A STRATIFICATION FOR VARIETIES OF MODULES

NICOLA J. RICHMOND

Abstract. We introduce a stratification of the variety of modules of a fixed dimension over a finite-dimensional algebra. For a large and interesting class of algebras which we call subfinite, we use this stratification to compute the dimension of this variety and give bounds on the number of its irreducible components.

1. Introduction and Main Results

Let K be an algebraically closed field and let A be an associative K-algebra of dimension $d < \infty$ with a unit 1_A. We identify the affine variety $\mathrm{Mod}(A, n)$ of left A-module structures on K^n with the set of K-algebra homomorphisms $A \longrightarrow \mathrm{M}_n(K)$, where $\mathrm{M}_n(K)$ denotes the set of $n \times n$ matices over K. By variety, we mean a locally closed subset of affine space. If $x \in \mathrm{Mod}(A, n)$, then K_x will denote the corresponding A-module.

The connected algebraic group $\mathrm{GL}_n(K)$ of invertible $n \times n$ matrices over K acts on $\mathrm{Mod}(A, n)$ by conjugation. So the set of $\mathrm{GL}_n(K)$ orbits $\mathcal{O}(x)$ in $\mathrm{Mod}(A, n)$ is in one to one correspondence with the set of isomorphism classes $[K_x]$ of n-dimensional A-modules. If $y \in \mathrm{Mod}(A, n)$ such that $\mathcal{O}(y)$ is contained in $\overline{\mathcal{O}(x)}$ then K_x is said to degenerate to K_y. In this case, $\dim \mathcal{O}(y) \leq \dim \mathcal{O}(x)$, with equality if and only if $\mathcal{O}(y) = \mathcal{O}(x)$ and so the notion of degeneration is a partial order on the set of isomorphism classes of n-dimensional A-modules. We denote this degeneration order by $\leq_{\deg}$.

Many varieties arise as varieties of modules over a finite-dimensional algebra and since the module categories for these algebras are well understood, it is important to establish links between geometric properties of $\mathrm{Mod}(A, n)$ and the representation theory of A. In this paper, we take steps towards achieving this.

Let L be an $n(d-1)$-dimensional A-module. We define $\mathrm{Mod}(A, n)_L$ to be the set of points $x \in \mathrm{Mod}(A, n)$ such that there exists a short exact sequence of the form

$$0 \longrightarrow L \longrightarrow A^n \longrightarrow K_x \longrightarrow 0.$$

If $x \in \mathrm{Mod}(A, n)_L$ then we say that K_x has *type* L. By Schanuel's lemma and Krull-Remak-Schmidt's theorem, the type of an A-module is unique up to isomorphism. Furthermore, every n-dimensional A-module is a quotient of A^n. Hence it follows that $\mathrm{Mod}(A, n)$ decomposes as the disjoint union $\dot\bigcup_{[L] \in \mathcal{S}_A(n)} \mathrm{Mod}(A, n)_L$, where $\mathcal{S}_A(n)$ denotes the set of isomorphism classes $[L]$ of $n(d-1)$-dimensional submodules of A^n. We occasionally refer to L as an *index module*. Our main results are as follows.

Theorem 1. *Suppose* $[L] \in \mathcal{S}_A(n)$. *Then* $\mathrm{Mod}(A, n)_L$ *is a smooth, irreducible locally closed subset of* $\mathrm{Mod}(A, n)$ *of dimension* $\dim \mathrm{Hom}_A(L, A^n) - \dim \mathrm{End}_A(L)$.

The results in this paper form part of the author's Ph.D. Thesis, University of Leeds. The author would like to thank her supervisor W. Crawley-Boevey for his advice, J. Schröer for many helpful discussions and the Sonderforschungsbereich 343 (Universität Bielefeld, Germany) for financial support.

Theorem 2. *Let* $[L]$ *and* $[L'] \in \mathcal{S}_A(n)$. *If* $\mathrm{Mod}(A,n)_{L'}$ *is contained in the closure of* $\mathrm{Mod}(A,n)_L$ *then* $L \leq_{\mathrm{deg}} L'$. *If* $L \leq_{\mathrm{deg}} L'$ *and* $\dim \mathrm{Hom}_A(L,A) = \dim \mathrm{Hom}_A(L',A)$ *then* $\mathrm{Mod}(A,n)_{L'}$ *is contained in the closure of* $\mathrm{Mod}(A,n)_L$.

We would like to point out that the second part of Theorem 2 is similar to a result in [3].

We say that an algebra A is *subfinite* if every projective A-module has only finitely many isomorphism classes of submodules. Examples of subfinite algebras include hereditary algebras, finite representation type algebras, radical square zero algebras, tame concealed algebras and tubular algebras. The subfiniteness of these algebras follows either trivially or from [7]. In [8], J. Schröer has shown that string algebras are subfinite and in [1], together with M. Barot, has classified subfinite canonical algebras. If A is a subfinite algebra, then for all $n \in \mathbb{N}$, the set $\mathcal{S}_A(n)$ is finite. Therefore as corollaries to Theorems 1 and 2 respectively, we have the following results.

Corollary 3. *Suppose A is a subfinite algebra. Then the dimension of* $\mathrm{Mod}(A,n)$ *is* $\max_{[L]\in\mathcal{S}_A(n)}\{\dim \mathrm{Hom}_A(L,A^n) - \dim \mathrm{End}_A(L)\}$.

Corollary 4. *Suppose A is a subfinite algebra. Then the number of irreducible components of* $\mathrm{Mod}(A,n)$ *is at most the cardinality of* $\mathcal{S}_A(n)$ *and at least the number of minimum elements in* $\mathcal{S}_A(n)$ *with respect to* $\leq_{\mathrm{deg}}$.

2. Properties of a Stratum

The main aim of this section is to prove Theorem 1. Our strategy is as follows. Let $\{q_1,\dots,q_n\}$ be the free module basis of A^n, where $q_i \in A^n$ is the tuple with a 1_A in the ith place and zeros elsewhere. We define $\mathrm{Span}(A,n)$ to be the set of points $\phi \in \mathrm{Surj}_K(A^n,K^n)$ such that $\mathrm{Ker}\,\phi$ is a submodule of A^n and the set $\{\phi(q_1),\dots,\phi(q_n)\}$ spans K^n. Suppose $[L] \in \mathcal{S}_A(n)$. Then we define $\mathrm{Span}(A,n)_L$ to be the set of points $\phi \in \mathrm{Span}(A,n)$ such that $\mathrm{Ker}\,\phi \cong L$ and $\mathrm{Inj}\,\mathrm{Span}(A,n)_L$ to be the set of points $(\theta,\phi) \in \mathrm{Inj}_A(L,A^n) \times \mathrm{Span}(A,n)$ such that $\phi\theta = 0$.

We prove that $\mathrm{Span}(A,n)$ and $\mathrm{Mod}(A,n) \times \mathrm{GL}_n(K)$ are isomorphic varieties and that $\mathrm{Span}(A,n)_L$ is a locally closed subset of $\mathrm{Span}(A,n)$ which is isomorphic to $\mathrm{Mod}(A,n)_L \times \mathrm{GL}_n(K)$. We then determine the properties of $\mathrm{Inj}\,\mathrm{Span}(A,n)_L$ and show that the projection map $\mathrm{Inj}\,\mathrm{Span}(A,n)_L \to \mathrm{Span}(A,n)_L$ is an open morphism of varieties with fibres isomorphic to $\mathrm{Aut}_A(L)$. We use this morphism to determine properties of $\mathrm{Span}(A,n)_L$ and hence of $\mathrm{Mod}(A,n)_L$.

Lemma 5. *The set* $\mathrm{Span}(A,n)$ *is a locally closed subset of* $\mathrm{Surj}_K(A^n,K^n)$.

Proof. If $\phi \in \mathrm{Span}(A,n)$, then any vector $v \in K^n$ can be expressed as a K-linear combination, say $v = \sum k_i\phi(q_i)$ and as ϕ is K-linear, $v = \phi(\sum k_i q_i)$, hence $\phi \in \mathrm{Surj}_K(A^n,K^n)$. Since $\{\phi(q_1),\dots,\phi(q_n)\}$ is a spanning set for K^n, the matrix $(\phi(q_1)\cdots\phi(q_n))$ whose ith column is given by $\phi(q_i)$ is invertible and therefore has a non zero determinant. This is an open condition in $\mathrm{Surj}_K(A^n,K^n)$. Let $\mathrm{Gr}\binom{A^n}{n}$ denote the Grassmannian of subspaces of A^n of codimension n and let $\mathrm{Gr}_A\binom{A^n}{n}$ denote the subset of $\mathrm{Gr}\binom{A^n}{n}$ containing the points U endowed with the structure of an A-submodule of A^n. Now $\mathrm{Ker}\,\phi$ is an A-submodule if and only if ϕ is in the pre-image of $\mathrm{Gr}_A\binom{A^n}{n}$ under the morphism $\mathrm{Surj}_K(A^n,K^n) \to \mathrm{Gr}\binom{A^n}{n}$, sending ϕ to $\mathrm{Ker}\,\phi$. This is a closed condition in $\mathrm{Surj}_K(A^n,K^n)$ since $\mathrm{Gr}_A\binom{A^n}{n}$ is a closed subset of $\mathrm{Gr}\binom{A^n}{n}$. We conclude that $\mathrm{Span}(A,n)$ is the intersection of an open subset of $\mathrm{Surj}_K(A^n,K^n)$ with a closed subset of $\mathrm{Surj}_K(A^n,K^n)$ and is therefore locally closed in $\mathrm{Surj}_K(A^n,K^n)$. $\square$

Suppose $(x,g) \in \mathrm{Mod}(A,n) \times \mathrm{GL}_n(K)$, then there is a surjective K-linear map $\phi_{x,g}\colon A^n \to K^n$ sending a_iq_i to $x(a_i)ge_i$, where e_i is the n-tuple with a 1 in the

ith place and zeros elsewhere. In fact, $\phi_{x,g} \in \mathrm{Surj}_A(A^n, K_x)$ so $\mathrm{Ker}\,\phi_{x,g}$ is an $n(d-1)$-dimensional submodule of A^n. Furthermore, since x is a K-algebra homomorphism, $x(1_A) = I_n$, where I_n denotes the identity element of $\mathrm{M}_n(K)$. Hence $\{\phi_{x,g}(q_1), \ldots, \phi_{x,g}(q_n)\}$ is a spanning set for K^n. So $\phi_{x,g} \in \mathrm{Span}(A, n)$. Suppose $\phi \in \mathrm{Span}(A, n)$, then there is a point $(\bar{\phi}, \phi^*) \in \mathrm{Mod}(A, n) \times \mathrm{GL}_n(K)$, where $\bar{\phi}(a)v = \phi a \phi^{-1}(v)$, $a \in A$, $v \in K^n$ and ϕ^* is the $n \times n$ matrix $(\phi(q_1) \cdots \phi(q_n))$. By $\phi^{-1}(v)$, we mean any preimage of v. It is straighforward to show that $\bar{\phi}$ is well defined. Moreover, ϕ has the structure of a surjective A-module homomorphism from A^n to $K_{\bar{\phi}}$.

Proposition 6. *The varieties* $\mathrm{Mod}(A, n) \times \mathrm{GL}_n(K)$ *and* $\mathrm{Span}(A, n)$ *are isomorphic.*

To prove the proposition, we define maps $\rho\colon \mathrm{Mod}(A, n) \times \mathrm{GL}_n(K) \to \mathrm{Span}(A, n)$ and $\mu\colon \mathrm{Span}(A, n) \to \mathrm{Mod}(A, n) \times \mathrm{GL}_n(K)$ based on the above constructions and show that they are morphisms of varieties with the property $\rho\mu$ and $\mu\rho$ are identities.

Lemma 7. *The map* $\rho : \mathrm{Mod}(A, n) \times \mathrm{GL}_n(K) \to \mathrm{Span}(A, n)$ *sending* (x, g) *to* $\phi_{x,g}$ *is a morphism of varieties.*

Proof. It suffices to show that ρ can be expressed as a polynomial map. We adopt the following notation. The respective K-bases for A^n and $M_n(K)$ will be given by $\{e_i q_j\}$ and $\{E_{jk}\}$, $i = 1, \ldots, d$ and $j, k = 1, \ldots, n$, where E_{ij} is the matrix with a 1 in the ijth place and zeros elsewhere. The element $a_i \in A$ will be the d-tuple $(a_{i1}, \ldots, a_{id})$ and so $(a_1, \ldots, a_n)$ is the nd-tuple $(a_{11}, \ldots, a_{1d}, \ldots, a_{n1}, \ldots, a_{nd})$. The points $x \in \mathrm{Mod}(A, n)$ and $g \in \mathrm{GL}_n(K)$ will be the matrices with ijth entries x_{ij} and g_{ij} respectively and $\phi_{x,g} \in \mathrm{Span}(A, n)$ will be expressed as an n-tuple of matrices $(\phi_{x,g_1}, \ldots, \phi_{x,g_n})$, where each $\phi_{x,g_i}\colon A \to K^n$ is an $n \times d$ matrix.

So the matrix $x(a_i)$ will have kmth entry $\sum_{j=1}^{d} x_{(k-1)n+m,j} a_{ij}$. Thus $x(a_i)g$ will have klth entry $\sum_{m=1}^{n} \sum_{j=1}^{d} x_{(k-1)n+m,j} a_{ij} g_{ml}$ and ith column $\phi_{x,g}(a_i q_i)$ given by

$$\begin{pmatrix} \sum_{j=1}^{d} \sum_{m=1}^{n} x_{mj} g_{mi} a_{ij} \\ \vdots \\ \sum_{j=1}^{d} \sum_{m=1}^{n} x_{(k-1)n+m,j} g_{mi} a_{ij} \\ \vdots \\ \sum_{j=1}^{d} \sum_{m=1}^{n} x_{(n-1)n+m,j} g_{mi} a_{ij} \end{pmatrix}.$$

So ϕ_{x,g_i} is the matrix with klth entry $\sum_{m=1}^{n} x_{(k-1)n+m,l} g_{mi}$. We conclude that ρ is a morphism of varieties. □

Lemma 8. *The map* $\mu\colon \mathrm{Span}(A, n) \to \mathrm{Mod}(A, n) \times \mathrm{GL}_n(K)$ *sending* ϕ *to* $(\bar{\phi}, \phi^*)$ *is a morphism of varieties.*

Proof. We extend the free module basis $\{q_1, \ldots, q_n\}$ to a K-basis for A^n. Thus ϕ can be expressed as an $n \times nd$ matrix, where the first n columns form the invertible matrix ϕ^*. We define left multiplication of an element in A^n by an element in A by a K-algebra homomorphism $L\colon A \to \mathrm{End}_A(A^n)$. So $\bar{\phi}(a) = \phi L(a) \begin{pmatrix} I_n \\ 0 \end{pmatrix} \phi^{*-1}$ for $a \in A$. The rest of the proof is now similar to that of Lemma 7. □

Proof of Proposition 6. It suffices to show that $\rho\mu$ and $\mu\rho$ are identity morphisms. Suppose $\phi \in \mathrm{Span}(A, n)$. Then $\mu(\phi) = (\bar{\phi}, \phi^*)$. So

$$\rho\mu(\phi)(a_i q_i) = \rho\bar{\phi}(a_i)\phi^* e_i = \phi a_i \phi^{-1} \phi(q_i) = \phi(a_i q_i).$$

On the other hand, suppose $(x, g) \in \mathrm{Mod}(A, n) \times \mathrm{GL}_n(K)$. Then $\rho(x, g) = \phi_{x,g}$, where $\phi_{x,g}(a_i q_i) = x(a_i) g e_i$. So $\mu(\phi_{x,g}) = (\bar{\phi}_{x,g}, \phi^*_{x,g})$. Now $\{ge_1, \ldots, ge_n\}$ is a

K-basis for K^n, so we can express every vector $v \in K^n$ as a K-linear combination $\sum k_i g(e_i) = \sum k_i \phi_{x,g}(q_i)$. Therefore

$$\bar{\phi}_{x,g}(a)v = \phi_{x,g} a {\phi_{x,g}}^{-1}(\sum_{i=1}^{n} k_i \phi_{x,g}(q_i)) = \phi_{x,g}(\sum_{i=1}^{n} a k_i q_i) = \sum_{i=1}^{n} x(a) k_i g e_i = x(a)v$$

and $\phi_{x,g}(q_i) = ge_i$ so $\phi^* = g$. $\square$

Suppose $(x,g) \in \mathrm{Mod}(A,n)_L \times \mathrm{GL}_n(K)$, then $\rho(x,g) = \phi \in \mathrm{Surj}_A(A^n, K_x)$. So by definition, $\mathrm{Ker}\,\phi \cong L$ and hence $\phi \in \mathrm{Span}(A,n)_L$. Conversely, suppose $\phi \in \mathrm{Span}(A,n)_L$. Then there exists an exact sequence of the form

$$0 \longrightarrow L \longrightarrow A^n \xrightarrow{\phi} K_{\bar{\phi}} \longrightarrow 0$$

so $\bar{\phi} \in \mathrm{Mod}(A,n)_L$. We conclude that $\mathrm{Mod}(A,n)_L \times \mathrm{GL}_n(K)$ and $\mathrm{Span}(A,n)_L$ are isomorphic as sets.

In order to prove that $\mathrm{Span}(A,n)_L$ is locally closed in $\mathrm{Span}(A,n)$, we define $\mathrm{Inj}\,\mathrm{Span}(A,n)$ to be the closed set of points $(\theta,\phi) \in \mathrm{Inj}_K(K^{n(d-1)}, A^n) \times \mathrm{Span}(A,n)$ such that $\phi\theta = 0$. If $(\theta,\phi) \in \mathrm{Inj}\,\mathrm{Span}(A,n)$, then there exists a short exact sequence of the form

$$0 \longrightarrow K^{n(d-1)} \xrightarrow{\theta} A^n \xrightarrow{\phi} K^n \longrightarrow 0,$$

where θ is K-linear and $\phi \in \mathrm{Span}(A,n)$. Note that since $\phi \in \mathrm{Span}(A,n)$, there is an induced A-module structure $\bar{\theta}$ on $K^{n(d-1)}$, where $\bar{\theta}(a)v = \theta^{-1}a\theta(v)$. Again, it is straightforward to check that $\bar{\theta}$ is well defined. Moreover, θ has the structure of an injective A-module homomorphism from $K_{\bar{\theta}}$ to A^n.

Lemma 9. *The map* $\Theta\colon \mathrm{Inj}\,\mathrm{Span}(A,n) \to \mathrm{Mod}(A,n(d-1))$ *sending* (θ,ϕ) *to* $\bar{\theta}$ *is a morphism of varieties.*

Proof. We fix K-bases $\{q_1,\dots,q_n,b_{n+1},\dots,b_{nd}\}$ for A^n as in the proof of Lemma 8 and $\{b_{n+1},\dots,b_{nd}\}$ for $K^{n(d-1)}$. So every $(\theta,\phi) \in \mathrm{Inj}\,\mathrm{Span}(A,n)$ can be expressed as a tuple of matrices of the form $\left(\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, (\phi_1\ \phi_2)\right)$, where $\theta_{1,2}$ and $\phi_{1,2}$ are matrices of sizes $n \times n(d-1)$, $n(d-1) \times n(d-1)$, $n \times n$ and $n \times n(d-1)$ respectively. Since $\phi \in \mathrm{Span}(A,n)$, by choice of basis, ϕ_1 is invertible. Furthermore, $\phi_1\theta_1 + \phi_2\theta_2 = 0$, i.e. $\theta_1 = -\phi^{-1}\phi_2\theta_2$ and as θ is injective, θ_2 is invertible. So for all $a \in A$, it follows that $\bar{\theta}(a) = \theta_2^{-1}(0\ I_{n(d-1)})L(a)\theta$, where $L\colon A \to \mathrm{End}_A(A^n)$ is defined in the proof of Lemma 8. The rest of the proof is similar to that of Lemma 7. $\square$

There is an action of $\mathrm{GL}_{n(d-1)}(K)$ on $\mathrm{Inj}\,\mathrm{Span}(A,n)$ via $g \bullet (\theta,\phi) = (\theta g^{-1}, \phi)$ and the set of $\mathrm{GL}_{n(d-1)}(K)$-orbits in $\mathrm{Inj}\,\mathrm{Span}(A,n)$ are in one to one correspondence with the set of $\mathrm{GL}_{n(d-1)}(K)$-orbits in $\mathrm{Mod}(A,n(d-1))$.

Lemma 10. *The projection map* $\pi\colon \mathrm{Inj}\,\mathrm{Span}(A,n) \to \mathrm{Span}(A,n)$ *sends closed* $\mathrm{GL}_{n(d-1)}(K)$*-stable sets to closed sets.*

Proof. Let X denote the set of points $(\theta,\phi) \in \mathrm{Hom}_K(K^{n(d-1)}, A^n) \times \mathrm{Span}(A,n)$ such that $\phi\theta = 0$. The projection map $X \to \mathrm{Span}(A,n)$, is a vector bundle of rank $n^2(d-1)^2$ and so π is locally trivial with fibres isomorphic to $\mathrm{GL}_{n(d-1)}(K)$. The result now follows. $\square$

Suppose $x \in \mathrm{Mod}(A,n(d-1))$ such that $[K_x] \in \mathcal{S}_A(n)$. Then $\mathrm{Span}(A,n)_{K_x \leq_{\deg}}$ and $\mathrm{Span}(A,n)_{K_x <_{\deg}}$ will denote the set of points $\phi \in \mathrm{Span}(A,n)$ such that $\mathrm{Ker}\,\phi$ is a degeneration respectively strict degeneration of K_x.

Lemma 11. *Suppose* $x \in \mathrm{Mod}(A,n(d-1))$ *such that* $[K_x] \in \mathcal{S}_A(n)$. *Then the sets* $\mathrm{Span}(A,n)_{K_x \leq_{\deg}}$ *and* $\mathrm{Span}(A,n)_{K_x <_{\deg}}$ *are closed in* $\mathrm{Span}(A,n)$.

Proof. We note the following equalities:

1. $\mathrm{Span}(A,n)_{K_x \leq_{\deg}} = \pi\Theta^{-1}(\overline{\mathcal{O}(x)})$;

2. $\operatorname{Span}(A,n)_{K_x <_{\deg}} = \pi\Theta^{-1}(\overline{\mathcal{O}(x)}\backslash\mathcal{O}(x))$.

It is straightforward to show that the preimages $\Theta^{-1}(\overline{\mathcal{O}(x)})$ and $\Theta^{-1}(\overline{\mathcal{O}(x)}\backslash\mathcal{O}(x))$ are $\mathrm{GL}_{n(d-1)}(K)$-stable closed subsets of $\operatorname{Inj}\operatorname{Span}(A,n)$. The result now follows from Lemma 10. □

Lemma 12. *If $x \in \operatorname{Mod}(A,n(d-1))$ such that $[K_x] \in \mathcal{S}_A(n)$, then $\operatorname{Span}(A,n)_{K_x}$ is a non empty, locally closed subset of $\operatorname{Span}(A,n)$.*

Proof. $\operatorname{Span}(A,n)_{K_x} = \pi\Theta^{-1}(\mathcal{O}(x)) = \operatorname{Span}(A,n)_{K_x \le_{\deg}} \backslash \operatorname{Span}(A,n)_{K_x <_{\deg}}$. The result now follows from Lemma 11. □

The next few results will be used to determine several geometric properties of $\operatorname{Span}(A,n)_L$ for any $[L] \in \mathcal{S}_A(n)$.

Lemma 13. *Suppose $[L] \in \mathcal{S}_A(n)$. Then $\operatorname{Inj}\operatorname{Span}(A,n)_L$ is a smooth, irreducible variety of dimension $\dim \operatorname{Hom}_A(L,A^n) + n^2$.*

Proof. Let X be the subset of points $(\theta,\phi) \in \operatorname{Inj}_A(L,A^n) \times \operatorname{Hom}_K(A^n,K^n)$ such that $\phi\theta = 0$. Now X is a closed subset of $\operatorname{Inj}_A(L,A^n) \times \operatorname{Hom}_K(A^n,K^n)$ and any fibre over the projection map $\pi\colon X \to \operatorname{Inj}_A(L,A^n)$ can be identified with $\mathrm{M}_n(K)$. So π is a vector bundle of rank n^2 and hence X is a smooth, irreducible variety of dimension $\dim \operatorname{Hom}_A(L,A^n) + n^2$. Since $\operatorname{Inj}\operatorname{Span}(A,n)_L$ is a dense open subset of X, the result follows. □

Lemma 14. *Let E and X be irreducible varieties and $\pi\colon E \to X$ a vector bundle. Suppose U is a smooth open subset of E, such that $\pi(U) = X$. Then X is smooth.*

Proof. Without loss of generality, we can assume that $\pi\colon X \times V \to X$ is trivial for some vector space V. Suppose $(x,v) \in U$. Then π induces a K-linear map $\mathrm{d}_{(x,v)}\,\pi\colon \mathrm{T}_{(x,v)}\,U \to \mathrm{T}_x\,X$ on tangent spaces. Since U is a smooth open subset of $X \times V$, it follows that $\mathrm{T}_{(x,v)}\,U \cong \mathrm{T}_x\,X \oplus V$ and $\dim \mathrm{T}_x\,X \oplus V = \dim X + \dim V$. We conclude that $\dim \mathrm{T}_x\,X = \dim X$. Hence X is smooth. □

Lemma 15. *Let $[L] \in \mathcal{S}_A(n)$. The projection map $\operatorname{Inj}\operatorname{Span}(A,n)_L \to \operatorname{Span}(A,n)_L$ is an open morphism of smooth varieties with fibres isomorphic to $\operatorname{Aut}_A(L)$.*

Proof. Let Y be the subset of points $(\theta,\phi) \in \operatorname{Hom}_A(L,A^n) \times \operatorname{Span}(A,n)$ such that $\phi\theta = 0$. Now Y is a closed subset of $\operatorname{Hom}_A(L,A^n) \times \operatorname{Span}(A,n)$ and any fibre over the projection morphism $\pi'\colon Y \to \operatorname{Span}(A,n)_L$ can be identified with $\operatorname{End}_A(L)$. So π' is a vector bundle of rank $\dim \operatorname{End}_A(L)$ and is thus an open morphism of varieties. The result now follows from Lemma 14 and the fact that $\operatorname{Inj}\operatorname{Span}(A,n)_L$ is a dense open subset of Y. □

As a consequence of Proposition 6 and Lemmas 12, 13 and 15, we have the following result from which Theorem 1 follows.

Lemma 16. *Suppose $[L] \in \mathcal{S}_A(n)$. Then $\operatorname{Span}(A,n)_L$ is a smooth, irreducible variety isomorphic to $\operatorname{Mod}(A,n)_L \times \mathrm{GL}_n(K)$ of dimension*

$$\dim \operatorname{Hom}_A(L,A^n) - \dim \operatorname{End}_A(L) + n^2.$$

3. The Closure of a Stratum

It is an interesting problem to characterize the closure of $\operatorname{Mod}(A,n)_L$. As in the case of orbit closures, one would hope that the closure of $\operatorname{Mod}(A,n)_L$ can be expressed as a disjoint union of some $\operatorname{Mod}(A,n)_{L'}$'s. Unfortunately, trivial examples show that this is not necessarily the case.

Proof of Theorem 2. Let $[L]$ and $[L'] \in \mathcal{S}_A(n)$ such that $\operatorname{Mod}(A,n)_{L'}$ is contained in the closure of $\operatorname{Mod}(A,n)_L$. By Propsition 6, it follows that $\overline{\operatorname{Mod}(A,n)_L} \times \mathrm{GL}_n(K) \cong \overline{\operatorname{Span}(A,n)_L}$ and also that $\operatorname{Span}(A,n)_{L'}$ is contained in the closure of

$\mathrm{Span}(A,n)_L$. By Lemma 11, $\mathrm{Span}(A,n)_{L\leq_{\deg}}$ is closed and since $L \leq_{\deg} L$, it contains $\mathrm{Span}(A,n)_L$, $\overline{\mathrm{Span}(A,n)_L}$ and hence $\mathrm{Span}(A,n)_{L'}$. Therefore $L \leq_{\deg} L'$. This proves the first part of Theorem 2.

Let $X \subseteq \mathrm{Mod}(A, n(d-1))$ be the set of points x such that $L \leq_{\deg} K_x$ and $\dim \mathrm{Hom}(L,A) = \dim \mathrm{Hom}(K_x, A)$ and let $D \subseteq X \times \mathrm{Hom}_K(K^{n(d-1)}, A^n)$ be the closed subset of (x,θ) such that $\theta \in \mathrm{Hom}_A(K_x, A^n)$.

The conditions $L \leq_{\deg} K_x$ and $\dim \mathrm{Hom}(K_x, A) = \dim \mathrm{Hom}(L,A)$ for all $x \in X$ are closed and locally closed in $\mathrm{Mod}(A, n(d-1))$ respectively, the latter since the function $\mathrm{Mod}(A, n(d-1)) \to \mathbb{N}$ sending x to $\dim \mathrm{Hom}(K_x, A)$ is upper semicontinuous. Therefore X is a locally closed subset of $\mathrm{Mod}(A, n(d-1))$. Furthermore, X contains $\mathcal{O}(L)$ as a dense open subset so is irreducible. It is straightforward to show that the projection map $\pi\colon D \to X$ is a vector bundle of rank $\dim \mathrm{Hom}_A(L, A^n)$ and so D is also an irreducible variety.

Let $U \subseteq D$ be the open subset of points (x,θ) such that $\theta \in \mathrm{Inj}_A(K_x, A^n)$. Now let $Z \subseteq U \times \mathrm{Hom}_K(A^n, K^n)$ be the closed subset of points $((x,\theta),\phi)$ such that $\phi\theta = 0$. Similarly the projection map $\xi : Z \to U$ is a vector bundle and so Z is an irreducible variety. Suppose $\mathrm{Inj}_A(L', A^n)$ is non empty. Then U is a dense open subset of D. Since $\pi^{-1}(\mathcal{O}(L))$ is open and dense in D, the intersection $U \cap \pi^{-1}(\mathcal{O}(L))$ is non empty. So $\mathrm{Inj}_A(L, A^n)$ is non empty. Let $U' \subseteq Z$ be the set of points $((x,\theta),\phi)$ such that $\{\phi(q_i)\}$ spans K^n, hence $\phi \in \mathrm{Span}(A,n)_{K_x}$. This is an open condition so U' is a dense open subset of Z and thus irreducible. We consider the projection map $\xi|_{U'}\colon U' \to \overline{\xi(U')}$, sending $((x,\theta),\phi)$ to ϕ. As U' is irreducible, it follows that $\overline{\xi(U')}$ is also irreducible. Therefore the set $U'' \subseteq U'$ of $((x,\theta),\phi)$, such that $(x,\theta) \in \pi^{-1}(\mathcal{O}(L))$, is open and dense in U'. Hence the restriction map $\xi|_{U''}$ is dominant with image $\mathrm{Span}(A,n)_L$. Thus $\overline{\xi(U')}$ is the closure of $\mathrm{Span}(A,n)_L$. Furthermore $\overline{\xi(U')} \supseteq \mathrm{Span}(A,n)_{K_x}$ for all $x \in X$. We conclude that if L' and L satisfy the hypothesis, then the closure of $\mathrm{Span}(A,n)_L$ contains $\mathrm{Span}(A,n)_{L'}$. □

4. Applications

In this section, we use the stratification introduced in Section 1 to determine properties of $\mathrm{Mod}(A,n)$, where A is a subfinite algebra. Similarly, for an arbitrary algebra, we determine properties of the variety of n-dimensional A-modules with projective dimension at most 1, which we denote by $\mathrm{Pdone}(A,n)$. If this variety is non empty, then we can obtain lower bounds on the dimension and the number of irreducible components of $\mathrm{Mod}(A,n)$.

4.1. Subfinite Algebras.

Recall that A is a subfinite algebra if and only if every projective A-module has only finitely many isomorphism classes of submodules.

If A is a subfinite algebra, then the set $\mathcal{S}_A(n)$ is finite. Therefore, $\mathrm{Mod}(A,n)$ decomposes as the finite disjoint union $\dot\bigcup_{[L]\in\mathcal{S}_A(n)} \mathrm{Mod}(A,n)_L$. The dimension of $\mathrm{Mod}(A,n)$ is given by the maximum dimension of the irreducible components of $\mathrm{Mod}(A,n)$ and since A is subfinite, each irreducible component is the closure of one stratum. Thus $\dim \mathrm{Mod}(A,n) = \max_{[L]\in\mathcal{S}_A(n)}\{\dim \mathrm{Hom}_A(L, A^n) - \dim \mathrm{End}_A(L)\}$.

Let $[L] \in \mathcal{S}_A(n)$. The isomorphism class $[L]$ is a minimum element of $\mathcal{S}_A(n)$ if for all $[M] \in \mathcal{S}_A(n)$ such that $M \leq_{\deg} L$ then $[M] = [L]$. Likewise, $[L]$ is a maximum element of $\mathcal{S}_n$ if for all $[M] \in \mathcal{S}_A(n)$ such that $L \leq_{\deg} M$ then $[M] = [L]$. Of course there may be several minimum and maximum elements and they may even coincide. For example, if A is hereditary, then every element of $\mathcal{S}_A(n)$ is both a minimum and a maximum.

Let $\mathcal{S}_A(n) = \{[L_1], \dots, [L_r]\}$. Without loss of generality, we assume that the elements of $\mathcal{S}_A(n)$ are ordered in such a way that if $L_i \leq_{\deg} L_j$ then $i \leq j$.

Lemma 17. *Suppose that L_i degenerates to $L_{i+1}, \dots, L_{i+r}$ and no other L_j. Then the union $\mathrm{Mod}(A,n)_{L_i} \,\dot\cup \cdots \dot\cup\, \mathrm{Mod}(A,n)_{L_{i+r}}$ is closed in $\mathrm{Mod}(A,n)$.*

Proof. By Lemma 11 $\mathrm{Span}(A,n)_{L_i \leq_{\deg}} = \mathrm{Span}(A,n)_{L_i} \,\dot\cup \cdots \dot\cup\, \mathrm{Span}(A,n)_{L_{i+r}}$ is closed in $\mathrm{Span}(A,n)$. The result now follows from Proposition 6. □

As consequences of the previous lemma, we have the following results.

Lemma 18. *Suppose* $[L_i]$ *is a maximum element of* $\mathcal{S}_n$*. Then* $\mathrm{Mod}(A,n)_{L_i}$ *is a closed subset of* $\mathrm{Mod}(A,n)$*.*

Lemma 19. *Suppose* $[L_i]$ *is a minimum element of* $\mathcal{S}_n$*. Then* $\mathrm{Mod}(A,n)_{L_i}$ *is an open subset of* $\mathrm{Mod}(A,n)$*.*

Proof. By Lemma 17, both $\dot\bigcup_{j=1}^{i-1} \mathrm{Mod}(A,n)_{Lj}$ and $\dot\bigcup_{j=i+1}^{s} \mathrm{Mod}(A,n)_{L_j}$ are closed subsets of $\mathrm{Mod}(A,n)$. So $\mathrm{Mod}(A,n)_{L_i}$ is open. □

If $\mathrm{Mod}(A,n)_{L_i}$ is open in $\mathrm{Mod}(A,n)$, then $\overline{\mathrm{Mod}(A,n)_{L_i}}$ is an irreducible component. Hence the previous lemma (and also Theorem 2) gives a lower bound on the number of irreducible components of $\mathrm{Mod}(A,n)$. Recall that if A is hereditary, then every element of $\mathcal{S}_n$ is a minimum. Therefore the upper and lower bounds on the number of irreducible components of $\mathrm{Mod}(A,n)$ are equal. In other words, these bounds are attainable.

4.2. **Existence of Modules of a given Type.** In this section, we give an iterative method for computing the elements of $\mathcal{S}_A(n)$ in the case where A is a subfinite algebra.

Suppose A is subfinite. Then we let $\mathcal{M}$ be the class of A-modules consisting of the projectives and any submodule of an A-module in $\mathcal{M}$. We also insist that up to isomorphism there are only finitely many modules in $\mathcal{M}$ of any given dimension vector. Since A is subfinite, the number of isomorphism classes in $\mathcal{M}$ is finite. Let M, N and $L \in \mathcal{M}$. We define $\mathrm{Hom}(M,N)_L$ to be the set of $\theta \in \mathrm{Hom}_A(M,N)$ such that $\mathrm{Im}\,\theta \cong L$.

Lemma 20. $\mathrm{Hom}(M,N)_L$ *is a constructible set with irreducible closure and if non empty, has dimension* $\dim \mathrm{Hom}_A(M,L) + \dim \mathrm{Hom}_A(L,N) - \dim \mathrm{End}_A(L)$*.*

Proof. The map $\mathrm{Surj}_A(M,L) \times \mathrm{Inj}_A(L,N) \to \mathrm{Hom}_A(M,N)$, sending (θ,ϕ) to $\phi\theta$ is a morphism of varieties with image $\mathrm{Hom}(M,N)_L$. Therefore $\mathrm{Hom}(M,N)_L$ is constructible with irreducible closure. The fibre over $\psi \in \mathrm{Hom}(M,N)_L$ can be identified with $\mathrm{Inj}_A(L, \mathrm{Im}\,\psi)$ which is identifiable with $\mathrm{Aut}_A(L)$. The result now follows. □

Of course if either $\mathrm{Surj}_A(M,L)$ or $\mathrm{Inj}_A(L,N)$ is empty, then $\mathrm{Hom}(M,N)_L$ is also empty. In this case we define the dimension of $\mathrm{Hom}(M,N)_L$ to be $-\infty$. If A is a subfinite algebra, then N has a finite number of submodules up to isomorphism. We will denote the set of isomorphism classes of submodules $L \subseteq N$ such that $\mathrm{Surj}_A(M,L)$ is non empty by $\mathcal{S}$. Then $\mathrm{Hom}_A(M,N)$ decomposes as the finite disjoint union $\dot\bigcup_{[L]\in\mathcal{S}} \mathrm{Hom}(M,N)_L$. So there is precisely one element $[L] \in \mathcal{S}$ such that $\mathrm{Hom}(M,N)_L$ contains a dense open subset of $\mathrm{Hom}_A(M,N)$.

Lemma 21. *Suppose A is a subfinite algebra and $M, N \in \mathcal{M}$. Then*

1. *The set* $\mathrm{Surj}_A(M,N)$ *is non empty if and only if for all A-modules L such that* $\underline{\dim}\, L < \underline{\dim}\, N$*, then* $\dim \mathrm{Hom}(M,N)_L < \dim \mathrm{Hom}_A(M,N)$*.*
2. *The set* $\mathrm{Inj}_A(M,N)$ *is non empty if and only if for all A-modules L such that* $\underline{\dim}\, L < \underline{\dim}\, M$*, then* $\dim \mathrm{Hom}(M,N)_L < \dim \mathrm{Hom}_A(M,N)$*.*

Proof. Since $\mathrm{Hom}_A(M,N)$ is irreducible, every non empty open subset is dense. In particular, $\mathrm{Surj}_A(M,N)$ is dense. Therefore $\mathrm{Surj}_A(M,N) = \mathrm{Hom}(M,N)_N$ is non empty if and only if $\dim \mathrm{Hom}(M,N)_N = \dim \mathrm{Hom}_A(M,N)$. This equality holds if and only if $\dim \mathrm{Hom}(M,N)_L < \dim \mathrm{Hom}(M,N)$ for all A-modules L such that $\underline{\dim}\, L < \underline{\dim}\, N$. This proves part 1. The proof of part 2. is similar. □

The previous lemma gives an iterative method for determining the existence of an injective A-module homomorphism from L to A^n and hence a method for determining the elements of $\mathcal{S}_A(n)$.

4.3. **The Variety of Modules with Projective Dimension at most 1.** We let $\operatorname{Pdone}(A,n)$ denote the set of points $x \in \operatorname{Mod}(A,n)$ such that pdim $K_x \leq 1$. $\operatorname{Pdone}(A,n)$ can also be described as the set of points $x \in \operatorname{Mod}(A,n)$ such that $\operatorname{Ext}^2(K_x, A/\operatorname{Rad} A) = 0$ and since the function $\operatorname{Mod}(A,n) \to \mathbb{N}$, sending x to $\dim \operatorname{Ext}^2(K_x, A/\operatorname{Rad} A)$ is upper semicontinuous, it follows that $\operatorname{Pdone}(A,n)$ is an open subset of $\operatorname{Mod}(A,n)$. Moreover, for any point $x \in \operatorname{Pdone}(A,n)$, we have that $\operatorname{Ext}^2(K_x, K_x) = 0$ so by Proposition 3.7 in [5], $\operatorname{Pdone}(A,n)$ is smooth. We shall see that these facts follow trivially from the stratification of $\operatorname{Mod}(A,n)$.

If $x \in \operatorname{Pdone}(A,n)$, then by definition, there exists a short exact sequence of the form

$$0 \longrightarrow P \longrightarrow A^n \longrightarrow K_x \longrightarrow 0,$$

for some projective A-module of dimension $n(d-1)$. So $x \in \operatorname{Mod}(A,n)_P$. Let $\mathcal{P}_A(n)$ denote the set of $[P] \in \mathcal{S}_A(n)$ such that P is projective. Since A is finite-dimensional, the set $\mathcal{P}_A(n)$ is finite. So $\operatorname{Pdone}(A,n)$ decomposes as the finite disjoint union $\dot\bigcup_{[P]\in\mathcal{P}_A(n)} \operatorname{Mod}(A,n)_P$.

Lemma 22. *Suppose P is an $n(d-1)$-dimensional projective submodule of A^n. Then $\operatorname{Mod}(A,n)_P$ is an open subvariety of $\operatorname{Mod}(A,n)$.*

Proof. Since P is projective, $\operatorname{Ext}^1(P,P) = 0$. So the $\operatorname{GL}_{n(d-1)}(K)$-orbit containing the point p corresponding to P say, is open in $\operatorname{Mod}(A, n(d-1))$. Therefore the preimage of $\mathcal{O}(p)$ under $\Theta\colon \operatorname{Inj}\operatorname{Span}(A,n) \to \operatorname{Mod}(A,n(d-1))$, sending (θ,ϕ) to $\bar\theta$, defined in Lemma 9, is an open subset of $\operatorname{Inj}\operatorname{Span}(A,n)$. The result now follows from Lemma 15. $\square$

Suppose $\mathcal{P}_A(n) = \{[P_1], \dots, [P_r]\}$. Then using the stratification of $\operatorname{Pdone}(A,n)$, we prove the following result.

Proposition 23. *The variety $\operatorname{Pdone}(A,n)$ is a smooth open subset of $\operatorname{Mod}(A,n)$ of dimension $\max_i\{\dim \operatorname{Hom}_A(P_i, A^n) - \dim \operatorname{End}_A(P_i)\}$ and the number of irreducible components of $\operatorname{Pdone}(A,n)$ is r, the cardinality of $\mathcal{P}_A(n)$.*

Proof. By Theorem 1 and Lemma 22, $\operatorname{Pdone}(A,n) = \dot\bigcup_i \operatorname{Mod}(A,n)_{P_i}$ is a smooth open subset of $\operatorname{Mod}(A,n)$ of dimension $\max_i\{\dim \operatorname{Hom}_A(P_i, A^n) - \dim \operatorname{End}_A(P_i)\}$. For any $1 \leq i \leq r$, both $\dot\bigcup_{j=1}^{i-1} \operatorname{Mod}(A,n)_{P_j}$ and $\dot\bigcup_{j=1+1}^{r} \operatorname{Mod}(A,n)_{P_j}$ are open subsets of $\operatorname{Mod}(A,n)$. So $\operatorname{Mod}(A,n)_{P_i}$ must also be a closed subset of $\operatorname{Mod}(A,n)$. $\square$

Since each $\overline{\operatorname{Mod}(A,n)_P}$ is an irreducible component of $\operatorname{Mod}(A,n)$, if there exists an n-dimensional A-module with projective dimension at most 1, then the dimension of $\operatorname{Mod}(A,n)$ is at least the dimension of $\operatorname{Pdone}(A,n)$ and the number of irreducible components of $\operatorname{Mod}(A,n)$ is at least the cardinality of $\mathcal{P}_A(n)$.

5. Examples

In this section we demonstrate that for many subfinite algbras, one can compute the index modules, the dimension of each stratum and in some cases the number of irreducible components of the variety of modules. We illustrate with examples of modules over a path algebras of a quiver with relations of a fixed dimension vector. This will require a reformulation of Theorems 1 and 2.

Let A be a basic K-algebra with $\{e_1, \dots, e_r\}$ a complete set of primitive orthogonal idempotents. If M is an A-module, then the dimension vector of M is the r-tuple (α_i), where $\alpha_i = \dim {}_K e_i M$. Let $\alpha = (\alpha_i)$ be a dimension vector with $\sum \alpha_i = n$. We identify the affine variety $\operatorname{Mod}(A,\alpha)$ of left A-module structures on $K^{\alpha_1} \oplus \cdots \oplus K^{\alpha_r}$ where e_i acts as projection onto the ith factor with the set of points $x \in \mathrm{Mod}(A,n)$ such that $x(e_i)$ is the relevant projection matrix.

The connected algebraic group $\mathrm{GL}(\alpha) = \prod_{i=1}^{r} \mathrm{GL}_{\alpha_i}(K)$ acts on $\mathrm{Mod}(A,\alpha)$ via conjugation. So the set of $\mathrm{GL}(\alpha)$-orbits in $\mathrm{Mod}(A,\alpha)$ is in one to one correspondence with the set of isomorphism classes of A-modules of dimension vector α.

Let L be an A-module of dimension vector $n\delta - \alpha$, where δ is the dimension vector of A. We define $\mathrm{Mod}(A,\alpha)_L$ to be the set of points $x \in \mathrm{Mod}(A,\alpha)$ such that there exists a short exact sequence of the form

$$0 \longrightarrow L \longrightarrow A^n \longrightarrow K_x \longrightarrow 0.$$

So $\mathrm{Mod}(A,\alpha)$ decomposes as the disjoint union $\dot\bigcup_{[L]\in\mathcal{S}_A(\alpha)} \mathrm{Mod}(A,\alpha)_L$, where $\mathcal{S}_A(\alpha)$ denotes the set of isomorphism classes of submodules of A^n of dimension vector $n\delta - \alpha$. As a direct consequence of Theorems 1, 2 and Theorem 2 (Bongartz) in [2], we have the following results.

Proposition 24. *Suppose* $[L] \in \mathcal{S}_A(\alpha)$. *Then* $\mathrm{Mod}(A,\alpha)_L$ *is a smooth, irreducible locally closed subset of* $\mathrm{Mod}(A,\alpha)$ *of dimension*

$$\dim \mathrm{Hom}_A(L, A^n) - \dim \mathrm{End}_A(L) - n^2 + \sum \alpha_i^2.$$

Proposition 25. *Let* $[L]$ *and* $[L'] \in \mathcal{S}_A(\alpha)$. *If* $\mathrm{Mod}(A,\alpha)_{L'}$ *is contained in the closure of* $\mathrm{Mod}(A,\alpha)_L$ *then* $L \leq_{\deg} L'$. *If* $L \leq_{\deg} L'$ *and* $\dim \mathrm{Hom}_A(L,A) = \dim \mathrm{Hom}_A(L',A)$ *then* $\mathrm{Mod}(A,\alpha)_{L'}$ *is contained in the closure of* $\mathrm{Mod}(A,\alpha)_L$.

Since A is basic, it can be written as the quotient of a path algebra $K\mathcal{Q}$ of a quiver $\mathcal{Q}$ by some admissable ideal I in $K\mathcal{Q}$ generated by a set of minimal relations $\mathcal{R}$, see [7] for details. We denote the set of vertices and arrows in $\mathcal{Q}$ by $\mathcal{I}$ and $\mathcal{A}$ respectively. Suppose $i \in \mathcal{I}$. Then P_i and S_i will denote the indecomposable projective A-module, respectively simple A-module at i. Suppose $a \in \mathcal{A}$, then $i(a)$ and $t(a)$ will denote the initial and terminal vertices of a respectively. Note that $\mathrm{Mod}(A,\alpha)$ is isomorphic to the affine variety $\mathrm{Rep}(A,\alpha)$ of representations of $\mathcal{Q}$ satisfying the relations in $\mathcal{R}$.

Let $i,j \in \mathcal{I}$. We denote by r_{ij} the number of relations from i to j in $\mathcal{R}$. Suppose α is a dimension vector for $\mathcal{Q}$. Then $a(\alpha) = \sum_{a\in\mathcal{A}} \alpha_{i(a)}\alpha_{t(a)} - \sum_{i,j\in\mathcal{I}} r_{ij}\alpha_i\alpha_j$. By a generalisation of Krull's Principal Ideal Theorem, it follows that each irreducible component of $\mathrm{Mod}(A,\alpha)$ has dimension at least $a(\alpha)$. Consequently, if A is a subfinite algebra with $\dim \mathrm{Mod}(A,\alpha) = a(\alpha)$, then one can use the stratification to determine the number of irreducible components of $\mathrm{Mod}(A,\alpha)$. For example, in [1], there are many subfinite canonical algebras C and dimension vectors α such that $\dim \mathrm{Mod}(C,\alpha) = a(\alpha)$.

5.1. **Example 1.** This trivial example illustrates that the closure of a stratum is not a union of strata and that degenerations between index modules do not give all the irreducible components.

Let $A = K\mathcal{Q}/I$, where $\mathcal{Q}$ is the Dynkin quiver of type A_3: $1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3$ and $I = \langle a_2a_1 \rangle$. Let $\alpha = (1,1,1)$. There are three isomorphism classes of A-modules of dimension vector α, $\mathcal{M}_1 = [S_1 \oplus S_2 \oplus S_3]$, $\mathcal{M}_2 = [S_1 \oplus P_2]$ $\mathcal{M}_3 = [P_1 \oplus P_3]$ and two strata in $\mathrm{Mod}(A,\alpha)$, indexed by $\mathcal{S}_A(\alpha) = \{[L_1],[L_2]\}$, where $L_1 = A^2 \oplus P_2$ and $L_2 = A^2 \oplus P_3 \oplus S_2$.

With abuse of notation, $\mathrm{Mod}(A,\alpha)_{L_1} = \mathcal{M}_3$ and $\mathrm{Mod}(A,\alpha)_{L_2} = \mathcal{M}_1 \cup \mathcal{M}_2$, but the closure of $\mathrm{Mod}(A,\alpha)_{A^2\oplus P_2}$ is $\mathcal{M}_1 \cup \mathcal{M}_3$ which is not a union of any strata in $\mathrm{Mod}(A,\alpha)$. Note also that $L_1 \leq_{\deg} L_2$ but $\mathrm{Mod}(A,\alpha)_{L_2}$ is not contained in the closure of $\mathrm{Mod}(A,\alpha)_{L_1}$.

5.2. **Example 2.** The purpose of this example is to demonstrate that for some algebras, the stratification gives not only the dimension but also the number of irreducible components.

Let $A = K\mathcal{Q}/I$, where $\mathcal{Q}$ is the double Kronecker quiver $1 \underset{a_2}{\overset{a_1}{\rightrightarrows}} 2 \underset{a_4}{\overset{a_3}{\rightrightarrows}} 3$ and $I = \langle a_3a_1, a_4a_1, a_3a_2, a_4a_2\rangle$. Let $\alpha = (1,2,2)$. There are three strata in $\mathrm{Mod}(A,\alpha)$, indexed by $\mathcal{S}_A(\alpha) = \{[L_i]\}$, $i = 0,1,2$, where $L_i = P_1^4 \oplus P_2^{5-i} \oplus P_3^{3+2i} \oplus S_2^i$. The respective dimensions of $\mathrm{Mod}(A,\alpha)_{L_0}$, $\mathrm{Mod}(A,\alpha)_{L_1}$ and $\mathrm{Mod}(A,\alpha)_{L_2}$ are 4, 7 and 8. Since $L_0 \leq_{\deg} L_1 \leq_{\deg} L_2$, we conclude that $\mathrm{Mod}(A\alpha)$ has dimension 8 and 3 irreducible components.

5.3. **Example 3.** Suppose $A = K[x,y]/(x^3, xy, y^3)$ and $\alpha = 3$. Then $\mathrm{Mod}(A,3)$ is the variety of so called 3-dimensional Gelfand-Ponomarev-modules. Note that A can be described as the quotient of the path algbera of the quiver with one vertex and two loops x,y modulo the admissable ideal generated by the set $\{x^3, xy, yx, y^2\}$ of relations. So A has one indecomposable projective A-module P and one simple A-module S. Note that A is a string algebra and is therefore subfinite.

There are seven strata in $\mathrm{Mod}(A,\alpha)$ which are illustrated in the following table together with the generic structure of their modules and dimensions.

	Index Module	**Generic Structure**	**Dimension**
L_1	$M(x) \oplus P^2$	$M(x^2y^{-1}, \lambda, 1)$	7
L_2	$M(y) \oplus P^2$	$M(xy^{-2}, \lambda, 1)$	7
L_3	$S^2 \oplus P^2$	$M(xy^{-1})$	6
L_4	$M(x) \oplus M(y) \oplus M(xy^{-1}) \oplus P$	$M(x^{-1}y)$	6
L_5	$S \oplus M(x) \oplus M(y)^2 \oplus P$	$S \oplus M(x)$	4
L_6	$S \oplus M(y) \oplus M(x)^2 \oplus P$	$S \oplus M(y)$	4
L_7	$M(x)^3 \oplus M(y)^3$	S^3	0

An explanation of the notation may be found in [4]. The index modules degenerate as follows $\begin{smallmatrix} L_1 \\ L_2 \end{smallmatrix} \leq_{\deg} L_3 \leq_{\deg} L_4 \leq_{\deg} \begin{smallmatrix} L_5 \\ L_6 \end{smallmatrix} \leq_{\deg} L_7$. Thus L_1 and L_2 are the minimum elements in $\mathcal{S}_A(3)$. By careful examination of the diagram in [6], Section 2.9, one can see that the closures of these strata are the only irreducible components of $\mathrm{Mod}(A,3)$.

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Department of Pure Mathematics, University of Leeds, Leeds, LS2 9JT

E-mail address: nicola@amsta.leeds.ac.uk