A SUPPORT PROBLEM FOR K-THEORY OF NUMBER FIELDS

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Abstract. In this paper we consider a support problem for the reduction map on the odd dimensional algebraic K-theory of number fields.

1. Introduction.

Pál Erdös stated the following problem for the integers in 1988.

Support Problem. *For two positive integers x and y the following two statements are equivalent.*

(1) *For every $n \in \mathbb{N}$ and every prime p we have* $x^n \equiv 1 \mod p$ *if and only if* $y^n \equiv 1 \mod p$.
(2) x *equals* y

This problem was solved by Corrales-Rodrigáñez and Schoof in the paper [C-RS], which also contains generalization of the support problem for all number fields and elliptic curves. Some other related support problems the reader can find in papers [Ba] and [S].

In this paper we consider the support problem for algebraic K-theory of a number field F. Let L/F be a finite field extension. Let n be a positive integer and l an odd prime. Consider the Dwyer-Friedlander map [D-F]

$$\psi_l\colon \quad K_{2n+1}(L) \to K_{2n+1}(L) \otimes \mathbb{Z}_l \to H^1(G_L; \mathbb{Z}_l(n+1)).$$

Let us define C_L to be the subgroup of $K_{2n+1}(L)$ which is generated by the l-parts of kernels of maps ψ_l for all primes l. We define the group $B(L)$ by putting

$$B(L) = K_{2n+1}(L)/C_L.$$

Note that the group C_L is finite by [DF] and it should vanish if the Quillen-Lichtenbaum conjecture is true. Let k_v be the residue field for a prime v in F. Put $g_v = G(\overline{k_v}/k_v)$ to be the absolute Galois group for the finite field k_v. For every

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l as above and for any L we have the following commutative diagram with the map ψ_l as the left vertical arrow.

$$
\begin{array}{ccc}
K_{2n+1}(F)/C_L & \xrightarrow{\pi_v} & \prod_{v\nmid l} K_{2n+1}(k_v)_l \\
\psi_l \big\downarrow & & = \big\downarrow \\
H^1(G_F;\, \mathbb{Z}_l(n+1)) & \xrightarrow{\pi_v} & \prod_{v\nmid l} H^1(g_v;\, \mathbb{Z}_l(n+1))
\end{array} \tag{1.1}
$$

The horizontal maps of the diagram are induced by the reduction maps at the prime ideals of $\mathcal{O}_F$ in K-theory and cohomology respectively. Note that

$$K_{2n+1}(\mathcal{O}_F)_l \cong K_{2n+1}(F)$$

and

$$H^1(G_F;\, \mathbb{Z}_l(n+1)) \cong H^1(G_{F,\Sigma_l};\, \mathbb{Z}_l(n+1)),$$

where Σ_l is the set of primes v of F which divide l. Observe that

$$K_{2n+1}(k_v)_l \cong H^1(g_v;\, \mathbb{Z}_l(n+1)) \cong H^0(g_v;\, \mathbb{Q}_l/Z_l(n+1)). \tag{1.2}$$

It follows by the definition of $B(L)$ that

$$B(L) \otimes \mathbb{Z}_l \cong H^1(G_L;\, \mathbb{Z}_l(n+1)).$$

We prove the following theorem.

Theorem. *Let P, Q be two nontorsion elements of the group $K_{2n+1}(F)$. Assume that for almost every v and almost every integer m the following condition holds*

$$m\pi_v(P) = 0 \qquad \textit{whenever} \qquad m\pi_v(Q) = 0.$$

Then the elements P and Q of $K_{2n+1}(F)$ are linearly dependent over $\mathbb{Z}$.

Let $\mathbb{K}(\mathcal{O}_F)$ and $\mathbb{K}(k_v)$ be the K-theory spectra of $\mathcal{O}_F$ and of the finite field k_v, respectively cf. [DM]. Consider the stable Hurewicz map for the K-theory spectrum $X = \mathbb{K}(\mathcal{O}_F)$. It follows by [A], Theorem 4.1 that after tensoring with $\mathbb{Q}$ the Hurewicz map gives an isomorphism

$$K_{2n+1}(\mathcal{O}_F) \otimes \mathbb{Q} \cong H_{2n+1}(\mathbb{K}(\mathcal{O}_F);\, \mathbb{Q}).$$

Hence, the group $H_{2n+1}(\mathbb{K}(\mathcal{O}_F);\, \mathbb{Z})$ is finitely generated, because the K-theory of $\mathcal{O}_F$ is finitely generated [Q]. The reduction map at the prime v induces the natural map on homology

$$\pi'_v\colon H_{2n+1}(\mathbb{K}(\mathcal{O}_F);\, \mathbb{Z}) \longrightarrow H_{2n+1}(\mathbb{K}(k_v);\, \mathbb{Z}).$$

Using the proof of the Theorem for the reduction map π'_v we obtain the following result.

Corollary. *Let P', Q' be two nontorsion elements in the group $H_{2n+1}(\mathbb{K}(\mathcal{O}_F);\mathbb{Z})$. Assume that for almost every prime ideal v and for almost every integer m the following condition holds*

$$m\pi'_v(P')=0 \qquad \textit{whenever} \qquad m\pi'_v(Q')=0.$$

Then the elements P' and Q' are linearly dependent in the group $H_{2n+1}(\mathbb{K}(\mathcal{O}_F);\mathbb{Z})$.

Remark. Our proof of the Theorem is based on the methods of the paper [C-RS]. In [BGK] we consider the support problem for the Galois cohomology of l-adic representations and associated intermediate Jacobians.

2. Proof of the theorem.
Let S_l be the finite set of primes which contains the prime l and all primes for which the assumptions of the theorem do not hold.

Lemma 1. *Let L be a number field.*

(1) *The natural map $H^1(G_L;\mathbb{Z}_l(n{+}1)) \longrightarrow \varinjlim_{L'/L} H^1(G_{L'};\mathbb{Z}_l(n{+}1))$ is an embedding, where the limit is over all finite extensions L'/L.*
(2) *The discrete Galois module $\varinjlim_{L'/L} H^1(G_{L'};\mathbb{Z}_l(n{+}1))$ is a divisible group.*
(3) *For every k, the l^k-torsion subgroup of $\varinjlim_{L'/L} H^1(G_{L'};\mathbb{Z}_l(n{+}1))$ is naturally isomorphic to $\mathbb{Z}/l^k(n{+}1)$.*

This follows by [T], Proposition 2.3. We leave details of the proof for the reader.

We denote by $\hat{P}$, $\hat{Q}\in H^1(G_F;\mathbb{Z}_l(n{+}1))$ the images of P and Q via the Dwyer-Friedlander map. Let H_l denote the kernel of the action of G_F on the module $\mathbb{Z}/l(n{+}1)$ and we put $F_l=\bar{F}^{H_l}$. Let $\hat{R}$ and $\hat{R}'$ be elements of the module $\varinjlim_{L/F} H^1(G_L;\mathbb{Z}_l(n{+}1)$ such that

$$\hat{P}=l\hat{R} \qquad \text{and} \qquad \hat{P}'=l\hat{R}'.$$

The elements $\hat{R}$, $\hat{R}'$ exist by Lemma 1 (2). Let H'_l denote the kernel of the G_F-action on the subgroup of $\varinjlim_{L/F} H^1(G_L;\mathbb{Z}_l(n{+}1))$ generated by $\mathbb{Z}/l^k(n{+}1)$ and $\hat{R}$. Observe that this subgroup is G_F-invariant and does not depend on the choice of $\hat{R}$, due to Lemma 1 (3). Put $F_l(\frac{1}{l}\hat{P})=\bar{F}^{H'_l}$. In the similar way we define the field $F_l(\frac{1}{l}\hat{Q})$. We choose primes over v in the corresponding fields as shown on the picture below. Let $v\notin S_l$ be a prime which splits completely in $F_l(\frac{1}{l}\hat{P})$. Let us put

$$t_v=\sharp K_{2n+1}(k_v)_l.$$

(2.1)

$$\begin{tikzcd}[column sep=small]
& & K \arrow[dll, dash] \arrow[drr, dash] & & \\
F_l(\frac{1}{l}\hat{P}) \arrow[d, dash] \arrow[drr, dash] & & & & F_l(\frac{1}{l}\hat{Q}) \arrow[dll, dash] \arrow[d, dash] \\
F(\hat{R}) \arrow[drr, dash] & & F_l \arrow[d, dash] & & F(\hat{R}') \arrow[dll, dash] \\
& & F & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& w \arrow[dl, dash] & & w' \arrow[dr, dash] & \\
\beta \arrow[d, dash] \arrow[drr, dash] & & & & \beta' \arrow[dll, dash] \arrow[d, dash] \\
u \arrow[drr, dash] & & {} \arrow[d, dash] & & u' \arrow[dll, dash] \\
& & v & &
\end{tikzcd}$$

Consider the following commutative diagram.

$$
\begin{array}{ccc}
H^1(G_K;\, \mathbb{Z}_l(n+1)) & \xrightarrow{\pi_w} & H^1(g_w;\, \mathbb{Z}_l(n+1)) \\
\uparrow & & \uparrow \\
H^1(G_{F_l(\frac{1}{l}\hat{P})};\, \mathbb{Z}_l(n+1)) & \xrightarrow{\pi_\beta} & H^1(g_\beta;\, \mathbb{Z}_l(n+1)) \\
\uparrow & & \uparrow \\
H^1(G_{F(\hat{R})};\, \mathbb{Z}_l(n+1)) & \xrightarrow{\pi_u} & H^1(g_u;\, \mathbb{Z}_l(n+1)) \\
\uparrow & & \uparrow \\
H^1(G_F;\, \mathbb{Z}_l(n+1)) & \xrightarrow{\pi_v} & H^1(g_v;\, \mathbb{Z}_l(n+1))
\end{array}
\tag{2.2}
$$

Since v splits completely in $F_l(\frac{1}{l}\hat{P})$, the two right, bottom, vertical arrows in the diagram (2.2) are isomorphisms. In addition this means that $l|t_v$ (observe that $t_v = \sharp K_{2n+1}(k_\beta)_l$, cf. (1.2)). Hence, by the diagram (2.2) we have the following equality

$$\frac{t_v}{l}\pi_v(\hat{P}) = \pi_\beta(\frac{t_v}{l}\hat{P}) = t_v\pi_\beta(\hat{R}) = 0.$$

By the assumption of the theorem this implies that $\frac{t_v}{l}\pi_v(\hat{Q}) = 0$. We want to prove that v splits completely in the field $F_l(\frac{1}{l}\hat{Q})$. Consider the diagram obtained from (2.2) by replacing P by Q, R by R' and primes u, β, w by primes u', β', w', respectively. Then there exists $\hat{P}_0 \in \mathbb{Z}/l(n+1)$ such that

$$Fr_{w'}(\hat{R}') = \hat{R}' + \hat{P}_0,$$

where $Fr_{w'} \in g_{w'} \subset G(K/F)$.
Applying the reduction at w' to the last equality we obtain

$$Fr_{w'}\pi_{w'}(\hat{R}') = \pi_{w'}Fr_{w'}(\hat{R}') = \pi_{w'}(\hat{R}') + \pi_{w'}(\hat{P}_0). \tag{2.3}$$

Since

$$t_v\pi_{w'}(\hat{R}') = \frac{t_v}{l}\pi_{w'}(l\hat{R}') = \frac{t_v}{l}\pi_{w'}(\hat{Q}) = 0,$$

it follows that $\pi_{w'}(\hat{R}')$ belongs to the group $H^1(g_v;\, \mathbb{Z}_l(n+1))$, hence it is fixed by the Frobenius element $Fr_{w'}$. Together with the equality (2.3) this gives

$$\pi_{w'}(\hat{P}_0) = 0.$$

Using [T], Proposition 2.3, p. 261 we prove that the reduction map $\pi_{w'}$ is injective on l-torsion parts, hence $\hat{P}_0 = 0$. This implies that the action of the Frobenius $Fr_{w'}$ on $\hat{R}'$ is trivial. Since $\mathbb{Z}/l(n+1) \subset H^1(g_v;\, \mathbb{Z}_l(n+1))$, it follows that v splits completely in $F_l(\frac{1}{l}\hat{Q})$. By the Frobenius density theorem, [J], Corollary 5.5, p. 136, we conclude that

$$F_l(\frac{1}{l}\hat{Q}) \subset F_l(\frac{1}{l}\hat{P}). \tag{2.4}$$

Consider the map $\phi_{\hat{P}}\colon G_{F_l} \to \mathbb{Z}/l(n{+}1)$ which is given by the formula

$$\phi_{\hat{P}}(\sigma) = \sigma(\hat{R}) - \hat{R}$$

and the analogous map $\phi_{\hat{Q}}$ defined for the element $\hat{Q}$. The maps $\phi_{\hat{P}}$, $\phi_{\hat{Q}}$ are well defined by Lemma 1 (3). It is clear by (2.4) that

$$ker(\phi_{\hat{P}}) \subset ker(\phi_{\hat{Q}}).$$

In this way we get the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker(\phi_{\hat{P}}) & \longrightarrow & G(\bar{F}/F_l) & \xrightarrow{\phi_{\hat{P}}} & \mathbb{Z}/l(n+1) & \longrightarrow & 0 \\ & & \Big\downarrow & & =\Big\downarrow & & \gamma\Big\downarrow & & \\ 0 & \longrightarrow & ker(\phi_{\hat{Q}}) & \longrightarrow & G(\bar{F}/F_l) & \xrightarrow{\phi_{\hat{Q}}} & \mathbb{Z}/l(n+1) & \longrightarrow & 0 \end{array} \tag{2.5}$$

with imbedding as the left vertical arrow. Since $Im\,\phi_{\hat{P}}$ is a cyclic group, the map γ is a multiplication by an integer d, say

$$\phi_{\hat{Q}} = d\phi_{\hat{P}}. \tag{2.6}$$

There is a natural embedding

$$\theta\colon H^1(F_l;\, \mathbb{Z}_l(n+1))/l \to H^1(F_l;\, \mathbb{Z}/l(n+1)) = Hom(G_{F_l};\, \mathbb{Z}/l(n+1))$$

which is given by the formula $\theta(\hat{X}) = \phi_{\hat{X}}$. By Lemma 2 (note that $F_l \subset F(\mu_l)$) below and by (2.6) we obtain the following equality

$$\hat{Q} = d\hat{P} \tag{2.7}$$

in the group $H^1(G_F;\, \mathbb{Z}_l(n+1))/l = B(F)/lB(F)$. Hence by (2.7), the image of Q in $B_0 = B(F)/\{kP; k \in \mathbb{Z}\}$ is contained in lB_0 for all primes l which are not in S_l. Since by results of Quillen the group $B(F)$ is finitely generated it follows that B_0 is also finitely generated. Therefore $\bigcap_{l \notin S_l} lB_0$ is finite. Hence,

$$bQ = aP$$

in the group $K_{2n+1}(F)$ for some integers a and b.

Lemma 2. *Assume that l does not divide the number $w_{n+1}(F)$ whose l-part by definition is $\#H^0(G_F;\, \mathbb{Q}_l/\mathbb{Z}_l(n+1))$. Then the natural map*

$$H^1(G_F;\, \mathbb{Z}_l(n+1))/l^k \to H^1(G_{F(\mu_{l^k})};\, \mathbb{Z}_l(n+1))/l^k$$

is an imbedding.

Proof. cf. [AB] or [BG]. □

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