \neq (0)$. Then, by (9), this Hom-space is one-dimensional. Thus, there exists an element $\nu \in X(T)$ so that

$$\mathrm{Hom}_{G_1}(Z(\mu), X(\mu')) = \mathrm{Hom}_{G_1T}(Z(\mu), X(\mu') \otimes_k k_{p\nu}),$$

cf. [22, (I.6.9.(5))]. The Loewy series of $Z(\mu)$ and $Z'(\mu')$ show that any nonzero homomorphism from $Z(\mu)$ to $X(\mu') \otimes_k k_{p\nu}$ induces an isomorphism from $Z(\mu)/L(\mu')$ onto $X(\mu') \otimes_k k_{p\nu}$.

Observe that $Z(\mu)$ and $Z'(\mu)$ have the same G_1T-composition factors with the same multiplicities [22, (II.9.7)]. The G_1T-Loewy series of $Z(\mu)$ and $Z'(\mu)$ can be computed explicitly from [25, p. 8]. Viewed as a G_1T-module, $Z(\lambda_n)$ has Loewy series

$$L(p-1, n)/L(p+n, -n-2) \oplus L(n-p, p-n-2)/L(p-n-2, -1),$$

while $X(\lambda'_n)$ has Loewy series

$$L(-1, n)/L(n-p, 2p-n-2) \oplus L(n, -n-2).$$

Thus, $L(p-1, n)$ only occurs as a G_1T-composition factor of $X(\lambda'_n) \otimes_k k_{p\nu}$ provided $\nu = (1, 0)$. But for this ν the G_1T-composition factors of $X(\lambda'_n) \otimes_k k_{p\nu}$ are $L(p-1, n)$, $L(n, 2p-n-2)$, and $L(p+n, -n-2)$, and these are distinct from those of $Z(\lambda_n)/L(\lambda'_n)$, a contradiction.

For $\mu = \gamma_n$ we argue analogously. As a G_1T-module $Z(\gamma_n)$ has Loewy series

$$L(n, p-n-2)/L(p-1, n-p) \oplus L(-n-2, p-1)/L(n-p, -n-2),$$

while $X(\gamma_n)$ has Loewy series

$$L(n-p, -n-2)/L(p-1, n-p) \oplus L(-n-2, p-1).$$

Consequently, $L(n, p-n-2)$ only occurs as a G_1T-composition factor of $X(\gamma_n) \otimes_k k_{p\nu}$ provided $\nu = (1, 1)$. However, for this particular choice of ν the G_1T-composition factors of $X(\gamma_n) \otimes_k k_{p\nu}$

are $L(n, p-n-2)$, $L(2p-1, n)$, and $L(p-n-2, 2p-1)$, and these are distinct from those of $Z(\gamma_n)/L(\gamma_n)$, a contradiction.

Consequently, $\mathrm{Hom}_{u(\mathfrak{g})}(Z(\mu), X(\mu')) = (0)$, in both instances, as claimed. □

Proposition 5.4. *For $\mu = \lambda_n$ or λ'_n the module $Z(\mu)$ has exactly one predecessor in $\Theta(\mu)$.*

Proof. We are going to verify the conditions of Lemma 4.3(b). By Proposition 3.4(ii), condition (i) holds. As $Z(\mu)$ is a non-projective Verma module, (ii) and (iii) obtain. Using (8), (9), and Lemma 5.3, we see that $\mathrm{Soc}(\Omega^2(Z(\mu))) \cong \mathrm{Top}(\Omega(Z(\mu))) \cong L(\gamma_n)$ is simple and not isomorphic to $\mathrm{Soc}(Z(\mu)) \cong L(\mu')$. Moreover, by [1, (4.5)], $\mathrm{Soc}(Z(\mu))$ does not admit any self-extensions. Accordingly, (iv) and (b) are also valid. □

Proposition 5.5. *The module $Z(\gamma_n)$ has exactly one predecessor in $\Theta(\gamma_n)$ for $0 \le n \le p-2$.*

Proof. For a fixed n set $\eta_1 := \lambda_n$ and $\eta_2 := \lambda'_n$. For $i = 1, 2$ set $X_i := L(\eta_i)$. According to (9) and Lemma 5.3, these are the two isomorphism types of simple constituents of $\mathrm{Soc}(\Omega^2(Z(\gamma_n)))$. Let $P_i := P_{\{\sigma_i\}}$ be the two maximal standard parabolic subgroups of G, and $\mathrm{Lie}\, P_i = \mathfrak{p}_i = \mathfrak{l}_i \oplus \mathfrak{n}_i$ the standard Levi decomposition of $\mathrm{Lie}\, P_i$. We abbreviate $e_i := e_{\sigma_i}$, $f_i := e_{-\sigma_i}$, and $h_i := [e_i, f_i]$. Then for $z_i := 2h_{3-i} + h_i$ we have $\mathfrak{l}_i = (kf_i \oplus kh_i \oplus ke_i) \oplus kz_i$, where z_i is central.

We denote a simple $u(\mathfrak{l}_i)$-module by $L(m)_t$, where $0 \le m \le p-1$ is the highest weight for the semisimple part $\mathfrak{l}'_i := [\mathfrak{l}_i, \mathfrak{l}_i] \cong \mathfrak{sl}(2)$ of $\mathfrak{l}_i$ and the suffix $t \in k$ indicates the scalar of the action of z_i. The simple $u(\mathfrak{l}_i)$-modules $L(m)_t$ and $L(p-2-m)_t$ belong to the same block, denoted by $\mathcal{B}(m)_t$. By the above we have

$$L_{\mathfrak{l}_i}(\mu) \cong L(\mu(h_i))_{\mu(z_i)}. \tag{10}$$

By Corollary 1.4, the module

$$L(\eta_i)^{\mathfrak{n}_i} \cong L_{\mathfrak{l}_i}(\eta_i) \cong L(p-1)_{\eta_i(z_i)}$$

is $u(\mathfrak{l}_i)$-projective. As a consequence, we obtain

$$\mathrm{Ext}^1_{u(\mathfrak{l}_i)}(M, L(\eta_i)^{\mathfrak{n}_i}) = (0), \tag{11}$$

for any $u(\mathfrak{l}_i)$-module M. Recall that for A_1 the Verma module $Z(n)$ has a filtration $L(n)/L(p-2-n)$ for $0 \le n \le p-2$. By Lemma 1.2(iii)

$$Y_i := u(\mathfrak{g}) \otimes_{u(\mathfrak{p}_i)} L(p-2-\gamma_n(h_i))_{\gamma_n(z_i)}$$

is a submodule of $Z(\gamma_n)$. We have

$$\begin{aligned} \mathrm{Ext}^1_{u(\mathfrak{g})}(Y_i, X_i) &\cong \mathrm{Ext}^1_{u(\mathfrak{p}_i)}(L(p-2-\gamma_n(h_i))_{\gamma_n(z_i)}, L(\eta_i)) \\ &\cong \mathrm{H}^1(u(\mathfrak{p}_i), L(p-2-\gamma_n(h_i))^*_{\gamma_n(z_i)} \otimes_k L(\eta_i)). \end{aligned} \tag{12}$$

By applying the cohomology five-term sequence to $S_i := L(p-2-\gamma_n(h_i))^*_{\gamma_n(z_i)} \otimes_k L(\eta_i)$, we get

$$(0) \longrightarrow \mathrm{H}^1(u(\mathfrak{l}_i), S_i^{\mathfrak{n}_i}) \longrightarrow \mathrm{H}^1(u(\mathfrak{p}_i), S_i) \longrightarrow \mathrm{H}^1(u(\mathfrak{n}_i), S_i)^{\mathfrak{l}_i}. \tag{13}$$

It follows from (11) that

$$\begin{aligned} \mathrm{H}^1(u(\mathfrak{l}_i), S_i^{\mathfrak{n}_i}) &\cong \mathrm{H}^1(u(\mathfrak{l}_i), L(p-2-\gamma_n(h_i))^*_{\gamma_n(z_i)} \otimes_k L(\eta_i)^{\mathfrak{n}_i}) \\ &\cong \mathrm{Ext}^1_{u(\mathfrak{l}_i)}(L(p-2-\gamma_n(h_i))_{\gamma_n(z_i)}, L(\eta_i)^{\mathfrak{n}_i}) = (0). \end{aligned} \tag{14}$$

Considering the term $\mathrm{H}^1(u(\mathfrak{n}_i), S_i)^{\mathfrak{l}_i}$, we have

$$\begin{aligned} \mathrm{H}^1(u(\mathfrak{n}_i), S_i) &= \mathrm{H}^1(u(\mathfrak{n}_i), L(p-2-\gamma_n(h_i))^*_{\gamma_n(z_i)} \otimes_k L(\eta_i)) \\ &\cong \mathrm{H}^1(u(\mathfrak{n}_i), L(\eta_i)) \otimes_k L(p-2-\gamma_n(h_i))^*_{\gamma_n(z_i)}, \end{aligned}$$

as $u(\mathfrak{l}_i)$-modules. There result isomorphisms

$$\begin{aligned} \mathrm{H}^1(u(\mathfrak{n}_i), S_i)^{\mathfrak{l}_i} &\cong \mathrm{Hom}_{u(\mathfrak{l}_i)}(k, L(p-2-\gamma_n(h_i))^*_{\gamma_n(z_i)} \otimes_k \mathrm{H}^1(u(\mathfrak{n}_i), L(\eta_i))) \\ &\cong \mathrm{Hom}_{u(\mathfrak{l}_i)}(L(p-2-\gamma_n(h_i))_{\gamma_n(z_i)}, \mathrm{H}^1(u(\mathfrak{n}_i), L(\eta_i))). \end{aligned} \tag{15}$$

Set $V(\mu) := \mathrm{Hom}_{u(\mathfrak{p}_i^-)}(u(\mathfrak{g}), L_{\mathfrak{l}_i}(\mu))$. It follows from Lemma 1.3 that

$$\mathrm{Soc}_{u(\mathfrak{g})}(V(\mu)) = \mathrm{Soc}_{u(\mathfrak{g})}(Z'(\mu)) \cong L(\mu). \tag{16}$$

As $Z'_{\mathfrak{l}_i}(\mu)$ has a filtration $L(p-2-\mu(h_i))_{\mu(z_i)}/L(\mu(h_i))_{\mu(z_i)}$, we have a filtration of $Z'(\mu)$ of the form:

$$\mathrm{Hom}_{u(\mathfrak{p}_i^-)}(u(\mathfrak{g}), L(p-2-\mu(h_i))_{\mu(z_i)})/\mathrm{Hom}_{u(\mathfrak{p}_i^-)}(u(\mathfrak{g}), L(\mu(h_i))_{\mu(z_i)}). \tag{17}$$

Since $\dim V(\mu) = p^2 \dim L_{\mathfrak{l}_i}(\mu) \neq \dim L(\mu)$, we see that $V(\mu)$ is not simple. As $Z'(\mu)$ has a filtration of the form $V(\hat{\mu})/V(\mu)$, by (17), and $\ell(Z'(\mu)) = 4$, the module $V(\mu)$ has length 2. Because $L(\eta_i)$ does not admit any self-extensions [1, (4.5)] and the extension groups between $L(\eta_i)$ and $L(\eta_{3-i})$ are trivial [32, (3.3.2)], from (16) we obtain an exact sequence

$$(0) \longrightarrow L(\eta_i) \longrightarrow V(\eta_i) \longrightarrow L(\gamma_n) \longrightarrow (0).$$

Since $V(\eta_i)|_{u(\mathfrak{n}_i)}$ is projective, there results an exact sequence

$$(0) \longrightarrow L(\eta_i)^{\mathfrak{n}_i} \longrightarrow V(\eta_i)^{\mathfrak{n}_i} \longrightarrow L(\gamma_n)^{\mathfrak{n}_i} \longrightarrow \mathrm{H}^1(u(\mathfrak{n}_i), L(\eta_i)) \longrightarrow (0).$$

It follows from the proof of Corollary 1.4 that we have an isomorphism $L(\eta_i)^{\mathfrak{n}_i} \cong V(\eta_i)^{\mathfrak{n}_i}$. Hence, Corollary 1.4 yields

$$\mathrm{H}^1(u(\mathfrak{n}_i), L(\eta_i))) \cong L(\gamma_n)^{\mathfrak{n}_i} \cong L_{\mathfrak{l}_i}(\gamma_n).$$

Accordingly, (15) implies

$$\begin{aligned} \mathrm{H}^1(u(\mathfrak{n}_i), S_i)^{\mathfrak{l}_i} &\cong \mathrm{Hom}_{u(\mathfrak{l}_i)}(L(p-2-\gamma_n(h_i))_{\gamma_n(z_i)}, L_{\mathfrak{l}_i}(\gamma_n)) \\ &\cong \mathrm{Hom}_{u(\mathfrak{l}_i)}(L(p-2-\gamma_n(h_i))_{\gamma_n(z_i)}, L(\gamma_n(h_i))_{\gamma_n(z_i)}) = (0). \end{aligned} \tag{18}$$

By (12), (13), (14), and (18) we see that $\mathrm{Ext}^1_{u(\mathfrak{g})}(Y_i, X_i) = (0)$ for each simple submodule X_i of $\mathrm{Soc}(\Omega^2(Z(\gamma_n)))$ and $Y_i \subset Z(\gamma_n)$ as above. Consequently, our result follows from Proposition 3.4(ii) and Lemma 4.3(a). $\square$

6. The Location of Verma Modules

We adopt the conventions of the preceding sections, and assume that $\mathfrak{g}$ is the Lie algebra of a connected, reductive group G of characteristic $p \geq 5$.

Let $\Theta \subset \Gamma_s(\mathfrak{g})$ be a component of tree class A_∞. A module $M \in \Theta$ that has only one predecessor is called *quasi-simple*. We are now in a position to prove the main result of this paper.

Theorem 6.1. *Let $\lambda \in X(T)$.*

(i) *The middle term of the almost split sequence terminating in $Z(\lambda)$ is indecomposable.*
(ii) *If $\mathfrak{g}$ contains no direct summand of type A_2, then $Z(\lambda)$ is quasi-simple.*

Proof. Suppose that $\dim \mathcal{V}_{\mathfrak{g}}(\Theta(\lambda)) \geq 3$. Then, by [11, (2.1)], $\Theta(\lambda) \cong \mathbb{Z}[A_\infty]$. Being a vertex of minimal dimension, by Proposition 3.4, $Z(\lambda)$ is quasi-simple. If $\dim \mathcal{V}_{\mathfrak{g}}(\Theta(\lambda)) = 1$, then $\Theta(\lambda)$ is periodic and hence isomorphic to $\mathbb{Z}[A_\infty]/(\tau)$. As before, we see that $Z(\lambda)$ is quasi-simple. Finally, suppose that $\dim \mathcal{V}_{\mathfrak{g}}(\Theta(\lambda)) = 2$. By Theorem 3.5 there exists a direct summand $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ or $\mathfrak{sl}(3)$, and that there is a quiver isomorphism $\Theta_{\mathfrak{s}}(\lambda) \cong \Theta(\lambda)$, originating in the component $\Theta_{\mathfrak{s}}(\lambda)$ of $\Gamma_s(\mathfrak{s})$ containing the Verma module $Z_{\mathfrak{s}}(\lambda)$. Moreover, this isomorphism maps $Z_{\mathfrak{s}}(\lambda)$ to $Z(\lambda)$. If $\mathfrak{s} \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, then Proposition 5.2 implies $\Theta(\lambda) \cong \mathbb{Z}[A_\infty]$, and thus the result follows. If $\mathfrak{s} \cong \mathfrak{sl}(3)$, then assertion (i) follows from a consecutive application of Propositions 5.4 and 5.5. $\square$

We expect that the exclusion of direct summands of type A_2 in Theorem 6.1(ii) is superfluous. It follows from Propositions 5.4 and 5.5 that for $\mathfrak{g}$ of type A_2 the component $\Theta(\lambda)$ of $Z(\lambda)$ is isomorphic to $\mathbb{Z}[A_\infty]$ or $\mathbb{Z}[D_\infty]$. It is likely that there are no instances of components of tree class D_∞.

7. Appendix: Varieties for B_2

Throughout this section $\mathfrak{g}$ is the simple Lie algebra of type B_2. Let σ, γ and $\tilde{\alpha}$ be the long, short simple roots and the highest root, respectively. We put $\alpha := \sigma + \gamma$.

Let $\mathfrak{l} = ke_{-\sigma} \oplus \mathfrak{t} \oplus ke_\sigma$ be the Levi subalgebra of $\mathfrak{g}$ associated to σ, and

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}$$

the corresponding triangular decomposition, i.e.,

$$\mathfrak{n} = ke_\gamma \oplus ke_\alpha \oplus ke_{\tilde{\alpha}} \quad \text{and} \quad \mathfrak{n}^- = ke_{-\gamma} \oplus ke_{-\alpha} \oplus ke_{-\tilde{\alpha}}.$$

We also consider the parabolic subalgebras $\mathfrak{p} := \mathfrak{l} \oplus \mathfrak{n}$ and $\mathfrak{p}^- := \mathfrak{l} \oplus \mathfrak{n}^-$ of $\mathfrak{g}$.

By ordering the roots according to their heights, we see that $\omega := \prod_{\beta \in \Psi(\mathfrak{n})} e_\beta^{p-1}$ is a left *integral* of the Hopf algebra $u(\mathfrak{n})$, i.e., $k\omega$ is the trivial $u(\mathfrak{n})$-module, cf. [31, (5.1.6)]. Since $\mathfrak{n}$ is p-nilpotent, $k\omega$ is a two-sided integral.

Next, we observe $u(\mathfrak{p}) : u(\mathfrak{l})$ is a ν-Frobenius extension, where $\nu : \mathfrak{l} \longrightarrow \mathfrak{l}$ is the automorphism given by $\nu(x) = x - \operatorname{tr}(\operatorname{ad}_{\mathfrak{n}}(x))1$ for all $x \in \mathfrak{l}$. It follows from [13, (1.2)] that there exists a subspace $X \subset u(\mathfrak{p})$ not containing ω such that $a\omega \equiv \omega\nu(a) \mod (X)$ for every $a \in u(\mathfrak{l})$. Given $x \in \mathfrak{l}$, the element $[x, \omega]$ is easily seen to be a left integral of $u(\mathfrak{n})$. Consequently, $[x, \omega] \in k\omega$, and we conclude

$$a\omega = \omega\nu(a) \text{ for all } a \in u(\mathfrak{l}). \tag{19}$$

Recall the Cartan-Weyl formula [30, (I.1.3)]:

$$y^m x = \sum_{i=0}^{m} \binom{m}{i} (\operatorname{ad} y)^{m-i}(x) y^i,$$

whence for all x and y in $u(\mathfrak{g})$

$$[x, y^m] = -\sum_{i=0}^{m-1} \binom{m}{i} (\operatorname{ad} y)^{m-i}(x) y^i.$$

Now consider the element $e_\alpha \in \mathfrak{n}$, and observe that $[e_\alpha, e_\sigma] = 0$, as $2\sigma + \gamma$ is not a root. The following formulae are direct consequences of the Cartan-Weyl identity, the fact that ω is a two-sided integral, and the structure of a root system of type B_2:

(i) Let $a \in u(\mathfrak{n}^-)$. Then

$$\omega[e_\alpha, e_{-\sigma}^m a] = \omega e_{-\sigma}^m [e_\alpha, a].$$

(ii) Put $\mathfrak{n}_2^- := ke_{-\alpha} \oplus ke_{-\tilde{\alpha}}$. Given $b \in u(\mathfrak{n}_2^-)$ and $0 \leq m \leq p-1$, there exists $\zeta \in k$ such that

$$[e_\alpha, e_{-\gamma}^m b] = \zeta e_\sigma e_{-\gamma}^{m-1} b + e_{-\gamma}^m [e_\alpha, b].$$

(iii) Let $c \in u(ke_{-\tilde{\alpha}})$ and $0 \leq m \leq p-1$. Then there exist ζ_1, ζ_2 such that

$$[e_\alpha, e_{-\alpha}^m c] = (\zeta_1 h_\alpha + \zeta_2 1) e_{-\alpha}^{m-1} c + e_{-\alpha}^m [e_\alpha, c].$$

(iv) Let $0 \leq m \leq p-1$. Then there exists $\zeta \in k$ with
$$[e_\alpha, e_{-\tilde{\alpha}}^m] = \zeta e_{-\gamma} e_{-\tilde{\alpha}}^{m-1}.$$

Let $\mathfrak{t} := kh_\sigma \oplus kh_\gamma$ be the maximal torus of $\mathfrak{g}$. For $\xi \in \mathbb{F}_p$ we define
$$u_\xi = \prod_{\eta \in \mathbb{F}_q \setminus \{\xi\}} (h_\sigma - \eta 1) \ \in u(\mathfrak{t}).$$
Further, for an algebra homomorphism $\lambda : u(\mathfrak{t}) \to k$ we define the element
$$u_1 := \prod_{\eta \in \mathbb{F}_p \setminus \{\lambda(h_\sigma + h_\gamma)\}} (h_\sigma + h_\gamma - \eta 1) \ \in u(\mathfrak{t}).$$
For notational convenience we set $u_2 := u_{\lambda(h_\sigma)+2}$, $u_3 := u_1 u_2$, and $u_4 := u_{\lambda(h_\sigma)}$. Since $h_\sigma + h_\gamma$ lies in the center of $\mathfrak{l}$, we have $u_1 \in \mathfrak{z}(u(\mathfrak{l}))$.

Let $y := e_\sigma + e_\alpha$, so that $y^{[p]} = 0$. We consider the induced module
$$A := u(\mathfrak{g}) \otimes_{u(ky)} k$$
along with the $u(\mathfrak{t})$-submodules
$$M := \{u_1 \omega a \otimes 1 \mid a \in u(\mathfrak{u}^- \oplus kh_\sigma)\}, \quad \text{and}$$
$$\tilde{M} := \langle \{u_3 \omega e_{-\sigma}^i a \otimes 1 \ ; \ a \in u(\mathfrak{n}^-), \ 0 \leq i \leq p-2\} \rangle \oplus \langle \{u_3 \omega e_{-\sigma}^{p-1} a \otimes 1 \ ; \ a \in u(\mathfrak{n}^-)^\dagger\} \rangle,$$
where $u(\mathfrak{n}^-)^\dagger$ is the augmentation ideal of the Hopf algebra $u(\mathfrak{n}^-)$. Owing to (19) we have $\tilde{M} \subseteq M$.

Lemma 7.1. *We have $u_3 \omega [e_\sigma, a] \otimes 1 \in \tilde{M}$ for all $a \in u(\mathfrak{u}^-)$.*

Proof. First observe that $u(\mathfrak{u}^-) = \bigoplus_{i=0}^{p-1} e_{-\sigma}^i \, u(\mathfrak{n}^-)$. From the Cartan-Weyl formula we obtain
$$[e_\sigma, e_{-\sigma}^m] = m h_\sigma e_{-\sigma}^{m-1} + m(m-1) e_{-\sigma}^{m-1}.$$
Since $\mathfrak{n}^- \subset \mathfrak{p}^-$ is an ideal, we have $[e_\sigma, u(\mathfrak{n}^-)] \subseteq u(\mathfrak{n}^-)^\dagger$. The assertion now follows by observing (19), the equality $u_2 h_\sigma = (\lambda(h_\sigma) + 2) u_2$, and
$$[e_\sigma, e_{-\sigma}^m b] = [e_\sigma, e_{-\sigma}^m] b + e_{-\sigma}^m [e_\sigma, b],$$
for every $b \in u(\mathfrak{n}^-)$. $\square$

We denote the differential of $\lambda \in X(T)$ and the resulting algebra homomorphism $u(\mathfrak{t}) \to k$ again by λ for simplicity.

Lemma 7.2. *Let $\lambda \in X(T)$ such that $\Psi_\lambda^+ = \{\gamma\}$. Then $e_\sigma M_{\lambda(h_\sigma)} \subseteq \tilde{M}$.*

Proof. We consider an element
$$a := e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} e_{-\alpha}^{m_\alpha} e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}}} \in u(\mathfrak{u}^-)$$
and claim that $q := e_\sigma u_1 u_4 \omega a \otimes 1 \in \tilde{M}$. As $e_\sigma u_4 = u_2 e_\sigma$ and $\nu(e_\sigma) = e_\sigma$, we get, observing (19),
$$q = u_3 \omega [y, a] \otimes 1.$$
By Lemma 7.1 we also have
$$q \equiv u_3 \omega [e_\alpha, a] \otimes 1 \qquad \text{mod } (\tilde{M}).$$
We set $q_1 := u_3 \omega [e_\alpha, a] \otimes 1$, and $a_1 := e_{-\gamma}^{m_\gamma} e_{-\alpha}^{m_\alpha} e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}}} \in u(\mathfrak{n}^-)$. From (i) we obtain
$$q_1 = u_3 \omega e_{-\sigma}^{m_\sigma} [e_\alpha, a_1] \otimes 1.$$

Setting $a_2 := e_{-\alpha}^{m_\alpha} e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}}}$, we obtain from (ii) the existence of $\zeta_1 \in k$ such that

$$\begin{aligned}(20)\quad q_1 &= \zeta_1 u_3 \omega e_{-\sigma}^{m_\sigma} e_\sigma e_{-\gamma}^{m_\gamma - 1} a_2 \otimes 1 + u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} [e_\alpha, a_2] \otimes 1 \\ &= \zeta_1 u_3 \omega [e_{-\sigma}^{m_\sigma}, e_\sigma] e_{-\gamma}^{m_\gamma - 1} a_2 \otimes 1 + \zeta_1 u_3 \omega e_\sigma e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma - 1} a_2 \otimes 1 + u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} [e_\alpha, a_2] \otimes 1.\end{aligned}$$

Since $[e_\sigma, e_{-\sigma}^{m_\sigma}] = m_\sigma h_\sigma e_{-\sigma}^{m_\sigma - 1} + m_\sigma(m_\sigma - 1) e_{-\sigma}^{m_\sigma - 1}$, the first summand is contained in $\tilde{M}$. Now we consider the third summand

$$q_2 := u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} [e_\alpha, a_2] \otimes 1.$$

Owing to (iii) there exist scalars $\zeta_2, \zeta_3 \in k$ such that

$$q_2 = u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} (\zeta_2 h_\alpha + \zeta_3 1) e_{-\alpha}^{m_\alpha - 1} e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}}} \otimes 1 + u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} e_{-\alpha}^{m_\alpha} [e_\alpha, e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}}}] \otimes 1.$$

Now apply (iv) to obtain $\zeta_4 \in k$ with

$$q_2 = u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} (\zeta_2 h_\alpha + \zeta_3 1) e_{-\alpha}^{m_\alpha - 1} e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}}} \otimes 1 + \zeta_4 u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} e_{-\alpha}^{m_\alpha} e_{-\gamma} e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}} - 1} \otimes 1.$$

From the Cartan-Weyl formula we get for some $\zeta_5 \in k$

$$e_{-\alpha}^{m_\alpha} e_{-\gamma} = e_{-\gamma} e_{-\alpha}^{m_\alpha} + \zeta_5 e_{-\tilde{\alpha}} e_{-\alpha}^{m_\alpha - 1} = e_{-\gamma} e_{-\alpha}^{m_\alpha} + \zeta_5 e_{-\alpha}^{m_\alpha - 1} e_{-\tilde{\alpha}}.$$

Since $u_3 h_\alpha$ is a scalar multiple of u_3, there exists an element $\zeta_6 \in k$ such that

$$q_2 = \zeta_6 u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} e_{-\alpha}^{m_\alpha - 1} e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}}} \otimes 1 + \zeta_4 u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma + 1} e_{-\alpha}^{m_\alpha} e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}} - 1} \otimes 1 + \zeta_4 \zeta_5 u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} e_{-\alpha}^{m_\alpha - 1} e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}}} \otimes 1.$$

Observe that the second summand belongs to $\tilde{M}$. Accordingly, we consider the element q_3 given by the first and third summand. By combining these, we find $\zeta_7 \in k$ such that

$$q_3 = \zeta_7 u_3 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma} e_{-\alpha}^{m_\alpha - 1} e_{-\tilde{\alpha}}^{m_{\tilde{\alpha}}} \otimes 1.$$

The element q_3 belongs to $\tilde{M}$ unless $m_\sigma = p - 1$, $m_\gamma = 0$, $m_\alpha = 1$, and $m_{\tilde{\alpha}} = 0$. In that case the original element a has the form

$$a = e_{-\sigma}^{p-1} e_{-\alpha}.$$

Repeating the previous steps we obtain, observing (i),

$$q_1 = u_3 \omega [e_\alpha, a] \otimes 1 = u_3 \omega e_{-\sigma}^{p-1} [e_\alpha, e_{-\alpha}] \otimes 1 = u_3 \omega e_{-\sigma}^{p-1} h_\alpha \otimes 1.$$

From the Cartan-Weyl formula we obtain, observing that -2σ is not a root,

$$\begin{aligned} e_{-\sigma}^{p-1} h_\alpha &= h_\alpha e_{-\sigma}^{p-1} + (p-1)(\mathrm{ad}\, e_{-\sigma})(h_\alpha) e_{-\sigma}^{p-2} \\ &= h_\alpha e_{-\sigma}^{p-1} - [e_{-\sigma}, h_\alpha] e_{-\sigma}^{p-2} \\ &= h_\alpha e_{-\sigma}^{p-1} - \sigma(h_\alpha) e_{-\sigma}^{p-1}. \end{aligned}$$

Since the duality $^\vee : \Psi \longrightarrow \Psi^\vee$ interchanges long and short roots (cf. [20, (2.9)]), we have $h_\alpha = 2h_\sigma + h_\gamma$ and $h_{\tilde{\alpha}} = h_\sigma + h_\gamma$. We now have

$$q_1 = u_3 \omega h_\alpha e_{-\sigma}^{p-1} \otimes 1 - \sigma(h_\alpha) u_3 \omega e_{-\sigma}^{p-1} \otimes 1.$$

Recall from (19) that $\omega h_\alpha = \nu^{-1}(h_\alpha)\omega = (h_\alpha + \mathrm{tr}[(\mathrm{ad}_{\mathfrak{n}})(h_\alpha)]1)\omega$. Moreover, we have

$$u_3 h_\alpha = (\lambda(h_\gamma) + 2(\lambda(h_\sigma) + 1)) u_3.$$

Consequently,

$$q_1 = r_q u_3 \omega e_{-\sigma}^{p-1} \otimes 1,$$

where

$$\begin{aligned} r_q &= \lambda(h_\gamma) + 2(\lambda(h_\sigma)+1) + \gamma(h_\alpha) + \alpha(h_\alpha) + \tilde{\alpha}(h_\alpha) - \sigma(h_\alpha) \\ &= \lambda(h_\gamma) + 2\lambda(h_\sigma) + 2 + 3\gamma(h_\alpha) + \alpha(h_\alpha) \\ &= \lambda(h_\gamma) + 2\lambda(h_\sigma) + 4, \end{aligned}$$

since $\gamma(h_\alpha) = 0$. Setting $\lambda(h_\gamma) = -1$, we get

$$r_q = 3 + 2\lambda(h_\sigma).$$

We can find λ such that $\lambda(h_\sigma) \neq -1$ and $r_q = 0$. With this choice of λ we see that

$$\lambda(h_\alpha) = 2\lambda(h_\sigma) + \lambda(h_\gamma) = -4 \neq -3 \quad \text{and} \quad \lambda(h_{\tilde{\alpha}}) = \lambda(h_\sigma + h_\gamma) = \lambda(h_\sigma) - 1.$$

Since $\lambda(h_\sigma) \neq -1$, we have $\lambda(h_{\tilde{\alpha}}) \neq -2$. Consequently, $\Psi^+_\lambda = \{\gamma\}$, as desired.

As an upshot of the above we obtain for this particular choice of λ from (20)

$$q \equiv \zeta_1 u_3 \omega e_\sigma e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma - 1} a_2 \otimes 1 \equiv \zeta_1 e_\sigma u_1 u_4 \omega e_{-\sigma}^{m_\sigma} e_{-\gamma}^{m_\gamma - 1} a_2 \otimes 1 \mod (\tilde{M}).$$

Observe that this last element is again of the form we considered initially. The proof of the lemma can now be completed by induction on m_γ. □

Proposition 7.3. *Suppose that $p \geq 5$. Let G be of type B_2 and $\lambda \in X(T)$ such that $\Psi^+_\lambda = \{\gamma\}$. Then $\mathcal{V}_{\mathfrak{g}}(Z(\lambda)) \supseteq \mathfrak{n}_2 = ke_\sigma + ke_\alpha + ke_{\tilde{\alpha}}$.*

Proof. We begin by showing that there exists a character λ such that $\Psi^+_\lambda = \{\gamma\}$ and $e_\sigma + e_\alpha \in \mathcal{V}_{\mathfrak{g}}(Z(\lambda))$. Choose λ as in Lemma 7.2 and extend it to $\mathfrak{b}$ by setting $\lambda(\mathfrak{u}) = \{0\}$.

Let $y := e_\sigma + e_\alpha$. Then we have $y^{[p]} = 0$, and the $u(\mathfrak{g})$-module $A := u(\mathfrak{g}) \otimes_{u(ky)} k$ has dimension p^9 and rank variety $\mathcal{V}_{\mathfrak{g}}(A) = ky$.

Since $u(\mathfrak{g}) : u(ky)$ is a Frobenius extension of first kind, [13, (1.6)] provides an isomorphism

$$\mathrm{Hom}_{u(ky)}(u(\mathfrak{g}), k) \cong A$$

of $u(\mathfrak{g})$-modules. Frobenius reciprocity then yields

$$\mathrm{Ext}^1_{u(ky)}(Z(\lambda), k) \cong \mathrm{Ext}^1_{u(\mathfrak{g})}(Z(\lambda), A) \cong \mathrm{Ext}^1_{u(\mathfrak{b})}(k_\lambda, A) \cong \mathrm{H}^1(u(\mathfrak{b}), A \otimes_k k_{-\lambda}).$$

Note that $\mathfrak{b} = (\mathfrak{t} \oplus ke_\sigma) \oplus \mathfrak{n}$ and $\mathfrak{b}^- = (\mathfrak{t} \oplus ke_{-\sigma}) \oplus \mathfrak{n}^-$ are semidirect products of restricted Lie algebras. We denote their first summands by $\mathfrak{h}$ and $\mathfrak{h}^-$, respectively.

Since $\mathcal{V}_{\mathfrak{g}}(A) \cap \mathfrak{n} = ky \cap \mathfrak{n} = \{0\}$, the module $A|_{u(\mathfrak{n})}$ is projective, and the Grothendieck spectral sequence

$$\mathrm{H}^p(u(\mathfrak{h}), \mathrm{H}^q(u(\mathfrak{n}), A \otimes_k k_{-\lambda})) \Rightarrow \mathrm{H}^n(u(\mathfrak{b}), A \otimes_k k_{-\lambda})$$

collapses to isomorphisms

$$\mathrm{H}^n(u(\mathfrak{h}), (A \otimes_k k_{-\lambda})^{\mathfrak{n}}) \cong \mathrm{H}^n(u(\mathfrak{b}), A \otimes_k k_{-\lambda})$$

for all $n \geq 0$. Since $\lambda(\mathfrak{n}) = \{0\}$, we have

$$x \cdot (a \otimes 1) = xa \otimes 1 \text{ for all } x \in \mathfrak{n}, a \in A,$$

whence

$$(A \otimes_k k_{-\lambda})^{\mathfrak{n}} \cong A^{\mathfrak{n}} \otimes_k k_{-\lambda}.$$

As an upshot of the above we have

$$\mathrm{Ext}^1_{u(ky)}(Z(\lambda), k) \cong \mathrm{Ext}^1_{u(\mathfrak{b})}(k_\lambda, A) \cong \mathrm{Ext}^1_{u(\mathfrak{h})}(k_\lambda, A^{\mathfrak{n}}).$$

We continue by investigating the structure of the $u(\mathfrak{h})$-module $A^{\mathfrak{n}}$. Observe that

$$\{\omega u \otimes 1 \mid u \in u(\mathfrak{p}^-)\} \subseteq A^{\mathfrak{n}},$$

where ω is the integral of $u(\mathfrak{n})$. The decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{n}^- \oplus \mathfrak{t} \oplus ke_{-\sigma} \oplus ky = \mathfrak{n} \oplus \mathfrak{b}^- \oplus ky$$

shows, via the PBW-Theorem, that $u \mapsto \omega u \otimes 1$ induces an isomorphism from $u(\mathfrak{b}^-)$ onto the left-hand space. In particular, this space has dimension p^6. On the other hand, the algebra $u(\mathfrak{n})$ is local, so the dimension of the free $u(\mathfrak{n})$-module A is given by

$$\dim A = (\dim u(\mathfrak{n}))(\dim A^{\mathfrak{n}}).$$

Thus, $\dim A^{\mathfrak{n}} = p^6$, and consequently

$$A^{\mathfrak{n}} = \{\omega u \otimes 1 \mid u \in u(\mathfrak{p}^-)\} = \{\omega u \otimes 1 \mid u \in u(\mathfrak{b}^-)\}.$$

Since $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{k}$, where $\mathfrak{k} = kh_\sigma \oplus ke_\sigma$ and $\mathfrak{z}(\mathfrak{h}) = k(h_\sigma + h_\gamma)$, we obtain

$$u(\mathfrak{h}) \cong u(\mathfrak{z}(\mathfrak{h})) \otimes_k u(\mathfrak{k}) \cong \bigoplus_{i=1}^{p} u(\mathfrak{k}) = \bigoplus_{\zeta \in \mathbb{F}_p} v_\zeta u(\mathfrak{k}),$$

where $v_\zeta = \prod_{\eta \in \mathbb{F}_p \setminus \{\zeta\}} (h_\sigma + h_\gamma - \eta 1)$. Since $u(\mathfrak{k})$ is connected, it follows that $u(\mathfrak{h})$ has exactly p blocks, each of which is isomorphic to $u(\mathfrak{k})$. Recall the $u(\mathfrak{t})$-module

$$M = u_1 A^{\mathfrak{n}} = \{u_1 \omega a \otimes 1 \ ; \ a \in u(\mathfrak{n}^- \oplus \mathfrak{k}^-)\}$$

introduced before Lemma 7.1, where $\mathfrak{k}^- := kh_\sigma \oplus ke_{-\sigma}$. Since $\mathfrak{n}$ is an ideal of $\mathfrak{p}$, we see that M is e_σ-invariant and thus a $u(\mathfrak{h})$-module. Observing that k_λ belongs to the block $v_{\lambda(h_\sigma + h_\gamma)} u(\mathfrak{k}) = u_1 u(\mathfrak{k})$, the foregoing block decomposition entails

$$\mathrm{Ext}^1_{u(ky)}(Z(\lambda), k) \cong \mathrm{Ext}^1_{u(\mathfrak{h})}(k_\lambda, A^{\mathfrak{n}}) \cong \mathrm{Ext}^1_{u(\mathfrak{k})}(k_\lambda, M) \cong \underline{\mathrm{Hom}}_{u(\mathfrak{k})}(\Omega_{u(\mathfrak{k})}(k_\lambda), M).$$

The algebra $u(\mathfrak{k})$ is a Nakayama algebra, and we denote by $X_i(\zeta)$ the i-dimensional indecomposable module with top k_ζ, where $1 \le i \le p$ and $\zeta \in \mathbb{F}_p$.

Let q_λ be a generator of $X_{p-1}(\lambda(h_\sigma) + 2) \cong \Omega_{u(\mathfrak{k})}(k_\lambda)$. We are going to show the existence of a homomorphism $\varphi : X_{p-1}(\lambda(h_\sigma) + 2) \longrightarrow M$ that sends q_λ to the following element

$$z_\lambda := u_1 u_2 \omega e_{-\sigma}^{p-1} \otimes 1.$$

From the Cartan-Weyl identities we obtain:

$$\begin{aligned} e_\sigma e_{-\sigma}^m &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} e_{-\sigma}^j (\mathrm{ad}\, e_{-\sigma})^{m-j}(e_\sigma) \\ &= e_{-\sigma}^m e_\sigma + m h_\sigma e_{-\sigma}^{m-1} + m(m-1) e_{-\sigma}^{m-1}, \end{aligned}$$

whence for each m with $0 \le m \le p-1$ we get

$$\begin{aligned} e_\sigma u_1 u_\xi \omega e_{-\sigma}^m \otimes 1 &= u_1 u_{\xi+2} e_\sigma \omega e_{-\sigma}^m \otimes 1 \\ &= u_1 u_{\xi+2} \omega e_\sigma e_{-\sigma}^m \otimes 1 \\ &= u_1 u_{\xi+2} \omega e_{-\sigma}^m e_\sigma \otimes 1 \\ &\quad + m u_1 u_{\xi+2} \omega h_\sigma e_{-\sigma}^{m-1} \otimes 1 \\ &\quad + m(m-1) u_1 u_{\xi+2} \omega e_{-\sigma}^{m-1} \otimes 1. \end{aligned}$$

From (i) above we obtain $0 = \omega[e_\alpha, e_{-\sigma}^m] = \omega e_\alpha e_{-\sigma}^m - \omega e_{-\sigma}^m e_\alpha$. Since $e_\alpha \in \mathfrak{n}$ and ω is a two-sided integral of $\mathfrak{n}$, we have $\omega e_\alpha e_{-\sigma}^m = 0$, so that $\omega e_{-\sigma}^m e_\alpha = 0$. As $y = e_\sigma + e_\alpha$, we have $\omega e_{-\sigma}^m e_\sigma \otimes 1 = -\omega e_{-\sigma}^m e_\alpha \otimes 1$, and the first summand above vanishes.

Directly from the definition we obtain $\nu(h_\sigma) = h_\sigma$, whence we get $mu_1u_{\xi+2}\omega h_\sigma e_{-\sigma}^{m-1} \otimes 1 = m(\xi+2)u_1u_{\xi+2}\omega e_{-\sigma}^{m-1} \otimes 1$. Consequently, we obtain

$$e_\sigma u_1 u_\xi \omega e_{-\sigma}^m \otimes 1 = m(m+\xi+1)u_1u_{\xi+2}\omega e_{-\sigma}^{m-1} \otimes 1.$$

Thus in particular,

$$\begin{aligned} e_\sigma u_1 u_\xi \omega e_{-\sigma}^{p-i-1} \otimes 1 &= (p-i-1)(\xi-i)u_1u_{\xi+2}\omega e_{-\sigma}^{p-i-2} \otimes 1 \\ &= (i+1)(i-\xi)u_1u_{\xi+2}\omega e_{-\sigma}^{p-i-2} \otimes 1 \end{aligned}$$

for each $0 \le i \le p-1$. Let $W := u(\mathfrak{k})z_\lambda$. The above identities show that

$$W = \bigoplus_{i=0}^{c_\sigma} ku_1u_{2i+\lambda(h_\sigma)+2}\omega e_{-\sigma}^{p-i-1} \otimes 1 \cong X_{-\lambda(h_\sigma)-1}(\lambda(h_\sigma)+2),$$

where $0 \le c_\sigma \le p-1$ and $c_\sigma \equiv -\lambda(h_\sigma) - 2 \mod p$. Consequently, $\dim W \equiv -\lambda(h_\sigma) - 1 \mod p$, and since $\lambda(h_\sigma) \neq -1$, we have $\dim W \le p-1$. Accordingly, there exists a homomorphism $\varphi : X_{p-1}(\lambda(h_\sigma)+2) \longrightarrow W \subseteq M$ sending q_λ to z_λ.

We continue by showing that φ does not factor through a projective module. Since $u(\mathfrak{k})$ is self-injective this is equivalent to saying that φ does not factor through the injective hull $X_p(\lambda(h_\sigma))$ of $X_{p-1}(\lambda(h_\sigma)+2)$. As $X_{p-1}(\lambda(h_\sigma)+2) = e_\sigma X_p(\lambda(h_\sigma))$, the map $\varphi : X_{p-1}(\lambda(h_\sigma)+2) \longrightarrow M$ does not factor through $X_p(\lambda(h_\sigma))$ if $\varphi(q_\lambda) \notin e_\sigma M \cap M_{\lambda(h_\sigma)+2} = e_\sigma M_{\lambda(h_\sigma)}$. By Lemma 7.2 the element $\varphi(q_\lambda) = z_\lambda$ does not lie in $\tilde{M} \supseteq e_\sigma M_{\lambda(h_\sigma)}$. Accordingly, φ defines a nonzero element of $\underline{\mathrm{Hom}}_{u(\mathfrak{k})}(X_{p-1}(\lambda(h_\sigma)+2), M) \cong \mathrm{Ext}^1_{u(ky)}(Z(\lambda), k)$. In particular, $Z(\lambda)$ is not a projective $u(ky)$-module, and $y = e_\sigma + e_\alpha \in \mathcal{V}_\mathfrak{g}(Z(\lambda))$.

Direct computation shows that $B \cdot (e_\sigma + e_\alpha) = B \cdot (e_\sigma + e_{\tilde{\alpha}})$. Since $\dim B \cdot (e_\sigma + e_{\tilde{\alpha}}) = 3$ (cf. [5, p. 31]), the closure of this orbit coincides with $\mathfrak{n}_2$. Finally, by Lemma 2.1 the inclusion $\mathfrak{n}_2 = \overline{B \cdot (e_\sigma + e_{\tilde{\alpha}})} \subseteq \mathcal{V}_\mathfrak{g}(Z(\lambda))$ follows, as required.

Without loss we may assume that λ is p-restricted. If $\Psi_\lambda^+ = \{\gamma\}$, then any such λ belongs to the closure of one of two walls separated by the hyperplane $\{\mu \in X(T) \otimes \mathbb{R} \mid \langle \mu + \varrho, \alpha^\vee \rangle = p\}$, where $\alpha = \sigma + \gamma$. Then for any two $\lambda, \mu \in X_1(T)$ with $\Psi_\lambda^+ = \Psi_\mu^+ = \{\gamma\}$ we apply translation functors [23, (11.21)] (twice in case λ and μ belong to different walls) in order to see that $\mathcal{V}_\mathfrak{g}(Z(\lambda)) = \mathcal{V}_\mathfrak{g}(Z(\mu))$.

This completes the proof of the proposition. □

Remark. One can also show that the element y in the proof of Proposition 7.3 is subregular. Consequently, it is a Richardson element relative to the parabolic subgroup $P_{\{\gamma\}}$ (of semisimple rank 1). Since $\mathcal{V}_\mathfrak{g}(Z(\lambda))$ is $P_{\{\gamma\}}$-invariant (cf. Lemma 2.5), this implies $\mathfrak{n}_2 \subseteq \mathcal{V}_\mathfrak{g}(Z(\lambda))$.

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