Estimates of Semigroups and Eigenvalues Using Functional Inequalities

Feng-Yu Wang*

Department of Mathematics, Beijing Normal University,

Beijing 100875, P. R. China (E-mail: wangfy@bnu.edu.cn)

Present address: Universität Bielefeld, Fakultät für Mathematik, Postfach 100131,

D-33501 Bielefeld, Germany (E-mail: fwang@mathematik.uni-bielefeld.de)

November 1999

Abstract

Boundedness properties of semigroups are studied by using general Poincaré-Sobolev type inequalities, from which Gross' theorem on hyperboundedness and log-Sobolev inequality is extended. Some results hold also for nonsymmetric semigroups. For instance, a super-Poincaré inequality always imply an estimate of the corresponding semigroup. In particular, the log-Sobolev inequality implies the hyperboundedness of the corresponding (possibly non-symmetric) semigroup. Conversely, functional inequalities are derived from boundedness properties of semigroups which are either symmetric or satisfy a "gradient estimate". Especially, the log-Sobolev inequality follows from hyperboundedness for such semigroups.

According to our previous work, under some mild assumptions, the super-Poincaré inequality implies empty essential spectrum of the generator. In this paper we present some lower bound estimates of high order eigenvalues using functional inequalities. The resulting estimates are sharp in the sense of order. Some examples of diffusion processes and birth-death processes are presented.

Finally, functional inequalities are obtained by using concentration of invariant measures for diffusions on a noncompact Riemannian manifold. This provides a converse of Herbst type results.

AMS 1991 Subject classification: 35P15, 47D07.
Keywords and phrases: Super-Poincaré inequality, F-Sobolev inequality, semigroup, eigenvalue, concentration.

*Supported in part by Alexander von Humboldt Foundation, NNSFC(19631060) and Fok Ying-Tung Educational Foundation.

1 Introduction

Let $(E, \mathcal{F}, \mu)$ be a measure space, and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ a nonnegative coercive closed bilinear form on $L^2(\mu)$. Let P_t denote the associated strongly continuous contraction semigroup on $L^2(\mu)$ with generator $(L, \mathcal{D}(L))$, see e.g. [24] for general theory in this literature. We always assume that P_t is contractive in $L^p(\mu)$ for any $p \geq 1$.

The hyperboundedness of P_t in the sense that $\|P_t\|_{2\to 4} < \infty$ for some $t > 0$, was first connected by Gross [19] with the log-Sobolev inequality

$$\mu(f^2 \log f^2) \leq C_1 \mathcal{E}(f,f) + C_2, \quad f \in \mathcal{D}(\mathcal{E}), \mu(f^2) = 1. \tag{1.1}$$

In particular, when $\mathcal{E}$ is symmetric, (1.1) is equivalent to the hyperboundedness of P_t, see [20], [3], [14],[15] and [16]. As has been shown by Bakry (see Proposition 3.1 and Theorem 3.2 in [3]), (1.1) implies the hyperboundedness provided P_t is a (not necessarily symmetric) diffusion semigroup. Proofs of above mentioned results are based on the following inequality:

$$\mathcal{E}(g^{p/2}, g^{p/2}) \leq -\frac{p^2}{4(p-1)} \mu(g^{p-1} L g), \quad g \in \mathcal{D}(L) \cap L^1(\mu) \cap L^\infty(\mu), g \geq 0. \tag{1.2}$$

This inequality was proved by Stroock [27] and Varopoulos [29] for symmetric case (see e.g. also [14], [16] and [20]), and by Bakry [3] for diffusion case. In this paper, we deduce estimates of P_t from Poincaré-Sobolev type inequalities for general nonsymmetric case without using (1.2). In particular, we prove the hyperboundedness of P_t using the log-Sobolev inequality for nonsymmetric $\mathcal{E}$.

We shall work with the following F-Sobolev inequality:

$$\mu(f^2 F(f^2)) \leq C_1 \mathcal{E}(f,f) + C_2, \quad f \in \mathcal{D}(\mathcal{E}), \mu(f^2) = 1, \tag{1.3}$$

where F is an increasing function on $(0, \infty)$ with $\sup_{r \in (0,1]} |rF(r)| < \infty$ and $F(r) \uparrow \infty$ as $r \uparrow \infty$. Without loss of generality, we may assume that F is increasing by using $\inf_{[r,\infty)} F$ in place of $F(r)$. A typical example is that $F(r) = [\log(1+r)]^\delta$ for $\delta > 0$. As a generalization of Gross' theorem on log-Sobolev inequality, the corresponding property of P_t is obtained in section 2 for the F-Sobolev inequality.

It has been shown in [18] that the F-Sobolev inequality implies the L^2-uniform integrability of P_t, which was also presented earlier by Wu [35] and [36] for the case where $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is symmetric and E is a Polish space. If P_t is symmetric, [18] also proved the inverse result. To obtain concrete estimates of the uniform integrability, it is convenient to use the following super-Poincaré inequality instead of the F-Sobolev inequality:

$$\mu(f^2) \leq r\mathcal{E}(f,f) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}), \tag{1.4}$$

where β is a positive and decreasing function on $(0,\infty)$. The reason is that (1.4) is equivalent to the following semigroup inequality by a standard argument (see Wang [33]):

$$\mu((P_tf)^2) \le \mathrm{e}^{-2\lambda t}\mu(f^2) + \beta(1/\lambda)(1-\mathrm{e}^{-2\lambda t})\mu(|f|)^2, \quad \lambda, t>0, f\in L^2(\mu). \tag{1.5}$$

Actually, we obtain (1.4) by taking derivative over t in (1.5) at $t=0$, and (1.4) implies (1.5) since $\frac{\mathrm{d}}{\mathrm{d}t}\mu((P_tf)^2) = -2\mathcal{E}(P_tf, P_tf)$.

It is easy for one to exchange inequalities (1.3) and (1.4), since they are equivalent when $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form according to [18] and [33]. Moreover, F in (1.3) and β in (1.4) can be estimated by each other. In particular, for $\delta>0$, (1.3) with $F(r)=[\log(1+r)]^\delta$ is equivalent to (1.4) with $\beta(r)=\exp[c(1+r^{-1/\delta})]$ for some $c>0$, while (1.3) with $F(r)=r^\delta$ is equivalent to (1.4) with $\beta(r)=c(1+r^{-1/\delta})$ for some $c>0$. In the later case, (1.3) and (1.4) are also equivalent to the classical Sobolev inequality with dimension $p=2/\delta$ whenever $\delta<1$:

$$\|f\|^2_{2p/(p-2)} \le C_1\mathcal{E}(f,f)+C_2, \quad f\in\mathcal{D}(\mathcal{E}), \mu(f^2)=1.$$

For general $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, the F-Sobolev inequality always implies (1.4) with the same estimate of β as in the Dirichlet form case. See [18] and [33] for general estimates of β and F.

To conclude this part of the paper, we present below a simple result which is a consequence of the main results in section 2.

Proposition 1.1. *Let $\delta\in(0,1]$. If (1.3) holds with $F(r)=[\log(1+r)]^\delta$, or (1.4) holds with $\beta(r)=\exp[c(1+r^{-1/\delta})]$ for some $c>0$, then for any $t>0$ there exists $c_t>0$ such that*

$$\sup_{\mu(f^2)=1}\int (P_tf)^2\exp\Big\{c_t\big[\log(1+(P_tf)^2)\big]^\delta\Big\}\mathrm{d}\mu<\infty. \tag{1.6}$$

Consequently, the log-Sobolev inequality implies the hyperboundedness of P_t.

On the other hand, if $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is symmetric and (1.6) holds for some $t>0$ and $c_t>0$, then (1.4) holds with $\beta(r)=\exp[c(1+r^{-1/\delta})]$ for some $c>0$, which is equivalent to (1.3) for $F(r)=[\log(1+r)]^\delta$ whenever $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form.

In general, to deduce a functional inequality from a semigroup estimate, we have to assume that P_t is symmetric. This assumption is removed in section 3 whenever P_t satisfies a "gradient estimate". In section 4 we study two stronger properties called supercontractivity and ultracontractivity ([15]). See Theorem 4.2 for the case $\delta>1$ in Proposition 1.1.

Besides semigroup properties, we also study eigenvalue estimates for L. To this end, we assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is symmetric and μ is a probability measure. By

Theorem 2.1 in this paper and Lemma 3.1 in [18] (see also [35, 36] for an ealier related result), if P_t has kernel with respect to μ, then (1.4) implies the empty essential spectrum of L. Denote by

$$(0 \leq)\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

all eigenvalues of $-L$ counted with multiplicity. Our aim is to estimate λ_n by using β in the super-Poincaré inequality. Our estimates are sharp in the order of n. To see this, we present below some simple examples.

Example 1.1. Let $E = M$ be a compact complete Riemannian manifold of dimension d, and let $L = \Delta + \nabla V$ for some $V \in C^2(M)$ with $\mu(\mathrm{d}x) = \mathrm{e}^V \mathrm{d}x$ a probability measure. Then the super-Poincaré inequality holds with $\beta(r) = c(1 + r^{-d/2})$ for some $c > 0$, see Corollary 3.3 in [33]. By Corollary 5.3, we have $\lambda_n \geq \lambda n^{2/d}$ for some $\lambda > 0$ and all n. This is sharp in the order of n as is well known by Weyl's asymptotic formula for $V = 0$, see e.g. [8].

Example 1.2. Let $E = M$ be a noncompact, connected complete Riemannian manifold with Ricci curvature bounded from below. Let $L = \Delta + \nabla V$ with $V(x) = -\alpha\rho(x)^\delta(\alpha > 0, \delta > 1)$, where $\rho(x)$ denotes the Riemannian distance between x and a fixed point $o \in M$. By [33] we know that (1.4) holds with $\beta(r) = \exp[c(1 + r^{-\delta/2(\delta-1)})]$ for some $c > 0$. Therefore, by Corollary 5.3, we have (noting that $\lambda_1 > 0$ since $\lambda_0 = 0$ has single multiplicity)

$$\lambda_n \geq \lambda[\log(n+1)]^{2(\delta-1)/\delta} \tag{1.7}$$

for some $\lambda > 0$ and all $n \geq 0$ provided $\delta > 2$. If in addition $\mathrm{Ric}-\mathrm{Hess}_V$ is bounded from below, by Theorem 5.5, (1.7) holds for all $\delta > 1$.

We note that (1.7) is sharp in the order of n. To see this, let us consider the O-U process on $\mathbb{R}^d(d > 1)$ where $L = \Delta - \frac{1}{2}\nabla|x|^2$. In this case (1.7) holds for $\delta = 2$. It is well known that all eigenvalues of L are $\{0, 1, \cdots\}$, and the multiplicity of $\lambda = k$ is d^k. Actually, the unique eigenfunction for $\lambda = 0$ is constant, and the eigenspace of $\lambda = k+1$ is expanded by $\{-\frac{\partial}{\partial x_i}g + x_i g : 1 \leq i \leq d, g \text{ is an eigenfunction of } k\}$. Therefore $\lambda_{\sum_{i=0}^k d^i} = k$, hence the best estimate is

$$\lambda_n \geq \frac{\log[n(d-1)+1]}{\log d}$$

which gives the same order of n as in (1.7) for $\delta = 2$.

Our results also enable us to obtain eigenvalue estimates for jump processes in terms of functional inequalities derived in [9], [10] and [32]. The following is a simple example for birth-death processes.

Example 1.3. Let $E = \mathbb{Z}_+$. Consider the birth-death process with a_i (death rate) $= b_i$ (birth rate) $= i^\alpha[\log(1+i)]^\delta (\alpha > 1, \delta \in \mathbb{R}), i \geq 1$, and $a_0 = 0, b_0 = 1$. We have $\mu(i) = ci^{-\alpha}[\log(1+i)]^{-\delta}$ for $i \geq 1$ and $\mu(0) = c$, where $c > 0$ is such that μ is a probability measure.

1) If $\alpha = 2, \delta > 0$, then by Example 3.1 in [32], the F-Sobolev inequality holds for $F(r) = [\log(1+r)]^\delta$. By Corollary 5.3, when $\alpha > 1$ we have $\lambda_n \geq \lambda[\log(n+1)]^\alpha$ for some $\lambda > 0$ and all $n \geq 0$. Actually, this estimate holds for all $\delta > 0$ by Corollary 5.4. To see this we observe that $cs < i^2[\log(1+i)]^\delta$ implies $i > \frac{c_1\sqrt{s}}{[\log(1+s)]^{\delta/2}}$ for some $c_1 > 0$. Then we may take in Corollary 5.4 $h(s) = \frac{c_2}{\sqrt{s}[\log(1+s)]^{\delta/2}}$ for some $c_2 > 0$ and all $s > 0$. We have $s_n \geq c_3(1+n)^{2/3}[\log(1+n)]^{-\delta/3}$ for some $c_3 > 0$. Since (1.4) holds with $\beta(r) = \exp[c'(1 + r^{-1/\delta})]$ for some $c' > 0$, by Corollary 5.4 we obtain

$$\lambda_n \geq \frac{1}{2\beta^{-1}((n+1)/(8s_n))} \geq c_4[\log(1+n)]^\delta$$

for some $c_4 > 0$ and big n. Therefore, there exists $\lambda > 0$ such that $\lambda_n \geq \lambda[\log(1+n)]^\delta$ for all $n \geq 0$ since $\lambda_1 > 0$ by [12] and [32].

2) If $\alpha > 2$ and $\delta \geq 0$, by Example 3.1 in [32], the classical Sobolev inequality holds with dimension $p = \frac{2(\alpha-1)}{\alpha-2}$, which is equivalent to (1.4) for $\beta(r) = c(1 + r^{-(\alpha-1)/(\alpha-2)})$ for some $c > 0$ as mentioned above. Moreover, we have $\lambda_1 > 0$ by [12] and [32]. Then, by Corollary 5.3, $\lambda_n \geq \lambda n^{(\alpha-2)/(\alpha-1)}$ for some $\lambda > 0$ and all $n \geq 0$.

Finally, we show in section 6 how can concentrations of μ imply functional inequalities. We shall restrict ourselves to diffusions on a complete connected Riemannian manifold so that the Harnack inequalities in [30] are available. Let $L = \Delta + Z$ for some C^1-vector field Z such that the corresponding diffusion process has an invariant probability measure μ. Let $\rho(x)$ denote the Riemannian distance between x and a fixed point, and let $\mathcal{E}(f,f) = \mu(|\nabla f|^2)$. If $\text{Ric} - \langle \nabla . Z, \cdot \rangle$ is bounded from below, then for $\delta \in (1,2)$, the following are equivalent (see Theorem 6.4 and Corollary 6.5): (1) there exists $c > 0$ such that $\mu(\exp[c\rho^{2/(2-\delta)}]) < \infty$; (2) (1.3) holds with $F(r) = [\log(1+r)]^\delta$; (3) $\|P_t\|_{1\to\infty} \leq \exp[\lambda(1 + t^{-1/(\delta-1)})]$ for some $\lambda > 0$ and all $t > 0$. It was shown in [33] that, (1.3) with $F(r) = [\log(1+r)]^\delta$ for some $\delta > 2$ implies compactness of the manifold.

It should be useful to note that if $\mathcal{E}$ is symmetric and irreducible, P_t has kernel, then (1.3) or (1.4) implies Poincaré inequality, hence we may take in (1.4) $\beta(r) = 1$ for big r. Actually in this case we know that the essential spectrum of L is empty, since $\mathcal{E}$ is irreducible, 0 is a single multiplicity eigenvalue, hence there exists a spectral gap.

Acknowledgments. The author would like to thank Prof. T. Coulhon, Prof. M. Röckner and Dr. F. Z. Gong for useful discussions.

2 Boundedness properties of semigroups

The aim of this section is to study semigroup properties using super-Poincaré inequalities.

Theorem 2.1. *Assume that* (1.4) *holds. For fixed* $t > 0$, *let*

$$\Gamma_t(s) = \inf\{r \geq 0 : \beta(1/r)(e^{2rt} - 1) \geq s^2\}, \quad s \geq 0.$$

For any probability measure ν *on* $[0, \infty)$ *we have*

$$\int d\mu \int_0^{|P_t f|} \big(|P_t f| - s\big)^2 \exp[2t\Gamma_t(s)]\nu(ds) \leq 1, \quad \mu(f^2) = 1. \tag{2.1}$$

Consequently, P_t *is uniformly integrable in* $L^2(\mu)$ *with*

$$\sup_{\mu(f^2)=1} \mu((P_t f)^2 1_{\{|P_t f| > r\}}) \leq \exp[-2t\Gamma_t(\varepsilon r)]/(1-\varepsilon)^2, \quad r > 0, \ \varepsilon \in (0,1).$$

Proof. Let $t > 0$ be fixed. For any $f \geq 0$ with $\mu(f^2) = 1$, define

$$g_s = (P_t f - s)^+, \quad h_s = P_t(f - s)^+, \quad s \geq 0.$$

Since the function $j(r) = (r - s)^+$ is convex, we have $h_s \geq g_s$. By (1.5),

$$\begin{aligned}
\mu(h_s^2) &\leq e^{-2rt}\mu((f-s)^{+2}) + \beta(1/r)(1 - e^{-2rt})\mu((f-s)^+)^2 \\
&\leq \big(1 - s^2\mu(f > s)\big)\big(e^{-2rt} + \beta(1/r)(1 - e^{-2rt})\mu(f > s)\big) \\
&\leq \sup_{x \in (0, s^{-2})} (1 - s^2 x)\big(e^{-2rt} + \beta(1/r)(1 - e^{-2rt})x\big) \\
&= e^{-2rt}, \quad \text{if } \beta(1/r)(1 - e^{-2rt}) \leq s^2 e^{-2rt}.
\end{aligned}$$

Taking $r = \Gamma_t(s)$, we obtain

$$\mu(g_s^2) \leq \mu(h_s^2) \leq \exp[-2t\Gamma_t(s)].$$

Therefore,

$$\begin{aligned}
&\int d\mu \int_0^{P_t f} (P_t f - s)^2 \exp[2t\Gamma_t(s)]\nu(ds) \\
&= \int_0^\infty \exp[2t\Gamma_t(s)]\mu(g_s^2)\nu(ds) \leq \int_0^\infty \nu(ds) = 1.
\end{aligned}$$

To see that P_t is uniformly integrable, for any $r > 0$ we take $\nu = \delta_{r/2}$. Then (2.1) implies that

$$(1-\varepsilon)^2\mu((P_t f)^2 1_{\{|P_t f| > r\}}) \leq 4\mu([(P_t f - \varepsilon r)^+]^2) \leq \exp[-2t\Gamma_t(\varepsilon r)].$$

$\square$

To prove the converse of Theorem 2.1, we need the following lemma.

Lemma 2.2. *If* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is symmetric, then*

$$\mu((P_s f)^2) \le \mu((P_t f)^2)^{s/t} \mu(f^2)^{1-s/t}, \quad s < t, \ f \in L^2(\mu). \tag{2.2}$$

Proof. Since the form is symmetric, the generator L is self-adjoint on $L^2(\mu)$. Let $\{E_\lambda\}$ denote the spectral family of L, we have

$$P_t f = \int_{-\infty}^0 \mathrm{e}^{\lambda t} \mathrm{d}E_\lambda(f).$$

Therefore,

$$\mu((P_t f)^2) = \int_{-\infty}^0 \mathrm{e}^{2\lambda t} \mathrm{d}\langle E_\lambda(f), E_\lambda(f)\rangle_{L^2(\mu)}.$$

If $\mu(f^2) = 1$, then $\mathrm{d}\langle E_\lambda(f), E_\lambda(f)\rangle_{L^2(\mu)}$ is a probability measure on $(-\infty, 0]$. We obtain

$$\mu((P_t f)^2) \ge \left[\int_{-\infty}^0 \mathrm{e}^{2\lambda s} \mathrm{d}\langle E_\lambda(f), E_\lambda(f)\rangle_{L^2(\mu)} \right]^{t/s} = \mu((P_s f)^2)^{t/s}.$$

This proves (2.2). $\square$

Taking derivative over s at $s = 0$, we obtain from (2.2) that

$$-\frac{2t\mathcal{E}(f,f)}{\mu(f^2)} \le \log \frac{\mu((P_t f)^2)}{\mu(f^2)} \tag{2.3}$$

which coincides with an inequality on page 514 in Coulhon [13].

Proposition 2.3. *Assume that* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is symmetric. If there exist* $t > 0$ *and positive decreasing function* β_t *on* $(0, \infty)$ *such that*

$$\mu((P_t f)^2) \le r\mu(f^2) + \beta_t(r)\mu(|f|)^2, \quad r > 0, \ f \in L^2(\mu), \tag{2.4}$$

then (1.4) *holds with*

$$\beta(r) = \sup_{x \in (\mathrm{e}^{-2t/r}, 1)} \inf_{r_1 \in (0,x)} \frac{\beta_t(r_1)[\frac{r}{2t}\log x + 1]}{x - r_1}.$$

Proof. Let $\mu(f^2)=1$. By (2.4),

$$\mu(|f|)^2 \geq \frac{\mu((P_tf)^2)-r_1}{\beta_t(r_1)}, \qquad r_1<\mu((P_tf)^2).$$

Combining this with (2.2), we obtain

$$\mu((P_sf)^2) \leq \mathrm{e}^{-s/r} + \frac{\beta_t(r_1)(\mu((P_tf)^2)^{s/t}-\mathrm{e}^{-s/r})^+}{\mu((P_tf)^2)-r_1}\mu(|f|)^2 \tag{2.5}$$

for $0<r_1<\mu((P_tf)^2)$ and all $r>0, s\geq 0$, where the equality holds for $s=0$. Taking derivative in (2.5) over s at $s=0$, we arrive at

$$-2\mathcal{E}(f,f) \leq -\frac{1}{r} + \frac{\beta_t(r_1)[\frac{1}{t}\log\mu((P_tf)^2)+\frac{1}{r}]^+}{\mu((P_tf)^2)-r_1}\mu(|f|)^2, \quad 0<r_1<\mu((P_tf)^2), r>0.$$

Therefore,

$$\mu(f^2) \leq 2r\mathcal{E}(f,f) + \frac{\beta_t(r_1)[\frac{r}{t}\log\mu((P_tf)^2)+1]^+}{\mu((P_tf)^2)-r_1}\mu(|f|)^2, \quad 0<r_1<\mu((P_tf)^2), r>0.$$

This proves the assertion. □

Theorem 2.4. *Assume that $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is symmetric and P_t is uniformly integrable in $L^2(\mu)$ for some $t>0$. Let $h_t(s)=\sup\{\mu((P_tf)^2 1_{\{|P_tf|>s\}}) : \mu(f^2)=1\}$ which goes to zero as $s\to\infty$. Then (1.4) holds with*

$$\beta(r) = \sup_{x\in(\mathrm{e}^{-2t/r},1)} \inf_{r_1\in(0,x)} \frac{[h_t^{-1}(r_1/2)]^2}{2r_1(x-r_1)}\Big[\frac{r}{2t}\log x+1\Big],$$

where $h_t^{-1}=\inf\{s>0 : h_t(s)\leq r\}$.

Proof. For f with $\mu(f^2)=1$, we have

$$\begin{aligned}\mu((P_tf)^2) &\leq \mu((P_tf)^2 1_{\{|P_tf|\leq s\}}) + \mu((P_tf)^2 1_{\{|P_tf|>s\}})\\ &\leq s\mu(|f|)+h_t(s) \leq \frac{r}{2}+\frac{s^2\mu(|f|)^2}{2r}+h_t(s).\end{aligned}$$

Taking $s=h_t^{-1}(r/2)$ we prove (2.4) for $\beta_t(r)=[h_t^{-1}(r/2)]^2/(2r)$. The proof is completed by Proposition 2.4. □

Theorem 2.5. *Assume that $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is symmetric. If there exist $t>0$ and increasing positive function G_t on $(0,\infty)$ such that $G_t(\infty)=\infty$ and*

$$\sup_{\mu(f^2)=1} \mu\big((P_tf)^2G_t((P_tf)^2)\big) := C_t < \infty, \tag{2.6}$$

then (1.4) *holds for*

$$\beta(r) = \sup_{x\in(\mathrm{e}^{-2t/r},1)} \inf_{r_1\in(0,x)} \frac{G_t^{-1}((1+C_t)/r_1)(1+C_t)}{4r_1(x-r_1)}\Big[\frac{r}{2t}\log x + 1\Big],$$

where $G_t^{-1}(r) = \sup\{s \geq 0 : G_t(s) \leq r\}, r > 0$. *As usual we set* $\sup\emptyset = 0$.

Proof. Let $f \in L^2(\mu)$ be such that $\mu(|f|) = 1$, and denote $a = \mu(f^2)$. We have

$$\sup_{r\geq 0}\{rs - rG_t(r^2/a)\} \leq s\sqrt{aG_t^{-1}(s)}, \quad s > 0. \tag{2.7}$$

Actually, if $rs \geq s\sqrt{aG_t^{-1}(s)}$, then $rG_t(r^2/a) \geq rs$. Hence (2.7) holds since the left-hand side of (2.7) is nonpositive. By (2.6) and (2.7),

$$\begin{aligned} aC_t &\geq \mu\big((P_tf)^2G_t((P_tf)^2/a)\big) \geq s\mu((P_tf)^2) - s\sqrt{aG_t^{-1}(s)}\mu(|P_tf|) \\ &\geq s\mu((P_tf)^2) - s\sqrt{aG_t^{-1}(s)}, \quad s > 0. \end{aligned}$$

Therefore,

$$\mu((P_tf)^2) \leq \frac{aC_t}{s} + \sqrt{aG_t^{-1}(s)} \leq \frac{a(1+C_t)}{s} + \frac{sG_t^{-1}(s)}{4}, \quad s > 0. \tag{2.8}$$

This implies (2.4) for $\beta_t(r) = (1+C_t)G_t^{-1}((1+C_t)/r)\big/(4r)$. The proof is completed by Proposition 2.3. $\square$

Proof of Proposition 1.1. By Corollary 3.3 in [33], (1.3) with $F(r) = [\log(1+r)]^\delta$ implies (1.4) with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ for some $c > 0$. When $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form, the converse holds by Theorem 2.3 in [18]. If (1.4) holds for such β with some $\delta \in (0,1]$, by Theorem 2.1 we obtain

$$\int_0^\infty \gamma(s)\mathrm{d}s \geq \int \mathrm{d}\mu \int_0^{|P_tf|} (|P_tf| - s)^2\gamma(s)\exp[2t\Gamma_t(s)]\mathrm{d}s \tag{2.9}$$

for any positive $\gamma \in C[0,\infty)$, where

$$\begin{aligned} \Gamma_t(s) &= \inf\{r \geq 0 : \exp[c(1+r^{1/\delta})](\mathrm{e}^{2rt} - 1) \geq s^2\} \\ &\geq c_1[\log(1+\mathrm{e}^{-c}s^2)]^\delta - c_2 \end{aligned}$$

for some $c_1, c_2 > 0$. Taking

$$\gamma(s) = \frac{1}{1+s} \exp\big\{ - c_1 t[\log(1+\mathrm{e}^{-c}s^2)]^\delta\big\},$$

we obtain

$$\begin{aligned}
&\int_0^{|P_tf|} (|P_tf|-s)^2\gamma(s)\exp[2t\Gamma_t(s)]\mathrm{d}s\\
&\geq \frac{(P_tf)^2}{4+2|P_tf|}\int_{|P_tf|/4}^{|P_tf|/2} \exp\big\{c_1t[\log(1+\mathrm{e}^{-c}s^2)]^\delta - 2c_2t\big\}\mathrm{d}s\\
&\geq \frac{|P_tf|^3\mathrm{e}^{-2c_2t}}{16+8|P_tf|}\exp\big\{c_1t[\log(c_3|P_tf|^2+1)]^\delta\big\}
\end{aligned}$$

for $c_3 = \frac{\exp[-c]}{16}$. This proves (1.6) by (2.9).

On the other hand, assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is symmetric and (1.6) holds for some $t > 0$ and $c_t > 0$. Then (2.6) holds for $G_t(r) = \exp\{c_t[\log(1+r^2)]^\delta\}$ and some $C_t > 0$. It is easy to see that

$$G_t^{-1}((1+C_t)/r) \leq \exp\big\{c_4[\log(1+1/r)]^{1/\delta} + c_4\big\}$$

for some $c_4 = c_4(t) > 0$. Therefore, by Theorem 2.5, there exists $c_5 = c_5(t) > 0$ such that (1.4) holds for

$$\beta(r) = c_5 \sup_{x\in(\mathrm{e}^{-2t/r},1)} \inf_{r_1\in(0,x)} \frac{\exp\{c_4[\log(1+1/r_1)]^{1/\delta}\}(\frac{r}{2t}\log x + 1)}{r_1(x-r_1)}.$$

Taking $r_1 = x/2$, we see that

$$\begin{aligned}
\beta(r) &\leq \frac{c_5}{4}\sup_{x\in(\mathrm{e}^{-2t/r},1)} \frac{1}{x^2}\exp\big\{c_4[\log(1+2/x)]^{1/\delta}\big\}\\
&= \frac{c_5}{4}\exp\big\{4t/r + c_4[\log(1+2\mathrm{e}^{2t/r})]^{1/\delta}\big\} \leq \exp[c(1+r^{-1/\delta})]
\end{aligned}$$

for some $c = c(t) > 0$ since $\delta \in (0,1]$. $\square$

Finally, we study the semigroup property for the following weaker inequality:

$$\mu(f^2) \leq r\mathcal{E}(f,f) + \beta(r)\mu(|f|)^2, \quad r > r_0, \tag{2.10}$$

where $r_0 > 0$ and β is a positive decreasing function on (r_0, ∞).

Theorem 2.6. *If* (2.10) *holds, then*

$$\lim_{r\to\infty}\sup_{\mu(f^2)\le 1}\mu((P_tf)^2 1_{\{|P_tf|>r\}})\le \exp[-2t/r_0] \tag{2.11}$$

for all $t>0$. *On the other hand, if* L *is normal and there exists* $t>0$ *such that* (2.11) *holds, then* (2.10) *holds for some decreasing positive* β *on* (r_0,∞).

Proof. Let Γ_t be defined in Theorem 2.1, where β is a decreasing positive function on (r_0,∞). Then $\Gamma_t(\infty)\ge r_0^{-1}$ for any $t>0$. If (2.10) holds, then the proof of Theorem 2.1 implies

$$\sup_{\mu(f^2)\le 1}\mu((P_tf)^2 1_{\{|P_tf|>r\}})\le \frac{\exp[-2t\Gamma_t(\varepsilon s)]}{(1-\varepsilon)^2}$$

for all $r>0$ and any $\varepsilon\in(0,1)$. This prove (2.11).

On the other hand, if L is normal and (2.11) holds for some $t>0$, then for any $r>r_0$ there exists $R_r>0$ such that $\mu((P_tf)^2 1_{\{|P_tf|>r\}})\le \exp[-2t/r_1]$ for all $\mu(f^2)\le 1$, where $r_1=r_0+(r-r_0)/2$. Hence

$$\mu((P_tf)^2)\le \exp[-2t/r_1]+R_r\mu(|f|)^2,\quad \mu(f^2)=1.$$

Therefore, there exists positive decreasing β_t on $(\exp[-2t/r_0],\infty)$ such that (2.4) holds for all $r>\exp[-2t/r_0]$ in place of $r>0$. Then the proof of Proposition 2.3 implies (2.10) for

$$\beta(r)=\inf_{r_1\in(\exp[-2t/r_0],\exp[-2t/r])}\frac{\beta_t(r_1)}{\exp[-2t/r]-r_1},\quad r>r_0.$$

$\square$

3 Nonsymmetric case

Theorem 2.1 enables one to obtain estimates of P_t from the super-Poincaré inequality. But in general, to obtain an inverse result, one need to assume that $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is symmetric. In this section we consider a class of semigroups for which we are able to work out inverse results of Theorem 2.1 without the symmetry assumption. The main idea comes from [25] in which the log-Sobolev inequality was studied for nonsymmetric diffusions on a Riemannian manifold.

We simply follow the notion of Bakry and Ledoux [4] (section2). Assume that μ is an invariant probability measure of P_t. Let $\mathcal{A}$ be a nice algebra of bounded functions on E dense in $L^2(\mu)$ and in $\mathcal{D}(\mathcal{E})$ (under the Sobolev norm), stable by L and product, i.e. $Lf, fg\in\mathcal{A}$ if $f,g\in\mathcal{A}$. We introduce the so called "carré du champ" on $\mathcal{A}\times\mathcal{A}$:

$$\Gamma(f,g) = \frac{1}{2}\{L(fg) - fLg - gLf\}, \quad \Gamma_2(f,f) = \frac{1}{2}L\Gamma(f,f) - \Gamma(f,Lf), \quad f,g \in \mathcal{A}.$$

We say that the curvature of L is bounded from below by $-K$ for some $K \geq 0$, if

$$\Gamma_2(f,f) \geq -K\Gamma(f,f), \qquad f \in \mathcal{A}. \tag{3.1}$$

This definition of curvature lower bound coincides with that of Ricci curvature on a Riemannian manifold when $L = \Delta$. According to [4] (page 266), (3.1) is equivalent to

$$\Gamma(P_t f, P_t f) \leq \mathrm{e}^{2Kt} P_t \Gamma(f,f), \quad t \geq 0, f \in \mathcal{A}. \tag{3.2}$$

Actually, we may work with the following general "gradient estimate": for some positive $h \in C[0,\infty)$,

$$\Gamma(P_t f, P_t f) \leq h(t) P_t \Gamma(f,f), \quad t \geq 0, f \in \mathcal{A}. \tag{3.3}$$

Lemma 3.1. *Assume that* (3.3) *holds. We have the following semigroup Poincaré inequality:*

$$P_t f^2 \leq 2 P_t \Gamma(f,f) \int_0^t h(s)\mathrm{d}s + (P_t f)^2, \quad t > 0, f \in \mathcal{A}.$$

Proof. The proof is standard as was shown in [4]. For $f \in \mathcal{A}$,

$$P_t f^2 - (P_t f)^2 = 2\int_0^t P_s \Gamma(P_{t-s} f, P_{t-s} f)\mathrm{d}s \leq 2 P_t \Gamma(f,f) \int_0^t h(s)\mathrm{d}s.$$

□

Theorem 3.2. *Assume that μ is an invariant measure of P_t. If* (3.3) *holds for some positive $h \in C[0,\infty)$, then* (1.4) *holds for some β if and only if P_t is uniformly integrable in $L^2(\mu)$ for each $t > 0$. More precisely, for each $t > 0$, let $G_t \in C[0,\infty)$ be positive and increasing with $G_t(\infty) = \infty$ such that*

$$\mu((P_t f)^2 G_t((P_t f)^2)) \leq 1, \qquad \mu(f^2) = 1, \tag{3.4}$$

then (1.4) *holds for $\beta(r) = 2G_{t(r)}^{-1}(4)$, where $t(r) = \sup\{t \geq 0 : 4\int_0^t h(s)\mathrm{d}s \leq r\}$. In particular, if* (3.1) *holds, then $t(r) = \frac{1}{2K}\log(1 + \frac{rK}{2})$ for $K > 0$ and $t(r) = \frac{r}{4}$ for $K = 0$.*

Proof. By Theorem 2.1, it suffices to prove the second assertion. By (3.4) and the proof of Theorem 2.5, we have (see (2.8))

$$\mu((P_tf)^2) \le \frac{2}{s}\mu(f^2) + \frac{s}{4}G_t^{-1}(s)\mu(|f|)^2, \quad s>0, f\in\mathcal{A}.$$

Combining this with Lemma 3.1, we obtain

$$\begin{aligned}\mu(f^2) &\le 2\mathcal{E}(f,f)\int_0^t h(r)\mathrm{d}r + \mu((P_tf)^2)\\ &\le 2\mathcal{E}(f,f)\int_0^t h(r)\mathrm{d}r + \frac{2}{s}\mu(f^2) + \frac{s}{4}G_t^{-1}(s)\mu(|f|)^2, \quad s>0, f\in\mathcal{A}.\end{aligned}$$

Taking $s=4, t=t(r)$ and noting that $\mathcal{A}$ is dense in $\mathcal{D}(\mathcal{E})$ under the Sobolev norm, we complete the proof. □

Remark. 1) Obviously, (3.3) in Theorem 3.2 can be replaced by $\mathcal{E}(P_tf,P_tf)\le h(t)\mathcal{E}(f,f), f\in\mathcal{A}, t>0$. Assume that L is a diffusion operator (see [4]). Then (3.1) is equivalent to ([4])

$$\sqrt{\Gamma(P_tf,P_tf)} \le \mathrm{e}^{Kt}P_t\sqrt{\Gamma(f,f)}$$

which implies the following log-Sobolev inequality:

$$P_t(f^2\log f^2) \le \frac{2(\mathrm{e}^{2Kt}-1)}{K}P_t\Gamma(f,f) + P_tf^2\log P_tf^2. \tag{3.5}$$

To see this, let $\psi(s) = P_s(P_{t-s}f^2\log P_{t-s}f^2), s\in[0,t]$. Then

$$\begin{aligned}\psi'(s) &= P_s\frac{\Gamma(P_{t-s}f^2,P_{t-s}f^2)}{P_{t-s}f^2} \le \mathrm{e}^{2K(t-s)}P_s\frac{\left(P_{t-s}\sqrt{\Gamma(f^2,f^2)}\right)^2}{P_{t-s}f^2}\\ &\le 4\mathrm{e}^{2K(t-s)}P_t\Gamma(f,f).\end{aligned}$$

This proves (3.5) immediately. It follows from (3.5) that

$$\mu(f^2\log f^2) \le \frac{2(\mathrm{e}^{2Kt}-1)}{K}\mathcal{E}(f,f) + \mu\big(P_tf^2\log P_tf^2\big). \tag{3.6}$$

By (3.6) and using the argument of [25] (section 3), we obtain the following result: if $\|P_t\|_{p\to q}<\infty$ for some $t>0$ and $q>p>1$, then

$$\mu(f^2\log f^2) \le \frac{2(\mathrm{e}^{2Kt}-1)p(q-1)}{K(q-p)}\mathcal{E}(f,f) + \frac{pq}{q-p}\log\|P_t\|_{p\to q}. \tag{3.7}$$

Combining this with Theorem 4.1 and the paragraph after its proof, we see that P_t is supercontractive if and only if the super log-Sobolev inequality holds, i.e. there exists positive decreasing function α on $(0,\infty)$ such that

$$\mu(f^2\log f^2)\le r\mathcal{E}(f,f)+\alpha(r),\quad r>0, f\in\mathcal{D}(\mathcal{E}),\mu(f^2)=1. \tag{3.8}$$

2) Consider $L=\Delta+Z$ on a complete connected Riemannian manifold, where Z is a C^1-vector field. Let $k\in C(M)$ be such that

$$\mathrm{Ric}(X,X)-\langle\nabla_X Z,X\rangle\ge k(x)|X|$$

for any $x\in M$ and $X\in T_xM$. We have (see [17])

$$|\nabla P_t f|\le E^x\Big\{|\nabla f|(x_t)\mathrm{e}^{\int_0^t k(x_s)\mathrm{d}s}\Big\},$$

where x_t is the L-diffusion process. Noting that $\Gamma(f,g)=\langle\nabla f,\nabla g\rangle$, we see that (3.3) holds for

$$h(t)=\sup_{x\in M}E^x\exp\left\{2\int_0^t k(x_s)\mathrm{d}s\right\}$$

whenever the right-hand side is finite.

3) We may also consider general diffusions on $\mathbb{R}^d$. Let

$$L=\sum_{i,j=1}^d a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i},$$

where $a(x)=(a_{ij}(x))=\sigma(x)\sigma(x)^*$ is positive definite and a_{ij},b_i are smooth. Assume that

$$\lambda:=\sup\Big\{\frac{\langle a(x)y,y\rangle}{|y|^2}:y\ne 0,x\in\mathbb{R}^d\Big\}<\infty,$$
$$K=\sup_{x\ne y}\frac{1}{|x-y|^2}\big[\|\sigma(x)-\sigma(y)\|^2+\langle b(x)-b(y),x-y\rangle\big]<\infty,$$

where $\|A\|=\mathrm{tr}(AA^*)$ for a matrix A. By Theorem 4.13 in [11] we have

$$P_tf^2\le\frac{\lambda}{K}(\mathrm{e}^{2Kt}-1)P_t|\nabla f|^2+(P_tf)^2.$$

If there exists $c>0$ such that $a\ge cI$, then Theorem 3.2 holds for $t(r)=\frac{1}{2K}\log(1+\frac{rKc}{2\lambda})$.

4 Supercontractivity and ultracontractivity

As a complement of Theorems 2.1 and 2.4, we study the supercontractivity and ultracontractivity of P_t, where supercontractivity means that $\|P_t\|_{2\to4}<\infty$ for any

$t > 0$ while ultracontractivity reads $\|P_t\|_{1\to 2} < \infty$ for any $t > 0$. When P_t is symmetric, this definition of ultracontractivity is equivalent to the one in Davies and Simon [15], i.e. $\|P_t\|_{2\to\infty} < \infty$ for any $t > 0$.

Ultracontractivity was studied by Davies and Simon [15] using (super) log-Sobolev inequalities, then by Carlen, Kusuoka and Stroock [7], Tomisaki [28] and Coulhon [13] using Nash type inequalities. In this section, we study the above two contractivity properties using the super-Poincaré inequality which is a general form of all inequalities in the literature. Actually, part work in the direction has been done in [33]. For instance, if (1.4) holds with β such that

$$\Psi(t) := \int_t^\infty \frac{\beta^{-1}(r)}{r} \mathrm{d}r < \infty, \quad t > \inf \beta,$$

then (Theorem 5.1 in [33])

$$\sqrt{\|P_t\|_{1\to\infty}} \le \max\{\varepsilon^{-1} \inf \beta, \Psi^{-1}((1-\varepsilon)t)\}, \quad \varepsilon \in (0,1), t > 0. \tag{4.1}$$

Actually, the proof of Theorem 5.1 in [33] implies $\|P_t\|_{1\to 2} \le \max\{\varepsilon^{-1} \inf \beta, \Psi^{-1}(2(1-\varepsilon)t)\}$ and the same estimate for P_t^* (the adjoint of P_t) in place of P_t. Then we obtain (4.1) by the facts $\|P_t\|_{1\to\infty} \le \|P_{t/2}\|_{1\to 2} \cdot \|P_{t/2}\|_{2\to\infty}$ and $\|P_{t/2}^*\|_{1\to 2} = \|P_{t/2}\|_{2\to\infty}$.

The following is a result on supercontractivity.

Theorem 4.1. *If* (1.4) *holds for some* β *with* $\lim_{r\to 0} r \log \beta(r) = 0$, *then* P_t *is supercontractive. On the other hand, if* P_t *is supercontractive and is symmetric in* $L^2(\mu)$, *then* (1.4) *holds for*

$$\beta(r) = \inf_{s \le r} \left(\frac{s}{3\mathrm{e}} \wedge 2 \right) \inf_{t>0} \frac{(1 + \|P_t\|_{2\to 4}^2)^2}{t} \exp[6t/s]. \tag{4.2}$$

Consequently, $\lim_{r\to 0} r \log \beta(r) = 0$.

Proof. Let $t > 0$ be fixed. If $\lim_{r\to 0} r\beta(r) = 0$, then for any $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that $\beta(1/r) \le c(\varepsilon)\mathrm{e}^{\varepsilon r}$. Therefore,

$$\Gamma_t(s) \ge \frac{1}{2t+\varepsilon} \log \Big[1 + \frac{s^2}{c(\varepsilon)}\Big], \quad s > 0.$$

Taking
$nu(\mathrm{d}s) = \varepsilon(1+s)^{-(1+\varepsilon)}\mathrm{d}s$, (2.1) implies that

$$\int \mathrm{d}\mu \int_0^{|P_t f|} (|P_t f| - s)^2 (1+s)^{-(1+\varepsilon)} \Big(1 + \frac{s^2}{c(\varepsilon)}\Big)^{2t/(2t+\varepsilon)} \mathrm{d}s \le \frac{1}{\varepsilon}.$$

This implies that $\|P_t\|_{2\to 3} < \infty$ since ε is arbitrary. By Riesz-Thorin's interpolation theorem, we obtain $\|P_t\|_{3\to 4} < \infty$. Therefore, $\|P_{2t}\|_{2\to 4} \le \|P_t\|_{2\to 3}\|P_t\|_{3\to 4} < \infty$. This proves the first assertion since $t > 0$ is arbitrary.

Conversely, assume that P_t is symmetric and supercontractive. Let $c_t = \|P_t\|_{2\to 4}^2$. By taking $r_1 = x/2$ in Theorem 2.4, we obtain (1.4) with

$$\beta(r) = 2\inf_{t>0}\sup_{x\in(e^{-2t/r},1)} \frac{(1+c_t)^2}{x^3}\Big[\frac{r}{2t}\log x + 1\Big] \le \inf_{t>0}\frac{r(1+c_t)^2}{3et}\exp[6t/r].$$

Since $\log x \le 0$, it is obvious that $\beta(r) \le 2\inf_{t>0}(1+c_t)^2\exp[6t/r]$. The proof is completed by noting that if (1.4) holds for some β, then it holds with $\beta(r)$ replaced by $\inf_{s\le r}\beta(s)$. $\square$

It is easy to see from [18] and [33] that when $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a Dirichlet form, (1.4) with some β satisfying $\lim_{r\to\infty} r\log\beta(r) = 0$ is equivalent to (1.3) with some F satisfying $\log_{r\to\infty}\frac{F(r)}{\log r} = \infty$, they are also equivalent to the super log-Sobolev inequality (3.8) for some positive decreasing function α on $(0,\infty)$. Therefore, when $\mathcal{E}$ is a symmetric Dirichlet form, Theorem 4.1 implies the equivalence of the supercontractivity and the super log-Sobolev inequality, which is well-known by Gross' theorem (see e.g. [15]).

Next, we go to study ultracontractivity, for which a sufficient condition is given by (4.1).

Theorem 4.2. *Assume that P_t is symmetric and ultracontractive, then* (1.4) *holds for*

$$\beta(r) = \inf_{s\le r, t>0}\frac{s\|P_t\|_{1\to\infty}}{t}\exp\big[t/s - 1\big]. \tag{4.3}$$

Consequently, we have

1) *Let $\delta > 1$. (1.4) with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ for some $c > 0$ is equivalent to*

$$\|P_t\|_{1\to\infty} \le \exp[\lambda(1+t^{-1/(\delta-1)})], \qquad t>0, \tag{4.4}$$

for some $\lambda > 0$. They are also equivalent to (1.3) *with $F(r) = [\log(1+r)]^\delta$ if $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a Dirichlet form.*

2) *Let $p > 0$. (1.4) with $\beta(r) = c(1+r^{-p/2})$ for some $c > 0$ is equivalent to*

$$\|P_t\|_{1\to\infty} \le \lambda(1+t^{-p/2}) \tag{4.5}$$

for some $\lambda > 0$ and all $t > 0$. When $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is a Dirichlet form, they are also equivalent to (1.3) *with $F(r) = r^{2/p}$, and the classical Sobolev inequality with dimension p whenever $p > 2$.*

Proof. Let $c_t = \|P_t\|_{1\to 2}^2 = \|P_{2t}\|_{1\to\infty}$. For any $f \in L^1(\mu) \cap L^2(\mu) \cap \mathcal{D}(L)$ with $\mu(|f|) = 1$, we have $\mu((P_t f)^2) \le c_t$. By lemma 2.2,

$$\mu((P_s f)^2) \le \mu((P_t f)^2)^{s/t} \mu(f^2)^{1-s/t} \le c_t^{s/t} \mu(f^2)^{1-s/t}, \qquad s \in [0, t],$$

where equality holds when $s = 0$. By taking derivative at $s = 0$ for both sides of the above inequality, we obtain

$$\mu(f^2) \log \mu(f^2) \le 2t\mathcal{E}(f, f) + \mu(f^2) \log c_t. \tag{4.6}$$

Noting that $x \log x \ge Nx - \mathrm{e}^{N-1}$ for any $x, N > 0$, we obtain from (4.6) that

$$\mu(f^2) \le \frac{2t}{N - \log c_t} \mathcal{E}(f, f) + \frac{\mathrm{e}^{N-1}}{N - \log c_t}$$

for any $N > \log c_t$. The first assertion follows by taking $N = \frac{2t}{r} + \log c_t$.

Consequences 1) and 2) follow from the first assertion and (4.1) and results on functional inequalities mentioned in the paragraph following (1.5). $\square$

Finally, we present the following result to show that $\|P_t\|_{1\to 2}$ and $\|P_t\|_{2\to\infty}$ can be estimated by each other. We note that when P_t is symmetric, these two things are equal.

Theorem 4.3. *For any $t > 0$,*

$$\|P_t\|_{2\to\infty} \le \inf\left\{ \prod_{n=1}^{\infty} \|P_{t_n}\|_{1\to 2}^{2^{-n}} : t_n > 0, \ \sum_{n=1}^{\infty} t_n = t \right\}, \tag{4.7}$$

which is also true when $\|\cdot\|_{2\to\infty}$ and $\|\cdot\|_{1\to 2}$ are exchanged. In particular, for $\delta > 0$, $\|P_t\|_{1\to 2}$ is bounded from above by $\exp[\lambda(1 + t^{-\delta})]$ for some $\lambda > 0$ and all $t > 0$ if and only if so is $\|P_t\|_{2\to\infty}$.

Proof. Let $\{t_n\} \subset (0, t)$ be such that $\sum_{n=1}^{n} t_n = t$. By Riesz-Thorin's interpolation theorem (see e.g. [14]), for any $n \ge 1$ we have (recall that P_t is contractive in $L^1(\mu)$ and $L^\infty(\mu)$)

$$\|P_{t_n}\|_{2^n \to 2^{n+1}} \le \|P_{t_n}\|_{1\to 2}^{2^{-n}}, \quad \|P_{t_n}\|_{1/(1-2^{-1-n})\to 1/(1-2^{-n})} \le \|P_{t_n}\|_{2\to\infty}^{2^{-n}}.$$

This proves (4.7) and the inequality with $\|\cdot\|_{1\to 2}$ and $\|\cdot\|_{2\to\infty}$ exchanged since

$$\|P_t\|_{1\to 2} = \prod_{n=1}^{\infty} \|P_{t_n}\|_{1/(1-2^{-1-n})\to 1/(1-2^{-n})}, \quad \|P_t\|_{2\to\infty} = \prod_{n=1}^{\infty} \|P_{t_n}\|_{2^n\to 2^{n+1}}.$$

For the second assertion, without loss of generality we assume that $\|P_t\|_{1\to 2} \le \exp[\lambda(1+t^{-\delta})]$. Let $t_n = 2^{-n/(1+\delta)}t/c$, where $c = \sum_{n=1}^\infty 2^{-n/(1+\delta)}$. We obtain from (4.7) that

$$\|P_t\|_{2\to\infty} \le \exp\left[\lambda \sum_{n=1}^\infty 2^{-n}(1+t_n^{-\delta})\right] \le \exp[\lambda(1+c^{1+\delta}t^{-\delta})].$$

$\square$

5 Estimates of eigenvalues

Let $(E, \mathcal{F}, \mu)$ be a probability space and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be symmetric. We assume that the essential spectrum of L is empty. According to [18, 33], if P_t has kernel $p_t(x,y)$ with respect to μ and (1.4) holds for some β, then P_t is compact for any $t > 0$ and the essential spectrum of L is empty. Denote by $(0 \le)\lambda_0 \le \lambda_1 \le \cdots \le \lambda_n \le \cdots$ all eigenvalues of $-L$ counted with multiplicity. The aim of this section is to estimate λ_n using β.

Let f_i be the normalized eigenfunction of $\lambda_i, i \ge 0$. Since P_t is compact, by spectral theorem (see e.g. [22], page 28) we have

$$P_t f = \sum_{i=0}^\infty \mathrm{e}^{-\lambda_i t}\mu(ff_i)f_i, \quad f \in L^2(\mu). \tag{5.1}$$

When $p_t \in L^2(\mu\times\mu)$, P_t is a Hilbert-Schmidt operator (see [22], page 39), hence

$$p_t(x,y) = \sum_{i=0}^\infty \mathrm{e}^{-\lambda_i t} f_i(x) f_i(y), \quad t > 0, x, y \in E. \tag{5.2}$$

In general, we have

$$p_t(x,x) \ge \mathrm{e}^{-\lambda_n t}\sum_{i=0}^n f_i(x)^2, \quad x \in E, t > 0, n \ge 0. \tag{5.3}$$

Actually, for fixed $x \in E$ and $n \ge 0$, let $f(y) = \sum_{i=0}^n f_i(x)f_i(y)$. Then $f \in L^2(\mu)$. By (5.1) and Schwartz inequality,

$$\begin{aligned}
&\mathrm{e}^{-\lambda_n t/2}\sum_{i=0}^n f_i(x)^2 \le \sum_{i=0}^n \mathrm{e}^{-\lambda_i t/2} f_i(x)^2 = P_{t/2}f(x) \\
&= \int p_{t/2}(x,y)f(y)\mu(\mathrm{d}y) \le \sqrt{p_t(x,x)\mu(f^2)} = \sqrt{p_t(x,x)\sum_{i=0}^n f_i(x)^2}.
\end{aligned}$$

This implies (5.3).

For any $s, t > 0$, let $A_{s,t} = \{x : p_t(x,x) \le s\}$ and $\delta_{s,t} = \mu(A_{s,t}^c)$. We assume that for any $t > 0$, $p_t(x,x)$ is finite for a.e.-μ $x \in E$. Then $\delta_{s,t} \downarrow 0$ as $s \uparrow \infty$ for each $t > 0$.

Theorem 5.1. *Assume that $\mathcal{E}$ is symmetric and the essential spectrum of L is empty, and (1.4) holds for some β. Let $\beta^{-1}(r) = \inf\{s > 0 : \beta(s) \le r\}$ for $r > 0$ and set $\inf \emptyset = \infty$. We have*

$$\lambda_n \ge \sup\Big\{\Big(\frac{1}{t}\log\frac{(n+1)\varepsilon}{s}\Big) \wedge \frac{1}{2\beta^{-1}(1/[4(\varepsilon+\delta_{s,t})])} : \varepsilon \in (0,1), s, t > 0\Big\}.$$

Proof. By (5.3),

$$\mathrm{e}^{-\lambda_n t}\sum_{i=0}^n (1_{A_{s,t}} f_i^2) \le s.$$

This implies

$$\lambda_n \ge \frac{1}{t}\log\Big\{\frac{1}{s}\sum_{i=0}^n \mu(1_{A_{s,t}} f_i^2)\Big\} \ge \frac{1}{t}\log\frac{(n+1)\varepsilon}{s} \tag{5.4}$$

provided $\mu(1_{A_{s,t}} f_i^2) \ge \varepsilon$ for any $0 \le i \le n$. On the other hand, if there exists $0 \le i \le n$ such that $\mu(1_{A_{s,t}} f_i^2) < \varepsilon$, we have

$$\mu(|f_i|)^2 = \big\{\mu(1_{A_{s,t}}|f_i|) + \mu(1_{A_{s,t}^c}|f_i|)\big\}^2 \le 2(\varepsilon + \delta_{s,t}).$$

Combining this with (1.4) we obtain

$$1 = \mu(f_i^2) \le r\lambda_i \mu(f_i^2) + \beta(r)\mu(|f_i|)^2 \le r\lambda_i + 2\beta(r)(\varepsilon+\delta_{s,t}), \quad s, t, r > 0.$$

Therefore,

$$\lambda_n \ge \lambda_i \ge \frac{1}{r}\big[1 - 2\beta(r)(\varepsilon+\delta_{s,t})\big], \quad r, s, t > 0.$$

Taking $r = \beta^{-1}(1/[4(\varepsilon+\delta_{s,t})])$, we obtain

$$\lambda_n \ge \frac{1}{2\beta^{-1}(1/[4(\varepsilon+\delta_{s,t})])}$$

provided there exists $0 \le i \le n$ such that $\mu(1_{A_{s,t}} f_i^2) \le \varepsilon$. The proof is completed by combining this with (5.4). $\square$

We remark that Theorem 5.1 gives lower bounds of λ_n which go to infinity as $n \to \infty$. Such bounds can be obtained by taking concrete choices of s, t, ε, for instance, $t = 1, s = n^{1/4}$ and $\varepsilon = n^{-1/4}$. The estimate given in Theorem 5.1 depends on not only the super-Poincaré inequality but also the heat kernel. Actually, eigenvalues could not be determined completely by a functional inequality. For instance, the best possible inequality for the O-U process on $\mathbb{R}^d (d \geq 1)$ is the log-Sobolev inequality which is dimension-free, but we have $\lambda_n = n$ for $d = 1$ and $\lambda_n = O(\log n)$ for $d > 1$. Explicit lower bounds are available whenever P_t is ultracontractive.

Theorem 5.2. *Assume that $p_t(x, x)$ is integrable with respect to μ for some $t > 0$. We have*

$$\lambda_n \geq \sup_{t>0} \frac{1}{t} \log \frac{n+1}{\int p_t(x,x)\mu(\mathrm{d}x)}, \quad n \geq 0.$$

Especially, $\int p_t(x,x)\mu(\mathrm{d}x)$ can be replaced by $\|P_t\|_{1\to\infty}$ whenever P_t is ultracontractive. Consequently, if (1.4) holds with β satisfying

$$\Psi(t) := \int_t^\infty \frac{1}{r}\beta^{-1}(r)\mathrm{d}r < \infty, \quad t > \inf \beta,$$

then there exists $c > 0$ such that

$$\lambda_n \geq \sup_{t>0} \frac{c}{t} \log \frac{n+1}{\Psi^{-1}(t)}, \quad n \geq 0.$$

Proof. By (5.3) (in fact, (5.2) holds in this case),

$$\int p_t(x,x)\mu(\mathrm{d}x) \geq \mathrm{e}^{-\lambda_n t} \sum_{i=0}^n \mu(f_i^2) = (n+1)\mathrm{e}^{-\lambda_n t}.$$

This proves the first assertion. The second assertion follows from the first one and (4.1). □

Corollary 5.3. *Let $\delta > 1$. If the F-Sobolev inequality holds for $F(r) = [\log(1+r)]^\delta$, or (1.4) holds with $\beta(r) = \exp[c(1 + r^{-1/\delta})]$ for some $c > 0$, then there exists $\lambda > 0$ such that*

$$\lambda_n \geq \frac{(\delta-1)^{\delta-1}}{\delta^\delta \lambda^{\delta-1}} \big([\log(n+1) - \lambda]^+\big)^\delta, \quad n \geq 0.$$

Next, let $p > 0$. If (1.3) holds for $F(r) = r^p$, or the super-Poincaré inequality holds with $\beta(r) = c(1 + r^{-1/p})$ for some $c > 0$, then there exists $\lambda > 0$ such that

$$\lambda_n \geq \big([\lambda(n+1) - 1]^+\big)^p, \quad n \geq 0.$$

Proof. Simply note that in the first case we have

$$\|P_t\|_{1\to\infty} \le \exp\left[\lambda(1+t^{-1/(\delta-1)})\right]$$

while in the second case

$$\|P_t\|_{1\to\infty} \le \lambda(1+t^{-1/p})$$

for some $\lambda > 0$ and all $t > 0$, see [33]. □

To obtain explicit estimates from Theorem 5.1, we need to study the kernel p_t. Below we consider two concrete cases, namely, countable Markov chains and diffusions on a Riemannian manifold.

Corollary 5.4. *Let $E = S$ be a countable set. Consider symmetric Markov chain on S with reversible probability measure μ, where $\mu(i) > 0$ for any $i \in S$. Let $h \in C[0,\infty)$ be decreasing with $h(0) = 1$ and $h(s) \ge \sum_{\mu(i)<1/s} \mu(i)$. Let $s_n > 0$ be such that $\frac{1}{s_n}h(s_n) = \frac{1}{n+1}$. If (1.4) holds, then*

$$\lambda_n \ge \frac{1}{2\beta^{-1}((n+1)/(8s_n))}, \quad n \ge 0.$$

Proof. We note that $p_t(i,i) \le 1/\mu(i)$ for any $t > 0$ and $i \in S$. Then $\delta_{s,t} \le h(s)$ for any $s,t > 0$. For any $\varepsilon > h(s_n)$, we have $\log \frac{(n+1)\varepsilon}{s_n} > 0$. By Theorem 5.1 we obtain

$$\lambda_n \ge \frac{1}{2\beta^{-1}(1/[4(\varepsilon + h(s_n))])}.$$

The proof is completed by letting $s \downarrow h(s_n)$. □

In the remainder of this section, we restrict ourselves to the manifold case. Let $E = M$ be a connected complete Riemannian manifold. Consider $L = \Delta + \nabla V$ for some $V \in C^2(M)$ with $R := \int \mathrm{e}^V \mathrm{d}x < \infty$. Let $\mathrm{d}\mu = R^{-1}\mathrm{e}^V \mathrm{d}x$. Assume that there exists $K \ge 0$ such that

$$(\mathrm{Ric} - \mathrm{Hess}_V)(X,X) \ge -K|X|^2, \quad X \in TM. \tag{5.5}$$

By the dimension-free Harnack inequality (Lemma 2.1 in [30]),

$$[P_t f(x)]^2 \le P_t f^2(y) \exp\left[\frac{K\rho(x,y)^2}{1-\mathrm{e}^{-2Kt}}\right], \quad f \in L^2(\mu),\ t > 0,$$

where $\rho(x,y)$ denotes the Riemannian distance between x and y. Then for any f with $\mu(f^2) = 1$,

$$\begin{aligned} 1 &\geq [P_t f(x)]^2 \int \exp\Big[-\frac{K\rho(x,y)^2}{1-\mathrm{e}^{-2Kt}}\Big]\mu(\mathrm{d}y) \\ &\geq [P_t f(x)]^2 \mu(B(o,1)) \exp\Big[-\frac{K(\rho(x)+1)^2}{1-\mathrm{e}^{-2Kt}}\Big], \end{aligned} \tag{5.6}$$

where $o \in M$ is a fixed point, $B(o,1)$ is the unit geodesic ball with center x, and $\rho(x) = \rho(x,o)$. Taking $f(y) = p_t(x,y)/\sqrt{p_{2t}(x,x)}$, we obtain

$$p_{2t}(x,x) \leq \frac{1}{\mu(B(o,1))} \exp\Big[\frac{K(\rho(x)+1)^2}{1-\mathrm{e}^{-2Kt}}\Big], \quad t > 0, x \in M. \tag{5.7}$$

Therefore,

$$A_{s,1} = \{x : p_1(x,x) \leq s\} \supset \Big\{x : \rho(x) \leq \sqrt{((1-\mathrm{e}^{-K}) \log[s\mu(B(o,1))])/K} - 1\Big\}.$$

Hence, there exist $c_1, c_2 > 0$ such that

$$\delta_{s,1} \leq \mu\Big(\rho \geq c_1\sqrt{\log s} - c_2\Big), \quad s > 1. \tag{5.8}$$

Corollary 5.5. *Let $E = M$ such that* (5.5) *holds. The estimate in Theorem* 5.1(*taking $t = 1$*) *holds with $\delta_{s,1}$ replaced by $\mu(\rho \geq c_1\sqrt{\log s} - c_2)$ for some $c_1, c_2 > 0$. In particular, if* (1.4) *holds with $\beta(r) = \exp[c(1 + r^{-1/\delta})]$ for some $c, \delta > 0$, or* (1.3) *holds for $F(r) = [\log(1 + r)]^\delta$, then*

$$\lambda_n \geq \lambda[\log(1+n)]^\delta \tag{5.9}$$

for some $\lambda > 0$ and all $n \geq 0$.

Proof. The first assertion follows immediately from (5.8). In the present case, the Dirichlet form is irreducible and hence $\lambda_0 = 0$ has single multiplicity, this means that $\lambda_1 > 0$. Therefore, it suffices to prove (5.9) for big n. Noting that (5.9) for $\delta > 1$ follows from Corollary 5.3, we need only to consider the case where $\delta \leq 1$. In this case, by the concentration of μ (see Corollary 6.3 in [33]),

$$\mu(\rho > r) \leq \alpha_1 \exp[-\alpha_2 r^{2/(2-\delta)}]$$

for some $\alpha_1, \alpha_2 > 0$ and all $r > 0$. Then

$$\mu(\rho \geq c_1\sqrt{\log s} - c_2) \leq c_3 \exp[-c_4(\log s)^{1/(2-\delta)}]$$

for some $c_3, c_4 > 0$ and all $s > 1$. Taking $\varepsilon = \varepsilon(s) = c_3 \exp[-c_4(\log s)^{1/(2-\delta)}]$ for big s, we obtain

$$\Big\{2\beta^{-1}\Big(\frac{1}{4(\varepsilon+\delta_{s,1})}\Big)\Big\}^{-1}\geq \frac{1}{2}\Big\{\frac{c_4}{c}[\log s]^{1/(2-\delta)}-\frac{1}{c}\log(8c_3)-1\Big\}^{\delta}\geq c_5[\log s]^{\delta/(2-\delta)}$$

for some $c_5>0$ and big s. For big n, let $s>1$ be such that

$$\log\frac{(n+1)\varepsilon(s)}{s}=c_5[\log s]^{\delta/(2-\delta)},$$

we have $\log s\geq c_6[\log(n+1)]^{2-\delta}$ for some $c_6>0$ and big n. Therefore, by Theorem 5.1,

$$\lambda_n\geq c_5[\log s]^{\delta/(2-\delta)}\geq \lambda[\log(n+1)]^{\delta}$$

for some $\lambda>0$ and big n. $\square$

Finally, (1.4) also enables us to estimate λ_n and $N(n)$ by each other, where $N(n)$ is the nodal number of f_n, i.e., the number of connected components of $\{f_n\neq 0\}$. By Courant's nodal domain theorem, we have $N(n)\leq n+1$ for the Laplacian on a compact Riemannian manifold, see e.g. [8]. The following is a relationship between λ_n and $N(n)$.

Theorem 5.6. *Let $E=M$ and $L=\Delta+\nabla V$ as above. If* (1.4) *holds, then*

$$N(n)\leq \inf_{r>0}\frac{\beta(r)}{1-r\lambda_n},\quad \lambda_n\geq \sup_{r>0}\frac{N(n)-\beta(r)}{rN(n)},\quad n\geq 0.$$

Proof. It suffices to prove the first inequality. If $N(n)\geq m$, then there exists a nodal set A of f_n such that $\mu(A)\leq 1/m$. Let $f=1_Af_n$. By (1.4), Green's formula and Schwartz inequality,

$$\mu(f^2)\leq r\lambda_n\mu(f^2)+\beta(r)\mu(f^2)/m.$$

Since $\mu(A)>0$ and f_n is an eigenfunction, we have $\mu(f^2)>0$. Therefore, $m\leq \beta(r)/(1-r\lambda_n)$ for any $r>0$. $\square$

6 Functional inequalities by concentration

Concentration of the reference measure μ has been studied in [33] using functional inequalities. The basic idea in this study goes back to Herbst's argument. See [1], [2], [21], [23], and [26] for earlier results in the literature. In this section, we aim to seek for converse of such results, say, to prove functional inequalities using concentration. Such results have been obtained in [30], [34], and [25] for logarithmic Sobolev inequalities in the context of diffusions on a manifold. The basic tool is the dimension-free Harnack inequality established in [30].

For simplicity, we also restrict ourselves to the manifold case, although it is natural to extend everything below to the case where E is a length space, see e.g. [5].

Let $E = M$ be a noncompact, connected, complete Riemannian manifold. Consider $L = \Delta + Z$ for some C^1-vector field Z such that the L-diffusion process has an invariant probability measure μ. Let $\mathcal{E}(f, g) = -\mu(fLg)$ with $\mathcal{D}(\mathcal{E})$ the complement of $C_0^\infty(M)$ under the Sobolev norm. We assume that there exists $K \geq 0$ such that

$$\mathrm{Ric}(X, X) - \langle \nabla_X Z, X\rangle \geq -K|X|^2, \qquad X \in TM. \tag{6.1}$$

Let $\rho(x)$ denote the Riemannian distance between x and a fixed point $o \in M$. The following result is essentially taken from [34] and [25].

Theorem 6.1. *Assume* (6.1).

1) *If there exists $c > K/2$ such that $\mu(\exp[c\rho^2]) < \infty$, then the log-Sobolev inequality* (1.1) *holds for $C_2 = 0$.*

2) *The super log-Sobolev inequality* (3.8) *holds if and only if $\mu(\exp[c\rho^2]) < \infty$ for all $c > 0$.*

Proof. If there exists $c > K/2$ such that $\mu(\exp[c\rho^2]) < \infty$, then there exist $p > 2$ and $t > 0$ such that $\|P_t\|_{2\to p} < \infty$, see the proof of Theorem 1.1 in [34]. By (3.7) we know that (1.1) holds. By the regularity of μ due to [6], there exists $V \in W^{2,1}_{loc}(M)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$, where $\mathrm{d}x$ denotes the Riemannian volume element on M. Then $\mathcal{E}'(f, g) := \mu(\langle \nabla f, \nabla g\rangle)$ is irreducible. By the observation in the last paragraph in section 1, the Poincaré inequality holds. Therefore, (1.1) holds for some $c_1 > 0$ and $C_2 = 0$.

Next, by (5.6), P_t is supercontractive if $\mu(\exp[c\rho^2]) < \infty$ for any $c > 0$. Then the second assertion follows immediately from (3.7) and concentration results in [21] and [25]. $\square$

Theorem 6.2. *Assume* (6.1). *If $h(r) := \mu(\exp[r\rho^2]) < \infty$ for all $r > 0$, then* (3.8) *holds with*

$$\alpha(r) = \log\big[ch(2K + 12/r)\big]$$

for some $c > 0$. Consequently, (1.4) *holds with*

$$\beta(r) = c \inf_{0<r_1\leq r} r_1 \inf_{s>0} \frac{1}{s} h(2K + 12/s)\exp[s/r_1 - 1]$$

for the same constant $c > 0$.

Proof. By (5.6) we have

$$\|P_t f\|_{2\to 4}^4 \leq c_1 h\big(2K/(1 - \exp[-2Kt])\big)$$

for some $c_1 > 0$ and all $t > 0$. Combining this with (3.7) we prove the first assertion.

To prove the second assertion, let $f \in L^2(\mu)$ be such that $\mu|f|) = 1$. Denote $a = \mu(f^2)$. By (3.8)

$$r\mathcal{E}(f,f) \geq \mu\big(f^2 \log(f^2/a)\big) - \alpha(r)a. \tag{6.2}$$

It is easy to check that for any $N > 0$,

$$f^2 \log f^2 - f^2 \log a - Nf^2 + 2\exp[N/2 - 1]\sqrt{a}|f| \geq 0.$$

Actually, we have $\inf\big\{2r\log r - r\log a - Nr : r \geq 0\big\} = -2\exp[N/2-1]$. Then, it follows from (6.2) that

$$\mu(f^2) \leq \frac{s}{N - \alpha(s) - 1}\mathcal{E}(f,f) + \frac{\exp[N-2]}{N - \alpha(s) - 1} \tag{6.3}$$

holds for all $s > 0, N > \alpha(s) + 1$ and all f with $\mu(|f|) = 1$. This proves the second assertion by letting $N = s/r + 1 + \alpha(s)$ for α given by the first assertion. □

Corollary 6.3. *Assume* (6.1). *Let $\Phi \in C[0,\infty)$ be positive such that $r^{-1}\Phi(r) \to \infty$ as $r \uparrow \infty$. If $\mu(\exp[\Phi(\rho^2)]) < \infty$, then* (1.4) *holds with*

$$\beta(r) = \inf_{0<r_1\leq r} cr_1 \inf_{s>0} \frac{1}{s}\exp\big[\phi(2K + 12/s) + s/r_1\big] \tag{6.4}$$

for some $c > 0$, where $\phi(s) = \sup_{r>0}\{sr - \Phi(r)\}$.

Proof. Noting that $\Phi(\rho^2) \geq s\rho^2 - \phi(s)$ for all $s > 0$, we complete the proof by Theorem 6.2. □

Theorem 6.4. *Assume* (6.1). *If $\|P_t\|_{1\to 2} < \infty$ for any $t > 0$, then* (1.4) *holds with*

$$\beta(r) = \inf_{0<r_1\leq r} r_1 \inf_{s>0}\frac{1}{s}\exp[s/r_1 - 1]\|P_{t(s)}\|_{1\to 2}^2,$$

where $t(s) = \frac{1}{2K}\log(1 + Ks/2)$. Consequently, let $\delta > 1$, (1.3) *holds with $F(r) = [\log(1+r)]^\delta$ if and only if there exists $\lambda > 0$ such that* (4.4) *holds.*

Proof. By (3.7),

$$\mu(f^2\log f^2) \leq \frac{2}{K}(\exp[2Kt] - 1)\mathcal{E}(f,f) + \log\|P_t\|_{1\to 2}^2, \quad t > 0.$$

Therefore, (3.8) holds for $\alpha(r) = \log\|P_{t(r)}\|_{1\to 2}^2$. Then the first assertion follows from (6.3) by taking $N = s/r + 1 + \alpha(s)$.

The necessity in the second assertion follows from (4.1), since (1.3) with $F(r) = [\log(1+r)]^\delta$ is equivalent to (1.4) with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ for some $c > 0$. On the other hand, if (4.4) holds then

$$\exp[s/r]\|P_{t(s)}\|_{1\to 2} \le \exp\left\{\frac{s}{r} + 2\lambda + 2\lambda\Big[\frac{1}{2K}\log\Big(1+\frac{Ks}{2}\Big)\Big]^{-1/(\delta-1)}\right\}.$$

Taking $s = r^{(\delta-1)/\delta}$ and noting that $\log(1+r) = O(r)$ for small r, we know by the first assertion that (1.4) holds with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ for some $c > 0$. □

Obviously, we may replace $\|P_t\|_{1\to 2}$ by $\|P_t\|_{2\to\infty}$ in Theorem 6.4 according to its proof and (3.7), but the function $t(s)$ becomes $t(s) = \frac{1}{2K}\log(1+Ks/4)$.

Corollary 6.5. *Assume* (6.1).

1) *For* $\delta \in (1,2)$, (1.3) *with* $F(r) = [\log(1+r)]^\delta$ *is equivalent to* $\mu(\exp[c\rho^{2/(2-\delta)}]) < \infty$ *for some* $c > 0$.

2) (1.3) *with* $F(r) = [\log(1+r)]^2$ *implies* $\mu(\exp[\exp(c\rho)]) < \infty$ *for some* $c > 0$. *Conversely,* $\mu(\exp[\exp(c\rho)]) < \infty$ *for some* $c > 0$ *implies* (1.4) *with*

$$\beta(r) = \exp\left[c'(1 + r^{-1/2}\log(1+r^{-1}))\right]$$

for some $c' > 0$, *which is slightly weaker than* (1.3) *with* $F(r) = [\log(1+r)]^2$.

3) *If* $\delta > 2$, *then* (1.3) *implies that* M *is compact.*

Proof. The last assertion and concentration results for (1.3) are contained by Corollary 6.3 in [33], we need only to prove (1.3) or (1.4) using concentration.

Let $\delta \in (1,2)$. For $\Phi(r) = cr^{1/(2-\delta)}$, we see $\phi(s) = c_1 s^{1/(\delta-1)}$ for some $c_1 > 0$. By Corollary 6.3 with the choice $s = r_1^{(\delta-1)/\delta}$, (1.4) holds with

$$\beta(r) \le \inf_{0<r_1\le r} r_1^{1/\delta}\exp[c_2(1+r_1^{-1/\delta})] \le \exp[c_3(1+r^{-1/\delta})]$$

for some $c_2, c_3 > 0$. Therefore (1.3) holds for $F(r) = [\log(1+r)]^\delta$.

For the case where $\Phi(r) = \exp[cr^{1/2}], \phi(s) \le c_4 s[\log(1+s)]^2$ for some $c_4 > 0$. Then (1.4) with β given in 3) follows by taking $s = \sqrt{r_1}\log(1+r_1^{-1})$ in (6.4). □

References

[1] S. Aida, T. Masuda and I. Shigekawa, *Logarithmic Sobolev inequalities and exponential integrability*, J. Funct. Anal. 126(1994), 83–101.

[2] S. Aida and D. Stroock, *Moment estimates derived from Poincaré and logarithmic Sobolev inequalities*, Math. Res. Lett. 1(1994), 75–86.

[3] D. Bakry, *L'hypercontractivité et son utilisation en théorie des semigroups*, Lecture Notes in Math. 158(1994), 1–114.

[4] D. Bakry and M. Ledoux, *Lévy-Gromov's isoperimetric inequality for an infinite-dimensional diffusion generator*, Invent. Math. 123(1996), 259–281.

[5] D. Bakry, M. Ledoux and Z. Qian, *Logarithmic Sobolev inequalities, Poincaré inequalities and heat kernel bounds,* 1997 Preprint.

[6] V. I. Bogachev, N. V. Krylov and M. Röckner, *Elliptic regularity and essential self-adjointness of Dirichlet operators on* $\mathbb{R}^n$*,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. 24(1997), 451–461.

[7] E. A. Carlen, S. Kusuoka and D. Stroock, *Upper bounds for symmetric Markov transition functions,* Ann. Inst. H. Poincaré Probab. Statist. 23(1987), 245–287.

[8] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.

[9] M. F. Chen, *Nash inequalities for general symmetric forms*, 15(1999), 353–370.

[10] M. F. Chen, *Log-Sobolev inequality for symmetric forms,* to appear in Sci. Sin.(A).

[11] M. F. Chen and F. Y. Wang, *Estimation of spectral gap for elliptic operators*, Trans. Amer. Math. Soc. 349(1997), 1239–1267.

[12] M. F. Chen and F. Y. Wang, *Cheeger's inequalities for general symmetric forms and existence criteria for spectral gap,* to appear in Ann. Probab.

[13] T. Coulhon, *Ultracontractivity and Nash type inequalities*, J. Funct. Anal. 141(1996), 510–539.

[14] E. B. Davies, *Heat Kernels and Spectral Theory,* Cambridge Univ. Press, 1989.

[15] E. B. Davies and B. Simon, *Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians,* J. Funct. Anal. 59(1984), 335–395.

[16] J. D. Deuschel and D. Stroock, *Large Deviations*, Pure and Appl. Math. Ser. 137, Academic Press, 1989.

[17] K. D. Elworthy and X. M. Li, *Formulae for the derivatives of heat semigroups,* J. Funct. Anal. 125(1994), 252–286.

[18] F. Z. Gong and F. Y. Wang, *Functional inequalities for uniformly integrable semigroups and applications,* SFB-343 preprint 99-80, Bielefeld.

[19] L. Gross, *Logarithmic Sobolev inequalities,* Amer. J. Math. 97(1976), 1061–1083.

[20] L. Gross, *Logarithmic Sobolev inequalities and contractivity properties of semigroups,* Lecture Notes in Math. 1563(1993), 54–88.

[21] L. Gross and O. Rothaus, *Herbst inequalities for supercontractive semigroups*, J. Math. Kyoto Univ. 38(1998), 295–318.

[22] H. König, *Eigenvalues distribution of compact operators*, Birkhäuser, 1986.

[23] M. Ledoux, *Concentration of measures and logarithmic Sobolev inequalities*, Lecture Notes in Math., to appear.

[24] Z. M. Ma and M. Röckner, *Introduction to Theory of (Non-symmetric) Dirichlet Forms,* Springer-Verlag, Berlin, 1992.

[25] M. Röckner and F. Y. Wang, *Ultracontractivity and supercontractivity of Markov semigroups,* SFB-343 preprint 99-88, Bielefeld.

[26] O. R. Rothaus, *Logarithmic Sobolev inequalities and the growth of L^p norms*, Proc. Amer. Math. Soc. 126(1998), 2309–2314.

[27] D. Stroock, *An Introduction to the Theory of Large Deviations,* Springer-Verlag, 1984.

[28] M. Tomisaki, *Comparison theorems on Dirichlet norms and their applications,* Forum. Math. 2(1990), 277–295.

[29] N. Th. Varopoulos, *Hardy-Littlewood theory for semigroups,* J. Funct. Anal. 63(1985), 240–260.

[30] F. Y. Wang, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds,* Probability Theory Relat. Fields 109(1997), 417–424.

[31] F. Y. Wang, *Existence of spectral gap for elliptic operators,* Arkiv för Matematik 37(1999), 395–407.

[32] F. Y. Wang, *Sobolev type inequalities for general symmetric forms,* to appear in Proc. Amer. Math. Soc.

[33] F. Y. Wang, *Functional inequalities for empty essential spectrum*, J. Funct. Anal. 170(2000), to appear.

[34] F. Y. Wang, *Logarithmic Sobolev inequalities: conditions and counterexamples,* J. Operator Theory, to appear.

[35] L. M. Wu, *Large deviations for Markov processes under superboundedness*, C. R. Acad. Sci. Paris Série I 324(1995), 777–782.

[36] L. M. Wu, *Uniformly integrable operators and large deviations for Markov processes*, to appear in J. Funct. Anal.