Lattice chain models for affine buildings of classical type

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Abstract. A concrete lattice chain model for the buildings of the classical groups over non archimedean complete skew fields is given. The building axioms are proved in a uniform way using hereditary orders with involution instead of lattice chains.

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1 Introduction

By the theory of Bruhat and Tits (see [BrT 72], [BrT 84a]), there is an affine building associated to each reductive group over a local field. This group theoretic construction of affine buildings is, in general, highly complicated. However, in special cases "concrete" descriptions for these buildings were developed (and applied), which do not refer to the general Bruhat-Tits theory. The most well-known example of this type is the model of the affine building associated to $SL_n(\mathcal{D})$, where we denote by $\mathcal{D}$ a discretely valuated skew field and by $\mathfrak{M}$ the corresponding valuation ring. This model, which for $n = 2$ was introduced in [Ser 77], uses $\mathfrak{M}$-lattices in the vector space $\mathcal{D}^n$. The simplices of the building are either described (as in [Ser 77]) as certain sets of (homothety) classes of $\mathfrak{M}$-lattices or alternatively as chains of $\mathfrak{M}$-lattices which are stable under the action of $\mathcal{D}^*$ (cf. for instance [Gra 80] or [Gar 97, Chapter 19]).

The other classical groups are described as unitary groups of an algebra with involution, see [Wei 61], and hence naturally act on a hermitian vector space. From this point of view it is natural to look for lattice class (or chain) models in the case of affine buildings associated to other classical groups over local fields as well. This is partly done in [Gar 97, Chapter 20]. However, Garrett does not consider (proper) unitary groups, his treatment of the non-split orthogonal groups looks a little bit ad hoc (it is based on "primitive", i. e. unimodular lattices, which do not always exist) and the verification of the building axioms is incomplete (he just claims that two chambers are always contained in an apartment; cf. on the other hand the non-trivial proof of Theorem 25 below). In [BrT 84b], [BrT 87], Bruhat and Tits discuss in detail a reinterpretation of their affine buildings for classical groups over complete, discretely valuated (and even more general) skew fields. This discussion is based on the fact that the points of the Bruhat-Tits building (viewed as a metric space) associated to $SL_n(\mathcal{D})$ are in one-to-one correspondence with the homothety classes of the "normes scindables" (cf. [BrT 84b, 2.13 and 5.1]), which in their turn are closely related to

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lattice chains (cf. [BrT 84b, 1.7 and 1.8]). In [BrT 87, Section 2] it is then shown that the other classical affine buildings can always be obtained as subspaces (with "normes maximinorantes" as points) of this model for the affine building of $SL_n(\mathcal{D})$. In order to do this, the general Bruhat-Tits theory, which à priori guarantees the existence of the affine buildings associated to the groups in question, has to be presupposed.

In the present paper, we develop an approach towards "concrete" models for classical buildings which is less general than that of Bruhat and Tits but (as we hope) easier accessible and, this being our main point, completely independent of the Bruhat-Tits theory. In fact, we shall show that certain simplicial complexes, on which the corresponding unitary groups act in a natural way, are affine buildings, and we shall deduce from this the existence of affine BN-pairs in these groups. The simplicial complexes in question can again be described by lattice chains. In the case of unitary groups (and we include symplectic as well as orthogonal groups here), they arise as fixed point structures (but not necessarily as subcomplexes!) of the standard model for the affine building of $SL_n(\mathcal{D})$, this latter building being naturally acted upon by an involution which is induced by the hermitian form (cf. Section 3). Using a criterion of Muehlherr (cf. [Mue 94, Theorem 1.8.22]), we show that these fixed point structures are (weak) buildings. At the same time, we are exhibiting concrete (and very natural) systems of apartments for these buildings. All this depends on a new description - given in Section 6 - of our simplicial complexes, where the simplices are interpreted as certain orders with involution instead of lattice chains. In particular this allows to treat all the unitary groups in a uniform way, without distinguishing the different types of hermitian forms.

The paper is organized as follows. In Section 2, we briefly recall some facts concerning the affine building $\hat{\Delta}$ associated to $SL_n(\mathcal{D})$. The simplicial complex Δ consisting of all "admissible" chains of lattices in a given hermitian space V is introduced in Section 3. First properties of the thin subcomplexes associated to "hyperbolic frames" as well as of the natural action of the unitary group $U(V)$ on Δ are derived in Section 5. As a preparation, we deduce some facts about maximal lattices in anisotropic spaces in Section 4. The heart of this paper is Section 6, where important properties of Δ are derived by using orders with involution and in particular by studying their idempotents. The main result is Theorem 25, which yields that Δ is a weak building (of type $\tilde{C}_r$, where r is the Witt index of V) admitting the thin subcomplexes introduced in Section 5 as a system of apartments. In order to obtain thick buildings also in those cases when there are panels in Δ which are contained in two chambers only, we apply a generalization of the well-known "oriflamme construction" (cf. [Tit 74, Chapter 7]) in Section 7. It follows that $U(V)$ possesses a normal subgroup $U(V)_0$ with $[U(V) : U(V)_0] \leq 4$, which acts strongly transitively on a thick building Δ_0 of type $\tilde{C}_r$, $\tilde{B}_r$, or $\tilde{D}_r$ associated to Δ ($\Delta_0 = \Delta$ and $U(V)_0 = U(V)$ if Δ is already thick). This is used in order to construct an affine BN-pair in $U(V)_0$, and it also shows that Δ_0 can be described algebraically as the (Bruhat-Tits) building associated to this BN-pair.

2 The affine building associated to V

This paper gives concrete models for certain abstract buildings. By a building we always understand here a chamber complex possessing a system of subcomplexes, called apartments, which satisfy the usual conditions as stated in [Bro 89, Section IV.1]. A building is called thick if each of its panels (:= codimension-1-simplices) is contained in at least three chambers. We definitely do not only consider thick buildings in this paper. If we want to stress that the building need not be thick, we also use the expression weak building.

Let $\mathcal{D}$ be a discretely valuated skew field with valuation ν, valuation ring $\mathfrak{M}$ and π a prime element of $\mathfrak{M}$. Let V be an n-dimensional left vector space over $\mathcal{D}$.

Definition 1 *An $\mathfrak{M}$-lattice L in V is a free $\mathfrak{M}$-module $L \leq V$ with $\mathcal{D}L = V$. If $L = \mathfrak{M}b_1 \oplus \ldots \oplus \mathfrak{M}b_n$ then the $\mathcal{D}$-basis $(b_1, \ldots, b_n)$ of V is called a lattice basis of L.*

A chain $\ldots \subseteq L_s \subseteq L_{s+1} \subseteq \ldots$ of $\mathfrak{M}$-lattices L_i is called admissible, if the set $\{L_i\}$ is closed under multiplication by integral powers of π.

A $\mathcal{D}$-basis $(b_1, \ldots, b_n)$ of V is called a chain basis for the chain $(L_i)_{i \in \mathbb{Z}}$ if for each L_i there are $\alpha_{ij} \in \mathbb{Z}$ such that $(\pi^{\alpha_{ij}} b_j, 1 \leq j \leq n)$ is a lattice basis for L_i.

Definition 2 *Denote by $\hat{\Delta}$ the partially ordered (by inclusion) set of all admissible chains of lattices in V.*

A frame of V is a collection $\langle v_1 \rangle, \ldots, \langle v_n \rangle$ of 1-dimensional $\mathcal{D}$-subspaces spanning V. The subcomplex $\hat{\Sigma}(v_1, \ldots, v_n)$ of Δ associated with the frame $\langle v_1 \rangle, \ldots, \langle v_n \rangle$ of V is the set of all admissible chains admitting $(v_1, \ldots, v_n)$ as a chain basis.

It is well known that $\hat{\Delta}$ is a thick affine building of type $\tilde{A}_{n-1}$ and that $\{\hat{\Sigma}(v_1, \ldots, v_n)\}$, where $\langle v_1 \rangle, \ldots, \langle v_n \rangle$ runs through all frames of V, forms a system of apartments of $\hat{\Delta}$, see for instance [Gar 97], [BrT 84b]. Also one has a natural numbering on $\hat{\Delta}$ which can be described as follows: Fix a lattice L_0 in V. Then for an arbitrary lattice $L =: L_0 g$ for $g \in GL(V)$, the vertex $\{\pi^m L \mid m \in \mathbb{Z}\}$ receives the number $\nu(det(g))$ (mod n) $\in \{0, \ldots, n-1\}$, where $det(g)$ is the Dieudonné determinant of g.

From now on we assume that $\mathcal{D}$ is finite dimensional over its center $K := Z(\mathcal{D})$. Let $A := \mathrm{End}_{\mathcal{D}}(V)$ be the endomorphism ring of V and $R := \mathfrak{M} \cap K$. There is a close connection between lattices in V and certain R-orders in A. An R-order Λ in A is an R-lattice that is closed under multiplication such that $K\Lambda = A$. By [Rei 75, Corollary (1.7.4)] the maximal R-orders in A are endomorphism rings $\mathrm{End}_{\mathfrak{M}}(L) := \{\lambda \in A \mid L\lambda = L\}$ of $\mathfrak{M}$-lattices L in V. An R-order Λ of A is called a chain order or hereditary order, if Λ is the intersection $\cap_{L \in \mathcal{L}} \mathrm{End}_{\mathfrak{M}}(L)$ of all endomorphism rings of lattices in an admissible chain $\mathcal{L}$. Its invariant lattices are precisely the ones in this chain (cf. [Rei 75, Theorem 39.23]). Therefore one gets the following remark (which seems to hold without the assumption that $\dim_K(\mathcal{D}) < \infty$).

Remark 3 *The map $\mathcal{L} \mapsto \cap_{L \in \mathcal{L}} \mathrm{End}(L)$ defines an inclusion reversing isomorphism between Δ and the poset of all chain orders in A.*

The projections $e_1, \ldots, e_n$ with respect to the decomposition

$$V = \langle v_1 \rangle \oplus \ldots \oplus \langle v_n \rangle$$

are orthogonal idempotents of A. The condition that an admissible chain $\mathcal{L}$ lies in the apartment $\hat{\Sigma}(v_1, \ldots, v_n)$ is equivalent to $e_1, \ldots, e_n \in \cap_{L \in \mathcal{L}} \mathrm{End}_{\mathfrak{M}}(L)$.

3 The affine building of the hermitian space V

Now let $K := Z(\mathcal{D})$ and $R = K \cap \mathfrak{M}$. We shall assume from now on that $\mathcal{D}$ is complete and finite dimensional over K and that $k := \mathfrak{M}/\pi\mathfrak{M}$ has characteristic $\neq 2$.

Let $^-$ be an involution on $\mathcal{D}$ with $\overline{\mathfrak{M}} = \mathfrak{M}$, $\epsilon = \pm 1$ and

$$(,) : V \times V \to \mathcal{D}$$

be a non degenerate $(^-, \epsilon)$-hermitian form on V with Witt index r.

Then $(,)$ induces an involution $\ ^\circ$ on $A = \mathrm{End}_{\mathcal{D}}(V)$: Let $a \in A$. Since $(,)$ is non degenerate, there is a unique $a^\circ \in A$ with $(va, w) = (v, wa^\circ)$ for all $v, w \in V$. Then the unitary group

$$U(V) = \{a \in A \mid aa^\circ = 1\}$$

is the full isometry group of $(,)$, that is the full orthogonal, symplectic, respectively unitary group of V.

Definition 4 *If L is an $\mathfrak{M}$-lattice in V then its dual lattice is*

$$L^{\#} := \{v \in V \mid (v, l) \in \mathfrak{M} \text{ for all } l \in L\}.$$

Note that $L^{\#}$ is again an $\mathfrak{M}$-lattice since $\mathfrak{M}$ is stable under $^-$. Moreover $\mathrm{End}_{\mathfrak{M}}(L^{\#}) = \mathrm{End}_{\mathfrak{M}}(L)^\circ$. Since $(L^{\#})^{\#} = L$ and the duals of the lattices in an admissible chain $\mathcal{L}$ form again an admissible chain $\mathcal{L}^{\#}$, $\#$ induces an automorphism of order 2 on the affine building $\hat{\Delta}$. We call an admissible chain $\mathcal{L}$ #-admissible, if $\mathcal{L} = \mathcal{L}^{\#}$.

Definition 5 *Denote by Δ the partially ordered (by inclusion) set of all #-admissible chains of lattices in V.*

The isometry group $U(V)$ acts on Δ. For a #-admissible chain $\mathcal{L} = \{L_i \mid i \in \mathbb{Z}\}$ and $g \in U(V)$ let $\mathcal{L}g$ be the #-admissible chain $\mathcal{L}g := \{L_i g \mid i \in \mathbb{Z}\}$.

Remark 6 *Let $c \in \mathcal{D}^*$ with $\overline{c} = \pm c$. Then the form f_c defined by $f_c(v, w) := (v, w)c$ for all $v, w \in V$ is an $(^{-\prime}, \epsilon')$-hermitian form on V, where $\epsilon' = \epsilon \overline{c}^{-1} c$ and the involution $^{-\prime}$ on $\mathcal{D}$ is defined by $\overline{a}' := c^{-1} \overline{a} c$. The group $U(V)$ as well as the set Δ remain the same.*

Remark 7 *As a set Δ clearly consists of the fixed points of $\#$ in $\hat{\Delta}$. As one easily shows, the partially ordered set Δ is in fact a simplicial complex. However much more holds true according to [Mue 94, Theorem 1.8.22] which implies that Δ is a totally gated chamber complex.*

In Section 6 below, we shall show that Δ is in fact a weak building.

Definition 8 *(i) A* hyperbolic frame *of V is a collection $\langle v_1\rangle, \dots, \langle v_{2r}\rangle$ of 1-dimensional subspaces of V such that $(v_i, v_j) \neq 0$ if and only if $j = 2r + 1 - i$.*

(ii) The subcomplex $\Sigma(v_1, \dots, v_{2r})$ of Δ associated with the frame $\langle v_1\rangle, \dots, \langle v_{2r}\rangle$ of V is the simplicial complex consisting of all $\#$-admissible chains that admit a chain basis which is contained in $\{v_1, \dots, v_{2r}\} \cup \langle v_1, \dots, v_{2r}\rangle^{\perp}$.

Since r is the Witt index of $(,)$, the hermitian space $\langle v_1, \dots, v_{2r}\rangle^{\perp}$ is anisotropic. To show that the $\Sigma(v_1, \dots, v_{2r})$ are apartments, we need some general facts about anisotropic spaces.

4 Maximal lattices in anisotropic spaces

Let $\mathcal{D}$, $^-$, $\mathfrak{M}$ be as in section 3, in particular $\bar{\mathfrak{M}} = \mathfrak{M}$ and $2 \in \mathfrak{M}^*$.

Lemma 9 *Let W be a vector space over $\mathcal{D}$, $\epsilon \in Z(\mathcal{D}) = K$ with $\epsilon\bar{\epsilon} = 1$ and $\phi : W \times W \to \mathcal{D}$ a $(^-, \epsilon)$-hermitian form. Assume that there exist $v_0, w \in W$ satisfying $\phi(v_0, v_0) \in \pi\mathfrak{M}$, $\phi(w, w) \in \mathfrak{M}$ and $\phi(w, v_0) \in \mathfrak{M}^*$. Then W contains an isotropic vector.*

Proof: We construct inductively $v_m \in \mathcal{D}w + \mathcal{D}v_0$ satisfying

$$\phi(v_m, v_m) \in \pi^{2^m}\mathfrak{M}, \ \phi(w, v_m) \in \mathfrak{M}^*, \ v_{m+1} - v_m \in \pi^{2^m}\mathfrak{M}w$$

for all $m \in \mathbb{N}_0$. Given v_m, we set

$$\alpha := \phi(v_m, v_m) = \epsilon\overline{\alpha} \in \pi^{2^m}\mathfrak{M}, \ \eta := \phi(w, v_m) \in \mathfrak{M}^*, \ \beta := \phi(w, w) \in \mathfrak{M}.$$

With $\lambda := -\frac{1}{2}\alpha\eta^{-1} \in \pi^{2^m}\mathfrak{M}$ let $v_{m+1} := v_m + \lambda w$. Then

$$\phi(w, v_{m+1}) = \phi(w, v_m) + \phi(w, w)\overline{\lambda} \in \mathfrak{M}^* + \pi^{2^m}\mathfrak{M} \subseteq \mathfrak{M}^*$$

and

$$\phi(v_{m+1}, v_{m+1}) = \alpha + \lambda\eta + \epsilon\overline{\lambda\eta} + \lambda\beta\overline{\lambda} = \alpha - \frac{1}{2}(\alpha + \epsilon\overline{\alpha}) + \lambda\beta\overline{\lambda} = \lambda\beta\overline{\lambda} \in \pi^{2^{m+1}}\mathfrak{M}.$$

Since $\mathcal{D}$ is complete, the Cauchy sequence $(v_m)_{m\in\mathbb{N}_0}$ converges in $\mathcal{D}v_0 + \mathcal{D}w$. Then $v := \lim_{m\to\infty} v_m$ obviously satisfies $\phi(v, v) = 0$ and $\phi(w, v) \in \mathfrak{M}^*$. In particular $v \neq 0$ and W is isotropic. □

Corollary 10 *Assume that W is anisotropic and finite dimensional and set $X := \{x \in W \mid \phi(x,x) \in \mathfrak{M}\}$. Then the following holds*

(i) $\phi(X,X) \subseteq \mathfrak{M}$.

(ii) *X is an $\mathfrak{M}$-lattice in W.*

(iii) $\pi X^{\#} \subseteq X \subseteq X^{\#}$.

(iv) *If L is an $\mathfrak{M}$-lattice in W satisfying $\pi L^{\#} \subseteq L \subseteq L^{\#}$ then $L = X$.*

(v) *X has an orthogonal basis.*

Proof: (i) Suppose there are $x, y \in X$ such that $\phi(x,y) \in \pi^{-m}\mathfrak{M}^*$ for some $m \geq 1$. Then $v_0 := \pi^m x$ and $w := y$ satisfy the conditions of Lemma 9 implying that W is isotropic in contradiction to our assumption.

(ii) From (i) it follows that X is closed under addition. So X is obviously an $\mathfrak{M}$-module. Clearly $\mathcal{D}X = W$. Now choose an $\mathfrak{M}$-lattice L in W which is contained in X. Applying (i) once more, we obtain $X \subseteq L^{\#}$. Since $L^{\#}$ is an $\mathfrak{M}$-lattice, X is an $\mathfrak{M}$-lattice as well.

(iii) $X \subseteq X^{\#}$ follows from (i). In order to show $\pi X^{\#} \subseteq X$, we shall verify $\phi(\pi x, \pi x) \in \mathfrak{M}$ for all $x \in X^{\#}$. In fact, we even get $\phi(x,x) \in \pi^{-1}\mathfrak{M}$: Assume that $x \in X^{\#}$ with $\phi(x,x) \in \pi^{-m}\mathfrak{M}^*$ for some $m \geq 2$. Let $l := \left[\frac{m+1}{2}\right]$ and $x' := \pi^l x$. Then $\phi(x',x') \in \mathfrak{M}$, hence $x' \in X$. Now $x \in X^{\#}$ implies $\phi(x',x) \in \mathfrak{M}$, in contradiction to $\phi(x',x) = \pi^l \phi(x,x) \in \pi^{l-m}\mathfrak{M}^*$.

(iv) In view of (iii) and the definition of X, $\pi L^{\#} \subseteq L \subseteq L^{\#}$ implies

$$\pi X^{\#} \subseteq \pi L^{\#} \subseteq L \subseteq X \subseteq X^{\#} \subseteq L^{\#}.$$

Suppose $L \neq X$. Then also $\pi L^{\#} \neq \pi X^{\#}$, and we can choose $v_0 \in \pi L^{\#} - \pi X^{\#}$. It follows that $\phi(v_0, v_0) \in \phi(\pi L^{\#}, L) \subseteq \pi\mathfrak{M}$. Furthermore, since $v_0 \notin \pi X^{\#}$, there is $w \in X$ such that $\phi(w, v_0) \in \mathfrak{M}^*$. Now Lemma 9 implies that W is isotropic which contradicts our assumption.

(v) First we observe that there exists $v \in X$ satisfying $\phi(X,X) = \phi(v,v)\mathfrak{M}$. This is clear if there is a $v' \in X$ such that $\phi(v',v') \in \mathfrak{M}^*$. If such a v' does not exist, then there must be a $v \in X$ such that $\phi(v,v) \in \pi\mathfrak{M}^*$. ($\phi(x,x) \in \pi^2\mathfrak{M}$ for all $x \in X$ would lead to the contradiction $\pi^{-1}X \subseteq X$). Furthermore, Lemma 9 immediately implies $\phi(X,X) \subseteq \pi\mathfrak{M} = \phi(v,v)\mathfrak{M}$ in this case. Now if $x \in X$ is given arbitrarily, $x - \lambda v$ is in the orthogonal complement of v for $\lambda := \phi(x,v)\phi(v,v)^{-1} \in \mathfrak{M}$. This shows that $X = \mathfrak{M}v + (X \cap v^{\perp})$ is an orthogonal decomposition of the lattice X. The claim now follows by induction on $\dim_{\mathcal{D}}(W)$. □

5 Apartments and the action of $U(V)$

Lemma 11 *Let $\langle v_1\rangle, \ldots, \langle v_{2r}\rangle$ be a hyperbolic frame of V. Then there is a basis $(v_1, \ldots, v_n)$ of V such that the apartment $\hat{\Sigma}(v_1, \ldots, v_n)$ (cf. Section 2) is invariant under # and*

$$\hat{\Sigma}(v_1, \ldots, v_n) \cap \Delta = \Sigma(v_1, \ldots, v_{2r}).$$

Proof: Let $(v_{2r+1}, \ldots, v_n)$ be an orthogonal basis of the maximal integral lattice X in the anisotropic space $W := \langle v_1, \ldots, v_{2r}\rangle^{\perp}$ (see Corollary 10). Then it is clear that $\hat{\Sigma}(v_1, \ldots, v_n)$ is invariant under #. By definition $\hat{\Sigma}(v_1, \ldots, v_n) \cap \Delta \subseteq \Sigma(v_1, \ldots, v_{2r})$. To see the other inclusion, let $L \in \mathcal{L} \in \Sigma(v_1, \ldots, v_{2r})$. Then

$$L = \oplus_{i=1}^{2r} \pi^{m_i} \mathfrak{M} v_i \perp Y$$

for some $m_i \in \mathbb{Z}$ and a lattice Y in W. Since $\mathcal{L}^{\#} = \mathcal{L}$, there is some $m \in \mathbb{Z}$ such that

$$\pi^m Y \subseteq Y^{\#} \subseteq \pi^{m-1} Y.$$

One easily shows using Corollary 10 that either Y or $Y^{\#}$ is a multiple of the maximal integral lattice X. Therefore $\mathcal{L} \in \hat{\Sigma}(v_1, \ldots, v_n)$. □

5.1 A combinatorial model for $\Sigma(v_1, \ldots, v_{2r})$

Let $\mathcal{L} \in \Delta$ be a vertex, which means that $\mathcal{L}$ is a minimal non-empty #-admissible chain. Then it is straightforward to see that there is a unique lattice $L \in \mathcal{L}$ which satisfies $L \subseteq L^{\#} \subseteq \pi^{-1}L$. This lattice is called the standard representative of $\mathcal{L}$. One obtains an identification of the set of vertices in Δ with the set of standard lattices L in V, which are the lattices $L \subset V$ with $L \subseteq L^{\#} \subseteq \pi^{-1}L$.

Let $\langle v_1\rangle, \ldots, \langle v_{2r}\rangle$ be a hyperbolic frame of V such that $(v_i, v_{2r+1-i}) \in \mathfrak{M}^*$ for all $1 \leq i \leq r$ and let $W := \langle v_1, \ldots, v_{2r}\rangle^{\perp}$. Let L be a lattice of the form $L = \oplus_{i=1}^{2r} \pi^{n_i} \mathfrak{M} v_i \perp Y$ with $n_i \in \mathbb{Z}$ and a lattice $Y \subset W$.

Then $L^{\#} = \oplus_{i=1}^{2r} \pi^{-n_{2r+1-i}} \mathfrak{M} v_i \perp Y^{\#}$, and hence L is a standard lattice if and only if firstly $n_i + n_{2r+1-i} \in \{0, 1\}$ for all $1 \leq i \leq r$ and secondly $Y \subseteq Y^{\#} \subseteq (1/\pi)Y$. By Corollary 10 the second condition is equivalent to $Y = X$, where X is the maximal integral lattice in W. The vertices $\mathcal{L}, \mathcal{L}' \in \Sigma(v_1, \ldots, v_{2r})$ defined by the standard lattices $L = \oplus_{i=1}^{2r} \pi^{n_i} \mathfrak{M} v_i \perp X$ and $L' = \oplus_{i=1}^{2r} \pi^{n'_i} \mathfrak{M} v_i \perp X$ are adjacent, if and only if the multiples of L, $L^{\#}$, L', and $L'^{\#}$ form a chain. In view of $L \subseteq L^{\#} \subseteq (1/\pi)L$ and $L' \subseteq L'^{\#} \subseteq (1/\pi)L'$ this is equivalent to $L \subseteq L'$ (i.e. $n_i \geq n'_i$ for all $1 \leq i \leq 2r$) or $L' \subseteq L$ (i.e. $n_i \leq n'_i$ for all $1 \leq i \leq 2r$). Summarizing this, we get the following proposition.

Proposition 12 *Denote by N_{2r} the set*

$$N_{2r} := \{(n_1, \ldots, n_{2r}) \in \mathbb{Z}^{2r} \mid n_i + n_{2r+1-i} \in \{0, 1\} \text{ for all } 1 \leq i \leq r\}$$

and endow it with a partial ordering by

$$(n_1,\ldots,n_{2r}) \le (n'_1,\ldots,n'_{2r}) :\Leftrightarrow n_i \le n'_i \text{ for all } 1 \le i \le 2r.$$

Let Σ_{2r} be the simplicial complex with set of vertices N_{2r} and set of simplices equal to the set of all subsets of N_{2r} which are totally ordered. Then $\Sigma(v_1,\ldots,v_{2r})$ is isomorphic to Σ_{2r}.

The isomorphism mentioned in Proposition 12 depends on the choice of $(v_1,\ldots,v_{2r})$ and is obtained by identifying a vertex $\mathcal{L} \in \Sigma(v_1,\ldots,v_{2r})$ with standard representative $L = \oplus_{i=1}^{2r} \pi^{n_i}\mathfrak{M}v_i \perp X$ with the $2r$-tupel $(n_1,\ldots,n_{2r})$. By a slight abuse of notation, we will also speak of "the standard lattice $L = (n_1,\ldots,n_{2r})$" or of "the vertex $\mathcal{L} = (n_1,\ldots,n_{2r})$".

Proposition 12 can be used in order to derive properties of $\Sigma(v_1,\ldots,v_{2r})$ in a combinatorial way. For instance, it is clear that all maximal simplices in Σ_{2r} contain precisely $r+1$ vertices. One could also show directly that Σ_{2r} is a thin chamber complex with "sufficiently many" foldings and hence a Coxeter complex. However, we shall not carry out the corresponding computations here, since this also follows from a general building theoretic result (see Corollary 13).

5.2 The standard chamber

Now we fix $b_1,\ldots,b_{2r} \in V$ with $(b_i,b_j) = \delta_{j,2r+1-i}$ for all $1 \le i \le j \le 2r$ and consider the standard apartment $\Sigma_0 = \Sigma(b_1,\ldots,b_{2r})$. Let X_0 be the maximal integral lattice in the anisotropic space $W_0 := \langle b_1,\ldots,b_{2r}\rangle^\perp$. For $j = 0,\ldots r$ we consider the standard lattices

$$L_j := \oplus_{i=1}^{j} \pi\mathfrak{M}b_i \oplus \oplus_{i=j+1}^{2r}\mathfrak{M}b_i \oplus X_0 = (\underbrace{1,\ldots,1}_{j \text{ times}},0\ldots,0)$$

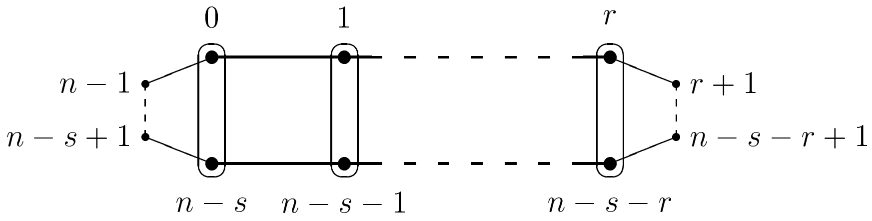
and the corresponding vertices $\mathcal{L}_j \in \Sigma_0$. Then $\mathcal{C}_0 := \cup_{j=0}^{r}\mathcal{L}_j$ is a maximal #-admissible chain in Σ_0, called the standard chamber.

The type of $\mathcal{C}_0 \in \hat{\Delta}$ is $J := \{0,\ldots,r\} \cup \{n-s,\ldots,n-s-r\} \subseteq \mathbb{Z}/n\mathbb{Z} = \{0,\ldots,n-1\}$ where $s = \nu(\det(g)) \in \{0,\ldots,n-2r\}$ for some $g \in GL(W_0)$ with $X_0^\# g = X_0$.

The types of the Δ-vertices in $\mathcal{C}_0$, which are vertices or edges in $\hat{\Delta}$, are the elements of

$$\{\{j, n-s-j\} \mid j = 0,\ldots,r\} \leftrightarrow \{0,\ldots,r\} =: \tilde{J}.$$

This can be visualized by the following Tits diagram:



In particular one immediately checks that the condition concerning types in [Mue 94, Proposition 2.3.1 (7.)] is satisfied in our situation. By Lemma 11 $\Sigma_0 := \Sigma(b_1,\ldots,b_{2r})$

can be considered as the fixed complex of # acting on the Coxeter complex $\hat{\Sigma}(b_1, \ldots, b_n)$. Therefore [Mue 94, Proposition 2.3.1 (7.)] immediately implies the following Corollary.

Corollary 13 *Σ_0 is a Coxeter complex with numbering $\tau : \Sigma_0 \to P(\tilde{J})$, induced by the numbering of $\hat{\Sigma}(b_1, \ldots, b_n)$.*

Applying Proposition 12 it is easy to determine the Coxeter diagram of $\Sigma_0 \cong \Sigma_{2r}$. We first assume $r \geq 2$ and calculate the diameter m_{ij} of the link of the facet of $\mathcal{C}_0$ of type $\tilde{J} - \{i,j\}$ $(0 \leq i < j \leq r)$ by examining the vertices $\mathcal{L} = (n_1, \ldots, n_{2r})$ in this link. If $\{i,j\} = \{0,1\}$ then $(n_1, \ldots, n_{2r}) < (1,1,0,\ldots,0)$. Therefore $n_3 = \ldots = n_{2r-2} = 0$ and $(n_1, n_2; n_{2r-1}, n_{2r}) \in \{(0,0;0,0),\ (1,0;0,0),\ (1,0;0,-1),\ (0,1;0,0),\ (0,1;-1,0),\ (1,1;0,-1),\ (1,1;-1,0),\ (1,1;-1,-1)\}$. Hence the Coxeter number $m_{01} = 4$. Similarly, $m_{r-1,r} = 4$. If $1 \leq i \leq r-2$ and $j = i+1$, then the standard representative L of $\mathcal{L}$ satisfies $L_{i-1} \supset L \supset L_{i+2}$. Therefore $n_1 = \ldots = n_{i-1} = 1$, $n_{i+3} = \ldots = n_{2r} = 0$, and $(n_i, n_{i+1}, n_{i+2}) \in \{0,1\}^3 - \{(0,0,0),(1,1,1)\}$. Hence $m_{i,i+1} = 3$ for all $1 \leq i \leq r-2$. Finally one easily verifies $m_{ij} = 2$ for $j - i \geq 2$, since the corresponding rank 2 links contain precisely 4 vertices.

If $r = 1$, one easily checks that every vertex has precisely 2 neighbors. Hence, since Σ_2 is infinite, it is a Coxeter complex of type $\tilde{A}_1$. Thus we have shown the following remark.

Remark 14 *The Coxeter complex Σ_0 is of affine type $\tilde{C}_r$ if $r \geq 2$ and $\tilde{A}_1$ if $r = 1$.*

5.3 The action of $U(V)$

Remark 15 *The same arguments used in [Mue 94, Proposition 2.3.1] in order to derive a numbering on Σ_0 show that the natural numbering on $\hat{\Delta}$ induces a numbering $\tau : \Delta \to P(\tilde{J})$. Since the determinants of the elements in $U(V)$ are units in $\mathfrak{M}$, $U(V)$ acts type preservingly on $\hat{\Delta}$ and hence on Δ.*

Lemma 16 *$U(V)$ acts transitively on the set of pairs $(\mathcal{C}, \Sigma(v_1, \ldots, v_{2r}))$ where $\langle v_1 \rangle$, …, $\langle v_{2r} \rangle$ is a hyperbolic frame of V and $\mathcal{C} \in \Sigma(v_1, \ldots, v_{2r})$ a maximal #-admissible chain.*

<u>Proof:</u> First we note that $U(V)$ clearly acts transitively on the set of hyperbolic frames of V and hence on the set of apartments of Δ. So it remains to show that the stabilizer N in $U(V)$ of Σ_0 acts transitively on the chambers in Σ_0. For $j = 1, \ldots, r-1$ let $g_j \in N$ be the element of $U(V)$ that interchanges b_j with b_{j+1} and b_{2r-j} with b_{2r-j+1} and is the identity on the orthogonal complement $\langle b_1, b_{j+1}, b_{2r-j}, b_{2r-j+1} \rangle^{\perp}$. Let $g_0 \in N$ be the element mapping b_1 to $\epsilon\pi^{-1} b_{2r}$ and b_{2r} to $\overline{\pi} b_1$ and fixing the orthogonal complement of $\langle b_1, b_{2r} \rangle$ and $g_r \in N$ be the element mapping b_r to ϵb_{r+1} and b_{r+1} to b_r and fixing the orthogonal complement of $\langle b_r, b_{r+1} \rangle$. Then

$$\mathcal{C}_0 g_0, \ldots, \mathcal{C}_0 g_r$$

are precisely the neighbours of $\mathcal{C}_0$ in Σ_0 and $\mathcal{C}_0 g_j^2 = \mathcal{C}_0$ for all j. This shows that the g_j act as generating reflections of the Weyl group of the Coxeter complex Σ_0. □

6 Orders with involution

The automorphism $\#$ of $\hat{\Delta}$ is induced by the involution on A, as one easily sees identifying $\hat{\Delta}$ with the poset of all chain orders in A (cf. Remark 3).

A chain order Λ is called admissible if $\Lambda = \Lambda^\circ$, or equivalently if the chain of Λ-invariant lattices is $\#$-admissible.

Remark 17 *The map $\mathcal{L} \mapsto \cap_{L\in\mathcal{L}}\mathrm{End}_{\mathfrak{M}}(L)$ defines an inclusion reversing isomorphism between Δ and the poset of all admissible chain orders in A.*

Next we want to study some connections between hyperbolic frames of V and certain idempotents in A (or in admissible chain orders).

Definition 18 *Given a hyperbolic frame $\langle v_1\rangle, \dots, \langle v_{2r}\rangle$ of V the set of orthogonal idempotents $\{e_1, \dots, e_{2r}\}$ of A defined by $v_i e_j = \delta_{ij} v_i$, $\langle v_1, \dots, v_{2r}\rangle^\perp e_j = 0$ for all $1 \leq i, j \leq 2r$ is said to be* associated *to this frame.*

Lemma 19 *(i) If the set $\{e_1, \dots, e_{2r}\}$ of orthogonal idempotents of A is associated to the hyperbolic frame $\langle v_1\rangle, \dots, \langle v_{2r}\rangle$ of V then $e_i^\circ = e_{2r+1-i}$ for all $1 \leq i \leq 2r$.*

(ii) If the set $\{e_1, \dots, e_{2r}\}$ of orthogonal idempotents of A satisfies $e_i^\circ = e_{2r+1-i}$ for all $1 \leq i \leq 2r$, then $Ve_1, \dots, Ve_{2r}$ is a hyperbolic frame of V to which $\{e_1, \dots, e_{2r}\}$ is associated.

(iii) If $\mathcal{L}$ is a $\#$-admissible chain, $\langle v_1\rangle, \dots, \langle v_{2r}\rangle$ a hyperbolic frame of V and $\{e_i \mid 1 \leq i \leq 2r\}$ the associated set of idempotents of A, then the following equivalence holds:

$$\mathcal{L} \in \Sigma(v_1, \dots, v_{2r}) \Leftrightarrow \{e_1, \dots, e_{2r}\} \subseteq \mathrm{End}_{\mathfrak{M}}(L) \text{ for all } L \in \mathcal{L}$$

Proof: (i) This follows from the Definitions 8 (i) and 18, from the fact that $(ve_i^\circ, w) = (v, we_i)$ for all $v, w \in V$ and from the non-degeneracy of $(\cdot,\cdot)$.

(ii) Setting $e_0 := 1 - \Sigma_{i=1}^{2r} e_i$, we obtain the partition $1 = \Sigma_{i=0}^{2r} e_i$ of 1 into orthogonal idempotents. Hence $V = \oplus_{i=0}^{2r} Ve_i$. From $(Ve_i, Ve_j) = (V, Ve_j e_i^\circ) = \delta_{i,2r+1-i}\mathcal{D}$ for all $1 \leq i, j \leq 2r$ we deduce that the subspaces $\oplus_{i=1}^{r} Ve_i$ and $\oplus_{i=r+1}^{2r} Ve_i$ are totally isotropic. Since r is the Witt index of V, it follows that $\dim_{\mathcal{D}} Ve_i = 1$ for all $1 \leq i \leq 2r$. Hence $\{Ve_i \mid 1 \leq i \leq 2r\}$ is a hyperbolic frame of V. In order to see that $\{e_i \mid 1 \leq i \leq 2r\}$ is associated to this frame, we still have to verify $(\Sigma_{i=1}^{2r} Ve_i)^\perp e_j = 0$ for all $1 \leq j \leq 2r$. But this follows from $e_0^\circ = e_0$ which implies that $(\Sigma_{i=1}^{2r} Ve_i)^\perp = Ve_0$.

(iii) If $\mathcal{L} \in \Sigma(v_1, \dots, v_{2r})$ and $L \in \mathcal{L}$ then by Definition 8 (ii)

$$L = \oplus_{i=1}^{2r}(L \cap \mathcal{D}v_i) \oplus (L \cap \langle v_1, \dots, v_{2r}\rangle^\perp).$$

Hence the projection $Le_j = L \cap \mathcal{D}v_j \subseteq L$ lies again in L, or equivalently $e_j \in \mathrm{End}_{\mathfrak{M}}(L)$ for all $1 \leq j \leq 2r$. Now assume that $\{e_1, \ldots, e_{2r}\} \subseteq \mathrm{End}_{\mathfrak{M}}(L)$ and set again $e_0 := 1 - \Sigma_{i=1}^{2r} e_i$. Then

$$L = \oplus_{i=0}^{2r} Le_i = \oplus_{i=1}^{2r} (L \cap \mathcal{D}v_i) \oplus (L \cap \langle v_1, \ldots, v_{2r} \rangle^{\perp})$$

for all L in $\mathcal{L}$, and this shows that we can choose a chain basis for $\mathcal{L}$ in $\{v_1, \ldots, v_{2r}\} \cup \langle v_1, \ldots, v_{2r} \rangle^{\perp}$. □

Chain orders are examples of graduated orders. An R-order Λ in A is called graduated, if Λ contains a full system $x_1, \ldots, x_n$ of orthogonal primitive idempotents of A such that $x_i \Lambda x_i \cong \mathfrak{M}$ is the maximal order in $x_i A x_i \cong \mathcal{D}$. The lattices of graduated orders are well understood and described in [Ple 83, Remark (II.4)]: Namely let L be a lattice in V with $L\Lambda = L$ for some graduated order Λ in A and let $x_1, \ldots, x_n \in \Lambda$ be a system of orthogonal primitive idempotents of A. Then $L = L \cdot 1 = L \cdot (\Sigma_{i=1}^{n} x_i) = \oplus_{i=1}^{n} Lx_i$. Let $b_i \in V$ be a generator of Vx_i $(1 \leq i \leq n)$. Then for each Λ-lattice L there are integers $\alpha_1, \ldots, \alpha_n$ such that $L = \mathfrak{M}\pi^{\alpha_1} b_1 \oplus \ldots \oplus \mathfrak{M}\pi^{\alpha_n} b_n$.

Lemma 20 *Let $\Gamma_1 \supseteq \Gamma_2$ be two graduated orders in A. Let $S_1, \ldots, S_a$ represent the isomorphism classes of simple Γ_1-modules. Then the Γ_2-composition factors of the S_i $(1 \leq i \leq a)$ form a system of representatives of the isomorphism classes of the simple Γ_2-modules.*

Proof: Let $x_1, \ldots, x_n \in \Gamma_2$ be a full system of orthogonal primitive idempotents in A. Let L be a Γ_1-lattice in the simple A-module V and $\mathcal{L}$: $L = L_0 > L_1 > \ldots > L_b = \pi L$ a chain of Γ_1-lattices in V such that $M_i := L_{i-1}/L_i$ $(1 \leq i \leq b)$ represent all isomorphism classes of simple Γ_1-modules. Choosing a composition series of M_i as Γ_2-module, we can refine the chain $\mathcal{L}$ to a chain $\mathcal{L}'$: $L = L'_0 > L'_1 > \ldots > L'_c = \pi L$ of Γ_2-lattices such that $M'_i := L'_{i-1}/L'_i$ $(1 \leq i \leq c)$ represent all isomorphism classes of simple Γ_2-modules. Since $x_1, \ldots, x_n \in \Gamma_2$, $L'_i = \oplus_{j=1}^{n} L'_i x_j$ for all $0 \leq i \leq c$ and $(b_1, \ldots, b_n)$ is a chain basis of $\mathcal{L}'$, where $\mathfrak{M}b_i = Lx_i$. So for each $1 \leq i \leq n$ there is exactly one $j \in \{1, \ldots, c\}$ with $M'_j x_i \neq 0$. In particular the composition factors of $L/\pi L$ are pairwise non isomorphic Γ_2-modules. □

Lifting idempotents

Let Λ be an R-order in A that is stable under the involution, $\Lambda^\circ = \Lambda$.

It is well known that one may lift idempotents of $\Lambda/J(\Lambda)$ to idempotents of Λ (cf. [Rei 75, Theorem 6.18]). Here $J(\Lambda)$ is the Jacobson radical of Λ, i.e. the intersection of all maximal (left or right) ideals of Λ. One important property of $J(\Lambda)$ is that $\Lambda/J(\Lambda)$ is the biggest semisimple quotient of Λ. It follows that the isomorphism classes of the simple Λ-modules (regarded as $\Lambda/J(\Lambda)$ modules) are in bijection with the central primitive idempotents of $\Lambda/J(\Lambda)$.

Since $\Lambda^\circ = \Lambda$, the involution $\,^\circ$ preserves $J(\Lambda)$ and therefore permutes the central primitive idempotents of $\Lambda/J(\Lambda)$. We want to lift these idempotents to a $\,^\circ$-invariant set of orthogonal idempotents of Λ.

Remark 21 *(i) If $\epsilon \in \Lambda$ maps onto an idempotent of $\Lambda/J(\Lambda)$ then the Newton-Hensel-iteration $e_0 := \epsilon$, $e_{i+1} := e_i + (e_i^2 - e_i)(1-2e_i) = 3e_i^2 - 2e_i^3$, $i = 0,1,2,\dots$ yields an idempotent $e = \lim(e_i)$ in Λ, with $e \equiv \epsilon\ (J(\Lambda))$. (cf. [Rei 75, Theorem 6.18])*

(ii) If $e_0 = e_0^\circ$, then it holds for all e_i that $e_i^\circ = e_i$. In particular $e^\circ = e$.

(iii) If $\epsilon \equiv \epsilon^\circ\ (J(\Lambda))$, then $e_0 := \epsilon\epsilon^\circ$ is equivalent to ϵ modulo $J(\Lambda)$ and invariant under the involution ${}^\circ$.

The remark above says that one may lift involution invariant idempotents of $\Lambda/J(\Lambda)$ to involution invariant idempotents of Λ. Since $2 \in R^*$, one gets a similar result if the involution ${}^\circ$ interchanges two orthogonal idempotents of $\Lambda/J(\Lambda)$.

Lemma 22 *Let $\epsilon \in \Lambda$ be congruent to an idempotent of $\Lambda/J(\Lambda)$ such that $\epsilon\epsilon^\circ$ and $\epsilon^\circ\epsilon \in J(\Lambda)$. Then there is an idempotent e' in Λ with $e' \equiv \epsilon\ (J(\Lambda))$ and $e'e'^\circ = 0$.*

Proof: Let $e_0 := \epsilon + \epsilon^\circ$. Then $e_0^2 = \epsilon^2 + (\epsilon^\circ)^2 + \epsilon\epsilon^\circ + \epsilon^\circ\epsilon \equiv e_0\ (J(\Lambda))$. The Newton-Hensel-iteration in Remark 21 (i) yields an involution invariant idempotent $e = e^2 \equiv e_0\ (J(\Lambda))$.

Now we are looking for an element $f \in \Lambda$ satisfying $f \equiv \epsilon - \epsilon^\circ\ (J(\Lambda))$, $f^\circ = -f$, $f^2 = e$ and $ef = fe = f$. If we find such an f, then $e' := \frac{1}{2}(e+f)$ obviously proves the lemma. So let us start by setting $f_0 := e(\epsilon - \epsilon^\circ)e$. Observe that $e \equiv \epsilon + \epsilon^\circ$, $\epsilon\epsilon^\circ \equiv \epsilon^\circ\epsilon \equiv 0$, and $\epsilon^2 \equiv \epsilon\ (J(\Lambda))$ imply that $f_0^\circ = -f_0$ and $f_0^2 \equiv e\ (J(\Lambda))$. For $i \geq 0$ let $f_{i+1} := f_i - \frac{1}{2}(f_i^2 - e)f_i$. Then $f_{i+1}^2 - e = \frac{1}{4}(f_i^2 - e)^2(f_i^2 - 4e)$ and hence, by induction, $f_i^2 - e \in (J(\Lambda))^{2^i}$ for all i. In particular $(f_i)_{i\in\mathbb{N}_0}$ is a Cauchy sequence, therefore converging to an $f := \lim(f_i) \in \Lambda$. Since we have $f_i \equiv f_0 \equiv \epsilon - \epsilon^\circ\ (J(\Lambda))$, $f_i^\circ = -f_i$, $f_i^2 \equiv e\ (J(\Lambda)^{2^i})$, $ef_i = f_ie = f_i$ for all $i \in \mathbb{N}_0$, f possesses the desired properties. □

Corollary 23 *Assume that $\epsilon_1, \dots, \epsilon_s \in \Lambda$ satisfy the following conditions:*

(i) $\epsilon_1 + J(\Lambda), \dots, \epsilon_s + J(\Lambda)$ is a system of orthogonal idempotents in $\Lambda/J(\Lambda)$.

(ii) There is a permutation ${}^* : \{1,\dots,s\} \to \{1,\dots,s\}$ such that $\epsilon_{i^*} \equiv \epsilon_i^\circ\ (J(\Lambda))$ $1 \leq i \leq s$.

Then there exist orthogonal idempotents $e_1, \dots, e_s \in \Lambda$ satisfying $e_i \equiv \epsilon_i\ (J(\Lambda))$ and $e_{i^*} = e_i^\circ$ for all $1 \leq i \leq s$.

Proof: We remark that the assumptions imply $i^{**} = i$ for all i and deduce the assertion by induction on s. If $1^* = 1$, Remark 21 yields an idempotent $e_1 \in \Lambda$ satisfying $e_1 \equiv \epsilon_1\ (J(\Lambda))$ and $e_1^\circ = e_1$. If $1^* \neq 1$, Lemma 22 provides idempotents $e_1, e_{1^*} := e_1^\circ \in \Lambda$ satisfying $e_1 \equiv \epsilon_1, e_{1^*} \equiv \epsilon_1^\circ \equiv \epsilon_{1^*}\ (J(\Lambda))$, with $e_1e_{1^*} = e_{1^*}e_1 = 0$. Set $f := 1 - e_1$ in the first case, $f = 1 - e_1 - e_{1^*}$ in the second and consider the involution invariant R-order $\Lambda' := f\Lambda f$. Setting $\epsilon_i' := f\epsilon_i f$ for all $i \in \{1,\dots,s\} - \{1,1^*\}$, we deduce $\epsilon_i' \equiv \epsilon_i(J(\Lambda))$ from condition (i). Furthermore, our assumptions together with the identity $J(\Lambda) \cap f\Lambda f = fJ(\Lambda)f = J(\Lambda')$ (which is proved in [Jac 64, Ch. III, §7, Proposition 1]) imply that Λ' and the ϵ_i' again satisfy the conditions (i) and (ii). Now the induction hypothesis yields a system of orthogonal idempotents $e_i \in \Lambda'$ such that $e_i \equiv \epsilon_i'(J(\Lambda'))$ and $e_i^\circ = e_{i^*}$ $(i \in \{1, \ldots, s\} - \{1, 1^*\})$. Since f is orthogonal to e_1 and e_{1^*}, $\{e_i \mid 1 \leq i \leq s\}$ is a system of orthogonal idempotents as well, and it has the required properties. □

Lemma 24 *Let $\Gamma \subseteq \Lambda$ be graduated $^\circ$-invariant R-orders in A. Assume that there exists a system of orthogonal idempotents $\{f_i \mid 1 \leq i \leq 2r\}$ in Λ such that $f_i^\circ = f_{2r+1-i}$ and $f_i + J(\Lambda)$ is central primitive in $\Lambda/J(\Lambda)$ for all i. Then Γ also contains a system of orthogonal idempotents $\{e_i \mid 1 \leq i \leq 2r\}$ satisfying $e_i^\circ = e_{2r+1-i}$ and $e_i + J(\Lambda) = f_i + J(\Lambda)$.*

Proof: Let S_i be the simple Λ-module associated with f_i. Since $\dim(Vf_i) = 1$ the module S_i is of dimension 1 over $\mathfrak{M}/\pi\mathfrak{M}$. Now the simple components of $\Gamma/J(\Gamma)$ are isomorphic to matrix rings over $\mathfrak{M}/\pi\mathfrak{M}$, so the S_i stay simple Γ-modules. By Lemma 20 the S_i are pairwise non isomorphic as Γ-modules. Let $\epsilon_i + J(\Gamma)$ be the central primitive idempotent of $\Gamma/J(\Gamma)$ that induces the identity on S_i and 0 on the other S_j $(j \neq i \in \{1, \ldots, 2r\})$. The involution $^\circ$ preserves Γ and $J(\Gamma)$ and therefore permutes the central primitive idempotents of $\Gamma/J(\Gamma)$. Since $\epsilon_i^\circ + J(\Lambda)$ acts like $\epsilon_{2r+1-i} + J(\Lambda)$ on the simple Λ-modules one gets $\epsilon_i^\circ + J(\Gamma) = \epsilon_{2r+1-i} + J(\Gamma)$.

By Lemma 22 there are orthogonal idempotents $e_i \in \Gamma$ with $e_i^\circ = e_{2r+1-i}$ and $e_i + J(\Gamma) = \epsilon_i + J(\Gamma)$. Since e_i induces the identity on S_i and annihilates the other simple Γ-modules one gets $e_i + J(\Lambda) = f_i + J(\Lambda)$. □

Theorem 25 *For any two simplices in Δ there is an apartment $\Sigma(v_1, \ldots, v_{2r})$ of Δ containing both.*

Proof. We first show that for any simplex $\mathcal{L}$ in Δ, there is an apartment $\Sigma(v_1, \ldots, v_{2r})$ of Δ containing $\mathcal{L}$ and the standard chamber $\mathcal{C}_0$. It is convenient to use the language of orders. So let $\Lambda := \bigcap_{L \in \mathcal{L}} \mathrm{End}_{\mathfrak{M}}(L)$ be an admissible chain order in A and $\Lambda_0 := \bigcap_{L \in \mathcal{C}_0} \mathrm{End}_{\mathfrak{M}}(L)$ be the standard minimal admissible chain order in A. It is well known (cf. for instance [Gar 97, p. 326-327]) that there is a basis of V that is a chain basis for both chains of Λ and Λ_0. This is the same as saying that $\Gamma := \Lambda \cap \Lambda_0$ is a graduated order. Since Λ_0 contains a system $\{f_1, \ldots, f_{2r}\}$ of orthogonal idempotents associated to a hyperbolic frame of V, Lemma 24 implies that there are orthogonal idempotents $e_1, \ldots, e_{2r} \in \Gamma$ with $e_i^\circ = e_{2r+1-i}$ $(1 \leq i \leq 2r)$. Hence Λ and Λ_0 both belong to the apartment defined by the frame Ve_i, $i = 1, \ldots, 2r$.

In particular every chamber in Δ corresponds to a minimal admissible chain order Λ in A that contains orthogonal idempotents $f_1, \ldots, f_{2r} \in \Lambda$ belonging to a hyperbolic frame of V such that $f_i + J(\Lambda)$ are central in $\Lambda/J(\Lambda)$. Now the same proof as above shows the Theorem for an arbitrary pair of simplices in Δ. □

This readily implies the following

Corollary 26 *Δ is a weak building of type $\tilde{C}_r$ (if $r > 1$), respectively $\tilde{A}_1$ (if $r = 1$), and*

$$\mathcal{A} := \{\Sigma(v_1, \dots, v_{2r}) \mid \langle v_1 \rangle, \dots, \langle v_{2r} \rangle \text{ is a hyperbolic frame of } V\}$$

is a system of apartments in Δ. The group $U(V)$ acts strongly transitively on $(\Delta, \mathcal{A})$.

Proof: The last claim follows from the first and Lemma 16. In view of Theorem 25 and the fact that the $\Sigma(v_1, ..., v_{2r})$ are Coxeter complexes of type $\tilde{C}_r$, respectively $\tilde{A}_1$ (cf. Corollary 13 and Remark 14), we just have to verify the following building axiom

(*) If Σ and Σ' are elements of $\mathcal{A}$ with a common chamber, then there exists an isomorphism $\alpha : \Sigma \to \Sigma'$ which fixes $\Sigma \cap \Sigma'$ pointwise.

Choose a chamber $\mathcal{C} \in \Sigma \cap \Sigma'$ and a type preserving isomorphism α between Σ and Σ' fixing $\mathcal{C}$ (note that Σ and Σ' are isomorphic by Proposition 12 and that their Weyl groups are transitive on their chamber sets). From the fact that the apartments $\hat{\Sigma}$ and $\hat{\Sigma}'$ of Lemma 11 are convex in $\hat{\Delta}$ and [Mue 94, Theorem 1.8.22] it follows that the elements of $\mathcal{A}$ are convex chamber subcomplexes of Δ. This implies that $\Sigma \cap \Sigma'$ is also a convex chamber subcomplex of Δ. Now, since Σ and Σ' are thin, an easy induction on the gallery distance from $\mathcal{C}$ shows that any chamber $\mathcal{C}'$ in $\Sigma \cap \Sigma'$ is fixed by α, thus proving (*). $\square$

Remark 27 *One can also use orders with involution again in order to derive the above building axiom (*). One essentially shows that two sets of orthogonal idempotents as in Lemma 19 in a graduated order $\Gamma = \Gamma^\circ$ that lift the same central primitive idempotents of $\Gamma/J(\Gamma)$ are already conjugate in $U(\Gamma) = \Gamma^* \cap U(V)$.*

If one is only interested in the fact that the simplicial complex Δ is a building (and not in an apartment system), then this can already be derived from the fact that any element of Δ is contained in a subcomplex of type $\Sigma(v_1, ..., v_{2r})$ by applying [Mue 94, Theorem 1.8.22], [SchR 85] and our Lemma 16, which implies the "homogeneity" assumption of [SchR 85].

7 Thick buildings of type $\tilde{B}_r$, $\tilde{C}_r$, $\tilde{D}_r$ and associated BN-pairs

In the previous sections we have shown that Δ is a (weak) building of type $\tilde{C}_r$. We now analyse, when this building is already thick. If it is not, one has to apply a building theoretical construction, a generalization of the well known "oriflamme" construction in [Tit 74, 7.12] for the C_r-case, in order to obtain thick buildings.

7.1 The thin panels

First we determine the thin panels of Δ, i.e. those panels which are contained in exactly two chambers. Since $U(V)$ acts transitively on the chambers in Δ, it is enough to consider the panels of the standard chamber $\mathcal{C}_0$ described on page 8.

The cotype of a simplex in Δ is defined as the complement of its type. The chambers that have a panel of cotype $i \in \{0, \dots, r\}$ in common with the standard chamber $\mathcal{C}_0$ are the maximal admissible lattice chains, in which all multiples of L_i and $L_i^\#$ are replaced by multiples of suitable lattices L and $L^\#$. For $1 \le i \le r-1$ these i-neighbours of $\mathcal{C}_0$ are in bijection with the 1-dimensional subspaces in the 2-dimensional space L_{i-1}/L_{i+1}. In particular the panels of cotype i with $1 \le i \le r-1$ are thick. So the crucial types are $i = 0$ and $i = r$. The chambers in Δ, that contain the panel of type 0 of $\mathcal{C}_0$, are in bijection with the isotropic subspaces of $L_1^\#/L_1$: Since $\mathrm{char}(k) \neq 2$ there is a prime element π of $\mathfrak{M}$ satisfying $\overline{\pi} = \epsilon'\pi$ for some $\epsilon' = \pm 1$. Then we define $f : L_1^\#/L_1 \times L_1^\#/L_1 \to k = \mathfrak{M}/\pi\mathfrak{M}$ via $f(x+L_1, y+L_1) := (x,y)\pi + \pi\mathfrak{M}$. One easily checks, that f is a $(\epsilon\epsilon', \iota)$-hermitian form for the involution $\iota : k \to k, x \mapsto \pi^{-1}\overline{x}\pi$. Therefore the panels of cotype 0 are thin, iff this hermitian space $L_1^\#/L_1$ is a hyperbolic plane with a symmetric bilinear form, i.e. $X_0 = X_0^\#$, $\epsilon\epsilon' = 1$ and $\iota = id$.

For cotype r one has to consider $L_{r-1}/\pi L_{r-1}^\#$ and finds that the panels of cotype r are thin, iff $X_0 = \pi X_0^\#$, $\epsilon = 1$ and $^-$ induces the identity on k.

If $^- \neq id$, then there exists $c \in \mathcal{D}^*$ with $\overline{c} = -c$, because $\mathrm{char}(\mathcal{D}) \neq 2$. Rescaling the form as described in Remark 6, we assume that $\epsilon = -1$ if $^- \neq id$. If the involution is the identity on $\mathcal{D}$, then $\mathcal{D}$ is commutative. Choosing $c = \pi^{-1}$ in Remark 6 if necessary, we may also assume that the maximal lattice X_0 in the anisotropic kernel satisfies $X_0 \neq \pi X_0^\#$, if $n > 2r$. Hence the panels of cotype r are thin, iff $^- = id$, $\epsilon = 1$ and $n = 2r$. Notice that in this case also the panels of cotype 0 are thin.

7.2 A general "oriflamme" construction

In this subsection let Δ be a weak building of rank $r+1$ endowed with a numbering of its vertices by the integers in $I := \{0, 1, \dots, r\}$ and a type function type : $\Delta \to \mathcal{P}(I)$. We assume that the following conditions are satisfied:

(1) Any vertex of type 1 is incident (i.e. joined by an edge) with precisely two vertices of type 0.

(2) The Coxeter number m_{01} (i.e. the diameter of the link $\mathrm{lk}_\Delta(a)$ for any $a \in \Delta$ of cotype $\{0,1\}$) is even and ≥ 4.

Then we associate to Δ a simplicial complex $\tilde{\Delta}$. The simplices (considered as sets of vertices) in $\tilde{\Delta}$ are the simplices $\tilde{a}$ where a is a simplex in Δ and

$$\tilde{a} := \begin{cases} a & 1 \notin \mathrm{type}(a) \\ (a - \{x_1\}) \cup \{x_0', x_0''\} & 1 \in \mathrm{type}(a) \end{cases}$$

where $x_1 \in a$ is of type 1 and x_0', x_0'' are the two neighbors of x_1 of type 0. Geometrically, $\tilde{\Delta}$ is obtained from Δ by deleting all panels of cotype 0 so that the two Δ-chambers sharing such a panel become one $\tilde{\Delta}$-chamber. In particular, one may identify the geometric realizations of Δ and $\tilde{\Delta}$.

Let $\mathcal{A}$ be a system of apartments for Δ.

Lemma 28 *(i)* $\tilde{\Delta}$ *is a weak building and* $\tilde{\mathcal{A}} := \{\tilde{\Sigma} \mid \Sigma \in \mathcal{A}\}$ *is a system of apartments for* $\tilde{\Delta}$.

(ii) $\tilde{\Delta}$ *possesses a numbering* $\widetilde{\text{type}} : \tilde{\Delta} \to \mathcal{P}(\tilde{I})$, *where* $\tilde{I} := \{0', 0'', 2, 3, \ldots, r\}$ *such that the* $\tilde{\Delta}$*-vertices of type* i *are the* Δ*-vertices of type* i *for* $i \in \{2, \ldots, r\}$ *and the* Δ*-vertices of type* 0 *are the* $\tilde{\Delta}$*-vertices of type* $0'$ *or* $0''$.

(iii) The links in Δ *and* $\tilde{\Delta}$ *are related as follows:*
$\mathrm{lk}_{\tilde{\Delta}}(\tilde{a}) = \mathrm{lk}_\Delta(a)$ *for* $a \in \Delta$, $0, 1 \in \mathrm{type}(a)$;
$\mathrm{lk}_{\tilde{\Delta}}(a) \cong \mathrm{lk}_\Delta(a)$ *for* $a \in \Delta$, $0 \in \mathrm{type}(a)$, $1 \notin \mathrm{type}(a)$;
$\mathrm{lk}_{\tilde{\Delta}}(a) \cong \widetilde{\mathrm{lk}_\Delta(a)}$ *for* $a \in \Delta$, $0, 1 \notin \mathrm{type}(a)$.

(iv) If $M = (m_{ij})$ *is the Coxeter matrix of* Δ, *then the Coxeter matrix* $\tilde{M} = (\tilde{m}_{ij})$ *of* $\tilde{\Delta}$ *is given by* $\tilde{m}_{ij} = m_{ij}$ for all $2 \le i, j \le r$; $\tilde{m}_{0'j} = \tilde{m}_{0''j} = m_{1j}$ for all $2 \le j \le r$; $\tilde{m}_{0'0''} = m_{01}/2$.

(v) If a group G *acts strongly transitively on* $(\Delta, \mathcal{A})$, *then its index 2 subgroup* $\tilde{G} := \{g \in G \mid \widetilde{\text{type}}(g \cdot a) = \widetilde{\text{type}}(a)$ for all *vertices* $a \in \tilde{\Delta}\}$ *acts strongly transitively on* $(\tilde{\Delta}, \tilde{\mathcal{A}})$.

Proof: First of all condition (1) implies

$$(1')\quad m_{0i} = 2 \text{ for all } 2 \le i \le r.$$

In fact, if $a \in \Delta$ has cotype $\{0, i\}$, then $\mathrm{lk}_\Delta(a)$ contains exactly two vertices of type 0 by (1) and therefore must have diameter 2.

Now we can prove the following:

($\star$) Let $x_1 \in \Delta$ be a vertex of type 1 and x_0', x_0'' its neighbors in Δ of type 0. Then for any vertex $y \in \Delta$ of type $i \neq 0$, $\{x_1, y\}$ is a simplex in Δ if and only if $\{x_0', y\}$ and $\{x_0'', y\}$ are.

If $\{x_1, y\}$ is a simplex, then it is contained in a panel p of cotype 0. Then $p \cup \{x_0'\}$ and $p \cup \{x_0''\}$ are the only chambers in Δ that contain p, which shows that $\{y, x_0'\}$ and $\{y, x_0''\}$ are simplices in Δ. To prove the reverse conclusion, we first show that $x_1 \in \mathrm{proj}_{x_0'}(x_0'')$ which is the intersection of all chambers $c \ni x_0'$ in the minimal galeries between x_0' and x_0'' (cf. [DrS 87]). Let $a \in \Delta$ be of cotype $\{0, 1\}$ such that $c' := \{x_0', x_1\} \cup a$ (and hence also $c'' := \{x_0'', x_1\} \cup a$) is a Δ-chamber. Choose an apartment $\Sigma \in \mathcal{A}$ which contains c' and c'' and a root $\alpha \in \Sigma$ such that $c' \in \alpha$ and $\{x_0'\} \cup a \in \partial\alpha$ (:= wall bounding α). Since $\mathrm{lk}_\Sigma(a)$ has diameter $m_{01} \ge 4$ by condition (2), x_0'' is not opposite x_0' in $\mathrm{lk}_\Sigma(a)$ and therefore $x_0'' \notin \partial\alpha$. This implies $\mathrm{proj}_{x_0'}(x_0'') \notin \partial\alpha$. On the other hand, (c', c'') is a minimal gallery stretched between x_0' and x_0'' and hence $\mathrm{proj}_{x_0'}(x_0'') \subseteq c'$. Since $\mathrm{proj}_{x_0'}(x_0'') \nsubseteq \{x_0'\} \cup a$, we have shown that $x_1 \in \mathrm{proj}_{x_0'}(x_0'')$. This proves that the vertex x_1 is the unique vertex of Δ of type 1 that is incident with x_0' and x_0''. Now let y be a vertex of type $i > 1$ such that x_0' and x_0'' are both contained in $st_\Delta(y)$. Since this star is a convex subcomplex of Δ, also $\mathrm{proj}_{x_0'}(x_0'')$ and therefore x_1 is contained in $st_\Delta(y)$ which shows ($\star$).

We are now in a position to prove the claims of the lemma.

(i) Let $\Sigma \in \mathcal{A}$ be an apartment of Δ. We first show that $\tilde{\Sigma}$ is a Coxeter complex. It is clear that the chambers of $\tilde{\Sigma}$ are of the form $\tilde{c}$ with $c \in \mathrm{cham}(\Sigma)$. It is also obvious that $\tilde{c}$ and $\tilde{d}$ are adjacent (i.e. share a panel), if c and d are. Hence $\tilde{\Sigma}$ is a chamber complex. In view of $(\star)$ it is thin. Any folding φ of Σ induces a folding of $\tilde{\Sigma}$ provided that the wall $\partial\alpha$ corresponding to φ does not contain any panels of cotype 0. However, the conditions (1') and (2) imply (in view of the fact that $\partial\alpha$ is itself a chamber complex, cf. [Abr 94, Proposition 1]) that all Δ-panels in $\partial\alpha$ have the same cotype if $\partial\alpha$ possesses at least one panel of cotype 0. Since all $\tilde{\Sigma}$-panels contain vertices of type 0, foldings along all $\tilde{\Sigma}$-panels of a $\tilde{\Sigma}$-chamber exist and $\tilde{\Sigma}$ is a Coxeter complex.

Let $c, d \in \mathrm{cham}(\Delta)$. Then there is an apartment $\Sigma \in \mathcal{A}$ containing both chambers. Hence also $\tilde{c}, \tilde{d} \in \tilde{\Sigma}$, because for any vertex x_1 of type 1 in Σ, both neighbors of type 0 must also be contained in Σ. Finally, let $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ be such that $\tilde{\Sigma}_1 \cap \tilde{\Sigma}_2$ contains a $\tilde{\Delta}$-chamber $\tilde{c}$. Then $\Sigma_1 \cap \Sigma_2$ contains the Δ-chamber c, and there exists an isomorphism $f : \Sigma_1 \to \Sigma_2$ fixing $\Sigma_1 \cap \Sigma_2$ pointwise. In particular f is type preserving and thus induces an isomorphism $\tilde{f} : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$, which fixes $\tilde{\Sigma}_1 \cap \tilde{\Sigma}_2$ pointwise.

(ii) Choose a $\tilde{\Delta}$-chamber $\tilde{c}$ and define $\widetilde{\mathrm{type}}$ on its vertices such that the vertices of Δ-type $i > 1$ obtain the same $\tilde{\Delta}$-type and the two vertices of $\tilde{c}$ of Δ-type 0 are numbered $0'$ and $0''$ respectively. There exists a unique retraction $\rho_{\tilde{c}}$ of $\tilde{\Delta}$ onto the complex consisting of $\tilde{c}$ and all its faces (cf. [Tit 74, 3.8]). Define $\widetilde{\mathrm{type}}(x) := \widetilde{\mathrm{type}}(\rho_{\tilde{c}}(x)) \in \mathcal{P}(\tilde{I})$ for all simplices $x \in \tilde{\Delta}$. In the way $\rho_{\tilde{c}}$ is constructed (cf. [Tit 74, 2.3, 2.4, 3.3 and 3.8]) it is clear that $\rho_{\tilde{c}}$ preserves the Δ-type of all vertices in $\tilde{\Delta}$ which proves (ii).

(iii) Firstly let $a \in \Delta$ be such that $0, 1 \in \mathrm{type}(a)$, x_1 its vertex of type 1, x_0 its vertex of type 0 and $y_0 \neq x_0$ the other vertex of type 0 in Δ which is incident with x_1. Then $\tilde{a} = a - \{x_1\} \cup \{y_0\}$, and both $b \in \mathrm{lk}_{\tilde{\Delta}}(\tilde{a})$ as well as $b \in \mathrm{lk}_{\Delta}(a)$ imply that $b \in \Delta$ and $\mathrm{type}(b) \subseteq \mathrm{cotype}(a) (\subseteq I - \{0, 1\})$. Since the buildings $\mathrm{lk}_{\tilde{\Delta}}(\tilde{a})$ and $\mathrm{lk}_{\Delta}(a)$ are flag complexes (cf. [Tit 74, 3.16]), it now suffices to show that any vertex $y \in \Delta$ with $\mathrm{type}(y) \subset \mathrm{cotype}(a)$ is incident with all vertices of $\tilde{a}$ if and only if y is incident with all vertices of a. However, this follows from $(\star)$.
Secondly we assume that $a \in \Delta$, $0 \in \mathrm{type}(a)$, $1 \notin \mathrm{type}(a)$. Let x_0 be the vertex of a of type 0. We define a map $f : \{\text{vertices of } \mathrm{lk}_{\Delta}(a)\} \to \{\text{vertices of } \mathrm{lk}_{\tilde{\Delta}}(a)\}$ as follows. f is the identity on the vertices in $\mathrm{lk}_{\Delta}(a)$ of some type $i > 0$. For all vertices x_1 of type 1 that satisfy $\{x_1\} \cup a \in \Delta$ we set $f(x_1) = y_0$, where $y_0 \neq x_0$ is the unique other neighbor of x_1 of type 0. Now again $(\star)$ implies that f induces a simplicial map $\mathrm{lk}_{\Delta}(a) \to \mathrm{lk}_{\tilde{\Delta}}(a)$ which is an isomorphism.
Finally let $a \in \Delta$ be such that $0, 1 \in \mathrm{cotype}(a)$. Then the weak building $\Theta := \mathrm{lk}_{\Delta}(a)$ also satisfies the conditions (1) and (2) above. Hence we already know, that there is a building $\tilde{\Theta}$ associated to Θ as described before. Looking at the definitions, one immediately sees that the buildings $\tilde{\Theta}$ and $\mathrm{lk}_{\tilde{\Delta}}(a)$ are canonically isomorphic.

(iv) This statement follows by specializing (iii) to simplices $\tilde{a}, a \in \tilde{\Delta}$ of codimension 2. Note that the diameter of $\tilde{\Theta}$ is m, if Θ is a generalized $(2m)$-gon.

(v) Let $\tilde{\Sigma}_1, \tilde{\Sigma}_2 \in \tilde{\mathcal{A}}$ and $\tilde{c}_i \in \mathrm{cham}(\tilde{\Sigma}_i)$ be given, where $\Sigma_i \in \mathcal{A}$, $c_i \in \mathrm{cham}(\Sigma_i)$, $i = 1, 2$. Denote by d_i the unique Δ-chamber $d_i \neq c_i$ such that $\tilde{d}_i = \tilde{c}_i$. Then d_i is adjacent to c_i and also contained in Σ_i. Since G acts strongly transitively on $(\Delta, \mathcal{A})$, there exists a $g \in G$ such that $g \cdot \Sigma_1 = \Sigma_2$ and $g \cdot c_1 = c_2$. This implies $g \cdot \tilde{\Sigma}_1 = \tilde{\Sigma}_2$ and $g \cdot \tilde{c}_1 = \tilde{c}_2$. Note that G preserves all Δ-types (this is part of the definition of the strongly transitive action), but g may interchange the $\tilde{\Delta}$-types $0'$ and $0''$. If this is the case, we choose an element $n \in Stab_G(\Sigma_2)$ which induces a reflection on Σ_2 such that $n \cdot c_2 = d_2$ and put $\tilde{g} := ng$, otherwise $\tilde{g} := g$. Then $\tilde{g}$ preserves all $\tilde{\Delta}$-types, $\tilde{g} \cdot \tilde{\Sigma}_1 = \tilde{\Sigma}_2$ and $\tilde{g} \cdot \tilde{c}_1 = \tilde{c}_2$. □

7.3 The thick buildings and their BN-pairs

In our situation the building Δ of admissible lattice chains clearly satisfies the conditions (1) and (2) of the previous subsection, if Δ is not thick.

A lattice class model for the thick building is obtained as follows: Assume that the panels of cotype 0 are thin. Then $L_1^{\#}/L_1$ is a hyperbolic plane. The two maximal isotropic subspaces of $L_1^{\#}/L_1$ correspond to the two lattices $L_{0'} := L_0 = (0, \ldots, 0)$ and $L_{0''} := (1, 0, \ldots, 0, -1)$ in the notation of Subsection 5.1. The set of vertices in the new standard apartment $\tilde{\Sigma}(b_1, \ldots, b_{2r})$ corresponds to

$$\tilde{N}_{2r} := \{(n_1, \ldots, n_{2r}) \in N_{2r} \mid \sum_{i=1}^{2r} n_i \neq 1\}.$$

The type of $(n_1, \ldots, n_{2r}) \in \tilde{N}_{2r}$ is $t := \sum_{i=1}^{2r} n_i$ if $t > 0$. If $t = 0$ then the vertex has type $0'$ if $\frac{1}{2} \sum_{i=1}^{2r} |n_i|$ is even and $0''$ if it is odd. Since for any standard lattice L satisfying $L = L^{\#}$ there is an apartment $\Sigma(v_1, \ldots, v_{2r})$ containing L as well as $L_{0'}$, one can verify that L is of type $0'$ (resp. type $0''$) if $L \cap L_{0'} = L_{0'} g$ for some $g \in GL(V)$ with $\nu(det(g))$ even (resp. odd). The simplicial complex $\tilde{\Sigma}_{2r}$ is obtained from Σ_{2r} by the general procedure described in Subsection 7.2: If $a \subset N_{2r}$ is a simplex in Σ_{2r} then the corresponding simplex in $\tilde{\Sigma}_{2r}$ is $\tilde{a} = a$, if $a \subset \tilde{N}_{2r}$. Otherwise let $x := (n_1, \ldots, n_{2r}) \in a$ be the vertex of type 1 and $i \in \{1, \ldots, r\}$ such that $n_i + n_{2r+1-i} = 1$ and $n_j = -n_{2r+1-j}$ for all $i \neq j \in \{1, \ldots, r\}$. Let x_1 (resp. x_2) be the vertex in $\tilde{N}_{2r}$ replacing n_i by $n_i - 1$ (resp. n_{2r+1-i} by $n_{2r+1-i} - 1$). Then $\tilde{a} := a - \{x\} \cup \{x_1, x_2\}$ is a simplex in $\tilde{\Sigma}_{2r}$. Since $|n_i - 1| + |n_{2r+1-i}| - |n_i| - |n_{2r+1-i} - 1| = \pm 2$ the vertices $x_1, x_2 \in \tilde{N}_{2r}$ have different types (in $\{0', 0''\}$) and the labelling defines a labelling of $\tilde{\Sigma}_{2r}$.

In the lattice chain model, the chambers in $\tilde{\Delta}$ correspond to the sets of lattices $\tilde{\mathcal{C}}$ in V that are closed under multiplication by π and taking duals, which contain lattices $L_{0'} = L_{0'}^{\#}, L_{0''} = L_{0''}^{\#} \in \tilde{\mathcal{C}}$ such that $L_1 := L_{0'} \cap L_{0''} \notin \tilde{\mathcal{C}}$ is a maximal $\mathfrak{M}$-sublattice of $L_{0'}$ (and $L_{0''}$) and $\mathcal{C} := \tilde{\mathcal{C}} - \{\pi^i L_{0'} \mid i \in \mathbb{Z}\} \cup \{\pi^i L_1, \pi^i L_1^{\#} \mid i \in \mathbb{Z}\}$ is a maximal admissible chain of lattices in V.

If also the panels of cotype r are thin, one deals with them analogously (working with the basis $(\pi b_1, \dots, \pi b_r, b_{r+1}, \dots, b_{2r})$ and multiplying the bilinear form by π^{-1}) where we assume that $r > 2$. Then the chambers in $\tilde{\tilde{\Delta}}$ correspond to the sets of lattices $\tilde{\tilde{\mathcal{C}}}$ in V that are closed under multiplication by π and taking duals, which contain lattices $L_{0'} = L_{0'}^{\#}, L_{0''} = L_{0''}^{\#} L_{r'} = \pi^{-1} L_{r'}^{\#}, L_{r''} = \pi^{-1} L_{r''}^{\#} \in \tilde{\tilde{\mathcal{C}}}$ such that $L_1 := L_{0'} \cap L_{0''}, L_{r-1} := L_{r'} + L_{r''} \notin \tilde{\tilde{\mathcal{C}}}$ satisfy $L_{0'}/L_1 \cong L_{r-1}/L_{r'} \cong \mathfrak{M}/\pi\mathfrak{M}$ and $\mathcal{C} := \tilde{\tilde{\mathcal{C}}} - \{\pi^i L_{0'}, \pi^i L_{r'} \mid i \in \mathbb{Z}\} \cup \{\pi^i L_1, \pi^i L_1^{\#}, \pi^i L_{r-1}, \pi^i L_{r-1}^{\#} \mid i \in \mathbb{Z}\}$ is a maximal admissible chain of lattices in V.

The corresponding subgroups of $U(V)$ that act type preservingly on the thick building can be constructed as follows: Let B be the stabilizer in $U(V)$ of the standard chamber $\mathcal{C}_0$ and let N be the stabilizer in $U(V)$ of the standard apartment Σ_0. In the notation of Subsection 5.2 let $d := n - 2r$ be the dimension of the anisotropic kernel W_0 and $(x_1, \dots, x_d)$ be a orthogonal $\mathfrak{M}$-basis of the maximal integral lattice X_0 in W_0 such that $(x_1, x_1), \dots, (x_t, x_t) \in \mathfrak{M}^*$ and $(x_{t+1}, x_{t+1}), \dots, (x_d, x_d) \in \pi\mathfrak{M}^*$, where $t := d - s$. With respect to the basis $(b_1, \dots, b_r, x_1, \dots, x_t, b_{r+1}, \dots, b_{2r}, x_{t+1}, \dots, x_d)$

$$B = \begin{pmatrix} \mathfrak{M} & \pi\mathfrak{M} & \dots & \pi\mathfrak{M} & \pi\mathfrak{M}^{1\times s} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \mathfrak{M}^{t\times t} & \ddots & \vdots \\ \mathfrak{M} & \dots & \dots & \mathfrak{M} & \pi\mathfrak{M}^{1\times s} \\ \mathfrak{M}^{s\times 1} & \dots & \dots & \mathfrak{M}^{s\times 1} & \mathfrak{M}^{s\times s} \end{pmatrix} \cap U(V).$$

The group N acts monomially on $(b_1, \dots, b_{2r})$, i.e.

$$N = \{g \in U(V) \mid b_i g = \pi^{e(g)_i} u_i b_{\sigma_g(i)} \text{ with } e(g)_i \in \mathbb{Z}, u_i \in \mathfrak{M}^* (i = 1, \dots, r) \sigma_g \in S_{2r}\}.$$

Since $g \in U(V)$, the associated exponent vector satisfies $\sum_{i=1}^{2r} e(g)_i = 0$. Let

$$\tilde{N} := \{g \in N \mid \sum_{i=1}^{2r} |e(g)_i| \equiv 0 \pmod 4\}$$

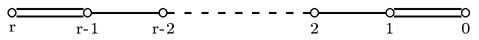
and

$$\tilde{\tilde{N}} := \{g \in \tilde{N} \mid |\{1, \dots, r\}\sigma_g \cap \{r+1, \dots, 2r\}| \equiv 0 \pmod 2\}.$$

Then $\tilde{\tilde{N}} \le \tilde{N} \le N$ are subgroups with $[N : \tilde{N}] = [\tilde{N} : \tilde{\tilde{N}}] = 2$. We define subgroups $U(V)_0$, N_0 of $U(V)$, N as follows: If Δ is thick, set $U(V)_0 := U(V)$ and $N_0 := N$. If only the panels of cotype 0 are thin, let $U(V)_0 := \tilde{U}(V) := \widetilde{U(V)}$ (cf. the definition in Lemma 28 (v)) and $N_0 := N \cap U(V)_0 = \tilde{N}$. If the panels of cotype 0 and cotype r are thin (and $r > 2$), set $U(V)_0 := \tilde{\tilde{U}}(V) := \widetilde{\widetilde{U(V)}}$ and $N_0 := N \cap U(V)_0 = \tilde{\tilde{N}}$. Observe that in each case $U(V)_0$ acts on a thick building now (cf. Lemma 28 (iii) and the discussion in Subsection 7.1). Therefore [Tit 74, Proposition 3.11], Corollary 26 and Lemma 28 immediately imply the following

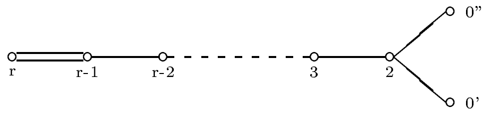
Proposition 29 *The pair (B, N_0) is a BN-pair of the group $U(V)_0$.* □

To deduce properties of the associated BN-pairs for the possible thick buildings, it is helpful to know their Coxeter diagrams. By Remark 14, the weak building Δ has always type $\tilde{C}_r$ (if $r > 1$):

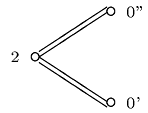


If the panels of cotype 0 are thin (i.e. $X_0 = X_0^{\#}$, $\epsilon\epsilon' = 1$ and $\iota = id$), then one obtains a building $\tilde{\Delta}$ of type $\tilde{B}_r$, if $r \geq 3$ and of type $\tilde{C}_r$, if $r = 2$:

$r > 2$

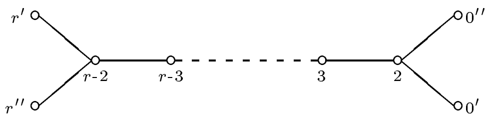


$r = 2$

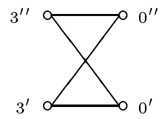


If also the panels of cotype r are thin (i.e. $^{-} = id$, $\epsilon = 1$ and $n = 2r$) then the associated thick building $\tilde{\tilde{\Delta}}$ is of type $\tilde{D}_r$, if $r \geq 4$ and of type $\tilde{A}_3$, if $r = 3$:

$r > 3$



r=3



If $r = 2$, then the oriflamme construction described in Section 7.2 cannot be applied any longer to the still not thick building $\tilde{\Delta}$ of type $\tilde{C}_2$. In this case, the appropriate "thickening procedure" would be as follows. Take the original building Δ of type $\tilde{C}_2$, and delete in any star of any vertex of type 1 this vertex and the four edges containing it. What one obtains is a thick polysimplicial (in our case: quadratic) complex of type $\tilde{A}_1 \times \tilde{A}_1$. This is precisely the Bruhat-Tits building associated to the non-simple split group SO_4, which is of type $A_1 \times A_1$, over the discretely valuated field K.

Finally, we recall that Δ is of type $\tilde{A}_1$, i. e. a tree without vertices of valency 1, if $r = 1$ (cf. Remark 14). Then either Δ is a thick semihomogeneous tree, or $\tilde{\Delta}$ is a thick homogeneous tree (obtained from Δ by removing the vertices of valency 2) or Δ is a thin tree. The latter happens precisely for two-dimensional hyperbolic quadratic spaces, to which no thick buildings can be associated in a natural way.

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