1-twinnings of Buildings

Peter Abramenko[1*] Hendrik Van Maldeghem[2†]

[1] *Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D – 33501 Bielefeld,* abramenk@mathematik.uni-bielefeld.de
[2] *Department of Pure Mathematics and Computer Algebra, University of Ghent, Galglaan 2, B – 9000 Ghent,* hvm@cage.rug.ac.be

Abstract

In this paper, we characterize twinnings of buildings by 1-twinnings and various additional conditions. Specialized to the spherical case, we obtain new characterizations of the opposition relation in such buildings. Also, special attention is given to the rank 2 case.

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1 Introduction and Statement of the Main Results

Twin buildings were introduced by Ronan and Tits (see [18]) as the natural geometries of groups of Kac-Moody type. Roughly speaking, a twin building consists of a pair of buildings (Δ_+, Δ_-) of the same type, which are linked by a certain function, i.e., a map δ^* associating to every pair of chambers $(c, d) \in (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+)$ an element of the Weyl group W corresponding with Δ_+ and Δ_-, and satisfying some additional conditions (for precise definitions, see below). When $\delta^*(c, d) = 1$, then we call c and d *opposite*. For classification purposes, this opposition relation is of crucial importance (it plays the same role as the opposition relation in spherical buildings; in fact it can be seen as a generalization of the latter, as every spherical building can be viewed in a canonical

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†The second author is a Research Director of the Fund for Scientific Research – Flanders (Belgium)

way as a twin building). However, a reasonable classification is only possible for those twin buildings which are 2-*spherical*, i.e. with only generalized polygons (and no trees) as rank 2 residues. The uniqueness part of the classification of 2-spherical twin buildings is settled by Tits [18], Mühlherr and Ronan [7], and Ronan [9].

However, in the classification programme outlined by Mühlherr in [6], the proof of existence of certain 2-spherical twin buildings is, unlike the spherical case, a major part of the job. To that end, one tries to find the most suitable definition of twin building, and in the context of the classification it turns out that axiomatization of the opposition relation rather than the full codistance function makes things easier. This was done by Mühlherr in [5], where a characterization of twinning was given by a local condition on opposition in rank 2 residues, the so-called 2-*twinning*. These 2-twinnings are stronger than 1-twinnings (see below for precise definitions), and a question of Mühlherr if 1-twinnings would already characterize twinnings was answered negatively by Abramenko (who constructed a counterexample for trees) and Van Maldeghem (who constructed counterexamples for generalized polygons). In Section 6 of the present paper, we present these counterexamples (in a modified and uniform version). But the main purpose of our work is to give a characterization of twin buildings by means of 1-twinnings. This will be done by adding just one global condition involving (twin) apartments. This new characterization is less technical and complicated than those known before, and it allows to introduce twin buildings very directly even without presupposing any knowledge of the defining properties of the opposition relation in rank 2 twin buildings (in fact, we also obtain a new characterization of twin trees, see Proposition 5.1). In particular, some (standard) examples of twin buildings can be treated in a new "group-free" way, as was done by the first author in the course "Introduction to and Applications of Twin Buildings" at the international conference *(Moufang) Polygons and (Twin) Buildings* held in Gent (Belgium) in June 1999. We will demonstrate this again in Section 4 below.

Specialized to the spherical case, our results imply new characterizations of the natural opposition relation in spherical buildings (see Corollary 3.6, where they are stated in the language of twinnings). Particular attention shall be paid to the rank 2 case, where the one-dimensionality of the apartments admits a couple of further variations and applications of our general results (see Section 5). Eventually, we also obtain a new characterization of generalized polygons amongst the rank 2 geometries (cf. Corollary 5.9), and we answer an open question concerning near n-gons (cf. Corollary 5.10).

In order to state our Main Result, we need some preliminaries. We assume the reader is familiar with the notion of *buildings*, the *Weyl group* related to a building, and *Coxeter systems* (see [15]).

Let (W, S) be a Coxeter system and denote by $\ell : W \to \mathbb{N}$ the usual length function with respect to S. Let Δ_+ and Δ_- be two buildings of type (W, S), i.e., with apartments isomorphic to the standard Coxeter complex $\Sigma(W, S)$ associated to (W, S). Denote by

$\mathcal{C}_\epsilon$ the set of chambers of Δ_ϵ, $\epsilon \in \{+,-\}$. Associated with Δ_ϵ there is a Weyl distance function $\delta_\epsilon : \mathcal{C}_\epsilon \times \mathcal{C}_\epsilon \to W$ (see for instance [17]). Two chambers c_ϵ, d_ϵ of Δ_ϵ are s-adjacent, $s \in S$, precisely if $\delta_\epsilon(c_\epsilon, d_\epsilon) = s$. A *panel of type* $\{s\}$ is a maximal set of chambers which are pairwise s-adjacent or equal.

With this set-up, we first recall the definition of *twin building*, as stated in [18].

Definition 1.1 Let there be given a function $\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$. Then $(\Delta_-, \Delta_+, \delta^*)$ is a *twin building* if the following three axioms are satisfied, for all $x_\epsilon \in \mathcal{C}_\epsilon$, and all $y_{-\epsilon}, z_{-\epsilon} \in \mathcal{C}_{-\epsilon}$, $\epsilon \in \{+,-\}$.

(Tw1) $\delta^*(x_\epsilon, y_{-\epsilon}) = \delta^*(y_{-\epsilon}, x_\epsilon)^{-1}$,

(Tw2) if $\delta^*(x_\epsilon, y_{-\epsilon}) = w \in W$, $\delta_{-\epsilon}(y_{-\epsilon}, z_{-\epsilon}) = s \in S$ and $\ell(ws) < \ell(w)$, then $\delta^*(x_\epsilon, z_{-\epsilon}) = ws$,

(Tw3) if $\delta^*(x_\epsilon, y_{-\epsilon}) = w \in W$ and $s \in S$, then there exists $z_{-\epsilon} \in \mathcal{C}_{-\epsilon}$ satisfying $\delta_{-\epsilon}(y_{-\epsilon}, z_{-\epsilon}) = s$ and $\delta^*(x_\epsilon, z_{-\epsilon}) = ws$.

The mapping δ^* is called the *codistance*.

Next, with the foregoing set-up, we repeat the definition of a 1-twinning, due to Mühlherr [5].

Definition 1.2 A non-empty symmetric relation $\mathcal{O} \subseteq (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+)$ is called a 1-*twinning* of (Δ_+, Δ_-) if the following axiom holds.

(1Tw) Given a pair of chambers $(c_+, c_-) \in \mathcal{O}$ and two panels P_+, P_- of the same type and belonging to Δ_+, Δ_-, respectively, such that $c_\epsilon \in P_\epsilon$, $\epsilon \in \{+,-\}$, then for any $\epsilon \in \{+,-\}$ and $x_\epsilon \in P_\epsilon$, there exists a unique $y_{-\epsilon} \in P_{-\epsilon}$ such that $(x_\epsilon, y_{-\epsilon}) \notin \mathcal{O}$.

We say that a 1-twinning $\mathcal{O}$ of (Δ_+, Δ_-) *induces* a twin building $(\Delta_+, \Delta_-, \delta^*)$ if $\mathcal{O}$ is precisely the set of pairs of chambers at codistance 1 from each other.

We now may state our Main Result as follows.

Main Result. *A* 1*-twinning* $\mathcal{O}$ *of* (Δ_+, Δ_-) *induces a (necessarily unique) twin building* $(\Delta_+, \Delta_-, \delta^*)$ *if and only if the following condition is satisfied for some* $\epsilon \in \{+,-\}$.

(TA) *There exists a chamber* $c_{-\epsilon} \in \mathcal{C}_{-\epsilon}$ *such that for any chamber* x_ϵ*, with* $(c_{-\epsilon}, x_\epsilon) \in \mathcal{O}$*, there is an apartment* Σ_ϵ *of* Δ_ϵ *satisfying* $\{x_\epsilon\} = \{y_\epsilon \in \mathcal{C}_\epsilon \,|\, y_\epsilon \in \Sigma_\epsilon \text{ and } (c_{-\epsilon}, y_\epsilon) \in \mathcal{O}\}$.

For consequences and applications we refer to Sections 3 and 5 below. Note that in Section 2, we collect and prove many statements that can be derived from a 1-twinning. We also make some efforts to include the case of non-thick buildings, which were excluded in the main result of [5].

2 1-twinnings and the Weyl Codistance

In this section, we are given two buildings Δ_+ and Δ_- of type (W, S) and a 1-twinning $\mathcal{O}$ of (Δ_+, Δ_-). We introduce some further notation and conventions.

If not explicitly stated otherwise, any definition and assertion containing the symbol ϵ must be read as: for ϵ equal to $+$ and $-$ respectively. If we mention chambers $c_\epsilon, d_\epsilon, \ldots$, we always automatically mean that they belong to $\mathcal{C}_\epsilon$. Also, we shall write δ for the Weyl distance functions δ_+ and δ_- whenever it is clear which of the two buildings Δ_+ or Δ_- we are considering. For any $T \subset S$ and any chamber c_ϵ, we set $W_T := \langle T \rangle \leq W$ and define the *T-residue of* c_ϵ by

$$R_T(c_\epsilon) := \{d_\epsilon \in \mathcal{C}_\epsilon \mid \delta(c_\epsilon, d_\epsilon) \in W_T\}.$$

The cardinality $|T|$ is called the *rank* of $R_T(c_\epsilon)$, while T itself is called the *type* of $R_T(c_\epsilon)$. A panel (see above) is then just a rank 1 residue.

For any pair $(c_\epsilon, c_{-\epsilon}) \in \mathcal{O}$, we say that the chambers c_ϵ and $c_{-\epsilon}$ are *opposite* and we use the notation $c_\epsilon \mathcal{O} c_{-\epsilon}$. We write $c_\epsilon \not\!\mathcal{O} d_{-\epsilon}$ if the chambers c_ϵ and $d_{-\epsilon}$ are not opposite. Two residues are *opposite* if they have the same type and contain opposite chambers. Condition (1Tw) of Definition 1.2 can be stated as follows: given two opposite panels P_+, P_-, any chamber $x_\epsilon \in P_\epsilon$ is opposite all chambers of $P_{-\epsilon}$ but (exactly) one.

For any chambers $c_\epsilon, d_{-\epsilon}$, we define

$$\begin{aligned}
c_\epsilon^{\mathcal{O}} &:= \{x_{-\epsilon} \in \mathcal{C}_{-\epsilon} \mid x_{-\epsilon} \mathcal{O} c_\epsilon\}, \\
\ell^*(c_\epsilon, d_{-\epsilon}) &:= \min\{\ell(\delta(x_{-\epsilon}, d_{-\epsilon})) \mid x_{-\epsilon} \in c_\epsilon^{\mathcal{O}}\}, \\
D^*(c_\epsilon, d_{-\epsilon}) &:= \{\delta(x_{-\epsilon}, d_{-\epsilon}) \mid x_{-\epsilon} \in c_\epsilon^{\mathcal{O}} \text{ and } \ell(\delta(x_{-\epsilon}, d_{-\epsilon})) = \ell^*(c_\epsilon, d_{-\epsilon})\}.
\end{aligned}$$

We remark that $c_\epsilon^{\mathcal{O}}$ is necessarily non-empty (cf. [5], Lemma 5.1).

If $|D^*(c_\epsilon, d_{-\epsilon})| = 1$, then we define the *Weyl codistance* $\delta^*(c_\epsilon, d_{-\epsilon})$ as the unique element of $D^*(c_\epsilon, d_{-\epsilon})$. We say in this case that δ^* is well-defined at $(c_\epsilon, d_{-\epsilon})$.

Denoting the *sphere with center a chamber* c_ϵ *and radius* $w \in W$ by

$$\mathcal{S}_w(c_\epsilon) := \{x_\epsilon \in \mathcal{C}_\epsilon \mid \delta(c_\epsilon, x_\epsilon) = w\},$$

we can restate the following lemma due to Mühlherr ([5], Lemmas 5.2 and 5.3).

Lemma 2.1 *Given chambers $x_\epsilon, y_{-\epsilon}$ and an element $w \in D^*(x_\epsilon, y_{-\epsilon})$, we have*

(i) $\ell^*(x_\epsilon, y_{-\epsilon}) = \ell^*(y_{-\epsilon}, x_\epsilon) = \ell(w)$,

(ii) $\mathcal{S}_w(x_\epsilon) \subseteq y_{-\epsilon}^{\mathcal{O}}$ *and* $\mathcal{S}_{w^{-1}}(y_{-\epsilon}) \subseteq x_\epsilon^{\mathcal{O}}$.

This lemma has some interesting consequences.

Corollary 2.2 *Given chambers $x_\epsilon, y_{-\epsilon}, z_{-\epsilon}$ and an element $w \in W$, we have*

(i) $w \in D^*(x_\epsilon, y_{-\epsilon})$ *if and only if* $w^{-1} \in D^*(y_{-\epsilon}, x_\epsilon)$,

(ii) *if* $w \in D^*(x_\epsilon, y_{-\epsilon})$, $\delta(y_{-\epsilon}, z_{-\epsilon}) = s \in S$ *and* $\ell(ws) < \ell(w)$, *then* $ws \in D^*(x_\epsilon, z_{-\epsilon})$.

Proof.

(i) It suffices of course to verify one implication. So assume $w \in D^*(x_\epsilon, y_{-\epsilon})$, and choose $y_\epsilon \in \mathcal{S}_w(x_\epsilon)$. Lemma 2.1 yields $y_\epsilon \mathcal{O} y_{-\epsilon}$ as well as $\ell^*(y_{-\epsilon}, x_\epsilon) = \ell(w) = \ell(w^{-1})$, and hence $w^{-1} \in D^*(y_{-\epsilon}, x_\epsilon)$.

(ii) Set $w' := ws$, then $\ell(w') = \ell(w) - 1$ by assumption. Applying Lemma 2.1(*ii*) again, we obtain

$$\mathcal{S}_{w'^{-1}}(z_{-\epsilon}) \subseteq \mathcal{S}_{w^{-1}}(y_{-\epsilon}) \subseteq x_\epsilon^{\mathcal{O}}.$$

Therefore, we have

$$\ell(w) - 1 = \ell^*(x_\epsilon, y_{-\epsilon}) - 1 \leq \ell^*(x_\epsilon, z_{-\epsilon}) \leq \ell(w') = \ell(w) - 1.$$

Hence $w' = ws \in D^*(x_\epsilon, z_{-\epsilon})$. □

Provided that the function $\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ is *globally well-defined*, i.e., well-defined at every element of the source of δ^*, Corollary 2.2 shows that δ^* already satisfies Axioms (Tw1) and (Tw2) of the introduction. It is slightly more complicated to deal with (Tw3). We are going to study this axiom locally, i.e., for a fixed chamber $c_\epsilon = x_\epsilon$ and $y_{-\epsilon}$ running through the elements of a panel $P_{-\epsilon} \subseteq \mathcal{C}_{-\epsilon}$.

Lemma 2.3 *Let c_ϵ be a chamber and let $P_{-\epsilon}$ be a panel of type $\{s\}$ such that $|D^*(c_\epsilon, y_{-\epsilon})| = 1$, for all $y_{-\epsilon} \in P_{-\epsilon}$. Then*

$$\delta^*(c_\epsilon, P_{-\epsilon}) := \{\delta^*(c_\epsilon, y_{-\epsilon}) \,|\, y_{-\epsilon} \in P_{-\epsilon}\} = \{w, w'\},$$

for two distinct elements $w, w' \in W$, and either

(i) $w' = ws$,

or else

(ii) $|P_{-\epsilon}| = 2$ *and* $\ell(w) = \ell(w') < \ell(ws) = \ell(w's)$.

Proof. Without loss of generality, we may put $\epsilon = +$. We choose an element $w \in \delta^*(c_+, P_-)$ of minimal length and an $x_- \in P_-$ with $\delta^*(c_+, x_-) = w$. We first state and show three claims

(1) $\ell(ws) > \ell(w)$.
If on the contrary we had $\ell(ws) < \ell(w)$, then Corollary 2.2(ii) would imply $\ell(\delta^*(c_+, y_-)) = \ell(ws) < \ell(w)$, for all $y_- \in P_- \setminus \{x_-\}$, contradicting our choice of w.

(2) *If $z_- \in P_-$ and $\ell^*(c_+, z_-) > \ell(w)$, then $\delta^*(c_+, z_-) = ws$.*
Choose $c_- \in c_+^{\mathcal{O}}$ with $\delta(c_-, x_-) = w$. Since $\ell(ws) > \ell(w)$ by (1), we have $\delta(c_-, z_-) = ws$. Therefore $\ell^*(c_+, z_-) > \ell(w)$ implies $ws \in D^*(c_+, z_-)$, hence $\delta^*(c_+, z_-) = ws$.

(3) *There is a unique $z_- \in P_-$ such that $\delta^*(c_+, y_-) = w$, for all $y_- \in P_- \setminus \{z_-\}$.*
Choose $x_+ \in \mathcal{S}_w(c_+)$. Then $x_+ \mathcal{O} x_-$ by Lemma 2.1(ii). Let $z_- \in P_-$ be the unique chamber satisfying $z_- \not{\mathcal{O}} x_+$. For any $y_- \in P_- \setminus \{z_-\}$, the fact that $y_- \mathcal{O} x_+$ together with $\ell^*(y_-, c_+) = \ell^*(c_+, y_-) \geq \ell(w)$ implies that w^{-1} belongs to $D^*(y_-, c_+)$. Hence $\delta^*(c_+, y_-) = w$ by Corollary 2.2(i). Now if we had $\delta^*(c_+, z_-) = w$ as well, then Lemma 2.1 would imply that also z_- is opposite x_+, contradicting the choice of z_-.

Set $w' := \delta^*(c_+, z_-)$ with z_- as in (3); in particular $w' \neq w$. If $\ell(w') > \ell(w)$, then $w' = ws$ by (2) and we are in Case (i). So suppose $\ell(w') = \ell(w)$. Then x_- and w can be replaced by z_- and w', respectively, in (1) — yielding $\ell(w's) > \ell(w)$ — and in (3). This shows that $P_- = \{x_-, z_-\}$, for if there were a chamber $y_- \in P_- \setminus \{x_-, z_-\}$, we would obtain $w = \delta^*(c_+, y_-) = w'$, clearly a contradiction. So $\ell(w') = \ell(w)$ implies that we are in Case (ii). □

The counter-example that we discuss below shows that Case (ii) of Lemma 2.3 in fact does occur. However, this is of course impossible for thick buildings, as well as for buildings of rank 2 (since $\ell(w) = \ell(w') < \ell(ws) = \ell(w's)$ already implies $w = w'$ for $|S| = 2$). Therefore, Lemma 2.3 in combination with Corollary 2.2 immediately implies:

Corollary 2.4 *Suppose that δ^* is globally well-defined (in other words, $|D^*(x_\epsilon, y_{-\epsilon})| = 1$ for any chambers x_ϵ and $y_{-\epsilon}$). Assume further that the buildings Δ_+ and Δ_- are thick or of rank 2. Then $(\Delta_+, \Delta_-, \delta^*)$ is a twin building.* □

Counter-example. We want to demonstrate that the single fact of δ^* being well-defined does not necessarily imply that $(\Delta_+, \Delta_-, \delta^*)$ is a twin building. So take an arbitrary Coxeter system (W, S) of rank 3, set $S = \{r, s, t\}$ and consider the standard Coxeter complex $\Sigma = \Sigma(W, S)$ with the elements of W being the chambers of Σ. Take two copies Σ_+, Σ_- of Σ, and write w_+, w_- if $w \in W$ is considered as a chamber of Σ_+, Σ_-, respectively. Now we define a symmetric relation $\mathcal{O}$ as follows:

$$w_+ \mathcal{O} v_- :\Longleftrightarrow wv^{-1} \in \{1, rst\},$$

$$w_-\mathcal{O}v_+ :\Longleftrightarrow wv^{-1} \in \{1, tsr\},$$

for $w, v \in W$. Noting that rst is not a conjugate in W of an element of S, it is easily checked that $\mathcal{O}$ indeed defines a 1-twinning of (Σ_+, Σ_-). Also, δ^* is globally well-defined in this situation because, for instance, $w_+^{\mathcal{O}} = \{w_-, (tsrw)_-\}$, and the gallery distances $\ell(w^{-1}v), \ell(w^{-1}rstv)$ are not congruent modulo 2, so that for any chamber $v_- \in \Sigma_-$, there is a nearest chamber in $w_+^{\mathcal{O}}$. However, $(\Sigma_+, \Sigma_-, \delta^*)$ violates (Tw3) and certainly does not define a twin building.

Next we will show that the question whether δ^* is well-defined or not, as well as the question whether (Tw3) is satisfied can be reduced to an analysis of $\delta^*(c_\epsilon, \cdot)$ for a fixed chamber c_ϵ.

Proposition 2.5 *Suppose there exists a chamber c_ϵ satisfying the following two conditions:*

(a) $|D^*(c_\epsilon, x_{-\epsilon})| = 1$ *for all* $x_{-\epsilon} \in \mathcal{C}_{-\epsilon}$,

(b) *for any panel* $P_{-\epsilon} \subseteq \mathcal{C}_{-\epsilon}$ *of type* $\{t\}$ *(with* $t \in S$*), we have* $\delta^*(c_\epsilon, P_{-\epsilon}) = \{w, wt\}$ *for some* $w \in W$.

Then δ^* is globally well-defined and $(\Delta_+, \Delta_-, \delta^*)$ is a twin building.

Proof. Set ϵ equal to $+$. Let d_+ be an s-neighbour of c_+ (with $s \in S$), i.e. $\delta(c_+, d_+) = s$. For an arbitrary chosen chamber x_- we set $v := \delta^*(c_+, x_-)$, and we choose $w \in D^*(d_+, x_-)$. We will show that w is already uniquely determined by v, more precisely,

(1) If $\ell^*(d_+, x_-) \neq \ell^*(c_+, x_-)$ $(= \ell(v))$, then $w = sv$; if $\ell^*(d_+, x_-) = \ell^*(c_+, x_-)$, then $w = v$.

First we note that the identities $\ell(\delta(c_+, d_+)) = 1$, $\ell^*(c_+, x_-) = \ell^*(x_-, c_+)$, $\ell^*(d_+, x_-) = \ell^*(x_-, d_+)$ together with the definition of ℓ^* immediately imply $|\ell^*(c_+, x_-) - \ell^*(d_+, x_-)| \leq 1$.

We first consider the case $\ell^*(c_+, x_-) = \ell^*(d_+, x_-) + 1$. Since $w^{-1} \in D^*(x_-, d_+)$ by assumption and by Corollary 2.2(i), we obtain $w^{-1}s \in D^*(x_-, c_+)$ (and $\ell(w^{-1}s) > \ell(w^{-1})$). Hence $w^{-1}s = \delta^*(x_-, c_+) = v^{-1}$, implying $w = sv$.

Now the case $\ell^*(c_+, x_-) \leq \ell(d_+, x_-)$ will be settled by using induction on $\ell^*(d_+, x_-)$ (the claim being trivial for $\ell(v) \leq \ell(w) = 0$). Choose a decomposition $w = w't$ with $t \in S$ and $\ell(w') = \ell(w) - 1$. Using assumption (b), we find a chamber x'_- satisfying $\delta(x_-, x'_-) = t$ and $\delta^*(c_+, x'_-) = vt := v'$. By Corollary 2.2($ii$) we also have $w' = wt \in D^*(d_+, x'_-)$, in particular $\ell^*(d_+, x'_-) = \ell(w') < \ell(w) = \ell^*(d_+, x_-)$. We have to distinguish two cases now.

1. $\ell(v) = \ell(w)$.
   Then $\ell(v') = \ell(v) \pm 1 = \ell(w) \pm 1$. However, the inequality $\ell(v') = \ell^*(c_+, x'_-) \leq \ell^*(d_+, x'_-) + 1 = \ell(w') + 1 = \ell(w)$ excludes the possibility $\ell(v') = \ell(w) + 1$. Therefore $\ell(v') = \ell(w) - 1 = \ell(w')$, and the induction hypothesis yields $w' = v'$, hence $w = v$.

2. $\ell(v) = \ell(w) - 1$.
   Then $\ell(v') = \ell(w) - 1 \pm 1 = \ell(w') \pm 1$ and both possibilities might occur. For $\ell(v') = \ell(w') + 1$, the case already treated implies $w' = sv'$. For $\ell(v') = \ell(w') - 1$, this follows from the induction hypothesis. Hence we have $w = sv$ in both cases.

From (1) immediately follows:

(2) For any chamber x_- we have $|D^*(d_+, x_-)| = 1$ and $\delta^*(d_+, x_-) \in \{v, sv\}$, where $v := \delta^*(c_+, x_-)$.

Next we show:

(3) For any panel $P_- \subseteq \mathcal{C}_-$ of type $\{t\}$ (with $t \in S$), we have $\delta^*(d_+, P_-) = \{w, wt\}$ for some $w \in W$.

Choose $w \in \delta^*(d_+, P_-)$ of minimal length and a chamber $x_- \in P_-$ satisfying $\delta^*(d_+, x_-) = w$. Note that $\ell(wt) > \ell(w)$ (cf. (1) in the proof of Lemma 2.3). Again set $v := \delta^*(c_+, x_-)$ and choose, using Assumption (b), a chamber $z_- \in P_-$ satisfying $\delta^*(c_+, z_-) = vt$. Finally, we set $u := \delta^*(d_+, z_-)$. By (2), we have $w \in \{v, sv\}$ and $u \in \{vt, svt\}$. We will show that $u \neq wt$ implies $u = w$, thus ruling out Case (ii) of Lemma 2.3. Note that $u \neq wt$ already yields $\ell(u) = \ell(w)$ by Lemma 2.3. We again distinguish between two cases.

1. $w = v$ and $u \neq wt$.
   In view of Corollary 2.2, $\delta^*(d_+, x_-) = v$ is only possible if $\ell(sv) > \ell(v)$, hence $\ell(sw) > \ell(w)$. We already deduced $\ell(wt) > \ell(w)$ and $\ell(u) = \ell(w)$ above. Note that $u \in \{vt, svt\} \setminus \{vt\}$, so $u = svt$. Now an easy argument on Coxeter systems gives the implication
$$\left.\begin{array}{r} \ell(sw) > \ell(w), \\ \ell(wt) > \ell(w), \\ \ell(swt) = \ell(w) \end{array}\right\} \Longrightarrow w = swt = u.$$

2. $w = sv$ and $u \neq wt$.
   We already established $\ell(wt) > \ell(w)$ and $\ell(w) = \ell(u)$. Since $u \in \{vt, svt\}$ and $u \neq wt = svt$, we have $u = vt = swt$, hence also $\ell(w) = \ell(swt)$. Now $\delta^*(d_+, z_-) = vt$ and $\delta^*(d_+, x_-) = sv \neq v$ imply, using Corollary 2.2, that $\ell(v) > \ell(vt)$, consequently $\ell(sw) > \ell(swt) = \ell(w)$. So we are again in the situation $\ell(sw) > \ell(w)$, $\ell(wt) > \ell(w)$ and $\ell(swt) = \ell(w)$, which yields $w = swt = u$.

So we have shown that $wt \in \delta^*(d_+, P_-)$ or that $\delta^*(d_+, x_-) = \delta^*(d_+, z_-)$ for the two distinct elements x_-, z_- of P_-. Therefore $\delta^*(d_+, P_-) = \{w, wt\}$ by Lemma 2.3.

Now we can show that

(4) δ^* is globally well-defined and satisfies (Tw3).

An obvious induction using (2) and (3) yields $|D^*(y_+, x_-)| = 1$ for all chambers y_+, x_-. Also, (3) gives (Tw3) for the sign ϵ being $+$ (i.e., when searching for appropriate neighbours in Δ_-). Finally, (2) and Corollary 2.2(i) imply $\delta^*(x_-, P_+) \subseteq \{w, ws\}$ for some $w \in W$, where $x_- \in \mathcal{C}_-$ and $P_+ \subseteq \mathcal{C}_+$ is an arbitrary panel of type $\{s\}$. Combined with Lemma 2.3, this yields $\delta^*(x_-, P_+) = \{w, ws\}$, thus proving (Tw3) completely.

Now (4) and Corollary 2.2 imply that $(\Delta_+, \Delta_-, \delta^*)$ is a twin building. $\square$

Corollary 2.6 *Suppose Δ_+ and Δ_- are thick or of rank 2. If there exists a chamber c_ϵ satisfying $|D^*(c_\epsilon, x_{-\epsilon})| = 1$ for all $x_{-\epsilon} \in \mathcal{C}_{-\epsilon}$, then δ^* is globally well-defined and $(\Delta_+, \Delta_-, \delta^*)$ is a twin building.* $\square$

Corollary 2.6 for thick buildings was independently proved by Valery Vermeulen (private communication; his — much longer — proof will be included in his doctoral thesis).

We end this section by noting that Proposition 5.1 of Section 5 is an application of Corollary 2.6. Other applications will follow in Section 3.

3 1-twinnings and (twin) Apartments

One of the fundamental notions in building theory is the one of an apartment. If 1-twinnings are supposed to lead to something reasonable, meaning twin buildings, then the opposition relation $\mathcal{O}$ must be linked in a natural way to the apartments of the buildings Δ_+, Δ_-. It turns out that the global condition which distinguishes the twin buildings amongst the 1-twinnings is the existence of "sufficiently many twin apartments" (in a sense which will be made precise soon) coming along with $\mathcal{O}$.

In this section we keep all notions and notations introduced in the previous section. In particular, $\mathcal{O}$ will again be a 1-twinning of a pair of buildings (Δ_+, Δ_-) of type (W, S). When we speak of apartments in Δ_ϵ, we always mean members of the maximal system of apartments of Δ_ϵ.

Definition 3.1 (i) A chamber c_ϵ is said to be *twinned* with an apartment $\Sigma_{-\epsilon}$ of $\Delta_{-\epsilon}$ if there is exactly one chamber in $\Sigma_{-\epsilon}$ opposite c_ϵ.

(*ii*) A pair (Σ_+, Σ_-) of apartments is called a *twin apartment* of $(\Delta_+, \Delta_-, \mathcal{O})$ if each chamber of Σ_ϵ is twinned with $\Sigma_{-\epsilon}$ for $\epsilon \in \{+, -\}$.

Remark 3.2 It follows immediately from Lemma 2.1(*ii*) that $|\Sigma_{-\epsilon} \cap c_\epsilon^{\mathcal{O}}| \geq 1$ for any chamber c_ϵ and any apartment $\Sigma_{-\epsilon}$ of $\Delta_{-\epsilon}$.

Remark 3.3 The apparently weaker notion of a chamber twinned with an apartment is in fact equivalent to that of a twin apartment because one can show the following. An apartment Σ_ϵ of Δ_ϵ is part of a twin apartment if and only if $\Sigma_\epsilon^* := \{c_{-\epsilon} \in \mathcal{C}_{-\epsilon} \mid |c_{-\epsilon}^{\mathcal{O}} \cap \Sigma_\epsilon| = 1\}$ is non-empty. In that case, Σ_ϵ^* is the set of chambers of an apartment $\Sigma_{-\epsilon}$ of $\Delta_{-\epsilon}$, and $(\Sigma_\epsilon, \Sigma_{-\epsilon})$ is a twin apartment. However, we shall not use this statement in the sequel.

The following easy lemma will prove to be very useful in the rest of this section. It provides a link between the Weyl codistance and twin apartments.

Lemma 3.4 *Suppose the chamber c_ϵ is twinned with the apartment $\Sigma_{-\epsilon} \subseteq \Delta_{-\epsilon}$. Denote by $c_{-\epsilon}$ the unique chamber of $\Sigma_{-\epsilon}$ opposite c_ϵ. Then $|D^*(c_\epsilon, x_{-\epsilon})| = 1$ and $\delta^*(c_\epsilon, x_{-\epsilon}) = \delta(c_{-\epsilon}, x_{-\epsilon})$ for any chamber $x_{-\epsilon}$ of $\Sigma_{-\epsilon}$.*

Proof. Let $x_{-\epsilon}$ be a chamber of $\Sigma_{-\epsilon}$ and $w \in D^*(c_\epsilon, x_{-\epsilon})$. Since $\Sigma_{-\epsilon}$ is an apartment, $\mathcal{S}_{w^{-1}}(x_{-\epsilon}) \cap \Sigma_{-\epsilon} \neq \emptyset$. However, $\mathcal{S}_{w^{-1}}(x_{-\epsilon}) \subseteq c_\epsilon^{\mathcal{O}}$ by Lemma 2.1(*ii*). Therefore necessarily $\mathcal{S}_{w^{-1}}(x_{-\epsilon}) \cap \Sigma_{-\epsilon} = \{c_{-\epsilon}\}$ and hence $w^{-1} = \delta(x_{-\epsilon}, c_{-\epsilon})$. This means that $w = \delta(c_{-\epsilon}, x_{-\epsilon})$ is the only element of $D^*(c_\epsilon, x_{-\epsilon})$. □

Using this lemma and the results of Section 2, twin buildings can now be characterized as follows.

Theorem 3.5 *A 1-twinning $\mathcal{O}$ of (Δ_+, Δ_-) induces a twin building $(\Delta_+, \Delta_-, \delta^*)$ if and only if the following condition is satisfied.*

(TA) *There exists a chamber $c_{-\epsilon}$ such that for any $x_\epsilon \in c_{-\epsilon}^{\mathcal{O}}$, there exists an apartment Σ_ϵ of Δ_ϵ satisfying $c_{-\epsilon}^{\mathcal{O}} \cap \Sigma_\epsilon = \{x_\epsilon\}$.*

Proof. It is well known that any twin building satisfies (TA). In fact, if $(\Delta_+, \Delta_-, \delta^*)$ is a twin building, then for any two given chambers x_+, y_-, there exists a twin apartment (Σ_+, Σ_-) with $x_+ \in \Sigma_+$ and $y_- \in \Sigma_-$ (see for instance [1], Lemma 2).

Now let $\mathcal{O}$ be a 1-twinning of (Δ_+, Δ_-). Then we claim the following assertion.

($*$) Let $c_{-\epsilon}$ be any given chamber and let $\mathcal{M}$ be any set of apartments of Δ_ϵ such that $c_{-\epsilon}$ is twinned with each $\Sigma_\epsilon \in \mathcal{M}$. Denote by U_ϵ the union of all members of $\mathcal{M}$. If $c_{-\epsilon}^{\mathcal{O}} \subseteq U_\epsilon$, then $U_\epsilon = \Delta_\epsilon$.

Indeed, we shall show that $y_\epsilon \in U_\epsilon$ for any given chamber $y_\epsilon \in \Delta_\epsilon$ by induction on $\ell^*(c_{-\epsilon}, y_\epsilon)$, the start of the induction being the assumption $c_{-\epsilon}^{\mathcal{O}} \subseteq U_\epsilon$. Choose $w \in D^*(c_{-\epsilon}, y_\epsilon)$, where $\ell(w) = \ell^*(c_{-\epsilon}, y_\epsilon) > 0$. Write $w = w's$ with $s \in S$ and $\ell(w') = \ell(w) - 1$, and let y'_ϵ be such that $\delta(y_\epsilon, y'_\epsilon) = s$. Then $w' \in D^*(c_{-\epsilon}, y'_\epsilon)$ by Corollary 2.2(ii). Applying the induction hypothesis, we find an apartment $\Sigma_\epsilon \in \mathcal{M}$ containing y'_ϵ. Let c_ϵ be the unique chamber of Σ_ϵ opposite $c_{-\epsilon}$, and let z_ϵ be the chamber of Σ_ϵ satisfying $\delta(y'_\epsilon, z_\epsilon) = s$. Now Lemma 3.4 yields $w' = \delta^*(c_{-\epsilon}, y'_\epsilon) = \delta(c_\epsilon, y'_\epsilon)$ and $\delta^*(c_{-\epsilon}, z_\epsilon) = \delta(c_\epsilon, z_\epsilon) = w's = w$. Applying Corollary 2.2(ii) again, we see that $\ell^*(c_{-\epsilon}, x_\epsilon) = \ell(w')$, for all x_ϵ in $R_{\{s\}}(z_\epsilon) \backslash \{z_\epsilon\}$. However, $y_\epsilon \in R_{\{s\}}(z_\epsilon)$ and $\ell^*(c_{-\epsilon}, y_\epsilon) = \ell(w) > \ell(w')$. Hence $y_\epsilon = z_\epsilon \in \Sigma_\epsilon \subseteq U_\epsilon$. This shows our claim.

Now suppose that (TA) is satisfied. Then ($*$) shows that for any chamber $x_\epsilon \in \mathcal{C}_\epsilon$, there exists an apartment Σ_ϵ of Δ_ϵ containing x_ϵ such that $c_{-\epsilon}$ is twinned with Σ_ϵ. So Lemma 3.4 first of all implies $|D^*(c_{-\epsilon}, x_\epsilon)| = 1$, for all $x_\epsilon \in \mathcal{C}_\epsilon$. However, Lemma 3.4 secondly implies that $\delta^*(c_{-\epsilon}, z_\epsilon) = wt$, where $w = \delta^*(c_{-\epsilon}, x_\epsilon)$ and z_ϵ is the unique t-neighbour of x_ϵ in Σ_ϵ. This shows that the assumptions of Proposition 2.5 are satisfied here (concerning Assumption (b), see also Lemma 2.3). Hence this proposition implies that $(\Delta_+, \Delta_-, \delta^*)$ is a twin building. $\square$

In the spherical case, Condition (TA) can still be weakened considerably.

Corollary 3.6 *Let Δ_+ and Δ_- be spherical buildings of type (W, S) (so W is finite). Then the following statements are equivalent for a 1-twinning $\mathcal{O}$ of (Δ_+, Δ_-).*

(i) *There exists an apartment $\Sigma_\epsilon \subseteq \Delta_\epsilon$ and a chamber $c_{-\epsilon}$ which is twinned with Σ_ϵ.*

(ii) *There exist chambers $c_{-\epsilon}$ and d_ϵ with $\ell^*(c_{-\epsilon}, d_\epsilon) = \ell(w_0)$, where w_0 denotes the unique elements of maximal length in (W, S).*

(iii) *The codistance δ^* is well-defined and $(\Delta_+, \Delta_-, \delta^*)$ is a twin building.*

Proof. Put ϵ equal to $+$.

(i) $\Rightarrow$ (ii) : Let c_+ be the chamber in Σ_+ which is opposite c_-, and let d_+ be the chamber in Σ_+ satisfying $\delta(c_+, d_+) = w_0$. Then also $\delta^*(c_-, d_+) = w_0$ by Lemma 3.4.

$(ii) \Rightarrow (iii)$: First of all, because of the uniqueness of w_0, we must have $\delta^*(c_-, d_+) = w_0 = w_0^{-1}$. Then Lemma 2.1$(ii)$ implies $\mathcal{S}_{w_0}(d_+) \subseteq c_-^{\mathcal{O}}$. However, since $\ell(\delta(y_+, d_+)) < \ell(w_0)$ for any chamber $y_+ \in \mathcal{C}_+ \setminus \mathcal{S}_{w_0}(d_+)$, we also have $c_-^{\mathcal{O}} \subseteq \mathcal{S}_{w_0}(d_+)$, hence $c_-^{\mathcal{O}} = \mathcal{S}_{w_0}(d_+)$. So for any $x_+ \in c_-^{\mathcal{O}}$, there is precisely one apartment Σ_+ of Δ_+ which contains x_+ and d_+ (namely the convex hull of x_+ and d_+ in Δ_+). Since $c_-^{\mathcal{O}} \cap \Sigma_+ = \mathcal{S}_{w_0}(d_+) \cap \Sigma_+ = \{x_+\}$, we see that c_- is twinned with Σ_+. Hence (TA) is satisfied and Theorem 3.5 yields statement (iii).

$(iii) \Rightarrow (i)$: This is clear. $\square$

4 The standard twin building of type $\widetilde{A}_{n-1}$

Twin buildings naturally arise with Kac-Moody groups and more generally with certain group theoretic axiomatic settings like twin BN-pairs and RGD-systems, introduced by Jacques Tits(cf. [18]). Though it is desirable to have at least some concrete descriptions for (certain) twin buildings which do not refer to groups already in the very definition (similar to the well-known standard model for affine buildings of type $\widetilde{A}_{n-1}$ using lattices), almost no examples of this sort have been discussed in the literature so far. The only exception we are aware of is the detailed treatment of the twin tree associated to $\mathrm{SL}_2(k[t,t^{-1}])$ in Section 2 of [10]. In the way it is dealt with there, this example cannot easily be generalized to twin buildings of type $\widetilde{A}_{n-1}$ for $n > 2$ since one would have to consider W-codistances, where W is the affine Weyl group of type $\widetilde{A}_{n-1}$, instead of numerical codistances between vertices which are sufficient in the twin tree case. However, the new characterization of twin buildings derived in Section 3 allows a technically uncomplicated uniform description of the twin buildings associated to $\mathrm{SL}_n(k[t,t^{-1}])$ for arbitrary n without referring to groups or W-codistances. Lattice class models for twin buildings associated to classical groups over $k[t,t^{-1}]$ can probably be deduced similarly, though in a technically more complicated way (involving $k(t)$-vector spaces endowed with forms defined over k).

In order to derive our description of the twin building associated to $\mathrm{SL}_n(k[t,t^{-1}])$, we need some notation.

Let k be a commutative field, $\mathbb{K} = k(t)$ the rational function field over k, let v_+, v_- be the discrete valuations on $\mathbb{K}$ determined by $v_+(k^*) = v_-(k^*) = \{0\}$ and $v_+(t) = v_-(t^{-1}) = 1$, and let $\mho_\epsilon = \{\lambda \in \mathbb{K} \,|\, v_\epsilon(\lambda) \geq 0\}$, $\epsilon \in \{+,-\}$, be the corresponding discrete valuation rings. Denote by A the Laurent polynomial ring $A = k[t,t^{-1}]$, by V an n-dimensional $\mathbb{K}$-vector space and by M a free A-submodule of V of rank n (hence $M \otimes_A \mathbb{K} \cong V$). In order to use coordinates, we fix an A-basis $b_1, b_2, \ldots, b_n$ of M.

We recall the construction of the affine building Δ_ϵ of type $\widetilde{A}_{n-1}$ naturally associated with (V, v_ϵ), $\epsilon \in \{+,-\}$. Firstly, we set

$$\mathcal{L}_\epsilon := \{L \,|\, L \text{ is an } \mho_\epsilon - \text{lattice in } V\},$$

where "$\mho_\epsilon$-lattice" means "free $\mho_\epsilon$-submodule of rank n". Secondly, we define the equivalence relation

$$L \sim L' :\Leftrightarrow \exists \lambda \in \mathbb{K}^* : L' = \lambda L, \text{ for } L, L' \in \mathcal{L}_\epsilon$$

and we set $\overline{\mathcal{L}}_\epsilon := \mathcal{L}/\sim$, denoting the class of L in $\overline{\mathcal{L}}_\epsilon$ by $[L]$. Thirdly, we introduce an adjacency (or "incidence") relation in $\overline{\mathcal{L}}_\epsilon$ by calling $\Lambda, \Lambda' \in \overline{\mathcal{L}}_\epsilon$ adjacent if there exist representatives $L \in \Lambda$ and $L' \in \Lambda'$ such that $t^\epsilon L < L' < L$, where $t^+ := t$ and $t^- := t^{-1}$. Now Δ_ϵ is by definition the flag complex with respect to this incidence relation, i.e., the simplices of Δ_ϵ are precisely the (finite) subsets of $\overline{\mathcal{L}}_\epsilon$ of the form $\{\Lambda_1, \ldots, \Lambda_r\}$, where Λ_i and Λ_j are adjacent for all $1 \le i < j \le r$. To any $\mathbb{K}$-basis $e_1, \ldots, e_n$ of V, we associate a full subcomplex $\Sigma_\epsilon(e_1, \ldots, e_n)$ of Δ_ϵ with set of vertices

$$\overline{\mathcal{L}}_\epsilon(e_1, \ldots, e_n) := \{[L] \mid L = \bigoplus_{i=1}^n t^{m_i} \mho_\epsilon e_i \text{ for some } m_1, \ldots, m_n \in \mathbb{Z}\}$$

and with set of simplices those which have all their vertices in $\overline{\mathcal{L}}_\epsilon(e_1, \ldots, e_n)$. The following statements are folklore (cf. [12], Chapter II,§1 for $n = 2$ and [4], Chapter 19, for the general case).

Facts 4.1 (*i*) *Each subcomplex $\Sigma_\epsilon(e_1, \ldots, e_n)$ of Δ_ϵ is a Coxeter complex of type $\widetilde{A}_{n-1}$, i.e., isomorphic to the Coxeter complex of the affine Weyl group $W = W_{aff}(\widetilde{A}_{n-1})$ associated to a root system of type A_{n-1}.*

(*ii*) *Δ_ϵ is a thick building of rank n and*

$$\mathcal{A}'_\epsilon := \{\Sigma_\epsilon(e_1, \ldots, e_n) \mid e_1, \ldots, e_n \text{ is a } \mathbb{K}\text{-basis of } V\}$$

is a system of apartments for Δ_ϵ.

(*iii*) *A well-defined numbering* type: $\overline{\mathcal{L}}_\epsilon \to \mathbb{Z}/n\mathbb{Z}$ *of the vertices of Δ_ϵ can be obtained as follows. Given $\Lambda \in \overline{\mathcal{L}}_\epsilon$, choose a representative $L \in \Lambda$ and an element $g \in GL(V)$ such that $L = g(\bigoplus \mho_\epsilon b_i)$. Then the congruence class modulo n of $v_\epsilon(detg)$ is independent of the choice of L and g, and we set* $\mathrm{type}(\Lambda) := \epsilon v_\epsilon(detg) \bmod n$. *This function canonically extends to a numbering* type: $\Delta_\epsilon \longrightarrow 2^{\mathbb{Z}/n\mathbb{Z}}$ *of the whole building Δ_ϵ.*

The sign ϵ in the definition of $\mathrm{type}(\Lambda)$ is chosen in such a way to make sure that opposite vertices Λ_+, Λ_- (which by definition must have the same type) enjoy a certain natural property (see 4.3(i) below).

To any ordered $\mathbb{K}$-basis $e_1, \ldots, e_n$ of V, and any number $0 \le j < n$, we associate

(1) the lattices

$$L_\epsilon^j := L_\epsilon^j(e_1, \ldots, e_n) := \langle te_1, \ldots, te_j, e_{j+1}, \ldots, e_n \rangle_{\mho_\epsilon},$$

where $\langle x_1, \ldots, x_n \rangle_{\mho_\epsilon}$ denotes the $\mho_\epsilon$-lattice generated by $x_1, \ldots, x_n$;

(2) the lattice classes

$$\Lambda_\epsilon^j := \Lambda_\epsilon^j(e_1, \dots, e_n) := [L_\epsilon^j]$$

;

(3) the chambers

$$c_\epsilon(e_1, \dots, e_n) := \{\Lambda_\epsilon^0, \dots, \Lambda_\epsilon^{n-1}\} \in \Delta_\epsilon.$$

Definition 4.2 Two chambers $c_+ \in \Delta_+$, $c_- \in \Delta_-$ are called *opposite* — and we again use the notation $c_+\mathcal{O}c_-$ and $c_-\mathcal{O}c_+$ — if there exists an A-basis $e_1, \dots, e_n$ of M such that $c_+ = c_+(e_1, \dots, e_n)$ and $c_- = c_-(e_1, \dots, e_n)$. Two vertices $\Lambda_+ \in \Delta_+$ and $\Lambda_- \in \Delta_-$ are called *opposite* if they are contained in opposite chambers and type(Λ_+) =type(Λ_-).

It is clear that $\mathcal{O}$ is a non-empty symmetric relation. In the rest of this section we shall show that $\mathcal{O}$ is a 1-twinning of (Δ_+, Δ_-) which induces a twin building $(\Delta_+, \Delta_-, \delta^*)$.

Lemma 4.3 *(i) Two vertices Λ_+ and Λ_- are opposite if and only if there exists an A-basis $f_1, \dots, f_n$ of M such that $\Lambda_\epsilon = [\langle f_1, \dots, f_n\rangle_{\mho_\epsilon}]$, for $\epsilon \in \{+, -\}$.*

(ii) Let $e_1, \dots, e_n$ be an A-basis of M and $\Sigma = \Sigma_\epsilon(e_1, \dots, e_n)$, $\epsilon \in \{+, -\}$. Then for any vertex $\Lambda_\epsilon \in \Sigma_\epsilon$, respectively any chamber $c_\epsilon \in \Sigma_\epsilon$, there is a unique vertex $\Lambda_{-\epsilon} \in \Sigma_{-\epsilon}$ which is opposite Λ_ϵ, respectively a unique chamber $c_{-\epsilon} \in \Sigma_{-\epsilon}$ which is opposite c_ϵ.

Proof. (i). Let $f_1, \dots, f_n$ be an A-basis of M such that $\Lambda_\epsilon = [\langle f_1, \dots, f_n\rangle_{\mho_\epsilon}]$, for $\epsilon \in \{+, -\}$. Then Λ_+ and Λ_- are contained in the opposite chambers $c_+(f_1, \dots, f_n)$ and $c_-(f_1, \dots, f_n)$, respectively. Furthermore, there exists a $g \in$GL(M) such that $gb_i = f_i$, for all $i \in \{1, \dots, n\}$, hence, in particular, det$g \in A^* = \{\lambda t^m \mid \lambda \in k^*, m \in \mathbb{Z}\}$ and $v_+($det$g) = -v_-($det$g)$. So by the definition of the numberings of Δ_+ and Δ_-, we obtain

$$\text{type}(\Lambda_+) = v_+(\det g) \bmod n = -v_-(\det g) \bmod n = \text{type}(\Lambda_-).$$

Now suppose that Λ_+ and Λ_- are opposite vertices, contained in the opposite chambers $c_+(e_1, \dots, e_n)$ and $c_-(e_1, \dots, e_n)$, respectively, for an appropriate A-basis $e_1, \dots, e_n$ of M. Then there are numbers $j, \ell \in \{0, \dots, n-1\}$ such that $\Lambda_+ = \Lambda_+^j(e_1, \dots, e_n)$ and $\Lambda_- = \Lambda_-^\ell(e_1, \dots, e_n)$. By assumption, we also have type(Λ_+) =type(Λ_-). This obviously implies $j = \ell$, and so the A-basis $te_1, \dots, te_j, e_{j+1}, \dots, e_n$ of M generates an $\mho_\epsilon$-lattice which represents Λ_ϵ for $\epsilon \in \{+, -\}$.

(ii). Suppose that the vertices $\Lambda_+ \in \Sigma_+$ and $\Lambda_- \in \Sigma_-$ are opposite. In view of (i), we can find representatives $L_+ \in \Lambda_+$ and $L_- \in \Lambda_-$ such that $L_+ \cap L_- \cap M$ is an n-dimensional k-vector space (generated by $f_1, \dots, f_n$ in the notation of (i)) which also generates V as a $\mathbb{K}$-vector space. On the other hand, all representatives $L_+ \in \Lambda_+$ and $L_- \in \Lambda_-$ are of the form $L_+ = \langle t^{m_1}e_1, \dots, t^{m_n}e_n\rangle_{\mho_+}$ and $L_- = \langle t^{\ell_1}e_1, \dots, t^{\ell_n}e_n\rangle_{\mho_-}$, respectively, with all $m_i, \ell_i \in \mathbb{Z}$. Hence our requirement concerning $L_+ \cap L_- \cap M$ immediately implies $m_i = \ell_i$ for all i, which shows that $\Lambda_\epsilon \in \Sigma_\epsilon$ uniquely determines its opposite $\Lambda_{-\epsilon} \in \Sigma_{-\epsilon}$. This shows in particular that the set of vertices of a chamber $c_\epsilon \in \Sigma_\epsilon$ uniquely determines the set of vertices of any chamber $c_{-\epsilon} \in \Sigma_{-\epsilon}$ opposite c_ϵ. Hence there can be at most one such $c_{-\epsilon}$, and of course there exists one, namely $c_{-\epsilon} = c_{-\epsilon}(t^{q_1}e_{\sigma(1)}, \dots, t^{q_n}e_{\sigma(n)})$ if the $q_1, \dots, q_n \in \mathbb{Z}$ and the permutation σ of $\{1, \dots, n\}$ are such that $c_\epsilon = c_\epsilon(t^{q_1}e_{\sigma(1)}, \dots, t^{q_n}e_{\sigma(n)})$ $\square$

Proposition 4.4 *The opposition relation $\mathcal{O}$ introduced in* Definition 4.2 *is a 1-twinning of (Δ_+, Δ_-) which induces a twin building $(\Delta_+, \Delta_-, \delta^*)$. The set of twin apartments of $(\Delta_+, \Delta_-, \delta^*)$ is*

$$\mathcal{A} := \{(\Sigma_+(e_1, \dots, e_n), \Sigma_-(e_1, \dots, e_n)) \mid e_1, \dots, e_n \textit{ is an } A\textit{-basis of } M\}.$$

Proof. Lemma 4.3 implies that any element $(\Sigma_+, \Sigma_-) \in \mathcal{A}$ is a twin apartment with respect to $\mathcal{O}$. Hence by Definition 4.2, any two opposite chambers are contained in a twin apartment, which in particular shows that Condition (TA) of Theorem 3.5 is satisfied. So in order to prove that $(\Delta_+, \Delta_-, \mathcal{O})$ induces a twin building, we just have to verify that $\mathcal{O}$ is a 1-twinning.

Given two opposite chambers c_+, c_-, we choose an A-basis $e_1, \dots, e_n$ of M such that $c_\epsilon = c_\epsilon(e_1, \dots, e_n) = \{\Lambda_\epsilon^0, \dots, \Lambda_\epsilon^{n-1}\}$ and we set $\Sigma_\epsilon = \Sigma_\epsilon(e_1, \dots, e_n)$ for each $\epsilon \in \{+, -\}$. Each index $0 \le j < n$ determines a panel P_ϵ^j which consists of all chambers containing $c_\epsilon \setminus \{\Lambda_\epsilon^j\}$. Note that P_+^j and P_-^j are panels of the same type since type(Λ_+^j) =type(Λ_-^j) (cf. Lemma 4.3(i)). We denote by d_ϵ^j the unique chamber of Σ_ϵ contained in P_ϵ^j and distinct from c_ϵ. We first show

(1) $\{x_\epsilon \in P_\epsilon^j \mid x_\epsilon \mathcal{O} c_{-\epsilon}\} = P_\epsilon^j \setminus \{d_\epsilon^j\}$.

We set ϵ equal to $+$ and distinguish two cases.

1. Case $0 < j < n$.
Here we have $d_+^j = (c_+ \setminus \{\Lambda_+^j\}) \cup \{M_+^j\}$, where $M_+^j := [\langle te_1, \dots, te_{j-1}, e_j, te_{j+1}, e_{j+2}, \dots, e_n\rangle_{\mho_+}]$. By Lemma 4.3 d_+^j is not opposite c_-. Any chamber $x_+ \in P_+^j \setminus \{d_+^j\}$ is of the form $x_+ = (c_+ \setminus \{\Lambda_+^j\}) \cup \{\Lambda\}$ with $\Lambda = [\langle te_1, \dots, te_{j-1}, te_j, e'_{j+1}, e_{j+2}, \dots, e_n\rangle]$, where $e'_{j+1} = \lambda e_j + e_{j+1}$ for some $\lambda \in k$. For $i \ne j+1$, $1 \le i \le n$, we define $e'_i = e_i$. Observe that $\langle e'_j, e'_{j+1}\rangle_{\mho_\epsilon} = \langle e_j, e_{j+1}\rangle_{\mho_\epsilon}$ for any $\epsilon \in \{+, -\}$, and also $\langle te'_j, e'_{j+1}\rangle_{\mho_-} = \langle te_j, e_j\rangle_{\mho_-}$ (since $t^{-1} \in \mho_-$). This implies $c_+(e'_1, \dots, e'_n) = x_+$ and $c_-(e'_1, \dots, e'_n) = c_-(e_1, \dots, e_n) = c_-$, showing $x_+ \mathcal{O} c_-$.

2. Case $j = 0$.
If we set $M_+^0 := [\langle te_1, e_2, \dots, e_{n-1}, t^{-1}e_n\rangle_{\mho_+}]$, then $d_+^0 = (c_+ \setminus \{\Lambda_+^0\}) \cup \{M_+^0\}$. Again

by Lemma 4.3, d_+^0 is not opposite c_-. Any chamber $x_+ \in P_+^0 \setminus \{d_+^0\}$ is of the form $x_+ = (c_+ \setminus \{\Lambda_+^0\}) \cup \{\Lambda\}$ with $\Lambda = [\langle e_1', e_2, \dots, e_n\rangle]$, where $e_1' = e_1 + \lambda t^{-1} e_n$ for some $\lambda \in k$. We set $e_i' = e_i$ for $2 \leq i \leq n$ and observe that $\langle te_1', e_n'\rangle_{\mho_\epsilon} = \langle te_1, e_n\rangle_{\mho_\epsilon}$ for any $\epsilon \in \{+,-\}$, as well as $\langle e_1', e_n'\rangle_{\mho_-} = \langle e_1, e_n\rangle_{\mho_-}$. Hence $c_+(e_1', \dots, e_n') = x_+$ and $c_-(e_1', \dots, e_n') = c_-$, showing $x_+ \mathcal{O} c_-$.

Next we observe

(2) For any $z_\epsilon \in P_\epsilon^j$, there exists a $z_{-\epsilon} \in P_{-\epsilon}^j$ such that $z_\epsilon \mathcal{O} z_{-\epsilon}$.

Indeed, if $z_\epsilon \neq d_\epsilon^j$, we already saw that $z_\epsilon \mathcal{O} c_{-\epsilon}$. So it suffices to verify $d_+^j \mathcal{O} d_-^j$. However, this follows immediately from $d_\epsilon^j = c_\epsilon(e_1, \dots, e_{j-1}, e_{j+1}, e_j, e_{j+2} \dots, e_n)$ for $0 < j < n$ and from $d_\epsilon^0 = c_\epsilon(t^{-1}e_n, e_2, \dots, e_{n-1}, te_1)$, for any $\epsilon \in \{+,-\}$.

Now we apply statement (1) to the pair $z_+ \mathcal{O} z_-$ occurring in (2) as well, thus obtaining

(3) Any chamber $z_\epsilon \in P_\epsilon^j$ is opposite all chambers of $P_{-\epsilon}^j$ but exactly one.

Since the opposite panels P_+^j and P_-^j were given arbitrarily, we now have verified Condition (1Tw) of Definition 1.2.

Since we now know that $(\Delta_+, \Delta_-, \mathcal{O})$ induces a twin building, we also know that any twin apartment (Σ_+, Σ_-) is uniquely determined by each pair of chambers $c_+ \in \Sigma_+$ and $c_- \in \Sigma_-$ satisfying $c_+ \mathcal{O} c_-$. Therefore the set of all twin apartments cannot be larger than $\mathcal{A}$.

The proposition is proved. □

Remark 4.5 It follows from Proposition 4.4 and general properties of twin buildings (cf. [1], Lemma 2) that the set

$$\mathcal{A}_\epsilon := \{\Sigma_\epsilon(e_1, \dots, e_n) \mid e_1, \dots, e_n \text{ is an } A\text{-basis of } M\}$$

is an apartment system of Δ_ϵ. So given any two chambers $x, y \in \Delta_\epsilon$, we can find an A-basis $e_1, \dots, e_n$ of M, integers $m_1, \dots, m_n$ and a permutation σ of the set $\{1, \dots, n\}$ such that $x = c_\epsilon(e_1, \dots, e_n)$ and $y = c_\epsilon(t^{m_1} e_{\sigma(1)}, \dots, t^{m_n} e_{\sigma(n)})$. We have proved this here in passing (almost) without any calculations.

Remark 4.6 It is now clear that the group $G = \mathrm{SL}_n(k[t, t^{-1}])$ — identified with $\mathrm{SL}(M)$ via the basis $b_1, \dots, b_n$ — acts (type-preserving) on $(\Delta_+, \Delta_-, \mathcal{O})$ and hence on $(\Delta_+, \Delta_-, \delta^*)$. It is also easily checked that G acts "strongly transitively", i.e., transitively on the set $\{(c_+, c_-) \in \Delta_+ \times \Delta_- \mid c_+, c_- \text{ are opposite chambers}\}$. So again the general theory of twin buildings (cf. [18],§3.2, or [2], Section 2) yields a twin BN-pair (G, B_+, B_-, N, S) in G, and $(\Delta_+, \Delta_-, \delta^*)$ is canonically isomorphic to the twin buildings associated to this twin BN-pair.

5 Rank 2 Buildings and Rank 2 Geometries

In this section we consider the special case of rank 2 spherical buildings. Historically it was exactly this case that started the research presented in the present paper. It turns out that some other nice conditions on a potential opposition relation are sufficient to conclude that this relation is really an opposition. Further, we will generalize these results to arbitrary rank 2 geometries.

5.1 Definitions and notation

A *rank 2 geometry* $\Gamma = (\mathcal{P}, \mathcal{L}, \mathcal{F})$ (we will briefly talk about *geometry* from now on) consists of a set $\mathcal{P}$ of *points*, a set $\mathcal{L}$ of *lines*, with $\mathcal{P} \cap \mathcal{L} = \emptyset$, and a set of *flags* $\mathcal{F} \subseteq \{\{p, L\} \,|\, p \in \mathcal{P} \text{ and } L \in \mathcal{L}\}$. It is called *thick* (*firm*) if every point and every line is contained in at least three (two) flags. We only deal with firm geometries. If $\{p, L\} \in \mathcal{F}$, then we usually say that p and L *are incident*, or that p (respectively L) *is incident with* L (respectively p). For a given geometry Γ, the *incidence graph* $G(\Gamma) = (X(\Gamma), E(\Gamma))$ is the graph obtained from Γ by putting $X(\Gamma) = \mathcal{P} \cup \mathcal{L}$ (the set of vertices of $G(\Gamma)$) and $E(\Gamma) = \mathcal{F}$ (the set of edges of $G(\Gamma)$). The *(local) diameter* of Γ is by definition the (local) diameter of $G(\Gamma)$ (in a certain vertex v; i.e., the distance in $G(\Gamma)$ from v to an element at maximal distance from v) and the *girth* of Γ is the girth of $G(\Gamma)$. Note that the girth, if finite, is always an even natural number (the girth is ∞ if there are no circuits); therefore one sometimes considers the *gonality* of Γ, which is half of the girth. When the diameter of Γ is finite and equal to the gonality of Γ, then we say that Γ is a *generalized polygon* (see [16]; these objects were introduced by Tits in [14]). This, in fact, is equivalent with saying that the simplicial complex with set of vertices $\mathcal{P} \cup \mathcal{L}$ and set of maximal simplices $\mathcal{F}$ is a spherical rank 2 building (for an explicit proof, see e.g. [19]). We denote this simplicial complex, endowed with the natural type function, by $\Delta(\Gamma)$. The question that we treat in this section is the following.

Given two geometries $\Gamma_+ = (\mathcal{P}_+, \mathcal{L}_+, \mathcal{F}_+)$ and $\Gamma_- = (\mathcal{P}_-, \mathcal{L}_-, \mathcal{F}_-)$, a natural number n (n possibly infinite; for generalized polygons, n will usually be the diameter; for trees n will be infinite) and a symmetric relation $\mathcal{O} \subseteq ((\mathcal{F}_+ \times \mathcal{F}_-) \cup (\mathcal{F}_- \times \mathcal{F}_+))$ (the complementary relation of which we denote by $\not{\mathcal{O}}$), what conditions are needed in order to be able to define a codistance function δ^* such that the triple $(\Delta(\Gamma_+), \Delta(\Gamma_-), \delta^*)$ is a twin building (related to a generalized n-gon, a generalized ∞-gon being a connected tree without finite endpoints) with opposition relation precisely $\mathcal{O}$? We will answer this question first for generalized polygons, and then afterwards for more general geometries. In order to do so, we consider Conditions (A), (B_{f_ϵ}), (C_{f_ϵ}) and ($\mathrm{D}_{f\epsilon}$), with respect to Γ_+, Γ_-, $\mathcal{O}$ and a flag $f_\epsilon \in \mathcal{F}_\epsilon$, $\epsilon \in \{+, -\}$. Condition (A) is just a rephrasing for geometries of a 1-twinning.

(A) Given a pair of flags $(f_+, f_-) \in \mathcal{O}$ ($f_\epsilon \in \mathcal{F}_\epsilon$, $\epsilon \in \{+, -\}$) and two elements $x_+ \in f_+$

and $x_- \in f_-$ of the same type, then for any $\epsilon \in \{+,-\}$ and any flag $g_\epsilon \ni x_\epsilon$, there exists a unique flag $h_{-\epsilon} \ni x_{-\epsilon}$ such that $(g_\epsilon, h_{-\epsilon}) \notin \mathcal{O}$.

A type-free corollary of (A) is the following much weaker statement (in the formulation, adjacent flags are not equal).

(A$'$) Given a pair of flags $(f_+, f_-) \in \mathcal{O}$ ($f_\epsilon \in \mathcal{F}_\epsilon$, $\epsilon \in \{+,-\}$) and for each $\epsilon \in \{+,-\}$, a flag x_ϵ adjacent to f_ϵ. Then there exists a flag $y_{-\epsilon}\mathcal{O}x_\epsilon$ adjacent to $f_{-\epsilon}$.

A finite sequence $(f_0, f_1, \ldots, f_k)$ of flags of Γ_ϵ, $\epsilon \in \{+,-\}$, with $|f_{i-1} \cap f_i| = 1$, $i \in \{1, 2, \ldots, k\}$, and $f_{i-1} \cap f_i \cap f_{i+1} = \emptyset$, $i \in \{1, 2, \ldots, k-1\}$, will be called a *simple path (of flags)*.

(B$_{f_\epsilon}$) Let $(f_0, f_1, \ldots, f_k)$ be a simple path of flags of $\Gamma_{-\epsilon}$ with $f_0\mathcal{O}f_\epsilon\mathcal{O}f_k$ and $f_\epsilon \not\mathcal{O} f_i$, $i \in \{1, 2, \ldots, k-1\}$. If $k < 2n$, then k is odd.

A very particular case of Condition (B$_{f_\epsilon}$) is the following condition.

(B$'_{f_\epsilon}$) Let (f_0, f_1, f_2) be a simple path of flags of $\Gamma_{-\epsilon}$ with $f_0\mathcal{O}f_\epsilon\mathcal{O}f_2$. Then $f_1\mathcal{O}f_\epsilon$.

The following four conditions only make sense if n is finite.

(C$_{f_\epsilon}$) There exists a simple path $(f_0, f_1, \ldots, f_{2n})$ in $\Gamma_{-\epsilon}$ with $f_\epsilon \not\mathcal{O} f_i$, $i \in \{1, 2, \ldots, 2n-1\}$.

(C$'_{f_\epsilon}$) There exists a simple path $(f_0, f_1, \ldots, f_{2n})$ in $\Gamma_{-\epsilon}$ with $f_0\mathcal{O}f_\epsilon\mathcal{O}f_{2n}$ and $f_\epsilon \not\mathcal{O} f_i$, $i \in \{1, 2, \ldots, 2n-1\}$.

(C$''_{f_\epsilon}$) For every pair of flags $f', f'' \in f_\epsilon^{\mathcal{O}}$, there exists a simple path $(f' = f_0, f_1, \ldots, f_{2n} = f'')$ with $f_\epsilon \not\mathcal{O} f_i$, $i \in \{1, 2, \ldots, 2n-1\}$.

(D$_{f_\epsilon}$) For every simple path $(f_0, f_1, \ldots, f_{2n-1})$ in $\Gamma_{-\epsilon}$, there exists $i \in \{0, 1, \ldots, 2n-1\}$ such that $f_\epsilon\mathcal{O}f_i$.

For generalized polygons, we may state a slightly weaker form of (D$_{f_\epsilon}$).

(D$'_{f_\epsilon}$) For every apartment Σ of the building $\Delta(\Gamma_{-\epsilon})$ associated with the generalized polygon $\Gamma_{-\epsilon}$ we have $\mathrm{Ch}(\Sigma) \cap f_\epsilon^{\mathcal{O}} \neq \emptyset$.

All the above conditions are satisfied for twinnings of generalized polygons. Indeed, (A) follows immediately from (1Tw); (B$_{f_\epsilon}$) follows from Lemma 2.1; for (C$''_{f_\epsilon}$), consider the unique chamber f_n in $\Gamma_{-\epsilon}$ such that $\ell^*(f_\epsilon, f_n) = n$ (and choose minimal paths $(f_n, f_{n-1}, \ldots, f')$ and $(f_n, f_{n+1}, \ldots, f'')$ from f_n to f' and f'' respectively in such a way that f_{n-1} and f_{n+1} are neither adjacent nor equal); (D$_{f_\epsilon}$) follows from the 1-twinning and Lemma 2.1.

5.2 Generalized polygons

Our first result characterizes all geometries related to rank 2 twin buildings. Essentially, we keep the hypothesis of the 1-twinning (Condition (A)), but we replace the conditions of Proposition 2.5 and Corollary 2.6 with combinatorial ones.

Proposition 5.1 *Let $\Gamma_\epsilon = (\mathcal{P}_\epsilon, \mathcal{L}_\epsilon, \mathcal{F}_\epsilon)$, $\epsilon \in \{+,-\}$, be two generalized n-gons ($n = \infty$ is allowed). If $\mathcal{O} \subseteq ((\mathcal{F}_+ \times \mathcal{F}_-) \cup (\mathcal{F}_- \times \mathcal{F}_+))$ is a non-empty symmetric relation satisfying* (A)*, and if there exists some flag $f_\epsilon \in \mathcal{F}_\epsilon$, for some $\epsilon \in \{+,-\}$, such that $\Gamma_+, \Gamma_-, \mathcal{O}, n, f_\epsilon$ satisfies* (B_{f_ϵ})*, then δ^* is globally well-defined and $\mathcal{O}$ is the natural opposition relation associated with the twin building $(\Delta(\Gamma_+), \Delta(\Gamma_-), \delta^*)$.*

Proof. We use the notation of Section 2 (considering the buildings $\Delta(\Gamma_\epsilon)$). In view of Corollary 2.6 we just have to show that $|D^*(f_\epsilon, x_{-\epsilon})| = 1$, for all $x_{-\epsilon} \in \mathcal{F}_{-\epsilon}$. Set $k := \ell^*(f_\epsilon, x_{-\epsilon}) \leq n$. We may assume $k < n$, since there is only one element of length n in W (for finite n). Denote by w, w' the two elements of W of length k, and assume $D^*(f_\epsilon, x_{-\epsilon}) = \{w, w'\}$. So we find two simple paths $\beta = (x_0, \ldots, x_k = x_{-\epsilon})$ and $\beta' = (x'_0, \ldots, x'_k = x_{-\epsilon})$ in $\Gamma_{-\epsilon}$ with x_{k-1} and x'_{k-1} neither adjacent nor equal, and with $f_\epsilon^{\mathcal{O}} \cap \{x_i, x'_i \,|\, 0 \leq i \leq k\} = \{x_0, x'_0\}$. But now the simple path $\gamma = (x_0, \ldots, x_k, x'_{k-1}, \ldots, x'_0)$, which is of length $2k \leq 2n-2$, contradicts our assumption (B_{f_ϵ}). Hence either $D^*(c_\epsilon, x_{-\epsilon}) = \{w\}$ or $D^*(c_\epsilon, x_{-\epsilon}) = \{w'\}$. $\square$

The second characterization only holds in the case of finite diameter, and it slightly improves Corollary 3.6 in the rank 2 case.

Proposition 5.2 *Let $\Gamma_\epsilon = (\mathcal{P}_\epsilon, \mathcal{L}_\epsilon, \mathcal{F}_\epsilon)$, $\epsilon \in \{+,-\}$, be two generalized n-gons ($n < \infty$). If $\mathcal{O} \subseteq ((\mathcal{F}_+ \times \mathcal{F}_-) \cup (\mathcal{F}_- \times \mathcal{F}_+))$ is a non-empty symmetric relation satisfying* (A)*, and if there exists some flag $f_\epsilon \in \mathcal{F}_\epsilon$, for some $\epsilon \in \{+,-\}$, such that $\Gamma_+, \Gamma_-, \mathcal{O}, n, f_\epsilon$ satisfies* (C_{f_ϵ})*, then δ^* is globally well-defined and $\mathcal{O}$ is the natural opposition relation associated with the twin building $(\Delta(\Gamma_+), \Delta(\Gamma_-), \delta^*)$.*

Proof. By assumption, there exists a simple path $\gamma = (f_1, \ldots, f_n, \ldots, f_{2n-1})$ with $f_\epsilon \not{\mathcal{O}} f_i$, for all $i \in \{1, 2, \ldots, 2n-1\}$. Since γ is simple, we have $\{\delta(f_n, x_i) \,|\, 1 \leq i \leq 2n-1\} = W \setminus \{w_0\}$ (w_0 is the longest element in the Coxeter system related to $\Delta(\Gamma_+)$ and $\Delta(\Gamma_-)$). Now Lemma 2.1(*ii*) implies $D^*(f_{-\epsilon}, f_n) = \{w_0\}$. Hence Corollary 3.6 yields the twinning $(\Delta(\Gamma_+), \Delta(\Gamma_-), \delta^*)$ as desired. $\square$

The next result characterizes twinnings of rank 2 buildings without the assumption of a 1-twinning.

Proposition 5.3 *Let $\Gamma_\epsilon = (\mathcal{P}_\epsilon, \mathcal{L}_\epsilon, \mathcal{F}_\epsilon)$, $\epsilon \in \{+,-\}$, be two thick generalized n-gons $(n < \infty)$. Let $\mathcal{O} \subseteq ((\mathcal{F}_+ \times \mathcal{F}_-) \cup (\mathcal{F}_- \times \mathcal{F}_+))$ be a symmetric relation. If for all flags $f_\epsilon \in \mathcal{F}_\epsilon$, $\epsilon \in \{+,-\}$, the objects $\Gamma_+, \Gamma_-, \mathcal{O}, n, f_\epsilon$ satisfy either* $(\mathrm{B}_{f_\epsilon})$, $(\mathrm{C}'_{f_\epsilon})$ *and* $(\mathrm{D}'_{f_\epsilon})$, *or* $(\mathrm{B}_{f_\epsilon})$, $(\mathrm{C}_{f_\epsilon})$ *and* $(\mathrm{D}_{f_\epsilon})$, *then δ^* is globally well-defined and the types in Γ_- can be chosen such that $\mathcal{O}$ is the natural opposition relation associated with the twin building $(\Delta(\Gamma_+), \Delta(\Gamma_-), \delta^*)$.*

Proof. Let f_+ be an arbitrary flag of Γ_+ and let $(f_0, f_1, \ldots, f_{2n})$ be a simple path in Γ_- with $f_+ \not\mathcal{O} f_i$, $i \in \{1, 2, \ldots, 2n-1\}$, as in (C_{f_+}). If (D_{f_+}) holds, then it implies that $f_0 \mathcal{O} f_+ \mathcal{O} f_{2n}$, as in (C'_{f_+}). Hence from now on we may assume the Conditions $(\mathrm{B}_{f_\epsilon})$, $(\mathrm{C}'_{f_\epsilon})$ and $(\mathrm{D}'_{f_\epsilon})$. We first claim that every flag $g_- \in \mathcal{F}_-$ which satisfies $g_- \mathcal{O} f_+$ is necessarily opposite f_n in Γ_-. Suppose by way of contradiction that $g_- \mathcal{O} f_+$ and g_- is not opposite f_n. Let $(f_n, g_1, g_2, \ldots, g_k, g_-)$ be the unique simple path in Γ_- connecting f_n and g_-, with $k < n-1$. We choose k minimal, i.e., we assume that $f_+ \not\mathcal{O} g_i$ for all $i \in \{1, 2, \ldots, k\}$. Put $g_0 = f_n$ and let $\ell \in \{0, 1, \ldots, k\}$ be maximal with respect to the property $g_\ell \in \{f_0, f_1, \ldots, f_{2n}\}$ (note that certainly g_- does not belong to the latter set) and let $m \in \{0, 1, \ldots, 2n\}$ be such that $g_\ell = f_m$. By symmetry we may assume that $m \leq n$. Then $(f_0, f_1, \ldots, f_{m-1}, g_{\ell+1}, g_{\ell+2}, \ldots, g_k, g_-)$ is a simple path in Γ_- with the property that $f_0 \mathcal{O} f_+ \mathcal{O} g_-$ and all flags h_- in between — and there are exactly $m+k-\ell-1$ such — satisfy $h_- \not\mathcal{O} f_+$. Similarly, $(f_{2n}, f_{2n-1}, f_m, g_{\ell+1}, g_{\ell+2}, \ldots, g_k, g_-)$ is a simple path with the property that $f_{2n} \mathcal{O} f_+ \mathcal{O} g_-$ and all flags h_- in between — and this time there are exactly $2n+k-m-\ell$ such — satisfy $h_- \not\mathcal{O} f_+$. Note also that $2n+k-m-\ell < 2n-1$ since $\ell = n-m$. But $(2n+k-m-\ell) - (m+k-\ell-1) = 2n-2m+1$ is odd, which contradicts Condition (B_{f_+}). Our claim follows.

Now we claim that every flag g_- opposite f_n in Γ_- satisfies $g_- \mathcal{O} f_+$. But this follows immediately from (D'_{f_+}) and the previous paragraph as otherwise the apartment through f_n and g_- does not satisfy the condition in (D'_{f_+}).

Hence the set $f_+^{\mathcal{O}}$ is precisely the set of flags of Γ_- opposite f_n. Clearly f_n is unique with respect to that property. Consequently (and by interchanging the roles of "+" and "−"), we have a map $\theta : (\mathcal{F}_+ \cup \mathcal{F}_-) \to (\mathcal{F}_+ \cup \mathcal{F}_-) : f_\epsilon \mapsto f_\epsilon^\theta$ satisfying (and we write $x_\epsilon \operatorname{op} y_\epsilon$ for (x_ϵ, y_ϵ) belonging to the natural opposition relation in $\Delta(\Gamma_\epsilon)$, $\epsilon \in \{+,-\}$)

$$g_{-\epsilon} \operatorname{op} f_\epsilon^\theta \Leftrightarrow g_{-\epsilon} \mathcal{O} f_\epsilon \Leftrightarrow f_\epsilon \mathcal{O} g_{-\epsilon} \Leftrightarrow f_\epsilon \operatorname{op} g_{-\epsilon}^\theta.$$

Using the characterization of adjacency by the opposition relation in thick spherical buildings as derived in [3], Section 4, one shows that θ is bijective and that both θ and θ^{-1} preserve adjacency of flags (the proof is almost identical with the proof of Corollary 5.5 given in [3]). By Theorem 3.21 of [15], θ extends to simplicial isomorphisms $\alpha_1 : \Delta(\Gamma_+) \to \Delta(\Gamma_-)$ and $\alpha_2 : \Delta(\Gamma_-) \to \Delta(\Gamma_+)$. We can choose types in Γ_- such that α_1 is type-preserving for even n and type-reversing for odd n. For a fixed flag $f_\epsilon \in \mathcal{F}_\epsilon$ we have

$$x_\epsilon \operatorname{op} f_\epsilon \Leftrightarrow x_\epsilon^\theta \operatorname{op} f_\epsilon^\theta \Leftrightarrow x_\epsilon \operatorname{op} f_\epsilon^{\theta\theta},$$

for all flags $x_\epsilon \in \mathcal{F}_\epsilon$, $\epsilon \in \{+,-\}$. So $\alpha_1 = \alpha_2^{-1}$ and consequently, if α_1 preserves (reverses) types, then so does α_2. It is now clear that $\mathcal{O}$ is a 1-twinning of $(\Delta(\Gamma_+), \Delta(\Gamma_-))$ (or equivalently, Condition (A) is satisfied). The proposition follows from the previous propositions. □

5.3 Arbitrary rank 2 geometries

In this subsection we give characterizations of generalized polygons as rank 2 geometries endowed with a certain opposition relation. The first is a generalization of Propositions 5.2 and 5.3.

Proposition 5.4 *Let $\Gamma_\epsilon = (\mathcal{P}_\epsilon, \mathcal{L}_\epsilon, \mathcal{F}_\epsilon)$, $\epsilon \in \{+,-\}$, be two geometries and let n be a positive integer. Let $\mathcal{O} \subseteq ((\mathcal{F}_+ \times \mathcal{F}_-) \cup (\mathcal{F}_- \times \mathcal{F}_+))$ be a symmetric relation. If for all flags $f_\epsilon \in \mathcal{F}_\epsilon$, $\epsilon \in \{+,-\}$, the objects $\Gamma_+, \Gamma_-, \mathcal{O}, n, f_\epsilon$ satisfy (A′), $(\mathrm{B}'_{f_\epsilon})$, $(\mathrm{C}''_{f_\epsilon})$ and $(\mathrm{D}_{f_\epsilon})$, then both Γ_+ and Γ_- are generalized n-gons. Also, δ^* is globally well-defined and the types of Γ_- can be chosen such that $\mathcal{O}$ is the natural opposition relation associated with the twin building $(\Delta(\Gamma_+), \Delta(\Gamma_-), \delta^*)$, if one of the following two conditions is satisfied.*

- *(i) The geometries Γ_+ and Γ_- are thick and for all flags $f_\epsilon \in \mathcal{F}_\epsilon$, $\epsilon \in \{+,-\}$, the objects $\Gamma_+, \Gamma_-, \mathcal{O}, n, f_\epsilon$ satisfy $(\mathrm{B}_{f_\epsilon})$;*
- *(ii) The triple $(\Gamma_+, \Gamma_-, \mathcal{O})$ satisfies (A).*

Proof. Let $f_\epsilon \in \mathcal{F}_\epsilon$ be an arbitrary flag of Γ_ϵ. Condition $(\mathrm{D}_{f_\epsilon})$ implies that the set $f_\epsilon^{\mathcal{O}}$ is non-empty. So choose $f_{-\epsilon} \in f_\epsilon^{\mathcal{O}}$. By Condition $(\mathrm{C}''_{f_\epsilon})$, there is a simple path $\gamma = (f_{-\epsilon} = f_0, f_1, \dots, f_n, \dots, f_{2n} = f_{-\epsilon})$ with $f_\epsilon \not\mathcal{O} f_i$, for all $i \in \{1, 2, \dots, 2n-1\}$. Note that f_1 and f_{2n-1} cannot be adjacent or equal since otherwise there is a closed circuit contained in γ all members of which belong to $f_\epsilon^{\mathcal{O}}$, and this contradicts Condition $(\mathrm{D}_{f_\epsilon})$ by considering the juxtaposition of sufficiently many copies of that circuit.

We claim that, if $x_{-\epsilon}, y_{-\epsilon} \in f_\epsilon^{\mathcal{O}}$, and if $(x_{-\epsilon} = g_0, g_1, \dots, g_n, \dots, g_{2n} = y_{-\epsilon})$ is a simple path with $g_i \notin f_\epsilon^{\mathcal{O}}$ for all $i \in \{1, 2, \dots, 2n-1\}$, then $g_n = f_n$.

Indeed, let $(x_{-\epsilon} = h_0, h_1, \dots, h_n, \dots, h_{2n} = f_{-\epsilon})$ be a simple path in $\Gamma_{-\epsilon}$ with $h_i \not\mathcal{O} f_\epsilon$ for $i \in \{1, 2, \dots, 2n-1\}$ (this path exists by Condition $(\mathrm{C}''_{f_\epsilon})$). Without loss of generality we may assume that h_{2n-1} and f_{2n-1} are adjacent (if none of f_{2n-1} and f_1 were adjacent to h_{2n-1}, then f_{2n-1} would be adjacent or equal to f_1, which is impossible as remarked above). Let $j \geq 1$ be minimal with respect to the property $h_j = f_j$. If $j > n$, then the simple path $(h_1, h_2, \dots, h_{j-1}, f_{j-1}, f_{j-2}, \dots, f_1)$ contradicts Condition $(\mathrm{D}_{f_\epsilon})$. Hence $j \leq n$

and so $h_n = f_n$. Now one similarly shows that h_n is the middle element of any simple closed path of length $2n$ with extremities $x_{-\epsilon}$ no members of which lie in $f_\epsilon^{\mathcal{O}}$, except for $x_{-\epsilon}$. And that middle element is on its turn equal to g_n by the same token. The claim follows.

Next we claim that, with the above notation, all neighbours of $f_{-\epsilon}$ except exactly two (which are mutually not adjacent) belong to $f_\epsilon^{\mathcal{O}}$. Indeed, clearly f_1 and f_{2n-1} do not belong to $f_\epsilon^{\mathcal{O}}$. Now let f' be adjacent to $f_{-\epsilon}$ with $f_1 \neq f' \neq f_{2n-1}$. Without loss of generality we may assume that f' and f_1 are adjacent. If $f' \notin f_\epsilon^{\mathcal{O}}$, then the simple path $(f', f_1, \ldots, f_{2n-1})$ contradicts Condition $(\mathrm{D}_{f_\epsilon})$. Our second claim is proved.

So by our first claim, we have a well-defined map

$$\alpha_\epsilon : \mathcal{F}_\epsilon \to \mathcal{F}_{-\epsilon} : f_\epsilon \mapsto f_n.$$

We show that α_ϵ is surjective. It suffices to prove that every flag adjacent to f_n is in the image of α_ϵ, and that $\Gamma_{-\epsilon}$ is connected.

We first show that $\Gamma_{-\epsilon}$ is connected. By Condition $(\mathrm{D}_{f_\epsilon})$, all connected components of $\Gamma_{-\epsilon}$ contain elements of $f_\epsilon^{\mathcal{O}}$. By Condition $(\mathrm{C}''_{f_\epsilon})$, all these elements lie in the same connected component.

Now let $f'_{-\epsilon} \in \mathcal{F}_{-\epsilon}$ be adjacent to f_n, and choose a simple path $(g_0, g_1, \ldots, g_n)$ with $g_n = f_n$ and $g_{n-1} = f'_{-\epsilon}$ (by the firmness assumption, this is possible). We show that $g_0 \in f_\epsilon^{\mathcal{O}}$. If none of the g_i, $i = 0, \ldots, n$, belong to $f_\epsilon^{\mathcal{O}}$, then, as above, we derive a contradiction against $(\mathrm{D}_{f_\epsilon})$. Let j be maximal with the property that $g_j \mathcal{O} f_\epsilon$. By $(\mathrm{C}''_{f_\epsilon})$, there is a simple path $(g_j = g'_0, g'_1, \ldots, g'_{n-1}, g'_n = f_n)$ such that g'_1 and g_{j+1} are adjacent or equal, and such that none of g'_i, $i = 1, 2, \ldots, n-1$ belongs to $f_\epsilon^{\mathcal{O}}$. If $j \neq 0$, then we can combine the elements of that path with the g_i, $i = j+1, \ldots, n$, (because the two paths $(g'_0, \ldots, g'_{n-1})$ and $(g_j, \ldots, g_{n-1})$ cannot coincide) to get a closed simple path no element of which belongs to $f_\epsilon^{\mathcal{O}}$, and this contradicts Condition $(\mathrm{D}_{f_\epsilon})$. Hence $j = 0$ and $g_0 \mathcal{O} f_\epsilon$.

Moreover, we have shown above that, if $(g_0, g'_1, \ldots, g'_n, g_{n+1}, \ldots, g_{2n} = g_0)$ is a simple path with the property that $g'_i, g_j \notin f_\epsilon^{\mathcal{O}}$, for $1 \leq i \leq n < j < 2n$, then (assuming without loss of generality that g_1 and g'_1 are adjacent or equal) $g_i = g'_i$, for all $i \in \{1, 2, \ldots, n\}$. Since $g_{2n} \mathcal{O} f_\epsilon$, Condition (A') implies that there exists a flag f'_ϵ adjacent to f_ϵ with $f'_\epsilon \mathcal{O} g_{2n-1}$. It follows from our second claim above that all flags of Γ_ϵ adjacent to both f_ϵ and f'_ϵ, but distinct from f_ϵ, belong to $g_{2n-1}^{\mathcal{O}}$. By the same claim, there is exactly one flag g_ϵ amongst them that does not belong to $g_{2n}^{\mathcal{O}}$. Suppose that $g_\epsilon \mathcal{O} g_i$, $i \in \{1, 2, \ldots, 2n\}$. Then by (A'), there exists a neighbour $g'_{-\epsilon}$ of g_i which belongs to $f_\epsilon^{\mathcal{O}}$, hence $i \in \{1, 2n-1, 2n\}$. By construction $i \neq 0$ and $g_\epsilon \mathcal{O} g_{2n-1}$. If $g_\epsilon \mathcal{O} g_1$, then the simple path (g_{2n-1}, g_{2n}, g_1) contradicts $(\mathrm{B}'_{g_\epsilon})$. Hence it is now clear that $g_\epsilon^{\alpha_\epsilon} = f'_{-\epsilon}$. Consequently α_ϵ is surjective.

There is no closed simple path through f_n of length $k < 2n$, since such a path could not contain any element of $f_\epsilon^{\mathcal{O}}$, contradicting $(\mathrm{D}_{f_\epsilon})$ by considering the juxtaposition of enough copies of that path. Also, there are no flags at distance $n+1$ from f_n, since such a flag would be adjacent to a flag $f_{-\epsilon} \in f_\epsilon^{\mathcal{O}}$, and hence either itself belongs to $f_\epsilon^{\mathcal{O}}$ (and then it has distance $\leq n$ from f_n), or it is one of the two unique flags adjacent to $f_{-\epsilon}$ on a path of length n from $f_{-\epsilon}$ to f_n (and then it has distance $\leq n-1$ from f_n). These properties are consequences of f_n being in the image of α_ϵ. By the surjectivity of α_ϵ, we conclude that Γ_ϵ is a generalized n-gon.

Now, if on one hand additionally (i) is satisfied, then the result follows from Proposition 5.3. If on the other hand additionally (ii) is satisfied, then the result follows from Proposition 5.2. □

Corollary 5.5 *Let $\Gamma_\epsilon = (\mathcal{P}_\epsilon, \mathcal{L}_\epsilon, \mathcal{F}_\epsilon)$, $\epsilon \in \{+,-\}$, be two connected geometries and let n be a positive integer. Let $\mathcal{O} \subseteq ((\mathcal{F}_+ \times \mathcal{F}_-) \cup (\mathcal{F}_- \times \mathcal{F}_+))$ be a non-empty symmetric relation. If for all flags $f_\epsilon \in \mathcal{F}_\epsilon$, $\epsilon \in \{+,-\}$, the objects $\Gamma_+, \Gamma_-, \mathcal{O}, n, f_\epsilon$ satisfy* (A), $(\mathrm{B}'_{f_\epsilon})$ *and* $(\mathrm{C}''_{f_\epsilon})$, *then both Γ_+ and Γ_- are generalized n-gons, δ^* is globally well-defined and $\mathcal{O}$ is the natural opposition relation associated with the twin building $(\Delta(\Gamma_+), \Delta(\Gamma_-), \delta^*)$.*

Proof. Since Condition (A') is readily seen to be a direct consequence of (A), it suffices to show that for all flags f_ϵ Condition $(\mathrm{D}_{f_\epsilon})$ is satisfied. First we remark that (A) implies the following statement, the proof of which is completely similar to Lemma 5.3 of [5].

(*) *Let $f_\epsilon \in \mathcal{F}_\epsilon$ be any flag of Γ_ϵ and let $g_{-\epsilon} \in \mathcal{F}_{-\epsilon} \not\mathcal{O} f_\epsilon$ be arbitrary in $\Gamma_{-\epsilon}$. Let k be a natural number such that*

- (i) *for any simple path $(g_{-\epsilon} = g_0^*, \ldots, g_j^*)$, with $j < k$, we have $g_i^* \notin f_\epsilon^{\mathcal{O}}$, for all $i \in \{0, \ldots, j\}$;*
- (ii) *there exists a simple path $(g_{-\epsilon} = g_0, \ldots, g_k)$ with $g_k \mathcal{O} f_\epsilon$.*

Then for any simple path $(g_{-\epsilon} = g'_0, \ldots, g'_k)$ such that $g_{-\epsilon} \cap g_1 = g_{-\epsilon} \cap g'_1$, one has $g'_k \mathcal{O} f_\epsilon$.

Now let the simple path $(g_0, g_1, \ldots, g_{2n-1})$ be a counterexample to $(\mathrm{D}_{f_\epsilon})$, with $f_\epsilon \in \mathcal{F}_\epsilon$ (so $g_i \notin f_\epsilon^{\mathcal{O}}$, for all $i \in \{0, 1, \ldots, 2n-1\}$). Because of Condition (A') and the connectedness of Γ_+ and Γ_-, we see that $f_\epsilon^{\mathcal{O}}$ is non-empty. Hence we can extend the above simple path to a simple path $(g_0, g_1, \ldots, g_k)$, with $k \geq 2n$, such that $g_i \notin f_\epsilon^{\mathcal{O}}$, for all $i \in \{0, 1, \ldots, k-1\}$ and $g_k \mathcal{O} f_\epsilon$. By shifting indices we may assume $k = 2n$. Now let $(g_{2n} = f_0, \ldots, f_n, \ldots, f_{2n} = g_{2n})$ be a simple path as in $(\mathrm{C}''_{f_\epsilon})$. By Condition (A), one of f_1 or f_{2n-1} must be equal to g_{2n-1}. Without loss of generality, we may assume $f_{2n-1} = g_{2n-1}$. Now let $\ell \leq n$ be minimal with respect to the property "there exists a simple path $(f_n = h_0, \ldots, h_\ell)$ with $h_\ell \in f_\epsilon^{\mathcal{O}}$". If $\ell < n$, then (*) implies that some element of $\{f_1, \ldots, f_{2n-1}\}$ is in $f_\epsilon^{\mathcal{O}}$, a contradiction. Hence $\ell = n$. Consequently, if $\ell' \leq 2n-1$ is minimal with respect to the property "$f_i = g_i$ for all $i \in \{\ell', \ldots, 2n-1\}$", then $\ell' \leq n$ (otherwise $g_{2\ell'-2n-1} \mathcal{O} f_\epsilon$ by (*), a contradiction). But now (*) implies that $g_0 \mathcal{O} f_\epsilon$, again a contradiction. We conclude that $(\mathrm{D}_{f_\epsilon})$ holds, and the result follows from the previous proposition. $\square$

Definition 5.6 We now introduce the notion of 1-twinning in a single geometry Γ. Therefore, let $\mathcal{F}$ be the set of flags of Γ and suppose that $\mathcal{O}$ is a symmetric relation in $\mathcal{F}$. For Γ' a copy of either Γ or of its dual, with set of flags $\mathcal{F}'$, we can view $\mathcal{O}$ as a symmetric relation $\mathcal{O}^*$ between the sets $\mathcal{F}$ and $\mathcal{F}'$. We call $\mathcal{O}$ a 1-*twinning* if $\mathcal{O}^*$ satisfies Condition (A) for one of the choices of Γ' (so for either Γ or its dual).

The following result characterizes generalized polygons amongst all rank 2 geometries. In fact, it can be seen as a weakening of the so-called *gate property*, see [11].

Recall that two vertices v, v' of a graph are said to be at maximal distance if all vertices adjacent to v (respectively v') are not further away from v' (respectively v) than v (respectively v').

Proposition 5.7 *Let Γ be a (rank 2) geometry with finite diameter and with flag set $\mathcal{F}$. For $f, f' \in \mathcal{F}$ we define $f \mathcal{O} f'$ if f and f' are at maximal distance in the flag graph of Γ (i.e., the graph with vertex set $\mathcal{F}$ and adjacency is just adjacency of flags). If $\mathcal{O}$ is a 1-twinning of Γ, then Γ is a generalized polygon and $\mathcal{O}$ is the natural opposition relation.*

Proof. Put $\Gamma_+ := \Gamma$ and let $\Gamma_- := \Gamma'$ (with Γ' as in Definition 5.6 above). In the following, we regard $\mathcal{O}$ sometimes as a relation between the flag set of Γ_+ and the flag set of Γ_-. Note that, in this way, $(\Gamma_+, \Gamma_-, \mathcal{O})$ satisfies (A).

Let $f_+ \mathcal{O} f_-$. There is a minimal and hence simple path $(f_+ = f_0, f_1, \ldots, f_n = f_-)$ connecting f_+ with f_- and $f_i \notin f_+^{\mathcal{O}}$ for $i \in \{0, 1, \ldots, n-1\}$. Let $f_\epsilon = \{x_\epsilon, y_\epsilon\}$, with x_ϵ a point of Γ_ϵ and y_ϵ a line of Γ_ϵ, for all $\epsilon \in \{+, -\}$. Without loss of generality, we may assume that f_1 contains x_+. If f_{n-1} contained x_-, then the fact that the two different flags f_0, f_1 both are not contained in $f_{n-1}^{\mathcal{O}}$ would contradict Condition (A). Hence $y_- \in f_{n-1}$. We remark that this shows in particular that for any simple path $(h_0, h_1, \ldots, h_{m-1}, h_m)$ of length $m \equiv n \bmod 2$, the type in Γ_+ of the unique element of $h_0 \cap h_1$ is the same as the type in Γ_- of the unique element of $h_m \setminus h_{m-1}$. We now claim that there exists a flag f_1' containing y_+ at distance $n-1$ from f_- (in the flag graph of Γ). Let f' be the unique flag containing y_+ and such that $f' \notin f_-^{\mathcal{O}}$. If f' is at distance $n-1$ from f_-, then we set $f_1' = f'$ and the claim follows. The only alternative is that the distance between f' and f_- is equal to n (since a distance bigger than n would imply that f_+ and f_- are not at maximal distance). Clearly, a path of length n between f' and f_- cannot contain a flag containing y_+ other than f' as otherwise we can take this flag as f_1' and the claim follows. Hence there is a path $(f_- = g_0, g_1, \ldots, g_n = f')$, with $y_+ \notin g_{n-1}$

and hence $y_- \in g_1$ (by our remark above). Clearly, $g_1 \notin f'^{\mathcal{O}}$. By assumption, also $f_- \notin f'^{\mathcal{O}}$. Now (A) implies $g_1 = f_-$, a contradiction. Our claim is proved. So we have a path $(f_+ = f'_0, f'_1, \ldots, f'_n = f_-)$ with f_1 and f'_1 not adjacent. It is now easy to see that every element of $f_+^{\mathcal{O}}$ must occur at distance $\geq n$ from f_+, otherwise, as in the previous proofs, (*), which follows from the 1-twinning, would imply $f_i \mathcal{O} f_+$ or $f'_i \mathcal{O} f_+$, for some $i \in \{1, \ldots, n-1\}$. By symmetry, all elements of $f_+^{\mathcal{O}}$ are at distance n from f_+. It is now clear that n is the local diameter of Γ_+ in f_+ and that (C''_{f_+}) holds for this n. Now, every flag f of Γ_+ occurs in a simple path of length n starting with f_+, and by (*), it is clear that the path in question ends in an element of $f_+^{\mathcal{O}}$. An easy induction argument on the distance from f_+ to f yields that the local diameter in f is at least the local diameter in f_+. Reversing the roles of f and f_+, we see that the local diameter is a constant. Hence (C''_f) holds for every flag f, with n the diameter of Γ_+.

We now show (B'_{f_+}). Let (f_0, f_1, f_2) be a simple path with $f_0 \mathcal{O} f_+ \mathcal{O} f_2$. If $f_1 \notin f_+^{\mathcal{O}}$, then let $(f_+ = g_0, g_1, \ldots, g_{n-2}, g_{n-1} = f_1)$ be a minimal path. Without loss of generality, we may assume that $f_1 \cap g_{n-2} = f_1 \cap f_0$. But then $(f_+ = g_0, g_1, \ldots, g_{n-2}, f_0)$ is path of length $n-1$ connecting f_+ with f_1, hence $f_0 \notin f_+^{\mathcal{O}}$ by definition. This is a contradiction.

The proposition is now a consequence of Corollary 5.5. $\square$

In the previous characterization of generalized polygons, at the one hand, one does not have to check whether "opposite" flags have a constant distance from each other. This is rather a consequence. On the other hand, calling two elements of Γ opposite if non-opposition defines a bijection between the respective sets of flags containing these elements, the 1-twinning forces us to check that opposition induces a well-defined map on the set of types (either the identity, or the unique involution interchanging types). Our next result is complementary to these remarks. Opposite flags will automatically have constant distance, and opposite elements should not have a priori predestinate types. Because of the latter, we cannot assume a 1-twinning anymore. But we can assume a consequence of the 1-twinning.

Proposition 5.8 *Let Γ be a (rank 2) geometry with finite diameter, flag set $\mathcal{F}$ and diameter of the flag graph equal to n. For $f, f' \in \mathcal{F}$ we define $f \mathcal{O} f'$ if f and f' are at distance n in the flag graph of Γ, and we call two flags f and g opposite if $f \mathcal{O} g$. For every two opposite flags f, g, and any element $x \in f$, we assume that there exist a unique flag $f' \ni x$ not opposite g, and at least one flag g' adjacent to g and opposite f'. Then Γ is a generalized n-gon and $\mathcal{O}$ is the natural opposition relation.*

Proof. Let f, g be opposite flags of Γ, and let $x \in f$. Let f' be the unique flag containing x not opposite g and let g' be any flag adjacent to g and opposite f'. Then the type of $g' \cap g$ only depends on the type x. Indeed, $g' \cap g$ is the unique element of g not contained in the second flag h of a minimal path γ connecting g with f'. But γ has length $n-1$, and hence if n is odd, then the type of $g \cap h$ is the same as the type of x; if n is even these types are distinct. Hence the map θ mapping the type of x to the type of $g' \cap g$ is well defined and involutory. Let now x and y be two elements of Γ such that their types correspond under θ and such that there exist flags $f \ni x$ and $g \ni y$ with $f\mathcal{O}g$. Our assumption implies that there is a unique flag f' containing x and not opposite g. It is now clear that $\mathcal{O}$ induces a 1-twinning of Γ as defined in Definition 5.6.

Furthermore, our assumption together with the fact that θ is well-defined implies the following. Given two opposite flags $f = \{x_1, x_2\}$ and $g = \{y_1, y_2\}$, where we assume that the type of y_i is the image under θ of the type of x_i, $i = 1, 2$, there exist two simple paths $(f = f_0, f_1, \ldots, f_n = g)$ and $(f = f'_0, f'_1, ..., f'_n = g)$ such that $f_0 \cap f_1 = \{x_1\}$, $f_{n-1} \cap f_n = \{y_2\}$ and $f'_0 \cap f'_1 = \{x_2\}$, $f'_{n-1} \cap f'_n = \{y_1\}$. This yields Condition (C''_f) for all $f \in \mathcal{F}$ (with respect to $\Gamma_+ := \Gamma$ and Γ_-, which is either Γ or its dual, depending on whether θ is the identity or not). Finally, Condition (B'_f) can be verified similarly as at the end of the proof of Proposition 5.7. Hence our claim follows again from Corollary 5.5. □

There is a nice immediate consequence of these propositions. For the definition of near $2d$-gon, we refer to [13]. Let us mention that it has been an open question since the introduction of near polygons whether a near polygon which is also a dual near polygon is necessarily a generalized polygon. A positive answer to this question for firm near n-gons with n even (see Corollary 5.10 below) will follow immediately from a stronger result, a new characterization of generalized polygons, which can be deduced from our last proposition.

Corollary 5.9 *Let Γ be a (rank 2) geometry with finite diameter n. If for every pair of elements x, y of Γ, with x and y at distance $n-1$ from each other, there is a unique element x' incident with x and nearest to y (in the incidence graph), then Γ is a generalized n-gon.*

Proof. Given flags $f = \{x, x'\}$ and $g = \{y, y'\}$, where without loss of generality we assume $d(x, y') < d(x, y)$ (d denotes the distance function in the incidence graph of Γ), we observe that f and g are at distance n in the flag graph of Γ if and only if $d(x, y) = d(x', y') = n$. Hence our assumption implies in particular that there exist flags which are at distance n from each other (note that we also need our general assumption of firmness here). Therefore the diameter of the flag graph (which is always smaller than or equal to the diameter of the incidence graph) is equal to n. It is now clear that the assumption of the corollary implies the condition about opposite flags in Proposition 5.8. Hence this proposition implies our claim. □

Corollary 5.10 *If Γ is a (firm) near n-gon for some positive even integer n, such that the dual of Γ is again a near n-gon, then Γ is a generalized polygon.*

6 Some Counter-Examples

In this section, we will construct 1-twinnings of rank 2 buildings which are not twinnings. We use the notation of the previous section and look at these buildings as point-line geometries. As before, a generalized ∞-gon is just a connected tree without finite end points.

We make the following observation. Let $(\Gamma_+, \Gamma_-, \mathcal{O})$ be a 1-twinning of two generalized n-gons, n possibly infinite. Then for every point p_+ of Γ_+, the set of points of Γ_- not opposite p_+ is a geometric hyperplane $\mathcal{H}_{p_+}$ of Γ_-, i.e., a set of points such that any line L either meets $\mathcal{H}_{p_+}$ in exactly 1 point, or every point of L is contained in $\mathcal{H}_{p_+}$. Also, if $p_- \in \mathcal{H}_{p_+}$, then $p_+ \in \mathcal{H}_{p_-}$ (with obvious notation). Similarly, for every line L_ϵ of Γ_ϵ. $\epsilon \in \{+,-\}$, there is a unique dual geometric hyperplane $\mathcal{H}_{L_\epsilon}$ in $\Gamma_{-\epsilon}$. Hence, if we want 1-twinnings which are not twinnings, then we have to look for suitable sets of geometric hyperplanes, i.e., we must find a set of hyperplanes and dual hyperplanes which intersect in the right way. The easiest way to make the intersections right is simply to choose all (dual) geometric hyperplanes disjoint (and it follows that no geometric hyperplane will contain the set of points of a line; hence all geometric hyperplanes are distance-2-ovoids in the sense of Chapter 7 of [19]).

For projective planes, the only geometric hyperplanes are the sets of points incident with a line, and the full point set. Hence there is no way to choose disjoint ones. Therefore, we exclude this case in the following proposition.

Proposition 6.1 *Let Γ_ϵ be a generalized n_ϵ-gon, with $n_\epsilon > 3$, $\epsilon \in \{+,-\}$. Suppose that there is a non-trivial partition Π_ϵ of the points set of Γ_ϵ into disjoint geometric hyperplanes, and a partition Π'_ϵ of the line set of Γ_ϵ into dual geometric hyperplanes. Suppose also that there are pairings $b : \Pi_+ \longleftrightarrow \Pi_-$ and $b' : \Pi'_+ \longleftrightarrow \Pi'_-$. For an element x_ϵ of Γ_ϵ, we denote by H_{x_ϵ} the unique (dual) geometric hyperplane of Π_ϵ (Π'_ϵ) containing x_ϵ. For chambers $\{p_\epsilon, L_\epsilon\}$ of Γ_ϵ, we define*

$$\{p_+, L_+\}\mathcal{O}\{p_-, L_-\} \Longleftrightarrow H^b_{p_+} \neq H_{p_-} \text{ and } H^{b'}_{L_+} \neq H_{L_-}.$$

Then $\mathcal{O}$ satisfies Condition (1Tw) *of* Definition 1.2 *and hence defines a 1-twinning of (Γ_+, Γ_-) if $n_+ = n_-$. In the latter case, $(\Gamma_+, \Gamma_-, \mathcal{O})$ does not induce a twin building.*

Proof. Obviously, to prove (1Tw) of Definition 1.2, all we have to show is that, up to point-line duality and plus-minus duality, if for the lines L_+ of Γ_+ and L_- of Γ_- the sets $H^{b'}_{L_+}$ and H_{L_-} are not the same, then for every point p_+ of L_+, there is a unique point p_- of L_- belonging to $H^b_{p_+}$. The existence of p_- follows from the fact that $H^b_{p_+}$ is a geometric hyperplane. If p_- were not unique, then every point of L_- would belong to $H^b_{p_+}$ and hence L_- would not meet any other member of Π_-, a contradiction.

Now, if $n_+ = n_-$, the function ℓ^* associated to the 1-twinning $(\Gamma_+, \Gamma_-, \mathcal{O})$ satisfies $\ell^*(f_+, g_-) < 3$ for all flags f_+, g_-, and hence $(\Gamma_+, \Gamma_-, \mathcal{O})$ has no twin apartments (cf. Lemma 3.4). $\square$

Let, with the notation of the previous proposition, (s_ϵ, t_ϵ) be the order of Γ_ϵ (i.e., each line is incident with $s_\epsilon + 1$ points and each point with $t_\epsilon + 1$ lines; both numbers can be infinite). If Π_ϵ and Π'_ϵ both exist for $\epsilon \in \{+,-\}$, then necessary and sufficient for the existence of the pairings b and b' is obviously that $s_+ = s_-$ and $t_+ = t_-$. In the finite case, there are no generalized polygons known admitting partitions of geometric hyperplanes and dual geometric hyperplanes other than the quadrangles $T^*(O)$ of Tits (see [8]) and hence the latter admit 1-twinnings which are not twinnings. In the infinite case, every generalized n-gon with $n > 3$ admits such partitions by transfinite induction. For the sake of simplicity, let us explain this in the case $s_+ = s_- = t_+ = t_- = |\mathbb{N}|$. Let us construct a partition Π_+ of the point set of Γ_+ into distance-2-ovoids, or briefly *ovoids*. Let $\{p_i \,|\, i \in \mathbb{N}\}$ be the set of points of Γ_+ and let $\{L_j \,|\, j \in \mathbb{N}\}$ be the set of lines of Γ_+. Our task is to construct a countable set $\{O_i \,|\, i \in \mathbb{N}\}$ of disjoint ovoids O_i covering the whole point set of Γ_+. Suppose we already constructed all ovoids O_i with $i < k$, for certain $k \in \mathbb{N}$ (this includes $k = 0$, which corresponds to the first step of the induction). We now construct O_k, also by induction. Let m be minimal with respect to the property "$p_m \notin O_0 \cup \ldots \cup O_{k-1}$" (for $k = 0$, just set $m = 0$) and let m' be minimal with respect to the property "$L_{m'}$ is not incident with p_m". Then we put $\Theta_{m'} = \{p_m\}$. Suppose now we have constructed a (finite) set Θ_ℓ of mutually non-collinear points such that every line L_i with $i < \ell$ is incident with a unique element of Θ_ℓ, but no element of Θ_ℓ is incident with L_ℓ, for some $\ell \in \mathbb{N}$. Then we can choose a point x on L_ℓ outside $O_1 \cup \cdots \cup O_{k-1}$ and not collinear with any element of Θ_ℓ. If ℓ' is the smallest positive integer with the property that $L_{\ell'}$ is not incident with any element of $\Theta_\ell \cup \{x\}$, then we put $\Theta_{\ell'} = \Theta_\ell \cup \{x\}$. Now we define O_k as the union of all the Θ_ℓ, and the construction is complete.

For trees, the construction is even simpler. Let T_+, T_- be two isomorphic semi-homogeneous trees with valencies a and b, more precisely, all vertices in T_+ and T_- of the first type are supposed to have valency b (and we call such vertices *points*) and all vertices of the second type to have valency a (and we call these vertices *lines*). Let I and J be two index sets with cardinalities a and b, respectively. For any vertex in T_+ or T_-, denote by $N(x)$ its set of neighbours. Now choose a function f_1 (which can be very easily constructed) from the set of all points in T_+ and T_- to I such that the restriction of f_1 to any set $N(L)$, where L is a line, is a bijection onto I. Analogously, construct a function f_2 from the set of all lines to J such that the restriction to any $N(p)$, where p is a point, is a bijection onto J. The $+$ and $-$ parts of the fibers of f_1 (f_2) are the (dual) geometric hyperplanes in T_+, T_- we wanted.

For projective planes, we can construct 1-twinnings which are not twinnings in the following way. Consider a pair of projective planes (Γ_+, Γ_-) of the same order q (possibly infinite). Let Π_ϵ, $\epsilon \in \{+,-\}$, be a partition of the flag set of Γ_ϵ such that every point and every line of Γ_ϵ is contained in exactly one element of each member of Π_ϵ. Then $|\Pi_+| = |\Pi_-| = q+1$ and we can choose a bijection b from Π_+ to Π_-. If we define $f_+\mathcal{O}f_-$, for two flags f_+, f_- in Γ_+, Γ_-, respectively, if the image under b of the class of Π_+ containing f_+ does not contain f_-, then we have a 1-twinning $(\Gamma_+, \Gamma_-, \mathcal{O})$ which is clearly not a twinning. It is again easy to construct such partitions for infinite projective planes. In the finite case, we consider for Γ_+ and Γ_- the Desarguesian projective plane of order q. There, such a partition can be defined as the set of orbits in the flag set under a Singer cycle. For the plane of order 2, this was exactly the counterexample alluded to in [5], attributed there to the second author.

References

[1] Abramenko P., *Twin buildings and Applications to S-Arithmetic Groups*, Lecture Notes in Math. **1641**, Springer, 1996.

[2] Abramenko P., Group actions on twin buildings, *Bull. Belg. Math. Soc. Simon Stevin* **3** (1996), 391 – 406.

[3] Abramenko P. and H. Van Maldeghem, On opposition in spherical buildings and twin buildings, *Preprint* 98-090, SFB Bielefeld (1998), to appear in *Annals of Combinatorics*.

[4] Garrett P., *Buildings and Classical Groups*, Chapman & Hall, London (1997).

[5] Mühlherr B., A rank 2 characterization of twinnings, *European J. Combin.* **19**, no. 5 (1998), 603 – 612.

[6] Mȟlherr B., *On the Existence of* 2*-Spherical Twin Buildings*, Habilitationsschrift, Dortmund 1999.

[7] Mühlherr B. and M. A. Ronan, Local to global structures in twin buildings, *Invent. Math.* **122** (1995), 71 – 81.

[8] Payne S. E. and J. A. Thas, *Finite Generalized Quadrangles*, Pitman, Boston, London, Melbourne, 1984.

[9] Ronan M. A., Local isometries of twin buildings, to appear in *Math. Z.*

[10] Ronan M. A. and J. Tits, Twin trees I., *Invent. Math.* **116** (1994), 463 – 479.

[11] Scharlau R., A characterization of Tits buildings by metrical properties, *J. London Math. Soc.* (2) **32** (1985), 317 – 327.

[12] SERRE J.-P., *Trees*, Translated from the french by John Stillwell; Springer, Berlin, Heidelberg, New York, 1980.

[13] SHULT E. E. and A. YANUSHKA, Near n-gons and line systems, *Geom. Dedicata* **9** (1980), 1 – 72.

[14] TITS J., Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13 – 60.

[15] TITS J., *Buildings of Spherical Type and Finite BN-Pairs*, Springer Verlag, Berlin, Heidelberg, New York, *Lecture notes in Math.* **386**, 1974

[16] TITS J., Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, *Invent. Math.* **43** (1977), 283 – 295.

[17] TITS J., A local approach to buildings, **in** *The Geometric Vein. The Coxeter Festschrift* (ed. D. Chandler *et al.*), Springer-Verlag (1981), 519 – 559.

[18] TITS J., Twin buildings and groups of Kac-Moody type, *London Math. Soc. Lecture Note Ser.* **165** (Proceedings of a conference on *Groups, Combinatorics and Geometry*, ed. M. Liebeck and J. Saxl, Durham 1990), Cambridge University Press (1992), 249 – 286.

[19] VAN MALDEGHEM H., *Generalized Polygons*, Birkhäuser Verlag, Basel, Boston, Berlin, *Monographs in Mathematics* **93**, 1998.