LÉVY–LEPAGE SERIES REPRESENTATION OF STABLE VECTORS. CONVERGENCE IN VARIATION

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Abstract. Multidimensional stable laws G_α admit a well-known Lévy–LePage series representation

$$G_\alpha = \mathcal{L}\big(\sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} X_j\big), \quad 0 < \alpha < 2,$$

where $\Gamma_1, \Gamma_2, \dots$ are the successive times of jumps of a standard Poisson process, and $X_1, X_2, \dots$ denote i.i.d. random vectors, independent of $\Gamma_1, \Gamma_2, \dots$ We present (asymptotically) optimal bounds for the total variation distance between a stable law and the distribution of a partial sum of the Lévy–LePage series. In the one-dimensional case similar results were obtained earlier by Bentkus, Götze and Paulauskas.

1 . Introduction and formulation of results

Let $\mathbb{R}^d$ denote a standard real Euclidean space with the norm $|x|^2 = x_1^2 + \cdots + x_d^2$ and the corresponding inner product $\langle x, x\rangle = |x|^2$. Let $X, X_1, X_2, \dots$ be a sequence of independent and identically distributed (i.i.d.) $\mathbb{R}^d$-valued random vectors with the distribution $F = \mathcal{L}\big(X\big)$. Introduce a sequence $\lambda_1, \lambda_2, \dots$ of i.i.d. random variables with exponential distribution, that is $\mathbf{P}\{\lambda > x\} = \exp\{-x\}$, for $x \geqslant 0$. It is well known that the sequence $\Gamma_n = \lambda_1 + \cdots + \lambda_n$ defines the successive times of the jumps of a standard Poisson process. Assume that the sequences $\lambda_1, \lambda_2, \dots$ and $X_1, X_2, \dots$ are independent. Denote by $F_n = \mathcal{L}\big(S_n\big)$ the distribution of the sum

$$S_n = \sum_{j=1}^{n} \Gamma_j^{-1/\alpha} X_j.$$

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It is well known that the limit law $G := \lim\limits_{n\to\infty} F_n$ exists (Theorem 1.1 shows that the convergence holds in variation) provided that X satisfies certain moments assumptions. Moreover, G is a stable distribution with the characteristic exponent $0 < \alpha < 2$ (see LePage 1989, 1981, LePage et al 1981; see Ferguson and Klass 1972 and Rosinski 1990 for similar representations of infinitely divisible distributions in Banach spaces). Such representations are related to simulations of stable random vectors, see Janicki and Weron (1994) for further references. In this paper we establish bounds for the error when G is approximated by F_n.

From now on, ϱ denotes a finite non-negative σ-additive measure on the unit sphere $s_{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. We write the characteristic function

$$\varphi(t) = \int_{\mathbb{R}^d} \exp\big\{ i\big\langle t, x\big\rangle \big\} \, G(dx)$$

of a stable law G in the following form

$$\varphi(t) = \exp\Big\{ - \int_{s_{d-1}} \big|\big\langle t, y\big\rangle\big|^{\alpha} N(y, t, \alpha)\, \varrho(dy)\Big\}, \tag{1}$$

where

$$N(y,t,\alpha) = 1 - i \,\operatorname{sign}\big(\big\langle t, y\big\rangle\big)\, \operatorname{tg}\frac{\pi\alpha}{2}, \qquad \alpha \neq 1,$$

$$N(y,t,1) = 1 + i\,\frac{2}{\pi}\,\operatorname{sign}\big(\big\langle t, y\big\rangle\big)\, \ln\big|\big\langle t, y\big\rangle\big|.$$

Up to a shift, (1) represents any stable law G, see Remark 1. The measure ϱ is called the spectral measure of G. We shall write $G_{\alpha,\varrho}$, φ_α, etc., in cases when we would like to emphasize the dependence on α or on ϱ. The density (if it exists) of G with respect to the Lebesgue measure on $\mathbb{R}^d$ we denote by g. For more information about multivariate stable laws we refer to Samorodnitsky and Taqqu (1994).

Remark 1. The assumption (1) does not restrict generality of our results since for any stable random vector W there exists a vector $\mu \in \mathbb{R}^d$ such that the characteristic function of $W + \mu$ has the form (1). In this paper we are interested in approximations of G by $\mathcal{L}\big(S_n\big)$. Hence, in cases when $\mu \neq 0$, we approximate G by the distributions of $\mathcal{L}\big(S_n + \mu\big)$.

By $\mathcal{B}(S)$ we denote the class of all Borel subsets of a metric space S, and $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$.

Theorem 2. *Let $0 < \alpha < 2$ and $\mathbf{E}\big|X\big|^{\alpha} < \infty$. For $\alpha \geqslant 1$, assume that $\mathbf{E}X = 0$. For non-symmetric X in the case $\alpha = 1$ assume in addition that $\mathbf{E}\big|X\big| \log\big(1 + \big|X\big|\big) < \infty$. Then the distributions $F_n = \mathcal{L}\big(S_n\big)$ converge in variation to a stable law G, that is, $\sup\limits_{B\in\mathcal{B}} \big|F_n\big(B\big) - G\big(B\big)\big| \to 0$, as $n \to \infty$. The spectral measure ϱ of G satisfies*

$$\varrho\big(B\big) = \varkappa\, \mathbf{E}\big|X\big|^{\alpha} \mathbf{I}\big\{ X \neq 0,\ X/\big|X\big| \in B\big\}, \qquad \text{for all } B \in \mathcal{B}(s_{d-1}), \tag{2}$$

with the constant

$$\varkappa = \begin{cases} \Gamma(2-\alpha)\cos(\pi\alpha/2)/(1-\alpha), & \alpha \neq 1, \\ \pi/2, & \alpha = 1. \end{cases}$$

Here Γ *is the well-known gamma function,* $\Gamma(y) = \int\limits_{x>0} x^{y-1}\exp(-y)\,dx$.

Remark 3. Formula (2) establishes a relation between the spectral measure ϱ and the distribution $\mathcal{L}(X)$. The distribution $\mathcal{L}(X)$ defines ϱ (and, hence, G) in a unique way, while the knowledge of ϱ does not allow to recover a unique $\mathcal{L}(X)$. The sums S_n may be used to generate i.i.d. samples, say $U_1, \dots, U_m$, such that $\mathcal{L}(U_1)$ approximately equals G. It follows from Theorem 2 that any G with $0 < \alpha < 1$ may be mimicked. However, for $\alpha \geqslant 1$, the condition $\mathbf{E}X = 0$ restricts a bit the generality. To see this consider the case $d = 1$. Then $s_0 = \{-1, 1\}$ is a two point set. If ϱ is given and both $\varrho(\{-1\})$ and $\varrho(\{1\})$ are positive, then it is easy to find a bounded X, $\mathbf{E}X = 0$, satisfying (2). But in the case that $\varrho(\{-1\}) = 0$ and $\varrho(\{1\}) > 0$, a random vector X, $\mathbf{E}X = 0$, satisfying (2) does not exist. A way to overcome this difficulty would be to use centerings of S_n such as in Theorem 1.4.5 in Samorodnitsky and Taqqu (1994). We would like to notice that it is possible to extend the results of this paper to the case $\mathbf{E}X \neq 0$. We are going to publish such generalizations elsewhere since they are considerable in size. Anyway, the results of this paper cover the case of symmetric G.

Our goal is to obtain bounds for the variation of the signed measure $G - F_n$ in the multidimensional case. One-dimensional case was considered in Janicki and Kokoszka (1992), Ledoux and Paulauskas (1996), Bentkus, Götze, Paulauskas (1996). Restricted to the one-dimensional case, our results are very close to those in Bentkus, Götze, Paulauskas (1996). However, the proofs differ considerably from those in the one-dimensional case. In order to cover the multidimensional case we had to develop a new method, the main ingredient of which is iterative integration by parts using specially constructed random vector fields, see Section 5, Lemma 9. Earlier a similar approach allowed to solve an old problem related to uniform bounds for stable measures of convex shells in $\mathbb{R}^d$, see Bentkus, Juozulynas, Paulauskas (1999).

In order to formulate bounds for $G - F_n$, we need additional notation. Let $\mathcal{A}$ denote a class of subsets of $\mathbb{R}^d$. For $n < N \leqslant \infty$, write

$$\delta_{n,N}(\mathcal{A}) = \sup_{A\in\mathcal{A}} \bigl|F_n(A) - F_N(A)\bigr|, \qquad \delta_{n,N} = \delta_{n,N}(\mathcal{B}),$$

as well as

$$\delta_n(\mathcal{A}) = \delta_{n,n+1}(\mathcal{A}), \qquad \delta_n = \delta_n(\mathcal{B}).$$

Here by $F_\infty = G$ we understand the limit law $G = \lim\limits_{n\to\infty} F_n$.

Without loss of generality we shall assume that the distribution $F = \mathcal{L}(X)$ is non-degenerate, since otherwise we may pass to a linear subspace of $\mathbb{R}^d$. A distribution F is called non-degenerate if $F(L) < 1$, for any linear subspace $L \subset \mathbb{R}^d$ such that $\dim L < d$.

Our main result is Theorem 5 below, and it yields all other results of the introduction. In particular, we have the following theorem.

Theorem 4. *Let ϱ be a spectral measure such that $\varrho(s_{d-1}) > 0$. Let a random vector V have the distribution such that $\mathbf{P}(V \in B) = \varrho(s_{d-1} \cap B)/\varrho(s_{d-1})$, for all $B \in \mathcal{B}$. Define $X = (\varkappa/\varrho(s_{d-1}))^{-1/\alpha} V$. Then:*

(i) *If $0 < \alpha < 1$ then the sums S_n converge to the stable distribution G with the spectral measure ϱ and*

$$\delta_{n,\infty} = O(n^{1-1/\alpha}); \tag{3}$$

(ii) *Assume that $\int_{s_{d-1}} s\,\varrho(ds) = 0$. Then the sums S_n converge to the stable distribution G with the spectral measure ϱ, for all $0 < \alpha < 2$, and*

$$\delta_{n,\infty} = O(n^{1-2/\alpha}). \tag{4}$$

The condition $\int_{s_{d-1}} s\,\varrho(ds) = 0$ on the spectral measure ϱ restricts somewhat the generality of Theorem 4, cf. Remark 3.

Theorem 4 improves a result of Davydov and Nagaev (1999), where for non-degenerate symmetric X such that $\mathbf{P}\{|X| = 1\} = 1$, the following bound

$$\delta_{n,\infty} = O(n^{-\min\{\alpha,2-\alpha\}/(d+\alpha)})$$

was obtained (see Theorem 3.2 in the aforementioned paper).

In the remaining part of the introduction we describe bounds for $\delta_{n,\infty}$, which hold for X such that the distributions $F_n = \mathcal{L}(S_n)$ converge to G, as $n \to \infty$. These bounds reflect the dependence of rates on (truncated) moments of X. In particular, the usage of truncated moments allows to obtain optimal conditions under which F_n converge to G (see Theorem 2).

By c we denote absolute constants, and, for example, we write $c(\alpha)$ if a constant depends on α. Throughout C_0, C_1, C_2 denote constants which may depend on α, d, r and the distribution $F = \mathcal{L}(X)$. The constants C_j exist if F is non-degenerate. See (13)–(17), where explicit upper bounds for C_j are provided.

Our main result—Theorem 5—shows that the Lévy–LePage series is *fundamental* in variation in a very strong sense. Notice that

$$\delta_{n,N}(\mathcal{A}) \leqslant \delta_n(\mathcal{A}) + \delta_{n+1}(\mathcal{A}) + \cdots + \delta_N(\mathcal{A}). \tag{5}$$

Introduce the normalized random vector

$$X_* = n^{-1/\alpha} X,$$

and indicator functions

$$\mathbf{I} = \mathbf{I}\{|X_*| \leqslant 1\}, \quad \mathbf{I}^c = 1 - \mathbf{I} = \mathbf{I}\{|X_*| > 1\}.$$

Theorem 5. *We have*

$$\delta_n \leqslant C_0 \Big(\mathbf{E}\,\mathbf{I}^c + \big|\mathbf{E} X_* \mathbf{I}\big| + \mathbf{E}\big|X_*\big|^2 \mathbf{I}\Big). \tag{6}$$

Write

$$\Lambda_r = \sup_{t>0} t^r\, \mathbf{P}\big\{ \big|X\big| > t \big\}.$$

Notice that $\Lambda_r \leqslant \mathbf{E}\big|X\big|^r$. Theorem 5 combined with (5) yields the following Corollaries 6–9.

Corollary 6. *Let* $0 < \alpha < 1$. *Then*

$$\delta_{n,\infty} \leqslant C_1\, n \Big(\mathbf{E}\big|X_*\big|^\alpha \mathbf{I}^c + \mathbf{E}\big|X_*\big| \mathbf{I}\Big). \tag{7}$$

In particular, we have:

(i) $\delta_{n,\infty} \leqslant C_2\, n^{1-1/\alpha} \mathbf{E}\big|X\big|$;
(ii) $\delta_{n,\infty} \leqslant C_2\, n^{1-r/\alpha} \Lambda_r$, *for* $\alpha < r < 1$;
(iii) $\delta_{n,\infty} = o\big(n^{1-r/\alpha}\big)$ *if* $\mathbf{E}\big|X\big|^r < \infty$, *for* $\alpha \leqslant r < 1$.

Corollary 7. *Assume that* $\mathbf{E}X = 0$ *and* $\alpha \neq 1$, $0 < \alpha < 2$. *Then*

$$\delta_{n,\infty} \leqslant C_1\, n \Big(\mathbf{E}\big|X_*\big|^\alpha \mathbf{I}^c + \mathbf{E}\big|X_*\big|^2 \mathbf{I}\Big). \tag{8}$$

In particular, we have:

(i) $\delta_{n,\infty} \leqslant C_2\, n^{1-2/\alpha} \mathbf{E}\big|X\big|^2$;
(ii) $\delta_{n,\infty} \leqslant C_2\, n^{1-r/\alpha} \Lambda_r$, *for* $1 < r < 2$, $r > \alpha$;
(iii) $\delta_{n,\infty} = o\big(n^{1-r/\alpha}\big)$ *if* $\mathbf{E}\big|X\big|^r < \infty$, *for* $1 < r < 2$, $r \geqslant \alpha$.

Corollary 8. *Assume that* $\mathbf{E}X = 0$ *and* $\alpha = 1$. *Then*

$$\delta_{n,\infty} \leqslant C_1\, n \Big(\mathbf{E}\big|X_*\big| \log\big(\mathbf{I} + \mathbf{I}^c \big|X_*\big|\big) + \mathbf{E}\big|X_*\big|^2 \mathbf{I}\Big). \tag{9}$$

In particular, the bounds (i)–(iii) of Corollary 7 hold for $\alpha = 1$, $r > 1$. *Furthermore, we have* $\delta_{n,\infty} = o(1)$ *if* $\mathbf{E}\big|X\big| \log\big(1 + \big|X\big|\big) < \infty$.

Corollary 9. *Assume that X is symmetric. Then (8) holds for $0<\alpha<2$ and $\alpha\neq 1$. In particular, we have*

$$\delta_{n,\infty}\leqslant C_2\, n^{1-1/\alpha}\Lambda_1, \quad \text{for} \quad \alpha<1.$$

If $\alpha=1$ and $\mathbf{E}\big|X\big|<\infty$, then $\delta_{n,\infty}\leqslant C_1\, n\Big(\mathbf{E}\big|X_*\big|\mathbf{I}^c+\mathbf{E}\big|X_*\big|^2\mathbf{I}\Big)$ and $\delta_{n,\infty}=o(1)$.

Remark 10. An example proposed by Ledoux and Paulauskas (1996, Proposition 9) shows that the bounds (7)–(9) in the one-dimensional case are asymptotically optimal, as $n\to\infty$, under given moment assumptions. In this example X is a symmetric stable random variable, say η, with the exponent r. The example extends to the multidimensional case as follows. Let $\eta_1,\dots,\eta_d$ be independent copies of η and $X=(\eta_1,\dots,\eta_d)$. Then $\Lambda_r<\infty$, and for the class $\mathcal{B}_h=\big\{\mathbb{R}^{d-1}\times(-\infty,x),\, x\in\mathbb{R}\big\}$ of half-spaces of $\mathbb{R}^d$ we have

$$\delta_{n,\infty}\big(\mathcal{B}\big)\geqslant\delta_{n,\infty}\big(\mathcal{B}_h\big)\geqslant C(\alpha,r)\, n^{-(r-\alpha)/\alpha},$$

which establishes the optimality of (7)–(9) in the multidimensional case.

Now we describe the constants C_0, C_1, C_2 used in Theorem 5 and Corollaries 6–9. Let $\mathcal{Q}=\big\{B_1,B_2,\dots,B_d\big\}$ be a collection of bounded closed sets $B_j\in\mathcal{B}$ such that any sequence of vectors $a_i\in B_i$, $i=1,\dots,d$ is linearly independent. Here and in the sequel, $B=\cup_{j=1}^d B_j$. Let

$$p=\prod_{j=1}^d F\big(B_j\big). \tag{10}$$

Write

$$a^*=\sup\big\{\,|a|\,:\,a\in B\,\big\},\ \ a_*=\inf\big\{\,|a|\,:\,a\in B\,\big\} \tag{11}$$

and

$$\tilde a=\inf_{|t|=1}\ \inf_{a}\ \max_{1\leqslant j\leqslant d}|a_j|^{-1}|\langle a_j,t\rangle|. \tag{12}$$

Throughout $\inf_a$ (resp. $\sup_a$) denotes the infimum (resp. supremum) over all $a_1,\dots,a_d$ such that $a_j\in B_j$, for all $1\leqslant j\leqslant d$. The condition $a^*<\infty$ (resp. $a_*>0$) means that the set B is bounded (resp. that B is disjoint with a ball of a positive radius centered at 0). The condition $\tilde a>0$ implies that B is not a subset of a proper linear subspace of $\mathbb{R}^d$. It is easy to see that for any non-degenerate F there exists a collection $\mathcal{Q}$ such that $p>0$, $\tilde a>0$, $a_*>0$ and $a^*<\infty$. For example, in the case when F has a positive density, we can choose

$$B_j=e_j+\big\{x\in\mathbb{R}^d\,:\,|x|<\varepsilon\big\},\quad 1\leqslant j\leqslant d,$$

where $e_1,\dots,e_d$ is an orthonormal system of vectors in R^d, and a number $\varepsilon>0$ is sufficiently small.

For the constants C_0 and C_1 of Theorem 5 and its corollaries the following explicit upper bounds hold

$$C_0\leqslant c(\alpha,d)\, C_3 \quad\text{and}\quad C_1\leqslant c\, C_0\, C_4^{-1}, \tag{13}$$

where

$$C_3 = p^{-1-2/\alpha}\big(1 + a_*^{-2} + C_5\,C_6\,C_7\big) \tag{14}$$

with

$$C_5 = (a_*\tilde{a})^{(1-2\alpha)d}, \quad C_6 = 1 + (a^*)^{2\alpha d}, \quad C_7 = 1 + (a_*\tilde{a})^{-2-d},$$

and

$$C_4 = \begin{cases} |1-\alpha|\,(2-\alpha), & \text{for } \ \alpha \neq 1, \\ 1, & \text{for } \ \alpha = 1. \end{cases} \tag{15}$$

The constant C_3 depends on α, d, and on F, $\mathcal{Q}$ through parameters p, α^*, α_*, $\tilde{a}$ defined by (10)–(12). The estimates (13) are rather rough. However, they reflect the dependence of our bounds for $\delta_{n,\infty}$ on the choice of sets $\mathcal{Q}$. The constant C_2 satisfies

$$C_2 \leqslant c\,C_1\,C_8^{-1} \tag{16}$$

with

$$C_8 = \begin{cases} (1-r)(r-\alpha), & \text{for } \ r < 1, \\ (r-\alpha), & \text{for } \ r = 1, \\ (2-r)(r-\alpha), & \text{for } \ 1 < r < 2,\, \alpha \neq 1, \\ (2-r)\Big(\int\limits_{|u|>1} u^{-r}\log(u)\,du\Big)^{-1}, & \text{for } \ 1 < r < 2,\, \alpha = 1, \\ 1, & \text{for } \ r = 2, \end{cases} \tag{17}$$

for $r > \alpha$.

The paper is organized as follows. In Section 2 we derive all results of the introduction from Theorem 5. In particular, Corollaries 6–9 follow combining (5) with Theorem 5. In Section 3 we provide a reduction which allows to replace S_n by a sum of independent random vectors. The reduction is based on conditioning and random selection method (see Lemma 12 below) introduced in Bentkus and Götze (1996). In Section 4 we provide some auxiliary lemmas, which we need for the proof of Theorem 5. In Section 5 we prove the main result—Theorems 5. The proof of this Theorem is based on integration by parts (see Lemma 16 below) using specially constructed random vector fields (cf. Bentkus, Juozulynas, Paulauskas (1999)).

2. Proofs of corollaries of the main Theorem 5

In this Section from Theorem 5 we derive all other results of the introduction. The proofs are in essence elementary.

Proof of Corollaries 6–9. We shall prove only Corollary 6. The proofs of other Corollaries are similar. Recall that

$$X_* \equiv X_*(n) = n^{-1/\alpha}X, \quad \mathbf{I} \equiv \mathbf{I}(n) = \mathbf{I}\big\{\,|X_*(n)|\leqslant 1\,\big\}, \quad \mathbf{I}^c \equiv \mathbf{I}^c(n) = \mathbf{I}\big\{\,|X_*(n)| > 1\,\big\}.$$

Let us start with the proof of (7). Recall that $\alpha < 1$. Combining Theorem 5 with (5) and letting $N \to \infty$, we obtain

$$\delta_{n,\infty} \leqslant C_0 \left(Z_1 + Z_2 + Z_3 \right), \tag{18}$$

where

$$Z_1 = \sum_{j \geqslant n} \mathbf{E}\, \mathbf{I}^c(j), \quad Z_2 = \sum_{j \geqslant n} \left| \mathbf{E} X_*(j) \mathbf{I}(j) \right|, \quad Z_3 = \sum_{j \geqslant n} \mathbf{E} \left| X_*(j) \right|^2 \mathbf{I}(j).$$

Interchanging the summation and expectation, we get

$$\begin{aligned} Z_3 &= \sum_{j \geqslant n} \mathbf{E} \left| X_*(j) \right|^2 \mathbf{I}(j) \\ &= \mathbf{E} \left| X \right|^2 \sum_{j \geqslant \max\{|X|^\alpha,\, n\}} j^{-2/\alpha} \\ &\leqslant c\, \mathbf{E} \left| X \right|^2 \int_{u \geqslant \max\{|X|^\alpha,\, n\}} u^{-2/\alpha}\, du \\ &\leqslant c (2-\alpha)^{-1}\, \mathbf{E} \left| X \right|^2 \min\left\{ \left| X \right|^{-2+\alpha}, n^{1-2/\alpha} \right\} \\ &= c(2-\alpha)^{-1} n \left(\mathbf{E} \left| X_* \right|^\alpha \mathbf{I}^c + \mathbf{E} \left| X_* \right|^2 \mathbf{I} \right). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} Z_2 &\leqslant \sum_{j \geqslant n} \mathbf{E} \left| X_*(j) \mathbf{I}(j) \right| \\ &= \mathbf{E} \left| X \right| \sum_{j \geqslant \max\{|X|^\alpha,\, n\}} j^{-1/\alpha} \\ &\leqslant c(1-\alpha)^{-1}\, \mathbf{E} \left| X \right| \min\left\{ \left| X \right|^{-1+\alpha}, n^{1-1/\alpha} \right\} \\ &= c(1-\alpha)^{-1} n \left(\mathbf{E} \left| X_* \right|^\alpha \mathbf{I}^c + \mathbf{E} \left| X_* \right| \mathbf{I} \right), \end{aligned}$$

and

$$Z_1 = \mathbf{E}\, \mathbf{I}^c(n) \sum_{n \leqslant j \leqslant |X|^\alpha} 1 \leqslant c n\, \mathbf{E} \left| X_* \right|^\alpha \mathbf{I}^c.$$

Collecting the bounds for Z_1, Z_2, Z_3 and using (18), we derive (7).

It remains to prove relations (i)–(iii) of the corollary. The bound (i) is obvious. Indeed, using $\left| X_* \right| \geqslant 1$ in cases when $\mathbf{I}^c = 1$, the bound (7) implies (i) since

$$\mathbf{E} \left| X_* \right|^\alpha \mathbf{I}^c \leqslant \mathbf{E} \left| X_* \right| \mathbf{I}^c \leqslant \mathbf{E} \left| X_* \right| = n^{-1/\alpha}\, \mathbf{E} \left| X \right|.$$

Let us prove that (7) yields (ii). In order to estimate $\mathbf{E}\big|X_*\big|^\alpha \mathbf{I}^c$ and $\mathbf{E}\big|X_*\big|\mathbf{I}$, it suffices to integrate by parts. We have

$$\begin{aligned}\mathbf{E}\big|X_*\big|^\alpha \mathbf{I}^c &\leqslant c n^{-1} \int\limits_{u>n^{1/\alpha}} u^{\alpha-1-r} u^r \mathbf{P}\big\{\big|X\big|>u\big\}\,du \\ &\leqslant c n^{-1}\Lambda_r \int\limits_{u>n^{1/\alpha}} u^{\alpha-1-r}\,du \\ &\leqslant c(r-\alpha)^{-1} n^{-r/\alpha}\Lambda_r\end{aligned}$$

and

$$\mathbf{E}\big|X_*\big|\mathbf{I}\leqslant c(1-r)^{-1} n^{-r/\alpha}\Lambda_r.$$

Hence (7) yields (ii).

Let us prove (iii). Assuming that $\mathbf{E}\big|X\big|^r<\infty$ and $\alpha\leqslant r<1$, we have to show that $\delta_{n,\infty}=o\big(n^{1-r/\alpha}\big)$. By (7) it suffices to verify that $\mathbf{E}\big|X_*\big|^\alpha\mathbf{I}^c=o\big(n^{-r/\alpha}\big)$ and $\mathbf{E}\big|X_*\big|\mathbf{I}=o\big(n^{-r/\alpha}\big)$. Using $\big|X_*\big|\geqslant 1$ when $\mathbf{I}^c=1$, we have

$$\mathbf{E}\big|X_*\big|^\alpha\mathbf{I}^c\leqslant\mathbf{E}\big|X_*\big|^r\mathbf{I}^c=n^{-r/\alpha}\mathbf{E}\big|X\big|^r\mathbf{I}\big\{\big|X\big|>n^{1/\alpha}\big\}=o\big(n^{-r/\alpha}\big).$$

Furthermore, $\big|X_*\big|\mathbf{I}\leqslant 1$ and

$$\mathbf{E}\big|X_*\big|\mathbf{I}=n^{-r/\alpha}\mathbf{E}\Big(\big|X_*\big|^{1-r}\mathbf{I}\Big)\big|X\big|^r=o\big(n^{-r/\alpha}\big)$$

by the Lebesgue dominated convergence Theorem, since $\big|X_*\big|^{1-r}\mathbf{I}\leqslant 1$ and $\big|X_*\big|^{1-r}\mathbf{I}\to 0$ with probability 1. Thus, (7) implies (iii).

Proofs of Corollaries 7–9 are similar. In particular, in those proofs it is useful to notice that

$$\big|\mathbf{E}X_*\mathbf{I}\big|=\big|\mathbf{E}X_*\mathbf{I}^c\big|,$$

for $r>1$, since we assume that $\mathbf{E}X=0$, for $r>1$. □

Proof of Theorem 2. In the case $r=\alpha$, relations (iii) of Corollaries 6–8 show that $\delta_{n,\infty}=o(1)$, as $n\to\infty$. This means that $F_n=\mathcal{L}\big(S_n\big)$ converge in variation to G.

To conclude the proof of the theorem, it remains to show that $\mathcal{L}\big(X\big)$ defines ϱ by (2). For $\alpha=1$ the proof is a bit more technically involved than for $\alpha\neq 1$. Therefore for simplicity we shall consider only the case $\alpha\neq 1$. For $d=1$ the formula (2) holds by Theorem 1.4.5 in Samorodnitsky and Taqqu (1994). We are going to reduce the multidimensional case to the case $d=1$. Write $e_t=t/|t|$, for $t\in\mathbb{R}^d$ such that $t\neq 0$. We have

$$\big\langle S_n, e_t\big\rangle=\sum_{j=1}^n \Gamma^{-1/\alpha}\omega_j \tag{19}$$

with i.i.d. real random variables $\omega_j=\big\langle X_j,e_t\big\rangle$ such that $\mathbf{E}\big|\omega_j\big|^\alpha\leqslant\mathbf{E}\big|X\big|^\alpha<\infty$. By the first part of the theorem the series in (19) converges to a stable distribution on $\mathbb{R}^1$.

Hence, using $\langle S_n, t\rangle = |t|\langle S_n, e_t\rangle$ and the representation (1) for the characteristic functions of one-dimensional stable distributions, we have

$$\varphi(t) = \lim_{n\to\infty} \mathbf{E}\exp\bigl\{ i\langle S_n, t\rangle \bigr\} = \exp\{-\sigma\,|t|^\alpha(1 - i\beta\,\mathrm{tg}\,\tfrac{\pi\alpha}{2})\} \tag{20}$$

with some σ and β. The aforementioned Theorem 1.4.5 in Samorodnitsky and Taqqu (1994) shows that

$$\sigma = \tilde{c}\mathbf{E}\bigl|\langle X, e_t\rangle\bigr|^\alpha, \quad \beta = \mathbf{E}\bigl|\langle X, e_t\rangle\bigr|^\alpha \operatorname{sign}\bigl(\langle X, e_t\rangle\bigr)/\mathbf{E}\bigl|\langle X, e_t\rangle\bigr|^\alpha. \tag{21}$$

It what follows $\mathbf{E}_{\{\dots\}}$ will denote the expectation taken with respect to those random vectors which are written in parenthesis as subscript to $\mathbf{E}$. Let $\mathbf{E}\bigl(V\bigm|W\bigr)$ be the conditional expectation of V given $W = X/\bigl|X\bigr|$. Introducing the measure

$$\varrho_0\bigl(B\bigr) = \mathbf{E}\bigl|X\bigr|^\alpha \mathbf{I}\bigl\{X \neq 0,\ X/\bigl|X\bigr| \in B\bigr\},$$

we may write

$$\begin{aligned}\mathbf{E}\bigl|\langle X, e_t\rangle\bigr|^\alpha &= \mathbf{E}\bigl|\langle X, e_t\rangle\bigr|^\alpha \mathbf{I}\bigl\{\bigl|X\bigr| \neq 0\bigr\} = \mathbf{E}_W\,\mathbf{E}\Bigl(\bigl|\langle W, e_t\rangle\bigr|^\alpha \bigl|X\bigr|^\alpha \Bigm| W\Bigr)\\ &= \mathbf{E}_W\bigl|\langle W, e_t\rangle\bigr|^\alpha\, \mathbf{E}\Bigl(\bigl|X\bigr|^\alpha \Bigm| W\Bigr) = \int_{s_{d-1}} \bigl|\langle y, e_t\rangle\bigr|^\alpha \varrho_0(dy),\end{aligned} \tag{22}$$

and

$$\mathbf{E}\bigl|\langle X, e_t\rangle\bigr|^\alpha \operatorname{sign}\bigl(\langle X, e_t\rangle\bigr) = \int_{s_{d-1}} \operatorname{sign}\bigl(\langle y, e_t\rangle\bigr)\bigl|\langle y, e_t\rangle\bigr|^\alpha \varrho_0(dy). \tag{23}$$

Collecting (20)–(23) we have

$$\varphi(t) = \exp\Bigl\{-\int_{s_{d-1}} \bigl|\langle t, y\rangle\bigr|^\alpha N(y, t, \alpha)\,\tilde{c}\,\varrho_0(dy)\Bigr\}. \tag{24}$$

The formula (24) is a representation of φ of type (1). Since we assume that $\mu = 0$, the representation (1) is unique in the sense that a stable distribution defines a unique ϱ. Hence, (24) yields $\tilde{c}\varrho_0 = \varrho$, and (2) follows. □

Proof of Theorem 4. Notice that the random vector from the condition of the Theorem satisfies $\mathbf{E}\bigl|X\bigr|^2 < \infty$. Therefore the bound (i) of Corollary 6 yields (3), and the bound (i) of Corollary 7 implies (4). Finally, (2) ensures that G has the spectral measure ϱ. □

3. The main reduction

For the sake of completeness we adapt the relevant material from Bentkus, Götze, Paulauskas (1996) where the one dimensional case was considered. The proof in the multidimensional case requires only a few natural changes.

In this section we show that instead of the sum S_n, which has relatively complicated structure due to the dependence of random variables Γ_j's, we can study a simpler sum of specially chosen independent random vectors. We obtain a counterpart of S_n replacing the random vectors X_j by some non-random vectors a_j and $\Gamma_j^{-1/\alpha}$ by random variables $n^{-1/\alpha}\xi_j$, where ξ_i, $i\geqslant 1$ are i.i.d. and ξ_1 has a Pareto distribution, that is,

$$\mathbf{P}\{\xi_1 > x\} = x^{-\alpha}, \quad \text{for} \quad x\geqslant 1. \tag{25}$$

Assume that sequences

$$\{X_i,\, i\geqslant 1\}, \quad \{\xi_i,\, i\geqslant 1\} \quad \text{and} \quad \{\lambda_i,\, i\geqslant 1\}$$

are independent. Let $\mathcal{M}$ denote a class of subsets of $\mathbb{R}^d$ invariant with respect to shifts and multiplication by positive constants, that is

$$A \in \mathcal{M},\, a > 0,\, b \in \mathbb{R}^d \quad \Rightarrow \quad aA \in \mathcal{M},\, A + b \in \mathcal{M}.$$

Note that the class $\mathcal{B}$ of Borel sets satisfies the aforementioned conditions.

Write

$$Z_n = \sum_{j=1}^{n} \xi_j\, X_j.$$

Lemma 11. *We have*

$$\delta_n\big(\mathcal{M}\big)\leqslant \sup_{A\in\mathcal{M}} \big|\mathbf{P}\big\{\, Z_n \in A\big\} - \mathbf{P}\big\{\, Z_n + X \in A\big\}\big|. \tag{26}$$

Proof. Let $\xi_{n,1}\geqslant\xi_{n,2}\geqslant\ldots\geqslant\xi_{n,n}$ be the ordered values of $\xi_1,\ldots,\xi_n$. Then the random vectors

$$\mathcal{P} = (\xi_{n,1},\ldots,\xi_{n,n}) \quad \text{and} \quad \mathcal{G} = \big(\Gamma_1^{-1/\alpha}\Gamma_{n+1}^{1/\alpha},\ldots,\Gamma_n^{-1/\alpha}\Gamma_{n+1}^{1/\alpha}\big)$$

satisfy $\mathcal{L}\big(\mathcal{P}\big) = \mathcal{L}\big(\mathcal{G}\big)$ (see, for example, LePage (1981)). Therefore we can write the following distributional equalities

$$S_n = \Gamma_{n+1}^{-1/\alpha} \sum_{j=1}^{n} \Gamma_j^{-1/\alpha}\Gamma_{n+1}^{1/\alpha}\, X_j = \Gamma_{n+1}^{-1/\alpha} \sum_{j=1}^{n} \xi_{n,j}X_j = \Gamma_{n+1}^{-1/\alpha} \sum_{j=1}^{n} \xi_j X_j = \Gamma_{n+1}^{-1/\alpha}\, Z_n.$$

Proceeding similarly, using the independence assumption and $\mathcal{L}\big(X_{n+1}\big)=\mathcal{L}\big(X_n\big)$, we obtain

$$S_{n+1}=S_n+\Gamma_{n+1}^{-1/\alpha}X_{n+1}=\Gamma_{n+1}^{-1/\alpha}\sum_{j=1}^{n}\xi_jX_j+\Gamma_{n+1}^{-1/\alpha}X_{n+1}=\Gamma_{n+1}^{-1/\alpha}\big(Z_n+X\big).$$

Finally, the definition of the class $\mathcal{M}$ combined with conditioning on $\Gamma_{n+1}^{-1/\alpha}$ implies the result. $\square$.

Let $\psi_1,\psi_2,\dots$ denote i.i.d. random variables such that

$$\psi_j=\xi_j,\ \text{ for }\ \alpha\leqslant 1,\quad \text{ and }\quad \psi_j=\xi_j-\mathbf{E}\xi_j,\ \text{ for }\ \alpha>1. \tag{27}$$

Let $B_1,\dots,B_d\subset\mathbb{R}^d$ be sets as in the introduction (see the text above (10)). For non-random

$$a_1,\dots,a_m\quad \text{such that}\ \ a_{j+id}\in B_j,\quad \text{for all}\ \ 1\leqslant j\leqslant d,\ 1\leqslant j+id\leqslant m \tag{28}$$

introduce the sum

$$V_m=m^{-1/\alpha}\sum_{j=1}^{m}a_j\psi_j. \tag{29}$$

Write

$$\tilde{\delta}_m\big(\mathcal{M}\big)=\sup_{A\in\mathcal{M}}\ \sup_{a}\big|\mathbf{P}\big\{V_m\in A\big\}-\mathbf{P}\big\{V_m+m^{-1/\alpha}X\in A\big\}\big|.$$

Here and in the sequel $\sup_a$ denotes supremum over all $a_1,\dots,a_m$ satisfying (28). Let $[b]$ denote the integer part of a real number b.

Lemma 12. *We have*

$$\delta_n\big(\mathcal{M}\big)\leqslant\tilde{\delta}_m\big(\mathcal{M}\big)+\exp\{-3pn/(4d)\}, \tag{30}$$

where $m=[p[n/d]/2]$.

Proof. If $p=1$, we derive the result conditioning on $X,X_1,\dots,X_n$. Thus, it remains to prove the bound (30) assuming that $0<p<1$.

Recall that (cf. (10))

$$p=\prod_{j=1}^{d}F\big(B_j\big)$$

with $B_1,\dots,B_d\subset\mathbb{R}^d$ such that $p>0$. Introduce the i.i.d. Bernoulli random variables

$$\eta_{j+1}=\mathbf{I}\big\{X_{1+jd}\in B_1,\dots,X_{d+jd}\in B_d\big\},\quad j\geqslant 0.$$

Let $q=[n/d]$. Consider the event

$$A_q=\{\eta_1+\dots+\eta_q>pq/2\}$$

ant its complement A_q^c. Since $\mathbf{E}\eta_i = p$ and $\left|\eta_i - \mathbf{E}\eta_i\right| \leqslant 1$, a well-known exponential inequality yields

$$\mathbf{P}\{A_q^c\} \leqslant \exp\{-3qp/4\}. \tag{31}$$

In what follows $\mathbf{E}_{\{\dots\}}$ will denote the expectation taken with respect to those random vectors which are written in parenthesis as subscript to $\mathbf{E}$. Combining (26) with (31), the definition of the class $\mathcal{M}$, conditioning on $X, X_1, \dots, X_n$ and $\xi_1, \dots, \xi_n$, we have

$$\begin{aligned}
\delta_n(\mathcal{M}) &\leqslant \sup_{A\in\mathcal{M}} \left|\mathbf{E}_{\{X, X_i, \xi_i,\, 1\leqslant i\leqslant q\}} \mathbf{E}_{\{X_j, \xi_j,\, q<j\leqslant n\}} \big(\mathbf{I}\{Z_n \in A\} - \mathbf{I}\{Z_n + X \in A\}\big)\right| \\
&= \mathbf{E}_{\{X_j, \xi_j,\, q<j\leqslant n\}} \sup_{A\in\mathcal{M}} \left|\mathbf{E}_{\{X, X_i, \xi_i,\, 1\leqslant i\leqslant q\}} \big(\mathbf{I}\{Z_n \in A\} - \mathbf{I}\{Z_n + X \in A\}\big)\right| \\
&\leqslant \sup_{A\in\mathcal{M}} \left|\mathbf{E}_{\{X, X_i, \xi_i,\, 1\leqslant i\leqslant q\}} \big(\mathbf{I}\{Z_q \in A\} - \mathbf{I}\{Z_q + X \in A\}\big)\right| \\
&\leqslant \sup_{A\in\mathcal{M}} \left|\mathbf{E}_{\{X, X_i, \xi_i,\, 1\leqslant i\leqslant q\}} \big(\mathbf{I}\{Z_q \in A\} - \mathbf{I}\{Z_q + X \in A\}\big)\mathbf{I}\{A_q\}\right| + \mathbf{P}\{A_q^c\} \\
&\leqslant \mathbf{E}_{\{X_i,\, 1\leqslant i\leqslant q\}} \delta_q'(\mathcal{M}) + \exp\{-3pq/4\},
\end{aligned}$$

where

$$\delta_q'(\mathcal{M}) = \mathbf{I}\{A_q\} \sup_{A\in\mathcal{M}} \left|\mathbf{E}_{\{X, \xi_i,\, 1\leqslant i\leqslant q\}} \big(\mathbf{I}\{Z_q \in A\} - \mathbf{I}\{Z_q + X \in A\}\big)\right|.$$

Thus, in order to prove the lemma, it is sufficient to verify that

$$\delta_q'(\mathcal{M}) \leqslant \tilde{\delta}_m(\mathcal{M}), \tag{32}$$

where $m = [pq/2]$. If $\mathbf{I}\{A_q\} = 0$, the inequality (32) is obviously fulfilled. Therefore we can assume that $\mathbf{I}\{A_q\} = 1$, that is, that the event A_n occurs. Consequently, there exist indices $l_1, \dots, l_k$ such that

$$k > pq/2 \geqslant m \text{ and } X_{l_i} \in B_i, \text{ for } l_i = i + jd,\ 1\leqslant i\leqslant d,\ j\geqslant 0,\ 1\leqslant i + jd\leqslant k.$$

Notice that k and $l_1, \dots, l_k$ are random (they depend on X's) but independent of $\xi_1, \dots, \xi_q$. Define the random sets

$$M_1 = \{l_1, \dots, l_m\} \quad \text{and} \quad M_2 = \{1, \dots, dq\} \backslash M_1.$$

Introducing the notation

$$U = \sum_{j\in M_1} \xi_j X_j, \quad V = \sum_{j\in M_2} \xi_j X_j \quad \text{and} \quad \mathbf{E}_{M_i} = \mathbf{E}_{\{\xi_j,\, j\in M_i\}},\ i = 1, 2,$$

we can write

$$\begin{aligned}
\delta'_q(\mathcal{M}) &= \sup_{A\in\mathcal{M}}\big|\mathbf{E}_{M_1}\mathbf{E}_{M_2}\mathbf{E}_{\{X\}}\big(\mathbf{I}\{U\in A-V\}-\mathbf{I}\{U+X\in A-V\}\big)\big|\\
&\leqslant \mathbf{E}_{M_2}\sup_{A\in\mathcal{M}}\big|\mathbf{E}_{M_1}\mathbf{E}_{\{X\}}\big(\mathbf{I}\{U\in A-V\}-\mathbf{I}\{U+X\in A-V\}\big)\big|\\
&\leqslant \sup_{A\in\mathcal{M}}\big|\mathbf{E}_{M_1}\mathbf{E}_{\{X\}}\big(\mathbf{I}\{U\in A\}-\mathbf{I}\{U+X\in A\}\big)\big|\\
&\leqslant \sup_{A\in\mathcal{M}}\sup_{a}\big|\mathbf{E}_{\{M_1\}}\mathbf{E}_{\{X\}}\big(\mathbf{I}\{U\in A\}-\mathbf{I}\{U+X\in A\}\big)\big|\\
&= \tilde{\delta}_m(\mathcal{M}).
\end{aligned}$$

In the last step we replaced the sum U by $m^{-1/\alpha}\sum\limits_{j\in M_1}\psi_j X_j$, where random variables ψ_j are defined in (25), (27), and used the obvious equality

$$\mathbf{E}_{\{M_1\}}f\big(\psi_j,\, j\in M_1\big)=\mathbf{E}f(\psi_1,\dots,\psi_m),$$

which is valid for any measurable function f of m arguments since $\psi_1,\psi_2\dots$ are identically distributed. $\square$

4. Auxiliary lemmas.

By ξ we denote a Pareto random variable, cf. (25). Define $\psi=\xi$, for $\alpha\leqslant 1$, and $\psi=\xi-\mathbf{E}\xi$, for $\alpha>1$. Let $f(t)=\mathbf{E}\exp\{it\psi\}$.

Lemma 13. *There exist positive constants $c_1(\alpha)$, $c_2(\alpha)$ and $c_3(\alpha)<1$ such that $\exp\{-c_1(\alpha)\}\geqslant c_3(\alpha)$ and*

$$|f(t)|\leqslant \exp\{-c_1(\alpha)|t|^\alpha\},\qquad \text{for } |t|\leqslant 1, \tag{33}$$

$$|f'(t)|\leqslant c_2(\alpha)|t|^{\alpha-1},\qquad \text{for all } t\neq 0, \tag{34}$$

$$|f(t)|\leqslant c_2(\alpha)|t|^{-1},\qquad \text{for all } t\neq 0, \tag{35}$$

$$|f(t)|\leqslant c_3(\alpha),\qquad \text{for } |t|>1. \tag{36}$$

For the proof of Lemma 13 we refer to Lemma 8 in Bentkus, Götze, Paulauskas (1996). Let m be an integer. We shall use an upper bound, say h, for $|f|$ such that

$$h(t)=\begin{cases}\exp\{-c_1(\alpha)\,|t|^\alpha\}, & \text{if } |t|\leqslant 1;\\ c_3(\alpha), & \text{if } 1<|t|<2\,c_2(\alpha)/c_3(\alpha);\\ c_2(\alpha)|t|^{-1}, & \text{if } |t|\geqslant 2\,c_2(\alpha)/c_3(\alpha).\end{cases} \tag{37}$$

Choosing the constant $c_2(\alpha)\geqslant 1$ sufficiently large and introducing $h_m(t)=h\big(m^{-1/\alpha}t\big)$, we have

$$|f(m^{-1/\alpha}t)|\leqslant h_m(t), \tag{38}$$

$$h_m(t)\leqslant 1 \tag{39}$$

and

$$h_m(t_1)\leqslant h_m(t_2),\quad \text{for } |t_2|\leqslant|t_1|. \tag{40}$$

Remark 14. Let $\beta>-1$ and $2\beta+2<\gamma,\ m/(4d^2)<\gamma<4m$. Then

$$\int\limits_{\mathbb{R}}|t|^{\beta}\,h_m^{\gamma}(t)\,dt\leqslant c(\alpha,\beta)<\infty, \tag{41}$$

and, for $\gamma>\beta\geqslant 0$, we have

$$|t|^{\beta}\,h_m^{\gamma}(t)\leqslant c(\alpha,\beta). \tag{42}$$

Remark 14 can be proved by a direct calculation using the definition of the function h_m.

Lemma 15. *Assume that $m>4d^2$ and $m/(4d^2)<\gamma<4m$. Let $C>0$ and let a vector $b\in\mathbb{R}^d$ satisfy $|b|\leqslant C$. Let $a_{1+id},\dots,a_{d+id}$ be a subset of the set of vectors $a_1,\dots,a_m$ defined by (28). Then there exist positive constants $c_5=c_5\big(\alpha,\mathcal{Q}\big)$ and $c_6=c_6\big(C,\mathcal{Q}\big)$ such that the inequality*

$$(1+|\langle t,w\rangle|^{\tau})\prod_{j=1}^{d}h_m^{2\gamma}\big(\langle a_{j+id},t\rangle\big)\leqslant c_5^{\tau}\,h_m^{(2-\tau)\gamma}\big(c_6\,\langle b,t\rangle\big), \tag{43}$$

holds with $\tau=0$ and $\tau=1$, for all $w,t\in\mathbb{R}^d$ such that $|w|=1$. Moreover, we may choose

$$c_5=c(\alpha)\,(1+\tilde a^{-1}\,a_*^{-1}),\quad c_6=C^{-1}\,\tilde a\,a_*.$$

Proof. We shall prove (43) with $\tau=1$ only. The bound (43) trivially holds if $t=0$. Let $t\neq 0$. The definition (2) of $\tilde a$, property $\tilde a>0$ and definition (28) of vectors a_j show that at least one of the following inequalities is satisfied

$$|\langle a_{j+id},t\rangle|\geqslant c_7\,|\langle b,t\rangle|/\,|b|,\quad 1\leqslant j\leqslant d, \tag{44}$$

where $c_7=\tilde a\,a_*$. In particular, for $w\in\mathbb{R}^d$ such that $|w|=1$, we have

$$|\langle t,w\rangle|\leqslant c_7^{-1}\,|\langle a_{j+id},t\rangle|, \tag{45}$$

for some j such that $1\leqslant j\leqslant d$. Combining (45),(42) and using (39), we get

$$(1+|\langle t,w\rangle|)\prod_{j=1}^{d}h_m^{2\gamma}\big(\langle a_{j+id},t\rangle\big)\leqslant c_5\prod_{j=1}^{d}h_m^{\gamma}\big(\langle a_{j+id},t\rangle\big). \tag{46}$$

Consider the sets

$$D_k = \{t \in \mathbb{R}^d : |\langle a_{k+id}, t\rangle| \geqslant c_6\, |\langle b, t\rangle|\},$$

for $1 \leqslant k \leqslant d$. We have $D_1 \cup \dots \cup D_d = \mathbb{R}^d$ since at least one of inequalities (44) is fulfilled. The sets

$$E_k = D_k \setminus \big(D_1 \cup \dots \cup D_{k-1}\big),$$

for $1 \leqslant k \leqslant d$, are disjoint and $E_1 \cup \dots \cup E_d = \mathbb{R}^d$. Let $t \in \mathbb{R}^d$. Then $t \in E_k$, for some k, and therefore $|\langle a_{k+id}, t\rangle| \geqslant c_6\, |\langle b, t\rangle|$. Hence, using (40) and (39), we obtain

$$\prod_{j=1}^{d} h_m\big(\langle a_{j+id}, t\rangle\big) \leqslant h_m\big(c_6 \langle b, t\rangle\big) \prod_{j=1,\, j\neq k}^{d} h_m\big(\langle a_{j+id}, t\rangle\big) \leqslant h_m\big(c_6 \langle b, t\rangle\big), \tag{47}$$

for all $t \in \mathbb{R}^d$. Combining (46) and (47), we obtain (43). $\square$

5. Proof of Theorem 5.

Let $w, w_1, \dots, w_s$ be vectors in $\mathbb{R}^d$. The directional derivatives $d_{w_1} \dots d_{w_s} p(x)$ we may define iteratively starting with $d_w p(x) := \lim_{h\to 0} \big(p(x+hw) - p(x)\big)/h$.

To prove Theorem 5 we shall use the following Lemma 16. Its proof is rather complicated and therefore it will be given later.

Lemma 16. *The distribution of the random vector V_m defined by (29) has a bounded density, say p_m, provided that $m \geqslant d$.*

If $m \geqslant 6\,d\,s$, then there exist bounded continuous derivatives $d_{w_1} \dots d_{w_s} p_m(x)$, for all $w_1, \dots, w_s \in \mathbb{R}^d$. The derivatives are square integrable as functions of x and vanish as $|x| \to \infty$.

Furthermore, the derivatives $d_{w_1} \dots d_{w_s} p_m(x)$ are integrable as functions of x for sufficiently large $s = s(m, d)$, and the quantities

$$K_w := \sup_{m \geqslant 50\, d^2} \sup_a \int_{\mathbb{R}^d} |d_w p_m(x)|\, dx, \quad K_{w_1,w_2} := \sup_{m \geqslant 50\, d^2} \sup_a \int_{\mathbb{R}^d} |d_{w_1} d_{w_2} p_m(x)|\, dx$$

satisfy

$$K_{w_1,w_2} \leqslant c\, K_{w_1} K_{w_2}, \tag{48}$$

$$K := \sup_{|w|=1} K_w \leqslant c(\alpha, d)\, C \tag{49}$$

with

$$C = (a_* \tilde{a})^{(1-2\alpha)d/2} \big(1 + (a^*)^{\alpha d}\big) \big(1 + (a_* \tilde{a})^{-1-d/2}\big).$$

Remark 17. The random vector $U_m = \sum\limits_{j=1}^{m} \xi_j\, a_j$ satisfies $U_m = V_m + b$, where b is a non-random shift such that $b = 0$, for $0 < \alpha \leqslant 1$, and $b = \sum\limits_{j=1}^{m} \mathbf{E}\xi_j\, a_j$, for $1 < \alpha < 2$. All assertions of Lemma 16 are independent of shifts of V_m. Hence, Lemma 16 remains true when we replace everywhere V_m by U_m.

Proof of Theorem 5. We have to prove the bound (6) for δ_n. Recall that

$$X_* = n^{-1/\alpha} X, \quad \mathbf{I} = \mathbf{I}\big\{\,|\,X_*\,|\leqslant 1\,\big\}, \quad \mathbf{I}^c = \mathbf{I}\big\{\,|\,X_*\,| > 1\,\big\}.$$

Notice that the definitions (10)–(11) of p and a_* yield $\mathbf{P}\big\{\,|\,X\,|\geqslant a_*\,\big\}\geqslant p$, $0 < p\leqslant 1$. Without loss of generality we may assume that $n\geqslant 102\, d^3/p$. Indeed, otherwise (6) is implied by the trivial bound $\delta_n\leqslant 1$ and

$$\begin{aligned}
&p^{-1-2/\alpha}(1 + a_*^{-2})\,\Big(\mathbf{E}\,\mathbf{I}^c + \mathbf{E}\big|\,X_*\,\big|^2\mathbf{I}\Big)\\
&\geqslant p^{-1-2/\alpha}(1 + a_*^{-2})\,\Big(\mathbf{P}\big\{\,|\,X\,| > n^{1/\alpha}\,\big\} + n^{-2/\alpha}\, a_*^2\, \mathbf{P}\big\{\, a_*\leqslant|\,X\,|\leqslant n^{1/\alpha}\,\big\}\Big)\\
&\geqslant p^{-2/\alpha} n^{-2/\alpha}\geqslant (102\, d^3)^{-2/\alpha},
\end{aligned}$$

for $a_*^\alpha < n < (102\, d^3/p)^\alpha$, and

$$p^{-1-2/\alpha}(1 + a_*^{-2})\,\Big(\mathbf{E}\,\mathbf{I}^c + \mathbf{E}\big|\,X_*\,\big|^2\mathbf{I}\Big)\geqslant 1,$$

for $n\leqslant a_*^\alpha$. Since $n\geqslant 102\, d^3/p$, and $m = [p[n/d]/2]$, it follows that proving the bound (6) we may assume that $m\geqslant 50\, d^2$.

Write

$$J = \big|\,\mathbf{P}\big\{\,V_m \in A\,\big\} - \mathbf{P}\big\{\,V_m + m^{-1/\alpha} X \in A\,\big\}\,\big|.$$

It follows from inequality (30) that it suffices to estimate the quantities

$$\tilde{\delta}_m\big(\mathcal{M}\big) = \sup_{A\in\mathcal{M}}\ \sup_{a}\ J \qquad \text{and} \qquad \exp\{-3m/2\}. \tag{50}$$

Introduce the truncated random vector $Y = m^{1/\alpha}\, X\mathbf{I}\big\{\,|\,X\,|\leqslant n^{1/\alpha}\,\big\}$. Then

$$J\leqslant J_t + \mathbf{P}\big\{\,|\,X\,| > n^{1/\alpha}\,\big\}, \tag{51}$$

where

$$J_t = \big|\,\mathbf{P}\big\{\,V_m \in A\,\big\} - \mathbf{P}\big\{\,V_m + Y \in A\,\big\}\,\big|.$$

Let us estimate J_t. Let θ be random variable with the density $p_\theta(x) = 3\,(1 - x)^2$, where $x \in [0, 1]$, independent of all other random variables. By Taylor's formula we have $p(x + y) = p(x) + d_y p(x) + d_y d_y p(x + \theta y)$, for any sufficiently smooth function p

and $x, y \in \mathbb{R}^d$. Hence, writing $e = \mathbf{E}Y/|\mathbf{E}Y|$, $w = Y/|Y|$ and using (48), (49) of Lemma 16, we have

$$\begin{aligned} J_t &= \Big|\mathbf{E}\int_A \big(p_m(x) - p_m\big(x - Y\big)\big)\,dx\Big| \\ &= \Big|\mathbf{E}\int_A \big(d_Y p_m(x) + 12^{-1}\mathbf{E}_\theta\, d_Y d_Y p_m\big(x - \theta Y\big)\big)dx\Big| \\ &\leqslant \big|\mathbf{E}Y\big|\Big|\int_A d_e p_m(x)dx\Big| + \mathbf{E}_\theta\,\mathbf{E}\big|Y\big|^2\Big|\int_A d_w d_w p_m\big(x - \theta Y\big)dx\Big| \\ &\leqslant c(d)\big(K\big|\mathbf{E}Y\big| + K^2\,\mathbf{E}\big|Y\big|^2\big). \end{aligned} \tag{52}$$

Let us estimate $\exp\{-3m/2\}$. Recall that $\mathbf{P}\{|X|\geqslant a_*\}\geqslant p$. Hence, we obtain

$$\begin{aligned} &\mathbf{P}\{|X| > n^{1/\alpha}\} + a_*^{-2}\,\mathbf{E}|Y|^2 \\ &\geqslant \mathbf{P}\{|X| > n^{1/\alpha}\} + m^{-2/\alpha}\,\mathbf{P}\{a_*\leqslant|X|\leqslant n^{1/\alpha}\} \\ &\geqslant p\,m^{-2/\alpha}\geqslant c(\alpha)\,p\,\exp\{-3m/2\}. \end{aligned} \tag{53}$$

Using (50)–(53), the inequality (30) implies

$$\delta_n \leqslant c(d)\,p^{-1-2/\alpha}\big(K\big|\mathbf{E}X_*\mathbf{I}\big| + \big(K^2 + a_*^{-2}\big)\mathbf{E}\big|X_*\big|^2\mathbf{I} + \mathbf{E}\mathbf{I}^c\big). \tag{54}$$

Thus, the inequality (54) and the bound (49) for K yield (6). $\square$

Proof of Lemma 16. We define an indicator function $\mathbf{I}\big(x; A\big)$ of a subset $A \subset \mathbb{R}^d$ as $\mathbf{I}\big(x; A\big) = 1$ if $x \in A$, and $\mathbf{I}\big(x; A\big) = 0$ otherwise. By $\widehat{f}$ we denote the Fourier transform of a function f.

Split V_m into a sum of two independent vectors,

$$V_m = Z_1 + Z_2, \text{ where } Z_1 = m^{-1/\alpha}\sum_{j=1}^{k} a_j\psi_j \text{ and } Z_2 = V_m - Z_1,$$

where $1\leqslant k\leqslant m$. Vectors a_j are defined by (28) and random variables ψ_j are defined in (27). Let z_1 and z_2 denote the densities (if they exist) of Z_1 and Z_2, respectively.

Let us start with a proof that the density p_m exist and is bounded for $m\geqslant d$. Let $k = d$. In order to simplify notation, assume for a while that $0 < \alpha < 1$. Then we have $V_m = m^{-1/\alpha}\sum_{j=1}^{m} a_j\xi_j$ with random variables ξ_j defined by (25). Introduce the matrix

$$\mathbb{A} := (m^{-1/\alpha}a_{j,i})_{i,j=1,\dots,d}, \qquad a_i = (a_{i,1}, \dots, a_{i,d}).$$

Note that the definition (28) of a_j implies that $\det \mathbb{A} \neq 0$. Let $\mathbb{A}_j(x)$ denote the matrix obtained from the matrix $\mathbb{A}$ replacing the j-th column by $x_1, \dots, x_d$. Then we may write the distribution function of Z_1 as

$$\mathbf{P}\big\{ Z_1 < x \big\} = \int_{\mathbb{R}^d} \mathbf{I}\big\{ y\mathbb{A} < x \big\} \alpha^d \prod_{i=1}^{d} y_i^{-\alpha-1}\, \mathbf{I}\big\{ y_i;\, (1,\infty) \big\}\, dy_i \tag{55}$$

($x < y$ for vectors $x, y \in \mathbb{R}^d$ means that the inequality is fulfilled coordinatewise). Changing in (55) variables $u = \mathbb{A}y$ and differentiating with respect to $x_1, \dots, x_d$, we get

$$\begin{aligned} z_1(x) = \alpha^d \big|\det \mathbb{A}\big|^{-1} \prod_{i=1}^{d} \Big(\big(\det \mathbb{A}^{-1} \, \det \mathbb{A}_i(x) \big)^{-\alpha-1} \\ \mathbf{I}\big\{ \det \mathbb{A}^{-1} \, \det \mathbb{A}_i(x);\, (1,\infty) \big\} \Big). \end{aligned}$$

Note that z_1 is a bounded function. Therefore, the convolution $p_m = z_1 * z_2$ is bounded as well.

Let us prove that the density p_m has other properties mentioned in Lemma 16, except the bound (49) for K_w. Let $k = 6\,dk_1 + k_2,\ k_1 \geqslant 1$ and $0 \leqslant k_2 < d$. Using the well-known properties of the Fourier transform, applying the geometric-arithmetic mean inequality and in the last step changing variables $t = u$ such that $u_i = \big\langle t, a_{ld+i} \big\rangle$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |\widehat{z}_1(t)|\, dt &\leqslant \int_{\mathbb{R}^d} \prod_{j=1}^{6dk_1} \big| f\big(m^{-1/\alpha} \langle a_j, t \rangle \big) \big|\, dt \\ &\leqslant (6\,k_1)^{-1} \sum_{l=0}^{k_1-1} \int_{\mathbb{R}^d} \prod_{j=ld+1}^{ld+d} \big| f\big(m^{-1/\alpha} \langle a_j, t \rangle \big) \big|^{6k_1}\, dt \\ &\leqslant c(\alpha, m, a_1, \dots, a_m) \Big(\int_{\mathbb{R}} \big| f(m^{-1/\alpha}\, u) \big|^{6k_1}\, du \Big)^d. \end{aligned} \tag{56}$$

The constant $c(\alpha, m, a_1, \dots, a_m)$ is finite, since by our choice (28) of $a_1, \dots, a_m$ all k_1 changes of variables in (56) have Jacobians with non-zero determinants. The bound (41) in Remark 14 holds without the assumption $m/(4d^2) < \gamma < 4m$ with a constant c depending on m. Combining (38) and (41), we obtain $\widehat{z}_1 \in L_1\big(\mathbb{R}^d\big)$. Thus, the inverse Fourier transform exists and is equal to z_1. In the same way we get

$$\big\langle t, w \big\rangle\, \widehat{z}_1 \in L_1\big(\mathbb{R}^d\big) \cap L_2\big(\mathbb{R}^d\big).$$

Therefore, the bounded continuous derivative $d_w z_1(x)$ exists and vanishes as $|x| \to \infty$. Using Parseval's equality

$$\int_{\mathbb{R}^d} \big| \langle t, w \rangle \widehat{z}_1(t) \big|^2\, dt = \int_{\mathbb{R}^d} |d_w z_1(x)|^2\, dx,$$

we obtain $d_w z_1 \in L_2(\mathbb{R}^d)$. Assume for a while that the bound (49) for K_w is fulfilled. Then $d_w z_1 \in L_1(\mathbb{R}^d)$, and the independence of Z_1 and Z_2 implies

$$|d_{w_1} d_{w_2} p_m| = |d_{w_1} z_1 * d_{w_2} z_2| \leqslant |d_{w_1} z_1| * |d_{w_2} z_2|,$$

where $*$ denotes the convolution of functions. Thus, integrating and using Fubini's Theorem, we obtain $K_{w_1,w_2} \leqslant c(\alpha)\, K_{w_1}\, K_{w_2}$ and easily derive all other assertions of the lemma.

Therefore, it remains to prove the bound (49) for K_w. For this we use techniques from Bentkus, Juozulynas, Paulauskas (1999). First, we describe the idea of the proof, which is based on integration by parts, repeating this integration d times. We construct a collection of vectors, say $b_1, \dots, b_d$. Each of the vectors b_j depends on the preceeding ones $b_1, \dots, b_{j-1}$ and on some other variables related to the construction. We may write

$$\widehat{\mathbf{I}}\big(t;\, A\big) = P(\partial)\, \widehat{\mathbf{I}}_*\big(t;\, A\big)$$

where

$$\mathbf{I}_*\big(x;\, A\big) := \mathbf{I}\big(x;\, A\big) \prod_{i=1}^{d} \big(1 + \big|\big\langle x, b_i \big\rangle\big|\big)^{-1},$$

$$P(\partial) := (1 + i\, d_{b_1}) \dots (1 + i\, d_{b_d}).$$

We remind that $\widehat{\mathbf{I}}(t;\, A)$ denotes the Fourier transform of the function $x \mapsto \mathbf{I}(x;\, A)$. The function $\mathbf{I}_*$ is integrable and $\big|\widehat{\mathbf{I}}_*(t)\big| \leqslant c_d$. Let ϑ_m denote the characteristic function of the random vector V_m and $f_j(t) := f\big(m^{-1/\alpha} \big\langle a_j,\, t \big\rangle\big)$. Note that $\vartheta_m(t) = \prod_{j=1}^{m} f_j(t)$. Integrating by parts we see that the estimate (49) will be proved if we can show that

$$\int_{\mathbb{R}^d} \big|P(\partial)\, \vartheta_m(t)\big|\, dt \leqslant c_d.$$

We choose the vectors b_j in such a way that $\mathbf{I}_* \in L_p\big(\mathbb{R}^d\big)$, for some $p > 1$. Furthermore, we have to construct the operator $P(\partial)$ such that the expression $P(\partial)\, \vartheta_m(t)$ depends only on the first order derivatives of $f_j(t)$.

Now we return to the proof of (49). It is clear that

$$\int_{\mathbb{R}^d} |d_w p_m(x)|\, dx = \sup_A \int_A |d_w p_m(x)|\, dx \leqslant 2 \sup_A \Big| \int_A d_w p_m(x)\, dx \Big| = 2\, J_* \tag{57}$$

with

$$J_* = \sup_A \Big| \int_{\mathbb{R}^d} \mathbf{I}(x;\, A)\, d_w p_m(x)\, dx \Big|,$$

where $\sup_A$ is taken over all bounded sets $A \subset \mathbb{R}^d$. By $\overline{z}$ we denote the complex conjugate of z. The Parseval's equality and the well-known properties of the Fourier transform imply

$$J_* = (2\pi)^d \sup_A \Big| \int_{\mathbb{R}^d} \overline{\widehat{\mathbf{I}}}(t;\, A)\, \big\langle t, w \big\rangle\, \vartheta_m(t)\, dt \Big|.$$

Introduce the class $D \subset C^{\infty}(\mathbb{R}^d)$ of functions v which satisfy

$$|d_{w_1} \dots d_{w_s} v| \leqslant 1 + |t|, \text{ for all } s = 0, 1, \dots \text{ and } |w_i| \leqslant 1.$$

Assuming that a set $A \subset \mathbb{R}^d$ is bounded, define

$$J_0(A) := \sup_{v \in D} \Big| \int_{\mathbb{R}^d} \widehat{\widehat{\mathbf{I}}}(t;\, A)\, v(t)\, \vartheta_m(t)\, dt \Big|. \tag{58}$$

Obviously, the function $t \mapsto \langle t, w \rangle$ belongs to the class D since $|w| = 1$. Hence,

$$J_* \leqslant (2\pi)^d \sup_{A} J_0(A). \tag{59}$$

Let us estimate the function $J_0(A)$. We integrate by parts d times. First, we explain the first integration by parts. Choose any vector $b_1 \in R^d$ such that $|b_1| = 1$. Splitting the set $A = A_1 \cup A_2$, where

$$A_1 = \{u :\ \langle b_1, u \rangle \geqslant 0\} \cap A \quad \text{and} \quad A_2 = \{u :\ \langle b_1, u \rangle < 0\} \cap A,$$

we can write the following obvious identity

$$\mathbf{I}(x;\, A) = \mathbf{I}_1(x;\, A_1)\big(1 + \langle b_1, x \rangle\big) + \mathbf{I}_1(x;\, A_2)\big(1 - \langle b_1, x \rangle\big), \tag{60}$$

where

$$\mathbf{I}_1(x;\, A_1) = \frac{\mathbf{I}(x;\, A_1)}{1 + |\langle b_1, x \rangle|}, \qquad \mathbf{I}_1(x;\, A_2) = \frac{\mathbf{I}(x;\, A_2)}{1 + |\langle b_1, x \rangle|}.$$

Using the well-known properties of the Fourier transform, the identity (60) yields

$$\widehat{\mathbf{I}}(t;\, A) = (1 - i d_{b_1})\, \widehat{\mathbf{I}}(t;\, A_1) + (1 + i d_{b_1})\, \widehat{\mathbf{I}}(t;\, A_2). \tag{61}$$

Using (58) and (61), we obtain

$$J_0(A) \leqslant 2 \max_{A_* = A_1,\, A_2} \sup_{v \in D} (Q_1 + Q_2), \tag{62}$$

where

$$Q_1 = \Big| \int_{\mathbb{R}^d} \widehat{\widehat{\mathbf{I}}}_1(t;\, A_*)\, v(t)\, \vartheta_m(t)\, dt \Big|, \qquad Q_2 = \Big| \int_{\mathbb{R}^d} d_{b_1} \widehat{\widehat{\mathbf{I}}}_1(t;\, A_*)\, v(t)\, \vartheta_m(t)\, dt \Big|.$$

Denote $\tilde{f}_j(t) = f'\big(m^{-1/\alpha} \langle a_j, t \rangle\big)$. Integrating by parts, using $|b_1| = 1$ and $d_{b_1} v \in D$ together with

$$d_{b_1} \vartheta_m(t) = m^{-1/\alpha} \sum_{j=1}^{m} \langle a_j, b_1 \rangle\, \tilde{f}_j(t) \prod_{i=1, i \neq j}^{m} f_i(t),$$

we have

$$Q_2 \leqslant m^{1-1/\alpha} \max_{1\leqslant j\leqslant m} \Big| \int_{\mathbb{R}^d} \hat{\bar{\bar{\mathbf{I}}}}_1\big(t;\, A_*\big)\, v(t) \big\langle a_j, b_1 \big\rangle\, \tilde{f}_j(t) \prod_{i=1, i\neq j}^{m} f_i(t)\, dt \Big| + \sup_{v\in D} Q_1. \tag{63}$$

In order to write the bounds (62) and (63) in a more compact form and to proceed with integration by parts, let us introduce additional notation. We shall denote by $\varOmega \subset \varOmega_j$ subsets of the set $\varOmega_j = \{1, 2, \dots, j\}$ and $\Upsilon \subset M$ subsets of the set $M = \{1, 2, \dots, m\}$. Combining (62), (63) and using $\big\langle b_1, b_1\big\rangle = 1$, we obtain

$$J_0\big(A\big) \leqslant 6\, J_1\big(A\big), \tag{64}$$

where

$$J_1\big(A\big) = \max_{\varOmega\subset\varOmega_1} \max_{\Upsilon\subset M,\ |\Upsilon|=|\varOmega|} \sup_{v\in D} \sup_{A_*\subset A} \big|J_1^*\big(A_*\big)\big|$$

with

$$J_1^*\big(A_*\big) = m^{(1-1/\alpha)|\varOmega|} \int_{\mathbb{R}^d} \hat{\bar{\bar{\mathbf{I}}}}_1\big(t;\, A_*\big)\, v(t)\, \gamma^{(1)}\, \beta^{(1)}\, \theta^{(1)}\, dt$$

and

$$\gamma^{(1)} = \prod_{i\in\varOmega_1\setminus\varOmega} \big\langle b_i, b_i\big\rangle, \quad \beta^{(1)} = \prod_{i\in\varOmega,\ \{l_i\colon i\in\varOmega\}=\Upsilon} \big\langle a_{l_i}, b_i\big\rangle\, \tilde{f}_{l_i}(t), \quad \theta^{(1)} = \prod_{k\in M\setminus\Upsilon} f_k(t)$$

(notice that $\gamma^{(1)} = 1$). The inequality (64) provides the result of the first integration by parts. Assume for a while that we have $b_1, \dots, b_j$ already chosen. Write

$$J_j\big(A\big) = \max_{\varOmega\subset\varOmega_j} \max_{\Upsilon\subset M,\ |\Upsilon|=|\varOmega|} \sup_{v\in D} \sup_{A_*\subset A} \big|J_j^*\big(A_*\big)\big|, \tag{65}$$

where

$$J_j^*\big(A_*\big) = m^{(1-1/\alpha)|\varOmega|} \int_{\mathbb{R}^d} \hat{\bar{\bar{\mathbf{I}}}}_j\big(t;\, A_*\big)\, v(t)\, \gamma^{(j)}\, \beta^{(j)}\, \theta^{(j)}\, dt \tag{66}$$

with

$$\mathbf{I}_j\big(x;\, A_*\big) = \mathbf{I}\big(x;\, A_*\big) \prod_{i=1}^{j} \big(1 + \big|\big\langle x, b_i\big\rangle\big|\big)^{-1} \tag{67}$$

and

$$\gamma^{(j)} = \prod_{i\in\varOmega_j\setminus\varOmega} \big\langle b_i, b_i\big\rangle, \tag{68}$$

$$\beta^{(j)} = \prod_{i\in\varOmega,\ \{l_i\colon i\in\varOmega\}=\Upsilon} \big\langle a_{l_i}, b_i\big\rangle\, \tilde{f}_{l_i}(t), \tag{69}$$

$$\theta^{(j)} = \prod_{k\in M\setminus\Upsilon} f_k(t). \tag{70}$$

Provided that $b_1, \ldots, b_j$ are chosen in an appropriate way, the integrals $J_j\big(A\big)$ defined by (65) satisfy

$$J_{j-1}\big(A\big) \leqslant 6\, J_j\big(A\big), \text{ for all } j = 1, \ldots, d. \tag{71}$$

To ensure that (71) holds we have to choose $b_1, \ldots, b_j$ in a special way. For $j = 1$ we choose b_1 as in (64). Assume that we have $b_1, \ldots, b_{j-1}$ already chosen, and that we have arrived to J_{j-1} with J^*_{j-1} of type (66) with $\mathbf{I}_{j-1}$, $\gamma^{(j-1)}$, $\beta^{(j-1)}$, $\theta^{(j-1)}$ defined by (67), (68), (69), (70), respectively. Then we choose b_j such that b_j, $|b_j| = 1$, is orthogonal to all b_i with $i \in \Omega_{j-1} \setminus \Omega$ and to all a_{l_i} with $i \in \Omega$ and $l_i \in \Upsilon$. Then

$$d_{b_j}\, \beta^{(j-1)}(t) = 0.$$

Hence, in order to prove (71) we can estimate $J_{j-1}\big(A\big)$ proceeding similarly as in (60)–(63), which led to the bound (64) for $J_0\big(A\big)$.

In particular, the bounds (71) yield

$$J_0\big(A\big) \leqslant 6^d\, J_d\big(A\big), \tag{72}$$

where $J_d\big(A\big)$ is defined by (65) with $j = d$ and

$$J^*_d\big(A_*\big) = m^{(1-1/\alpha)|\Omega|} \int_{\mathbb{R}^d} \bar{\bar{\mathbf{I}}}_d\big(t;\, A_*\big)\, v(t)\, \gamma^{(d)}\ \beta^{(d)}\ \theta^{(d)}\, dt. \tag{73}$$

We shall estimate $J^*_d\big(A_*\big)$. Write

$$\nu = \prod_{i \in \Omega,\, \{l_i :\, i \in \Omega\} = \Upsilon} \big\langle a_{l_i}, b_i \big\rangle \tag{74}$$

and

$$\chi = \prod_{i \in \Omega,\, \{l_i :\, i \in \Omega\} = \Upsilon} |\big\langle a_{l_i}, t \big\rangle|^{\alpha - 1}. \tag{75}$$

Using $|b_i| = 1$, the inequality

$$|f'(m^{-1/\alpha}\big\langle a, t \big\rangle)| \leqslant c_2(\alpha)\, |m^{-1/\alpha}\big\langle a, t\big\rangle|^{\alpha-1}$$

(see (34)) and (38), we obtain

$$\gamma^{(d)} = 1, \tag{76}$$

$$|\beta^{(d)}| \leqslant m^{-(1-1/\alpha)|\Omega|}\, |\nu|\, \chi \tag{77}$$

and

$$|\theta^{(d)}| \leqslant \prod_{k \in M \setminus \Upsilon} \big| h_m\big(\big\langle a_k, t\big\rangle\big)\big|. \tag{78}$$

If $\nu = 0$ then $J^*_d\big(B\big) = 0$ (cf. (73) and (77)). Hence, without loss of generality in the proof we may assume that $\nu \neq 0$. Let $\|f\|_p$ stand for the $L_p\big(\mathbb{R}^d\big)$ norm of a function f. Using Hölder's inequality with $1/p+1/q=1$ such that $1<p\leqslant 2,\ q\geqslant 2$, the inequality (73) with (74)–(77) yield

$$\big|\,J_d^*\big(A_*\big)\,\big| \leqslant\, c(\alpha)\,|\nu|\,\|\widehat{\mathbf{I}}_d\big(\,\cdot\,;A_*\big)\|_q\,\|\eta\|_p\,, \tag{79}$$

where

$$\eta(t) =\, |v(t)|\,\chi\,|\theta^{(d)}|.$$

To estimate $\|\widehat{\mathbf{I}}_d\big(\,\cdot\,;A_*\big)\|_q$ we shall use the fact that the Fourier transform is a bounded operator from $L_p\big(\mathbb{R}^d\big)$ to $L_q\big(\mathbb{R}^d\big)$, $1<p\leqslant 2$. More precisely, we have $\|\widehat{f}\|_q\leqslant\|f\|_p$ (see the classical Riesz-Thorin interpolation theorem in Triebel (1978)), whence

$$\|\widehat{\mathbf{I}}_d\big(\,\cdot\,;A_*\big)\|_q\leqslant\|\mathbf{I}_d\big(\,\cdot\,;A_*\big)\|_p\,.$$

Introducing the matrix

$$\mathbb{B}{:}=(b_{j,i})_{i,j=1,\dots,d},\quad b_j=(b_{j,1},\dots,b_{j,d}), \tag{80}$$

and changing the variables $u=\mathbb{B}\,t$, we obtain

$$\|\widehat{\mathbf{I}}_d\big(\,\cdot\,;A_*\big)\|_q\leqslant\|\mathbf{I}_d\big(\,\cdot\,;A_*\big)\|_p\leqslant\big|\det\mathbb{B}\big|^{-1/p}\,c_8 \tag{81}$$

with

$$c_8=\Big(\int_{\mathbb{R}}(1+|u|)^{-p}du\Big)^{d/p}=\big(2/(p-1)\big)^{d/p}<\,\infty.$$

Let us estimate $\|\eta\|_p$. The function v belongs to the class D. Therefore

$$|v|\leqslant 1+|t|\leqslant d\sup_{|w|=1}\big(1+|\langle t,w\rangle|\big).$$

Applying the arithmetic-geometric mean inequality, (39), (75) and (78) we have

$$\|\eta\|_p^p\leqslant c\sup_{|w|=1}\ \sup_{\{j:\,(j+1)d\leqslant m\}}\int_{\mathbb{R}^d}\big(1+|\langle t,w\rangle|\big)^p\Big(\prod_{l\in\Upsilon}|\langle a_l,t\rangle|^{p(\alpha-1)}\Big)$$
$$\Big(\prod_{1\leqslant i\leqslant d}h_m^{2\gamma}\big(\langle a_{i+jd},t\rangle\big)\Big)^d\,dt$$

with $\gamma=p[(m-d^2)/(2\,d^2)]$, whence, using d times the inequality (43) ($d-1$ times with $\tau=0$ and once with $\tau=1$), we obtain

$$\|\eta\|_p^p\leqslant c(\alpha)\,c_5^p\,c_6^{-p(\alpha-1)|\Omega|}\int_{\mathbb{R}^d}\Big(\prod_{l\in\Upsilon}|c_6\,\langle a_l,t\rangle|^{p(\alpha-1)}\,h_m^\gamma\big(c_6\,\langle a_l,t\rangle\big)\Big)$$
$$\Big(\prod_{k\in\Omega_d\setminus\Omega}h_m^\gamma\big(c_9\,\langle b_k,t\rangle\big)\Big)\,dt, \tag{82}$$

where $c_6 = c_6\big(a^*, \mathcal{Q}\big)$, $c_9 = c_6\big(1, \mathcal{Q}\big)$ and c_5, c_6 are defined in Lemma 15. Introduce the matrix

$$\mathbb{A} := \big(a_{l,i} \,|\, b_{k,i}\big)_{k\in\Omega_d\setminus\Omega}^{l\in\Upsilon} \tag{83}$$

such that the first $\big|\Upsilon\big|$ rows have the form $a_{l,1},\dots,a_{l,d}$ and and the remaining $\big|\Omega_d\setminus\Omega\big|$ rows have the form $b_{k,1},\dots,b_{k,d}$. Changing in (82) variables t into $u=(u_1,\dots,u_d)$ so that

$$u_l = c_6\,\big\langle a_l, t\big\rangle, \quad \text{for } l\in\Upsilon,$$

$$u_k = c_9\,\big\langle b_k, t\big\rangle, \quad \text{for } k\in\Omega_d\setminus\Omega,$$

we get (note that $a^\beta\, b^{\beta_1} \leqslant a+b$, for nonnegative β and β_1 such that $\beta+\beta_1=1$)

$$\begin{aligned}\|\eta\|_p &\leqslant \big|\det\mathbb{A}\big|^{-1/p} c(\alpha)\, c_{10}^{|\Omega|/p}\, c_{11}^{(d-|\Omega|)/p}\, c_5\, c_6^{-(\alpha-1+1/p)|\Omega|}\, c_9^{-(d-|\Omega|)/p},\\ &\leqslant \big|\det\mathbb{A}\big|^{-1/p} c(\alpha)\,\big(c_{10}^{d/p}+c_{11}^{d/p}\big)\, c_5\, c_6^{-(\alpha-1+1/p)|\Omega|}\, c_9^{-(d-|\Omega|)/p},\end{aligned} \tag{84}$$

where

$$c_{10} = \int_{\mathbb{R}} |u|^{p(\alpha-1)}\, h_m^\gamma(u)\,du\,, \qquad c_{11} = \int_{\mathbb{R}} h_m^\gamma(u)du.$$

The constant c_{10} is finite if $p(\alpha-1)>-1$, and the inequality (84) will be satisfied if we choose, for example,

$$p=2,\ \text{if}\,\alpha\geqslant 1, \qquad p=\frac{3}{2}\,,\ \text{if}\ \frac{1}{2}\leqslant\alpha<1, \quad p=\frac{1}{2}\big(1+\frac{1}{1-\alpha}\big),\ \text{if}\ 0<\alpha<\frac{1}{2}\,.$$

Multiplying the matrices $\mathbb{A}$ and $\mathbb{B}$ we see that, by our choice of the vectors b_j, all entries above the diagonal of the matrix $\mathbb{A}\mathbb{B}$ are equal zero (changing the order of the rows if necessary). Moreover, $\det\mathbb{A}\mathbb{B}=\nu$. In particular, both changes (80) and (83) of variables are well defined since $\det\mathbb{A}\,\det\mathbb{B}=\nu$ and we assume that $\nu\neq 0$. The relation $\det\mathbb{A}\,\det\mathbb{B}=\nu$ combined with the inequalities (79), (81), (84) yields

$$\big|\,J_d^*\big(A_*\big)\,\big| \leqslant |\nu|^{1-1/p}\, c(\alpha,d)\, c_5\, c_6^{-(\alpha-1+1/p)|\Omega|}\, c_9^{-(d-|\Omega|)/p}. \tag{85}$$

Using $|a_j|\leqslant a^*,\ j=1,\dots,m$, we obtain

$$|\nu|^{1-1/p}\leqslant (a^*)^{(1-1/p)|\Omega|} \tag{86}$$

since $1-1/p>0$. Thus, using (85) and (86), we get

$$\big|\,J_d^*\big(A_*\big)\,\big| \leqslant c(\alpha,d)\, C\big(\alpha,d,a_*,a^*,\tilde a\big), \tag{87}$$

where

$$C\big(\alpha,d,a_*,a^*,\tilde a\big) = (a_*\tilde a)^{(1-2\alpha)d/2}\,\big(1+(a^*)^{\alpha d}\big)\,\big(1+(a_*\tilde a)^{-1-d/2}\big).$$

Collecting bounds (57), (59), (72), (87) and using the definition (65) of $J_d\big(A\big)$, we obtain estimate (49) of K_w. $\square$

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