TENSOR PRODUCT MULTIPLICITIES, CANONICAL BASES AND TOTALLY POSITIVE VARIETIES

ARKADY BERENSTEIN AND ANDREI ZELEVINSKY

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1. Introduction

This paper continues and to some extent concludes the project initiated twelve years ago in [5]. The original goal of this project was to find explicit "polyhedral" combinatorial expressions for multiplicities in the tensor product of two simple finite-dimensional modules over a complex semisimple Lie algebra $\mathfrak{g}$. Here "polyhedral" means that the multiplicity in question is to be expressed as the number of lattice points in some convex polytope, or in more down-to-earth terms, the number of integer solutions of a system of linear equations and inequalities. The tensor product multiplicities are often called generalized Littlewood-Richardson coefficients because for type A_r they are given by the classical Littlewood-Richardson rule (a polyhedral version of the rule was given in [14], and a much more symmetric version was given in [6]). Conjectural polyhedral expressions for these multiplicities in the case of classical Lie algebras were given in [5]. A uniform combinatorial description of these multiplicities (even in a more general context of Kac-Moody algebras) has been obtained by P. Littelmann in [17]; his answer is in terms of the so-called path model. It is however not an easy task to transform this description into a polyhedral one; in particular, it is still not clear how to deduce the conjectural expressions in [5] from Littelmann's results.

In this paper, we explicitly construct a family of polyhedral expressions for tensor product multiplicities for an arbitrary semisimple Lie algebra $\mathfrak{g}$. To be more precise, we associate two such expressions to every reduced word of w_o, the longest element of the Weyl group, and also produce two "universal" expressions which we call *tropical Plücker models*. It can be shown that these expressions include as special cases the conjectural expressions in [5]. As another application, we obtain a family of polyhedral expressions for the multiplicities that occur when one restricts a simple finite-dimensional $\mathfrak{g}$-module to the Levi subalgebra of some parabolic subalgebra in $\mathfrak{g}$. Our answers use a new combinatorial concept of $\mathbf{i}$-trails which resembles Littelmann's paths but seems to be more tractable.

From the beginning, another (and maybe more important) aim of our project was to develop combinatorial understanding of "good" bases in finite-dimensional representations of $\mathfrak{g}$. A preliminary concept of a "good" or "canonical" basis was introduced in [13, 14] and independently in [1] but it was only given a firm mathematical foundation in pioneering works of G. Lusztig and M. Kashiwara on canonical bases in quantum groups. Thus the aim of our project can be more precisely formulated as follows: understand the combinatorial structure of Lusztig's canonical bases or, equivalently of Kashiwara's global bases. Although Lusztig's and Kashiwara's approaches were shown by Lusztig to be equivalent to each other, they lead to different combinatorial parametrizations of the canonical bases. One of the main results of the present paper is an explicit description of the relationship between these parametrizations.

In Lusztig's approach, every reduced word $\mathbf{i}$ for w_o gives rise to a parametrization ("of the PBW-type") of the canonical basis $\mathcal{B}$ by the set $\mathbb{Z}_{\geq 0}^m$ of all m-tuples of nonnegative integers, where $m = \ell(w_o)$ is the number of positive roots. Kashiwara's approach is closely related to another family of parametrizations which we call string parametrizations; they were introduced and studied in [7]. String parametrizations are also associated to reduced words for w_o; but this time every such reduced word $\mathbf{i}$ gives rise to a bijection between $\mathcal{B}$ and the set of all lattice points of some rational polyhedral convex cone $C_\mathbf{i}$ in $\mathbb{R}^m$. In this paper we obtain an explicit description of these cones for an arbitrary semisimple Lie algebra $\mathfrak{g}$ and any reduced word $\mathbf{i}$. Such a description was previously known in some special cases only: in [7], it was given for a special reduced word $(1, 2, 1, 3, 2, 1, \dots)$ for type A; in [18] it was extended to a special choice of a reduced word for any semisimple $\mathfrak{g}$, while in [15] it was given for any reduced word for type A; a more general setting of Kac-Moody algebras was discussed in [23] but the results there were inconclusive.

Our approach to the above problems is based on a remarkable observation by G. Lusztig that combinatorics of the canonical basis is closely related to geometry of the totally positive varieties. We formulate this relationship in terms of two mutually inverse transformations: "tropicalization" and "geometric lifting." The starting point for this is an observation that different parametrizations of the canonical basis are related to each other by piecewise-linear transformations that involve only the operations of addition, subtraction, and taking the minimum of two integers. As in [3], we shall represent such expressions as "tropical" subtraction-free rational expressions. Recall from [3] that a *semifield* K is a set equipped with operations of addition, multiplication and division (but no subtraction!) satisfying the usual field axioms. Two main examples for us will be $\mathbb{R}_{>0}$, the set of positive real numbers with usual operations, and the *tropical semifield* $\mathbb{Z}_{\mathrm{trop}}$, the set of integers with multiplication and addition given by

$$a \odot b = a + b \ , \quad a \oplus b = \min(a, b) \ .$$

We shall write $[Q]_{\mathrm{trop}}$ if we need to emphasize that a subtraction-free rational expression Q is understood in a tropical sense. We call Q a *geometric lifting* of a piecewise-linear expression $[Q]_{\mathrm{trop}}$. Note that a geometric lifting is *not* unique; for example, if Q is a Laurent polynomial with nonnegative integer coefficients then $[Q]_{\mathrm{trop}}$ only depends on the Newton polytope of Q (cf. [3, Proposition 4.1.1]).

In this terminology, Lusztig's observation can be formulated as follows: for $\mathfrak{g}$ of simply-laced type, the transition map between any two Lusztig parametrizations of $\mathcal{B}$ has a geometric lifting which describes the transition between two natural parametrizations of the totally positive variety in the maximal unipotent subgroup of the semisimple group G corresponding to $\mathfrak{g}$. In this paper we extend this result to a non-simply-laced case; and we also find a similar geometric lifting for transition maps between string parametrizations. This geometric lifting allows us to deduce combinatorial properties of the canonical basis from geometric properties of totally positive varieties. To do this, we rely on (and further develop) the "calculus of generalized minors" and its applications to the study of totally positive varieties in Schubert cells and double Bruhat cells developed in [3, 9, 11, 12]. An intriguing feature of our results is that combinatorial parametrizations of the canonical basis and related expressions for tensor product multiplicities for a semisimple Lie algebra $\mathfrak{g}$ are expressed in terms of geometry of totally positive varieties in the Langlands dual semisimple group ${}^L G$. It would be very interesting to give a conceptual explanation of this phenomenon.

The paper is organized as follows. Section 2 provides background on semisimple Lie algebras and summarizes our main results on tensor product multiplicities (with the exception of Plücker models which are given in Theorems 5.15 and 5.16) and reduction multiplicities. Section 3 provides background on quantum groups and canonical bases, and summarizes our results on parametrizations of canonical bases. In Section 4 we provide needed background, and present some new results on generalized minors, double Bruhat cells, and totally positive varieties. Section

5 presents our main results on geometric lifting and tropicalization; among other things, we find geometric counterpart of Kashiwara's crystal operators which play an important part in our arguments. The remaining sections contain the proofs of all the results in this paper. The proofs are not very difficult because most of the needed geometric technique was developed in the preceding papers [3, 9, 11]. However we need a number of important modifications to the geometric setup developed in the previous papers: for example, we replace double Bruhat cells from [11] by reduced double Bruhat cells introduced in Section 4 below.

2. Tensor product multiplicities

2.1. **Background on semisimple Lie algebras.** We start by fixing the terminology and notation (mostly standard) related to semisimple Lie algebras. Throughout the paper, $\mathfrak{g}$ is a complex semisimple Lie algebra of rank r with Chevalley generators $e_i, \alpha_i^\vee$, and f_i for $i = 1, \dots, r$. The elements $\alpha_i^\vee$ are *simple coroots* of $\mathfrak{g}$; they form a basis of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The *simple roots* $\alpha_1, \dots, \alpha_r$ form a basis in the dual space $\mathfrak{h}^*$ such that $[h, e_i] = \alpha_i(h) e_i$, and $[h, f_i] = -\alpha_i(h) f_i$ for any $h \in \mathfrak{h}$ and $i \in [1, r]$. The structure of $\mathfrak{g}$ is uniquely determined by the *Cartan matrix* $A = (a_{ij})$ given by $a_{ij} = \alpha_j(\alpha_i^\vee)$.

The *weight lattice* P of $\mathfrak{g}$ consists of all $\gamma \in \mathfrak{h}^*$ such that $\gamma(\alpha_i^\vee) \in \mathbb{Z}$ for all i. Thus P has a $\mathbb{Z}$-basis $\omega_1, \dots, \omega_r$ of *fundamental weights* given by $\omega_j(\alpha_i^\vee) = \delta_{i,j}$. A weight $\lambda \in P$ is *dominant* if $\lambda(\alpha_i^\vee) \geq 0$ for any $i \in [1, r]$; thus $\lambda = \sum_i l_i \omega_i$ with all l_i nonnegative integers. Let V_λ denote the simple (finite-dimensional) $\mathfrak{g}$-module with highest weight λ. We denote by $c_{\lambda,\nu}^\mu$ the multiplicity of the simple module V_μ in the tensor product $V_\lambda \otimes V_\nu$.

Every finite-dimensional $\mathfrak{g}$-module V is known to be $\mathfrak{h}$-diagonalizable, and we denote by $V = \oplus_{\gamma \in P} V(\gamma)$ its weight decomposition. Let $P(V)$ denote the set of weights of V, i.e., the set of all weights $\gamma \in P$ such that $V(\gamma) \neq 0$.

The *Langlands dual* complex semisimple Lie algebra ${}^L\mathfrak{g}$ corresponds to the transpose Cartan matrix A^T. Thus we can identify the Cartan subalgebra of ${}^L\mathfrak{g}$ with $\mathfrak{h}^*$, and simple roots (resp. coroots) of ${}^L\mathfrak{g}$ with simple coroots (resp. roots) of $\mathfrak{g}$. We denote the fundamental weights for ${}^L\mathfrak{g}$ by $\omega_1^\vee, \dots, \omega_r^\vee$; they are elements of $\mathfrak{h}$ such that $\gamma(\omega_i^\vee)$ is the coefficient of α_i in the expansion of $\gamma \in \mathfrak{h}^*$ in the basis of simple roots.

The Weyl groups of $\mathfrak{g}$ and $\mathfrak{g}^\vee$ are naturally identified with each other, and we denote both groups by the same symbol W. As an abstract group, W is a finite Coxeter group generated by *simple reflections* $s_1, \dots, s_r$; it acts in $\mathfrak{h}^*$ and $\mathfrak{h}$ by

$$s_i(\gamma) = \gamma - \gamma(\alpha_i^\vee)\alpha_i, \ s_i(h) = h - \alpha_i(h)\alpha_i^\vee$$

for $\gamma \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$.

A *reduced word* for $w \in W$ is a sequence of indices $(i_1, \dots, i_l)$ that satisfies $w = s_{i_1} \cdots s_{i_l}$ and has the shortest possible length $l = \ell(w)$. The set of reduced words for w will be denoted by $R(w)$. As customary, w_o denotes the unique element of maximal length in W.

2.2. **Polyhedral expressions for tensor product multiplicities.** In this section, we present our first main results: two families of combinatorial expressions for the tensor product multiplicities.

Definition 2.1. Let V be a finite-dimensional $\mathfrak{g}$-module, γ and δ two weights in $P(V)$, and $\mathbf{i} = (i_1, \ldots, i_l)$ a sequence of indices from $[1, r] := \{1, \ldots, r\}$. An $\mathbf{i}$-trail from γ to δ in V is a sequence of weights $\pi = (\gamma = \gamma_0, \gamma_1, \ldots, \gamma_l = \delta)$ such that:
(1) for $k = 1, \ldots, l$, we have $\gamma_{k-1} - \gamma_k = c_k \alpha_{i_k}$ for some nonnegative integer c_k;
(2) $e_{i_1}^{c_1} \cdots e_{i_l}^{c_l}$ is a non-zero linear map from $V(\delta)$ to $V(\gamma)$.

Note that the numbers c_k in Definition 2.1 can be written as

$$c_k = c_k(\pi) = \frac{\gamma_{k-1} - \gamma_k}{2}(\alpha_{i_k}^\vee) \ . \tag{2.1}$$

Our expressions for the tensor product multiplicity $c_{\lambda,\nu}^\mu$ for $\mathfrak{g}$ will involve $\mathbf{i}$-trails in the fundamental modules $V_{\omega_i^\vee}$ of the Langlands dual Lie algebra ${}^L\mathfrak{g}$. We denote by m the length of w_o.

Theorem 2.2. *Let λ, μ, ν be three dominant weights for $\mathfrak{g}$. For any reduced word $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_\mathrm{o})$, the multiplicity $c_{\lambda,\nu}^\mu$ is equal to the number of integer m-tuples $(t_1, \ldots, t_m)$ satisfying the following conditions:*

(1) $t_k \geq 0$ *for any* $k = 1, \ldots, m$;

(2) $\sum_k t_k \cdot s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} = \lambda + \nu - \mu$;

(3) $\sum_k c_k(\pi) t_k \geq (s_i \lambda + \nu - \mu)(\omega_i^\vee)$ *for any i and any $\mathbf{i}$-trail π from $s_i \omega_i^\vee$ to $w_\mathrm{o} \omega_i^\vee$ in $V_{\omega_i^\vee}$;*

(4) $\sum_k c_k(\pi) t_k \geq (\lambda + s_i \nu - \mu)(\omega_i^\vee)$ *for any i and any $\mathbf{i}$-trail π from $\omega_i^\vee$ to $w_\mathrm{o} s_i \omega_i^\vee$ in $V_{\omega_i^\vee}$.*

To present our second family of polyhedral expressions, let us define, for every $\mathbf{i}$-trail $\pi = (\gamma_0, \ldots, \gamma_l)$ in a $\mathfrak{g}$-module V, and every $k = 1, \ldots, l$ (cf. (2.1)):

$$d_k = d_k(\pi) = \frac{\gamma_{k-1} + \gamma_k}{2}(\alpha_{i_k}^\vee) \ . \tag{2.2}$$

Theorem 2.3. *Let λ, μ, ν be three dominant weights for $\mathfrak{g}$. For any reduced word $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_\mathrm{o})$, the multiplicity $c_{\lambda,\nu}^\mu$ is equal to the number of integer m-tuples $(t_1, \ldots, t_m)$ satisfying the following conditions:*

(1) $\sum_k d_k(\pi) t_k \geq 0$ *for any i and any $\mathbf{i}$-trail π from $\omega_i^\vee$ to $w_\mathrm{o} s_i \omega_i^\vee$ in $V_{\omega_i^\vee}$;*

(2) $\sum_k t_k \alpha_{i_k} = \lambda + \nu - \mu$;

(3) $\sum_k d_k(\pi) t_k \geq -\lambda(\alpha_i^\vee)$ *for any i and any $\mathbf{i}$-trail π from $s_i \omega_i^\vee$ to $w_\mathrm{o} \omega_i^\vee$ in $V_{\omega_i^\vee}$;*

(4) $t_k + \sum_{l > k} a_{i_k, i_l} t_l \leq \nu(\alpha_{i_k}^\vee)$ *for any* $k = 1, \ldots, m$.

Explicit bijections between the m-tuples in Theorems 2.2 and 2.3 will be given in Theorem 3.7 below.

2.3. **Tensor product multiplicities for classical groups.** In this section we present most concrete expressions for the multiplicity $c_{\lambda,\nu}^\mu$ in the case when $\mathfrak{g}$ is one of the classical simple Lie algebras of types B, C and D. We start with the types B_r ($\mathfrak{g} = so_{2r+1}$) and C_r ($\mathfrak{g} = sp_{2r}$). Choose a (non-standard) numeration of simple roots of $\mathfrak{g}$ by the index set $[0, r-1]$ so that the roots $\alpha_1, \ldots, \alpha_{r-1}$ form a system of type A_{r-1} in the standard numeration, and $a_{01} = -2$, $a_{10} = -1$ for type B_r, and $a_{01} = -1$, $a_{10} = -2$ for type C_r. Let us denote $a = |a_{10}|$, i.e., $a = 1$ for $\mathfrak{g} = so_{2r+1}$, and $a = 2$ for $\mathfrak{g} = sp_{2r}$.

Theorem 2.4. *Let $\mathfrak{g}$ be a simple Lie algebra of type B_r or C_r, and let λ, μ, ν be three dominant weights for $\mathfrak{g}$. Then the multiplicity $c_{\lambda,\nu}^{\mu}$ is equal to the number of integer tuples $(t_i^{(j)} : 0 \leq |i| \leq j < r)$ satisfying the following conditions:*

(1) $2t_{-j}^{(j)} \geq \cdots \geq 2t_{-1}^{(j)} \geq at_0^{(j)} \geq 2t_1^{(j)} \geq \cdots \geq 2t_j^{(j)} \geq 0$ *for* $0 \leq j < r$;

(2) $\sum_{0 \leq |i| \leq j < r} t_i^{(j)} \alpha_{|i|} = \lambda + \nu - \mu$;

(3) $\lambda(\alpha_0^\vee) \geq t_0^{(0)}$, *and* $\lambda(\alpha_j^\vee) \geq \max\ \{t_j^{(j)}, at_0^{(j)} - t_1^{(j-1)} - t_{-1}^{(j)}, t_1^{(j-1)} + t_{-1}^{(j)} - at_0^{(j-1)}, \varphi_i^{(j)}(t)\, (1 \leq i < j)\}$ *for* $0 \leq j < r$, *where* $\varphi_i^{(j)}(t) = \max\{t_i^{(j)} + t_{-i}^{(j)} - t_{i+1}^{(j-1)} - t_{-i-1}^{(j)}, t_{i+1}^{(j-1)} + t_{-i-1}^{(j)} - t_{-i}^{(j-1)} - t_i^{(j-1)}, t_{\pm i}^{(j)} - t_{\pm i}^{(j-1)}\}$ *(with the convention that* $t_i^{(j)} = 0$ *unless* $0 \leq |i| \leq j < r$*);*

(4) $\nu(\alpha_0^\vee) \geq \max_{j \geq 0}\ (t_0^{(j)} + \frac{2}{a} \sum_{k>j} (at_0^{(k)} - t_{-1}^{(k)} - t_1^{(k-1)}))$,

$$\nu(\alpha_1^\vee) \geq \max_{j \geq 1}\ \max\Big(t_{-1}^{(j)} + \sum_{k>j} (2t_{-1}^{(k)} + 2t_1^{(k-1)} - at_0^{(k-1)} - t_{-2}^{(k)} - t_2^{(k-1)}),$$

$$t_1^{(j)} + \sum_{k>j} (2t_{-1}^{(k)} + 2t_1^{(k)} - at_0^{(k)} - t_{-2}^{(k)} - t_2^{(k-1)})\Big)\ ,$$

$$\nu(\alpha_i^\vee) \geq \max_{j \geq i}\ \max\Big(t_{-i}^{(j)} + \sum_{k>j} (2t_{-i}^{(k)} + 2t_i^{(k-1)} - t_{-i+1}^{(k-1)} - t_{i-1}^{(k-1)} - t_{-i-1}^{(k)} - t_{i+1}^{(k-1)}),$$

$$t_i^{(j)} + \sum_{k>j} (2t_{-i}^{(k)} + 2t_i^{(k)} - t_{-i+1}^{(k)} - t_{i-1}^{(k)} - t_{-i-1}^{(k)} - t_{i+1}^{(k-1)})\Big)$$

for $2 \leq i < r$.

Now consider the type D_r ($\mathfrak{g} = so_{2r}$). Choose a (non-standard) numeration of simple roots of $\mathfrak{g}$ by the index set $\{-1\} \cup [1, r-1]$ so that the roots $\alpha_1, \ldots, \alpha_{r-1}$ form a system of type A_{r-1} in the standard numeration, and $a_{-1,2} = a_{2,-1} = -1$.

Theorem 2.5. *Let $\mathfrak{g}$ be a simple Lie algebra of type D_r and let λ, μ, ν be three dominant weights for $\mathfrak{g}$. Then the multiplicity $c_{\lambda,\nu}^{\mu}$ is equal to the number of integer tuples $(t_i^{(j)} : 1 \leq |i| \leq j < r)$ satisfying the following conditions:*

(1) $t_{-j}^{(j)} \geq \cdots \geq t_{-2}^{(j)} \geq \max\ (t_{-1}^{(j)}, t_1^{(j)}) \geq \min\ (t_{-1}^{(j)}, t_1^{(j)}) \geq t_2^{(j)} \geq \cdots \geq t_j^{(j)} \geq 0$ *for* $1 \leq j < r$;

(2) $\sum_j (t_{-1}^{(j)} \alpha_{-1} + t_1^{(j)} \alpha_1) + \sum_{2 \leq |i| \leq j < r} t_i^{(j)} \alpha_{|i|} = \lambda + \nu - \mu$;

(3) $\lambda(\alpha_{\pm 1}^\vee) \geq t_{\pm 1}^{(1)}$, *and* $\lambda(\alpha_j^\vee) \geq \max\ \{t_j^{(j)}, t_{\pm 1}^{(j)} - t_{\mp 1}^{(j-1)}, t_1^{(j)} + t_{-1}^{(j)} - t_{-2}^{(j)} - t_2^{(j-1)}, t_{-2}^{(j)} + t_2^{(j-1)} - t_1^{(j-1)} - t_{-1}^{(j-1)}, \varphi_i^{(j)}(t)\, (2 \leq i < j)\}$ *for* $2 \leq j < r$, *where* $\varphi_i^{(j)}(t)$ *is the same as in Theorem 2.4;*

(4) $\nu(\alpha_{\pm 1}^\vee) \geq \max_{j \geq 1}\ (t_{\pm 1}^{(j)} + \sum_{k>j} (2t_{\pm 1}^{(k)} - t_{-2}^{(k)} - t_2^{(k-1)}))$,

$$\nu(\alpha_i^\vee) \geq \max_{j \geq i}\ \max\Big(t_{-i}^{(j)} + \sum_{k>j} (2t_{-i}^{(k)} + 2t_i^{(k-1)} - t_{-i+1}^{(k-1)} - t_{i-1}^{(k-1)} - t_{-i-1}^{(k)} - t_{i+1}^{(k-1)}),$$

$$t_i^{(j)} + \sum_{k>j} (2t_{-i}^{(k)} + 2t_i^{(k)} - t_{-i+1}^{(k)} - t_{i-1}^{(k)} - t_{-i-1}^{(k)} - t_{i+1}^{(k-1)})\Big)$$

for $2 \leq i < r$.

Remark 2.6. We shall show that Theorems 2.4 and 2.5 can be obtained by specializing Theorem 2.3 for a specific reduced word $\mathbf{i}$. One can also show that specializing Theorem 2.2 for the same $\mathbf{i}$ leads to the expressions for $c_{\lambda,\nu}^{\mu}$ that were conjectured in [5].

2.4. **Reduction multiplicities.** For a subset $I \subset [1, r]$, let $\mathfrak{g}(I)$ denote the corresponding *Levi subalgebra* in $\mathfrak{g}$ generated by the Cartan subalgebra $\mathfrak{h}$ and by all e_i and f_i for $i \in I$. A weight $\beta \in P$ is dominant for $\mathfrak{g}(I)$ if $\beta(\alpha_i^\vee) \geq 0$ for $i \in I$; for such a weight, let $V_\beta^{(I)}$ denote the simple (finite-dimensional) $\mathfrak{g}(I)$-module with highest weight β. In this section, we compute the multiplicity of $V_\beta^{(I)}$ in the reduction to $\mathfrak{g}(I)$ of a simple $\mathfrak{g}$-module V_ν. This includes the weight multiplicities in V_ν as a special case when $I = \emptyset$.

Let $w_\mathrm{o}(I)$ denote the longest element of the parabolic subgroup in W generated by all s_i with $i \in I$.

Theorem 2.7. *For any reduced word $\mathbf{i} = (i_1, \ldots, i_n) \in R(w_\mathrm{o}(I)^{-1} w_\mathrm{o})$, the multiplicity of $V_\beta^{(I)}$ in the reduction of V_ν to $\mathfrak{g}(I)$ is equal to the number of integer n-tuples $(t_1, \ldots, t_n)$ satisfying the following conditions:*

(1) $t_k \geq 0$ *for* $k = 1, \ldots, n$;

(2) $\sum_k t_k \cdot s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} = w_\mathrm{o}(I)(\nu - \beta)$;

(3) $\sum_k c_k(\pi) t_k \geq (s_i \nu - \beta)(\omega_i^\vee)$ *for any i and any $\mathbf{i}$-trail π from $w_\mathrm{o}(I)\omega_i^\vee$ to $w_\mathrm{o} s_i \omega_i^\vee$ in $V_{\omega_i^\vee}$.*

Theorem 2.8. *For any reduced word $\mathbf{i} = (i_1, \ldots, i_n) \in R(w_\mathrm{o}(I)^{-1} w_\mathrm{o})$, the multiplicity of $V_\beta^{(I)}$ in the reduction of V_ν to $\mathfrak{g}(I)$ is equal to the number of integer n-tuples $(t_1, \ldots, t_n)$ satisfying the following conditions:*

(1) $\sum_k d_k(\pi) t_k \geq 0$ *for any i and any $\mathbf{i}$-trail π from $w_\mathrm{o}(I)\omega_i^\vee$ to $w_\mathrm{o} s_i \omega_i^\vee$ in $V_{\omega_i^\vee}$;*

(2) $\sum_k t_k \alpha_{i_k} = \nu - \beta$;

(3) $t_k + \sum_{l > k} a_{i_k, i_l} t_l \leq \nu(\alpha_{i_k}^\vee)$ *for* $k = 1, \ldots, n$.

Explicit bijections between the n-tuples in Theorems 2.7 and 2.8 will be given in Theorem 3.7 below. We illustrate these theorems with the following example.

Example 2.9. Let $\mathfrak{g} = sl_{r+1}$ be of type A_r (with the standard numeration of simple roots). Let $I = [1, r] \setminus \{p\}$ for some $p \in [1, r]$. Let $q = r + 1 - p$; then the algebra $\mathfrak{g}(I)$ is the intersection of $sl_{r+1} = sl_{p+q}$ with the block-diagonal subalgebra $gl_p \times gl_p \subset gl_{p+q}$. Denote by $M_{p \times q}$ the set of all $p \times q$ matrices $T = (t_{ij})$ with integer entries (we shall also use the convention that $t_{ij} = 0$ unless $(i, j) \in [1, p] \times [1, q]$). Theorems 2.7 and 2.8 specialize to the following two expressions for the reduction multiplicity.

Corollary 2.10. *The multiplicity of $V_\beta^{(I)}$ in the reduction of V_ν to $\mathfrak{g}(I)$ is equal to the number of all $T \in M_{p \times q}$ satisfying the following conditions:*

(1) $t_{ij} \geq 0$ *for all i and j;*

(2) $\sum_{i \leq l; j \leq p+q-l} t_{ij} = (\nu - \beta)(\omega_l^\vee)$ *for any* $l \in [1, r]$;

(3) $\nu(\alpha_i^\vee) \geq \max_{j \in [1,q]} (\sum_{k \geq 0} (t_{i,j+k} - t_{i+1,j+k+1}))$ *for* $1 \leq i < p$; $\quad \nu(\alpha_p^\vee) \geq t_{pq}$; $\nu(\alpha_{p+q-j}^\vee) \geq \max_{i \in [1,p]} (\sum_{k \geq 0} (t_{i+k,j} - t_{i+k+1,j+1}))$ *for* $1 \leq j < q$.

Corollary 2.11. *The multiplicity of $V_\beta^{(I)}$ in the reduction of V_ν to $\mathfrak{g}(I)$ is equal to the number of all $T \in M_{p\times q}$ satisfying the following conditions:*

(1) $t_{ij} \geq \max(t_{i+1,j}, t_{i,j+1})$ for all i and j (that is, T is a plane partition of shape $p \times q$);

(2) $\sum_{j-i=l-p} t_{ij} = (\nu - \beta)(\omega_l^\vee)$ for any $l \in [1, r]$;

(3) $\nu(\alpha_l^\vee) \geq \max_{j-i=l-p}(t_{ij} + \sum_{k>0}(2t_{i+k,j+k} - t_{i+k-1,j+k} - -t_{i+k,j+k-1}))$ for any $l \in [1, r]$.

Remark 2.12. The polytope defined by conditions (1) - (3) in Corollary 2.10 (with the additional assumption that $\nu(\alpha_p^\vee) >> 0$ so that the inequality $\nu(\alpha_p^\vee) \geq t_{pq}$ in (3) is skipped) appeared in a different context in [10]. Comparing Corollary 2.10 with [10, Theorem 1], we conclude the following. If $\nu(\alpha_p^\vee) >> 0$ then the reduction multiplicity in Corollary 2.10 is equal to the inner product of two skew Schur functions $s_{\nu^{(1)}/\beta^{(1)}}$ and $s_{\nu^{(2)}/\beta^{(2)}}$, where for each dominant $\mathfrak{g}(I)$-weight μ we define the partition $\mu^{(1)}$ (resp. $\mu^{(2)}$) with $\leq p$ (resp. $\leq q$) parts by setting $\mu_i^{(1)} = \mu(\omega_i^\vee - \omega_{i-1}^\vee)$ (resp. $\mu_j^{(2)} = \mu(\omega_{p+q-j}^\vee - \omega_{p+q-j+1}^\vee)$). Note that a polyhedral expression for the inner product of any skew Schur functions was first obtained in [4]; the expression given there is close to the one in Corollary 2.11. A bijective correspondence between the tuples in Corollaries 2.10 and 2.11 was constructed in an unpublished work of one of the authors (A.B.) and Anatol Kirillov; they also observed that their bijection can be interpreted as the Robinson-Schensted-Knuth correspondence.

3. Canonical bases and their parametrizations

3.1. **Background on canonical bases and their Lusztig parametrizations.** Let us recall some basic facts about quantized universal enveloping algebras and their canonical bases. Unless otherwise stated, all the results in this section are due to G. Lusztig and can be found in [20]. The quantized universal enveloping algebra $U = U_q(\mathfrak{g})$ associated to $\mathfrak{g}$ is defined as follows. Fix positive integers $d_1, \ldots, d_r$ such that $d_i a_{ij} = d_j a_{ji}$, where (a_{ij}) is the Cartan matrix of $\mathfrak{g}$. The algebra U is a $\mathbb{C}(q)$-algebra with unit generated by the elements $E_i, K_i^{\pm 1}$, and F_i for $i = 1, \ldots, r$ subject to the relations

$$K_i K_j = K_j K_i,\ K_i E_j K_i^{-1} = q^{d_i a_{ij}} E_j,\ K_i F_j K_i^{-1} = q^{-d_i a_{ij}} E_j\ ,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}$$

for all i and j, and the *quantum Serre relations*

$$\sum_{k+l=1-a_{ij}} (-1)^k E_i^{(k)} E_j E_i^{(l)} = \sum_{k+l=1-a_{ij}} (-1)^k F_i^{(k)} F_j F_i^{(l)} = 0$$

for $i \neq j$. Here $E_i^{(k)}$ and $F_i^{(k)}$ stand for the *divided powers* defined by

$$E_i^{(k)} = \frac{1}{[1]_i [2]_i \cdots [k]_i} E_i^k,\ F_i^{(k)} = \frac{1}{[1]_i [2]_i \cdots [k]_i} F_i^k\ ,$$

where $[l]_i = \frac{q^{d_i l} - q^{-d_i l}}{q^{d_i} - q^{-d_i}}$. The algebra U is graded by the root lattice of $\mathfrak{g}$ via

$$\deg(K_i) = 0,\ \deg(E_i) = -\deg(F_i) = \alpha_i\ .$$

To each $i = 1, \dots, r$, Lusztig associates an algebra automorphism T_i of U uniquely determined by:

$$T_i(K_j) = K_j K_i^{-a_{ij}} \quad (j = 1, \dots, r) \ ,$$
$$T_i(E_i) = -K_i^{-1} F_i, \ T_i(F_i) = -E_i K_i \ ,$$

and, for all $j \neq i$,

$$T_i(E_j) = \sum_{k+l=-a_{ij}} (-1)^k q^{-d_i k} E_i^{(k)} E_j E_i^{(l)} \ ,$$
$$T_i(F_j) = \sum_{k+l=-a_{ij}} (-1)^k q^{d_i k} F_i^{(l)} F_j F_i^{(k)} \ .$$

(This automorphism was denoted by $T'_{i,-1}$ in [20].) The T_i satisfy the braid relations and so extend to an action of the braid group on U.

Let U^+ denote the subalgebra of U generated by $E_1, \dots, E_r$. We now recall Lusztig's definitions of the PBW-type bases and the canonical basis in U^+. For a reduced word $\mathbf{i} = (i_1, \dots, i_m) \in R(w_\mathrm{o})$, and an m-tuple $t = (t_1, \dots, t_m) \in \mathbb{Z}_{\geq 0}^m$, denote

$$p_\mathbf{i}^{(t)} := E_{i_1}^{(t_1)} T_{i_1}(E_{i_2}^{(t_2)}) \cdots (T_{i_1} \cdots T_{i_{m-1}})(E_{i_m}^{(t_m)}) \ .$$

As shown in [20], all these elements belong to U^+. For a given $\mathbf{i}$, the set of all $p_\mathbf{i}^{(t)}$ with $t \in \mathbb{Z}_{\geq 0}^m$ is called the *PBW type basis* corresponding to $\mathbf{i}$ and is denoted by $\mathcal{B}_\mathbf{i}$. This terminology is justified by the following proposition proved in [20, Corollary 40.2.2].

Proposition 3.1. *For every $\mathbf{i} \in R(w_\mathrm{o})$, the set $\mathcal{B}_\mathbf{i}$ is a $\mathbb{C}(q)$-basis of U_+.*

According to [21, Proposition 8.2], the canonical basis $\mathcal{B}$ of U^+ can now be defined as follows. Let $u \mapsto \overline{u}$ denote the $\mathbb{C}$-linear involutive algebra automorphism of U^+ such that $\overline{q} = q^{-1}$, $\overline{E_i} = E_i$.

Proposition 3.2. *For every $\mathbf{i} \in R(w_\mathrm{o})$ and $t \in \mathbb{Z}_{\geq 0}^m$, there is a unique element $b = b_\mathbf{i}(t)$ of U^+ such that $\overline{b} = b$, and $b - p_\mathbf{i}^{(t)}$ is a linear combination of the elements of $\mathcal{B}_\mathbf{i}$ with coefficients in $q^{-1}\mathbb{Z}[q^{-1}]$. For any fixed $\mathbf{i}$, the elements $b_\mathbf{i}(t)$ for all $t \in \mathbb{Z}_{\geq 0}^m$ constitute the canonical basis $\mathcal{B}$.*

In view of Proposition 3.2, any $\mathbf{i} \in R(w_\mathrm{o})$ gives rise to a bijection $b_\mathbf{i} : \mathbb{Z}_{\geq 0}^m \to \mathcal{B}$. We refer to these bijections as *Lusztig parametrizations* of $\mathcal{B}$. Let us summarize some of their properties. To do this, we need some more notation. Let $i \mapsto i^*$ denote the involution on $[1, r]$ defined by $w_\mathrm{o}(\alpha_i) = -\alpha_{i^*}$. For every sequence $\mathbf{i} = (i_1, \dots, i_m)$, we denote by $\mathbf{i}^*$ and $\mathbf{i}^\mathrm{op}$ the sequences

$$\mathbf{i}^* = (i_1^*, \dots, i_m^*), \ \mathbf{i}^\mathrm{op} = (i_m, \dots, i_1) \ ; \tag{3.1}$$

clearly, both operations $\mathbf{i} \mapsto \mathbf{i}^*$ and $\mathbf{i} \mapsto \mathbf{i}^\mathrm{op}$ preserve the set of reduced words $R(w_\mathrm{o})$.

Proposition 3.3. (i) *Any canonical basis vector $b_\mathbf{i}(t) \in \mathcal{B}$ is homogeneous of degree $\sum_k t_k \cdot s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$.*
(ii) *Every subspace of the form $E_i^n U^+$ in U^+ is spanned by a subset of $\mathcal{B}$. Furthermore, let $l_i(b)$ denote the maximal integer n such that $b \in E_i^n U^+$; then, for any $\mathbf{i} \in R(w_\mathrm{o})$ which begins with $i_1 = i$, we have $l_i(b_\mathbf{i}(t_1, \dots, t_m)) = t_1$.*
(iii) *The canonical basis $\mathcal{B}$ is stable under the involutive $\mathbb{C}(q)$-linear algebra anti-automorphism $E \to E^\iota$ of U^+ such that $E_i^\iota = E_i$ for all i. Furthermore, we have $b_\mathbf{i}(t)^\iota = b_{\mathbf{i}^{*\mathrm{op}}}(t^\mathrm{op})$.*

This proposition allows us to interpret the tensor product multiplicity $c_{\lambda,\nu}^{\mu}$ in terms of the canonical basis. Indeed it is well-known (see e.g., [8]) that $c_{\lambda,\nu}^{\mu}$ is equal to the dimension of the homogeneous component of degree $\lambda+\nu-\mu$ in $U^+/\sum_i(E_i^{\lambda(\alpha_i^\vee)+1}U^+ \ + \ U^+E_i^{\nu(\alpha_i^\vee)+1})$. Thus Proposition 3.3 has the following corollary.

Corollary 3.4. *The multiplicity $c_{\lambda,\nu}^{\mu}$ is equal to the number of vectors $b\in\mathcal{B}$ of degree $\lambda+\nu-\mu$ satisfying the following property: if $b=b_{\mathbf{i}}(t_1,\ldots,t_m)$, and $\mathbf{i}\in R(w_{\rm o})$ begins with i and ends with j then $t_1\le\lambda(\alpha_i^\vee)$ and $t_m\le\nu(\alpha_{j^*}^\vee)$.*

3.2. **Preliminaries on string parametrizations.** We now describe another way to parametrize $\mathcal{B}$ with certain strings of nonnegative integers, the so-called *string parametrizations* introduced in [7, 16]. As a general setup, consider any U^+-module V such that each E_i acts on V as a locally nilpotent operator, i.e., for every nonzero $v\in V$ there exists some positive integer n such that $E_i^n(v)=0$. Let $c_i(v)$ denote the maximal integer n such that $E_i^n(v)\neq 0$. For any sequence of indices $\mathbf{i}=(i_1,\ldots,i_m)$, the *string* of v in direction $\mathbf{i}$ is a sequence $c_{\mathbf{i}}(v)=(t_1,\ldots,t_m)$ of nonnegative integers defined recursively as follows:

$$t_1=c_{i_1}(v), t_2=c_{i_2}(E_{i_1}^{t_1}(v)),\ldots,t_m=c_{i_m}(E_{i_{m-1}}^{t_{m-1}}\cdots E_{i_1}^{t_1}(v)) \ .$$

We apply this construction to a U^+-module $\mathcal{A}$ defined as follows. As a $\mathbb{C}(q)$-vector space, $\mathcal{A}$ is the restricted dual vector space of U^+, i.e., the direct sum of dual spaces of all homogeneous components of U^+ (recall that U^+ is graded by the root lattice of $\mathfrak{g}$). The action $(E,f)\mapsto E(f)$ of U^+ on $\mathcal{A}$ is given by $E(f)(u)=f(E^\iota u)$ for all $u\in U^+$ (see Proposition 3.3 (iii)). Clearly, each E_i acts in $\mathcal{A}$ as a locally nilpotent operator, so the corresponding strings are well defined.

Now we consider the *dual canonical* basis $\mathcal{B}^{\rm dual}$ in $\mathcal{A}$: its element $b^{\rm dual}$ corresponding to $b\in\mathcal{B}$ is a linear form on U^+ such that $b^{\rm dual}(b')=\delta_{b,b'}$ for $b'\in\mathcal{B}$. The following proposition was essentially proved in [18] (our results below will provide an independent proof).

Proposition 3.5. *For any $\mathbf{i}\in R(w_{\rm o})$, the string parametrization $c_{\mathbf{i}}$ is a bijection of $\mathcal{B}^{\rm dual}$ onto the set of all lattice points $C_{\mathbf{i}}(\mathbb{Z})$ of some rational polyhedral convex cone $C_{\mathbf{i}}$ in $\mathbb{R}^m$.*

We call the cone $C_{\mathbf{i}}$ in Proposition 3.5 the *string cone* associated to $\mathbf{i}\in R(w_{\rm o})$ (it was called the cone of adapted strings in [18]).

Comparing the definition of strings with the definition of $l_i(b)$ in Proposition 3.3 (ii), we see that

$$l_i(b)=c_i(b^{\rm dual}) \tag{3.2}$$

for any $i\in[1,r]$ and $b\in\mathcal{B}$. We shall also need some basic properties of $\mathcal{B}$ related to Kashiwara's crystal structure. The following proposition is a consequence of results in [16] and [20] (see also [23]).

Proposition 3.6. (i) *There exist embeddings $\tilde f_i:\mathcal{B}\to\mathcal{B}$ for $i=1,\ldots,r$ which satisfy the following property: if $b=b_{\mathbf{i}}(t_1,\ldots,t_m)$ and $c_{\mathbf{i}}(b^{\rm dual})=(t'_1,\ldots,t'_m)$ for some $\mathbf{i}=(i_1,\ldots,i_m)\in R(w_{\rm o})$ with $i_1=i$ then $\tilde f_i(b)=b_{\mathbf{i}}(t_1+1,t_2,\ldots,t_m)$, and $c_{\mathbf{i}}(\tilde f_i(b)^{\rm dual})=(t'_1+1,t'_2,\ldots,t'_m)$.*
(ii) *Every $b\in\mathcal{B}$ has the form $b=\tilde f_{i_1}\cdots\tilde f_{i_N}(1)$ for some sequence of indices $i_1,\ldots,i_N$ from $[1,r]$.*

3.3. **New results on Lusztig and string parametrizations.** We start with an explicit formula for the relationship between Lusztig and string parametrizations.

Theorem 3.7. (i) *Let* $\mathbf{i}$ *and* $\mathbf{i}'$ *be two reduced words of* w_o, *and let* $t = c_{\mathbf{i}}(b_{\mathbf{i}'}(t')^{\mathrm{dual}})$ *be the string in direction* $\mathbf{i}$ *of the dual canonical basis vector with the Lusztig parameters* t' *relative to* $\mathbf{i}'$. *Then* t *and* t' *are related as follows: for any* $k = 1, \ldots, m$, *we have*

$$t_k = \min_{\pi_1} \big(\sum_{l=1}^{m} c_l(\pi_1) \cdot t'_l\big) - \min_{\pi_2} \big(\sum_{l=1}^{m} c_l(\pi_2) \cdot t'_l\big) \ , \tag{3.3}$$

where π_1 *(resp.* π_2*) runs over* $\mathbf{i}$*-trails from* $s_{i_1} \cdots s_{i_{k-1}} \omega_{i_k}^\vee$ *(resp. from* $s_{i_1} \cdots s_{i_k} \omega_{i_k}^\vee$*) to* $w_o \omega_{i_k}^\vee$ *in* $V_{\omega_{i_k}^\vee}$.

(ii) *For any three dominant weights* λ, μ *and* ν, *the correspondence* (3.3) *restricts to a bijection between the set of tuples* t *in Theorem* 2.3 *and the set of tuples* t' *in Theorem* 2.2 *with* $\mathbf{i}$ *replaced by* $\mathbf{i}'$.

Using Theorem 3.7, we shall obtain the following explicit expression for the functions l_i on $\mathcal{B}$ in terms of Lusztig parameters (see Proposition 3.3 (ii)).

Theorem 3.8. *For every* $i \in [1, r]$ *and* $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_o)$, *we have*

$$l_i(b_{\mathbf{i}}(t_1, \ldots, t_m)) = \max_{\pi} \sum_k (s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}(\omega_i^\vee) - c_k(\pi)) t_k \ , \tag{3.4}$$

where π *runs over all* $\mathbf{i}$*-trails from* $s_i \omega_i^\vee$ *to* $w_o \omega_i^\vee$ *in* $V_{\omega_i^\vee}$.

The inverse of the map in (3.3) which expresses Lusztig parameters of b in terms of strings of b^{dual} can be also computed explicitly but this expression is more involved. We shall only present the following two important special cases.

Theorem 3.9. *Let* t *and* t' *have the same meaning as in Theorem 3.7 (i). Then*

$$t'_1 = -\min_{\pi} \sum_k d_k(\pi) t_k \ , \tag{3.5}$$

where π *runs over all* $\mathbf{i}$*-trails from* $s_{i'_1} \omega_{i'_1}^\vee$ *to* $w_o \omega_{i'_1}^\vee$ *in* $V_{\omega_{i'_1}^\vee}$*; and also*

$$t'_m = \max_{k : i_k^* = i'_m} (t_k + \sum_{l > k} a_{i'_m, i_l^*} t_l) \ . \tag{3.6}$$

We conclude this section by an explicit description of the string cones (see Proposition 3.5).

Theorem 3.10. *For any reduced word* $\mathbf{i} \in R(w_o)$, *the string cone* $C_{\mathbf{i}}$ *is the cone in* $\mathbb{R}^m$ *given by the inequalities* (1) *in Theorem 2.3, i.e., it consists of all real* m*-tuples* $(t_1, \ldots, t_m)$ *such that* $\sum_k d_k(\pi) t_k \geq 0$ *for any* i *and any* $\mathbf{i}$*-trail* π *from* $\omega_i^\vee$ *to* $w_o s_i \omega_i^\vee$ *in* $V_{\omega_i^\vee}$.

3.4. **More on string cones.** In this section we describe some specializations of Theorem 3.10. Our first result shows that, under some conditions on a reduced word $\mathbf{i} \in R(w_o)$, the corresponding string cone $C_{\mathbf{i}}$ splits into the direct product of smaller cones. To formulate the result, we need some more notation. First, for any sequence $\mathbf{i} = (i_1, \ldots, i_l)$ of indices from $[1, r]$, and any two elements $u, v \in W$, let $C_{\mathbf{i}}(u, v)$ denote the cone of all real l-tuples $(t_1, \ldots, t_l)$ such that $\sum_k d_k(\pi) t_k \geq 0$ for any i and any $\mathbf{i}$-trail π from $u\omega_i^\vee$ to $vs_i\omega_i^\vee$ in $V_{\omega_i^\vee}$. (In particular, Theorem 3.10 claims that $C_{\mathbf{i}} = C_{\mathbf{i}}(e, w_o)$.) Second, as in Section 2.4, for any subset $I \subset [1, r]$ let

$w_o(I)$ denote the longest element of the parabolic subgroup in W generated by all s_i with $i \in I$.

Theorem 3.11. *Let $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_p = [1,r]$ be any flag of subsets in $[1,r]$. Suppose $\mathbf{i} \in R(w_o)$ is the concatenation $(\mathbf{i}^{(1)}, \ldots, \mathbf{i}^{(p)})$, where $\mathbf{i}^{(j)} \in R(w_o(I_{j-1})^{-1} w_o(I_j))$ for $j = 1, \ldots, p$. Then the string cone $C_\mathbf{i}$ is the direct product of cones:*

(3.7)
$$C_\mathbf{i} = C_{\mathbf{i}^{(1)}}(e, w_o(I_1)) \times C_{\mathbf{i}^{(2)}}(w_o(I_1)), w_o(I_2)) \times \cdots \times C_{\mathbf{i}^{(p)}}(w_o(I_{p-1}), w_o(I_p)) \ .$$

Under some additional assumptions, it is possible to describe the factors in (3.7) much more explicitly. Following [24], we call an element $w \in W$ *fully commutative* if any two reduced words for w can be obtained from each other by a series of switches $(i_k, i_{k+1}) \to (i_{k+1}, i_k)$ such that $a_{i_k, i_{k+1}} = 0$.

Theorem 3.12. *In the situation of Theorem 3.11, suppose that an index $j \in [1,p]$ is such that $|I_j| = |I_{j-1}|+1$, and the element $w_o(I_{j-1})^{-1} w_o(I_j)$ is fully commutative of length l. Then the corresponding cone $C_{\mathbf{i}^{(j)}}(w_o(I_{j-1}), w_o(I_j)) \subset \mathbb{R}^l$ is given by the following inequalities:*

(1) $t_l \geq 0$;

(2) $t_{k(1)} \geq t_{k(2)}$ for any pair of indices $k(1) < k(2)$ in $[1,l]$ such that $\alpha_{i_{k(1)}}$ and $\alpha_{i_{k(2)}}$ generate a root system of type A_2, and $i_k \notin \{i_{k(1)}, i_{k(2)}\}$ for $k(1) < k < k(2)$;

(3) $|a_{ij}| t_{k(1)} \geq t_{k(2)} \geq |a_{ij}| t_{k(3)}$ for any indices $k(1) < k(2) < k(3)$ in $[1,l]$ such that $i_{k(1)} = i_{k(3)} = j \neq i = i_{k(2)}$, α_i and α_j generate a root system of type B_2, and $a_{i_k, j} = 0$ for all $k \neq k(2)$ between $k(1)$ and $k(3)$;

(4) $2|a_{ij}| t_{k(1)} \geq 2t_{k(2)} \geq |a_{ij}| t_{k(3)} \geq 2t_{k(4)} \geq 2|a_{ij}| t_{k(5)}$ for any indices $k(1) < \cdots < k(5)$ in $[1,l]$ such that $i_{k(1)} = i_{k(3)} = i_{k(5)} = j \neq i = i_{k(2)} = i_{k(4)}$, and α_i and α_j generate a root system of type G_2.

Combining Theorems 3.11 and 3.12, we obtain the following refinement of the main result of [18].

Corollary 3.13. *Suppose that $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_r = [1,r]$ is a flag of subsets in $[1,r]$ such that $|I_j| = j$, and $w_o(I_{j-1})^{-1} w_o(I_j)$ is fully commutative for every $j \in [1,r]$. Suppose $\mathbf{i} \in R(w_o)$ is the concatenation $(\mathbf{i}^{(1)}, \ldots, \mathbf{i}^{(r)})$, where $\mathbf{i}^{(j)} \in R(w_o(I_{j-1})^{-1} w_o(I_j))$. Then the string cone $C_\mathbf{i}$ is the direct product of cones given by* (3.7)*, with every factor in the product given by inequalities in Theorem 3.12.*

Theorem 3.10 also implies a more explicit description of *all* string cones for the type A_r (another description of these cones was found in [15]). We need the following notation: for any $i \in [1,r]$, let $u^{(i)}$ denote the minimal representative of the coset $W_{\hat{i}} s_i w_0$ in W, where $W_{\hat{i}}$ is the maximal parabolic subgroup in W generated by all s_j with $j \neq i$.

Theorem 3.14. *Let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w_o)$. For any $i \in [1,r]$ and any subword $(i_{k(1)}, \ldots, i_{k(p)})$ of $\mathbf{i}$ which is a reduced word for $u^{(i)}$, all the points $(t_1, \ldots, t_m)$ in the string cone $C_\mathbf{i}$ satisfy the inequality*

$$\sum_{j=0}^{p} \sum_{k(j)<k<k(j+1)} (s_{i_{k(1)}} \cdots s_{i_{k(j)}} \alpha_{i_k})(\omega_i^\vee) \cdot t_k \geq 0 \tag{3.8}$$

(with the convention that $k(0) = 0$ and $k(p+1) = m+1$). Furthermore, if $\mathfrak{g} = sl_{r+1}$ then $C_\mathbf{i}$ is the set of all $t \in \mathbb{R}^m$ satisfying the inequalities (3.8).

Example 3.15. Let us illustrate Theorem 3.14 by an example when $\mathfrak{g} = sl_4$. We shall use the standard numeration of simple roots and corresponding simple reflections for type A_3, so that s_1 and s_3 commute with each other. In our case $m = \ell(w_o) = 6$, and the elements $u^{(i)}$ are given by:

$$u^{(1)} = s_1 s_2, \quad u^{(2)} = s_2 s_1 s_3 = s_2 s_3 s_1, \quad u^{(3)} = s_3 s_2 \ .$$

Let us consider a reduced word $\mathbf{i} = (2,1,3,2,1,3)$ of w_o. Here is the list of subwords of $\mathbf{i}$ which are reduced words for the elements $u^{(i)}$ (their entries are underlined), and the corresponding inequalities of the form (3.8):

$$\begin{aligned}
u^{(1)} \quad &: \quad (2,\underline{1},3,\underline{2},1,3) \to t_6 \geq 0;\\
u^{(2)} \quad &: \quad (\underline{2},\underline{1},\underline{3},2,1,3) \to t_4 - t_5 - t_6 \geq 0,\ (\underline{2},\underline{1},3,2,1,\underline{3}) \to t_3 - t_5 \geq 0,\\
&\quad\ \ (\underline{2},1,\underline{3},2,\underline{1},3) \to t_2 - t_6 \geq 0,\ (\underline{2},1,3,2,\underline{1},\underline{3}) \to t_2 + t_3 - t_4 \geq 0,\\
&\quad\ \ (2,1,3,\underline{2},\underline{1},\underline{3}) \to t_1 \geq 0;\\
u^{(3)} \quad &: \quad (2,1,\underline{3},\underline{2},1,3) \to t_5 \geq 0.
\end{aligned}$$

Therefore, $C_{\mathbf{i}}$ is a cone in $\mathbb{R}^6$ given by:

$$t_1 \geq 0,\ t_2 \geq t_6 \geq 0,\ t_3 \geq t_5 \geq 0,\ t_2 + t_3 \geq t_4 \geq t_5 + t_6 \ .$$

4. Reduced double Bruhat cells and totally positive varieties

4.1. **Background on semisimple groups.** Let G be a simply connected complex semisimple Lie group with the Lie algebra $\mathfrak{g}$. For $i \in [1,r]$, we denote by $x_i(t)$ and $y_i(t)$ the one-parameter subgroups in G given by

$$x_i(t) = \exp\,(te_i),\ y_i(t) = \exp\,(tf_i)$$

(note that in [11] the notation $x_{\overline{i}}(t)$ was used instead of $y_i(t)$). Let N (resp. N_-) be the maximal unipotent subgroup of G generated by all $x_i(t)$ (resp. $y_i(t)$). Let H be the maximal torus in G with the Lie algebra $\mathfrak{h}$. Let $B = HN$ and $B_- = HN_-$ be two opposite Borel subgroups, so that $H = B_- \cap B$. For every $i \in [1,r]$, let $\varphi_i : SL_2 \to G$ denote the canonical embedding corresponding to the simple root α_i ; thus we have

$$x_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ y_i(t) = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \ .$$

We also set

$$t^{\alpha_i^\vee} = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in H$$

for any i and any $t \neq 0$.

The Weyl group W of $\mathfrak{g}$ is naturally identified with $\mathrm{Norm}_G(H)/H$; this identification sends each simple reflection s_i to the coset $\overline{s_i}H$, where the representative $\overline{s_i} \in \mathrm{Norm}_G(H)$ is defined by

$$\overline{s_i} = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

The elements $\overline{s_i}$ satisfy the braid relations in W; thus the representative $\overline{w}$ can be unambiguously defined for any $w \in W$ by requiring that $\overline{uv} = \overline{u} \cdot \overline{v}$ whenever $\ell(uv) = \ell(u) + \ell(v)$.

The weight lattice P is identified with the group of multiplicative characters of H, here written in the exponential notation: a weight $\gamma \in P$ acts by $a \mapsto a^\gamma$. Under this identification, the fundamental weights $\omega_1, \ldots, \omega_r$ act in H by $(t^{\alpha_j^\vee})^{\omega_i} = t^{\delta_{ij}}$. The action of W on P can be now written as $a^{w(\gamma)} = (w^{-1}aw)^\gamma$ for $w \in W$, $a \in H$, $\gamma \in P$.

We denote by $G_0 = N_- HN$ the open subset of elements $x \in G$ that have Gaussian decomposition; this (unique) decomposition will be written as $x = [x]_-[x]_0[x]_+$.

Following G. Lusztig, we define the variety $G_{\geq 0}$ of totally nonnegative elements in G as the multiplicative monoid with unit generated by the elements $t^{\alpha_i^\vee}$, $x_i(t)$, and $y_i(t)$ for all i and all $t > 0$.

4.2. **Preliminaries on generalized minors.** We now recall some basic properties of *generalized minors* introduced in [11]. For $u, v \in W$ and $i \in [1, r]$, the *generalized minor* $\Delta_{u\omega_i, v\omega_i}$ is the regular function on G whose restriction to the open set $\overline{u} G_0 \overline{v}^{-1}$ is given by

$$\Delta_{u\omega_i, v\omega_i}(x) = (\left[\overline{u}^{\ -1} x \overline{v}\right]_0)^{\omega_i} \ . \tag{4.1}$$

As shown in [11], $\Delta_{u\omega_i, v\omega_i}$ depends on the weights $u\omega_i$ and $v\omega_i$ alone, not on the particular choice of u and v. In the special case $G = SL_n$, the generalized minors are nothing but the ordinary minors of a matrix.

Although we do not need it in this paper, we would like to mention the following characterization of the totally nonnegative variety obtained in [12]: an element $x \in G$ is totally nonnegative if and only if all generalized minors take nonnegative real values at x.

Generalized minors have the following properties: (see [11, (2.14), (2.25)]):

$$\Delta_{\gamma,\delta}(a_1 x a_2) = a_1^\gamma a_2^\delta \Delta_{\delta,\gamma}(x) \ (a_1, a_2 \in H;\ x \in G), \quad \Delta_{\gamma,\delta}(x) = \Delta_{\delta,\gamma}(x^T) \ , \tag{4.2}$$

where $x \mapsto x^T$ is the "transpose" involutive antiautomorphism of G given by

$$a^T = a \quad (a \in H) \ , \quad x_i(t)^T = y_i(t) \ , \quad y_i(t)^T = x_i(t) \ . \tag{4.3}$$

Later we shall need the involutive antiautomorphism $\tau_{w_\circ}$ of G introduced in [11, (2.56)]; it is defined by

$$\tau_{w_\circ}(x) = \overline{w_\circ}(x^{-1})^{\iota T} \overline{w_\circ}^{\ -1} \ , \tag{4.4}$$

where $x \mapsto x^\iota$ is the involutive antiautomorphism of G given by

$$a^\iota = a^{-1} \quad (a \in H) \ , \quad x_i(t)^\iota = x_i(t) \ , \quad y_i(t)^\iota = y_i(t) \ . \tag{4.5}$$

By [11, (2.25) and Lemma 2.25], we have

$$\Delta_{\gamma,\delta}(x) = \Delta_{-\delta,-\gamma}(x^\iota) = \Delta_{w_\circ\delta, w_\circ\gamma}(\tau_{w_\circ}(x)) \tag{4.6}$$

for any generalized minor $\Delta_{\gamma,\delta}$, and any $x \in G$.

Now we present some less obvious identities for generalized minors. The following identity was obtained in [11, Theorem 1.17].

Proposition 4.1. *Suppose $u, v \in W$ and $i \in [1, r]$ are such that $\ell(us_i) = \ell(u) + 1$ and $\ell(vs_i) = \ell(v) + 1$. Then*

$$\Delta_{u\omega_i, v\omega_i} \Delta_{us_i\omega_i, vs_i\omega_i} = \Delta_{us_i\omega_i, v\omega_i} \Delta_{u\omega_i, vs_i\omega_i} + \prod_{j \neq i} \Delta_{u\omega_j, v\omega_j}^{-a_{ji}} \ . \tag{4.7}$$

The next proposition presents some generalized Plücker relations; they follow from [9, Corollary 6.6] (see also [11, Theorem 1.16]).

Proposition 4.2. *Let $u, v \in W$ and $i, j \in [1, r]$.*
1. *If $a_{ij} = a_{ji} = -1$ and $\ell(vs_is_js_i) = \ell(v) + 3$, then*

$$\Delta_{u\omega_i, vs_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j} = \Delta_{u\omega_i, v\omega_i}\Delta_{u\omega_j, vs_is_j\omega_j} + \Delta_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_j, v\omega_j} .$$

2. *If $a_{ij} = -2$, $a_{ji} = -1$, and $\ell(vs_is_js_is_j) = \ell(v) + 4$, then*

$$\begin{aligned}&\Delta_{u\omega_i, vs_i\omega_i}\Delta_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j}\\&= \Delta^2_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_j, v\omega_j}\\&\qquad + \Delta_{u\omega_i, v\omega_i}(\Delta_{u\omega_i, v\omega_i}\Delta_{u\omega_j, vs_js_is_j\omega_j} + \Delta_{u\omega_i, vs_is_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j})\end{aligned}$$

and

$$\begin{aligned}&\Delta_{u\omega_j, vs_is_j\omega_j}\Delta^2_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j}\\&= \Delta_{u\omega_j, vs_js_is_j\omega_j}\Delta^2_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_j, v\omega_j}\\&\qquad + (\Delta_{u\omega_i, v\omega_i}\Delta_{u\omega_j, vs_js_is_j\omega_j} + \Delta_{u\omega_i, vs_is_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j})^2 .\end{aligned}$$

3. *If $a_{ij} = -3$, $a_{ji} = -1$, and $l(vs_is_js_is_js_is_j) = l(v) + 6$, then*

$$\begin{aligned}&\Delta_{u\omega_i, vs_i\omega_i}\Delta^2_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_i, vs_js_is_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j}\Delta_{u\omega_j, vs_js_is_j\omega_j}\\&\quad = \Delta^3_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_i, vs_js_is_js_i\omega_i}\Delta_{u\omega_j, v\omega_j}\Delta_{u\omega_j, vs_js_is_j\omega_j}\\&+ \big(\Delta_{u\omega_i, v\omega_i}\Delta_{u\omega_j, vs_js_is_j\omega_j} + \Delta_{u\omega_i, vs_js_is_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j}\big)^2\Delta_{u\omega_i, v\omega_i}\Delta_{u\omega_i, vs_js_is_js_i\omega_i}\\&+ \big(\Delta_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_j, vs_js_is_js_is_j\omega_j} + \Delta_{u\omega_i, vs_is_js_is_js_i\omega_i}\Delta_{u\omega_j, vs_js_is_j\omega_j}\big)\\&\quad \cdot\Delta_{u\omega_i, v\omega_i}\Delta^2_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j} ;\end{aligned}$$

$$\begin{aligned}&\Delta_{u\omega_i, vs_is_js_i\omega_i}\Delta^3_{u\omega_i, vs_js_i\omega_i}\Delta^2_{u\omega_i, vs_js_is_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j}\Delta^2_{u\omega_j, vs_js_is_j\omega_j}\\&\quad = \Delta^3_{u\omega_i, vs_js_i\omega_i}\Delta^3_{u\omega_i, vs_js_is_js_i\omega_i}\Delta_{u\omega_j, v\omega_j}\Delta^2_{u\omega_j, vs_js_is_j\omega_j}\\&+ \big(\Delta_{u\omega_i, v\omega_i}\Delta_{u\omega_j, vs_js_is_j\omega_j} + \Delta_{u\omega_i, vs_js_is_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j}\big)^3\Delta^3_{u\omega_i, vs_js_is_js_i\omega_i}\\&+ \big(\Delta_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_j, vs_js_is_js_is_j\omega_j} + \Delta_{u\omega_i, vs_is_js_is_js_i\omega_i}\Delta_{u\omega_j, vs_js_is_j\omega_j}\big)^2\\&\quad \cdot\Delta^4_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j}\\&+ \big(3\Delta_{u\omega_i, v\omega_i}\Delta_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_j, vs_js_is_j\omega_j}\Delta_{u\omega_j, vs_js_is_js_is_j\omega_j}\\&\qquad + 2\Delta_{u\omega_i, v\omega_i}\Delta_{u\omega_i, vs_is_js_is_js_i\omega_i}\Delta^2_{u\omega_j, vs_js_is_j\omega_j}\\&\qquad + 2\Delta_{u\omega_i, vs_js_is_js_i\omega_i}\Delta_{u\omega_i, vs_is_js_is_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j}\Delta_{u\omega_j, vs_js_is_j\omega_j}\\&+ 2\Delta_{u\omega_i, vs_js_i\omega_i}\Delta_{u\omega_i, vs_js_is_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j}\Delta_{u\omega_j, vs_js_is_js_is_j\omega_j}\big)\\&\quad \cdot\Delta^2_{u\omega_i, vs_js_i\omega_i}\Delta^2_{u\omega_i, vs_js_is_js_i\omega_i}\Delta_{u\omega_j, vs_j\omega_j} ;\end{aligned}$$

$$
\begin{aligned}
&\Delta_{u\omega_j,vs_is_js_is_j\omega_j}\Delta^3_{u\omega_i,vs_js_i\omega_i}\Delta^3_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\Delta^2_{u\omega_j,vs_js_is_j\omega_j}\\
&= \Delta^3_{u\omega_i,vs_js_i\omega_i}\Delta^3_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,v\omega_j}\Delta^2_{u\omega_j,vs_js_is_j\omega_j}\Delta_{u\omega_j,vs_js_is_js_is_j\omega_j}\\
&+\big(\Delta_{u\omega_i,v\omega_i}\Delta_{u\omega_j,vs_js_is_j\omega_j}+\Delta_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\big)^3\\
&\qquad\cdot\Delta^3_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_js_is_js_is_j\omega_j}\\
&+\big(\Delta_{u\omega_i,vs_js_i\omega_i}\Delta_{u\omega_j,vs_js_is_js_is_j\omega_j}+\Delta_{u\omega_i,vs_is_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_js_is_j\omega_j}\big)^3\\
&\qquad\cdot\Delta^3_{u\omega_i,vs_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\\
&+\big(3\Delta_{u\omega_i,v\omega_i}\Delta_{u\omega_i,vs_js_i\omega_i}\Delta_{u\omega_j,vs_js_is_j\omega_j}\Delta_{u\omega_j,vs_js_is_js_is_j\omega_j}\\
&\qquad\qquad+3\Delta_{u\omega_i,v\omega_i}\Delta_{u\omega_i,vs_is_js_is_js_i\omega_i}\Delta^2_{u\omega_j,vs_js_is_j\omega_j}\\
&\qquad\qquad+3\Delta_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_i,vs_is_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\Delta_{u\omega_j,vs_js_is_j\omega_j}\\
&+2\Delta_{u\omega_i,vs_js_i\omega_i}\Delta_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\Delta_{u\omega_j,vs_js_is_js_is_j\omega_j}\big)\\
&\qquad\cdot\Delta^2_{u\omega_i,vs_js_i\omega_i}\Delta^2_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\Delta_{u\omega_j,vs_js_is_js_is_j\omega_j}\ ;
\end{aligned}
$$

$$
\begin{aligned}
&\Delta_{u\omega_j,vs_is_j\omega_j}\Delta^6_{u\omega_i,vs_js_i\omega_i}\Delta^3_{u\omega_i,vs_js_is_js_i\omega_i}\Delta^2_{u\omega_j,vs_j\omega_j}\Delta^3_{u\omega_j,vs_js_is_j\omega_j}\\
&\quad= \Delta^3_{u\omega_i,vs_js_i\omega_i}\Delta^2_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,v\omega_j}\Delta^2_{u\omega_j,vs_js_is_j\omega_j}\\
&\cdot\big\{\Delta^3_{u\omega_i,vs_js_i\omega_i}\Delta_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,v\omega_j}\Delta^2_{u\omega_j,vs_js_is_j\omega_j}\\
&\quad+2\big(\Delta_{u\omega_i,v\omega_i}\Delta_{u\omega_j,vs_js_is_j\omega_j}+\Delta_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\big)^3\Delta_{u\omega_i,vs_js_is_js_i\omega_i}\\
&+\big(3\Delta_{u\omega_i,v\omega_i}\Delta_{u\omega_i,vs_js_i\omega_i}\Delta_{u\omega_j,vs_js_is_j\omega_j}\Delta_{u\omega_j,vs_js_is_js_is_j\omega_j}\\
&\qquad\qquad+3\Delta_{u\omega_i,v\omega_i}\Delta_{u\omega_i,vs_is_js_is_js_i\omega_i}\Delta^2_{u\omega_j,vs_js_is_j\omega_j}\\
&\qquad\qquad+3\Delta_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_i,vs_is_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\Delta_{u\omega_j,vs_js_is_j\omega_j}\\
&+2\Delta_{u\omega_i,vs_js_i\omega_i}\Delta_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\Delta_{u\omega_j,vs_js_is_js_is_j\omega_j}\big)\\
&\quad\cdot\Delta^2_{u\omega_i,vs_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\big\}\\
&+\big\{\big(\Delta_{u\omega_i,v\omega_i}\Delta_{u\omega_j,vs_js_is_j\omega_j}+\Delta_{u\omega_i,vs_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\big)^2\Delta_{u\omega_i,vs_js_is_js_i\omega_i}\\
&+\big(\Delta_{u\omega_i,vs_js_i\omega_i}\Delta_{u\omega_j,vs_js_is_js_is_j\omega_j}+\Delta_{u\omega_i,vs_is_js_is_js_i\omega_i}\Delta_{u\omega_j,vs_js_is_j\omega_j}\big)\\
&\quad\cdot\Delta^2_{u\omega_i,vs_js_i\omega_i}\Delta_{u\omega_j,vs_j\omega_j}\big\}^3\ .
\end{aligned}
$$

4.3. **Reduced double Bruhat cells.** The group G has two *Bruhat decompositions*, with respect to opposite Borel subgroups B and B_- :

$$G = \bigcup_{u\in W} BuB = \bigcup_{v\in W} B_-vB_-\ .$$

The *double Bruhat cells* $G^{u,v}$ are defined by $G^{u,v} = BuB\cap B_-vB_-$. These varieties were introduced and studied in [11].

In this paper we shall concentrate on the following subset $L^{u,v}\subset G^{u,v}$ which we call a *reduced double Bruhat cell*:

$$L^{u,v} = N\overline{u}N\cap B_-vB_-\ . \tag{4.8}$$

(In particular, if u is the identity element $e \in W$ then $L^{e,v} = N \cap B_- v B_-$ is the variety N^v studied in [9].) We also set

$$L^{u,v}_{>0} := L^{u,v} \cap G_{\geq 0} \ ,$$

and call $L^{u,v}_{>0}$ the *totally positive part* of $L^{u,v}$.

The equations defining $L^{u,v}$ inside $G^{u,v}$ look as follows.

Proposition 4.3. *An element $x \in G^{u,v}$ belongs to $L^{u,v}$ if and only if $[\overline{u}^{\ -1} x]_0 = 1$, or equivalently if $\Delta_{u\omega_i,\omega_i}(x) = 1$ for any $i \in [1, r]$.*

The maximal torus H acts freely on $G^{u,v}$ by left (or right) translations, and $L^{u,v}$ is a section of this action. Thus $L^{u,v}$ is naturally identified with $G^{u,v}/H$, and all properties of $G^{u,v}$ can be translated in a straightforward way into the corresponding properties of $L^{u,v}$. In particular, Theorem 1.1 in [11] implies the following.

Proposition 4.4. *The variety $L^{u,v}$ is biregularly isomorphic to a Zariski open subset of an affine space of dimension $\ell(u) + \ell(v)$.*

We now introduce a family of systems of local coordinates in $L^{u,v}$. For any nonzero $t \in \mathbb{C}$ and any $i \in [1, r]$, denote

$$x_{-i}(t) = y_i(t) t^{-\alpha_i^\vee} = \varphi_i \begin{pmatrix} t^{-1} & 0 \\ 1 & t \end{pmatrix} . \tag{4.9}$$

A *double reduced word* for a pair $u, v \in W$ is a reduced word for an element (u, v) of the Coxeter group $W \times W$. To avoid confusion, we will use the indices $-1, \ldots, -r$ for the simple reflections in the first copy of W, and $1, \ldots, r$ for the second copy. A double reduced word for (u, v) is nothing but a shuffle of a reduced word $\mathbf{i}$ for u written in the alphabet $[-1, -r]$ (we will denote such a word by $-\mathbf{i}$) and a reduced word $\mathbf{i}'$ for v written in the alphabet $[1, r]$. We denote the set of double reduced words for (u, v) by $R(u, v)$.

For any sequence $\mathbf{i} = (i_1, \ldots, i_m)$ of indices from the alphabet $[1, r] \cup [-1, -r]$, let us define the *product map* $x_{\mathbf{i}} : \mathbb{C}^m_{\neq 0} \to G$ by

$$x_{\mathbf{i}}(t_1, \ldots, t_m) = x_{i_1}(t_1) \cdots x_{i_m}(t_m) . \tag{4.10}$$

The following proposition is a modification of [11, Theorems 1.2, 1.3].

Proposition 4.5. *For any $u, v \in W$ and $\mathbf{i} = (i_1, \ldots, i_m) \in R(u, v)$, the map $x_{\mathbf{i}}$ is a biregular isomorphism between $\mathbb{C}^m_{\neq 0}$ and a Zariski open subset of $L^{u,v}$, and it restricts to a bijection between the set $\mathbb{R}^m_{>0}$ of m-tuples of positive real numbers and the totally positive part $L^{u,v}_{>0}$ of $L^{u,v}$.*

4.4. **Factorization problem for reduced double Bruhat cells.** In this section, we address the following *factorization problem* for $L^{u,v}$: for any double reduced word $\mathbf{i} \in R(u, v)$, find explicit formulas for the inverse birational isomorphism $x_{\mathbf{i}}^{-1}$ between $L^{u,v}$ and $\mathbb{C}^m_{\neq 0}$, thus expressing the *factorization parameters* t_k in terms of the product $x = x_{\mathbf{i}}(t_1, \ldots, t_m) \in L^{u,v}$.

Recall from [11, Section 1.5] that there is a biregular "shift" isomorphism $\zeta^{u,v} : G^{u,v} \to G^{u^{-1},v^{-1}}$. This isomorphism does not however send $L^{u,v}$ to $L^{u^{-1},v^{-1}}$, so we will use the following modification.

Definition 4.6. For any $u, v \in W$, the twist map $\psi^{u,v}$ is defined by (see (4.5))

$$\psi^{u,v}(x) = [(\overline{v} x^\iota)^{-1}]_+ \, \overline{v} \, ([\overline{u}^{\ -1} x]_+)^\iota \ . \tag{4.11}$$

Theorem 4.7. *The twist map $\psi^{u,v}$ is a biregular isomorphism between $L^{u,v}$ and $L^{v,u}$, and it restricts to a bijection between $L^{u,v}_{>0}$ and $L^{v,u}_{>0}$. The inverse isomorphism is $\psi^{v,u}$.*

Now let us fix a pair $(u,v) \in W \times W$ and a double reduced word $\mathbf{i} = (i_1, \ldots, i_m) \in R(u,v)$. Recall that $\mathbf{i}$ is a shuffle of a reduced word for u written in the alphabet $[-1,-r]$ and a reduced word for v written in the alphabet $[1,r]$. In particular, the length m of $\mathbf{i}$ is equal to $\ell(u) + \ell(v)$. For $k \in [1,m]$, we denote

$$k^- = \max\{l : l < k, |i_l| = |i_k|\}\ ,\ k^+ = \min\{l : l > k, |i_l| = |i_k|\}\ ,$$

so that k^- (resp. k^+) is the previous (resp. next) occurrence of an index $\pm i_k$ in $\mathbf{i}$; if k is the first (resp. last) occurrence of $\pm i_k$ in $\mathbf{i}$ then we set $k^- = 0$ (resp. $k^+ = m+1$).

For $k \in [1,m]$, denote

$$u_{\geq k} = \prod_{\substack{l=m,\ldots,k \\ i_l<0}} s_{-i_l}\ , \quad v_{<k} = \prod_{\substack{l=1,\ldots,k-1 \\ i_l>0}} s_{i_l}\ . \tag{4.12}$$

This notation means that in the first (resp. second) product in (4.12), the index l is decreasing (resp. increasing); for example, if $\mathbf{i} = (-2,1,-3,3,2,-1,-2,1,-1)$, then, say, $u_{\geq 7} = s_1 s_2$ and $v_{<7} = s_1 s_3 s_2$. Let us define a regular function $M_k = M_{k,\mathbf{i}}$ on $L^{u,v}$ by

$$M_k(x) = M_{k,\mathbf{i}}(x) = \Delta_{v_{<k}\omega_{|i_k|}, u_{\geq k}\omega_{|i_k|}}(\psi^{u,v}(x))\ ; \tag{4.13}$$

by convention, we set $M_{m+1}(x) = 1$ (by Theorem 4.7, $\psi^{u,v}(x) \in L^{v,u}$ for $x \in L^{u,v}$, hence $\Delta_{v\omega_i,\omega_i}(\psi^{u,v}(x)) = 1$ for all i (see Proposition 4.3)).

Now we are ready to state our solution to the factorization problem.

Theorem 4.8. *Let $\mathbf{i} = (i_1, \ldots, i_m)$ be a double reduced word for (u,v), and suppose an element $x \in L^{u,v}$ can be factored as $x = x_{i_1}(t_1) \cdots x_{i_m}(t_m)$, with all t_k nonzero complex numbers. Then the factorization parameters t_k are determined by the following formulas: if $i_k < 0$ then*

$$t_k = M_k(x)/M_{k^+}(x)\ ; \tag{4.14}$$

if $i_k > 0$ then

$$t_k = \frac{1}{M_k(x) M_{k^+}(x)} \prod_{l:\ l^- < k < l} M_l(x)^{-a_{|i_l|,i_k}}\ . \tag{4.15}$$

The following two special cases will be of particular importance for us: $(u,v) = (e, w_\mathrm{o})$, and $(u,v) = (w_\mathrm{o}, e)$. In these cases, Definition 4.6 and Theorem 4.7 can be simplified as follows.

Corollary 4.9. *The twist map $\psi^{w_\mathrm{o},e}$ is a biregular isomorphism between $L^{w_\mathrm{o},e}$ and L^{e,w_o} (and also a bijection between $L^{w_\mathrm{o},e}_{>0}$ and $L^{e,w_\mathrm{o}}_{>0}$) given by*

$$\psi^{w_\mathrm{o},e}(x) = ([\overline{w_\mathrm{o}}^{\ -1} x]_+)^\iota\ . \tag{4.16}$$

The inverse biregular isomorphism ψ^{e,w_o} is given by

$$\psi^{e,w_\mathrm{o}}(x) = ([x\overline{w_\mathrm{o}}]_-\ [x\overline{w_\mathrm{o}}]_0)^\iota\ . \tag{4.17}$$

The formulas (4.14) and (4.15) now take the following form.

Corollary 4.10. *Let $\mathbf{i} = (i_1, \ldots, i_m)$ be a reduced word for w_o, and $t_1, \ldots, t_m$ be non-zero complex numbers.*

(i) *If* $x = x_{-i_1}(t_1)\cdots x_{-i_m}(t_m)$ *then the factorization parameters* t_k *are given by*

$$t_k = \frac{\Delta_{\omega_{i_k}, s_{i_m}\cdots s_{i_k}\omega_{i_k}}(\psi^{w_o,e}(x))}{\Delta_{\omega_{i_k}, s_{i_m}\cdots s_{i_{k+1}}\omega_{i_k}}(\psi^{w_o,e}(x))} \ ,$$

where $\psi^{w_o,e}(x)$ *is given by* (4.16).

(ii) *If* $x = x_{i_1}(t_1)\cdots x_{i_m}(t_m)$ *then the factorization parameters* t_k *are given by*

$$t_k = \frac{1}{\Delta_{s_{i_1}\cdots s_{i_{k-1}}\omega_{i_k},\omega_{i_k}}(\psi^{e,w_o}(x))\Delta_{s_{i_1}\cdots s_{i_k}\omega_{i_k},\omega_{i_k}}(\psi^{e,w_o}(x))} \times \prod_{j\neq i_k} \Delta_{s_{i_1}\cdots s_{i_{k-1}}\omega_j,\omega_j}(\psi^{e,w_o}(x))^{-a_{j,i_k}} \ ,$$

where $\psi^{e,w_o}(x)$ *is given by* (4.17).

In general, Theorem 4.8 expresses the factorization parameters of an element $x \in L^{u,v}$ as monomials in generalized minors of the twisted element $\psi^{u,v}(x)$. However, there are two important special cases when taking the twist is unnecessary.

Proposition 4.11. *Let* $\mathbf{i} = (i_1, \ldots, i_m)$ *be a reduced word for* w_o.

(i) *If* $x = x_{-i_1}(t_1)\cdots x_{-i_m}(t_m)$ *then the first and the last factorization parameters of* x *are given by*

$$t_1 = \frac{1}{\Delta_{s_{i_1}w_o\omega_{i_1^*},\omega_{i_1^*}}(x)} \ , \ t_m = \Delta_{w_o\omega_{i_m}, s_{i_m}\omega_{i_m}}(x) \ , \tag{4.18}$$

where $\omega_{i^*} = -w_o\omega_i$.

(ii) *If* $x = x_{i_1}(t_1)\cdots x_{i_m}(t_m)$ *then the first and the last factorization parameters of* x *are given by*

$$t_1 = \frac{\Delta_{\omega_{i_1}, w_o\omega_{i_1}}(x)}{\Delta_{s_{i_1}\omega_{i_1}, w_o\omega_{i_1}}(x)} \ , \ t_m = \frac{\Delta_{\omega_{i_m^*}, w_o\omega_{i_m^*}}(x)}{\Delta_{\omega_{i_m^*}, s_{i_m}w_o\omega_{i_m^*}}(x)} \ . \tag{4.19}$$

5. Geometric lifting and tropicalization

5.1. **Transition maps.** Let $\mathbf{i}$ and $\mathbf{i}'$ be two reduced words of w_o. In view of Proposition 3.2, there is a bijective *transition map*

$$R_{\mathbf{i}}^{\mathbf{i}'} = (b_{\mathbf{i}'})^{-1} \circ b_{\mathbf{i}} : \mathbb{Z}_{\geq 0}^m \to \mathbb{Z}_{\geq 0}^m$$

between the corresponding Lusztig parametrizations of the canonical basis $\mathcal{B}$. Similarly, by Proposition 3.5, there is a bijective transition map

$$R_{-\mathbf{i}}^{-\mathbf{i}'} = c_{\mathbf{i}'} \circ (c_{\mathbf{i}})^{-1} : C_{\mathbf{i}}(\mathbb{Z}) \to C_{\mathbf{i}'}(\mathbb{Z})$$

between the two string parametrizations of the dual canonical basis (the reason for the notation $R_{-\mathbf{i}}^{-\mathbf{i}'}$ will become clear soon).

It turns out that each component of a tuple $R_{\mathbf{i}}^{\mathbf{i}'}(t)$ or $R_{-\mathbf{i}}^{-\mathbf{i}'}(t)$ can be expressed through the components of t as a "tropical" subtraction-free rational expression (see Introduction).

Example 5.1. Let $\mathfrak{g} = sl_3$ be of type A_2, and let $\mathbf{i} = (1, 2, 1)$ and $\mathbf{i}' = (2, 1, 2)$ be the two reduced words for w_o. The transition map $R_{\mathbf{i}}^{\mathbf{i}'}$ between two Lusztig parametrizations was computed in [20]: the components of $t' = R_{\mathbf{i}}^{\mathbf{i}'}(t)$ are given by

$$t_1' = t_2 + t_3 - \min\,(t_1, t_3), \ t_2' = \min\,(t_1, t_3), \ t_3' = t_1 + t_2 - \min\,(t_1, t_3) \ ,$$

which can also be written as

$$(5.1) \qquad t'_1 = \left[\frac{t_2t_3}{t_1+t_3}\right]_{\mathrm{trop}},\ t'_2 = [t_1+t_3]_{\mathrm{trop}},\ t'_3 = \left[\frac{t_1t_2}{t_1+t_3}\right]_{\mathrm{trop}} .$$

The transition map $R_{-\mathbf{i}}^{-\mathbf{i}'}$ between two string parametrizations was computed in [8]: the components of $t' = R_{-\mathbf{i}}^{-\mathbf{i}'}(t)$ are given by

$$t'_1 = \max\,(t_3, t_2 - t_1),\ t'_2 = t_1 + t_3,\ t'_3 = \min\,(t_1, t_2 - t_3)\ ,$$

which can also be written as

$$(5.2) \qquad t'_1 = \left[\frac{t_2t_3}{t_1t_3+t_2}\right]_{\mathrm{trop}},\ t'_2 = [t_1t_3]_{\mathrm{trop}},\ t'_3 = \left[\frac{t_1t_3+t_2}{t_3}\right]_{\mathrm{trop}} .$$

We now generalize this example by giving a geometric lifting for each of the transition maps $R_{\mathbf{i}}^{\mathbf{i}'}$ and $R_{-\mathbf{i}}^{-\mathbf{i}'}$. To do this, we notice that by Proposition 4.5, for any two reduced words $\mathbf{i}$ and $\mathbf{i}'$ of a pair (u,v) of elements of W, there is a bijective transition map

$$\tilde R_{\mathbf{i}}^{\mathbf{i}'} = (x_{\mathbf{i}'})^{-1} \circ x_{\mathbf{i}} : \mathbb{R}^m_{>0} \to \mathbb{R}^m_{>0}$$

that relates the corresponding parametrizations of the totally positive variety $L^{u,v}_{>0}$. In particular, any two reduced words $\mathbf{i}$ and $\mathbf{i}'$ for w_o give rise to transition maps $\tilde R_{\mathbf{i}}^{\mathbf{i}'}$ and $\tilde R_{-\mathbf{i}}^{-\mathbf{i}'}$ (for the varieties $L^{e,w_\mathrm{o}}_{>0}$ and $L^{w_\mathrm{o},e}_{>0}$, respectively). We shall use the notation $(\tilde R_{\mathbf{i}}^{\mathbf{i}'})^\vee$ for the transition maps defined in the same way but for the Langlands dual group ${}^LG$ instead of G.

Theorem 5.2. (i) *For any $\mathbf{i}, \mathbf{i}' \in R(u,v)$, each component of $\tilde R_{\mathbf{i}}^{\mathbf{i}'}(t)$ is a subtraction-free rational expression in the components of t.*
(ii) *For any $\mathbf{i}, \mathbf{i}' \in R(w_\mathrm{o})$, each component of $(\tilde R_{\mathbf{i}}^{\mathbf{i}'})^\vee(t)$ (resp. $(\tilde R_{-\mathbf{i}}^{-\mathbf{i}'})^\vee(t)$) is a geometric lifting of the corresponding component of $R_{\mathbf{i}}^{\mathbf{i}'}(t)$ (resp. of $R_{-\mathbf{i}}^{-\mathbf{i}'}(t)$).*

As in [3] and [9, Section 4], Theorem 5.2(i) allows us to extend the definition of the totally positive varieties $L^{u,v}_{>0}$ from the "ground semifield" $\mathbb{R}_{>0}$ to an arbitrary semifield K (see Introduction). To do this, we define $\mathcal{L}^{u,v}(K)$ as the set of all tuples $\mathbf{t} = (t^{\mathbf{i}})_{\mathbf{i}\in R(u,v)}$, where each $t^{\mathbf{i}} = (t^{\mathbf{i}}_1, \ldots, t^{\mathbf{i}}_m)$ is a "vector" in K^m (with $m = \ell(u)+\ell(v)$), and these vectors satisfy the relations $t^{\mathbf{i}'} = \tilde R_{\mathbf{i}}^{\mathbf{i}'}(t^{\mathbf{i}})$ for all $\mathbf{i}, \mathbf{i}' \in R(u,v)$.

Example 5.3. By Proposition 4.5, the map $\mathcal{L}^{u,v}(\mathbb{R}_{>0}) \to G$ given by $\mathbf{t} \mapsto x_{\mathbf{i}}(t^{\mathbf{i}})$ is well-defined, and it is a bijection between $\mathcal{L}^{u,v}(\mathbb{R}_{>0})$ and $L^{u,v}_{>0}$.

Example 5.4. By Proposition 3.2 and Theorem 5.2 (ii), the map that sends every canonical basis vector $b \in \mathcal{B}$ to a tuple $\mathbf{t}$ with $t^{\mathbf{i}} = b_{\mathbf{i}}^{-1}(b)$ is a bijection between $\mathcal{B}$ and the set of all $\mathbf{t} \in \mathcal{L}^{e,w_\mathrm{o}}(\mathbb{Z}_{\mathrm{trop}})^\vee$ such that $t^{\mathbf{i}} \in \mathbb{Z}^m_{\geq 0}$ for all $\mathbf{i} \in R(w_\mathrm{o})$ (as usual in this paper, $\mathcal{L}^{e,w_\mathrm{o}}(K)^\vee$ stands for the set defined in the same way as $\mathcal{L}^{e,w_\mathrm{o}}(K)$ but for the Langlands dual group).

Example 5.5. By Proposition 3.5 and Theorem 5.2 (ii), the map that sends every dual canonical basis vector $b^* \in \mathcal{B}^{\mathrm{dual}}$ to a tuple $\mathbf{t}$ with $t^{-\mathbf{i}} = c_{\mathbf{i}}(b^*)$ is a bijection between $\mathcal{B}^{\mathrm{dual}}$ and the set of all $\mathbf{t} \in \mathcal{L}^{w_\mathrm{o},e}(\mathbb{Z}_{\mathrm{trop}})^\vee$ such that $t^{-\mathbf{i}} \in C_{\mathbf{i}}(\mathbb{Z})$ for all $\mathbf{i} \in R(w_\mathrm{o})$.

Using Examples 5.4 and 5.5, we shall identify $\mathcal{B}$ (resp. $\mathcal{B}^{\mathrm{dual}}$) with a part of $\mathcal{L}^{e,w_\mathrm{o}}(\mathbb{Z}_{\mathrm{trop}})^\vee$ (resp. $\mathcal{L}^{w_\mathrm{o},e}(\mathbb{Z}_{\mathrm{trop}})^\vee$). It is easy to see that the correspondence $b \mapsto b^{\mathrm{dual}}$ extends to a piecewise-linear map $\mathcal{L}^{e,w_\mathrm{o}}(\mathbb{Z}_{\mathrm{trop}})^\vee \to \mathcal{L}^{w_\mathrm{o},e}(\mathbb{Z}_{\mathrm{trop}})^\vee$. Our next goal is to find a geometric lifting for this map. To do this, we consider the following modification of the twist maps $\psi^{w_o,e}$ and ψ^{e,w_o} in Corollary 4.9. Let us define the maps $\eta^{w_o,e}$ and η^{e,w_o} by setting (see (4.4))

$$\eta^{w_o,e} = \tau_{w_o} \circ \psi^{w_o,e}, \quad \eta^{e,w_o} = \psi^{e,w_o} \circ \tau_{w_o} \ ; \tag{5.3}$$

an easy calculation shows that these maps are given by (see (4.3))

$$\eta^{w_o,e}(x) = [(\overline{w_o} x^T)^{-1}]_+, \quad \eta^{e,w_o}(x) = ([\overline{w_o}^{\ -1} x^T]_0)^{-1}([\overline{w_o}^{\ -1} x^T]_-)^{\ -1} \ . \tag{5.4}$$

Theorem 4.7 immediately implies the following.

Corollary 5.6. *The map $\eta^{w_o,e}$ is a biregular isomorphism between $L^{w_o,e}$ and L^{e,w_o}, and it restricts to a bijection between $L^{w_o,e}_{>0}$ and $L^{e,w_o}_{>0}$. The inverse map is η^{e,w_o}.*

Now we are ready to state our geometric lifting result.

Theorem 5.7. *The map $\eta^{w_o,e} : (L^{w_o,e}_{>0})^\vee \to (L^{e,w_o}_{>0})^\vee$ is a geometric lifting of $b \mapsto b^{\mathrm{dual}}$. In other words, for any $\mathbf{i}, \mathbf{i}' \in R(w_o)$, we have*

$$c_{\mathbf{i}}(b_{\mathbf{i}'}(t')^{\mathrm{dual}}) = [(x_{-\mathbf{i}}^{-1} \circ \eta^{e,w_o} \circ x_{\mathbf{i}'})^\vee(t')]_{\mathrm{trop}} \tag{5.5}$$

(as before, the superscript $^\vee$ means that the corresponding varieties and maps are related to the group ^LG).

Theorems 5.2 and 5.7 play crucial role in our proofs of the results in Sections 2 and 3; in fact, they allow us to deduce combinatorial properties of the canonical basis from the properties of totally positive varieties. The appearance of $\mathbf{i}$-trails is explained by the following evaluation of generalized minors.

Theorem 5.8. *Let γ and δ be two weights in the W-orbit of the same fundamental weight ω_i of $\mathfrak{g}$, and let $\mathbf{i} = (i_1, \ldots, i_m)$ be any sequence of indices from $[1, r]$.*
(i) *$\Delta_{\gamma,\delta}(x_{\mathbf{i}}(t_1, \ldots, t_m))$ is a positive integer linear combination of the monomials $t_1^{c_1(\pi)} \cdots t_m^{c_m(\pi)}$ for all $\mathbf{i}$-trails π from γ to δ in V_{ω_i}.*
(ii) *$\Delta_{\gamma,\delta}(x_{-\mathbf{i}}(t_1, \ldots, t_m))$ is a positive integer linear combination of the monomials $t_1^{d_1(\pi)} \cdots t_m^{d_m(\pi)}$ for all $\mathbf{i}$-trails π from $-\gamma$ to $-\delta$ in $V_{\omega_{i^*}}$.*

Theorem 5.8 has an important "tropical" corollary.

Corollary 5.9. *In the situation of Theorem 5.8, we have*

$$[\Delta_{\gamma,\delta}(x_{\mathbf{i}}(t_1, \ldots, t_m))]_{\mathrm{trop}} = \min_{\pi} \ (\sum_{k=1}^m c_k(\pi) t_k) \ ,$$

$$[\Delta_{\gamma,\delta}(x_{-\mathbf{i}}(t_1, \ldots, t_m))]_{\mathrm{trop}} = \min_{\pi'} \ (\sum_{k=1}^m d_k(\pi') t_k) \ ,$$

where π (resp. π') runs over all $\mathbf{i}$-trails from γ to δ in V_{ω_i} (resp. from $-\gamma$ to $-\delta$ in $V_{\omega_{i^*}}$).

5.2. **Geometric lifting of Kashiwara's crystals.** Let us recall the crystal operators $\tilde{f}_i : \mathcal{B} \to \mathcal{B}$ introduced in Proposition 3.6. In this section we compute a geometric lifting of the twisted operators $\tilde{f}_i^\iota : \mathcal{B} \to \mathcal{B}$ defined by $\tilde{f}_i^\iota(b) = (\tilde{f}_i(b^\iota))^\iota$. In view of Proposition 3.3 (iii)), for every positive integer n, the operator $(\tilde{f}_i^\iota)^n$ acts as follows on Lusztig parameters corresponding to any reduced word $\mathbf{i}' \in R(w_o)$ with $i'_m = i^*$:

$$(\tilde{f}_i^\iota)^n(b_{\mathbf{i}'}(t_1, \ldots, t_{m-1}, t_m)) = b_{\mathbf{i}'}(t_1, \ldots, t_{m-1}, t_m + n) \ .$$

We consider the following geometric counterpart of this operator: for any $c > 0$ define the bijection $F_i^{(c)} : L_{>0}^{e,w_\circ} \to L_{>0}^{e,w_\circ}$ by

$$F_i^{(c)}(x_{\mathbf{i}'}(t'_1, \ldots, t'_{m-1}, t'_m)) = x_{\mathbf{i}'}(t'_1, \ldots, t'_{m-1}, ct'_m) \ , \tag{5.6}$$

where $\mathbf{i}' \in R(w_\circ)$ ends with $i'_m = i^*$. If $\mathbf{i}' \in R(w_\circ)$ does not end with i^* then we do not have a nice formula for $F_i^{(c)}(x_{\mathbf{i}'}(t'_1, \ldots, t'_m))$. Our next result shows that such a formula exists for the bijection $L_{>0}^{w_\circ,e} \to L_{>0}^{w_\circ,e}$ obtained by transferring $F_i^{(c)}$ with the help of the bijection $\eta^{w_\circ,e} : L_{>0}^{w_\circ,e} \to L_{>0}^{e,w_\circ}$ (see Corollary 5.6).

Theorem 5.10. *Let $\mathbf{i}$ be a reduced word for $R(w_\circ)$, and let $T_k = t_k^{-1} \prod_{l>k} t_l^{-a_{i_l,i_k}}$ for $k = 1, \ldots, m$. Then*

$$(\eta^{e,w_\circ} \circ F_i^{(c)} \circ \eta^{w_\circ,e})(x_{-\mathbf{i}}(t_1, \ldots, t_m)) = x_{-\mathbf{i}}(\tilde t_1, \ldots, \tilde t_m) \ ,$$

where $\tilde t_k = t_k$ unless $i_k = i$, and

$$\tilde t_k = t_k \frac{\sum\limits_{l<k:i_l=i} T_l + c \sum\limits_{l\geq k:i_l=i} T_l}{\sum\limits_{l\leq k:i_l=i} T_l + c \sum\limits_{l>k:i_l=i} T_l} \ , \tag{5.7}$$

whenever $i_k = i$.

In view of Theorem 5.7, one obtains an explicit formula for the action of $(\tilde f_i^\iota)^n$ on $\mathcal{B}$ in terms of the string parameters by tropicalizing (5.7) (and passing from G to ${}^LG$ as usual).

Corollary 5.11. *For a reduced word $\mathbf{i} \in R(w_\circ)$, define the linear forms $T_k^\vee : \mathbb{Z}^m \to \mathbb{Z}^m$ for $k = 1, \ldots, m$ by $T_k^\vee(t) = -t_k - \sum_{l>k} a_{i_k,i_l} t_l$. If $c_{\mathbf{i}}(b^{\mathrm{dual}}) = (t_1, \ldots, t_m)$ then $c_{\mathbf{i}}((\tilde f_i^\iota)^n(b)^{\mathrm{dual}}) = (\tilde t_1, \ldots, \tilde t_m)$, where $\tilde t_k = t_k$ unless $i_k = i$, and*

$$\tilde t_k = t_k + \min\left(\min_{l<k:i_l=i} T_l^\vee(t), n + \min_{l\geq k:i_l=i} T_l^\vee(t)\right)$$
$$- \min\left(\min_{l\leq k:i_l=i} T_l^\vee(t), n + \min_{l>k:i_l=i} T_l^\vee(t)\right)$$

whenever $i_k = i$ (with the agreement that minimum over the empty set is $+\infty$).

Remark 5.12. Theorem 5.10 is a starting point of a new concept of *geometric crystals* introduced and developed by one of the authors (A.B.) in a joint work in progress with D. Kazhdan.

5.3. **Plücker models.** Following [9, Section 4], we now consider the "variety" $\mathcal{M}^{w_\circ}(K)$ of all tuples $(M_{\omega_i,\gamma})$ of elements of the ground semifield K (for all $i \in [1, r]$, and $\gamma \in W\omega_i$) satisfying the relations in Proposition 4.2 (with $u = e$, and each generalized minor $\Delta_{\omega_i,\gamma}$ replaced with $M_{\omega_i,\gamma}$). We shall show that each of the varieties $\mathcal{L}^{e,w_\circ}(K)$ and $\mathcal{L}^{w_\circ,e}(K)$ is in a natural bijection with a part of $\mathcal{M}^{w_\circ}(K)$. To define these bijections, we use Theorem 5.8, which assures that both $t \mapsto \Delta_{\gamma,\delta}(x_{\mathbf{i}}(t_1, \ldots, t_m))$ and $t \mapsto \Delta_{\gamma,\delta}(x_{-\mathbf{i}}(t_1, \ldots, t_m))$ are well-defined mappings $K^m \to K \cup \{0\}$ for any semifield K.

Theorem 5.13. (i) *For every semifield K, the correspondence*

$$p^+ : \mathbf{t} \mapsto (M_{\omega_i,\gamma} = \Delta_{\omega_i,\gamma}(x_{\mathbf{i}}(t^{\mathbf{i}})))$$

(where $\mathbf{i} \in R(w_\circ)$) is an embedding of $\mathcal{L}^{e,w_\circ}(K)$ into $\mathcal{M}^{w_\circ}(K)$. The image of p^+ consists of all tuples $(M_{\omega_i,\gamma}) \in \mathcal{M}^{w_\circ}(K)$ such that $M_{\omega_i,\omega_i} = 1$ for all i.

(ii) *The map* $p^- : \mathbf{t} \mapsto (M_{\omega_i,\gamma} = \Delta_{\gamma,\omega_i}(x_{-\mathbf{i}}(t^{\mathbf{i}})))$ *is an embedding of* $\mathcal{L}^{w_\circ,e}(K)$ *into* $\mathcal{M}^{w_\circ}(K)$. *The image of* p^- *consists of all tuples* $(M_{\omega_i,\gamma}) \in \mathcal{M}^{w_\circ}(K)$ *such that* $M_{\omega_i,w_\circ\omega_i} = 1$ *for all* i.

We will refer to the maps p^+ and p^- as *Plücker models* of $\mathcal{L}^{e,w_\circ}(K)$ and $\mathcal{L}^{w_\circ,e}(K)$.

Specializing Theorem 5.13 to the tropical semifield $K = \mathbb{Z}_{\mathrm{trop}}$, we obtain embeddings $p^+_{\mathrm{trop}} : \mathcal{B} \to \mathcal{M}^{w_\circ}(\mathbb{Z}_{\mathrm{trop}})^\vee$ and $p^-_{\mathrm{trop}} : \mathcal{B}^{\mathrm{dual}} \to \mathcal{M}^{w_\circ}(\mathbb{Z}_{\mathrm{trop}})^\vee$. (As before, the variety $\mathcal{M}^{w_\circ}(K)^\vee$ corresponds to the Langlands dual group ${}^LG$.) Our next task is to describe the images of these embeddings. To do this, we notice that for every $\gamma \in W\omega_i \setminus \{\omega_i\}$, there is a naturally defined function $M_{s_i\omega_i,\gamma} : \mathcal{M}^{w_\circ}(K) \to K$; it can be obtained as a subtraction-free expression in the components $M_{\omega_j,\delta}$ by iterating the identity (4.7) (with Δ replaced by M) and using the boundary condition $M_{s_i\omega_i,\omega_i} = 0$. An explicit formula for $M_{s_i\omega_i,\gamma}$ can be given as follows: let $\gamma = u\omega_i = s_{i_1}\cdots s_{i_l}\omega_i$, where $(i_1,\ldots,i_l)$ is a reduced word for $u \in W$ such that $i_l = i$; then we have

$$M_{s_i\omega_i,\gamma} = M_{\omega_i,\gamma} \sum_{\substack{k \le l \\ i_k = i}} \frac{1}{M_{\omega_i,s_{i_1}\cdots s_{i_k}\omega_i} M_{\omega_i,s_{i_1}\cdots s_{i_{k-1}}\omega_i}} \prod_{j \ne i} M_{\omega_j,s_{i_1}\cdots s_{i_k}\omega_j}^{-a_{ji}} . \tag{5.8}$$

Theorem 5.14. (i) *The image of the embedding* $p^+_{\mathrm{trop}} : \mathcal{B} \to \mathcal{M}^{w_\circ}(\mathbb{Z}_{\mathrm{trop}})^\vee$ *consists of all integer tuples* $(M_{\omega_i^\vee,\gamma}) \in \mathcal{M}^{w_\circ}(\mathbb{Z}_{\mathrm{trop}})^\vee$ *such that* $M_{\omega_i^\vee,\omega_i^\vee} = 0$ *and* $M_{\omega_i^\vee,s_i\omega_i^\vee} \ge 0$ *for all* i.

(ii) *The image of the embedding* $p^-_{\mathrm{trop}} : \mathcal{B}^{\mathrm{dual}} \to \mathcal{M}^{w_\circ}(\mathbb{Z}_{\mathrm{trop}})^\vee$ *consists of all integer tuples* $(M_{\omega_i^\vee,\gamma}) \in \mathcal{M}^{w_\circ}(\mathbb{Z}_{\mathrm{trop}})^\vee$ *such that* $M_{\omega_i^\vee,w_\circ\omega_i^\vee} = 0$ *and* $M_{s_i\omega_i^\vee,w_\circ\omega_i^\vee} \ge 0$ *for all* i.

The tropical Plücker models just constructed allow us to give two "universal" polyhedral expressions for the tensor product multiplicities.

Theorem 5.15. *For any three dominant weights* λ,μ,ν *for* $\mathfrak{g}$, *the multiplicity* $c^\mu_{\lambda,\nu}$ *is equal to the number of integer tuples* $(M_{\omega_i^\vee,\gamma}) \in \mathcal{M}^{w_\circ}(\mathbb{Z}_{\mathrm{trop}})^\vee$ *satisfying the following conditions for any* $i \in [1,r]$:

(0) $M_{\omega_i^\vee,\omega_i^\vee} = 0$;

(1) $M_{\omega_i^\vee,s_i\omega_i^\vee} \ge 0$;

(2) $M_{\omega_i^\vee,w_\circ\omega_i^\vee} = (\lambda+\nu-\mu)(\omega_i^\vee)$;

(3) $M_{s_i\omega_i^\vee,w_\circ\omega_i^\vee} \ge (s_i\lambda+\nu-\mu)(\omega_i^\vee)$;

(4) $M_{\omega_i^\vee,w_\circ s_i\omega_i^\vee} \ge (\lambda+s_i\nu-\mu)(\omega_i^\vee)$.

Theorem 5.16. *For any three dominant weights* λ,μ,ν *for* $\mathfrak{g}$, *the multiplicity* $c^\mu_{\lambda,\nu}$ *is equal to the number of integer tuples* $(M_{\omega_i^\vee,\gamma}) \in \mathcal{M}^{w_\circ}(\mathbb{Z}_{\mathrm{trop}})^\vee$ *satisfying the following conditions for any* $i \in [1,r]$:

(0) $M_{\omega_i^\vee,w_\circ\omega_i^\vee} = 0$;

(1) $M_{s_i\omega_i^\vee,w_\circ\omega_i^\vee} \ge 0$;

(2) $M_{\omega_i^\vee,\omega_i^\vee} = -(\lambda+\nu-\mu)(\omega_i^\vee)$;

(3) $M_{\omega_i^\vee,w_\circ s_i\omega_i^\vee} \ge -\lambda(\alpha_{i^*}^\vee)$;

(4) $M_{\omega_i^\vee,s_i\omega_i^\vee} \ge -(s_i\lambda+\nu-s_i\mu)(\omega_i^\vee)$.

6. Proofs of results in Section 4

Proof of Proposition 4.3. Any element of the Bruhat cell BuB can be written as $x = n_1 \overline{u} a n_2$ with $a \in H$ and $n_1, n_2 \in N$. Here the element a is uniquely determined by x, and we have $a = [\overline{u}^{\ -1} x]_0$. Thus the condition that $x \in N\overline{u}N$ is equivalent to $[\overline{u}^{\ -1} x]_0 = 1$. This in turn is equivalent to the condition that $\Delta_{u\omega_i, \omega_i}(x) = ([\overline{u}^{\ -1} x]_0)^{\omega_i} = 1$ for all $i \in [1, r]$. $\square$

Proof of Proposition 4.5. In view of [11, Theorems 1.2, 1.3], we only need to prove that the image $x_{\mathbf{i}}(\mathbb{C}^m_{\neq 0})$ is contained in $N\overline{u}N$. Trivially, $x_i(t) \in N$ for any $i \in [1, r]$; using the commutation relation [11, (2.13)], we also see that

$$x_{-i}(t) = y_i(t) t^{-\alpha_i^\vee} = x_i(t^{-1}) \overline{s_i} x_i(t) \in N \overline{s_i} N \ .$$

Using induction on $\ell(u) + \ell(v)$, it only remains to show that $\overline{s_i} N \overline{w} \subset N \overline{s_i w} N$ for any $w \in W$ such that $\ell(s_i w) = \ell(w) + 1$. For any $n \in N$, we have

$$\overline{s_i} n \overline{s_i}^{\ -1} = n' y_i(t)$$

for some $n' \in N$ and $t \in \mathbb{C}$. Furthermore, the condition $\ell(s_i w) = \ell(w) + 1$ implies that

$$n'' := \overline{s_i w}^{\ -1} y_i(t) \overline{s_i w} \in N \ .$$

Therefore, we obtain

$$\overline{s_i} n \overline{w} = n' y_i(t) \overline{s_i} \overline{w} = n' \overline{s_i w} n'' \in N \overline{s_i w} N \ ,$$

as required. $\square$

Proof of Theorem 4.7. Comparing (4.11) with [11, Definition 1.5], it is easy to show that

$$\psi^{u,v}(x) = (\zeta^{u,v}(x))^T = \zeta^{u^{-1}, v^{-1}}(x^T) \tag{6.1}$$

for any $x \in L^{u,v}$; here $\zeta^{u,v}$ is a biregular isomorphism between $G^{u,v}$ and $G^{u^{-1}, v^{-1}}$ (see [11, Theorem 1.6]), and $x \mapsto x^T$ is the "transpose" antiautomorphism of G given by (4.3). It is also clear from (4.11) that $\psi^{u,v}(x) \in N\overline{v}N$. Therefore, $\psi^{u,v}$ sends $L^{u,v}$ to $L^{v,u}$, and our theorem becomes a consequence of [11, Theorems 1.6, 1.7]. $\square$

Proof of Theorem 4.8. In order to distinguish our present notation from that in [11], let us denote by $\tilde{x}_{\mathbf{i}}$ the product map $H \times \mathbb{C}^m_{\neq 0} \to G^{u,v}$ defined in [11, (1.3)]. Using (4.9) and commutation relations [11, (2.5)], we can rewrite $x = x_{\mathbf{i}}(t_1, \ldots, t_m)$ as $x = \tilde{x}_{\mathbf{i}}(a; t'_1, \ldots, t'_m)$, where

$$a = \prod_{k : i_k < 0} t_k^{-\alpha^\vee_{-i_k}} \ ,$$

and

$$t'_k = t_k^{\varepsilon(i_k)} \cdot \prod_{\substack{l > k \\ i_l < 0}} t_l^{\varepsilon(i_k) a_{|i_l|, |i_k|}} \ ; \tag{6.2}$$

here $\varepsilon(i)$ denotes the sign of i, i.e., $\varepsilon(i) = \pm 1$ for $i \in \pm[1, r]$.

Now to prove our theorem, one only has to substitute into (6.2) the expressions for the t_l given by (4.14) and (4.15), and to verify that the resulting expression for t'_k agrees with [11, (1.18)]. This is done by a straigtforward check that we omit (notice that the function $\Delta_{l,\mathbf{i}}(x')$ in [11, (1.18)] is our present $M_l(x)$, and that $\Delta_{m+j,\mathbf{i}}(x') = \Delta_{v\omega_j,\omega_j}(\psi^{u,v}(x)) = 1$ for any $j \in [1,r]$, since $\psi^{u,v}(x) \in L^{v,u}$ by Theorem 4.7). $\square$

Proof of Proposition 4.11. To compute minors on the right-hand sides of (4.18) and (4.19), we shall use [9, Lemma 6.4 (b)] which says that

$$\Delta_{\omega_i,w^{-1}\omega_i}(x_{j_1}(t_1)\cdots x_{j_l}(t_l)) = \prod_{k=1}^{l} t_k^{\omega_i(s_{j_1}\cdots s_{j_{k-1}}\alpha_{j_k}^\vee)} \tag{6.3}$$

for any $i \in [1,r]$, $w \in W$, and $\mathbf{j} = (j_1,\ldots,j_l) \in R(w)$. This formula (with $j = (i_1,\ldots,i_m)$) directly applies to the two minors in the numerators in (4.19). To compute the second denominator in (4.19), we notice that it is equal to

$$\Delta_{\omega_{i_m^*},s_{i_m}w_\circ\omega_{i_m^*}}(x) = \Delta_{\omega_{i_m^*},s_{i_m}w_\circ\omega_{i_m^*}}(x_{i_1}(t_1)\cdots x_{i_{m-1}}(t_{m-1}))$$

(since $\overline{s_{i_m}w_\circ}^{\,-1}x_{i_m}(t_m)\overline{s_{i_m}w_\circ} \in N$), and so it is given by (6.3) for $\mathbf{j} = (i_1,\ldots,i_{m-1})$. This implies the formula for t_m in (4.19).

To compute the first denominator in (4.19), we shall use the antiautomorphism $\tau_{w_\circ}$ of G introduced in (4.4). It is easy to see that

$$\tau_{w_\circ}(x_{i_1}(t_1)\cdots x_{i_m}(t_m)) = x_{i_m^*}(t_m)\cdots x_{i_1^*}(t_1) \tag{6.4}$$

for any sequence $(i_1,\ldots,i_m)$ of indices from $[1,r]$. Using (4.6), we can now rewrite the first denominator in (4.19) as

$$\begin{aligned}\Delta_{s_{i_1}\omega_{i_1},w_\circ\omega_{i_1}}(x) &= \Delta_{\omega_{i_1},w_\circ s_{i_1}\omega_{i_1}}(x_{i_m^*}(t_m)\cdots x_{i_1^*}(t_1))\\ &= \Delta_{\omega_{i_1},w_\circ s_{i_1}\omega_{i_1}}(x_{i_m^*}(t_m)\cdots x_{i_2^*}(t_2))\ ,\end{aligned}$$

and then compute it by using (6.3) with $\mathbf{j} = (i_m^*,\ldots,i_2^*)$. This implies the formula for t_1 in (4.19).

To prove (4.18), we shall use the following lemma.

Lemma 6.1. *For any sequence $(i_1,\ldots,i_m)$ of indices from $[1,r]$, and any generalized minor $\Delta_{\gamma,\delta}$, we have*

$$(x_{-i_1}(t_1)\cdots x_{-i_m}(t_m))^T = t_1^{-\alpha_{i_1}^\vee}\cdots t_m^{-\alpha_{i_m}^\vee}x_{i_m}(t'_m)\cdots x_{i_1}(t'_1)\ , \tag{6.5}$$

and

$$\Delta_{\gamma,\delta}(x_{-i_1}(t_1)\cdots x_{-i_m}(t_m)) = \prod_{k=1}^{m} t_k^{-\delta(\alpha_{i_k}^\vee)}\cdot\Delta_{\delta,\gamma}(x_{i_m}(t'_m)\cdots x_{i_1}(t'_1))\ , \tag{6.6}$$

where the t'_k are given by

$$t'_k = t_k\cdot\prod_{l<k} t_l^{a_{i_l,i_k}}\ . \tag{6.7}$$

Proof. The equality (6.5) follows from (4.9) and commutation relations [11, (2.5)]. The equality (6.6) is an immediate consequence of (6.5) and (4.2). $\square$

Using Lemma 6.1, we deduce the computation of the minors in (4.18) to the computation of "transpose minors" evaluated at the product $x_{i_m}(t'_m)\cdots x_{i_1}(t'_1)$. The latter minors are computed in the same way as above, and (4.18) follows. $\square$

Remark 6.2. Although we do not need it in this paper, we would like to give an explicit formula for the inverse transformation in (6.7):

$$t_k = t'_k \cdot \prod_{l<k} {t'_l}^{\,-\alpha_{i_k}(s_{i_{k-1}}\cdots s_{i_{l+1}}\alpha^\vee_{i_l})} \ . \tag{6.8}$$

The fact that transformations in (6.7) and (6.8) are inverse of each other is proved by a straightforward calculation.

7. Proofs of results in Section 5.1

We start by recalling the classical Tits theorem about reduced words in Coxeter groups. We call a *d-move* a transformation of a reduced word (for some $w \in W$) that replaces d consecutive entries $i, j, i, j, \ldots$ by $j, i, j, i, \ldots$, for some i and j such that d is the order of $s_i s_j$. Note that, for given i and j, the value of d can be determined from the Cartan matrix as follows: if $a_{ij}a_{ji} = 0$ (resp. $1, 2, 3$), then $d = 2$ (resp. $3, 4, 6$). The Tits theorem says that every two reduced words for the same element of a Coxeter group can be obtained from each other by a sequence of d-moves.

Applying this to the group $W \times W$, we conclude that every two double reduced words $\mathbf{i}, \mathbf{i}' \in R(u, v)$ can be obtained from each other by a sequence of the following operations: *positive* d-moves for the alphabet $[1, r]$, *negative* d-moves for the alphabet $[-1, -r]$, and *mixed* 2-moves that interchange two consecutive indices of opposite signs. Furthermore, if $\mathbf{i}$ and $\mathbf{i}'$ are related by one of these d-moves then the transition map $\tilde R_{\mathbf{i}}^{\mathbf{i}'}$ only affects the d entries involved in the move; and the restriction of $\tilde R_{\mathbf{i}}^{\mathbf{i}'}$ to the corresponding segment of $t \in \mathbb{R}^m_{>0}$ is the "local" transition map $\tilde R_{i,j,i,\ldots}^{j,i,j,\ldots}$. In this section, we give explicit formulas for these local transition maps.

We shall write $\tilde R_{i,j,i,\ldots}^{j,i,j,\ldots}(t_1, \ldots, t_d) = (p_1, \ldots, p_d)$; thus the tuples $t_1, \ldots, t_d$ and $p_1, \ldots, p_d$ are related by

$$x_i(t_1)x_j(t_2)x_i(t_3)\cdots = x_j(p_1)x_i(p_2)x_j(p_3)\cdots \ . \tag{7.1}$$

The transition maps for positive d-moves (i.e., for $i, j \in [1, r]$) were computed in [9, Theorem 3.1]. For the convenience of the reader, let us reproduce these results here.

Proposition 7.1. *Let $i, j \in [1, r]$, and let d be the order of $s_i s_j$ in W. Then the transition map in* (7.1) *is given as follows:*
(1) Type $A_1 \times A_1$: *if $a_{ij} = a_{ji} = 0$ then $d = 2$, and*

$$p_1 = t_2, \ p_2 = t_1 \ .$$

(2) Type A_2: *if $a_{ij} = a_{ji} = -1$ then $d = 3$, and*

$$p_1 = \frac{t_2 t_3}{t_1 + t_3}, \ p_2 = t_1 + t_3, \ p_3 = \frac{t_1 t_2}{t_1 + t_3} \ .$$

(3) Type B_2: *if $a_{ij} = -2$, $a_{ji} = -1$ then $d = 4$, and*

$$p_1 = \frac{t_2 t_3^2 t_4}{\pi_2}, \ p_2 = \frac{\pi_2}{\pi_1}, \ p_3 = \frac{\pi_1^2}{\pi_2}, \ p_4 = \frac{t_1 t_2 t_3}{\pi_1} \ ,$$

where

$$\pi_1 = t_1 t_2 + (t_1 + t_3)t_4, \ \pi_2 = t_1^2 t_2 + (t_1 + t_3)^2 t_4 \ .$$

(4) Type G_2: *if* $a_{ij} = -3$, $a_{ji} = -1$ *then* $d = 6$, *and*

$$p_1 = \frac{t_2 t_3^3 t_4^2 t_5^3 t_6}{\pi_3},\ p_2 = \frac{\pi_3}{\pi_2},\ p_3 = \frac{\pi_2^3}{\pi_3\pi_4},$$
$$p_4 = \frac{\pi_4}{\pi_1\pi_2},\ p_5 = \frac{\pi_1^3}{\pi_4},\ p_6 = \frac{t_1 t_2 t_3^2 t_4 t_5}{\pi_1}\ ,$$

where

$$\begin{aligned}
\pi_1 &= t_1 t_2 t_3^2 t_4 + t_1 t_2 (t_3+t_5)^2 t_6 + (t_1+t_3) t_4 t_5^2 t_6\ ,\\
\pi_2 &= t_1^2 t_2^2 t_3^3 t_4 + t_1^2 t_2^2 (t_3+t_5)^3 t_6 + (t_1+t_3)^2 t_4^2 t_5^3 t_6\\
&+ t_1 t_2 t_4 t_5^2 t_6 (3t_1t_3 + 2t_3^2 + 2t_3t_5 + 2t_1t_5)\ ,\\
\pi_3 &= t_1^3 t_2^2 t_3^3 t_4 + t_1^3 t_2^2 (t_3+t_5)^3 t_6 + (t_1+t_3)^3 t_4^2 t_5^3 t_6\\
&+ t_1^2 t_2 t_4 t_5^2 t_6 (3t_1t_3 + 3t_3^2 + 3t_3t_5 + 2t_1t_5)\ ,\\
\pi_4 &= t_1^2 t_2^2 t_3^3 t_4 \big(t_1 t_2 t_3^3 t_4 + 2 t_1 t_2 (t_3+t_5)^3 t_6 + (3t_1t_3 + 3t_3^2 + 3t_3t_5 + 2t_1t_5) t_4 t_5^2 t_6\big)\\
&+ t_6^2 \big(t_1 t_2 (t_3+t_5)^2 + (t_1+t_3) t_4 t_5^2\big)^3\ .
\end{aligned}$$

(5) *Furthermore, in each of the cases* (1)–(4) *above, if we interchange* a_{ij} *with* a_{ji} *then the corresponding transition map in* (7.1) *is obtained from the given one by the transformation* $p_k \to p_{d+1-k}$, $t_k \to t_{d+1-k}$.

The transition maps for mixed 2-moves are given by the following proposition which is an immediate consequence of commutation relations [11, (2.5), (2.11)].

Proposition 7.2. *For any* $i, j \in [1, r]$, *we have* $x_j(t_1)x_{-i}(t_2) = x_{-i}(p_1)x_j(p_2)$, *where*

$$p_1 = t_2,\ p_2 = t_1 t_2^{a_{ij}}$$

for $i \neq j$, *and*

$$\frac{1}{p_1} = t_1 + \frac{1}{t_2},\ \frac{1}{p_2} = \frac{1}{t_2}\left(1 + \frac{1}{t_1 t_2}\right)$$

for $i = j$.

Finally, the transition maps for negative d-moves are given as follows.

Proposition 7.3. *Let* $i, j \in [-1, -r]$, *and let* d *be the order of* $s_{|i|}s_{|j|}$ *in* W. *Then the transition map in* (7.1) *is given as follows:*
(1) Type $A_1 \times A_1$: *if* $a_{|i|,|j|} = a_{|j|,|i|} = 0$ *then* $d = 2$, *and*

$$p_1 = t_2,\ p_2 = t_1\ .$$

(2) Type A_2: *if* $a_{|i|,|j|} = a_{|j|,|i|} = -1$ *then* $d = 3$, *and*

$$\frac{1}{p_1} = \frac{1}{t_3} + \frac{t_1}{t_2},\ p_2 = t_1 t_3,\ p_3 = t_1 + \frac{t_2}{t_3}\ .$$

(3) Type B_2: *if* $a_{|i|,|j|} = -1$, $a_{|j|,|i|} = -2$ *then* $d = 4$, *and*

$$\frac{1}{p_1} = \frac{t_1}{t_2} + \frac{t_2}{t_3} + \frac{1}{t_4},,\ \frac{1}{p_2} = \frac{1}{t_1}\left(\frac{t_2}{t_3} + \frac{1}{t_4}\right)^2 + \frac{1}{t_3}\ ,$$
$$p_3 = t_2 + t_1 t_4 + \frac{t_2^2 t_4}{t_3},\ p_4 = t_1 + t_3\left(\frac{t_2}{t_3} + \frac{1}{t_4}\right)^2\ .$$

(4) Type G_2: *if* $a_{|i|,|j|} = -1$, $a_{|j|,|i|} = -3$ *then* $d = 6$, *and*

$$\begin{aligned}
\tfrac{1}{p_1} &= \tfrac{t_1}{t_2} + t_3(\tfrac{t_2}{t_3} + \tfrac{1}{t_4})^2 + \tfrac{t_4}{t_5} + \tfrac{1}{t_6},\\
\tfrac{1}{p_2} &= \tfrac{t_1}{t_3} + 2t_3\left(\tfrac{t_2}{t_3} + \tfrac{1}{t_4}\right)^3 + \tfrac{1}{t_1}\left(t_3(\tfrac{t_2}{t_3} + \tfrac{1}{t_4})^2 + \tfrac{t_4}{t_5} + \tfrac{1}{t_6}\right)^3\\
&+ \tfrac{3t_2t_4}{t_3t_5} + \tfrac{3t_2}{t_3t_6} + \tfrac{3}{t_4t_6} + \tfrac{2}{t_5},\\
p_5 &= t_1t_6 + t_3^2t_6(\tfrac{t_2}{t_3} + \tfrac{1}{t_4})^3 + t_4t_6(\tfrac{t_4}{t_5} + \tfrac{1}{t_6})^2 + 2t_2 + \tfrac{2t_3}{t_4} + \tfrac{3t_2t_4t_6}{t_5} + \tfrac{2t_3t_6}{t_5},\\
p_6 &= t_1 + t_3^2(\tfrac{t_2}{t_3} + \tfrac{1}{t_4})^3 + t_5(\tfrac{t_4}{t_5} + \tfrac{1}{t_6})^3 + \tfrac{3t_2t_4}{t_5} + \tfrac{3t_2}{t_6} + \tfrac{3t_3}{t_4t_6} + \tfrac{2t_3}{t_5}\ ;
\end{aligned}$$

the two middle components p_3 *and* p_4 *are determined from two additional relations*

$$p_1p_3p_5 = t_2t_4t_6\ , \quad p_2p_4p_6 = t_1t_3t_5\ .$$

(5) *Furthermore, in each of the cases* (1)–(4) *above, if we interchange* $a_{|i|,|j|}$ *with* $a_{|j|,|i|}$ *then the corresponding transition map in* (7.1) *is obtained from the given one by the transformation* $p_k \to 1/p_{d+1-k}$, $t_k \to 1/t_{d+1-k}$.

Proof. Each of the formulas in Proposition 7.3 follows from the corresponding formula in Proposition 7.1 by applying the map $x \mapsto x^T$ to both sides of (7.1) and using (6.5). $\square$

Proof of Theorem 5.2. Part (i) of Theorem 5.2 follows from the Tits theorem and Propositions 7.1, 7.2 and 7.3 since all rational expressions appearing there are subtraction-free. As for part (ii), it is enough to check it for the rank 2 case when the "geometric" transition maps $\tilde{R}^{j,i,j,\dots}_{i,j,i,\dots}$ are given by Propositions 7.1 and 7.3.

The case of $A_1 \times A_1$ is obvious, and the case of A_2 is obtained by comparing the expressions in (5.1) and (5.2) with the corresponding expressions in Propositions 7.1 and 7.3.

The transition maps for string parametrizations for the types B_2 (or C_2) and G_2 were found in [18, 22]; our theorem is then proved by direct comparison of these formulas with the ones given by Proposition 7.3.

For the Lusztig parametrizations, our proof is even less computational (but still mysterious). First of all, the statement for types $A_1 \times A_1$ and A_2 implies that it is true for any simply-laced type. The transition maps for Lusztig parametrizations for the type B_2 were found in [19] using the following strategy. Let $a_{12} = -2$ and $a_{21} = -1$, i.e., α_2 is the long simple root. Lusztig (implicitly) claims that the transition map $R^{1,2,1,2}_{2,1,2,1}$ for type B_2 is obtained from the transition map $R^{2,1,3,2,1,3}_{1,3,2,1,3,2}$ for type A_3 (with the standard numeration of simple roots) by the following procedure:

$$R^{1,2,1,2}_{2,1,2,1}(t_1, t_2, t_3, t_4) = (p_1, p_2, p_3, p_4)$$

$$\Leftrightarrow R^{2,1,3,2,1,3}_{1,3,2,1,3,2}(t_1, t_1, t_2, t_3, t_3, t_4) = (p_1, p_2, p_2, p_3, p_4, p_4)\ .$$

Using our statement for type A_3, we see that $R^{2,1,3,2,1,3}_{1,3,2,1,3,2}$ is the tropicalization of the geometric transition map $\tilde{R}^{2,1,3,2,1,3}_{1,3,2,1,3,2}$. But then the equality

$$\tilde{R}^{2,1,3,2,1,3}_{1,3,2,1,3,2}(t_1, t_1, t_2, t_3, t_3, t_4) = (p_1, p_2, p_2, p_3, p_4, p_4)$$

is easily seen to be equivalent to $\tilde{R}^{2,1,2,1}_{1,2,1,2}(t_1, t_2, t_3, t_4) = (p_1, p_2, p_3, p_4)$, where $\tilde{R}^{2,1,2,1}_{1,2,1,2}$ is the geometric transition map for B_2, with the same convention as above: α_2 is the long simple root. This proves our statement for the type B_2 (notice that the geometric lifting of $R^{1,2,1,2}_{2,1,2,1}$ is $\tilde{R}^{2,1,2,1}_{1,2,1,2} = (\tilde{R}^{1,2,1,2}_{2,1,2,1})^\vee$, which explains the necessity of passing to the Langlands dual group).

Now for the type G_2 one can use the same argument, with A_3 replaced by D_4. This concludes our proof. $\square$

Proof of Theorem 5.7. Let us denote the left hand side of (5.5) by $F_{\mathbf{i},\mathbf{i}'}(t')$. We consider each $F_{\mathbf{i},\mathbf{i}'}$ as a map from $\mathbb{Z}^m_{\geq 0}$ to $\mathbb{Z}^m$. The following properties of these maps are immediate from the definitions:

(1) $F_{\mathbf{i},\mathbf{i}'}(0,\dots,0) = (0,\dots,0)$ for any $\mathbf{i},\mathbf{i}' \in R(w_\mathrm{o})$.

(2) $F_{\mathbf{i},\mathbf{i}''} = R^{-\mathbf{i}}_{-\mathbf{i}'} \circ F_{\mathbf{i}',\mathbf{i}''} = F_{\mathbf{i},\mathbf{i}'} \circ R^{\mathbf{i}'}_{\mathbf{i}''}$ for any $\mathbf{i},\mathbf{i}',\mathbf{i}'' \in R(w_\mathrm{o})$.

The next property is less obvious:

(3) For any two reduced words $\mathbf{i}$ and $\mathbf{i}'$ for w_o such that $i'_1 = i_1$, the first component of $F_{\mathbf{i},\mathbf{i}'}(t'_1,\dots,t'_m)$ is equal to t'_1, while all other components only depend on $t'_2,\dots,t'_m$.

The statement about the first component follows from (3.2) and Proposition 3.3 (ii). The statement about other components follows from properties of the crystal operators $\tilde{f}_i : \mathcal{B} \to \mathcal{B}$ in Proposition 3.6 (i). Indeed, if $i'_1 = i_1$ then $F_{\mathbf{i},\mathbf{i}'}$ commutes with the shift operator T_1 which acts on $\mathbb{Z}^m$ by adding 1 to the first component of a vector.

We now claim that the above properties uniquely determine the family of maps $F_{\mathbf{i},\mathbf{i}'}$. More precisely: if a collection of maps $F_{\mathbf{i},\mathbf{i}'} : \mathbb{Z}^m_{\geq 0} \to \mathbb{Z}^m$ satisfy properties (1)–(3) then $F_{\mathbf{i},\mathbf{i}'}(t') = c_\mathbf{i}(b_{\mathbf{i}'}(t')^{\mathrm{dual}})$ for any $\mathbf{i},\mathbf{i}' \in R(w_\mathrm{o})$ and any $t' \in \mathbb{Z}^m_{\geq 0}$.

First of all, the equality $F_{\mathbf{i},\mathbf{i}''} = F_{\mathbf{i},\mathbf{i}'} \circ R^{\mathbf{i}'}_{\mathbf{i}''}$ in (2) implies that there exists a collection of maps $F_\mathbf{i} : \mathcal{B} \to \mathbb{Z}^m$ such that $F_{\mathbf{i},\mathbf{i}'} = F_\mathbf{i} \circ b_{\mathbf{i}'}$ for any $\mathbf{i},\mathbf{i}' \in R(w_\mathrm{o})$. It remains to show that $F_\mathbf{i}(b) = c_\mathbf{i}(b^{\mathrm{dual}})$ for any $\mathbf{i} \in R(w_\mathrm{o})$ and $b \in \mathcal{B}$. By (1), this is true for $b = 1$. If $b \neq 1$ then, by Proposition 3.6 (ii), $b = \tilde{f}_i(b')$ for some $i \in [1,r]$ and $b' \in \mathcal{B}$. Pick any $\mathbf{i}'$ with $i'_1 = i$. Using induction on the weight of b, we can assume that $F_{\mathbf{i}'}(b') = c_{\mathbf{i}'}(b'^{\mathrm{dual}})$. By (3), this implies that $F_{\mathbf{i}'}(b) = c_{\mathbf{i}'}(b^{\mathrm{dual}})$. Finally, the first equality in (2) implies that $F_\mathbf{i}(b) = c_\mathbf{i}(b^{\mathrm{dual}})$ for any $\mathbf{i} \in R(w_\mathrm{o})$, as required.

To complete the proof of Theorem 5.7, it remains to show that the functions $\tilde{F}_{\mathbf{i},\mathbf{i}'}(t)$ given by the right hand side of (5.5) satisfy the same properties (1)–(3). To prove (1) notice that $[Q]_{\mathrm{trop}}(0,\dots,0) = 0$ for any subtraction-free rational expression Q. Property (2) follows from Theorem 5.2 (ii). Let us prove (3).

Fix two reduced words $\mathbf{i}$ and $\mathbf{i}'$ for w_o such that $i'_1 = i_1$. Let $t'_1,\dots,t'_m \in \mathbb{R}_{>0}$, and let

$$x' = x_{\mathbf{i}'}(t'_1,\dots,t'_m) \in L^{e,w_\mathrm{o}}_{>0},\ x = \eta^{e,w_\mathrm{o}}(x') = x_{-\mathbf{i}}(t_1,\dots,t_m) \in L^{w_\mathrm{o},e}_{>0}\ .$$

We need to show that $t_1 = t'_1$, and, for $k = 2,\dots,m$, that t_k does not depend on t'_1. Remembering (5.3) and combining Corollary 4.10 (i) with (4.6), we obtain

$$t_k = \frac{\Delta_{s_{i_1}\cdots s_{i_{k-1}}\omega_{i_k},\, w_\mathrm{o}\omega_{i_k}}(x')}{\Delta_{s_{i_1}\cdots s_{i_k}\omega_{i_k},\, w_\mathrm{o}\omega_{i_k}}(x')} \tag{7.2}$$

for any $k = 1,\dots,m$. The equality $t_1 = t'_1$ now follows by comparing (7.2) for $k = 1$ with the first equality in (4.19). Furthermore, if $k > 1$ then both minors in (7.2) are invariant under the transformation $x' \mapsto x_{i_1}(t)x$ for any t since both elements $s_{i_{k-1}} \cdots s_{i_1}$ and $s_{i_k} \cdots s_{i_1}$ send α_{i_1} to a negative root; it follows that t_k does not depend on t'_1, and we are done. $\square$

Proof of Theorem 5.8. First let us show that (ii) follows from (i). Indeed, using (6.6) and the first equality in (4.6), we see that

$$\Delta_{\gamma,\delta}(x_{-\mathbf{i}}(t_1,\ldots,t_m)) = \prod_{k=1}^m t_k^{-\delta(\alpha_{i_k}^\vee)} \cdot \Delta_{-\gamma,-\delta}(x_{\mathbf{i}}(t'_1,\ldots,t'_m)) \ ,$$

where the t'_k are given by (6.7). Computing the minor on the right hand side with the help of (i), we obtain the sum of monomials corresponding to $\mathbf{i}$-trails π from $-\gamma$ to $-\delta$; the exponent of t_k in such a monomial is equal to

$$-\delta(\alpha_{i_k}^\vee) + c_k(\pi) + \sum_{l>k} a_{i_k,i_l} c_l(\pi)$$

$$= (-\delta + \sum_{l>k} c_l(\pi)\alpha_{i_l} + c_k(\pi)\frac{\alpha_{i_k}}{2})(\alpha_{i_k}^\vee) \ .$$

Substituting $c_l(\pi)\alpha_{i_l} = \gamma_{l-1} - \gamma_l$ and remembering the definition (2.2), we conclude that the latter exponent is equal to $d_k(\pi)$, as required.

For the proof of (i) we need a little preparation. Consider the ring of regular functions $\mathbb{C}[G]$ as a $G\times G$-representation under the action $(g_1,g_2)f(x) = f(g_1^T x g_2)$. We denote by $f \mapsto (u_1,u_2)f$ the corresponding action of $U(\mathfrak{g})\times U(\mathfrak{g})$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. For every $f \in \mathbb{C}[G]$, the function $f(x_{\mathbf{i}}(t_1,\ldots,t_m))$ is a polynomial in $t_1,\ldots,t_m$, and the coefficient of each monomial $t_1^{n_1}\cdots t_m^{n_m}$ is equal to $((1, e_{i_1}^{(n_1)}\cdots e_{i_m}^{(n_m)})f)(e)$, where e stands for the identity element of G, and $e_i^{(n)}$ stands for the divided power $e_i^n/n!$ (cf. [3, Lemma 3.7.5]). If f is (bi)-homogeneous of degree (γ,γ') then $f(e)$ can be nonzero only if $\gamma=\gamma'$. It follows that if degree of f is (γ,δ) then $f(x_{\mathbf{i}}(t_1,\ldots,t_m))$ contains only monomials $t_1^{n_1}\cdots t_m^{n_m}$ with $\sum_k n_k\alpha_{i_k} = \gamma-\delta$.

Returning to generalized minors, we notice that $\Delta_{\gamma,\delta}$ has degree (γ,δ) (see (4.2)), and belongs to the submodule V_{ω_i,ω_i} of $\mathbb{C}[G]$ generated by the highest weight vector $\Delta_{\omega_i,\omega_i}$. Furthermore, $\Delta_{\gamma,\delta}$ spans the weight subspace $V_{\omega_i,\omega_i}(\gamma,\delta)$, and we also have $\Delta_{\gamma,\gamma}(e)=1$. It follows that the coefficient c of $t_1^{n_1}\cdots t_m^{n_m}$ in $\Delta_{\gamma,\delta}(x_{\mathbf{i}}(t_1,\ldots,t_m))$ can be found from the equality $(1, e_{i_1}^{(n_1)}\cdots e_{i_m}^{(n_m)})\Delta_{\gamma,\delta} = c\Delta_{\gamma,\gamma}$. Applying the element $(\overline{u}^T, e) \in G\times G$ to both sides of this equality, we see that $(1, e_{i_1}^{(n_1)}\cdots e_{i_m}^{(n_m)})\Delta_{\omega_i,\delta} = c\Delta_{\omega_i,\gamma}$. Remembering Definition 2.1, we see that $\Delta_{\gamma,\delta}(x_{\mathbf{i}}(t_1,\ldots,t_m))$ consists precisely of the monomials $t_1^{c_1(\pi)}\cdots t_m^{c_m(\pi)}$ for all $\mathbf{i}$-trails π from γ to δ in V_{ω_i}. It only remains to show that, for every such $\mathbf{i}$-trail π, the corresponding coefficient c is a positive integer.

Let us consider the $U(\mathfrak{g})$-module structure on $\mathbb{C}[G]$ given by $uf = (1,u)f$. Under this action, $\Delta_{\omega_i,\omega_i}$ is a highest weight vector, and it generates the submodule isomorphic to V_{ω_i}. Each minor $\Delta_{\omega_i,\gamma}$ is an extremal vector of weight δ in this submodule normalized in such a way that $u\Delta_{\omega_i,\gamma} = \Delta_{\omega_i,\omega_i}$ for some $u \in U(\mathfrak{g})$ which is a monomial in the divided powers $e_j^{(n)}$ (see [11, Lemma 2.8]). We also have $\Delta_{\omega_i,\delta} = u'\Delta_{\omega_i,\omega_i}$ for some $u' \in U(\mathfrak{g})$ which is a monomial in the divided powers $f_j^{(n)}$. Let us abbreviate $e^{(\pi)} = e_{i_1}^{(c_1(\pi))}\cdots e_{i_m}^{(c_m(\pi))}$. The equality $e^{(\pi)}\Delta_{\omega_i,\delta} = c\Delta_{\omega_i,\gamma}$ can be now rewritten as $ue^{(\pi)}u'\Delta_{\omega_i,\omega_i} = c\Delta_{\omega_i,\omega_i}$. This shows that part (i) of Theorem 5.8 is a consequence of the following statement.

Lemma 7.4. *If $u \in U(\mathfrak{g})$ is a monomial of degree 0 in the divided powers of the elements e_j and f_j then $u\Delta_{\omega_i,\omega_i} = c\Delta_{\omega_i,\omega_i}$ for some nonnegative integer c.*

Proof. To see that $c \in \mathbb{Z}$ notice that the commutation relations in $U(\mathfrak{g})$ between divided powers of the f_j and e_j involve integer coefficients only. It remains to show that $c \geq 0$. First consider the case when $\mathfrak{g}$ is simply-laced, i.e., $|a_{ij}| \leq 1$ for $i \neq j$. Then the nonnegativity of c is a consequence of [20, Theorem 4.3.13]; to be more precise, one applies the dual version of this result that says that each generator E_j or F_j of $U_q(\mathfrak{g})$ acts in the dual canonical basis in V_{ω_i} by a matrix whose entries are nonnegative integer Laurent polynomials in q (cf. [7]).

If $\mathfrak{g}$ is not simply-laced, we use a well known embedding of $\mathfrak{g}$ into a simply-laced complex semisimple Lie algebra $\tilde{\mathfrak{g}}$. This embedding can be described as follows: if $\tilde{\mathfrak{g}}$ has Chevalley generators $f_{\tilde{i}}, \alpha_{\tilde{i}}^\vee$, and $e_{\tilde{i}}$ for $\tilde{i} \in I$ then the Chevalley generators of a subalgebra $\mathfrak{g}$ have the form

$$f_i = \sum_{\tilde{i} \in I_i} f_{\tilde{i}},\ \alpha_i^\vee = \sum_{\tilde{i} \in I_i} \alpha_{\tilde{i}}^\vee,\ e_i = \sum_{\tilde{i} \in I_i} e_{\tilde{i}}\ ,$$

where the subsets $I_i \subset I$ are disjoint, and no two indices from the same I_i are adjacent to each other in the Dynkin diagram of $\tilde{\mathfrak{g}}$.

The embedding $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ allows us to identify any fundamental $\mathfrak{g}$-module V_{ω_i} with the $\mathfrak{g}$-submodule generated by a highest vector of a fundamental $\tilde{\mathfrak{g}}$-module $V_{\omega_{\tilde{i}}}$ for any $\tilde{i} \in I_i$. Since any monomial in the e_i and f_i is a sum of monomials in the $e_{\tilde{i}}$ and $f_{\tilde{i}}$, the desired inequality $c \geq 0$ follows from the corresponding claim for $\tilde{\mathfrak{g}}$. This completes the proofs of Lemma 7.4 and Theorem 5.8. $\square$

Remark 7.5. In general, we do not know of a nice formula for the coefficients of the monomials in Theorem 5.8. In some special cases these coefficients can be found with the help of the following formulas which are easy consequences of the above proof: $\Delta_{\gamma,\delta}(xx_i(t)) = \Delta_{\gamma,\delta}(x)$ for any $x \in G$ and $t \in \mathbb{C}$ whenever $\delta(\alpha_i^\vee) \geq 0$; and $\Delta_{\gamma,\delta}(xx_i(t)) = \Delta_{\gamma,\delta}(x) + t\Delta_{\gamma,s_i\delta}(x)$ whenever $\delta(\alpha_i^\vee) = -1$.

8. Proofs of results in Sections 2.2, 3.3, 5.2 and 5.3

We start with the proof of Theorem 3.7; the rest of the results will follow quite easily.

Proof of Theorem 3.7. In view of Theorem 5.7 and Corollary 5.9, part (i) of Theorem 3.7 is obtained via the tropicalization of (7.2) (with each fundamental weight ω_i replaced by $\omega_i^\vee$)

To prove part (ii), let us first rewrite conditions (1) – (4) in Theorem 2.2 in terms of the element $x_{\mathbf{i}}^\vee(t_1, \ldots, t_m)$.

Lemma 8.1. *Each of the conditions* (1) – (4) *in Theorem 2.2 is equivalent to the corresponding condition in Theorem 5.15 with* $M_{\gamma,\delta} = [\Delta_{\gamma,\delta}(x_{\mathbf{i}}^\vee(t_1, \ldots, t_m))]_{\mathrm{trop}}$.

Proof. For conditions (3) and (4), the claim follows from Corollary 5.9. By the same theorem and the definition of $\mathbf{i}$-trails, we have

$$[\Delta_{\omega_i, s_i\omega_i}(x_{\mathbf{i}}(t_1, \ldots, t_m))]_{\mathrm{trop}} = \min_{i_k = i} t_k\ ,$$

which proves our claim for the condition (1). As for (2), our claim follows from (6.3) (see also Corollary 9.5 below). $\square$

Our second step is to rewrite the same conditions in terms of the element $\eta^{e,w_\circ}(x_{\mathbf{i}}^\vee(t_1, \ldots, t_m))$.

Lemma 8.2. *The four conditions in Lemma 8.1 are equivalent to conditions* (1) – (4) *in Theorem 5.16 with* $M_{\gamma,\delta} = [\Delta_{\delta,\gamma}(\eta^{e,w_\circ}(x^\vee_{\mathbf{i}}(t_1,\dots,t_m)))]_{\mathrm{trop}}$.

Proof. Let us establish some identities between generalized minors of an element $x' \in L^{e,w_\circ}$ and those of $x = \eta^{e,w_\circ}(x') \in L^{w_\circ,e}$: for any $i \in [1,r]$, we have

$$\Delta_{\omega_i,w_\circ\omega_i}(x') = \frac{1}{\Delta_{\omega_i,\omega_i}(x)} \ ; \tag{8.1}$$

$$\Delta_{\omega_i,s_i\omega_i}(x') = \Delta_{w_\circ\omega_{i^*},s_{i^*}\omega_{i^*}}(x) \ ; \tag{8.2}$$

$$\Delta_{s_i\omega_i,w_\circ\omega_i}(x') = \frac{\Delta_{w_\circ s_{i^*}\omega_{i^*},\omega_{i^*}}(x)}{\Delta_{\omega_i,\omega_i}(x)} \ ; \tag{8.3}$$

$$\Delta_{\omega_i,w_\circ s_i\omega_i}(x') = \Delta_{s_i\omega_i,\omega_i}(x)\prod_{j\neq i}\Delta_{\omega_j,\omega_j}(x)^{a_{ij}} \ . \tag{8.4}$$

The identities (8.1) are equivalent to $[x]_0 = ([x'\overline{w_\circ}]_0)^{-1}$, which is an immediate consequence of (5.4). As for (8.2) – (8.4), they follow by equating expressions (4.18) and (4.19) with the corresponding expressions in Corollary 4.10. For example, (8.4) is obtained by equating the first expression in (4.19) with the one in Corollary 4.10 (ii) for $k = 1$ (note that in these formulas, one has to replace x with $\tau_{w_\circ}(x)$).

Our lemma is now obtained by a straightforward calculation (in which one applies the above formulas to the group ${}^LG$). For example, (8.1) shows that conditions (2) in Theorems 5.15 and 5.16 are equivalent to each other. □

To complete the proof of Theorem 3.7 (ii), it remains to show the following: if in Lemma 8.2 we replace $x^\vee_{\mathbf{i}}(t_1,\dots,t_m)$ with $x^\vee_{\mathbf{i}'}(t'_1,\dots,t'_m)$ and write the element $\eta^{e,w_\circ}(x^\vee_{\mathbf{i}'}(t'_1,\dots,t'_m)$ as $x^\vee_{-\mathbf{i}}(t_1,\dots,t_m)$ then the conditions in Lemma 8.2 become equivalent to the corresponding conditions in Theorem 2.3. This is done in a straightforward way by expanding each expression $[\Delta_{\delta,\gamma}(x^\vee_{-\mathbf{i}}(t_1,\dots,t_m))]_{\mathrm{trop}}$ with the help of Corollary 5.9. □

Proof of Theorem 3.8. In view of (3.2), the function $l_i(b_{\mathbf{i}}(t_1,\dots,t_m))$ can be computed by using a special case of (3.3) with $k = 1$ (one also needs to interchange $\mathbf{i}$ with $\mathbf{i}'$, and t with t' there). It remains to notice that in this situation, the first minimum in (3.3) becomes $\sum_k s_{i_1}\cdots s_{i_{k-1}}\alpha_{i_k}(\omega_i^\vee)\,t_k$ by (6.3) (see also Corollary 9.5 below). □

Proof of Theorem 3.9. Let us again consider two elements x and x' given by

$$x' = x_{\mathbf{i}'}(t'_1,\dots,t'_m) \in L^{e,w_\circ}_{>0} \ ,$$
$$x = \eta^{e,w_\circ}(x') = x_{-\mathbf{i}}(t_1,\dots,t_m) \in L^{w_\circ,e}_{>0} \ .$$

Then each t'_k is a subtraction-free rational expression in $t_1,\dots,t_m$, and these expressions can be found with the help of Corollary 4.10 (i). In general, these expressions are a little cumbersome but for $k = 1$ or $k = m$ they can be simplified by using (4.19) and (8.1) – (8.4); this was essentially done in the above proof of Theorem 3.7. For example, here is the answer for t'_m:

$$(t'_m)^{-1} = \sum_{k:i^*_k=i'_m} t_k^{-1}\prod_{l>k} t_l^{-a_{i_l,i_k}} \ . \tag{8.5}$$

Formulas (3.5) and (3.6) are obtained by "tropicalizing" these expressions with the help of Corollary 5.9 (and passing from G to ${}^LG$ as usual). □

Proof of Theorem 3.10. In the course of the proof of Theorem 3.7, we have shown that if $t = c_{\mathbf{i}}(b_{\mathbf{i}'}(t')^{\mathrm{dual}})$ then the nonnegativity of all Lusztig parameters t'_k (i.e., condition (1) in Theorem 2.2) is equivalent to the fact that the string components t_k satisfy inequalities (1) in Theorem 2.3. This is precisely what we need to show. □

Remark 8.3. 1. The above argument not only provides an explicit description of the string cones but actually proves their existence thus providing an independent proof of Proposition 3.5.
2. The above argument also implies that the string cones $C_{\mathbf{i}}$ can be characterized in terms of the transition maps $R_{-\mathbf{i}}^{-\mathbf{i}'}$ as follows: $C_{\mathbf{i}}$ consists of all $t \in \mathbb{R}^m$ such that, for any $\mathbf{i}' \in R(w_{\mathrm{o}})$, the last component of $R_{-\mathbf{i}}^{-\mathbf{i}'}(t)$ is nonnegative. This characterization was used in [15], where the string cones were explicitly described for type A_r.

Proof of Theorem 2.2. Taking into account Proposition 3.3 and Corollary 3.4, our statement is an easy consequence of Theorem 3.8. □

Proof of Theorem 2.3. This is a consequence of Theorems 2.2 and 3.7 (ii). □

Proof of Theorem 5.10. Let $x = x_{-\mathbf{i}}(t_1, \ldots, t_m) \in L_{>0}^{w_{\mathrm{o}},e}$, and $\tilde{x} = (\eta^{e,w_{\mathrm{o}}} \circ F_i^{(c)} \circ \eta^{w_{\mathrm{o}},e})(x) = x_{-\mathbf{i}}(\tilde{t}_1, \ldots, \tilde{t}_m)$. Let $x' = \eta^{w_{\mathrm{o}},e}(x) \in L_{>0}^{e,w_{\mathrm{o}}}$. In view of (5.6), we have $x = \eta^{e,w_{\mathrm{o}}}(x')$ and $\tilde{x} = \eta^{e,w_{\mathrm{o}}}(x' x_{i^*}(t))$, where $t = (c-1)t'_m$. By (8.5), we have (in the notation of Theorem 5.10):

$$t^{-1} = (c-1)^{-1} \sum_{l : i_l = i} T_l \ . \tag{8.6}$$

Our goal is to express each $\tilde{t}_k$ in terms of $t_1, \ldots, t_m$. To do this, we combine (7.2) with its counterpart for $\tilde{x}$:

$$\tilde{t}_k = \frac{\Delta_{s_{i_1} \cdots s_{i_{k-1}} \omega_{i_k}, w_{\mathrm{o}} \omega_{i_k}}(x' x_{i^*}(t))}{\Delta_{s_{i_1} \cdots s_{i_k} \omega_{i_k}, w_{\mathrm{o}} \omega_{i_k}}(x' x_{i^*}(t))} \ .$$

Since $(w_{\mathrm{o}} \omega_{i_k})(\alpha_{i^*}^\vee) = -\delta_{i_k,i}$, it follows from Remark 7.5 that $\tilde{t}_k = t_k$ unless $i_k = i$; furthermore, if $i_k = i$ then

$$\begin{aligned}\tilde{t}_k &= \frac{\Delta_{s_{i_1} \cdots s_{i_{k-1}} \omega_i, w_{\mathrm{o}} \omega_i}(x') + t \Delta_{s_{i_1} \cdots s_{i_{k-1}} \omega_i, w_{\mathrm{o}} s_i \omega_i}(x')}{\Delta_{s_{i_1} \cdots s_{i_k} \omega_i, w_{\mathrm{o}} \omega_i}(x') + t \Delta_{s_{i_1} \cdots s_{i_k} \omega_i, w_{\mathrm{o}} s_i \omega_i}(x')} \\ &= t_k \cdot \frac{t^{-1} + \Delta_{s_{i_1} \cdots s_{i_{k-1}} \omega_i, w_{\mathrm{o}} s_i \omega_i}(x') / \Delta_{s_{i_1} \cdots s_{i_{k-1}} \omega_i, w_{\mathrm{o}} \omega_i}(x')}{t^{-1} + \Delta_{s_{i_1} \cdots s_{i_k} \omega_i, w_{\mathrm{o}} s_i \omega_i}(x') / \Delta_{s_{i_1} \cdots s_{i_k} \omega_i, w_{\mathrm{o}} \omega_i}(x')} \ .\end{aligned}$$

We claim that

$$\frac{\Delta_{s_{i_1} \cdots s_{i_{k-1}} \omega_i, w_{\mathrm{o}} s_i \omega_i}(x')}{\Delta_{s_{i_1} \cdots s_{i_{k-1}} \omega_i, w_{\mathrm{o}} \omega_i}(x')} = \sum_{l \geq k : i_l = i} T_l, \quad \frac{\Delta_{s_{i_1} \cdots s_{i_k} \omega_i, w_{\mathrm{o}} s_i \omega_i}(x')}{\Delta_{s_{i_1} \cdots s_{i_k} \omega_i, w_{\mathrm{o}} \omega_i}(x')} = \sum_{l > k : i_l = i} T_l \ ; \tag{8.7}$$

the desired equality (5.7) then follows by plugging these expressions together with the one in (8.6) into the above formula for $\tilde{t}_k$.

To prove (8.7), we use the identity (5.8) with each variable $M_{\gamma,\delta}$ replaced by $\Delta_{\gamma,\delta}(\tau_{w_\circ}(x')) = \Delta_{w_\circ\delta, w_\circ\gamma}(x')$ (see (4.6)). We thus obtain

$$\frac{\Delta_{s_{i_1}\cdots s_{i_{k-1}}\omega_i, w_\circ s_i\omega_i}(x')}{\Delta_{s_{i_1}\cdots s_{i_{k-1}}\omega_i, w_\circ\omega_i}(x')}$$

$$= \sum_{l\geq k: i_l=i} \frac{1}{\Delta_{s_{i_1}\cdots s_{i_{l-1}}\omega_i, w_\circ\omega_i}(x')\Delta_{s_{i_1}\cdots s_{i_l}\omega_i, w_\circ\omega_i}(x')} \prod_{j\neq i} \Delta_{s_{i_1}\cdots s_{i_l}\omega_j, w_\circ\omega_j}(x')^{-a_{ji}}\ .$$

As an easy consequence of (7.2), here the summand corresponding to each index l is equal to T_l; this proves the first equality in (8.7). The second equality is proved in the same way. This completes the proof of Theorem 5.10. □

Proof of Theorem 5.13. The proof follows that of [3, Theorem 2.7.1]. Let us sketch the proof of part (i); the proof of (ii) is the same. Let $\mathcal{M}^{w_\circ}(K)^+$ denote the set of all tuples $(M_{\omega_i,\gamma}) \in \mathcal{M}^{w_\circ}(K)$ such that $M_{\omega_i,\omega_i} = 1$ for all i. It is clear that p^+ is a well-defined map $\mathcal{L}^{e,w_\circ}(K) \to \mathcal{M}^{w_\circ}(K)^+$. To show that this is a bijection, it suffices to construct the inverse map in the case when $K = \mathbb{R}_{>0}$. Since the map $\mathbf{t} \mapsto x_{\mathbf{i}}(t^{\mathbf{i}})$ is a bijection between $\mathcal{L}^{e,w_\circ}(\mathbb{R}_{>0})$ and $L_{>0}^{e,w_\circ}$ (see Example 5.3), we only need to construct a bijective correspondence $(M_{\omega_i,\gamma}) \mapsto x$ between $\mathcal{M}^{w_\circ}(\mathbb{R}_{>0})^+$ and $L_{>0}^{e,w_\circ}$. This is done as follows: pick any $\mathbf{i} \in R(w_\circ)$; define the factorization parameters t_k as in Corollary 4.10 (i) with each minor $\Delta_{\omega_i,\gamma}(\psi^{w_\circ,e}(x))$ replaced by $M_{\omega_i,\gamma}$; form the corresponding product $x' = x_{-i_1}(t_1)\cdots x_{-i_m}(t_m)$; and finally define $x = \psi^{w_\circ,e}(x')$. □

Proof of Theorem 5.14. The theorem follows by combining Theorem 5.13 and Corollary 5.9 with Examples 5.4 and 5.5 and with Lemmas 8.1 and 8.2. □

Proof of Theorems 5.15 and 5.16. These theorems follow by combining Theorems 2.2 and 5.14 with Lemmas 8.1 and 8.2. □

9. Proofs of results in Section 3.4

To deduce Theorems 3.11, 3.12, and 3.14 from Theorem 3.10, we need to develop some properties of $\mathbf{i}$-trails. Although Theorem 3.10 involves trails in the fundamental modules over the Langlands dual Lie algebra ${}^L\mathfrak{g}$, we find it more convenient to deal with trails in $\mathfrak{g}$-modules (translating from $\mathfrak{g}$ to ${}^L\mathfrak{g}$ is automatic).

We start with some easy consequences of Definition 2.1.

Proposition 9.1. (i) *Condition* (2) *in Definition* 2.1 *is equivalent to the following:* $f_{i_l}^{c_l}\cdots f_{i_1}^{c_1}$ *is a non-zero linear map from* $V(\gamma)$ *to* $V(\delta)$.
(ii) *All the weights* γ_k *in a* $\mathbf{i}$*-trail are weights of* V*, and* $\gamma = \gamma_0 \geq \gamma_1 \geq \cdots \geq \gamma_l = \delta$*, where* $\gamma \geq \delta$ *has the usual meaning that* $\gamma - \delta$ *is a nonnegative integer linear combination of simple roots of* $\mathfrak{g}$.

Proof. Part (ii) is trivial; (i) follows from the well-known fact that there exists a nondegenerate bilinear form B on V such that $B(xv_1, v_2) = B(v_1, x^T v_2)$ for any $x \in \mathfrak{g}$ and $v_1, v_2 \in V$. □

Throughout the rest of this section, we assume that V, γ, δ, and $\mathbf{i}$ in Definition 2.1 satisfy the following conditions:
(1) $V = V_\lambda$, where λ is a dominant weight for $\mathfrak{g}$;
(2) γ and δ are two extremal weights in V_λ, i.e., they belong to the W-orbit $W\lambda$;
(3) $\mathbf{i} = (i_1, \ldots, i_l) \in R(w)$ for some $w \in W$.

Recall that the extremal weights in V_λ are precisely the vertices of the weight polytope $P(V_\lambda)$, and the corresponding weight subspaces $V_\lambda(\gamma)$ are one-dimensional. We call an $\mathbf{i}$-trail from γ to δ *extremal* if all γ_k are extremal weights of V_λ. Extremal $\mathbf{i}$-trails can be described as follows. Let W_λ denote the stabilizer of λ in W. Every $\gamma \in W\lambda$ has a unique *minimal presentation* $\gamma = u\lambda$, where $u \in W$ is the minimal (with respect to the Bruhat order) representative of its coset uW_λ.

Proposition 9.2. *Suppose that extremal weights γ and δ have minimal presentations $\gamma = u\lambda$ and $\delta = v\lambda$. There exists an extremal $\mathbf{i}$-trail from γ to δ in V_λ if and only if $\ell(uv^{-1}) = \ell(v) - \ell(u)$, and $uv^{-1} \leq w$ (in the Bruhat order). Under these conditions, the extremal $\mathbf{i}$-trails from γ to δ are in a bijection with subwords $(i_{k(1)}, \dots, i_{k(p)})$ of $\mathbf{i}$ which are reduced words for uv^{-1}; the $\mathbf{i}$-trail corresponding to a sequence $(k(1) < \cdots < k(p))$ is given by $\gamma_k = s_{i_{k(j)}} \cdots s_{i_{k(1)}}\gamma$, where j is the maximal index such that $k(j) \leq k$.*

Proof. First of all, the condition that an $\mathbf{i}$-trail $\pi = (\gamma = \gamma_0, \dots, \gamma_l = \delta)$ is extremal can be reformulated as follows: $\gamma_k \in \{\gamma_{k-1}, s_{i_k}\gamma_{k-1}\}$ for $k = 1, \dots, l$. Let $k(1)$ be the minimal index such that $\gamma_{k(1)} = s_{i_{k(1)}}\gamma \neq \gamma$. Since $s_{i_{k(1)}}\gamma = s_{i_{k(1)}}u\lambda < u\lambda$, it follows easily that $\ell(s_{i_{k(1)}}u) = \ell(u) + 1$. The standard properties of the Bruhat order in W/W_λ then imply that $\gamma_{k(1)} = s_{i_{k(1)}}u\lambda$ is the minimal presentation of the weight $\gamma_{k(1)}$. Replacing γ with $\gamma_{k(1)}$ and using induction on the length of $\mathbf{i}$, we obtain the desired statement. □

In particular, if λ is *minuscule* (i.e., the extremal weights $w\lambda$ are the only weights of the $\mathfrak{g}$-module V_λ) then *all* $\mathbf{i}$-trails from γ to δ are extremal, and so are given by Proposition 9.2.

Proof of Theorem 3.14. Remembering (2.2) and using Proposition 9.2, we see that the inequalities in (3.8) are precisely the inequalities in Theorem 3.10 corresponding to extremal $\mathbf{i}$-trails. Thus all these inequalities hold on $C_\mathbf{i}$. If $\mathfrak{g}$ is of type A_r then $\mathfrak{g}$ is isomorphic to ${}^L\mathfrak{g}$, and all its fundamental weights are known to be minuscule. So all the trails in Theorem 3.10 are extremal, and Theorem 3.14 follows. □

Our next result gives upper and lower bounds for every $\mathbf{i}$-trail (between two extremal weights). Let us fix V, γ, δ, and $\mathbf{i}$ as above. Define weights $\underline{\gamma}_0, \dots, \underline{\gamma}_l$ and $\overline{\gamma}_0, \dots, \overline{\gamma}_l$ by setting $\underline{\gamma}_0 = \gamma$, $\overline{\gamma}_l = \delta$, and

$$\underline{\gamma}_k = \min(\underline{\gamma}_{k-1}, s_{i_k}\underline{\gamma}_{k-1}),\ \overline{\gamma}_{k-1} = \max(\overline{\gamma}_k, s_{i_k}\overline{\gamma}_k)$$

for $k = 1, \dots, l$.

Proposition 9.3. *Any $\mathbf{i}$-trail $(\gamma_0, \gamma_1, \cdots, \gamma_l)$ from γ to δ in V_λ satisfies $\underline{\gamma}_k \leq \gamma_k \leq \overline{\gamma}_k$ for $k = 0, \dots, l$.*

Proof. Let us prove the inequalities $\gamma_k \leq \overline{\gamma}_k$; the remaining ones are proved in a similar way with the help of Proposition 9.1(i). Let v_γ denote a nonzero vector in $V_\lambda(\gamma)$. We proceed by induction on l to prove a little stronger statement:
(*) If a vector $v = e_{i_{k+1}}^{c_{k+1}} \cdots e_{i_l}^{c_l} v_\delta$ is nonzero for some nonnegative integers $c_{k+1}, \dots, c_l$ then its weight $\gamma_k = \gamma + c_{k+1}\alpha_{i_{k+1}} + \cdots + c_l\alpha_{i_l}$ satisfies $\gamma_k \leq \overline{\gamma}_k$.

We can assume that $l \geq 1$, and that our statement holds for any $\mathbf{i}'$-trail, where $\mathbf{i}' = (i_1, \dots, i_{l-1})$. Let us abbreviate $\overline{\delta} = \overline{\gamma}_{l-1}$. Consider two cases.

Case 1: $\overline{\delta} = \delta \geq s_{i_l}\delta$. In this case $\delta + \alpha_{i_l}$ is not a weight of V_λ. Therefore, $\delta = \gamma_{l-1} = \overline{\delta}$, and our statement follows by induction.

Case 2: $\delta < s_{i_l}\delta = \overline{\delta}$. Using the representation theory of sl_2, we see that $e_{i_l} v_{\overline{\delta}} = 0$, and $v_\delta = f_{i_l}^a v_{\overline{\delta}}$, where $a = \overline{\delta}(\alpha_{i_l}^\vee) > 0$. Thus we have

$$v = e_{i_{k+1}}^{c_{k+1}} \cdots e_{i_l}^{c_l} v_\delta = e_{i_{k+1}}^{c_{k+1}} \cdots e_{i_l}^{c_l} f_{i_l}^a v_{\overline{\delta}} \ .$$

Using the commutation relations between f_{i_l} and the elements e_i in $U(\mathfrak{g})$, we can express v as a linear combination of vectors of the form $f_{i_l}^{a'} e_{i_{k+1}}^{c'_{k+1}} \cdots e_{i_{l-1}}^{c'_{l-1}} v_{\overline{\delta}}$. Hence at least one of these vectors is nonzero, and we conclude by induction that

$$\gamma_k \leq \gamma_k + a'\alpha_{i_l} \leq \overline{\gamma}_k \ ,$$

as required. $\square$

Corollary 9.4. *Suppose that $\gamma \geq s_{i_1}\gamma \geq \cdots \geq s_{i_l} \cdots s_{i_1}\gamma = \delta$. Then there is a unique $\mathbf{i}$-trail π from γ to δ in V_λ; it is given by $\gamma_k = s_{i_k} \cdots s_{i_1}\gamma$, and it has $c_k(\pi) = \gamma(s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}^\vee)$, and $d_k(\pi) = 0$ for $k = 1, \ldots, l$.*

Proof. The uniqueness of π follows from Proposition 9.3 since, under the present assumptions, we have $\underline{\gamma}_k = \overline{\gamma}_k = s_{i_k} \cdots s_{i_1}\gamma$ for $k = 0, \ldots, l$. The claim about $c_k(\pi)$ and $d_k(\pi)$ follows at once from the definitions (2.1) and (2.2). $\square$

The following special case of Corollary 9.4 extends [2, Proposition 3.3].

Corollary 9.5. *Suppose that $\gamma = u\lambda$ for some $u \in W$ such that $\ell(w^{-1}u) = \ell(u) + \ell(w)$. For every $\mathbf{i} \in R(w)$, there is a unique $\mathbf{i}$-trail π from γ to $w^{-1}\gamma$; it is given by $\gamma_k = s_{i_k} \cdots s_{i_1}\gamma$ for $k = 0, \ldots, l$.*

Our proof of Theorem 3.11 relies on one more corollary of Proposition 9.3; to formulate it we need the following definition.

Definition 9.6. Let V, γ, δ, and $\mathbf{i}$ have the same meaning as in conditions (1)–(3) above. An index $k \in [0, l]$ is splitting if it satisfies the following conditions:
(1) $\gamma \geq s_{i_1}\gamma \geq \cdots \geq s_{i_k} \cdots s_{i_1}\gamma$;
(2) $\delta \leq s_{i_l}\delta \leq \cdots \leq s_{i_{k+1}} \cdots s_{i_l}\delta$;
(3) $s_{i_{k+1}} \cdots s_{i_l}\delta - s_{i_k} \cdots s_{i_1}\gamma$ is a simple root for $\mathfrak{g}$.

Corollary 9.7. *If an index $k \in [0, l]$ is splitting then every $\mathbf{i}$-trail $(\gamma_0, \ldots, \gamma_l)$ from γ to δ has either $\gamma_j = s_{i_j} \cdots s_{i_1}\gamma$ for $0 \leq j \leq k$, or $\gamma_j = s_{i_{j+1}} \cdots s_{i_l}\delta$ for $k \leq j \leq l$.*

Proof. Conditions (1) and (2) in Definition 9.6 imply that $\underline{\gamma}_j = s_{i_j} \cdots s_{i_1}\gamma$ for $0 \leq j \leq k$, and $\overline{\gamma}_j = s_{i_{j+1}} \cdots s_{i_l}\delta$ for $k \leq j \leq l$. Combining condition (3) with Proposition 9.3, we conclude that γ_k must be equal to $s_{i_k} \cdots s_{i_1}\gamma$ or $s_{i_{k+1}} \cdots s_{i_l}\delta$. In the former case, Corollary 9.4 guarantees the uniqueness of an $(i_1, \ldots, i_k)$-trail from γ to γ_k, and we conclude that $\gamma_j = s_{i_j} \cdots s_{i_1}\gamma$ for $0 \leq j \leq k$. Similarly, in the latter case, we have $\gamma_j = s_{i_{j+1}} \cdots s_{i_l}\delta$ for $k \leq j \leq l$, as required. $\square$

Let $C_{\mathbf{i}}(\gamma, \delta)$ denote the cone of tuples $(t_1, \ldots, t_l) \in \mathbb{R}^l$ such that $\sum_k d_k(\pi) t_k \geq 0$ for any $\mathbf{i}$-trail π from γ to δ. Our next result is an immediate consequence of Corollary 9.7.

Lemma 9.8. *Suppose an index k is splitting (for $\mathbf{i}$, γ, and δ), and let $\mathbf{i}^{(1)} = (i_1, \ldots, i_k)$, and $\mathbf{i}^{(2)} = (i_{k+1}, \ldots, i_l)$. Then we have*

$$C_{\mathbf{i}}(\gamma, \delta) = C_{\mathbf{i}^{(1)}}(\gamma, s_{i_{k+1}} \cdots s_{i_l}\delta) \times C_{\mathbf{i}^{(2)}}(s_{i_k} \cdots s_{i_1}\gamma, \gamma) \ .$$

Proof of Theorem 3.11. It is enough to prove (3.7) for the case when $p = 2$, i.e., when the flag $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_p = [1, r]$ has only one proper subset I_1. Thus $\mathbf{i}$ is the concatenation $(\mathbf{i}^{(1)}, \mathbf{i}^{(2)})$, where $\mathbf{i}^{(1)} \in R(w_\mathrm{o}(I_1))$, and $\mathbf{i}^{(2)} \in R(w_\mathrm{o}(I_1)^{-1} w_\mathrm{o})$; let $l = \ell(w_\mathrm{o})$ be the length of $\mathbf{i}$, and $k = \ell(w_\mathrm{o}(I_1))$ be the length of $\mathbf{i}^{(1)}$.

In our present notation, each cone $C_\mathbf{i}(u, v)$ is the intersection of the cones $C_\mathbf{i}(u\omega_i^\vee, vs_i\omega_i^\vee)$ for all $i \in [1, r]$; in particular, the string cone $C_\mathbf{i}$ in (3.7) is equal to

$$C_\mathbf{i} = \bigcap_{i \in [1,r]} C_\mathbf{i}(\omega_i^\vee, w_\mathrm{o} s_i \omega_i^\vee) \ .$$

Therefore, it suffices to show that

$$C_\mathbf{i}(\omega_i^\vee, w_\mathrm{o} s_i \omega_i^\vee) = C_{\mathbf{i}^{(1)}}(\omega_i^\vee, w_\mathrm{o}(I_1) s_i \omega_i^\vee) \times C_{\mathbf{i}^{(2)}}(w_\mathrm{o}(I_1)\omega_i^\vee, w_\mathrm{o} s_i \omega_i^\vee) \tag{9.1}$$

for every $i \in [1, r]$. Let us distinguish two cases.

Case 1: $i \in I_1$. We claim that in this case the index k is splitting for $\mathbf{i}$, $\gamma = \omega_i^\vee$, and $\delta = w_\mathrm{o} s_i \omega_i^\vee$. Condition (1) in Definition 9.6 is obvious. To prove (2), we need to show that $s_{i_{j+1}} \cdots s_{i_l} w_\mathrm{o} s_i \omega_i^\vee(\alpha_{i_{j+1}}) \geq 0$ for $k \leq j < l$. This follows from the fact that $s_i w_\mathrm{o}(I_1) s_{i_{k+1}} \cdots s_{i_j}(\alpha_{i_{j+1}})$ is a positive root for $\mathfrak{g}$ (this is clear since $(i_{k+1}, \ldots, i_{j+1})$ is a reduced word for a left factor of $w_\mathrm{o}(I_1)^{-1} w_\mathrm{o}$). Finally, to prove (3) notice that in the present situation we have

$$s_{i_{k+1}} \cdots s_{i_l} \delta - s_{i_k} \cdots s_{i_1} \gamma = w_\mathrm{o}(I_1) s_i \omega_i^\vee - w_\mathrm{o}(I_1)\omega_i^\vee = -w_\mathrm{o}(I_1)\alpha_i^\vee \ ,$$

which is a simple root for ${}^L\mathfrak{g}$ whenever $i \in I_1$.

The desired equality (9.1) now follows from Lemma 9.8.

Case 2: $i \notin I_1$. Since in this case $s_j \omega_i^\vee = \omega_i^\vee$ for any $j \in I_1$, it follows that every $\mathbf{i}^{(1)}$-trail from $\omega_i^\vee$ is trivial, i.e., all its components are equal to $\omega_i^\vee$. Thus in this case both sides of (9.1) are equal to $\mathbb{R}^k \times C_{\mathbf{i}^{(2)}}(\omega_i^\vee, w_\mathrm{o} s_i \omega_i^\vee)$. This concludes the proof of Theorem 3.11. □

Proof of Theorem 3.12. Without loss of generality, we can assume that the sets I_{j-1} and I_j in the formulation are equal to $[1, r-1]$ and $[1, r]$ respectively. Let us abbreviate $w'_\mathrm{o} = w_\mathrm{o}([1, r-1])$, and let $w = w'_\mathrm{o} w_\mathrm{o}$, and $\mathbf{i} = (i_1, \ldots, i_l) \in R(w)$. We need to show that if w is fully commutative then the cone

$$\bigcap_{i \in [1,r]} C_\mathbf{i}(w'_\mathrm{o}\omega_i^\vee, w_\mathrm{o} s_i \omega_i^\vee) \subset \mathbb{R}^l$$

is given by the inequalities in Theorem 3.12.

First, let us show that $C_\mathbf{i}(w'_\mathrm{o}\omega_r^\vee, w_\mathrm{o} s_r \omega_r^\vee)$ is given by the only inequality $t_l \geq 0$ (this part does not use the fact that w is fully commutative). Notice that $w'_\mathrm{o}\omega_r^\vee = \omega_r^\vee$. Notice also that the coroot $w\alpha_i^\vee = w'_\mathrm{o} w_\mathrm{o} \alpha_i^\vee = -w'_\mathrm{o}\alpha_{i^*}^\vee$ is negative for $i = r^*$, and positive otherwise. It follows that $i_l = r^*$ regardless of the choice of $\mathbf{i} \in R(w)$. Since $w_\mathrm{o} s_r \omega_r^\vee(\alpha_{r^*}) = -s_r\omega_r^\vee(\alpha_r) = -1$, it follows that any $\mathbf{i}$-trail $\pi = (\gamma_0, \ldots, \gamma_l)$ from $\omega_r^\vee$ to $w_\mathrm{o} s_r \omega_r^\vee$ must have $\gamma_{l-1} = \gamma_l = w_\mathrm{o} s_r \omega_r^\vee$. Using Corollary 9.5, we see that π is unique, and we have $\gamma_k = s_{i_k} \cdots s_{i_1}\gamma$ for $k = 0, \ldots, l-1$. It follows that $d_k(\pi) = 0$ for $k = 0, \ldots, l-1$, and $d_l(\pi) = 1$. The corresponding linear inequality defining the cone $C_\mathbf{i}(\omega_r^\vee, w_\mathrm{o} s_r \omega_r^\vee)$ is $t_l \geq 0$ as claimed.

Now let us show the following:

(*) Each of the cones $C_\mathbf{i}(w'_\mathrm{o}\omega_{i^*}^\vee, w_\mathrm{o} s_{i^*}\omega_{i^*}^\vee)$ for $i \neq r^*$ is given by some of the inequalities (2)–(4) in Theorem 3.12.

This can be done by analyzing the corresponding **i**-trails but we prefer another method using geometric lifting. By Corollary 5.9, the cone $C_{\mathbf{i}}(w'_\mathrm{o}\omega^\vee_{i^*}, w_\mathrm{o}s_{i^*}\omega^\vee_{i^*})$ consists of all integer l-tuples $(t_1, \ldots, t_l)$ satisfying the inequality

$$[\Delta_{w\omega_i^\vee, s_i\omega_i^\vee}(x_{-\mathbf{i}}(t_1, \ldots, t_l))]_{\mathrm{trop}} \geq 0 \ .$$

We shall deal with the minor $\Delta_{w\omega_i, s_i\omega_i}$ instead of $\Delta_{w\omega_i^\vee, s_i\omega_i^\vee}$ (so in the resulting formulas one will have to replace the Cartan matrix with its transpose).

A calculation in SL_2 shows that $\overline{s}_i = \lim_{t\to\infty} x_{-i}(t)x_i(-t)$. It follows that the minor in question can be written as

$$\Delta_{w\omega_i, s_i\omega_i}(x_{-\mathbf{i}}(t_1, \ldots, t_l)) = \lim_{t\to\infty} \Delta_{w\omega_i, \omega_i}(x_{-\tilde{\mathbf{i}}}(t_1, \ldots, t_l, t)) \ ,$$

where $\tilde{\mathbf{i}}$ is the word $(i_1, \ldots, i_l, i)$. Since $i \neq r^*$, we have $\tilde{\mathbf{i}} \in R(ws_i)$. We also have $ws_iw^{-1} = w'_\mathrm{o}s_{i^*}{w'_\mathrm{o}}^{-1} = s_{i'}$ for some $i' \in [1, r-1]$. Therefore, the word $\tilde{\mathbf{i}}' = (i', i_1, \ldots, i_l)$ is also a reduced word for ws_i. With the help of the transition maps in Proposition 7.3, we can express the product $x_{-\tilde{\mathbf{i}}}(t_1, \ldots, t_l, t)$ as $x_{-\tilde{\mathbf{i}}'}(p, p_1, \ldots, p_l)$, where p and all p_k are subtraction-free rational expressions in $t_1, \ldots, t_l, t$. Since $w^{-1}(\alpha_{i'}) = \alpha_i$, it easily follows from Propositions 4.3 and 4.5 that

$$\Delta_{w\omega_i, \omega_i}(x_{-\tilde{\mathbf{i}}}(t_1, \ldots, t_l, t)) = \Delta_{w\omega_i, \omega_i}(x_{-\tilde{\mathbf{i}}'}(p, p_1, \ldots, p_l)) =$$

$$p^{-1}\Delta_{w\omega_i, \omega_i}(x_{-\mathbf{i}}(p_1, \ldots, p_l)) = p^{-1} \ .$$

Thus it remains to compute p as a subtraction-free rational expression $p(t_1, \ldots, t_l, t)$ and take its limit as $t \to \infty$.

To compute p, we shall use a combinatorial lemma valid for arbitrary Coxeter groups. It uses the following notation: for any two distinct indices i and j from $[1, r]$, let $w_\mathrm{o}(i, j)$ denote the longest element in the parabolic subgroup of W generated by s_i and s_j. Thus $w_\mathrm{o}(i, j) = s_is_js_i \cdots = s_js_is_j \cdots$ (both products have d factors, where d is the order of s_is_j in W).

Lemma 9.9. *Suppose an element w of an arbitrary Coxeter group W, and an index i are such that $\ell(ws_i) = \ell(w) + 1 > 1$, and $ws_iw^{-1} = s_{i'}$ for some i'. Then there exist an index $j \neq i$ and a reduced word $\mathbf{i}'$ of w such that s_is_j has finite order d, and $\mathbf{i}'$ ends with the word $(\ldots, j, i, j) \in R(w_\mathrm{o}(i, j)s_i)$ of length $d - 1$.*

Proof. Pick any reduced word $(i_1, \ldots, i_l) \in R(w)$; clearly, $i_l \neq i$. Both words $(i_1, \ldots, i_l, i)$ and $(i', i_1, \ldots, i_l)$ are reduced words for ws_i. By the Tits theorem, the latter word can be obtained from the former one by a sequence of d-moves. Consider the first move in this sequence that involves the last letter. If this move is performed on a word $(\mathbf{i}', i)$ then the word $\mathbf{i}' \in R(w)$ has the desired property. □

Applying Lemma 9.9 several times if necessary, we obtain a reduced word $\mathbf{i}' \in R(w)$ with the following property: $\mathbf{i}'$ is a concatenation $\mathbf{i}^{(1)}, \ldots, \mathbf{i}^{(n)}$ such that each $\mathbf{i}^{(k)}$ consists of two alternating letters, and one obtains a reduced word $(\mathbf{i}', i)$ from $(i', \mathbf{i}')$ by a sequence of n d-moves, the k-th one involving $\mathbf{i}^{(k)}$ and the preceding index. Thus if we replace $\mathbf{i}$ with $\mathbf{i}'$ then the above rational function $p(t_1, \ldots, t_l, t)$ is easily computed by repeatedly using the formula for $1/p_1$ in Proposition 7.3. Taking its tropicalization, we conclude that the desired property (*) holds for $\mathbf{i}'$; more precisely, the cone $C_{\mathbf{i}'}(w'_\mathrm{o}\omega^\vee_{i^*}, w_\mathrm{o}s_{i^*}\omega^\vee_{i^*})$ is given by the inequalities (2)–(4) in Theorem 3.12 corresponding to all the intervals $\mathbf{i}^{(k)}$ that have length > 1.

On the other hand, the fact that $\mathbf{i}$ satisfies (*) is clearly preserved by switches $(i_k, i_{k+1}) \to (i_{k+1}, i_k)$ with $a_{i_k, i_{k+1}} = 0$. Therefore if w is fully commutative then (*) holds for any $\mathbf{i} \in R(w)$.

To complete the proof of Theorem 3.12, it remains to show that conversely each of the inequalities (2)–(4) appears as one of the defining inequalities for $C_{\mathbf{i}'}(w'_{\mathrm{o}}\omega^\vee_{i^*}, w_{\mathrm{o}}s_{i^*}\omega^\vee_{i^*})$ with some $i \neq r^*$. Without loss of generality, we can assume that our inequality corresponds to a subword $(i_{k+1}, \ldots, i_{k+d-1}) = (j, i, j, \ldots)$ of $\mathbf{i}$, where $d \in \{3, 4, 6\}$ is the order of $s_i s_j$. Let $u = s_{i_1} \cdots s_{i_k}$. In view of the above argument, to complete the proof it suffices to show that $\ell(us_i) = \ell(u) + 1$, and $us_iu^{-1} = s_{i'}$ for some $i' \in [1, r-1]$. In other words, it suffices to show that the root $\beta = u\alpha_i$ is one of the simple roots $\alpha_1, \ldots, \alpha_{r-1}$.

Notice that $i_k \neq i$ (otherwise $\mathbf{i}$ would contain a subword $(i_k, \ldots, i_{k+d-1}) \in R(w_{\mathrm{o}}(i,j))$ which is impossible since w is fully commutative). Therefore, the root β is positive. On the other hand, the word $(i_{k+1}, \ldots, i_l)$ is a reduced word for $u^{-1}w$. Since w (and hence $u^{-1}w$) is fully commutative, and $a_{i,i_{k+1}} = a_{ij} \neq 0$, it follows that no reduced word for $u^{-1}w$ can begin with i. Therefore, the root $w^{-1}u\alpha_i = w^{-1}\beta$ is also positive. Since $w^{-1}\beta = w_{\mathrm{o}}w'_{\mathrm{o}}\beta$, we conclude that the root $w'_{\mathrm{o}}\beta$ is negative. Since β is a positive root sent to a negative one by w'_{o}, it follows that β does not contain α_r, i.e., it is a positive integer linear combination of $\alpha_1, \ldots, \alpha_{r-1}$.

As a final step in our argument, notice that any reduced word for u begins with $i_1 = r$ (since $\ell(w'_{\mathrm{o}}s_{i'}) = \ell(w'_{\mathrm{o}}) - 1$ for any $i' \in [1, r-1]$). It follows that $u^{-1}\alpha_{i'}$ is a positive root for any $i' \in [1, r-1]$. Therefore, $u^{-1}\beta$ can be a simple root only if β is simple. This completes the proof of Theorem 3.12. □

10. Proofs of results in Sections 2.3 and 2.4

Proofs of Theorems 2.4 and 2.5. To prove Theorem 2.4, we apply Theorem 2.3 to the following reduced word:

$$\mathbf{i} = (\mathbf{i}^{(0)}, \ldots, \mathbf{i}^{(r-1)}) \in R(w_{\mathrm{o}}), \quad \mathbf{i}^{(j)} = (j, \ldots, 1, 0, 1, \ldots, j) \ .$$

We shall rename the variables t_k as follows: the variables corresponding to each interval $\mathbf{i}^{(j)}$ will be denoted $(t^{(j)}_{-j}, \ldots, t^{(j)}_{-1}, t^{(j)}_0, t^{(j)}_1, \ldots, t^{(j)}_j)$. We only need to show that for this choice of $\mathbf{i}$, each of the conditions (1)–(4) in Theorem 2.3 specializes to the corresponding condition in Theorem 2.4. For conditions (2) and (4) this is straightforward, and for (1) this is a special case of Theorem 3.12 (the corresponding string cone was already found in [18]). It remains to analyze condition (3).

Let $\pi = (\gamma^{(j)}_i)$ be an $\mathbf{i}$-trail from $s_j\omega^\vee_j$ to $w_{\mathrm{o}}\omega^\vee_j$ in an ${}^L\mathfrak{g}$-module $V_{\omega^\vee_j}$; here the components γ_k of π are renamed in the same way as the corresponding variables t_k. We use Proposition 9.3 to obtain upper and lower bounds for the weights $\gamma^{(j)}_i$. First, since $s_is_j\omega^\vee_j = s_j\omega^\vee_j$ for $i < j-1$, we have $\underline{\gamma}^{(j')}_i = s_j\omega^\vee_j$ and $d^{(j')}_i(\pi) = 0$ for all $|i| \leq j' < j-1$. Second, an easy calculation shows that $\underline{\gamma}^{(j)}_{-j} = \overline{\gamma}^{(j)}_j = s_j \cdots s_1s_0s_1 \cdots s_j\omega^\vee_j$. It follows that every component $\gamma^{(j')}_i$ of π such that $j' > j$ is obtained from the previous component by the action of $s_{|i|}$; therefore, $d^{(j')}_i(\pi) = 0$ whenever $j' > j$. Thus, the only part of π that can contribute to an inequality in (3) is a $(\mathbf{i}^{(j-1)}, \mathbf{i}^{(j)})$-trail from $s_j\omega^\vee_j$ to $s_j \cdots s_1s_0s_1 \cdots s_j\omega^\vee_j$ in $V_{\omega^\vee_j}$.

If $j = 0$ then $(\mathbf{i}^{(j-1)}, \mathbf{i}^{(j)}) = (0)$, so there is a unique trail π, and the only non-zero number $d_k(\pi)$ is $d_0^{(0)}(\pi) = -1$. The corresponding inequality in (3) is $\lambda(\alpha_0^\vee) \geq t_0^{(0)}$, as claimed.

Now let $j > 0$. Then we can assume without loss of generality that $j = r - 1$. Let us also assume that $\mathfrak{g} = so_{2r+1}$ is of type B_r. Then $V_{\omega_j^\vee}$ is the $2r$-dimensional standard (sometimes also called *vector*) representation of ${}^L\mathfrak{g} = sp_{2r}$. This ${}^L\mathfrak{g}$-module is minuscule, so the trails in question are extremal. Therefore, they are given by Proposition 9.2 with $\lambda = \omega_j^\vee$, $\mathbf{i} = (\mathbf{i}^{(j-1)}, \mathbf{i}^{(j)})$, $u = s_j$, and $v = s_j \cdots s_1 s_0 s_1 \cdots s_j$. We see that these trails correspond to all occurrences of a word $(j-1, \ldots, 1, 0, 1, \ldots, j)$ as a subword of $\mathbf{i} = (j-1, \ldots, 1, 0, 1, \ldots, j-1; j, \ldots, 1, 0, 1, \ldots, j)$. By inspection, there are $4j-1$ such subwords falling into the following 4 classes (in each case, we represent a subword by the list of variables $t_{i'}^{j'}$ corresponding to positions *not* belonging to this subword):

(1) $\{t_{i'}^{(j-1)}\ (i' > i); t_{i'}^{(j)}\ (i' \leq i)\}$ for $0 \leq i < j$;

(2) $\{t_{i'}^{(j-1)}\ (i' > i); t_{i'}^{(j)}\ (-i-1 \neq i' \leq i+1)\}$ for $0 \leq i < j$;

(3) $\{t_{i'}^{(j-1)}\ (i' \geq -i); t_{i'}^{(j)}\ (i' < -i)\}$ for $0 \leq i < j$;

(4) $\{t_{i'}^{(j-1)}\ (i+1 \neq i' \geq -i-1); t_{i'}^{(j)}\ (-i-1 \neq i' \leq i+1)\}$ for $0 \leq i < j-1$.

Computing the coefficients $d_{i'}^{(j')}(\pi)$ for the $4j-1$ extremal $\mathbf{i}$-trails π corresponding to these subwords, we see that the inequalities in Theorem 2.4(3) are indeed specializations of those in Theorem 2.3(3). This completes the proof of Theorem 2.4 for $\mathfrak{g}$ of type B_r.

Now suppose that $\mathfrak{g} = sp_{2r}$ is of type C_r. The extremal trails and corresponding linear inequalities are described in the same way as for the type B_r. The only difference is that the ${}^L\mathfrak{g}$-module $V_{\omega_j^\vee}$ for $j = r-1$ is now the ($(2r+1)$-dimensional) vector representation of so_{2r+1}, and it is no longer minuscule. But it is *quasi-minuscule*, that is, its only non-extremal weight is the zero weight. It follows easily that there is a unique non-extremal $(\mathbf{i}^{(j-1)}, \mathbf{i}^{(j)})$-trail π from $s_j \omega_j^\vee$ to $s_j \cdots s_1 s_0 s_1 \cdots s_j \omega_j^\vee$ in $V_{\omega_j^\vee}$; and the only non-zero coefficients $d_{i'}^{(j')}(\pi)$ are $d_0^{(j-1)}(\pi) = 1$ and $d_0^{(j)}(\pi) = -1$. The resulting linear inequality in (3) is $\lambda(\alpha_j^\vee) \geq t_0^{(j)} - t_0^{(j-1)}$. But we do not have to include this inequality since it is a consequence of the two inequalities $\lambda(\alpha_j^\vee) \geq 2t_0^{(j)} - t_1^{(j-1)} - t_{-1}^{(j)}$ and $\lambda(\alpha_j^\vee) \geq t_1^{(j-1)} + t_{-1}^{(j)} - 2t_0^{(j-1)}$ corresponding to extremal trails. This completes the proof of Theorem 2.4. $\square$

The proof of Theorem 2.5 is very similar, and we leave the details to the reader. Note only that when $\mathfrak{g} = so_{2r}$ is of type D_r, we apply Theorem 2.3 to the following reduced word:

$$\mathbf{i} = (\mathbf{i}^{(1)}, \ldots, \mathbf{i}^{(r-1)}) \in R(w_\mathrm{o}), \quad \mathbf{i}^{(j)} = (j, \ldots, 2, -1, 1, 2 \ldots, j)\ .$$

The variables t_k corresponding to each interval $\mathbf{i}^{(j)}$ are now denoted

$$(t_{-j}^{(j)}, \ldots, t_{-1}^{(j)}, t_1^{(j)}, \ldots, t_j^{(j)})\ .$$

The same argument as for the type B_r above shows that all the $\mathbf{i}$-trails in Theorem 2.3(3) are extremal, and the corresponding linear inequalities are precisely those in Theorem 2.5(3). $\square$

Proofs of Theorems 2.7 and 2.8. Let us recall a well-known relationship between the reduction multiplicities and tensor product multiplicities (see e.g., [5]): the multiplicity $c_{nu,\beta}$ of $V_\beta^{(I)}$ in the reduction of V_ν to $\mathfrak{g}(I)$ is equal to $c_{\lambda,\nu}^{\lambda+\beta}$ for any weight λ such that $\lambda(\alpha_i^\vee)=0$ for $i\in I$, and $\lambda(\alpha_i^\vee)\mathfrak{g}0$ for $i\in[1,r]\setminus I$. (This follows from the interpretation of $c_{nu,\beta}$ as the dimension of the subspace of vectors $v\in V_\nu$ of weight β such that $e_iv=0$ for $i\in I$.) Thus, an expression for $c_{nu,\beta}$ can be obtained by computing $c_{\lambda,\nu}^{\lambda+\beta}$ using either of the theorems 2.2 and 2.3. In doing so we choose a reduced word $\tilde{\mathbf{i}}\in R(w_\mathrm{o})$ as a concatenation $(\mathbf{i}',\mathbf{i})$, where $\mathbf{i}'=(i_1',\dots,i_{m'}')\in R(w_\mathrm{o}(I))$ and $\mathbf{i}=(i_1,\dots,i_n)\in R(w_\mathrm{o}(I)^{-1}w_\mathrm{o})$. Let us write the corresponding variables t_k that appear in Theorems 2.2 and 2.3 as $\tilde t=(t_1',\dots,t_{m'}',t_1,\dots,t_n)$.

We now claim that conditions (1)–(3) in each of Theorems 2.2 and 2.3 imply that $t_1'=\cdots=t_{m'}'=0$. It should be possible to deduce this directly but we prefer another argument. Let us deal with Theorem 2.2; Theorem 2.3 can be treated in the same way. Recall that condition (3) (combined with (1) and (2)) in Theorem 2.2 was obtained as a reformulation of the following (see Proposition 3.3 (ii) and Corollary 3.4): $l_i(b_{\tilde{\mathbf{i}}}(\tilde t))\le\lambda(\alpha_i^\vee)$ for all $i\in[1,r]$. With the choice of λ as above, this just means that $l_i(b_{\tilde{\mathbf{i}}}(\tilde t))=0$ for $i\in I$. However if $i\in I$ then $l_i(b_{\tilde{\mathbf{i}}}(\tilde t))=l_i(b_{\mathbf{i}'}(t_1',\dots,t_{m'}'))$, in view of Proposition 3.3(ii); here $b_{\mathbf{i}'}(t_1',\dots,t_{m'}')$ is understood as a canonical basis vector for the (semisimple part of) the Lie algebra $\mathfrak{g}(I)$ instead of $\mathfrak{g}$. Remembering the definition of $l_i(b)$, we conclude that $b_{\mathbf{i}'}(t_1',\dots,t_{m'}')=1$ hence $t_1'=\cdots=t_{m'}'=0$, as claimed.

Now the conditions on $t_1,\dots,t_m$ in Theorem 2.7 (resp. Theorem 2.8) are easily seen to be equivalent to conditions (1), (2) and (4) in Theorem 2.2 (resp. Theorem 2.3); for conditions Theorem 2.8 (1) and Theorem 2.7 (3), this follows from the splitting property (3.7) in Theorem 3.11. □

Proofs of Corollaries 2.10 and 2.11. These corollaries are obtained by specializing Theorems 2.7 and 2.8 to the following choice of a reduced word $\mathbf{i}\in R(w_\mathrm{o}(I)^{-1}w_\mathrm{o})$:

$$\mathbf{i}=(p,p+1,\dots,p+q-1;p-1,p,\dots,p+q-2;\dots;1,2,\dots,q)\ .$$

We rename the corresponding variables t_k as follows:

$$t=(t_{11},\dots,t_{1q};t_{21},\dots,t_{2q};\dots;t_{p1},\dots,t_{pq})\ .$$

It is now shown by a direct check that all the conditions in Corollaries 2.10 and 2.11 are specializations of the corresponding conditions in Theorems 2.7 and 2.8 (for conditions Corollary 2.11(1) and Corollary 2.10(3), this follows from Theorem 3.12). We leave the details of this check to the reader. □

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Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: arkadiy@math.harvard.edu

Department of Mathematics, Northeastern University, Boston, MA 02115
E-mail address: andrei@neu.edu