PURE INJECTIVES AND THE SPECTRUM OF THE COHOMOLOGY RING OF A FINITE GROUP

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1. Introduction

Let G be a finite group and k be a field of characteristic p. The connection between the modular representations and the cohomology of G has been a subject of major interest over the last several years. For example, recent study of the stable category of kG-modules $\mathsf{StMod}(kG)$ led Rickard [27] to introduce certain idempotent modules and functors. This has led to a theory of varieties for infinitely generated modules [3, 4], and the classification of thick subcategories of the stable category $\mathsf{stmod}(kG)$ of finitely generated modules, at least in the case of a p-group [5].

The cohomology of the group G is intimately related to the properties of the trivial kG-module k and its shifted copies $\Omega^n k$, $n \in \mathbb{Z}$. It is therefore natural to ask for a description of the kG-modules which arise as a direct summand of a (possibly infinite) direct product of modules of the form $\Omega^n k$, $n \in \mathbb{Z}$. Note that such a module is always pure injective because any module over a finite dimensional algebra is pure injective if and only if it is isomorphic to a direct summand of a direct product of finite dimensional modules.

Pure injective modules have been studied for some time in representation theory of finite dimensional algebras, mostly because certain infinitely generated pure injectives (so-called generic modules) control the representation type of an algebra [11]. For example, generic modules have been used to show that the representation type of an algebra is an invariant of the stable module category [20].

More recently, the notion of a phantom map in $\mathsf{StMod}(kG)$ was introduced in [16, 6] as an analogue of a classical concept from stable homotopy theory. In this context, it became apparent as well that pure injective modules play an important role, as the modules which receive no phantom maps. In particular, under rather restrictive hypotheses, some of Rickard's idempotent modules were observed to be pure injective. This paper grew out of an attempt to understand this phenomenon.

The purpose of this paper is to investigate a certain functor T from injective modules I over the cohomology ring $H^*(G,k)$ to pure injective modules in the stable category $\mathsf{StMod}(kG)$. Most of the arguments work when kG is replaced by any finite dimensional cocommutative Hopf algebra A over k, and so this is the generality in which we work, although at some stages we need to make an assumption about the behavior of negative Tate cohomology which we only know to be true in the finite group context. Namely, either Tate cohomology is periodic, or negative Tate cohomology is nilpotent, in the sense that there exists an integer $n > 0$ such that every product of at least n elements is zero. We do not know of an example of a finite dimensional cocommutative Hopf algebra for which this fails.

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The construction of the functor T is described in Section 3, and is basically given by applying Brown representability to obtain an isomorphism

$$\mathrm{Hom}_{H^*(A,k)}(\hat{H}^*(A,M),I) \cong \underline{\mathrm{Hom}}_A(M,T(I)),$$

functorial for M in $\mathsf{StMod}(A)$. On injective modules I having no simple submodules, the functor T is fully faithful, and this allows us to understand the endomorphism ring of $T(I)$.

We begin in Section 2 by classifying the injective modules over the Tate cohomology ring $\hat{H}^*(A,k)$. We prove that under the condition that either Tate cohomology is periodic or negative Tate cohomology is nilpotent, every injective module over Tate cohomology is coinduced from an injective module over the ordinary cohomology ring. A module of the form $T(I)$ is determined by its Tate cohomology which is the coinduced module $\hat{I} = \mathrm{Hom}^*_{H^*(A,k)}(\hat{H}^*(A,k),I)$. This explains the relevance of coinduced modules for our study of the functor T.

Under the assumption on Tate cohomology mentioned above, the modules of the form $T(I)$ are precisely the direct summands of direct products of modules of the form $\Omega^n k$, $n \in \mathbb{Z}$. This is proved in Theorem 4.7. We show in Section 5 that the modules $T(I)$ are pure injective, and usually but not always Σ-pure injective. In Section 6, we show how Theorem 4.7 can be interpreted in terms of spectral categories. This establishes a complete classification of the modules which arise as a direct summand of a direct product of modules of the form $\Omega^n k$, $n \in \mathbb{Z}$.

In Section 7, we specialize to the case of a finite group, and describe the variety in the sense of Benson, Carlson and Rickard [4] of the module $T(I)$ for every indecomposable injective module I. This is essentially a question of translating a proof from Hovey, Palmieri and Strickland [18] into the language of group representation theory.

In Section 8, we investigate T as a map from the projective variety $\mathrm{Proj}(H^*(A,k))$ to the Ziegler spectrum $\mathrm{Ind}(A)$ of indecomposable pure injectives, endowed with the Zariski topology. We prove that this map induces a homeomorphism between $\mathrm{Proj}(H^*(A,k))$ and its image in $\mathrm{Ind}(A)$. The image of a point of $\mathrm{Proj}(H^*(A,k))$ is a generic module in the sense of Crawley-Boevey [11] if and only if the point is a generic point for a component of the variety, or in other words, it corresponds to a minimal prime in $H^*(A,k)$. An interesting consequence is the fact that A is of strongly unbounded representation type whenever $\hat{H}^*(A,k)$ is nonperiodic and the field k is infinite. This is a special case of the Second Brauer–Thrall Conjecture which asserts that every finite dimensional algebra over an infinite field and of infinite representation type has strongly unbounded representation type.

We show in Section 9 that in case $A = kG$, the generic modules corresponding to minimal primes are, up to translation, the modules investigated in our recent paper [7]. More generally, it turns out that for a nonmaximal prime ideal $\mathfrak{p}$ of $H^*(G,k)$, the module obtained by applying T to the injective hull of $H^*(G,k)/\mathfrak{p}$ is closely related to the corresponding Rickard idempotent module $\kappa(V)$ introduced in [4]. The proof of this involves a detailed analysis of the Greenlees–Lyubeznik spectral sequence [17], and will be the subject of a separate paper.

Conventions and notation. When we talk about modules and maps over a $\mathbb{Z}$-graded ring R, we always mean graded modules unless otherwise specified. If M and N are R-modules and $n \in \mathbb{Z}$, we write $M[n]$ for the graded module with $M[n]_i = M_{i+n}$. We write $\mathrm{Hom}_R(M,N)$ for the degree zero maps, and $\mathrm{Hom}^*_R(M,N)$ for the graded maps. In other words,

$$\mathrm{Hom}^n_R(M,N) = \mathrm{Hom}_R(M[n],N).$$

If R is graded commutative, then $\mathrm{Hom}^*_R(M,N)$ is a graded R-module in an obvious way.

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The modules $T(I)$ were first defined by Hovey, Palmieri and Strickland [18], definition 6.0.8 (c), but they are not systematically investigated in that paper.

2. Injective modules over the cohomology ring

Before introducing and investigating the functor T, we need to study the injective modules over the ordinary cohomology ring and also over the Tate cohomology ring. In this section we classify the injective modules over the Tate cohomology ring, under a hypothesis which we know to hold in the case of the group algebra of a finite group.

Let A be a finite dimensional cocommutative Hopf algebra over a field k. We denote by k the trivial A-module and the cohomology ring $H^*(A,k)$ is by definition $\mathrm{Ext}^*_A(k,k)$. This is a finitely generated graded commutative k-algebra by a theorem of Friedlander and Suslin [13]; in particular, it is a Noetherian ring.

We write $\mathsf{Mod}(A)$ for the category of all A-modules and homomorphisms, and $\mathsf{mod}(A)$ for the full subcategory whose objects are the finitely generated A-modules. We write $\mathsf{StMod}(A)$ for the stable category, which has the same objects as $\mathsf{Mod}(A)$, but where the homomorphisms are given by

$$\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)/\mathrm{PHom}_A(M,N)$$

where $\mathrm{PHom}_A(M,N)$ denotes the subspace of $\mathrm{Hom}_A(M,N)$ consisting of maps which factor through a projective module. This is a triangulated category, in which the triangles come from short exact sequences in $\mathsf{Mod}(A)$. The translation in this category is obtained by taking the cokernel of an injective hull, and is written Ω^{-1}. Its inverse, Ω, is obtained by taking the kernel of a projective cover. We write $\mathsf{stmod}(A)$ for the full subcategory of finitely generated modules in $\mathsf{StMod}(A)$.

Given an A-module M, the Tate cohomology $\hat{H}^*(A,M)$ of M is defined as the cohomology of the complex $\mathrm{Hom}_A(\hat{P}_*,M)$ where

$$\hat{P}_*: \qquad \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\delta} P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots$$

is a complete resolution of the trivial module, namely an exact sequence of projective modules such that $\mathrm{Im}(\delta) \cong k$. The Tate cohomology of M is the same as the graded homomorphisms $k \to M$ in the category $\mathsf{StMod}(A)$. More precisely,

$$\hat{H}^n(A,M) \cong \underline{\mathrm{Hom}}_A(\Omega^n k, M)$$

for all $n \in \mathbb{Z}$. The nonnegative part P_* of $\hat{P}_*$ is a projective resolution of the trivial module, and the canonical map $\hat{P}_* \to P_*$ induces a homomorphism $H^*(A,M) \to \hat{H}^*(A,M)$ since $H^*(A,M)$ is the cohomology of the complex $\mathrm{Hom}_A(P_*,M)$. This gives an exact sequence of $H^*(A,k)$-modules

$$0 \longrightarrow \mathrm{PHom}_A(k,M) \longrightarrow H^*(A,M) \longrightarrow \hat{H}^*(A,M) \longrightarrow \hat{H}^-(A,M) \longrightarrow 0 \qquad (2.1)$$

where $\hat{H}^-(A,M)$ denotes the negatively graded part of Tate cohomology. In particular, the cohomology ring $H^*(A,k)$ can be viewed as a subring of the Tate cohomology ring $\hat{H}^*(A,k)$.

The elements of positive degree in $H^*(A,k)$ form a maximal ideal $\mathfrak{m} = H^+(A,k)$ and we have $H^*(A,k)/\mathfrak{m} \cong k$. By Tate duality (see for example Cartan and Eilenberg [10], section XII.6), the negative cohomology $\hat{H}^-(A,k)$ is the k-dual of $H^*(A,k)$, and is hence isomorphic to the injective hull of k as an $H^*(A,k)$-module. The negative Tate cohomology $\hat{H}^-(A,M)$ of any A-module M is an $\mathfrak{m}$-torsion module, in the sense that given $x \in \mathfrak{m}$, and $a \in \hat{H}^-(A,M)$, there exists $n \geq 1$ such that $x^n a = 0$. This is because elements from $\mathfrak{m}$ have positive degree while elements from negative Tate cohomology have negative degree.

If M is a module over $H^*(A,k)$, then we obtain a module over $\hat{H}^*(A,k)$ by the recipe

$$\hat{M} = \operatorname{Hom}^*_{H^*(A,k)}(\hat{H}^*(A,k), M). \tag{2.2}$$

Note that the coinduction functor which sends M to $\hat{M}$ is right adjoint to restriction from $\hat{H}^*(A,k)$-modules to $H^*(A,k)$-modules. Therefore, it is left exact and takes injective $H^*(A,k)$-modules to injective $\hat{H}^*(A,k)$-modules. We denote by $\tau_M \colon \hat{M} \to M$ the canonical $H^*(A,k)$-module homomorphism given by evaluation on the identity.

Lemma 2.1. *Let I be an injective $H^*(A,k)$-module and suppose $\operatorname{Hom}^*_{H^*(A,k)}(k,I) = 0$. Then $\tau_I \colon \hat{I} \to I$ is an isomorphism of $H^*(A,k)$-modules. If M is another $H^*(A,k)$-module then the natural map*

$$\operatorname{Hom}_{H^*(A,k)}(I,M) \longrightarrow \operatorname{Hom}_{\hat{H}^*(A,k)}(\hat{I},\hat{M})$$

is an isomorphism.

Proof. Every quotient of $\hat{H}^-(A,k)$ is $\mathfrak{m}$-torsion and every nonzero $\mathfrak{m}$-torsion module has a submodule isomorphic to $k[n]$ for some n. Therefore we have

$$\operatorname{Hom}^*_{H^*(A,k)}(\hat{H}^-(A,k), I) = 0.$$

So applying $\operatorname{Hom}^*_{H^*(A,k)}(-,I)$ to the exact sequence (2.1) (with $M = k$) gives the first part. The second part follows easily because one gets an inverse map

$$\operatorname{Hom}_{\hat{H}^*(A,k)}(\hat{I},\hat{M}) \longrightarrow \operatorname{Hom}_{H^*(A,k)}(I,M)$$

by sending a map ϕ to the composite $\tau_M \circ \phi \circ (\tau_I)^{-1}$. □

We denote by $I_\mathfrak{m}$ the injective envelope of the $H^*(A,k)$-module $k \cong H^*(A,k)/\mathfrak{m}$.

Lemma 2.2. *Let $I = I_\mathfrak{m}[n]$, $n \in \mathbb{Z}$. Then $\hat{I} \cong \hat{H}^*(A,k)[n-1]$ and $\operatorname{End}_{\hat{H}^*(A,k)}(\hat{I}) \cong k$. In particular, the ring $\hat{H}^*(A,k)$ is self-injective.*

Proof. We have

$$I_\mathfrak{m}[n] \cong \operatorname{Hom}^*_k(H^*(A,k), k[n]).$$

So by Tate duality we have

$$\begin{aligned}\hat{I} &\cong \operatorname{Hom}^*_{H^*(A,k)}(\hat{H}^*(A,k), \operatorname{Hom}^*_k(H^*(A,k),k[n])) \\ &\cong \operatorname{Hom}^*_k(\hat{H}^*(A,k),k[n]) \cong \hat{H}^*(A,k)[n-1].\end{aligned}$$

In order to compute the endomorphism ring of $\hat{I}$, observe that

$$\operatorname{End}^*_{\hat{H}^*(A,k)}(\hat{I}) = \operatorname{End}^*_{\hat{H}^*(A,k)}(\hat{H}^*(A,k)) \cong \hat{H}^*(A,k).$$

Therefore $\operatorname{End}_{\hat{H}^*(A,k)}(\hat{I}) = \hat{H}^0(A,k) = k$. □

The classification of injective modules over a graded commutative Noetherian ring [1] R is well known, and can be found in Bruns and Herzog [9]. An arbitrary direct sum of injective modules is injective, and every injective module decomposes essentially uniquely as a direct sum of injective indecomposables. Given a homogeneous prime ideal $\mathfrak{p}$, the injective hull $I_\mathfrak{p} = E(R/\mathfrak{p})$ of the quotient $R/\mathfrak{p}$ is indecomposable, and each injective indecomposable is isomorphic to a shifted copy $I_\mathfrak{p}[n]$ for some prime $\mathfrak{p}$ and some $n \in \mathbb{Z}$. A theorem of Matlis [23] describes the endomorphism ring of $I_\mathfrak{p}[n]$. It is isomorphic to the completion $R_\mathfrak{p}^\wedge$ of the localization $R_\mathfrak{p}$ with respect to the powers of the maximal ideal $\mathfrak{p}_\mathfrak{p}$. Each element of $R_\mathfrak{p}^\wedge$ acts locally nilpotently on $I_\mathfrak{p}[n]$.

In terms of this classification of injective modules, if an injective $H^*(A,k)$-module I has no summand isomorphic to $I_\mathfrak{m}[n]$ for any n, then $\mathrm{Hom}^*_{H^*(A,k)}(k,I) = 0$, and so by Lemma 2.1, we have $I = \hat{I}$.

Next we shall investigate the injective modules over the Tate cohomology ring. Note that this ring is usually not Noetherian. However, in the case of the group algebra of a finite group, negative Tate cohomology is usually nilpotent, and this allows us to classify all injective modules.

Lemma 2.3. *If G is a finite group of p-rank greater than one, and H is a subgroup of p-rank one, then the restriction to H of any element of nonzero degree in $\hat{H}^*(G,k)$ is nilpotent.*

Proof. Let $x \in H^n(G,k)$. If $\mathrm{res}_{G,H}(x)$ is not nilpotent, then it is invertible, since $\hat{H}^*(H,k)$ is periodic. So there exists $y \in \hat{H}^{-n}(H,k)$ such that

$$\mathrm{res}_{G,H}(x\mathrm{Tr}_{H,G}(y)) = \mathrm{res}_{G,H}(x)y = 1.$$

Thus $x\mathrm{Tr}_{H,G}(y) = 1$. But $\hat{H}^*(G,k)$ is not periodic, so this implies that x is a multiple of the identity, and hence has degree zero. □

Proposition 2.4. *Let G be a finite group and k be a field of characteristic p. If the p-rank of G is at least two (i.e., $H^*(G,k)$ is not periodic) then there exists $n > 0$ such that $(\hat{H}^-(G,k))^n = 0$.*

Proof. Restriction to a Sylow p-subgroup is injective in cohomology, so without loss of generality, G is a p-group. Since G has p-rank at least two, and the center of a nontrivial p-group is necessarily nontrivial, it follows that every maximal elementary abelian p-subgroup of G has rank at least two. If G is elementary abelian, it follows from theorem 3.1 of [2] that $(\hat{H}^-(G,k))^2 = 0$, so we may assume that G is not elementary abelian. The rest of the proof is based on an argument of Quillen and Venkov [26]. By induction and Lemma 2.3, we may assume that there exists $n > 0$ such that every proper subgroup H of G has the property that $(\hat{H}^-(H,k))^n = 0$. Then every element of $(\hat{H}^-(G,k))^{2n}$ is divisible by the Bockstein $\beta(x)$ for each $0 \neq x \in H^1(G,\mathbb{F}_p)$, by the Quillen–Venkov lemma. By a theorem of Serre [28], there exist nonzero elements $x_1, \ldots, x_m \in H^1(G,\mathbb{F}_p)$ such that $\beta(x_1)\ldots\beta(x_m) = 0$. Therefore $(\hat{H}^-(G,k))^{2nm} = 0$. □

For a ring R we denote by $\mathsf{Inj}(R)$ the full subcategory of injective R-modules, and we denote by $\mathsf{Inj}_0(R)$ the full subcategory of injective R-modules I satisfying $\mathrm{Hom}_R(S,I) = 0$ for every simple R-module S. For example, $\mathsf{Inj}_0(H^*(A,k))$ is the full subcategory of injective $H^*(A,k)$-modules I satisfying $\mathrm{Hom}^*_{H^*(A,k)}(k,I) = 0$ since every simple $H^*(A,k)$-module is isomorphic to a shifted copy

[1] Note that in a graded commutative ring, $ab = (-1)^{|a||b|}ba$, so that it is not, strictly speaking, commutative. However, it is routine but tedious to check that the standard theorems about strictly commutative graded rings apply equally well to graded commutative rings.

of the trivial module k. If $\hat{H}^*(A,k)$ is periodic then we have

$$\mathsf{Inj}_0(\hat{H}^*(A,k)) = 0 = \mathsf{Inj}_0(H^*(A,k))$$

since $\hat{H}^*(A,k)$ and $H^*(A,k)$ both are graded Artinian. Next we study $\mathsf{Inj}_0(\hat{H}^*(A,k))$ in the non-periodic case.

Lemma 2.5. *Suppose that $\hat{H}^*(A,k)$ is not periodic. Then k is an $\hat{H}^*(A,k)$-module and every simple $\hat{H}^*(A,k)$-module is isomorphic to $k[n]$ for some $n \in \mathbb{Z}$.*

Proof. We claim that $\hat{H}^-(A,k) \oplus \hat{H}^+(A,k)$ is an ideal in $\hat{H}^*(A,k)$. This is because otherwise, we have elements $x \in \hat{H}^-(A,k)$ and $y \in \hat{H}^+(A,k)$ such that $xy = 1$. In this case, $\hat{H}^*(A,k)$ is periodic. Thus

$$k = \hat{H}^*(A,k)/(\hat{H}^-(A,k) \oplus \hat{H}^+(A,k))$$

is an $\hat{H}^*(A,k)$-module.

Now suppose that S is a simple $\hat{H}^*(A,k)$-module. The shifted copies of $I_{\mathfrak{m}}$ cogenerate the category of $H^*(A,k)$-modules. Using Lemma 2.2, we get therefore

$$\mathrm{Hom}_{\hat{H}^*(A,k)}(S, \hat{H}^*(A,k)[n-1]) \cong \mathrm{Hom}_{H^*(A,k)}(S, I_{\mathfrak{m}}[n]) \neq 0$$

for some $n \in \mathbb{Z}$. The $\hat{H}^*(A,k)$-module $\hat{H}^*(A,k)[n-1]$ is indecomposable injective with a simple socle which is isomorphic to $k[n]$. Thus $S \cong k[n]$. $\square$

Lemma 2.6. *Suppose that $(\hat{H}^-(A,k))^n = 0$ for some $n > 0$.*

(i) *If M is an $H^*(A,k)$-module such that $\hat{M} = 0$ then $M = 0$.*
(ii) *If N is an $\hat{H}^*(A,k)$-module such that $\mathrm{Hom}^*_{\hat{H}^*(A,k)}(k,N) = 0$ then $\mathrm{Hom}^*_{H^*(A,k)}(k,N) = 0$.*

Proof. Let $\mathfrak{r}$ be the ideal in $\hat{H}^*(A,k)$ generated by $\hat{H}^-(A,k)$ and let $\mathfrak{s} = \mathfrak{r} \cap H^*(A,k)$. Then $\mathfrak{r}$ is nilpotent by our assumption since the ring is graded commutative.

(i) Suppose M is a nonzero $H^*(A,k)$-module. Let $j > 0$ be the smallest integer such that $\mathfrak{s}^j.M = 0$. Then given any homogeneous nonzero element x in $\mathfrak{s}^{j-1}.M$, say of degree m, there is a nonzero $H^*(A,k)$-module homomorphism $\hat{H}^*(A,k)[m] \to M$ taking 1 to x. Thus $\hat{M} \neq 0$.

(ii) Suppose $\phi\colon k[m] \to N$ is a nonzero $H^*(A,k)$-module homomorphism. Let $j > 0$ be the smallest integer such that $\mathfrak{r}^j.\mathrm{Im}(\phi) = 0$. Then given any homogeneous nonzero element x in $\mathfrak{r}^{j-1}.\mathrm{Im}(\phi)$, say of degree m', there is a nonzero $\hat{H}^*(A,k)$-module homomorphism $k[m'] \to N$ taking 1 to x. $\square$

Proposition 2.7. *Suppose that $(\hat{H}^-(A,k))^n = 0$ for some $n > 0$. Then the functor*

$$\mathsf{Inj}(H^*(A,k)) \longrightarrow \mathsf{Inj}(\hat{H}^*(A,k))$$

sending I to $\hat{I}$ induces an equivalence between $\mathsf{Inj}_0(H^*(A,k))$ and $\mathsf{Inj}_0(\hat{H}^*(A,k))$.

Proof. First observe that $I \in \mathsf{Inj}_0(H^*(A,k))$ implies $\hat{I} \in \mathsf{Inj}_0(\hat{H}^*(A,k))$. This follows from the adjointness isomorphism

$$\mathrm{Hom}^*_{\hat{H}^*(A,k)}(k, \hat{I}) \cong \mathrm{Hom}^*_{H^*(A,k)}(k, I),$$

together with the description of the simple $\hat{H}^*(A,k)$-modules in Lemma 2.5. In Lemma 2.1 it is shown that the coinduction functor is fully faithful on $\mathsf{Inj}_0(H^*(A,k))$. Thus it remains to show that it is dense. To this end let J be a module in $\mathsf{Inj}_0(\hat{H}^*(A,k))$. Let $J \to E(J)$ be an injective envelope over $H^*(A,k)$ and observe that $E(J)$ belongs to $\mathsf{Inj}_0(H^*(A,k))$ by Lemma 2.6. The composite

$$J \longrightarrow \hat{J} \longrightarrow \widehat{E(J)}$$

is a split monomorphism so that $J \cong \operatorname{Ker}(\varepsilon)$ for some idempotent endomorphism ε of $\widehat{E(J)}$. Using again Lemma 2.1, we get an idempotent endomorphism ϕ of $E(J)$ such that $\hat{\phi} = \varepsilon$. Clearly, $I = \operatorname{Ker}(\phi)$ belongs to $\mathsf{Inj}_0(H^*(A,k))$ and $\hat{I} \cong J$. □

The following theorem gives a classification of injective modules over the Tate cohomology ring, under hypotheses which are satisfied at least in the case of the group algebra of a finite group.

Theorem 2.8. *Suppose that either*
(i) $(\hat{H}^-(A,k))^n = 0$ *for some* $n > 0$, *or*
(ii) $\hat{H}^*(A,k)$ *is periodic.*

Then an $\hat{H}^*(A,k)$*-module is injective if and only if it is isomorphic to*

$$\hat{I} = \operatorname{Hom}^*_{H^*(A,k)}(\hat{H}^*(A,k), I)$$

for some injective $H^*(A,k)$*-module* I*. Moreover, if* $I = \bigoplus_{\lambda\in\Lambda} I_\lambda$ *is a decomposition into indecomposables over* $H^*(A,k)$ *then each* $\hat{I}_\lambda$ *is an indecomposable* $\hat{H}^*(A,k)$*-module, and*

$$\hat{I} \cong \Big(\bigoplus_{\lambda\in\Lambda_0} \hat{I}_\lambda\Big) \oplus E\Big(\bigoplus_{\lambda\in\Lambda\setminus\Lambda_0} \hat{I}_\lambda\Big)$$

where Λ_0 *denotes the set of* $\lambda \in \Lambda$ *such that* $\operatorname{Hom}^*_{H^*(A,k)}(k, I_\lambda) = 0$.

Proof. (i) Let J be an injective $\hat{H}^*(A,k)$-module, and let $S = \operatorname{Hom}^*_{\hat{H}^*(A,k)}(k, J)$. Write $E_{\hat{H}}(S)$ for the injective hull of S as an $\hat{H}^*(A,k)$-module, so that we have a decomposition

$$J \cong J_0 \oplus E_{\hat{H}}(S)$$

with $\operatorname{Hom}^*_{\hat{H}^*(A,k)}(k, J_0) = 0$. It follows from Proposition 2.7 that J_0 is injective over $H^*(A,k)$ and that $\hat{J}_0 \cong J_0$.

We claim that the other summand, $E_{\hat{H}}(S)$, is isomorphic to $\widehat{E_H(S)}$, where $E_H(S)$ is the injective hull of S as an $H^*(A,k)$-module. The map $S \to E_H(S)$ induces a map $\hat{S} \to \widehat{E_H(S)}$. Composing with the natural monomorphism $S \to \hat{S}$, we see that there are maps of $\hat{H}^*(A,k)$-modules $E_{\hat{H}}(S) \to \widehat{E_H(S)}$ and $\widehat{E_H(S)} \to E_{\hat{H}}(S)$ whose composite is the identity on $E_{\hat{H}}(S)$. So we obtain a decomposition

$$\widehat{E_H(S)} \cong E_{\hat{H}}(S) \oplus L.$$

Now $\operatorname{Hom}^*_{\hat{H}^*(A,k)}(k, L) = 0$, since

$$\operatorname{Hom}^*_{\hat{H}^*(A,k)}(k, \widehat{E_H(S)}) \cong \operatorname{Hom}^*_{H^*(A,k)}(k, E_H(S)) \cong S.$$

It follows from the first part of the proof that L is injective over $H^*(A,k)$ and that $\hat{L} \cong L$. The split monomorphism $\hat{L} \to \widehat{E_H(S)}$ is of the form $\hat{\alpha}$ for some map $\alpha\colon L \to E_H(S)$ by Lemma 2.1. We have $\widehat{\operatorname{Ker}(\alpha)} \cong \operatorname{Ker}(\hat{\alpha}) = 0$ and it follows from Lemma 2.6 that α is a monomorphism. Thus $L = 0$ since $\operatorname{Hom}^*_{H^*(A,k)}(k, L) = 0$ and $\operatorname{Hom}^*_{H^*(A,k)}(k, U) \neq 0$ for every nonzero submodule U of $E_H(S)$.

We now take $I = J_0 \oplus E_H(S)$ to see that $\hat{I} \cong J$. Furthermore,

$$\widehat{E_H(S)} \cong E_{\hat{H}}(S) \cong E\big(\bigoplus_{\lambda\in\Lambda\setminus\Lambda_0} \hat{I}_\lambda\big),$$

where the set $\Lambda \setminus \Lambda_0$ corresponds to a k-basis for the graded vector space S.

(ii) If $\hat{H}^*(A,k)$ is periodic, then $\mathfrak{m}$ is the only prime ideal in $H^*(A,k)$. In this case, $\hat{H}^*(A,k)$ is graded Artinian, and by Lemma 2.2 it is self-injective. So every injective module is a direct sum of modules of the form $\widehat{I_\mathfrak{m}[n]} = \hat{H}^*(A,k)[n-1]$. □

Remark. The hypothesis of the above theorem is only used via Lemma 2.6.

We end this section with a result which shows that ordinary and Tate cohomology are the same whenever we work away from the maximal prime in $H^*(A,k)$.

Proposition 2.9. *Let I be in $\mathsf{Inj}_0(H^*(A,k))$ and $\mathfrak{p}$ be a nonmaximal homogeneous prime ideal in $H^*(A,k)$. Then for every A-module M the canonical map $H^*(A,M) \to \hat{H}^*(A,M)$ induces isomorphisms*

$$\operatorname{Hom}_{H^*(A,k)}(\hat{H}^*(A,M),I) \xrightarrow{\cong} \operatorname{Hom}_{H^*(A,k)}(H^*(A,M),I)$$

and

$$H^*(A,M)_\mathfrak{p} \xrightarrow{\cong} \hat{H}^*(A,M)_\mathfrak{p}.$$

If $I = I_\mathfrak{p}[n]$ for some $n \in \mathbb{Z}$, this can be rewritten as

$$\operatorname{Hom}_{H^*(A,k)}(\hat{H}^*(A,M),I) \xrightarrow{\cong} \operatorname{Hom}_{H^*(A,k)_\mathfrak{p}}(H^*(A,M)_\mathfrak{p},I).$$

Proof. Every quotient of $\hat{H}^-(A,M)$ is $\mathfrak{m}$-torsion and every nonzero $\mathfrak{m}$-torsion module has a submodule isomorphic to $k[n]$ for some n. Therefore we have

$$\operatorname{Hom}^*_{H^*(A,k)}(\hat{H}^-(A,M),I) = 0.$$

The $H^*(A,k)$-module $\operatorname{PHom}_A(k,M)$ is isomorphic to a direct sum of copies of k and therefore

$$\operatorname{Hom}^*_{H^*(A,k)}(\operatorname{PHom}_A(k,M),I) = 0.$$

So applying $\operatorname{Hom}_{H^*(A,k)}(-,I)$ to the exact sequence (2.1) gives the first part.

For the second part, observe that $\mathfrak{p}$-localization is exact and annihilates an $H^*(A,k)$-module X if and only if $\operatorname{Hom}^*_{H^*(A,k)}(X,I_\mathfrak{p}) = 0$. So the isomorphism $H^*(A,M)_\mathfrak{p} \to \hat{H}^*(A,M)_\mathfrak{p}$ follows from the exact sequence (2.1). □

3. The construction

Let I be an injective module over $H^*(A,k)$. Then the contravariant functor from $\mathsf{StMod}(A)$ to the category of abelian groups which takes a module M to

$$\operatorname{Hom}_{H^*(A,k)}(\hat{H}^*(A,M),I)$$

takes triangles to exact sequences and coproducts to products. So by the contravariant version of Brown representability (which always holds, see Brown [8], Neeman [24], whereas covariant Brown representability depends on extra assumptions), there exists a representing A-module $T(I)$ satisfying

$$\operatorname{Hom}_{H^*(A,k)}(\hat{H}^*(A,-),I) \cong \underline{\operatorname{Hom}}_A(-,T(I)). \tag{3.1}$$

The assignment $I \mapsto T(I)$ extends via Yoneda's lemma to a functor

$$T\colon \mathsf{Inj}(H^*(A,k)) \longrightarrow \mathsf{StMod}(A).$$

Because of the special role of the modules $I_{\mathfrak{m}}[n]$ in the relationship between ordinary and Tate cohomology, we shall often work under the hypothesis that $\mathrm{Hom}^*_{H^*(A,k)}(k,I) = 0$. This does not hurt much because of the following lemma.

Lemma 3.1. *Let $I = I_{\mathfrak{m}}[n]$, $n \in \mathbb{Z}$. Then $T(I) \cong \Omega^{-n+1}(k)$.*

Proof. If M is an A-module, then using Tate duality, and the fact that

$$I_{\mathfrak{m}}[n] \cong \mathrm{Hom}^*_k(H^*(A,k), k[n]),$$

we have

$$\begin{aligned}\mathrm{Hom}_{H^*(A,k)}(\hat{H}^*(A,M), I_{\mathfrak{m}}[n]) &\cong \mathrm{Hom}_k(\hat{H}^*(A,M), k[n]) \\ &\cong \mathrm{Hom}_k(\hat{H}^{-n}(A,M), k) \\ &\cong \widehat{\mathrm{Ext}}^{n-1}_A(M,k) \cong \underline{\mathrm{Hom}}_A(M, \Omega^{-n+1}(k)).\end{aligned}$$ □

Next we study some of the basic properties of the functor $T\colon \mathsf{Inj}(H^*(A,k)) \to \mathsf{StMod}(A)$.

Lemma 3.2. *There is an isomorpism of $\hat{H}^*(A,k)$-modules*

$$\mathrm{Hom}^*_{H^*(A,k)}(\hat{H}^*(A,-), I) \cong \widehat{\mathrm{Ext}}^*_A(-, T(I)).$$

Proof. For $i \in \mathbb{Z}$ and an A-module M, we have

$$\begin{aligned}\mathrm{Hom}^i_{H^*(A,k)}(\hat{H}^*(A,M), I) &\cong \mathrm{Hom}_{H^*(A,k)}(\hat{H}^{*-i}(A,M), I) \\ &\cong \mathrm{Hom}_{H^*(A,k)}(\hat{H}^*(A,\Omega^i M), I) \\ &\cong \mathrm{Hom}_A(\Omega^i M, T(I)) \cong \widehat{\mathrm{Ext}}^i_A(-, T(I)).\end{aligned}$$

This isomorphism is functorial in M, and so it is an isomorphism of $\widehat{\mathrm{Ext}}^*_A(M,M)$-modules. But the action of $\hat{H}^*(A,k)$ on each side factors through the natural map $\hat{H}^*(A,k) \to \widehat{\mathrm{Ext}}^*_A(M,M)$. □

Lemma 3.3. *There is a functorial isomorphism $\hat{I} \cong \hat{H}^*(A, T(I))$ as modules over $\hat{H}^*(A,k)$.*

Proof. This follows by applying Lemma 3.2 to the trivial module k and using the definition (2.2) of $\hat{I}$. □

Lemma 3.4. *For any A-module M, Tate cohomology induces an isomorphism*

$$\underline{\mathrm{Hom}}_A(M, T(I)) \xrightarrow{\cong} \mathrm{Hom}_{\hat{H}^*(A,k)}(\hat{H}^*(A,M), \hat{H}^*(A,T(I))).$$

Proof. Using equation (3.1) and Lemma 3.3, we have

$$\begin{aligned}\underline{\mathrm{Hom}}_A(M, T(I)) &\cong \mathrm{Hom}_{H^*(A,k)}(\hat{H}^*(A,M), I) \\ &\cong \mathrm{Hom}_{\hat{H}^*(A,k)}(\hat{H}^*(A,M), \hat{I}) \\ &\cong \mathrm{Hom}_{\hat{H}^*(A,k)}(\hat{H}^*(A,M), \hat{H}^*(A,T(I))).\end{aligned}$$

To check that this map is induced by Tate cohomology, check first that the identity map on $T(I)$ goes to the identity map on $\hat{H}^*(A,T(I))$ and then use naturality. □

Lemma 3.5. *If I and J are injective $H^*(A,k)$-modules then the map*
$$\mathrm{Hom}_{H^*(A,k)}(I,J) \longrightarrow \underline{\mathrm{Hom}}_A(T(I),T(J))$$
induced by T factors as the map
$$\tau_I^*\colon \mathrm{Hom}_{H^*(A,k)}(I,J) \longrightarrow \mathrm{Hom}_{H^*(A,k)}(\hat{I},J)$$
induced by the natural map $\tau_I\colon \hat{I} \to I$, followed by an isomorphism
$$\mathrm{Hom}_{H^*(A,k)}(\hat{I},J) \xrightarrow{\cong} \underline{\mathrm{Hom}}_A(T(I),T(J)). \tag{3.2}$$

Proof. By Lemma 3.3, equation (3.1) defines an isomorphism as in (3.2). It is easy to check that the composite is the map induced by T. □

Proposition 3.6. *The functor T, restricted to a functor $\mathsf{Inj}_0(H^*(A,k)) \to \mathsf{StMod}(A)$, is fully faithful.*

Proof. This follows from Lemma 3.5 and Lemma 2.1. □

This allows us to calculate the stable endomorphism ring of $T(I_\mathfrak{p}[n])$. It is the degree zero part of Tate Ext, which is described as follows.

Corollary 3.7. *Let I be in $\mathsf{Inj}_0(H^*(A,k))$. Then $\widehat{\mathrm{Ext}}^*_A(T(I),T(I)) \cong \mathrm{End}^*_{H^*(A,k)}(I)$. In particular, if $\mathfrak{p} \neq \mathfrak{m}$ is a homogeneous prime ideal in $H^*(A,k)$ and $n \in \mathbb{Z}$, then $\widehat{\mathrm{Ext}}^*_A(T(I_\mathfrak{p}[n]),T(I_\mathfrak{p}[n]))$ is isomorphic to the completion of the local ring $H^*(A,k)_\mathfrak{p}$ with respect to the powers of its maximal ideal $\mathfrak{p}_\mathfrak{p}$.*

Proof. This follows from Proposition 3.6, together with the analysis by Matlis [23] of the endomorphism rings of injective indecomposable modules over Noetherian rings. □

Another consequence of Proposition 3.6 is the following lemma which we include for further reference.

Lemma 3.8. *Let I be in injective $H^*(A,k)$-module. Then $T(I)$ is indecomposable in $\mathsf{StMod}(A)$ if and only if I is indecomposable.*

Proof. If I decomposes, then obviously $T(I)$ decomposes, because T takes nonzero $H^*(A,k)$-modules to nonzero objects in $\mathsf{StMod}(A)$. On the other hand, if I is indecomposable, then I is of the form $I_\mathfrak{p}[n]$ for some $n \in \mathbb{Z}$ and some prime ideal $\mathfrak{p}$. If $\mathfrak{p} \neq \mathfrak{m}$ then the endomorphism rings of $I_\mathfrak{p}[n]$ and $T(I_\mathfrak{p}[n])$ are isomorphic by Proposition 3.6. In the case where $\mathfrak{p} = \mathfrak{m}$, we have $T(I_\mathfrak{m}[n]) \cong \Omega^{-n+1}(k)$ by Lemma 3.1. □

Next we investigate the property of T to preserve direct products and direct sums.

Proposition 3.9. *The functor $T\colon \mathsf{Inj}(H^*(A,k)) \to \mathsf{StMod}(A)$ preserves arbitrary direct products. The restriction of T to $\mathsf{Inj}_0(H^*(A,k))$ preserves arbitrary direct sums.*

Proof. It follows immediately from the defining isomorphism (3.1) that T preserves direct products. Now suppose that $I = \bigoplus_\lambda I_\lambda$ is a direct sum of injectives in $\mathsf{Inj}_0(H^*(A,k))$. In order to show that the canonical map $\bigoplus_\lambda T(I_\lambda) \to T(I)$ is an isomorphism it is sufficient to show that for every finitely generated A-module X the induced map
$$\phi\colon \underline{\mathrm{Hom}}_A(X, \bigoplus_\lambda T(I_\lambda)) \longrightarrow \underline{\mathrm{Hom}}_A(X,T(I))$$

is an isomorphism. We have an isomorphism

$$\psi\colon \bigoplus_{\lambda} \underline{\mathrm{Hom}}_A(X, T(I_\lambda)) \longrightarrow \underline{\mathrm{Hom}}_A(X, \bigoplus_{\lambda} T(I_\lambda))$$

since X is finitely generated, and the composite $\phi \circ \psi$ is isomorphic to the canonical map

$$\rho\colon \bigoplus_{\lambda} \mathrm{Hom}_{H^*(A,k)}(H^*(A,X), I_\lambda) \longrightarrow \mathrm{Hom}_{H^*(A,k)}(H^*(A,X), I)$$

by the definition of T and Proposition 2.9. However, $H^*(A,X)$ is a finitely generated $H^*(A,k)$-module by the main result of [13]. Thus ρ is an isomorphism and we conclude that ϕ is an isomorphism. □

4. A characterization via Tate cohomology

We have already seen that for every injective $H^*(A,k)$-module I the Tate cohomology of $T(I)$ is precisely $\hat{I}$. This property can be used to characterize the modules of the form $T(I)$.

Let $\mathcal{N}$ be the localizing subcategory of $\mathsf{StMod}(A)$ consisting of all A-modules N satisfying $\hat{H}^*(A,N) = 0$. A module L is said to be $\mathcal{N}$-*local* if $\underline{\mathrm{Hom}}_A(N,L) = 0$ for all modules N in $\mathcal{N}$.

Lemma 4.1. *If I is an injective $H^*(A,k)$-module then $T(I)$ is $\mathcal{N}$-local.*

Proof. This follows immediately from equation (3.1). □

Proposition 4.2. *Let I be an injective $H^*(A,k)$-module. Suppose that M is an A-module satisfying*

(i) *$\hat{H}^*(A,M) \cong \hat{I}$ as modules over $\hat{H}^*(A,k)$, and*
(ii) *M is $\mathcal{N}$-local.*

Then M is isomorphic to $T(I)$ in $\mathsf{StMod}(A)$.

Proof. By equation (3.1) and (i), the composite

$$\hat{H}^*(A,M) \xrightarrow{\cong} \hat{I} \xrightarrow{\tau_I} I$$

gives rise to a map $M \to T(I)$. By Lemma 3.4, this map induces an isomorphism

$$\hat{H}^*(A,M) \longrightarrow \hat{H}^*(A,T(I)).$$

Complete $M \to T(I)$ to a triangle in $\mathsf{StMod}(A)$,

$$M \longrightarrow T(I) \longrightarrow X \longrightarrow \Omega^{-1}M.$$

Then the long exact sequence in cohomology shows that $\hat{H}^*(A,X) = 0$, so X is in $\mathcal{N}$.

Now for any N in $\mathcal{N}$, $\underline{\mathrm{Hom}}_A(N,T(I)) = 0$ by Lemma 4.1, and $\underline{\mathrm{Hom}}_A(N,M) = 0$ by (ii). Applying the long exact sequence, we see that $\underline{\mathrm{Hom}}_A(N,X) = 0$. Applying this to X, we see that the identity map of X is zero, so $X \cong 0$. Thus $M \to T(I)$ is an isomorphism in $\mathsf{StMod}(A)$. □

Recall that a functor $F\colon \mathsf{StMod}(A) \to \mathsf{StMod}(A)$ is a *localization functor* with respect to $\mathcal{N}$ if $F(M)$ is $\mathcal{N}$-local, and there is a natural transformation from the identity functor to F such that for each module M, the map $M \to F(M)$ induces an isomorphism

$$\underline{\mathrm{Hom}}_A(F(M),L) \xrightarrow{\cong} \underline{\mathrm{Hom}}_A(M,L)$$

whenever L is $\mathcal{N}$-local.

Lemma 4.3. *There is a localization functor $F_{\mathcal{N}}\colon \mathsf{StMod}(A) \to \mathsf{StMod}(A)$ with respect to the localizing subcategory $\mathcal{N}$.*

Proof. It is true in general that localization with respect to a homology theory exists, see for example Margolis [22], chapter 7. The homology theory in question is just Tate cohomology.

For the convenience of the reader, we also provide a direct proof, for which we are indebted to Jeremy Rickard. Consider the localizing subcategory $\mathcal{K}$ of $\mathsf{StMod}(A)$ generated by the trivial module k and observe that $\mathcal{N}$ is the category of $\mathcal{K}$-local objects. There exists a localization triangle

$$E_{\mathcal{K}}(M) \longrightarrow M \longrightarrow F_{\mathcal{K}}(M) \longrightarrow \Omega^{-1}E_{\mathcal{K}}(M)$$

such that $E_{\mathcal{K}}(M)$ is in $\mathcal{K}$ and $F_{\mathcal{K}}(M)$ is $\mathcal{K}$-local [27]. In the formation of the quotient category of $\mathsf{StMod}(A)$ by $\mathcal{N}$, each morphism is represented by one of the form $M \xleftarrow{\sim} E_{\mathcal{K}}(M) \to N$, so the arrows form a set. This proves the existence of the quotient category, and hence the localization functor with respect to $\mathcal{N}$ via Brown representability. □

Corollary 4.4. *Let I be an injective $H^*(A,k)$-module. Suppose that M is an A-module such that $\hat{H}^*(A,M) \cong \hat{I}$ as modules over $\hat{H}^*(A,k)$. Then $F_{\mathcal{N}}(M)$ is isomorphic to $T(I)$ in $\mathsf{StMod}(A)$.* □

We are now in a position to describe the image of the functor T. This leads to a complete classification of all direct summands of direct products of shifted copies of the trivial module.

Proposition 4.5. *Let I be an injective $H^*(A,k)$-module. Then $T(I)$ is isomorphic to a direct summand of a direct product of modules of the form $\Omega^n k$, $n \in \mathbb{Z}$.*

Proof. Any injective $H^*(A,k)$-module I can be embedded as a direct summand of a direct product of modules of the form $I_{\mathfrak{m}}[m]$, $m \in \mathbb{Z}$. Now use Lemma 3.1 and the fact that T preserves direct products. □

Proposition 4.6. *Let I be an indecompoable injective $H^*(A,k)$-module. Then $T(I)$ is finitely generated if and only if $I \cong I_{\mathfrak{m}}[n]$ for some $n \in \mathbb{Z}$.*

Proof. One direction is Lemma 3.1. Now suppose that $T(I)$ is finitely generated. The module $T(I)$ is indecomposable by Lemma 3.8 and a direct summand of a direct product of modules of the form $\Omega^n k$, $n \in \mathbb{Z}$, by Proposition 4.5. It follows from a result of Auslander (see corollary 3.2 in [1]) that $T(I) \cong \Omega^n k$ for some $n \in \mathbb{Z}$. Thus $T(I) \cong T(J)$ for $J = I_{\mathfrak{m}}[-n+1]$, and therefore $\hat{I} \cong \hat{J}$ by Lemma 3.3. We conclude from Lemmas 2.1 and 2.2 that $I \cong J$. □

Theorem 4.7. *Suppose that either $(\hat{H}^-(A,k))^n = 0$ for some $n > 0$, or $\hat{H}^*(A,k)$ is periodic. Then an A-module is isomorphic in $\mathsf{StMod}(A)$ to $T(I)$ for some injective $H^*(A,k)$-module I if and only if it is isomorphic to a direct summand of a direct product of modules of the form $\Omega^n k$, $n \in \mathbb{Z}$.*

Proof. One direction is proved in Proposition 4.5. For the other direction, first note that by Lemma 3.1, $\Omega^n k$ is in the image of T. By Proposition 3.9, T preserves products, so an arbitrary product of modules of the form $\Omega^n k$ is in the image of T.

Now suppose that M is a direct summand of a direct product of modules of the form $\Omega^n k$, $n \in \mathbb{Z}$. Then $\hat{H}^*(A,M)$ is an injective $\hat{H}^*(A,k)$-module by Lemma 3.3. By Theorem 2.8, $\hat{H}^*(A,M) \cong \hat{I}$ for some injective $H^*(A,k)$-module I. Now M and $T(I)$ have isomorphic Tate cohomology, and are both $\mathcal{N}$-local. So by Proposition 4.2 they are isomorphic. □

Remark. The only way in which the hypothesis of the above theorem is used in the proof is in order to ensure that every injective $\hat{H}^*(A,k)$-module is of the form $\hat{I}$ for some injective $H^*(A,k)$-module I. We do not know whether this is true in general.

Corollary 4.8. *Let G be a finite group and k be a field. Then the conclusion of the above theorem holds in the case $A = kG$.*

Proof. By Proposition 2.4, $A = kG$ satisfies the hypotheses of the theorem. □

5. Purity

A short exact sequence of modules is said to be *pure exact* if it gives a short exact sequence on applying $\mathrm{Hom}(F, -)$ with F a finitely presented module. A module M is said to be *pure injective* if every pure exact sequence beginning with M splits. In other words, $\mathrm{Hom}(-, M)$ is exact on pure exact sequences. So for example, every injective module is pure injective. For a more detailed introduction to purity, see for example Jensen and Lenzing [19].

A module is said to be Σ-*pure injective* if an arbitrary direct sum of copies of the module is pure injective. So for example, a ring is left Noetherian if and only if every injective (left) module is Σ-pure injective.

Theorem 5.1. *Let I be an injective $H^*(A,k)$-module. Then $T(I)$ is a pure injective A-module. If $\mathrm{Hom}^*_{H^*(A,k)}(k, I)$ is finite dimensional then $T(I)$ is Σ-pure injective.*

Proof. We give two proofs of the first statement. The first proof consists of observing that a module over a finite dimensional algebra is pure injective if and only if it is isomorphic to a direct summand of a direct product of finite dimensional modules (see exercise 7.10 of [19]). So it follows from Theorem 4.7 that $T(I)$ is pure injective.

A second, more direct proof goes as follows. Applying $\hat{H}^*(A, -)$ to a pure exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

gives an exact sequence, and we get from equation (3.1) an exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}_A(M'', T(I)) \longrightarrow \underline{\mathrm{Hom}}_A(M, T(I)) \longrightarrow \underline{\mathrm{Hom}}_A(M', T(I)) \longrightarrow 0.$$

Projective A-modules are injective, and therefore the sequence

$$0 \longrightarrow \mathrm{Hom}_A(M'', T(I)) \longrightarrow \mathrm{Hom}_A(M, T(I)) \longrightarrow \mathrm{Hom}_A(M', T(I)) \longrightarrow 0$$

is exact. It follows that $T(I)$ is pure injective.

Since $H^*(A,k)$ is a Noetherian ring, an arbitrary direct sum of injective $H^*(A,k)$-modules is injective. By Proposition 3.9, T takes direct sums in $\mathsf{Inj}_0(H^*(A,k))$ to direct sums in $\mathsf{StMod}(A)$, which are then represented by the corresponding direct sums in $\mathsf{Mod}(A)$. It follows that arbitrary direct sums of copies of such a $T(I)$ are pure injective.

Now if $\mathrm{Hom}^*_{H^*(A,k)}(k, I)$ is finite dimensional, it follows that I is isomorphic to a direct sum of a finite set of modules of the form $I_{\mathfrak{m}}[n]$ and a module I' in $\mathsf{Inj}_0(H^*(A,k))$. It now follows from the previous paragraph and Lemma 3.1 that $T(I)$ is Σ-pure injective. □

Recall from [11] that a module M over a ring R is *endofinite* if M has finite length over its endomorphism ring $\mathrm{End}_R(M)$. An indecomposable R-module is *generic* if it is endofinite but not of finite length over R. Now suppose that R is commutative Noetherian. Given a prime ideal $\mathfrak{p}$, it is not hard to see that the injective envelope $E(R/\mathfrak{p})$ of $R/\mathfrak{p}$ is an indecomposable endofinite R-module if and only if $\mathfrak{p}$ is a minimal prime. This observation explains our next theorem. However, the corresponding statement about $T(I)$ is slightly different because of the special role of the maximal ideal $\mathfrak{m}$ of $H^*(A,k)$, and is as follows.

Theorem 5.2. *Let I be an injective $H^*(A,k)$-module. Then $T(I)$ is an indecomposable endofinite module if and only if I is isomorphic to $I_\mathfrak{p}[n]$ for some $n \in \mathbb{Z}$, where $\mathfrak{p}$ is either a minimal prime or the maximal prime $\mathfrak{m}$.*

Proof. By Lemma 3.8, I is indecomposable if and only if $T(I)$ is indecomposable. So we only need consider modules of the form $I_\mathfrak{p}[n]$.

If $\mathfrak{p} = \mathfrak{m}$ then Lemma 3.1 shows that $T(I_\mathfrak{m}[n])$ is an endofinite module, because it is even finite dimensional.

Otherwise, we use the characterization of endofinite modules given in proposition 2.1 of [7]. Namely, an A-module M is endofinite if and only if for every finitely generated A-module X, $\underline{\mathrm{Hom}}_A(X,M)$ has finite length as an $\underline{\mathrm{End}}_A(M)$-module.

Let $\mathfrak{p} \neq \mathfrak{m}$. Then by Proposition 2.9 we have

$$\mathrm{Hom}_{H^*(A,k)_\mathfrak{p}}(H^*(A,X)_\mathfrak{p}, I_\mathfrak{p}[n]) \cong \underline{\mathrm{Hom}}_A(X, T(I_\mathfrak{p}[n])).$$

By Proposition 3.6, the endomorphism rings of $I_\mathfrak{p}[n]$ and $T(I_\mathfrak{p}[n])$ are isomorphic, and the above is an isomorphism of modules over this ring. So it suffices to consider the length of the left hand side as a module over $\mathrm{End}_{H^*(A,k)}(I_\mathfrak{p}[n])$.

Now the degree zero part of $H^*(A,k)_\mathfrak{p}$ is a local ring, and is Artinian if and only if $\mathfrak{p}$ is minimal. The degree n part of $I_\mathfrak{p}[n]$ is equal to the injective hull of the quotient of this local ring by its maximal ideal.

If $\mathfrak{p}$ is not minimal, then consider the case $X = \Omega^n k$. Then the left hand side of the above isomorphism is equal to the degree n part of $I_\mathfrak{p}[n]$, and is not of finite length over the degree zero part of $H^*(A,k)_\mathfrak{p}$. Therefore, $T(I_\mathfrak{p}[n])$ is not endofinite.

If $\mathfrak{p}$ is minimal, then $I_\mathfrak{p}[n]$ is a finitely generated $H^*(A,k)_\mathfrak{p}$-module, whose endomorphism ring is the degree zero part of $H^*(A,k)_\mathfrak{p}$. So for any finitely generated A-module X, the left hand side has finite length over this Artinian ring. Therefore, $T(I_\mathfrak{p}[n])$ is endofinite. □

One reason for our interest in indecomposable endofinite modules is the Second Brauer–Thrall Conjecture, which asserts that if a finite dimensional algebra over an infinite field is of infinite representation type, then it has strongly unbounded representation type. Recall that an algebra is of *strongly unbouded representation type* if, for infinitely many d, there are infinitely many nonisomorphic indecomposable modules of dimension d. A result of Crawley-Boevey shows that strongly unbounded representation type is equivalent to the existence of a generic module.

Corollary 5.3. *Suppose that $\hat{H}^*(A,k)$ is not periodic. Then there exists a generic A-module and therefore A is of strongly unbounded representation type provided the field k is infinite.*

Proof. Take $T(I_\mathfrak{p})$ for any minimal prime $\mathfrak{p}$. It is an indecomposable endofinite A-module by Theorem 5.2 and not of finite length by Proposition 4.6, since $\hat{H}^*(A,k)$ is not periodic. Thus $T(I_\mathfrak{p})$ is a generic A-module. Strongly unbounded representation type for A now follows from theorem 7.3 in [12]. □

We end our discussion of pure injectives in $\mathsf{StMod}(A)$ with an alternative construction of the functor T which does not involve Brown representability. To this end we denote by $(\mathsf{mod}(A)^{\mathrm{op}}, \mathrm{Ab})$ the category of additive contravariant functors $\mathsf{mod}(A) \to \mathrm{Ab}$ into the category of abelian groups. It is an abelian Grothendieck category which as far as we are concerned means that it has injective envelopes [14]. Define a functor

$$T' \colon \mathsf{Mod}(H^*(A,k)) \longrightarrow (\mathsf{mod}(A)^{\mathrm{op}}, \mathrm{Ab})$$

via

$$T'(M) = \operatorname{Hom}_{H^*(A,k)}(\hat{H}^*(A,-), M)|_{\mathsf{mod}(A)}.$$

One should think of T' as an extension of T because it sends injective $H^*(A,k)$-modules to objects which are injective in the full subcategory $(\mathsf{stmod}(A)^{\mathrm{op}}, \mathrm{Ab})$ consisting of functors $\mathsf{mod}(A) \to \mathrm{Ab}$ vanishing on projectives, and the functor

$$\mathsf{StMod}(A) \longrightarrow (\mathsf{stmod}(A)^{\mathrm{op}}, \mathrm{Ab})$$

which takes N to $\underline{\operatorname{Hom}}_A(-, N)|_{\mathsf{mod}(A)}$ identifies the full subcategory of pure injective A-modules with the full subcategory of injective objects in $(\mathsf{stmod}(A)^{\mathrm{op}}, \mathrm{Ab})$.

In order to compare T' with T, we begin with a lemma.

Lemma 5.4. *For every A-module M there is a functorial monomorphism*

$$\alpha_M \colon \underline{\operatorname{Hom}}_A(-, \Omega M)|_{\mathsf{mod}(A)} \longrightarrow \operatorname{Hom}_k(-, k) \otimes_A M|_{\mathsf{mod}(A)}$$

in $(\mathsf{mod}(A)^{\mathrm{op}}, \mathrm{Ab})$ which is an injective envelope if M is pure injective without nonzero projective direct summands.

Proof. Auslander–Reiten theory gives for every finitely generated A-module X an isomorphism

$$\underline{\operatorname{Hom}}_A(\Omega^{-2}X, -) \cong \underline{\operatorname{Hom}}_A(\operatorname{Tr}(\operatorname{Hom}_k(X, k)), -) \cong \operatorname{Tor}_1^A(\operatorname{Hom}_k(X, k), -).$$

We combine this with the monomorphism $\operatorname{Tor}_1^A(-, \Omega^{-1}M) \to - \otimes_A M$ which is induced by an exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow \Omega^{-1}M \longrightarrow 0,$$

and this gives α_M. If M is pure injective then the functor

$$F_M = \operatorname{Hom}_k(-, k) \otimes_A M|_{\mathsf{mod}(A)}$$

is an injective object in $(\mathsf{mod}(A)^{\mathrm{op}}, \mathrm{Ab})$ and every direct summand of F_M is of the form F_N for some direct summand N of M (see theorem 7.12 of [19]). We conclude from the functoriality of α_M that the map α_M is an injective envelope provided that M is minimal. □

Theorem 5.5. *Let I be an injective $H^*(A,k)$-module and denote by $E(T'(I))$ the injective envelope of $T'(I)$ in $(\mathsf{mod}(A)^{\mathrm{op}}, \mathrm{Ab})$. Then $T(I) \cong \Omega(E(T'(I))(A))$ in $\mathsf{StMod}(A)$.*

Proof. The defining isomorphism (3.1) for $T(I)$ gives an isomorphism

$$T'(I) \cong \underline{\operatorname{Hom}}_A(-, T(I))|_{\mathsf{mod}(A)}$$

and Lemma 5.4 provides an explicit recipe for the injective envelope of $T'(I)$ since $T(I)$ is pure injective by Theorem 5.1. This enables us to get $T(I)$ from the functor $T'(I)$ by evaluating $E(T'(I))$ at A. □

6. Spectral categories

Let $\mathcal{T}(A,k)$ be the full subcategory of $\mathsf{StMod}(A)$ consisting of A-modules which are isomorphic to a direct summand of a direct product of modules of the form $\Omega^n k$, $n \in \mathbb{Z}$. We are interested in a classification of the objects in this category. The results of Section 4 show that the category $\mathcal{T}(A,k)$ is closely related to $\mathsf{Inj}(H^*(A,k))$ via the functor T. Spectral categories provide the appropriate framework to express this relationship. For example, the spectral category of a ring controls the injective modules and the invertible homomorphisms between them. In this section we show that the functor T induces an equivalence between the spectral category of $H^*(A,k)$ and the spectral category of $\mathcal{T}(A,k)$ under our assumption on the negative Tate cohomology. We obtain therefore a complete classification of the objects in $\mathcal{T}(A,k)$ since we have a classification of the injective $H^*(A,k)$-modules.

Let $\mathcal{C}$ be an additive category. We define the *spectral category* $\mathsf{Spec}(\mathcal{C})$ of $\mathcal{C}$ as follows: the objects are the same as in $\mathcal{C}$, but the morphisms are given by

$$\operatorname{Hom}(X,Y)/\operatorname{rad}\operatorname{Hom}(X,Y)$$

where $\operatorname{rad}\operatorname{Hom}(X,Y)$ denotes the subgroup of $\operatorname{Hom}(X,Y)$ consisting of maps $\phi\colon X\to Y$ such that for every $\psi\colon Y\to X$ the composite $\psi\circ\phi$ belongs to the Jacobson radical $\operatorname{rad}(\operatorname{End}(X))$ of the endomorphism ring of X. Note that every object X in $\mathcal{C}$ has

$$\operatorname{End}(X)/\operatorname{rad}(\operatorname{End}(X))$$

as endomorphism ring in $\mathsf{Spec}(\mathcal{C})$. The spectral category $\mathsf{Sp}(R)$ of a ring R is by definition the spectral category $\mathsf{Spec}(\mathsf{Inj}(R))$. In [15], Gabriel and Oberst studied the spectral category of a ring R: it is an abelian Grothendieck category where every object is projective and injective. Moreover, two injective R-modules are isomorphic if and only if they are isomorphic as objects in $\mathsf{Sp}(R)$.

Theorem 6.1. *Suppose that either $(\hat{H}^-(A,k))^n=0$ for some $n>0$, or $\hat{H}^*(A,k)$ is periodic. Then the functor T induces an equivalence $\mathsf{Sp}(H^*(A,k))\to\mathsf{Spec}(\mathcal{T}(A,k))$.*

The proof of this theorem is based on the fact that we can write $T\colon \mathsf{Inj}(H^*(A,k))\to\mathsf{StMod}(A)$ as the composite of two functors

$$\mathsf{Inj}(H^*(A,k))\xrightarrow{H}\mathsf{Inj}(\hat{H}^*(A,k))\xrightarrow{\hat{T}}\mathsf{StMod}(A).$$

The first functor is $H=\operatorname{Hom}^*_{H^*(A,k)}(\hat{H}^*(A,k),-)$, so it sends I to $\hat{I}$. Given an injective $\hat{H}^*(A,k)$-module J, the second functor is defined via Brown representability by the isomorphism

$$\operatorname{Hom}_{\hat{H}^*(A,k)}(\hat{H}^*(A,-),J)\cong\underline{\operatorname{Hom}}_A(-,\hat{T}(J)).$$

The adjointness isomorphism

$$\operatorname{Hom}_{H^*(A,k)}(M,I)\cong\operatorname{Hom}_{\hat{H}^*(A,k)}(M,\hat{I})$$

for every $\hat{H}^*(A,k)$-module M implies that $T(I)\cong\hat{T}(\hat{I})$ for every injective $H^*(A,k)$-module I.

We proceed by showing that H and $\hat{T}$ induce equivalences on the level of the spectral categories. To this end we need the following lemma about the spectral category of a ring R. We denote by $S\colon\mathsf{Inj}(R)\to\mathsf{Sp}(R)$ the canonical projection.

Lemma 6.2. *Let R be a ring. Then the indecomposable injective R-modules are precisely the simple objects in $\mathsf{Sp}(R)$. If $(I_\lambda)_{\lambda\in\Lambda}$ is a family of injective R-modules then the canonical map*

$$\bigoplus_\lambda S(I_\lambda)\longrightarrow S(E(\bigoplus_\lambda I_\lambda))$$

is an isomorphism in $\mathsf{Sp}(R)$. Therefore the following conditions are equivalent:

(i) *Every object in $\mathsf{Sp}(R)$ is semisimple.*
(ii) *Every injective R-module is the injective envelope of a direct sum of indecomposable injective R-modules.*

Proof. This follows from the discussion of spectral categories in [15]. □

Note that every Noetherian ring R satisfies the equivalent conditions in Lemma 6.2. In particular, this holds for $H^*(A,k)$.

Lemma 6.3. *Suppose that either $(\hat{H}^-(A,k))^n = 0$ for some $n > 0$, or $\hat{H}^*(A,k)$ is periodic. Then the functor $H\colon \mathsf{Inj}(H^*(A,k)) \to \mathsf{Inj}(\hat{H}^*(A,k))$ sending I to $\hat{I}$ induces an equivalence $\mathsf{Sp}(H^*(A,k)) \to \mathsf{Sp}(\hat{H}^*(A,k))$.*

Proof. The lemma is a consequence of Theorem 2.8 and Lemma 6.2. In deed, H induces a bijection between the isomorphism classes of indecomposable injectives, and for every family $(I_\lambda)_{\lambda\in\Lambda}$ of indecomposable injective $\hat{H}^*(A,k)$-modules, we have

$$H(\bigoplus_\lambda I_\lambda) \cong E(\bigoplus_\lambda H(I_\lambda))$$

and therefore

$$S(H(\bigoplus_\lambda I_\lambda)) \cong S(E(\bigoplus_\lambda H(I_\lambda))) \cong \bigoplus_\lambda S(H(I_\lambda)).$$

It remains to observe that H induces an isomorphism

$$\mathrm{End}_{H^*(A,k)}(I)/\mathrm{rad}(\mathrm{End}_{H^*(A,k)}(I)) \xrightarrow{\cong} \mathrm{End}_{\hat{H}^*(A,k)}(\hat{I})/\mathrm{rad}(\mathrm{End}_{\hat{H}^*(A,k)}(\hat{I}))$$

for every indecomposable injective I. $\square$

Lemma 6.4. *The functor $\hat{T}\colon \mathsf{Inj}(\hat{H}^*(A,k)) \to \mathsf{StMod}(A)$ is fully faithful; it induces an equivalence between $\mathsf{Inj}(\hat{H}^*(A,k))$ and $\mathcal{T}(A,k)$.*

Proof. It follows from Lemma 3.3 and Lemma 3.4 that $\hat{T}$ is fully faithful. The argument given in Proposition 4.5 shows that the image of $\hat{T}$ is contained in $\mathcal{T}(A,k)$. Conversely, given any module M in $\mathcal{T}(A,k)$, we have $\hat{T}(I) \cong M$ for $I = \hat{H}^*(A,M)$. Therefore $\mathsf{Inj}(\hat{H}^*(A,k))$ and $\mathcal{T}(A,k)$ are equivalent via $\hat{T}$. $\square$

Proof of Theorem 6.1. We combine Lemma 6.3 and Lemma 6.4. The functor H induces an equivalence $\mathsf{Sp}(H^*(A,k)) \to \mathsf{Sp}(\hat{H}^*(A,k))$, and $\hat{T}$ induces an equivalence $\mathsf{Sp}(\hat{H}^*(A,k)) \to \mathsf{Spec}(\mathcal{T}(A,k))$. It follows that T induces an equivalence $\mathsf{Sp}(H^*(A,k)) \to \mathsf{Spec}(\mathcal{T}(A,k))$ since T is the composite of H and $\hat{T}$. $\square$

Corollary 6.5. *Let I and J be injective $H^*(A,k)$-modules. Then $T(I)$ and $T(J)$ are isomorphic in $\mathsf{StMod}(A)$ if and only if I and J are isomorphic.*

Proof. Suppose that $T(I)$ and $T(J)$ are isomorphic. Then I and J are isomorphic as objects in $\mathsf{Sp}(H^*(A,k))$ by Theorem 6.1. We get homomorphisms $\alpha\colon I \to J$ and $\beta\colon J \to I$ sucht that

$$\mathrm{id}_I - \beta\circ\alpha \in \mathrm{rad}(\mathrm{End}_{H^*(A,k)}(I)) \quad \text{and} \quad \mathrm{id}_J - \alpha\circ\beta \in \mathrm{rad}(\mathrm{End}_{H^*(A,k)}(J)).$$

It follows that $\mathrm{Ker}(\mathrm{id}_I - \beta\circ\alpha)$ and $\mathrm{Ker}(\mathrm{id}_J - \alpha\circ\beta)$ are essential submodules of I and J. We conlude that $\beta\circ\alpha$ and $\alpha\circ\beta$ are isomorphisms, and therefore I and J are isomorphic as $H^*(A,k)$-modules. $\square$

We obtain the following classification of the objects in $\mathcal{T}(A,k)$ from the classification of the injective $H^*(A,k)$-modules.

Corollary 6.6. *The indecomposable objects in $\mathcal{T}(A,k)$ are, up to isomorphism, precisely the A-modules of the form $T(I_{\mathfrak{p}}[n])$ for some homogeneous prime ideal $\mathfrak{p}$ in $H^*(A,k)$ and some $n \in \mathbb{Z}$. An arbitrary A-module M belongs to $\mathcal{T}(A,k)$ if and only if M is a pure injective envelope of a direct sum $\bigoplus_{\lambda\in\Lambda} M_\lambda$ of indecomposable objects in $\mathcal{T}(A,k)$. If $(N_\kappa)_{\kappa\in K}$ is a second family of indecomposable objects in $\mathcal{T}(A,k)$ such that $\bigoplus_\lambda M_\lambda$ and $\bigoplus_\kappa N_\kappa$ have the same pure injective envelope, then there exists a bijection $\tau\colon \Lambda \to K$ such that M_λ and $N_{\tau(\lambda)}$ are isomorphic.*

To prove this result we need the following lemma about the full subcategory $\mathsf{Pinj}(A)$ of A-modules which are pure injective. We denote by $S\colon \mathsf{Pinj}(A) \to \mathsf{Spec}(\mathsf{Pinj}(A))$ the canonical projection.

Lemma 6.7. *Let $(M_\lambda)_{\lambda\in\Lambda}$ be a family of pure injective A-modules and denote by M the pure injective envelope of $\bigoplus_\lambda M_\lambda$ Then the canonical map $\bigoplus_\lambda S(M_\lambda) \to S(M)$ is an isomorphism.*

Proof. See theorem 8.25 of [19]. □

Proof of Corollary 6.6. We apply Theorem 6.1. It follows from this result that T induces a bijection between the isomorphism classes of indecomposable objects in $\mathsf{Inj}(H^*(A,k)$ and $\mathcal{T}(A,k)$. Therefore the indecomposables in $\mathcal{T}(A,k)$ are precisely the modules of the form $T(I_{\mathfrak{p}}[n])$.

To prove the second part of the assertion let M be an arbitrary A-module. Suppose first that M is a pure injective envelope of a direct sum $\bigoplus_\lambda M_\lambda$ of objects in $\mathcal{T}(A,k)$. The canonical map $\bigoplus_\lambda M_\lambda \to \prod_\lambda M_\lambda$ is a pure monomorphism and therefore M is isomorphic to a direct summand of $\prod_\lambda M_\lambda$. It follows that M belongs to $\mathcal{T}(A,k)$ since $\prod_\lambda M_\lambda$ belongs to $\mathcal{T}(A,k)$.

Now suppose that M belongs to $\mathcal{T}(A,k)$. We consider $\mathsf{Spec}(\mathcal{T}(A,k))$ as a full subcategory of $\mathsf{Spec}(\mathsf{Pinj}(A))$. It is closed under taking arbitrary direct sums by the argument given in the previous paragraph in combination with Lemma 6.7. In $\mathsf{Spec}(\mathcal{T}(A,k))$, we get a decomposition $S(M) = \bigoplus_{\lambda\in\Lambda} S(M_\lambda)$ into simple objects by Lemma 6.2. It follows from Lemma 6.7 that M is isomorphic to the pure injective envelope of the direct sum $\bigoplus_\lambda M_\lambda$. The uniqueness of the family $(M_\lambda)_{\lambda\in\Lambda}$ follows from the uniqueness of the decomposition of $S(M)$ into simple objects. This finishes the proof. □

7. Subgroups and varieties

If H is a subgroup of G, then restriction is a ring homomorphism from $H^*(G,k)$ to $H^*(H,k)$. This makes $H^*(H,k)$ into a finitely generated $H^*(G,k)$-module. If I is an injective $H^*(G,k)$-module, we write $r_{G,H}(I)$ for the injective $H^*(H,k)$-module

$$r_{G,H}(I) = \operatorname{Hom}_{H^*(G,k)}(H^*(H,k), I).$$

Thus for example

$$r_{G,H}(I_{\mathfrak{p}}[n]) \cong I_{\operatorname{res}^{-1}_{G,H}(\mathfrak{p})}[n].$$

Proposition 7.1. *Let $H \leq G$, and write T_G and T_H for the functor T with respect to kG and kH respectively. If I is an injective $H^*(G,k)$-module, we have*

$$T_G(I){\downarrow_H} \cong T_H(r_{G,H}(I)).$$

Proof. For any kH-module M, we have

$$\begin{aligned}\underline{\mathrm{Hom}}_{kH}(M, T_G(I){\downarrow_H}) &\cong \underline{\mathrm{Hom}}_{kG}(M{\uparrow^G}, T_G(I))\\ &\cong \mathrm{Hom}_{H^*(G,k)}(\hat{H}^*(G, M{\uparrow^G}), I)\\ &\cong \mathrm{Hom}_{H^*(H,k)}(\hat{H}^*(G, M{\uparrow^G}), r_{G,H}(I))\\ &\cong \mathrm{Hom}_{H^*(H,k)}(\hat{H}^*(H, M), r_{G,H}(I))\\ &\cong \underline{\mathrm{Hom}}_{kH}(M, T_H(r_{G,H}(I))).\end{aligned}$$ □

Next, we calculate the variety of the module $T(I_{\mathfrak{p}}[n])$, in the sense of Benson, Carlson and Rickard [4]. This is essentially done in theorem 6.1.8 (b) ⇒ (e″) of Hovey, Palmieri and Strickland [18]. For the convenience of the reader, we translate their argument into the language of group cohomology.

Recall that for every finitely generated kG-module M, the variety $V_G(M)$ of M is a closed homogeneous subvariety of the cohomology variety V_G. For an arbitrary kG-module M, the variety of M in the sense of [4] is a collection of closed homogeneous subvarieties of V_G which is denoted by $\mathcal{V}_G(M)$. Next, recall from Rickard [27] that for each closed homogeneous subvariety V of V_G, there is a triangle

$$E_V \longrightarrow k \longrightarrow F_V \longrightarrow \Omega^{-1}E_V$$

in $\mathsf{StMod}(kG)$ such that E_V is a colimit of finitely generated modules with variety contained in V, and F_V is local with respect to such modules. The modules E_V and F_V are idempotent in $\mathsf{StMod}(kG)$.

Lemma 7.2. *Let $\mathfrak{p}$ be a prime ideal of $H^*(G,k)$, corresponding to a closed homogeneous irreducible subvariety V of V_G. Suppose that M is a kG-module such that $\hat{H}^*(G,M)$ is $\mathfrak{p}$-torsion. Then $\hat{H}^*(G, F_V \otimes_k M) = 0$.*

Proof. Write $\mathcal{C}$ for the localizing subcategory of $\mathsf{StMod}(kG)$ generated by M. Every module in $\mathcal{C}$ has $\mathfrak{p}$-torsion cohomology. This applies in particular to $F_V \otimes_k M$.

If $0 \neq \zeta \in H^n(G,k)$ is represented by a cocycle $\hat{\zeta}\colon \Omega^n k \to k$, we write L_ζ for the kernel, so that there is a short exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^n k \xrightarrow{\hat{\zeta}} k \longrightarrow 0.$$

Choose elements $\zeta_1, \dots, \zeta_t$ in degrees $n_1, \dots, n_t$, generating an ideal whose radical is $\mathfrak{p}$; in other words

$$V_G\langle\zeta_1\rangle \cap \cdots \cap V_G\langle\zeta_t\rangle = V.$$

Then $L_{\zeta_1} \otimes_k \cdots \otimes_k L_{\zeta_t} \otimes_k F_V$ is projective. Assume, by way of contradiction, that $\hat{H}^*(G, F_V \otimes_k M) \neq 0$. Set

$$X_i = L_{\zeta_1} \otimes_k \cdots \otimes_k L_{\zeta_i} \otimes_k F_V \otimes_k M,$$

so that $\hat{H}^*(G, X_0) \neq 0$ but $\hat{H}^*(G, X_t) = 0$. Choose $i > 0$ so that $\hat{H}^*(G, X_{i-1}) \neq 0$ but $\hat{H}^*(G, X_i) = 0$. Then the short exact sequence

$$0 \longrightarrow X_i \longrightarrow \Omega^{n_i} k \otimes X_{i-1} \xrightarrow{\hat{\zeta}_i} X_{i-1} \to 0$$

shows that ζ_i acts isomorphically on $\hat{H}^*(G, X_{i-1})$. But X_{i-1} is in $\mathcal{C}$, and ζ_i is in $\mathfrak{p}$, so ζ_i acts nilpotently on $\hat{H}^*(G, X_{i-1})$. So $\hat{H}^*(G, X_{i-1}) = 0$. This contradiction completes the proof of the lemma. □

Theorem 7.3. *Let $\mathfrak{p}$ be a nonmaximal prime ideal in $H^*(G,k)$, corresponding to a closed homogeneous irreducible subvariety V of V_G. Then $\mathcal{V}_G(T(I_\mathfrak{p}[n])) = \{V\}$.*

Proof. Using Proposition 7.1 with H a Sylow p-subgroup of G, we may assume that G is a p-group. By Lemmas 2.1 and 3.3, $\hat{H}^*(G, T(I_\mathfrak{p}[n])) \cong \widehat{I_\mathfrak{p}[n]} \cong I_\mathfrak{p}[n]$ is a $\mathfrak{p}$-torsion $H^*(G,k)$-module. So by Lemma 7.2, $\hat{H}^*(G, F_V \otimes_k T(I_\mathfrak{p}[n])) = 0$. Since G is a p-group, this implies that $F_V \otimes_k T(I_\mathfrak{p}[n])$ is projective. So only subvarieties of V can occur as elements of $\mathcal{V}_G(T(I_\mathfrak{p}[n]))$. But if W is a proper subvariety of V then for any finitely generated kG-module M with $V_G(M) \subseteq W$, equation (3.1) shows that $\underline{\mathrm{Hom}}_{kG}(M, T(I_\mathfrak{p}[n])) = 0$. So W is not an element of $\mathcal{V}_G(T(I_\mathfrak{p}[n]))$. Since $T(I_\mathfrak{p}[n])$ is not projective, this implies that $\mathcal{V}_G(T(I_\mathfrak{p}[n])) = \{V\}$. □

8. Zariski topology

The isomorphism classes of indecomposable pure injective A-modules form a set which we denote by $\mathrm{Ind}(A)$. Recall that a functor $F\colon \mathsf{Mod}(A) \to \mathrm{Ab}$ into the category of abelian groups is *coherent* if there exists a presentation

$$\mathrm{Hom}_A(Y, -) \longrightarrow \mathrm{Hom}_A(X, -) \longrightarrow F \longrightarrow 0$$

which is induced by a map $X \to Y$ between finitely generated A-modules. Given a coherent functor F, we define

$$D(F) = \{M \in \mathrm{Ind}(A) \mid F(M) = 0\}.$$

The sets of the form $D(F)$ for some coherent F form a basis for the open subsets of the *Zariski topology* on $\mathrm{Ind}(A)$. This topology on $\mathrm{Ind}(A)$ has been introduced by Prest [25]; it is the analogue of the Zariski topology on the set of isomorphism classes of indecomposable injective quasi-coherent sheaves on a Noetherian scheme which has been considered by Gabriel (chapitre VI of [14]).

We need to work in the stable category. We write $\underline{\mathrm{Ind}}(A)$ for the set of nonprojective elements of $\mathrm{Ind}(A)$, or equivalently the pure injective indecomposables in the stable module category. The topology on this is determined in the stable category by the following two lemmas.

Lemma 8.1. *Let P be an indecomposable projective A-module. Then there exist coherent functors F and G such that*

$$D(F) = \{P\} = \mathrm{Ind}(A) \setminus D(G).$$

Proof. We use the fact that a projective A-module is injective. A nonzero map $A/\mathrm{rad}(A) \to P$ induces the following exact sequence of coherent functors

$$0 \longrightarrow \mathrm{Hom}_A(P/\mathrm{soc}(P), -) \longrightarrow \mathrm{Hom}_A(P, -) \longrightarrow \mathrm{Hom}_A(A/\mathrm{rad}(A), -) \xrightarrow{\pi} F \longrightarrow 0.$$

It is easily checked that $D(F) = \{P\}$ and $\mathrm{Ind}(A) \setminus D(G) = \{P\}$ for $G = \mathrm{Ker}(\pi)$. □

Lemma 8.2. *Given any coherent functor F on $\mathsf{Mod}(A)$, there exists a coherent functor G on $\mathsf{Mod}(A)$ such that the following hold.*

(i) *G vanishes on projective modules, and*
(ii) *$D(G) \cap \underline{\mathrm{Ind}}(A) = D(F) \cap \underline{\mathrm{Ind}}(A)$.*

Proof. The sets of the form $V(F) = \mathrm{Ind}(A) \setminus D(F)$ for some coherent F form a basis for the open subsets of the *Ziegler topology* on $\mathrm{Ind}(A)$ which has been introduced by Ziegler in [29]; see [21] for the definition of this topology in terms of functors. A Ziegler-open set is quasi-compact if and only if it is of the form $V(F)$ for some coherent functor F (cf. corollary 4.5 in [21]). Now let $\mathbf{P}$ be the finite set of projectives in $\mathrm{Ind}(A)$. Note that $\{P\}$ is Ziegler-open and Ziegler-closed for each P in $\mathbf{P}$ by Lemma 8.1. Given a coherent functor F, it follows that $V(F) \setminus \mathbf{P}$ is a quasi-compact Ziegler-open subset of $\mathrm{Ind}(A)$, hence of the form $V(G) = V(F) \setminus \mathbf{P}$ for some coherent functor G. By construction, G vanishes on every projective module and $D(G) \cap \underline{\mathrm{Ind}}(A) = D(F) \cap \underline{\mathrm{Ind}}(A)$. ☐

Note that a coherent functor F vanishes on projectives if and only if it has a presentation of the form

$$\underline{\mathrm{Hom}}_A(Y,-) \longrightarrow \underline{\mathrm{Hom}}_A(X,-) \longrightarrow F \longrightarrow 0$$

with X and Y finitely generated.

Projective modules are separated by coherent functors, so it follows from Lemma 8.1 that the space $\mathrm{Ind}(A)$ with the Zariski topology is just the disjoint union of the closed set $\underline{\mathrm{Ind}}(A)$ and a closed point for each projective indecomposable.

We denote by $\mathrm{Proj}(H^*(A,k))$ the set of homogeneous prime ideals in $H^*(A,k)$ which are different from the maximal ideal $\mathfrak{m}$. The assignment $\mathfrak{p} \mapsto I_\mathfrak{p}$ allows us to identify $\mathrm{Proj}(H^*(A,k))$ with a set of indecomposable injective $H^*(A,k)$-modules. Given a finitely generated $H^*(A,k)$-module K, we define

$$D_+(K) = \{\mathfrak{p} \in \mathrm{Proj}(H^*(A,k)) \mid \mathrm{Hom}_{H^*(A,k)}(K, I_\mathfrak{p}) = 0\}.$$

This gives a basis of open sets for the Zariski topology on $\mathrm{Proj}(H^*(A,k))$.

Theorem 8.3. *The map* $\mathrm{Proj}(H^*(A,k)) \to \mathrm{Ind}(A)$ *which sends a prime* $\mathfrak{p}$ *to* $T(I_\mathfrak{p})$ *induces a homeomorphism between* $\mathrm{Proj}(H^*(A,k))$ *and its image in* $\mathrm{Ind}(A)$.

Proof. The assignment $\mathfrak{p} \mapsto T(I_\mathfrak{p})$ defines an injective map $\mathrm{Proj}(H^*(A,k)) \to \mathrm{Ind}(A)$ by Proposition 3.6 and Theorem 5.1.

Given a coherent functor F on $\mathsf{StMod}(A)$ determined by a map of finitely generated modules $X \to Y$, let K denote the kernel of the induced map $H^*(A,X) \to H^*(A,Y)$. For $I \in \mathrm{Ind}_0(A)$, it follows from Proposition 2.9 that

$$F(T(I)) \cong \mathrm{Hom}_{H^*(A,k)}(K, I).$$

So the inverse image of $D(F)$ is the open set $D_+(K)$.

Conversely, if K is a finitely generated module over $H^*(A,k)$, then a finite presentation gives rise to a map from a finite sum X of modules of the form $\Omega^n k$ to another such module Y. In Tate cohomology, we then get a sequence

$$\hat{H}^*(A,X) \longrightarrow \hat{H}^*(A,Y) \longrightarrow \hat{H}^*(A,k) \otimes_{H^*(A,k)} K \longrightarrow 0.$$

Now recall from Lemma 2.1 that $\hat{I} \cong I$, so that we have

$$\mathrm{Hom}_{\hat{H}^*(A,k)}(\hat{H}^*(A,k) \otimes_{H^*(A,k)} K, I) = \mathrm{Hom}_{H^*(A,k)}(K, I).$$

So by Proposition 2.9 we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{H^*(A,k)}(K,I) \longrightarrow \mathrm{Hom}_{H^*(A,k)}(H^*(A,Y),I) \longrightarrow \mathrm{Hom}_{H^*(A,k)}(H^*(A,X),I).$$

Complete $X \to Y$ to a triangle $X \to Y \to Z \to \Omega^{-1}X$, and define a coherent functor F by

$$\underline{\mathrm{Hom}}_A(\Omega^{-1}X, -) \longrightarrow \underline{\mathrm{Hom}}_A(Z, -) \longrightarrow F \longrightarrow 0.$$

The long exact sequence induced by the triangle gives the following exact sequence

$$0 \longrightarrow F \longrightarrow \underline{\mathrm{Hom}}_A(Y, -) \longrightarrow \underline{\mathrm{Hom}}_A(X, -).$$

Again applying Proposition 2.9, we get

$$F(T(I)) \cong \mathrm{Hom}_{H^*(A,k)}(K, I).$$

It follows that the image of $D_+(K)$ is equal to $D(F)$. □

9. Minimal primes

In this section we compute the modules $T(I_{\mathfrak{p}}[n])$ for every minimal prime ideal $\mathfrak{p}$ in $H^*(A,k)$ and every $n \in \mathbb{Z}$. Our analysis is based on a discussion of various localization functors with respect to a prime $\mathfrak{p}$. We fix a homogeneous prime ideal $\mathfrak{p}$ in $H^*(A,k)$ and denote by S the set of homogeneous elements in $H^*(A,k) \setminus \mathfrak{p}$. Given an $H^*(A,k)$-module M, the *localization* of M with respect to S is denoted by

$$M_{\mathfrak{p}} = S^{-1}M.$$

Lemma 9.1. *Let $\mathfrak{p} \neq \mathfrak{m}$ and denote by $\hat{S}$ the set of homogeneous elements in $H^*(A,k) \setminus \mathfrak{p}$, viewed as a subset of $\hat{H}^*(A,k)$. Then the localization of an $\hat{H}^*(A,k)$-module M with respect to $\hat{S}$ is isomorphic to $M_{\mathfrak{p}}$ as a module over $H^*(A,k)$. In particular, $\hat{S}^{-1}\hat{H}^*(A,N) \cong H^*(A,N)_{\mathfrak{p}}$ for every A-module N.*

Proof. Localization with respect to S and $\hat{S}$ gives universal maps $M \to M_{\mathfrak{p}}$ and $M \to \hat{S}^{-1}M$. We need to show that there are maps $M_{\mathfrak{p}} \to \hat{S}^{-1}M$ and $\hat{S}^{-1}M \to M_{\mathfrak{p}}$ whose composite is the identity on $M_{\mathfrak{p}}$ and $\hat{S}^{-1}M$. Clearly,

$$(\hat{S}^{-1}M)_{\mathfrak{p}} = \hat{S}^{-1}M,$$

and this gives the map $M_{\mathfrak{p}} \to \hat{S}^{-1}M$. On the other hand, $M_{\mathfrak{p}}$ is in a canonical way an $\hat{H}^*(A,k)$-module. In fact, the injectives $I_{\mathfrak{p}}[n]$, $n \in \mathbb{Z}$, cogenerate the category of $H^*(A,k)_{\mathfrak{p}}$-modules and therefore $M_{\mathfrak{p}}$ fits into an exact sequence

$$0 \longrightarrow M_{\mathfrak{p}} \longrightarrow \prod_{\lambda} I_{\mathfrak{p}}[n_\lambda] \longrightarrow \prod_{\kappa} I_{\mathfrak{p}}[n_\kappa]$$

of $H^*(A,k)$-modules. Applying the functor $\mathrm{Hom}^*_{H^*(A,k)}(\hat{H}^*(A,k), -)$ and using Lemma 2.1, we get $\widehat{M_{\mathfrak{p}}} \cong M_{\mathfrak{p}}$ since $\mathfrak{p} \neq \mathfrak{m}$. This gives the map $\hat{S}^{-1}M \to M_{\mathfrak{p}}$, and we conclude that the localization of M with respect to $\hat{S}$ is isomorphic to $M_{\mathfrak{p}}$. Finally, we get for every A-module N

$$\hat{S}^{-1}\hat{H}^*(A,N) \cong \hat{H}^*(A,N)_{\mathfrak{p}} \cong H^*(A,N)_{\mathfrak{p}}$$

where the second isomorphism is taken from Proposition 2.9. □

Recall that a localization functor $F\colon \mathsf{StMod}(A) \to \mathsf{StMod}(A)$ with respect to a localizing subcategory $\mathcal{L}$ of $\mathsf{StMod}(A)$ is *finite* if $\mathcal{L}$ is generated (as a localizing subcategory) by a class of finitely generated A-modules.

Proposition 9.2. *Let $\mathfrak{p} \neq \mathfrak{m}$ be a homogeneous prime ideal in $H^*(A,k)$. Then there exists a finite localization functor $F_\mathfrak{p}\colon \mathsf{StMod}(A) \to \mathsf{StMod}(A)$ such that*

$$\hat{H}^*(A, F_\mathfrak{p}(M)) \cong H^*(A,M)_\mathfrak{p}$$

for every A-module M.

Proof. Suppose that U is a set of homogeneous elements in $\hat{H}^*(A,k)$ such that localization with respect to U is an exact functor on $\hat{H}^*(A,k)$-modules. Then it is shown in theorem 3.3.7 of [18] that there exists a localization functor $F\colon \mathsf{StMod}(A) \to \mathsf{StMod}(A)$ such that

$$\hat{H}^*(A, F(M)) \cong U^{-1}\hat{H}^*(A,M)$$

for every M in $\mathsf{StMod}(A)$. In fact, letting L_ζ denote the kernel of a homomorphism $\Omega^n k \to k$ representing ζ, one then takes the finite localization functor with respect to the localizing subcategory generated by $\{L_\zeta \mid \zeta \in U\}$. Taking for U the set $\hat{S}$ corresponding to the prime $\mathfrak{p}$, the assertion follows from Lemma 9.1. □

Remark. In case $A = kG$ for some finite group G, the functor $F_\mathfrak{p}$ can be expressed as a Rickard idempotent functor [27] as follows. Let $\mathcal{U}$ be the set of subvarieties of the cohomology variety V_G which do not contain the variety determined by $\mathfrak{p}$. Let $\mathcal{C}_\mathcal{U}$ be the thick subcategory of $\mathsf{stmod}(kG)$ consisting of finitely generated modules X satisfying $V_G(X) \in \mathcal{U}$. Then $F_\mathfrak{p}$ is the F-idempotent module for $\mathcal{C}_\mathcal{U}$.

From now on we assume that A is the group algebra of a finite group G. In [17], Greenlees and Lyubeznik have shown that the localization of the cohomology ring $H^*(G,k)$ at a minimal prime $\mathfrak{p}$ is Gorenstein. This allows us to identify for every $n \in \mathbb{Z}$ the corresponding module $T(I_\mathfrak{p}[n])$. Let us recall that $F_\mathcal{N}$ denotes the localization functor with respect to the localizing subcategory $\mathcal{N}$ of $\mathsf{StMod}(A)$ consisting of all kG-modules M satisfying $\hat{H}^*(G,M) = 0$.

Theorem 9.3. *Let G be a finite group of p-rank r and k be a field of characteristic p. If $\mathfrak{p} \neq \mathfrak{m}$ is a minimal homogeneous prime ideal in $H^*(G,k)$ and $n \in \mathbb{Z}$, then*

$$T(I_\mathfrak{p}[n]) \cong \Omega^{-n+r} F_\mathcal{N}(F_\mathfrak{p}(k))$$

as modules in $\mathsf{StMod}(kG)$.

Proof. In section 7 of Greenlees and Lyubeznik [17], it is shown that $H^*(G,k)_\mathfrak{p} \cong I_\mathfrak{p}[r]$. Thus we have

$$\hat{H}^*(G, F_\mathfrak{p}(k)) \cong H^*(G,k)_\mathfrak{p} \cong I_\mathfrak{p}[r] \cong \widehat{I_\mathfrak{p}[r]}$$

by Proposition 9.2 and Lemma 2.1. In Corollary 4.4 it is shown that Tate cohomology determines the modules of the form $T(I)$ if one localizes away from $\mathcal{N}$. Therefore $T(I_\mathfrak{p}[r]) \cong F_\mathcal{N}(F_\mathfrak{p}(k))$, and the assertion of the theorem follows by applying the appropriate shift. □

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