ON RINGEL-HALL ALGEBRAS OF TAME HEREDITARY ALGEBRAS

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In memory of our teacher Xie Yu

0. Introduction

Let k be a finite field with q elements, Λ be a finite dimensional hereditary k-algebra of tame type, $\mathcal{P}$ the set of isomorphism classes of Λ-modules and I the set of isomorphism classes of simple Λ-modules. Let R be a subfield of real numbers. We have the (extended) Ringel-Hall algebra $\mathcal{H}^+ = \mathcal{H}^+(\Lambda)$ with basis $\{K_\alpha u_\lambda^+ | \alpha \in \mathbb{Z}[I], \lambda \in \mathcal{P}\}$ which is a Hopf algebra (see [6], [1] or [9]). Dually, we have the negative part $\mathcal{H}^-$ with basis $\{K_\alpha u_\lambda^- | \alpha \in \mathbb{Z}[I], \lambda \in \mathcal{P}\}$ which is a Hopf algebra too. The pairing $\varphi :\ \mathcal{H}^+ \times \mathcal{H}^- \to R$ given by Ringel [6]

$$\varphi(K_\alpha u_\beta^+, K_{\alpha'} u_{\beta'}^-) = v^{-(\alpha,\alpha')-(\beta,\alpha')+(\alpha,\beta')} \frac{|V_\beta|}{a_\beta} \delta_{\beta\beta'}$$

is a skew-Hopf pairing, where $v = \sqrt{q}$. Therefore we have the reduced Drinfeld double $\mathcal{D}(\Lambda)$ of $(\mathcal{H}^+, \mathcal{H}^-, \varphi)$. By a theorem of Green [1], it is well-known that the composition algebra $\mathfrak{c}^+(\Lambda)$, which is a subalgebra of $\mathcal{H}^+(\Lambda)$ generated by simple Λ-modules, is canonically isomorphic to the positive part of the quantized enveloping algebra of the affine Kac-Moody algebra of the same type as Λ. So, a natural question is, how can one measure the difference between $\mathcal{H}^+(\Lambda)$ and $\mathfrak{c}^+(\Lambda)$. Some detailed investigations have been given by Zhang-Zhang [11] and Zhang [12]. In fact, we have a strong feeling for a long time that the Ringel-Hall algebra of a tame hereditary algebra must be a quantized object of affine type. After we understood Theorem 1.2 of Sevenhant-Van den Bergh [8], this becomes very clear. Now we state our main theorem as follows:

Theorem 0.1. *Let Λ be a finite dimensional hereditary k-algebra of tame type, Δ the Cartan datum corresponding to Λ. Then the reduced Drinfeld double $\mathcal{D}(\Lambda)$ of $(\mathcal{H}^+, \mathcal{H}^-, \varphi)$ is canonically isomorphic to the quantized enveloping algebra of the Lie algebra*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l}^- \oplus \mathfrak{h} \oplus \mathfrak{l}^+ \oplus \mathfrak{n}^+,$$

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where $\mathfrak{h}$ *is the ordinary Cartan subalgebra,* $\mathfrak{g}_1 = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ *is the affine Kac-Moody algebra of type* Δ*,* $\mathfrak{g}_2 = \mathfrak{l}^- \oplus \mathfrak{h} \oplus \mathfrak{l}^+$ *is an infinite dimensional Heisenberg Lie algebra and* $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$*.*

Moreover, if Δ is of type $\tilde{A}_n$, $\tilde{D}_n$ or $\tilde{E}_i, i = 6, 7, 8$, the numbers of the generators of the Heisenberg algebra $\mathfrak{g}_2$ are explicitly given at every dimension level, by using the results in [3].

Corollary 0.2. *Let* Λ *be a finite dimensional hereditary* k*-algebra of tame type,* $\mathfrak{h}^+(\Lambda)$ *the usual Ringel-Hall algebra, which is a subalgebra of* $\mathcal{H}^+(\Lambda)$ *with basis* $\{u_\lambda^+ | \lambda \in \mathcal{P}\}$*. Then we have the following decomposition*

$$\mathfrak{h}^+(\Lambda) = \mathfrak{c}^+(\Lambda) \otimes_R R[x_1, x_2, \cdots],$$

where $R[x_1, x_2, \cdots]$ *is a polynomial ring with infinitely many variables.*

1. The double Ringel-Hall algebras

1.1 In this paper, we let R be a commutative field of characteristic zero. An R-vector space H is called a *Hopf algebra* if H possesses the R-linear operations: (1) multiplication $\mu : H \otimes H \to H$, (2) unit $\eta : R \to H$, (3) comultiplication $\Delta : H \to H \otimes H$, (4) counit $\epsilon : H \to R$ and (5) antipode $\sigma : H \to H$, which satisfy the following conditions: (i) (H, μ, η) is an associative R-algebra, (ii) (H, Δ, ϵ) is an associative R-coalgebra, (iii) Δ, ϵ are algebra homomorphisms and (iv) $\mu(\sigma \otimes 1)\Delta = \eta\epsilon = \mu(1 \otimes \sigma)\Delta$.

Given Hopf R-algebras A and B, a *skew Hopf pairing* between A and B is an R-bilinear form $\varphi : A \times B \to R$ which satisfies

(a) $\varphi(1, b) = \epsilon_B(b)$, $\varphi(a, 1) = \epsilon_A(a)$, for all $a \in A$, $b \in B$,

(b) $\varphi(a, bb') = \varphi(\Delta_A(a), b \otimes b')$, for all $a \in A$, $b, b' \in B$,

(c) $\varphi(aa', b) = \varphi(a \otimes a', \Delta_B^{op}(b))$, for all $a, a' \in A$, $b, b' \in B$,

(d) $\varphi(\sigma_A(a), b) = \varphi(a, \sigma_B^{-1}(b))$, for all $a \in A$, $b, b' \in B$,

where $\varphi(a \otimes a', b \otimes b') = \varphi(a, b)\varphi(a', b')$, and Δ_B^{op} is defined by $\Delta_B^{op}(b) = \sum b_1 \otimes b_2$ if $\Delta_B(b) = \sum b_2 \otimes b_1$. Here we assume that σ_B is invertible.

If φ is a skew-Hopf pairing, it is well known that $A \otimes B$ has a Hopf structure, which is the so-called *Drinfeld double* of (A, B, φ). The algebra structure on $A \otimes B$ is given as follows

(a′) $(a \otimes 1)(a' \otimes 1) = aa' \otimes 1$, for all $a, a' \in A$,

(b′) $(1 \otimes b)(1 \otimes b') = 1 \otimes bb'$, for all $b, b' \in B$,

(c′) $(a \otimes 1)(1 \otimes b) = a \otimes b$, for all $a \in A, b \in B$,

(d′) $(1 \otimes b)(a \otimes 1) = \sum \varphi(a_1, \sigma_B(b_1))(a_2 \otimes b_2)\varphi(a_3, b_3)$, for all $a \in A, b \in B$,

where $\Delta_A^2(a) = \sum a_1 \otimes a_2 \otimes a_3$ and $\Delta_B^2(b) = \sum b_1 \otimes b_2 \otimes b_3$.

1.2 According to Lusztig [5], a Cartan datum is a pair $(I, (\ ,\ ))$ consisting of a finite set I and a symmetric bilinear form $(-, -)$ on the free abelian group $\mathbb{Z}[I]$ with values in $\mathbb{Z}$ such that the following conditions are satisfied:

(a) $(i,i) \in \{\,2,4,6,\cdots\}$ for all $i \in I$,

(b) $2(i,j)/(i,i) \in \{0,-1,-2,\cdots\}$ for any $i \neq j \in I$.

Then $\Gamma = (a_{ij})_{i,j\in I}$ is a symmetrizable generalized Cartan matrix, where $a_{ij} = 2(i,j)/(i,i)$ for all $i,j \in I$. Without loss of generality, we identify Γ with $(I,(\ ,\ ))$.

1.3 Let k be a finite field with q elements, and $v = \sqrt{q}$. Let Λ be a finite dimensional hereditary algebra over k. From now on, We assume that R is a subfield of $\mathbb{R}$, the field of real numbers, and $v \in R$. Let $\mathcal{P}$ be the set of isomorphism classes of finite dimensional Λ-modules, $I \subset \mathcal{P}$ the set of isomorphism classes of simple Λ-modules, and $\mathcal{P}_1 = \mathcal{P}\backslash\{0\}$. We fix a representative V_α for each $\alpha \in \mathcal{P}$. By abuse of notation, we write

$$\langle\alpha,\beta\rangle = \langle\underline{\dim}\, V_\alpha, \underline{\dim}\, V_\beta\rangle = \dim{}_k\mathrm{Hom}_\Lambda(V_\alpha,V_\beta) - \dim{}_k\mathrm{Ext}^1(V_\alpha,V_\beta),$$

and $(\alpha,\beta) = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle$, for all $\alpha,\beta \in \mathcal{P}$. Since Λ is a hereditary algebra, the Euler form $\langle -,-\rangle$ and its symmetrization $(-,-)$ are defined on $\mathbb{Z}[I]$.

It is well-known that the symmetric Euler form $(I,(\ ,\ ))$ of Λ is a Cartan datum and any Cartan datum can be obtained in this way.

Given $\alpha,\beta,\lambda \in \mathcal{P}$, let $g^\lambda_{\alpha,\beta}$ be the number of submodules M of V_λ such that $M \cong V_\beta$ and $V_\lambda/M \cong V_\alpha$. More generally, given $\alpha_1,\cdots,\alpha_t,\lambda \in \mathcal{P}$, we let $g^\lambda_{\alpha_1,\cdots,\alpha_t}$ be the number of the filtrations

$$0 = M_t \subseteq M_{t-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = V_\lambda,$$

such that $M_{i-1}/M_i \cong V_{\alpha_i}$ for all $1 \le i \le t$. For any $\lambda \in \mathcal{P}$, we let $a_\lambda = |\,\mathrm{Aut}(V_\alpha)\,|$, the order of the automorphism group of V_α.

Let $\mathcal{H}^+$ be an R-vector space with basis $\{\,K_\alpha u^+_\lambda \mid \alpha \in \mathbb{Z}[I], \lambda \in \mathcal{P}\,\}$. Then $\mathcal{H}^+$ is a Hopf algebra in the following sense:

(a) Multiplication (Ringel [6]):

$$\begin{aligned} u^+_\alpha u^+_\beta &= v^{\langle\alpha,\beta\rangle}\sum_{\lambda\in\mathcal{P}} g^\lambda_{\alpha,\beta}u^+_\lambda, \quad \text{for all } \alpha,\beta \in \mathcal{P},\\ K_\alpha u^+_\beta &= v^{(\alpha,\beta)}u^+_\beta K_\alpha, \quad \text{for all } \alpha \in \mathbb{Z}[I], \beta \in \mathcal{P},\\ K_\alpha K_\beta &= K_{\alpha+\beta}, \qquad \text{for all } \alpha,\beta \in \mathbb{Z}[I], \end{aligned}$$

with unit $1 = u^+_0 = K_0$.

(b) Comultiplication (Green [1]):

$$\begin{aligned} \Delta(u^+_\lambda) &= \sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle}\frac{a_\alpha a_\beta}{a_\lambda} g^\lambda_{\alpha,\beta}u^+_\alpha K_\beta \otimes u^+_\beta, \quad \text{for all } \lambda \in \mathcal{P},\\ \Delta(K_\alpha) &= K_\alpha \otimes K_\alpha, \quad \text{for all } \alpha \in \mathbb{Z}[I], \end{aligned}$$

with counit $\epsilon(u^+_\lambda) = 0$, for $\lambda \neq 0$ in $\mathcal{P}$, and $\epsilon(K_\alpha) = 1$.

(c) Antipode (see [9]):

$$\begin{aligned} \sigma(u^+_\lambda) =& \delta_{\lambda 0} + \sum_{m\ge1}(-1)^m \sum_{\pi,\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\times\\ & \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1,\cdots,\lambda_m} g^\pi_{\lambda_1,\cdots,\lambda_m} K_{-\lambda}u^+_\pi, \end{aligned}$$

for all $\lambda \in \mathcal{P}$, and $\sigma(K_\alpha) = K_{-\alpha}$ for all $\alpha \in \mathbb{Z}[I]$.

The Hopf algebra $\mathcal{H}^+$ is called the *(extended twisted) Ringel-Hall algebra* of Λ. The subspace of $\mathcal{H}^+$ with basis $\{ u_\lambda^+ \mid \lambda \in \mathcal{P} \}$ is an associative subalgebra (but not closed under comultiplication), we denoted it by $\mathfrak{h}^+$. We point out that, if V_i $(i \in I)$ is a simple Λ-module, then

$$\Delta(u_i^+) = u_i^+ \otimes 1 + K_i \otimes u_i^+, \text{ and } \sigma(u_i^+) = -K_{-i} u_i^+.$$

It is easy to see that $\mathcal{H}^+$ and $\mathfrak{h}^+$ have the canonical $\mathbb{N}[I]$-gradation.

1.4 Dually, one can define a Hopf algebra $\mathcal{H}^-$ with basis $\{ K_\alpha u_\lambda^- \mid \alpha \in \mathbb{Z}[I], \lambda \in \mathcal{P} \}$ and Hopf structure as follows:

(a) Multiplication:

$$\begin{aligned} u_\alpha^- u_\beta^- &= v^{\langle \alpha,\beta \rangle} \sum_{\lambda \in \mathcal{P}} g_{\alpha,\beta}^\lambda u_\lambda^-, \quad \text{for all } \alpha, \beta \in \mathcal{P}, \\ K_\alpha u_\beta^- &= v^{-(\alpha,\beta)} u_\beta^- K_\alpha, \quad \text{for all } \alpha \in \mathbb{Z}[I], \beta \in \mathcal{P}, \\ K_\alpha K_\beta &= K_{\alpha+\beta}, \quad \text{for all } \alpha, \beta \in \mathbb{Z}[I], \end{aligned}$$

with unit $1 = u_0^- = K_0$.

(b) Comultiplication:

$$\begin{aligned} \Delta(u_\lambda^-) &= \sum_{\alpha,\beta \in P} v^{\langle \alpha,\beta \rangle} \frac{a_\alpha a_\beta}{a_\lambda} g_{\alpha,\beta}^\lambda u_\beta^- \otimes u_\alpha^- K_{-\beta}, \quad \text{for all } \lambda \in \mathcal{P}\ , \\ \Delta(K_\alpha) &= K_\alpha \otimes K_\alpha, \quad \text{for all } \alpha \in \mathbb{Z}[I], \end{aligned}$$

with counit $\epsilon(u_\lambda^-) = 0$, for $\lambda \neq 0$ in $\mathcal{P}$, and $\epsilon(K_\alpha) = 1$.

(c) Antipode and its inverse:

$$\sigma(u_\lambda^-) = \delta_{\lambda 0} + \sum_{m \geq 1} (-1)^m \sum_{\pi, \lambda_1, \cdots, \lambda_m \in \mathcal{P}_1} \frac{a_{\lambda_1} \cdots a_{\lambda_m}}{a_\lambda} g_{\lambda_1,\cdots,\lambda_m}^\lambda g_{\lambda_m,\cdots,\lambda_1}^\pi u_\pi^- K_\lambda$$

for all $\lambda \in \mathcal{P}$, and $\sigma(K_\alpha) = K_{-\alpha}$ for all $\alpha \in \mathbb{Z}[I]$.

$$\begin{aligned} \sigma^{-1}(u_\lambda^-) =& \delta_{\lambda 0} + \sum_{m \geq 1} (-1)^m \sum_{\pi, \lambda_1, \cdots, \lambda_m \in \mathcal{P}_1} v^{2 \sum_{i<j} \langle \lambda_i, \lambda_j \rangle} \\ & \times \frac{a_{\lambda_1} \cdots a_{\lambda_m}}{a_\lambda} g_{\lambda_1,\cdots,\lambda_m}^\lambda g_{\lambda_1,\cdots,\lambda_m}^\pi K_\lambda u_\pi^- \end{aligned}$$

for all $\lambda \in \mathcal{P}$, and $\sigma^{-1}(K_\alpha) = K_{-\alpha}$, for all $\alpha \in \mathbb{Z}[I]$. In particular, we have

$$\Delta(u_i^-) = u_i^- \otimes K_{-i} + 1 \otimes u_i^-, \text{and } \sigma(u_i^-) = -u_i^- K_i, \text{ for all } i \in I.$$

Also, we use $\mathfrak{h}^-$ to denote the subalgebra of $\mathcal{H}^-$ with basis $\{ u_\lambda^- \mid \lambda \in \mathcal{P} \}$.

1.5 Following Ringel [6], we associate a bilinear form $\varphi :\ \mathcal{H}^+ \times \mathcal{H}^- \to R$, which is defined as follows:

$$\varphi(K_\alpha u_\beta^+, K_{\alpha'} u_{\beta'}^-) = v^{-(\alpha,\alpha') - (\beta,\alpha') + (\alpha,\beta')} \frac{|V_\beta|}{a_\beta} \delta_{\beta\beta'}.$$

Proposition 1.5.1. *The bilinear form φ is a skew Hopf-paring.*

Proof. See Proposition 5.3 of [9]. $\square$

It is easy to see that the restriction of φ on $\mathfrak{h}^+_\alpha \times \mathfrak{h}^-_\beta$ for any $\alpha, \beta \in \mathbb{N}[I]$ is zero unless $\alpha = \beta$ and the restriction of φ on $\mathfrak{h}^+_\alpha \times \mathfrak{h}^-_\alpha$ is non-degenerate for any $\alpha \in \mathbb{N}[I]$. Therefore we have the Drinfeld double of $(\mathcal{H}^+, \mathcal{H}^-, \varphi)$. Its ideal generated by $\{ K_\alpha \otimes K_{-\alpha} - 1 \mid \alpha \in \mathbb{Z}[I] \}$, or equivalently by $\{ K_\alpha \otimes 1 - 1 \otimes K_\alpha \mid \alpha \in \mathbb{Z}[I] \}$, is a Hopf ideal. The corresponding quotient inherits a Hopf structure, which is called the *reduced Drinfeld double of the Ringel-Hall algebra* of Λ and is denoted by $\mathcal{D} = \mathcal{D}(\Lambda, R)$. Obviously, we have the triangular decomposition

$$\mathcal{D} = \mathfrak{h}^- \otimes \mathcal{T} \otimes \mathfrak{h}^+,$$

where $\mathcal{T}$ is the torus, which is generated by $\{ K_\alpha \mid \alpha \in \mathbb{Z}[I] \}$.

The subalgebra of $\mathcal{D}$ generated by $\{ u_i^\pm, K_i \mid i \in I \}$ is called the *double composition algebra* of Λ and is denoted by $\mathfrak{c}$. It is easy to see that $\mathfrak{c}$ is a Hopf subalgebra of $\mathcal{D}$ and admits a triangular decomposition

$$\mathfrak{c} = \mathfrak{c}^- \otimes \mathcal{T} \otimes \mathfrak{c}^+,$$

where $\mathfrak{c}^+$ (resp. $\mathfrak{c}^-$) is the *composition algebra* of Λ, which is generated by $\{ u_i^+ \mid i \in I \}$ (resp. $\{ u_i^- \mid i \in I \}$) and is still $\mathbb{N}[I]$-graded. We have the following observation.

Lemma 1.5.2. *For any $\alpha \in \mathbb{N}[I]$, the restriction $\varphi : \mathfrak{c}^+_\alpha \times \mathfrak{c}^-_\alpha \to R$ is non-degenerate.*

Proof. By a complete symmetry between the associative algebras $\mathfrak{h}^+$ and $\mathfrak{h}^-$, if $x^+ = \sum_{\lambda \in \mathcal{P}} b_\lambda u_\lambda^+ \in \mathfrak{c}^+_\alpha$, where $b_\lambda \in R$, then $x^- = \sum_{\lambda \in \mathcal{P}} b_\lambda u_\lambda^- \in \mathfrak{c}^-_\alpha$. Thus $\varphi(x^+, x^-) = \sum_{\lambda \in \mathcal{P}} b_\lambda^2 \frac{|V_\lambda|}{a_\lambda} > 0$ unless $b_\lambda = 0$ for all $\lambda \in \mathcal{P}$. $\square$

2. The decomposition induced by the pairing φ

2.1 From now on, we assume that Λ is of tame representation type and is indecomposable as an algebra. Then its Cartan datum $\Gamma = (I, (\ ,\ ))$ is of affine type. Let Φ be the root system of Γ, δ the unique minimal positive imaginary root of Φ. Then $\Phi^{im} = \{ m\delta \mid m \in \mathbb{Z}, m \neq 0 \}$ is the set of imaginary roots of Φ. By a simple homological property of $\operatorname{mod} \Lambda$, it is not difficult to see that all monomials on the set $\{ u_\beta^+ \mid \beta \in \mathcal{P}$ and V_β is indecomposable$\}$ in any fixed order provide a universal PBW-basis for $\mathfrak{h}^+$ (see Theorem 3.1 of [2]).

It is well known that the dimension vectors of all indecomposable Λ-modules are bijective to the positive roots in Φ and, if α is a positive real root, then there exists only one indecomposable Λ-module (up to isomorphism) with dimension vector α.

2.2 For $\alpha, \beta \in \mathbb{N}[I]$, we use $\alpha \leq \beta$ to mean that $\beta - \alpha \in \mathbb{N}[I]$. Take a minimal $\alpha \in \mathbb{N}[I]$ such that $\mathfrak{c}^+_\alpha \subsetneq \mathfrak{h}^+_\alpha$. Because $\{ u_\beta^+ \mid \beta \in \mathcal{P}$ and V_β is indecomposable$\}$ gives a universal PBW-basis of $\mathfrak{h}^+$, it follows that $\alpha \in \Phi^+$. If $\alpha < \delta$, then V_α is an exceptional Λ-module, and so $u_\alpha^+ \in \mathfrak{c}^+$ by Zhang [10]. It follows that $\mathfrak{c}_\beta^+ = \mathfrak{h}_\beta^+$ provided $\beta < \delta$. Now we define

$$\mathcal{L}_\delta^+ = \{\, x^+ \in \mathfrak{h}_\delta^+ \,|\, \varphi(x^+, \mathfrak{c}_\delta^-) = 0 \,\}.$$

It is easy to see that $\mathcal{L}_\delta^+$ is nonzero, but we do not need it. Dually, let

$$\mathcal{L}_\delta^- = \{\, y^- \in \mathfrak{h}_\delta^- \,|\, \varphi(\mathfrak{c}_\delta^+, y^-) = 0 \,\}.$$

By the non-degenerate property of φ and Lemma 1.5.2, we have the following

$$\begin{aligned} \mathfrak{h}_\delta^+ &= \mathfrak{c}_\delta^+ \oplus \mathcal{L}_\delta^+, \quad \mathfrak{h}_\delta^- = \mathfrak{c}_\delta^- \oplus \mathcal{L}_\delta^-, \\ \mathfrak{c}_\delta^+ &= \{\, x^+ \in \mathfrak{h}_\delta^+ \,|\, \varphi(x^+, \mathcal{L}_\delta^-) = 0 \,\}, \\ \mathfrak{c}_\delta^- &= \{\, y^- \in \mathfrak{h}_\delta^- \,|\, \varphi(\mathcal{L}_\delta^+, y^-) = 0 \,\}. \end{aligned}$$

Now let $\mathcal{D}(1)$ be the subalgebra of $\mathcal{D}$ generated by $\mathfrak{c}$ and $\mathcal{L}_\delta^\pm$. We have the triangular decomposition $\mathcal{D}(1) = \mathcal{D}(1)^- \otimes \mathcal{T} \otimes \mathcal{D}(1)^+$ and $\mathcal{D}(1)^\pm$ inherit the $\mathbb{N}[I]$-gradation.

Take a minimal $\alpha \in \mathbb{N}[I]$ such that $\mathcal{D}^+(1)_\alpha \subsetneq \mathfrak{h}_\alpha^+$. Similarly, we see that $\alpha \in \Phi^+$. If $\alpha < 2\delta$, then α is a real root because $\alpha \neq \delta$, and there is only one indecomposable Λ-module with dimension vector α. It is easy to see that u_α^+ is generated by elements in $\mathfrak{c}^+$ and $\mathcal{L}_\delta^+$. Therefore, $\beta < 2\delta$ implies that $\mathcal{D}(1)_\beta^+ = \mathfrak{h}_\beta^+$. We now define

$$\mathcal{L}_{2\delta}^+ = \{\, x^+ \in \mathfrak{h}_{2\delta}^+ \,|\, \varphi(x^+, \mathcal{D}(1)_{2\delta}^-) = 0 \,\},$$

$$\mathcal{L}_{2\delta}^- = \{\, y^- \in \mathfrak{h}_{2\delta}^- \,|\, \varphi(\mathcal{D}(1)_{2\delta}^+, y^-) = 0 \,\}.$$

Then we have

$$\begin{aligned} \mathfrak{h}_{2\delta}^+ &= \mathcal{D}(1)_{2\delta}^+ \oplus \mathcal{L}_{2\delta}^+, \quad \mathfrak{h}_{2\delta}^- = \mathcal{D}(1)_{2\delta}^- \oplus \mathcal{L}_{2\delta}^-. \\ \mathcal{D}(1)_{2\delta}^+ &= \{\, x^+ \in \mathfrak{h}_{2\delta}^+ \,|\, \varphi(x^+, \mathcal{L}_{2\delta}^-) = 0 \,\}, \\ \mathcal{D}(1)_{2\delta}^- &= \{\, y^- \in \mathfrak{h}_{2\delta}^- \,|\, \varphi(\mathcal{L}_{2\delta}^+, y^-) = 0) \,\}. \end{aligned}$$

In general, for $m \geq 3$, we define the following inductively

$$\mathcal{L}_{m\delta}^+ = \{\, x^+ \in \mathfrak{h}_{m\delta}^+ \,|\, \varphi(x^+, \mathcal{D}(m-1)_{m\delta}^-) = 0 \,\},$$

$$\mathcal{L}_{m\delta}^- = \{\, x^- \in \mathfrak{h}_{m\delta}^- \,|\, \varphi(\mathcal{D}(m-1)_{m\delta}^+, y^-) = 0 \,\}.$$

And let $\mathcal{D}(m)$ be the subalgbra of $\mathcal{D}$ generated by $\mathcal{D}(m-1)$ and $\mathcal{L}_{m\delta}^\pm$. Then we have

$$\mathfrak{h}_{m\delta}^+ = \mathcal{D}(m-1)_{m\delta}^+ \oplus \mathcal{L}_{m\delta}^+, \quad \mathfrak{h}_{m\delta}^- = \mathcal{D}(m-1)_{m\delta}^- \oplus \mathcal{L}_{m\delta}^-.$$

and

$$\begin{aligned} \mathcal{D}(m-1)_{m\delta}^+ &= \{\, x^+ \in \mathfrak{h}_{m\delta}^+ \,|\, \varphi(x^+, \mathcal{L}_{m\delta}^-) = 0 \,\}, \\ \mathcal{D}(m-1)_{m\delta}^- &= \{\, y^- \in \mathfrak{h}_{m\delta}^- \,|\, \varphi(\mathcal{L}_{m\delta}^+, y^-) = 0 \,\}. \end{aligned}$$

2.3 We have the following lemmas.

Lemma 2.3.1. *The elements in $\mathcal{L}^+_{m\delta}$, $m \geq 1$, are primitive, i.e., for any $x^+ \in \mathcal{L}^+_{m\delta}$, there hold $\Delta(x^+) = x^+ \otimes 1 + K_{m\delta} \otimes x^+$ and $\sigma(x^+) = -K_{-m\delta}x^+$. Dually, for any $y^- \in \mathcal{L}^-_{m\delta}$, there hold $\Delta(y^-) = y^- \otimes K_{-m\delta} + 1 \otimes y^-$ and $\sigma(y^-) = -y^-K_{m\delta}$.*

Proof. By the Green formula for the comultiplication 1.3(b), we may assume that $\Delta(x^+) = x^+ \otimes 1 + K_{m\delta} \otimes x^+ + \sum_{\alpha,\beta\in\mathcal{P}_1} b_{\alpha\beta}u^+_\alpha K_\beta \otimes u^+_\beta$, where $b_{\alpha\beta} \in R$. If $b_{\alpha\beta} \neq 0$ for some $\alpha, \beta \in \mathcal{P}_1$, then $u^+_\alpha, u^+_\beta \in \mathcal{D}(m-1)^+$, hence $u^-_\alpha, u^-_\beta \in \mathcal{D}(m-1)^-$ since $\mathcal{D}(m-1)^+$ and $\mathcal{D}(m-1)^-$ are a symmetric pair under φ. Therefore,

$$\varphi(x^+, u^-_\alpha u^-_\beta) = \varphi(\Delta(x^+), u^-_\alpha \otimes u^-_\beta) = b_{\alpha\beta}\varphi(u^+_\alpha K_\beta, u^-_\alpha)\varphi(u^+_\beta, u^-_\beta) = b_{\alpha\beta}\frac{|V_\alpha||V_\beta|}{a_\alpha a_\beta},$$

which is nonzero. This contradicts the fact that $\varphi(\mathcal{L}^+_{m\delta}, \mathcal{D}(m-1)^-_{m\delta}) = 0$. So we have $\Delta(x^+) = x^+ \otimes 1 + K_{m\delta} \otimes x^+$. By $\mu(\sigma \otimes 1)\Delta(x^+) = \eta\epsilon(x^+) = 0$, then $\sigma(x^+) + \sigma(K_{m\delta})x^+ = 0$, hence $\sigma(x^+) = -K_{-m\delta}x^+$. The conclusions for $y^- \in \mathcal{L}^-_{m\delta}$ are dual. □

Lemma 2.3.2. *For any $x^+ \in \mathcal{L}^+_{m\delta}$ and $y^- \in \mathcal{L}^-_{m\delta}$, we have $x^+y^- - y^-x^+ = -\varphi(x^+, y^-)(K_{m\delta} - K_{-m\delta})$.*

Proof. It follows from Lemma 2.3.1 that $\Delta^2(x^+) = x^+ \otimes 1 \otimes 1 + K_{m\delta} \otimes x^+ \otimes 1 + K_{m\delta} \otimes K_{m\delta} \otimes x^+$, and $\Delta^2(y^-) = y^- \otimes K_{-m\delta} \otimes K_{-m\delta} + 1 \otimes y^- \otimes K_{-m\delta} + 1 \otimes 1 \otimes y^-$. Now, the assertion of this lemma follows from formula 1.1(d). □

2.4 For $i \in I$, we have in $\mathcal{D}$

$$u^+_i u^-_j - u^-_j u^+_i = -\frac{|V_i|}{a_i}(K_i - K_i^{-1})\delta_{ij} = \frac{\delta_{ij}}{v^{-(i,i)} - 1}(K_i - K_i^{-1}).$$

We set $E_i(0) = u^+_i$ and $F_i(0) = -v_i^{-1}u^-_i$, $i \in I$, where $v_i = v^{(i,i)/2}$. Then

$$E_i(0)F_j(0) - F_j(0)E_i(0) = \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}}\delta_{ij}, \quad \text{for } i, j \in I. \tag{2.4.1}$$

For any integer $m \geq 1$, we take an R-basis $E_1(m), \cdots, E_{t_m}(m)$ of $\mathcal{L}^+_{m\delta}$ and an R-basis $F_1(m), \cdots, F_{t_m}(m)$ of $\mathcal{L}^-_{m\delta}$ (where $t_m = \dim_R\mathcal{L}^+_{m\delta} = \dim_R\mathcal{L}^-_{m\delta}$) such that

$$\varphi(E_i(m), F_j(m)) = \frac{-1}{v^m - v^{-m}}\delta_{ij}.$$

It follows from Lemma 2.3.2, we have

$$E_i(m)F_j(m) - F_j(m)E_i(m) = \frac{K_{m\delta} - K_{m\delta}^{-1}}{v^m - v^{-m}}\delta_{ij}, \quad \text{for } 1 \leq i, j \leq t_m. \tag{2.4.2}$$

Therefore the algebra $\mathcal{D}$ is generated by $E_i(m), F_i(m)$ ($m \geq 1$, $1 \leq i \leq t_m$), and $E_i(0), F_i(0)$, K_i, K_i^{-1} ($i \in I$). The elements $E_i(m)$ and $F_i(m)$ with $m \geq 1$ are called *Heisenberg generators* of $\mathcal{D}$.

2.5 Since all elements $E_i(m)$ and $F_i(m)$ are primitive, according to formula 1.1(d′), it is easy to see that

$$E_i(m)F_j(n) - F_j(n)E_i(m) = 0, \quad \text{for } m \neq n, 1 \leq i \leq t_m, 1 \leq j \leq t_n. \tag{2.5.1}$$

Since $\mathcal{L}^-_{(m+n)\delta}$ is orthogonal to $\mathcal{L}^+_{m\delta}$ and $\mathcal{L}^+_{n\delta}$ with respect to φ and all elements in $\mathcal{L}^-_{(m+n)\delta}$ are primitive, we see that $\varphi\big(E_i(m)E_j(n), \mathcal{L}^-_{(m+n)\delta}\big) = 0$. Note that this is true even when $n = 0$. For any $x \in \mathcal{H}^-_\alpha, y \in \mathcal{H}^-_\beta$ such that $\alpha, \beta > 0$ and $\alpha + \beta = (m+n)\delta$, then

$$\begin{aligned}
&\varphi\big(E_i(m)E_j(n), xy\big) = \varphi\big(\Delta(E_i(m))\Delta(E_j(n)), x \otimes y\big) \\
=&\varphi\big((E_i(m) \otimes 1 + K_{m\delta} \otimes E_i(m))(E_j(n) \otimes 1 + K_{n\delta} \otimes E_j(n)), x \otimes y\big) \\
=&\varphi\big(E_i(m), x\big)\varphi\big(E_j(n), y\big) + \varphi\big(E_j(n), x\big)\varphi\big(E_i(m), y\big).
\end{aligned}$$

Similarly, it is easy to show that

$$\varphi\big(E_j(n)E_i(m), xy\big) = \varphi\big(E_i(m), y\big)\varphi\big(E_j(n), x\big) + \varphi\big(E_j(n), y\big)\varphi\big(E_i(m), x\big)$$

Therefore, we have the following relations:

$$E_i(m)E_j(n) - E_j(m)E_i(n) = 0, \text{ for } m, n \geq 1, 1 \leq i \leq t_m, 1 \leq j \leq t_n. \tag{2.5.2}$$

If $m \geq 1$ but $n = 0$, for any $x \in \mathcal{H}^-_\alpha, y \in \mathcal{H}^-_\beta$, such that $\alpha, \beta > 0$ and $\alpha + \beta = m + n\delta + e_j$, then

$$\begin{aligned}
&\varphi\big(E_i(m)E_j(0), xy\big) = \varphi\big(K_{m\delta}E_j(0), x\big)\varphi\big(E_i(m), y\big) + \varphi\big(E_i(m)K_j, x\big)\varphi\big(E_i(0), y\big) \\
&= \varphi\big((E_j(0), x\big)\varphi\big(E_i(m), y\big) + \varphi\big(K_jE_i(m), x\big)\varphi\big(E_j(0), y\big) \\
&= \varphi\big(E_j(0)E_i(m), xy\big).
\end{aligned}$$

This implies the following relations:

$$E_i(m)E_j(0) - E_j(0)E_i(m) = 0, \text{ for } m \geq 1, 1 \leq i \leq t_m, 1 \leq j \leq t_0. \tag{2.5.3}$$

Symmetrically, we have the following relations:

$$F_i(m)F_j(n) - F_j(n)F_i(m) = 0 \quad \text{for } m \geq 1, n \geq 0, 1 \leq i \leq t_m, 1 \leq j \leq t_n. \tag{2.5.4}$$

Of course, we have the quantum Serre relations (see Ringel [6] or [7])

$$\begin{aligned}
&\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{\epsilon_i} E_i(0)^t E_j(0) E_i(0)^{1-a_{ij}-t} = 0 \quad \text{for } \; i \neq j, i, j \in I, \\
&\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{\epsilon_i} F_i(0)^t F_j(0) F_i(0)^{1-a_{ij}-t} = 0 \quad \text{for } \; i \neq j, i, j \in I.
\end{aligned} \tag{2.5.5}$$

where $a_{ij} = 2(e_i, e_j)/(e_i, e_i)$ and $\epsilon_i = (e_i, e_i)/2$.

A more precise form of Theorem 0.1 is the following

Theorem 2.5.1. *The algebra $\mathcal{D}$ is generated by the generators $E_i(0)$, $F_i(0)$, K_i, K_i^{-1} ($i \in I$) and $E_i(m)$, $F_i(m)$ ($m \geq 1,\ 1 \leq i \leq t_m$) with the generating relations* (2.4.1), (2.4.2), (2.5.1), (2.5.2), (2.5.3), (2.5.4) *and* (2.5.5)

The proof of this result will be given at the end of Section 4.

3. Casimir operator and integrable modules

3.1 We first introduce two symmetries on $\mathcal{D}$. The operator ω of $\mathcal{D}$ defined by $\omega(u_\lambda^+) = u_\lambda^-, \omega(u_\lambda^-) = u_\lambda^+$ for all $\lambda \in \mathcal{P}$ and, $\omega(K_\alpha) = K_{-\alpha}$ for all $\alpha \in \mathbb{Z}[I]$, is an involution on $\mathcal{D}$. For any $\alpha \in \mathbb{Z}[I]$, we write $\alpha = \sum_i k_i e_i, k_i \in \mathbb{Z}$, and let $\mathrm{tr}\alpha = \sum_i k_i$, $\tau(\alpha) = \sum_i k_i(e_i, e_i)/2$. Then the operator τ on $\mathcal{D}$ given below is an anti-involution:

$$\begin{aligned}
&\tau(x_\alpha) = (-1)^{\mathrm{tr}\alpha+1} v^{-\tau(\alpha)} K_\alpha \sigma(x_\alpha), \quad \text{for } x_\alpha \in \mathcal{H}_\alpha^+, \alpha \in \mathbb{N}[I],\\
&\tau(y_\alpha) = (-1)^{\mathrm{tr}\alpha+1} v^{\tau(\alpha)} \sigma(y_\alpha) K_{-\alpha}, \quad \text{for } y_\alpha \in \mathcal{H}_\alpha, \alpha \in \mathbb{N}[I],\\
&\tau(K_\alpha) = K_{-\alpha}, \quad \text{for } \alpha \in \mathbb{Z}[I].
\end{aligned}$$

It is easy to see the following

$$\tau(E_i(m)) = (-1)^{m\mathrm{tr}(\delta)} v^{-m\tau(\delta)} E_i(m), \ \tau(F_i(m)) = (-1)^{m\mathrm{tr}(\delta)} v^{m\tau(\delta)} F_i(m).$$

The operators ω and τ respect to φ:

$$\text{(3.1.3)} \qquad \varphi(x,y) = \varphi(\omega(y), \omega(x)) = \varphi(\tau(x), \tau(y)), \quad \text{for any } x \in \mathcal{H}^+, y \in \mathcal{H}^-.$$

3.2 According to the comultiplication formula of Green, for any $x \in \mathfrak{h}_\alpha^+$, we may assume that

$$\Delta(x) = x \otimes 1 + r_{i,m}(x) K_{m\delta} \otimes E_i(m) + \text{ “other terms”},$$

and

$$\Delta(x) = K_\alpha \otimes x + E_i(m) K_{\alpha - m\delta} \otimes r'_{i,m}(x) + \text{ “other terms”}.$$

It can be shown that

$$r_{i,m}(E_j(n)) = \delta_{mn}\delta_{ij} = r'_{i,m}(E_j(n)).$$

We have the following properties. For all $x \in \mathfrak{h}_\alpha^+$ and $x' \in \mathfrak{h}_\beta^+$, there hold

$$r_{i,m}(xx') = r_{i,m}(x)x' + x r_{i,m}(x'), \quad r'_{i,m}(xx') = r'_{i,m}(x)x' + x r'_{i,m}(x').$$

And for all $x \in \mathfrak{h}^+$ and $y \in \mathfrak{h}^-$, there hold

$$\begin{aligned}
\varphi(x, F_i(m)y) &= \frac{-1}{v^m - v^{-m}} \varphi(r'_{i,m}(x), y),\\
\varphi(x, yF_i(m)) &= \frac{-1}{v^m - v^{-m}} \varphi(r_{i,m}(x), y),\\
r'_{i,m}(x) &= (-1)^{m\,\mathrm{tr}\delta} v^{m\tau(\delta)} \tau r_{i,m} \tau(x).
\end{aligned}$$

For any $y \in \mathfrak{h}_\alpha^-$, we may assume that

$$\Delta(y) = y \otimes K_{-\alpha} + r_{i,m}(y) \otimes F_i(m) K_{-(\alpha - m\delta)} + \text{ “other terms”}.$$

and

$$\Delta(y) = 1 \otimes y + F_i(m) \otimes r'_{i,m}(y) K_{-m\delta} + \text{ “other terms”}.$$

Then we have

$$r_{i,m}(F_j(n)) = \delta_{mn}\delta_{ij} = r'_{i,m}(F_j(n)).$$

Also we have the following properties

$$\begin{aligned}
\varphi(E_i(m)x, y) &= \frac{-1}{v^m - v^{-m}} \varphi(x, r_{i,m}(y)), \\
\varphi(xE_i(m), y) &= \frac{-1}{v^m - v^{-m}} \varphi(x, r'_{i,m}(y)), \\
r'_{i,m}(y) &= (-1)^{m\,\mathrm{tr}\delta} v^{-m\tau(\delta)} \tau r_{i,m} \tau(y).
\end{aligned}$$

It can be easily shown that

$$r_{i,m}(y) = \omega r'_{i,m}\omega(y) \text{ and } r'_{i,m}(y) = \omega r_{i,m}\omega(y) \quad \text{for all} \quad y \in \mathfrak{h}^-.$$

Lemma 3.2.1. *For any $x \in \mathfrak{h}^+$, $y \in \mathfrak{h}^+$ and $m \geq 1$, $1 \leq i \leq t_m$, we have*

$$E_i(m)y - yE_i(m) = \frac{K_{m\delta} r_{i,m}(y) - r'_{i,m}(y) K_{-m\delta}}{v^m - v^{-m}},$$

$$xF_i(m) - F_i(m)x = \frac{r_{i,m}(x) K_{m\delta} - K_{-m\delta} r'_{i,m}(x)}{v^m - v^{-m}}.$$

Proof. We verify the first only, the second can be verified dually. It holds if we take y to be the generators $F_j(n)$ with $n \geq 0$ and $1 \leq j \leq t_n$. By induction, we have

$$\begin{aligned}
&E_i(m)(yy') - (yy')E_i(m) \\
=&(E_i(m)y - yE_i(m))y' + y(E_i(m)y' - y'E_i(m)) \\
=&\frac{K_{m\delta} r_{i,m}(y) - r'_{i,m}(y) K_{-m\delta}}{v^m - v^{-m}} y' + y \frac{K_{m\delta} r_{i,m}(y') - r'_{i,m}(y') K_{-m\delta}}{v^m - v^{-m}} \\
=&\frac{K_{m\delta}(r_{i,m}(y)y' + yr_{i,m}(y')) - (r'_{i,m}(y)y' + yr'_{i,m}(y'))K_{-m\delta}}{v^m - v^{-m}} \\
=&\frac{K_{m\delta} r_{i,m}(yy') - r'_{i,m}(yy') K_{-m\delta}}{v^m - v^{-m}}.
\end{aligned}$$

□

3.3 For each $\alpha \in \mathbb{N}[I]$, we choose an R-basis $x_\alpha^1, \cdots, x_\alpha^r$ of $\mathfrak{h}^+(\Lambda)_\alpha$, by the Ringel pairing, there exists an R-basis $y_\alpha^1, \cdots, y_\alpha^r$ of $\mathfrak{h}_\alpha^-$ such that $\varphi(x_\alpha^i, y_\alpha^j) = \delta_{ij}$ for all i, j. For example, we may take $x_\alpha^i = \sqrt{\frac{a_{\alpha(i)}}{|V_{\alpha(i)}|}} u_{\alpha(i)}^+$, where $\dim V_{\alpha(i)} = \alpha$, and $y_\alpha^i = \sqrt{\frac{a_{\alpha(i)}}{|V_{\alpha(i)}|}} u_{\alpha(i)}^-$. Set

$$\theta_\alpha = \sum_i y_\alpha^i \otimes x_\alpha^i \in \mathcal{D} \otimes \mathcal{D}.$$

By linear algebra, θ_α does not depend on the choice of the basis x_α^i. Then by (3.1.3), we see that

$$(\tau \otimes \tau)\theta_\alpha = \theta_\alpha, \quad \text{and} \quad (\omega \otimes \omega)\theta_\alpha = \theta_\alpha^{op}.$$

where $\theta_\alpha^{op} = \sum_i x_\alpha^i \otimes y_\alpha^i$.

Lemma 3.3.1. *For any $\alpha \in \mathbb{N}[I]$ and $i \in I$, there hold*

(1) $(E_i(0)\otimes 1)\theta_\alpha + (K_i \otimes E_i(0))\theta_{\alpha-i} = \theta_\alpha(E_i(0)\otimes 1) + \theta_{\alpha-i}(K_{-i}\otimes E_i(0))$,

(2) $(1\otimes F_i(0))\theta_\alpha + (F_i(0)\otimes K_i^{-1})\theta_{\alpha-i} = \theta_\alpha(1\otimes F_i(0)) + \theta_{\alpha-i}(F_i(0)\otimes K_i)$,

(3) $(E_i(m)\otimes 1)\theta_\alpha + (K_{m\delta}\otimes E_i(m))\theta_{\alpha-m\delta} =$
$$\theta_\alpha(E_i(m)\otimes 1) + \theta_{\alpha-m\delta}(K_{-m\delta}\otimes E_i(m)),$$

(4) $(1\otimes F_i(m))\theta_\alpha + (F_i(m)\otimes K_{-m\delta})\theta_{\alpha-m\delta} =$
$$\theta_\alpha(1\otimes F_i(m)) + \theta_{\alpha-m\delta}(F_i(m)\otimes K_{m\delta}).$$

Proof. We prove (3) only, all others can be proved similarly. Note that for any $y \in \mathfrak{h}_\alpha^-$, $y = \sum_t \varphi(x_\alpha^t, y)y_\alpha^t$. In the following $c_m = (v^m - v^{-m})^{-1}$. Therefore,

$$\begin{aligned}
&(E_i(m)\otimes 1)\theta_\alpha - \theta_\alpha(E_i(m)\otimes 1) = \sum_j (E_i(m)y_\alpha^j - y_\alpha^j E_i(m))\otimes x_\alpha^j\\
=&c_m \sum_j \big(K_{m\delta} r_{i,m}(y_\alpha^j) - r'_{i,m}(y_\alpha^j)K_{-m\delta}\big)\otimes x_\alpha^j\\
=&c_m \sum_j \Big(K_{m\delta}\sum_t \varphi(x_{\alpha-m\delta}^t, r_{i,m}(y_\alpha^j))y_\alpha^t - \sum_t \varphi(x_{\alpha-m\delta}^t, r'_{i,m}(y_\alpha^j))y_\alpha^t K_{-m\delta}\Big)\otimes x_\alpha^j\\
=&\sum_j \Big(K_{m\delta}\sum_t(-\varphi(E_i(m)x_{\alpha-m\delta}^t, y_\alpha^j)y_\alpha^t) -\\
&\qquad \sum_t (-\varphi(x_{\alpha-m\delta}^t E_i(m), y_\alpha^j)y_\alpha^t)K_{-m\delta}\Big)\otimes x_\alpha^j\\
=&\sum_t\Big(\sum_j \varphi(x_{\alpha-m\delta}^t E_i(m), y_\alpha^j)y_\alpha^t K_{-m\delta} - \sum_j K_{m\delta}\varphi(E_i(m)x_{\alpha-m\delta}^t, y_\alpha^j)y_\alpha^t\Big)\otimes x_\alpha^j\\
=&\sum_t y_\alpha^t K_{-m\delta}\otimes \sum_j(\varphi(x_{\alpha-m\delta}^t E_i(m), y_\alpha^j)x_\alpha^j -\\
&\qquad \sum_t K_{m\delta}y_\alpha^t \otimes \sum_j \varphi(E_i(m)x_{\alpha-m\delta}^t, y_\alpha^j)x_\alpha^j\\
=&\sum_t y_\alpha^t K_{-m\delta}\otimes x_{\alpha-m\delta}^t E_i(m) - \sum_t K_{m\delta}y_\alpha^t \otimes E_i(m)x_{\alpha-m\delta}^t\\
=&\theta_{\alpha-m\delta}(K_{-m\delta}\otimes E_i(m)) - (K_{m\delta}\otimes E_i(m))\theta_{\alpha-m\delta}.
\end{aligned}$$

Note here that $\theta_0 = 1\otimes 1$ and $\theta_\beta = 0$ if $\beta \notin \mathbb{N}[I]$. □

3.4 We may define $\theta = \sum_{\alpha\in\mathbb{N}[I]}\theta_\alpha$ in the sense in the completion of $\mathcal{D}\otimes\mathcal{D}$ and $\theta_{\leq p} = \sum_{\alpha\in\mathbb{N}[I],\mathrm{tr}\alpha\leq p}\theta_\alpha$. For the twisted comultiplication ${}^\tau\Delta = (\tau\otimes\tau)\Delta\tau^{-1}$, then

$${}^\tau\Delta(E_i(0)) = E_i(0)\otimes 1 + K_{-i}\otimes E_i(0),\ {}^\tau\Delta(F_i(0)) = F_i(0)\otimes K_i + 1\otimes F_i(0),$$

$${}^\tau\Delta(E_i(m)) = E_i(m)\otimes 1 + K_{-m\delta}\otimes E_i(m),\ {}^\tau\Delta(F_i(m)) = F_i(m)\otimes K_{m\delta} + 1\otimes F_i(m),$$

for $m \geq 1, 1 \leq i \leq t_m$. ${}^\tau\Delta(K_\alpha) = K_\alpha\otimes K_\alpha$ for $\alpha \in \mathbb{Z}[I]$.

Lemma 3.3.1 implies that

$$\Delta(u)\theta = \theta\,{}^{\tau}\Delta(u) \tag{3.4.1}$$

for all $u \in \mathcal{D}$ (equality holds in the completion of $\mathcal{D}\otimes\mathcal{D}$).

(3.4.2)

(1) $(E_i(0)\otimes 1 + K_i\otimes E_i(0))\theta_{\le p} - \theta_{\le p}(E_i(0)\otimes 1 + K_{-i}\otimes E_i(0))$
$= \sum_{\alpha,\mathrm{tr}\alpha=p}(K_i\otimes E_i(0))\theta_\alpha - \sum_{\alpha,\mathrm{tr}\alpha=p}\theta_\alpha(K_{-i}\otimes E_i(0)),$

(2) $(1\otimes F_i(0) + F_i(0)\otimes K_{-i})\theta_{\le p} - \theta_{\le p}(1\otimes F_i(0)) + F_i(0)\otimes K_i),$
$= \sum_{\alpha,\mathrm{tr}\alpha=p}(F_i(0)\otimes K_{-i})\theta_\alpha - \sum_{\alpha,\mathrm{tr}\alpha=p}\theta_\alpha(F_i(0)\otimes K_i),$

(3) $(E_i(m)\otimes 1 + K_{m\delta}\otimes E_i(m))\theta_{\le p} - \theta_{\le p}(E_i(m)\otimes 1 + K_{-m\delta}\otimes E_i(m))$
$= \sum_{\alpha,\mathrm{tr}\alpha=p}(K_{m\delta}\otimes E_i(m))\theta_\alpha - \sum_{\alpha,\mathrm{tr}\alpha=p}\theta_\alpha(K_{-m\delta}\otimes E_i(m)),$

(4) $(1\otimes F_i(m) + F_i(m)\otimes K_{-m\delta})\theta_{\le p} - \theta_{\le p}(1\otimes F_i(m) + F_i(m)\otimes K_{m\delta})$
$= \sum_{\alpha,\mathrm{tr}\alpha=p}(F_i(m)\otimes K_{-m\delta})\theta_\alpha - \sum_{\alpha,\mathrm{tr}\alpha=p}\theta_\alpha(F_i(m)\otimes K_{m\delta}).$

3.5 Let Φ be the root system of $\Gamma = (I, (\ ,\ ))$, I be the set of simple roots, X the weight lattice of Φ and X_+ the set of dominant weights. We consider weight $\mathcal{D}$-modules M with $M = \oplus_{\lambda\in X}M_\lambda$ (as R-vector space) such that, for any $\alpha\in\mathbb{Z}[I]$ and $x \in M_\lambda$, we have $K_\alpha x = v^{(\alpha,\lambda)}x$. Then we define the category $\mathcal{O}$ consisting of weight modules M satisfying: (1) every weight space M_λ is finite dimensional, (2) for any $x\in M$ there exists $n_0 \ge 0$ such that $\mathfrak{h}^+_\alpha x = 0$ if $\mathrm{tr}\alpha \ge n_0$.

For any $\lambda \in X$, we can define the Verma module $M(\lambda)\in\mathcal{O}$; consider the left ideal $\mathcal{J} = \sum_{\lambda\in\mathcal{P}_1}\mathcal{D}u^+_\lambda + \sum_{\alpha\in\mathbb{Z}[I]}\mathcal{D}(K_\alpha - v^{(\alpha,\lambda)}1)$, it is clear that $\sum_{\lambda\in\mathcal{P}_1}\mathcal{D}u^+_\lambda = \sum_{m\ge 0, 1\le i\le t_m}\mathcal{D}E_i(m)$; then $M(\lambda) = \mathcal{D}/\mathcal{J}$ is naturally a $\mathcal{D}$-module. Using triangular decomposition we see that the R-linear map $\mathfrak{h}^- \to \mathcal{D}/\mathcal{J}$ with $f \to f + \mathcal{J}$ is naturally an isomorphism. Via this isomorphism, $\mathfrak{h}^-$ becomes the Verma module $M(\lambda)$. The module action for any $y\in\mathfrak{h}^-$, $\alpha\in\mathbb{Z}[I]$ is given as $K_\alpha y = v^{(\alpha,\lambda-\beta)}y$, $F_i(m)\cdot y = F_i(m)y$, $E_i(m)\cdot 1 = 0$.

A $\mathcal{D}$-module $M\in\mathcal{O}$ is said to be *integrable* if for any $x\in M$ and any $i\in I$, there exists $n_0\ge 1$ such that $E_i(0)^{n_0}x = F_i(0)^{n_0}x = 0$.

Lemma 3.5.1. *Let $\lambda\in X_+$. If $(\lambda, e_i) > 0$ for some $i\in I$, let*

$$\mathcal{L} = \sum_{i\in I}\mathfrak{h}^- F_i(0)^{\frac{2(i,\lambda)}{(i,i)}+1}.$$

If $(\lambda, e_i) = 0$ for all $i \in I$, let

$$\mathcal{L} = \sum_{i\in I}\mathfrak{h}^- F_i(0) + \sum_{m\ge 1, 1\le i\le t_m}\mathfrak{h}^- F_i(m).$$

Then $L(\lambda) = M(\lambda)/\mathcal{L}$ is integrable with highest weight λ. $\square$

3.6 Applying $\mu(\sigma \otimes 1)$ to the identities in (3.4.2), where $\mu : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}$ is the multiplication, for example, we obtain the following identity for any $p \geq 0$

$$\begin{aligned} &\sum_{\mathrm{tr}\alpha \leq p} \sum_j (\sigma(E_i(0) y_\alpha^j) x_\alpha^j + \sigma(K_{m\delta} y_\alpha^j) E_i(m)) \\ &\qquad - \sigma(y_\alpha^j E_i(m)) x_\alpha^j - \sigma(y_\alpha^j K_{-m\delta}) x_\alpha^j E_i(m) \\ = &\sum_{\mathrm{tr}\alpha = p} \sum_j (\sigma(K_{m\delta} y_\alpha^j) E_i(m) x_\alpha^j - \sigma(y_\alpha^j K_{-m\delta}) x_\alpha^j E_i(m)). \end{aligned}$$

We set

$$\Omega = \sum_{\alpha \in \mathbb{N}[I]} \sum_j \sigma(y_\alpha^j) x_\alpha^j \quad \text{and} \quad \Omega_{\leq p} = \sum_{\mathrm{tr}\alpha \leq p} \sum_j \sigma(y_\alpha^j) x_\alpha^j.$$

Since $\sigma(E_i(m)) = -K_{-m\delta} E_i(m)$ and $\sigma(K_{m\delta}) = K_{-m\delta}$, we have

$$\begin{aligned} \text{(3.6.1)} \qquad &K_{-m\delta} E_i(m) \Omega_{\leq p} - K_{m\delta} \Omega_{\leq p} E_i(m) \\ = &\sum_{\mathrm{tr}\alpha = p} \sum_j (\sigma(K_{m\delta} y_\alpha^j) E_i(m) x_\alpha^j - \sigma(y_\alpha^j K_{-m\delta}) x_\alpha^j E_i(m)). \end{aligned}$$

For $M \in \mathcal{O}$ and any $x \in M$, we have that $\Omega(x) = \Omega_{\leq p}(x)$ for large enough p. By (3.6.1), we obtain

$$K_{-m\delta} E_i(m) \Omega = K_{m\delta} \Omega E_i(m), \quad \text{for } m \geq 1,\ 1 \leq i \leq t_m,$$

and similarly,

$$\begin{aligned} &K_{-i} E_i(0) \Omega = K_i \Omega E_i(0), \\ &\Omega F_i(0) = F_i(0) K_i \Omega K_i, \\ &\Omega F_i(m) = F_i(m) K_{m\delta} \Omega K_{m\delta}, \\ &\Omega K_\alpha = K_\alpha \Omega. \end{aligned}$$

as operators on M.

3.7 Let t be an indeterminate. By giving degrees to the generators of $\mathcal{D}$ as follows

$$\begin{aligned} &\deg E_i(0) = i, \quad \deg F_i(0) = -i, \quad \text{for all } i \in I, \\ &\deg E_i(m) = m\delta t, \quad \deg F_i(m) = -m\delta t, \quad \text{for all } m \geq 1, \\ &\deg K_\alpha = 0, \quad \text{for all } \alpha \in \mathbb{Z}[I], \end{aligned}$$

it is natural to make $\mathcal{D}$ into a $\mathbb{Z}[I] \times \mathbb{Z}[I]$-graded algebra:

$$\text{(3.7.1)} \qquad \mathcal{D} = \oplus_{\alpha, \beta \in \mathbb{Z}[I]} \mathcal{D}_{\alpha + \beta t}.$$

If $\mathcal{D} = \oplus_{\alpha\in\mathbb{Z}[I]}\mathcal{D}_\lambda$ is the weight space decomposition, then $\mathcal{D}_\lambda = \oplus_{\alpha+\beta=\lambda}\mathcal{D}_{\alpha+\beta t}$. We call (3.8.1) *refined weight space decomposition.* Also the subalgebra $\mathfrak{h}^\pm$ inherit the refined weight space decomposition from $\mathcal{D}$.

Let X be the weight system. Set $X[t] = \{\,\alpha + \beta t \,|\, \alpha, \beta \in X\,\}$ and the scale product $(\ ,\ )$ defined on $X[t]$ is given by

$$(\alpha + \beta t, \alpha' + \beta' t) = (\alpha, \alpha') + (\beta, \beta'), \quad \text{for } \alpha, \alpha', \beta, \beta' \in X.$$

It is similar to consider $X_+[t]$ and $\mathbb{Z}[I][t]$.

Let $\lambda \in X$. Set $\hat{\lambda} \in X[t]$ to be a highest weight such that $(\hat{\lambda}, \alpha) = (\lambda, \alpha)$ and $(\hat{\lambda}, \alpha t) = (\lambda, \alpha)$ for any $\alpha \in \mathbb{Z}[I]$. Naturally we have a Verma module $M^t(\hat{\lambda})$ with highest weight $\hat{\lambda}$ with respect to the refined weight space decomposition. Then, if x is a highest weight vector, then $K_\alpha x = v^{(\hat{\lambda},\alpha)}x$. It is also true that, if $y \in M^t(\hat{\lambda})$ is a weight vector with weight $\tilde{\mu}$ then $K_\alpha y = v^{(\tilde{\mu},\alpha)}y$ since $\tilde{\mu} = \hat{\lambda} - (\beta + m\delta t), y \in \mathfrak{h}^-_{\beta+m\delta t}(x)$, and $(\alpha, m\delta) = 0$.

There exists a $\rho \in X_+$ such that $(\rho, e_i) = (e_i, e_i)/2$ for all $i \in I$. We can define the operator $\tilde{\Omega}$ on $M^t(\tilde{\mu})$ (hence on any $M \in \mathcal{O}$ which has a refined weight space decomposition): for $x \in M^t(\tilde{\mu})_{\tilde{\mu}-(\alpha+m\delta t)}$,

$$\tilde{\Omega}(x) = v^{(\tilde{\mu}-(\alpha+m\delta t)+\rho,\tilde{\mu}-(\alpha+m\delta t)+\rho)}\Omega(x).$$

Note that $(\beta t, \rho) = 0$, and $(\tilde{\mu}-e_i+\rho, \tilde{\mu}-e_i+\rho)-(\tilde{\mu}+\rho, \tilde{\mu}+\rho) = 2(e_i, \tilde{\mu}+\rho)+(e_i, e_i) = 2(e_i, \tilde{\mu}+e_i)$, $(\tilde{\mu}-\delta t+\rho, \tilde{\mu}-\delta t+\rho)-(\tilde{\mu}+\rho, \tilde{\mu}+\rho) = 2(\delta t, \tilde{\mu})$. In fact, it is not difficult to see that there exists a function $G : X[t] \to \mathbb{Z}$ such that $G(\tilde{\mu}-e_i)-G(\tilde{\mu}) = 2(e_i, \mu+e_i)$ and $G(\tilde{\mu} - \delta t) - G(\tilde{\mu}) = 2(\delta t, \tilde{\mu})$.

Lemma 3.7.1. *Let $\lambda \in X$, $\hat{\lambda}$ as above. If M^t is a highest weight $\mathcal{D}$-module with highest weight $\tilde{\mu} = \hat{\lambda} - (\alpha + m\delta t)$ with $\alpha \in \mathbb{N}[I]$, $m \in \mathbb{N}$, then $\tilde{\Omega}$ is central to the action of all elements of $\mathcal{D}$ on M^t and $\tilde{\Omega}$ acts on M^t as the scale multiplication $v^{(\tilde{\mu}+\rho,\tilde{\mu}+\rho)}$.*

Proof. It suffices to show the result for $M = M^t(\tilde{\mu})$ for any $x \in M^t(\hat{\mu})_\mu$, where $\mu = \tilde{\mu} - (\alpha + \beta t)$, $\alpha, \beta \in \mathbb{N}[I]$, then

$$\tilde{\Omega}E_i(0)(x) = v^{(\mu+e_i+\rho,\mu+e_i+\rho)}\Omega E_i(0)(x),$$

$$\begin{aligned} E_i(0)\tilde{\Omega}(x) &= v^{(\mu+\rho,\mu+\rho)}E_i(0)\Omega(x) = v^{(\mu+\rho,\mu+\rho)}K_{2i}\Omega E_i(0)(x) \\ &= v^{(\mu+\rho,\mu+\rho)}v^{2(e_i,e_i+\mu)}\Omega E_i(0)(x). \end{aligned}$$

Since $(\mu + e_i + \rho, \mu + e_i + \rho) = (\mu + \rho, \mu + \rho) + 2(e_i, \mu + \rho) + (e_i, e_i) = (\mu + \rho, \mu + \rho) + 2(e_i, e_i + \mu)$, then $\tilde{\Omega}E_i(0)(x) = E_i(0)\tilde{\Omega}(x)$. For $m \geq 1$,

$$\begin{aligned} &\tilde{\Omega}E_i(m)(x) = v^{(\mu+m\delta t+\rho,\mu+m\delta t+\rho)}\Omega E_i(m)(x), \\ &E_i(m)\tilde{\Omega}(x) = v^{(\mu+\rho,\mu+\rho)}E_i(m)\Omega(x) = v^{(\mu+\rho,\mu+\rho)}K_{2m\delta}\Omega E_i(m)(x) \\ &= v^{(\mu+\rho,\mu+\rho)}v^{(2m\delta,m\delta)}v^{(2m\delta,\mu)}\Omega E_i(m)(x) = v^{(\mu+\rho,\mu+\rho)}v^{(2m\delta,\mu)}\Omega E_i(m)(x). \end{aligned}$$

Since

$$\begin{aligned}&(\mu+m\delta t+\rho,\mu+m\delta t+\rho)=(\mu+\rho,\mu+\rho)+2(m\delta t,\mu+\rho)+(m\delta t,m\delta t)\\&=(\mu+\rho,\mu+\rho)+2(m\delta t,\mu)\end{aligned}$$

and $\mu=\hat{\lambda}-(\alpha+\beta t)$, $\alpha,\beta\in\mathbb{N}[I]$, but $(\delta,\alpha)=(\delta t,\rho)=0$, hence

$$\begin{aligned}&(\mu+m\delta t+\rho,\mu+m\delta t+\rho)=(\mu+\rho,\mu+\rho)+2(m\delta t,\hat{\lambda})\\&=(\mu+\rho,\mu+\rho)+2(m\delta,\lambda)\\&=(\mu+\rho,\mu+\rho)+2(m\delta,\mu)\end{aligned}$$

Therefore $\tilde{\Omega}E_i(m)(x)=E_i(m)\tilde{\Omega}(x)$. It is similar to prove $\tilde{\Omega}F_i(0)(x)=F_i(0)\tilde{\Omega}(x)$, $\tilde{\Omega}F_i(m)(x)=F_i(m)\tilde{\Omega}(x)$ and $K_\alpha\tilde{\Omega}=\tilde{\Omega}K_\alpha$. So the first assertion is proved. The second follows directly from the first. □

The operator $\tilde{\Omega}$ defined above is called the *quantized Casimir operator of* $\mathcal{D}$.

4. Weyl-Kac character formula

4.1 Let now M be a $\mathcal{D}$-module in the category $\mathcal{O}$, $M=\oplus_{\lambda\in X}M_\lambda$ be its weight space decomposition. The formal character of M is $\operatorname{ch}M=\sum_{\lambda\in X}\dim M_\lambda e(\lambda)$. For $\alpha\in\mathbb{N}[I]$, we let $M_\Gamma(\alpha)$ be the number of isomorphism classes of Λ-modules with dimension vector α; for $\alpha\in\Phi^+$,we let $I_\Gamma(\alpha)$ be the number of isomorphism classes of indecomposable Λ-modules with dimension vector α.

We first consider a Verma module $M(\lambda)$. If $M(\lambda)_\beta\neq 0$, then $\lambda-\beta\in\mathbb{N}[I]$ and $\dim M(\lambda)_\beta=m(\lambda-\beta)$. So

$$\operatorname{ch}M(\lambda)=\sum_\beta m(\lambda-\beta)e(\beta)=e(\lambda)\sum_{\alpha\in\mathbb{N}[I]}m(\alpha)e(-\alpha).$$

Since all monomials on the set $\{\,u_\beta^-\mid V_\beta$ is indecomposable Λ-module$\,\}$ in a fixed order provide a universal PBW-basis of $\mathfrak{h}^-$, we have

$$\operatorname{ch}M(\lambda)=e(\lambda)\prod_{\alpha\in\Phi^+}(1+e(-\alpha)+e(-2\alpha)+\cdots)^{I_\Gamma(\alpha)}.$$

Hence

$$\operatorname{ch}M(\lambda)=e(\lambda)\prod_{\alpha\in\Phi^+}(1-e(-\alpha))^{-I_\Gamma(\alpha)}.$$

4.2 Let $\lambda\in X_+$, as an integrable $\mathcal{D}$-module, it is easy to see that $\dim L(\lambda)_\beta=\dim L(\lambda)_{w(\beta)}$, for all $w\in W$ and each weight β of $L(\lambda)$. The action of the Weyl group W on the formal characters is given by

$$w\big(\sum_\lambda c_\lambda e(\lambda)\big)=\sum_\lambda c_\lambda e(w(\lambda)),\quad\text{for } w\in W.$$

Then

$$w(\mathrm{ch}\, L(\lambda)) = \mathrm{ch}\, L(\lambda), \quad \text{for} \quad w \in W, \lambda \in X_+. \tag{4.2.1}$$

Consider now the element

$$Q = \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))^{I_\Gamma(\alpha)},$$

and for $w \in W$, we set $\epsilon(w) = (-1)^{l(w)}$.

Lemma 4.2.1. *We have the W-skew invariant property*

$$w(e(\rho)Q) = \epsilon(w)w(\rho)Q, \quad \textit{for } w \in W.$$

Proof. We only need to check the property for each simple reflection r_i. Recall that the set $\Phi^+\backslash\{e_i\}$ is r_i-invariant and $I_\Gamma(\alpha)$ is independent of the orientation of Λ for each $\alpha \in \Phi^+$ and, by the BGP-reflection functor, it holds $I_\Gamma(r_i(\alpha)) = I_\Gamma(\alpha)$ for $\alpha \in \Phi^+\backslash\{e_i\}$. Hence

$$\begin{aligned}
r_i(e(\rho)Q) &= e(\rho - e_i) r_i(1 - e(-e_i)) r_i\Big(\prod_{\alpha \in \Phi^+\backslash\{e_i\}} (1 - e(-\alpha))^{I_\Gamma(\alpha)}\Big) \\
&= e(\rho)e(-e_i)(1 - e(e_i)) \prod_{\alpha \in \Phi^+\backslash\{e_i\}} (1 - e(-\alpha))^{I_\Gamma(\alpha)} \\
&= -e(\rho)Q = e(r_i)e(\rho)Q. \qquad \square
\end{aligned}$$

4.3 We have the following theorem whose proof will be given at the end of Section 4.6.

Theorem 4.3.1. *Let $L(\lambda)$ be an irreducible highest $\mathcal{D}$-module with highest weight $\lambda \in X_+$. Assume that $(\lambda, e_i) > 0$ for some $i \in I$. Then*

$$\mathrm{ch}\, L(\lambda) = \frac{\sum_{w \in W} \epsilon(w) e(w(\lambda + \rho) - \rho)}{\prod_{\alpha \in \Phi^+} (1 - e(-\alpha))^{I_\Gamma(\alpha)}}.$$

4.4 Now we consider $\mathcal{D}$-modules with a refined weight space decomposition $M^t = \oplus_{\tilde{\mu}} M^t_{\tilde{\mu}}$. The formal character is defined as

$$\mathrm{ch}\, M^t = \sum_{\tilde{\mu}} (\dim M^t_{\tilde{\mu}}) e(\tilde{\mu}),$$

and the Weyl group W acts on $\mathrm{ch}\, M^t$ in an obvious way. In particular,

$$we(\alpha + \beta t) = e(w(\alpha) + w(\beta)t), \quad \text{for} \quad \alpha, \beta \in \mathbb{Z}[I], w \in W.$$

For any $\lambda \in X_+$ with $(\lambda, e_i) > 0$ for some $i \in I$, we have

$$L^t(\hat{\lambda}) = M^t(\hat{\lambda}) / \sum_{i\in I} \mathfrak{h}^- F_i(0)^{\frac{2(\lambda,e_i)}{(e_i,e_i)}+1}, \quad (\text{note } (\hat{\lambda}, e_i) = (\lambda, e_i)),$$

and then

$$\operatorname{ch} M^t(\hat{\lambda}) = \sum_{\alpha,\beta\in\mathbb{Z}[I]} \dim M^t(\hat{\lambda})_{\hat{\lambda}-(\alpha+\beta t)} e(\hat{\lambda} - (\alpha + \beta t)),$$

$$\operatorname{ch} L^t(\hat{\lambda}) = \sum_{\alpha,\beta\in\mathbb{Z}[I]} \dim L^t(\hat{\lambda})_{\hat{\lambda}-(\alpha+\beta t)} e(\hat{\lambda} - (\alpha + \beta t)).$$

Moreover, $w(\operatorname{ch} L^t(\hat{\lambda})) = \operatorname{ch} L^t(\hat{\lambda})$ for $w \in W$ and $\lambda \in X_+$. It is easy to see that

$$\lim_{t\to 1} \operatorname{ch} M^t(\hat{\lambda}) = \operatorname{ch} M(\lambda), \quad \lim_{t\to 1} \operatorname{ch} L^t(\hat{\lambda}) = \operatorname{ch} L(\lambda).$$

Lemma 4.4.1. *Let $\lambda \in X_+$, $\hat{\lambda}$ as above, M^t be a highest weight $\mathcal{D}$-module with highest weight $\hat{\lambda}$ with respect to the refined weight space decomposition. Then*

$$\operatorname{ch} M^t = \sum_{\substack{\tilde{\beta}\le\hat{\lambda} \\ |\tilde{\beta}+\rho|^2=|\hat{\lambda}+\rho|^2}} c_{\tilde{\beta}} \operatorname{ch} M^t(\tilde{\beta}), \quad \textit{where } c_{\tilde{\beta}} \in \mathbb{Z}, c_{\hat{\lambda}} = 1, \tag{4.4.1}$$

and $\tilde{\beta} \le \hat{\lambda}$ means $\hat{\lambda} - \tilde{\beta} \in \mathbb{N}[I] + \mathbb{N}\delta t$.

Proof. All $\mathcal{D}$-modules in $\mathcal{O}$ with a refined weight space decomposition have good filtrations with irreducible highest weight modules $L^t(\tilde{\beta})$(see Kac [4]). It suffices to prove the formula for $M^t = L^t(\hat{\lambda})$. For a Verma module $M^t(\tilde{\beta})$ with $\tilde{\beta} \le \hat{\lambda}$, then

$$\operatorname{ch} M^t(\tilde{\beta}) = \sum_{\tilde{\mu}\le\tilde{\beta}} [M^t(\tilde{\beta}) : L^t(\tilde{\mu})] \operatorname{ch} L^t(\hat{\mu}),$$

and the action of $\tilde{\Omega}$ on $\mathrm{L}^t(\tilde{\mu})$ is the scale multiplication $v^{(\tilde{\mu}+\rho,\tilde{\mu}+\rho)}$, by Lemma 3.7.1. Moreover, if $[M^t(\tilde{\beta}) : L^t(\tilde{\mu})] \ne 0$ then $v^{(\tilde{\mu}+\rho,\tilde{\mu}+\rho)} = v^{(\tilde{\beta}+\rho,\tilde{\beta}+\rho)}$. This implies that $(\tilde{\mu} + \rho, \tilde{\mu} + \rho) = (\tilde{\beta} + \rho, \tilde{\beta} + \rho)$ since v is not a root of unity. Set $B(\hat{\lambda}) = \{ \tilde{\beta} \mid \tilde{\beta} \le \hat{\lambda}, (\tilde{\beta}+\rho, \tilde{\beta}+\rho) = (\hat{\lambda}+\rho, \hat{\lambda}+\rho) \}$, and order the elements of this set $\tilde{\beta}_1, \tilde{\beta}_2, \cdots$ such that $\tilde{\beta}_i \ge \tilde{\beta}_j$ implies that $i \le j$. By the filtration of $M^t(\tilde{\beta}_i)$, we have the following system of linear equations:

$$\operatorname{ch} M^t(\tilde{\beta}_i) = \sum_{\tilde{\beta}_j\in B(\hat{\lambda})} c_{ij} \operatorname{ch} L^t(\tilde{\beta}_j).$$

The coefficient matrix (c_{ij}) of this system is triangular with ones on the diagonal. The equation (4.4.1) is obtained by solving this system. □

4.5 It is clear to see that, for $\tilde{\beta} \le \hat{\lambda}$, there holds

$$\operatorname{ch} M^t(\tilde{\beta}) = e(\tilde{\beta}) \prod_{\alpha\in\Phi^+} (1 - e(-\alpha))^{-\operatorname{mult}(\alpha)} \prod_{m=1}^{\infty} (1 - e(-m\delta t))^{-t_m},$$

where $\operatorname{mult}(\alpha)$ is the multiplicity of the root α in the sense of Kac-Moody algebra (see Kac [4]).

Let $Q^t = \prod_{\alpha\in\Phi^+} (1 - e(-\alpha))^{\operatorname{mult}(\alpha)} \prod_{m=1}^{\infty} (1 - e(-m\delta t))^{t_m}$. Then $e(\rho)Q^t$ is skew W-invariant. Since $\operatorname{ch} M(\lambda) = \lim_{t\to 1} \operatorname{ch} M^t(\hat{\lambda})$, we have the following consequence.

Corollary 4.5.1. *For $m \geq 1$, there holds $t_m = I_\Gamma(m\delta) - \mathrm{mult}(m\delta)$.* □

4.6 Now we come to prove the following formula:

$$\mathrm{ch}\, L^t(\hat{\lambda}) = \mathrm{ch}\, M^t(\hat{\lambda}) \sum_{w \in W} \epsilon(w) e(w(\hat{\lambda} + \rho) - (\hat{\lambda} + \rho)), \tag{4.6.1}$$

for $\lambda \in X_+$ with $(\lambda, e_i) > 0$ for some $i \in I$

Proof. According to (4.4.1) and (4.5), we see that

$$e(\rho) Q^t \mathrm{ch}\, L^t(\hat{\lambda}) = \sum_{\substack{\tilde{\beta} \leq \hat{\lambda} \\ |\tilde{\beta}+\rho|^2 = |\tilde{\lambda}+\rho|^2}} c_{\tilde{\beta}} e(\tilde{\beta} + \rho),$$

where $c_{\tilde{\beta}} \in \mathbb{Z}$ and $c_{\hat{\lambda}} = 1$. The left-hand side is skew W-invariant, hence the coefficients in the right hand side have the property:

$$c_{\tilde{\beta}} = \epsilon(w) c_{\tilde{\mu}}, \quad \text{if } w(\tilde{\beta} + \rho) = \tilde{\mu} + \rho \text{ for some } w \in W.$$

Let $\tilde{\beta}$ be such that $c_{\tilde{\beta}} \neq 0$, then $c_{w(\tilde{\beta}+\rho)-\rho} \neq 0$ for any $w \in W$. Then $w(\tilde{\beta}+\rho) \leq \hat{\lambda} + \rho$. Let $\tilde{\mu} \in \{\, w(\tilde{\beta}+\rho) - \rho \mid w \in W \,\}$ be such that $\hat{\lambda} - \tilde{\mu}$ is minimal. Then $(\tilde{\mu}+\rho, e_i) \geq 0$ for any $i \in I$. Otherwise there exists $i \in I$ such that $\hat{\lambda} \geq r_i(\tilde{\mu} + \rho) - \rho > \tilde{\mu}$ and $r_i(\tilde{\mu} + \rho) - \rho \in \{\, w(\tilde{\beta} + \rho) - \rho \mid w \in W \,\}$. Let $\tilde{\mu} = \lambda - (\alpha + m\delta t)$, $\alpha \in \mathbb{N}[I]$, $m \geq 0$. Then

$$(\hat{\lambda} + \rho, \hat{\lambda} + \rho) = (\tilde{\mu} + \rho, \tilde{\mu} + \rho) + (\tilde{\mu} + \rho, \alpha + m\delta t) + (\hat{\lambda} + \rho, \alpha + m\delta t).$$

We have $(\tilde{\mu}+\rho, \alpha) \geq 0$, $(\hat{\lambda}+\rho, \alpha) \geq 0$, and $(\rho, m\delta t) = 0$. There exists $i \in I$ such that $(\lambda, e_i) > 0$, this guarantees $(\hat{\lambda} + \rho, \alpha) > 0$ if $\alpha > 0$, and $(\tilde{\mu} + \rho, m\delta t) = (\tilde{\mu}, m\delta t) = (\hat{\lambda}, m\delta t) = (\hat{\lambda} + \rho, m\delta t) > 0$ if $m > 0$. Therefore $(\hat{\lambda} + \rho, \hat{\lambda} + \rho) = (\tilde{\mu} + \rho, \tilde{\mu} + \rho)$ implies that $\hat{\lambda} = \tilde{\mu}$. Thus $c_{\tilde{\beta}} \neq 0$ implies that $\tilde{\beta} + \rho = w(\hat{\lambda} + \rho)$ for some $w \in W$ and in this situation $c_{\tilde{\beta}} = \epsilon(w)$. It is easy to see that $w(\hat{\lambda} + \rho) = \hat{\lambda} + \rho$ implies that $w(\lambda + \rho) = \lambda + \rho$, and so $w = 1$. Finally we obtain

$$e(\rho) Q^t \mathrm{ch}\, L^t(\hat{\lambda}) = \sum_{w \in W} \epsilon(w) e(w(\hat{\lambda} + \rho)). \qquad \square$$

Theorem 4.3 can be obtained by taking $t \to 1$ in (4.6.1).

4.7 Of course, if $(\lambda, e_i) = 0$ for all $i \in I$, we have the denominator identity

$$1 = \mathrm{ch}\, L(0) = \frac{\sum_{w \in W} \epsilon(w) e(w(\rho) - \rho)}{\prod_{\alpha \in \Phi^+} (1 - e(-\alpha))^{\mathrm{mult}(\alpha)}}.$$

4.8 We now give the proof of Theorem 2.5.1.

Proof of Theorem 2.5.1. Let $\mathcal{D}'$ be the algebra generated by the generators $E_i(0)$, $F_i(0)$, K_i, K_i^{-1} $(i \in I)$ and $E_i(m)$, $F_i(m)$ $(m \geq 1, 1 \leq i \leq t_m)$ with the generating relations (2.4.1), (2.4.2), (2.5.1), (2.5.2), (2.5.3), (2.5.4) and (2.5.5). There exists a canonical surjection $\pi : \mathcal{D}' \to \mathcal{D}$ sending the generators to the corresponding generators. If we admit the $\mathbb{Z}[I]$-gradation of $\mathcal{D}'$ by setting $\deg E_i(0) = -\deg F_i(0) = e_i$ and $\deg E_i(m) = -\deg F_i(m) = m\delta$ for $m \geq 1$, then $\mathcal{D}'$ still has a weight space decomposition and π preserves the weight spaces. We consider any Verma module $M(\lambda)$ over $\mathcal{D}$ with highest weight λ, then $M(\lambda)$ becomes a $\mathcal{D}'$-module by defining the action $a * x = \pi(a)x$ for any $a \in \mathcal{D}'$ and $x \in M(\lambda)$. Obviously, $M(\lambda)$ is also a highest weight module of $\mathcal{D}'$ with highest weight λ. We have the canonical $\mathcal{D}'$-module surjection $\pi' : M'(\lambda) \to M(\lambda)$, where $M'(\lambda)$ is the Verma module of $\mathcal{D}'$ with the highest weight λ. We have known that

$$\operatorname{ch} M(\lambda) = e(\lambda) \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))^{-I_\Gamma(\alpha)}$$

in (4.1) and

$$t_m = I_\Gamma(m\delta) - \operatorname{mult}(m\delta)$$

in Corollary 4.5.1. It is well-known that

$$\operatorname{ch} M'(\lambda) = e(\lambda) \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))^{-\operatorname{mult}(\alpha)} \prod_{m=1}^{\infty} (1 - e(-m\delta))^{-t_m}$$

So, π' is a bijection. This implies that π induces a bijection on the negative parts of $\mathcal{D}'$ and $\mathcal{D}$. Dually, it is also right on the positive parts of $\mathcal{D}'$ and $\mathcal{D}$. By the standard triangular decompositions of $\mathcal{D}'$ and $\mathcal{D}$, we see that $\pi : \mathcal{D}' \to \mathcal{D}$ is an isomorphism. The proof is finished. $\square$

5. Numbers of Heisenberg generators

In this section we assume that the Cartan matrix Γ of Λ is symmetric, in other words, Λ is one of the path algebras of $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$. We aim to determine the number t_m, which is the number of generators of $\mathcal{D}$ with degree $m\delta$.

5.1 Let n_0 be the rank of Γ. It is well known that $\operatorname{mult}(m\delta) = n_0$ (see Kac [4]) for any $m \geq 1$. For any $\alpha \in \Phi^+$, we define $A_\Gamma(\alpha)$ to be the number of isoclasses absolutely indecomposable Λ-modules with dimension vector α. It is known that

$$A_\Gamma(\alpha) = \begin{cases} 1, & \text{if } \alpha \in \Phi^+_{re}; \\ q + n_0, & \text{if } \alpha = m\delta \text{ for some } m \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

A consequence of the Krull-Schmidt Theorem is the following formal identity:

$$\sum_{\alpha \in \mathbb{N}[I]} M_\Gamma(\alpha) e(-\alpha) = \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))^{-I_\Gamma(\alpha)}. \tag{5.1.1}$$

It follows from Theorem 4.1.7 of [3] that
(5.1.2)

$$\sum_{\alpha\in\mathbb{N}[I]} M_\Gamma(\alpha)e(-\alpha) = \prod_{\alpha\in\Phi^+_{re}}(1-e(-\alpha))^{-1}\prod_{r=1}^{\infty}(1-e(-r\delta))^{-n_0}(1-qe(-r\delta))^{-1}.$$

For $d\geq 1$, we let $\hat{\phi}_d(q)$ be the number of monic irreducible polynomials over k of degree d. It is known that

$$\prod_{d=1}^{\infty}(1-X^d)^{-\hat{\phi}_d(q)} = (1-qX)^{-1}. \tag{5.1.3}$$

Thus (5.1.1), (5.1.2) and (5.1.3) imply the following

$$\begin{aligned}\prod_{\alpha\in\Phi^+}(1-e(-\alpha))^{-I_\Gamma(\alpha)} = &\prod_{\alpha\in\Phi^+_{re}}(1-e(-\alpha))^{-1}\prod_{r=1}^{\infty}(1-e(-r\delta))^{-n_0}\\ &\times\prod_{r=1}^{\infty}(1-e(-dr\delta))^{-\hat{\phi}_d(q)}.\end{aligned}$$

Therefore, we have the following

$$I_\Gamma(\alpha)=\begin{cases}1, & \text{if } \alpha\in\Phi^+_{re};\\ \sum_{d|m}\hat{\phi}_d(q)+n_0, & \text{if } \alpha=m\delta \text{ for some } m\geq 1;\\ 0, & \text{otherwise.}\end{cases}$$

Recall from Corollary 4.5.1 that $t_m = I_\Gamma(m\delta) - \mathrm{mult}(m\delta)$ and notice that $\mathrm{mult}(m\delta) = n_0$. Thus we have the following.

Proposition 5.1.1. *For any $m\geq 1$, there holds $t_m = \sum_{d|m}\hat{\phi}_d(q)$.* □

5.2 Another form of t_m can be deduced as follows. First we let φ be the Euler function, i.e., $\varphi(n)$ is the number of numbers up to n which are relatively prime to n, and let $G(n,q)=\sum_{d|n}\frac{1}{n}\varphi(n/d)q^d$. Proposition 4.2.2 of [3] states that

$$\prod_{n=1}^{\infty}(1-qX^n) = \prod_{n=1}^{\infty}(1-X^n)^{G(n,q)}.$$

It follows from (5.1.2) and (5.2.1) that $I_\Gamma(m\delta) = G(m,q)+n_0$ and hence $t_m = G(m,q)$. Notice that $G(m,q)$ is divisible by q and has no negative coefficients.

Remark. The numbers t_m's do not depend on the type of Γ.

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