On Green's matrices of trees

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Abstract

The inverse $C = [c_{i,j}]$ of an irreducible nonsingular symmetric tridiagonal matrix is a so-called Green's matrix, i.e. it is given by two sequences of real numbers $\{u_i\}$ and $\{v_i\}$ such that $c_{i,j} = u_i v_j$ for $i \leq j$. A similar result holds for nonsymmetric matrices. An open problem on nonsingular sparse matrices is whether there exists a similar structure for their inverses as in the tridiagonal case. Here we positively answer this question for acyclic matrices, i.e. matrices whose undirected graphs are trees. We prove that the inverses of acyclic symmetric matrices are given as the Hadamard product of three matrices, a type D matrix, a flipped type D matrix and a matrix of tree structure which is closely related to the graph of the original matrix itself. For nonsymmetric matrices we obtain a similar structure. Moreover, our result include the result for symmetric and nonsymmetric tridiagonal matrices.

1 Introduction

In many mathematical problems nonsingular sparse matrices arise, i.e. matrices for which just a few entries are nonzero. One important class of sparse matrices is the class of tridiagonal matrices. For tridiagonal matrices many theoretical results are known. One of the most important result is established by Gantmacher and Krein [GK1] and [GK2]. They proved that a symmetric, irreducible nonsingular matrix A is tridiagonal if and only if $A^{-1} =: C = [c_{i,j}]$ is given by two sequences $\{u_i\}_{i=1}^n$, $\{v_i\}_{i=1}^n$ of numbers such that

$$(1.1) \qquad C = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_1v_2 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_1v_n & u_2v_n & \cdots & u_nv_n \end{bmatrix} \quad \text{i.e.} \quad c_{i,j} = \begin{cases} u_iv_j & i \leq j, \\ u_jv_i & i \geq j. \end{cases}$$

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Matrices of the form (1.1) are called Green's matrices by Gantmacher and Krein [GK2] and Karlin [Ka]. It was observed in [MNNST] that Green's matrices can be described more elegant as the Hadamard product (elementwise product) of a so-called type D matrix and a flipped type D matrix:

$$C = \begin{bmatrix} u_1 & u_1 & \cdots & u_1 \\ u_1 & u_2 & \cdots & u_2 \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & \cdots & u_n \end{bmatrix} \circ \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ v_2 & v_2 & \cdots & v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n & v_n & \cdots & v_n \end{bmatrix}. \tag{1.2}$$

A similar result holds for the inverse of a nonsymmetric irreducible tridiagonal matrix. There the inverse $C = A^{-1}$ can be described by four sequences $\{u_i\}, \{v_i\}, \{x_i\}, \{y_i\}$ which satisfy $u_i v_i = x_i y_i$ ([I]). In detail we have

$$c_{i,j} = \begin{cases} u_i v_j & i \leq j, \\ x_i y_j & i \geq j. \end{cases} \tag{1.3}$$

As in the symmetric case matrices of the form (1.3) can be written nicely as the Hadamard product of two matrices:

$$C = \begin{bmatrix} u_1 & u_1 & \cdots & \cdots & u_1 \\ x_1 & u_2 & \cdots & \cdots & u_2 \\ x_1 & x_2 & u_3 & \cdots & u_3 \\ \vdots & \vdots & & \ddots & \vdots \\ x_1 & x_2 & \cdots & \cdots & u_n \end{bmatrix} \circ \begin{bmatrix} v_1 & v_2 & \cdots & \cdots & v_n \\ y_2 & v_2 & \cdots & \cdots & v_n \\ y_3 & y_3 & v_3 & \cdots & v_n \\ \vdots & \vdots & & \ddots & \vdots \\ y_n & y_n & \cdots & \cdots & v_n \end{bmatrix}. \tag{1.4}$$

A frequently asked question about nonsingular sparse matrices is whether there exists a similar structure for their inverses as in the tridiagonal case. Or in other words, what are generalizations of Green's matrices for arbitrary sparse matrices? Probably the answer of the first question in general is negative. However, here we establish the structure of the inverses of acyclic matrices, i.e. matrices whose undirected graphs are trees. It turns out that the Hadamard product form (1.2) and (1.4) is the most promising approach for generalizations. We prove that the inverses of acyclic symmetric matrices are given as the Hadamard product of three matrices, a type D matrix, a flipped type D matrix (as in (1.2)) and a matrix of tree structure which is closely related to the graph of A itself. A similar result holds for nonsymmetric matrices. (see Theorem 3.5 and Corollary 3.6). Moreover, our result include the result by Gantmacher and Krein and also the result for nonsymmetric matrices. We also extend some results by Kirkland, Neumann, and Shader [KNS] as well as by Fiedler [F]. There inverses of $n-1 \times n-1$ principal submatrices of combinatorically symmetric ([KNS]) or symmetric ([F]) singular M–matrices whose graphs are trees, are considered.

2 Notations and preliminary results

We start this section with some notations. An *undirected weighted graph* $G = (V,E)$ of $n+1$ vertices is a graph with vertex set $V = \{0, \ldots, n+1\}$, edges $e_{ij} \in E$ between the vertices ($i, j \in \{0, 1, \ldots, n\}$) labeled by weights $w_{i,j} \in \mathbb{R}$. Note that we allow negative weights.

A *tree* is a undirected graph for which there exists a unique path between any two vertices i and j. A rooted tree is a tree with a prominent vertex, the root. We always assume that trees have vertex set $\{1, \ldots, n\}$ while for rooted trees $\{0, 1, \ldots, n\}$ is the vertex set and the root is labeled by 0. Moreover we assume that the vertices of a rooted tree are numbered consistently with the tree.

For a given tree Γ we construct a rooted tree $\Gamma_{(0)}$ by adding a new vertex 0, the root, and a new edge $e_{0,1}$ to the old tree. A *branch* starting at vertex i of the tree Γ is the connected subgraph of Γ including vertex i obtained by deleting edge $e_{j,i}, j < i$.

A *path* from vertex i to vertex j, denoted by $P_{i,j}$, is the set of edges $\{(k_1, k_2), (k_2, k_3), \ldots (k_{r-1}, k_r)\}$, where $k_1 = i$ and $k_r = j$.

The *graph* $G(A)$ of an $n \times n$ matrix $A = [a_{i,j}]$ is the undirected graph consisting of n vertices $\{1, \ldots, n\}$ such that there is an edge between vertex i and vertex j if and only if $a_{i,j} \neq 0$ or $a_{j,i} \neq 0$. A matrix A is called *acyclic* if $G(A)$ is a forest, i.e. a collection of trees. A is called *treediagonal* by Klein [Kl] if $G(A)$ is a tree.

Here we consider irreducible acyclic matrices, i.e. matrices whose directed graphs are connected trees. In other words we assume that the matrices are combinatorically symmetric, i.e. $a_{i,j} = 0$ if and only if $a_{j,i} = 0$. Note that irreducible acyclic matrices are treediagonal but not vice versa.

We denote by e the vector with all ones and $e_1 = (1, 0, \ldots, 0)^T$. The dimension of these vectors is not fixed.

Klein established in [Kl] some results on acyclic matrices which we will use in the next section.

Definition 2.1 *An $n \times n$ matrix $C = [c_{i,j}]$ satisfies the treeangle property with respect to a given tree Γ if for every $i, j, k \in V$ with $P_{i,k} \subseteq P_{i,j}$*

$$c_{i,j} c_{k,k} = c_{i,k} c_{k,j}. \tag{2.5}$$

Klein then proved the following theorems

Theorem 2.2 *Let Γ be a tree. If a nonsingular matrix C satisfies the treeangle property with respect to Γ and if $c_{i,i} \neq 0$ for all interior vertices, then C^{-1} is treediagonal with respect to Γ, i.e. C^{-1} is acyclic.*

Theorem 2.3 *Let Γ be a tree. If A is a nonsingular treediagonal matrix with respect to Γ then A^{-1} satisfies the treeangle property with respect to Γ.*

The above Theorems describe the structure of inverses of acyclic matrices. However, it is not clear at all how one can describe the treeangle property in terms of matrices as in the tridiagonal case. Moreover, what are the generalizations of Green's matrices? To do so we need the following class of matrices.

Definition 2.4 *Let Γ be a weighted rooted tree. A matrix $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is of tree structure with respect to Γ, if for all $i, j = 1, \ldots, n$*

$$a(i,j) \quad := \sum_{\{r,s\} \in P_{i,0} \cap P_{j,0}} w_{r,s}, \tag{2.6}$$

here $\{r,s\} \in P_{i,0} \cap P_{j,0}$ denotes a common edge of the pathes $P_{i,0}$ and $P_{j,0}$.

Note that (2.6) defines a 'distance' or better an 'inverse distance' between the the vertices of the tree. This distance was already used by Nabben and Varga in [NV2] for leaves of trees.

We immediately obtain the following characterization of matrices of tree structure.

Theorem 2.5 *Let Γ be a weighted rooted tree. Then $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is of tree structure with respect to Γ if and only if A can be decomposed as*

$$A = \sum_{i=1}^{n} \tau_i u_i u_i^T \tag{2.7}$$

where $\tau_i \in \mathbb{R}$ and $\tau_i = w_{j,i}$, with $j < i$. The vectors $u_i \in \mathbb{R}^{n,n}$ satisfy $(u_i)_j = 1$ for all j (including i) belonging to the branch of Γ starting at vertex i, and $(u_i)_j = 0$ otherwise.

Proof. First assume that A is of tree structure. Consider the branches starting at the root 0. If there are $s, s > 1$, branches and i and j are vertices of different branches then $a(i,j) = 0$ and obviously A is a block diagonal matrix:

$$A = \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_s \end{bmatrix}$$

where $C_1, \ldots, C_s$ are square matrices. If there is just one branch, then $a(1,j) = w_{0,1}$ for all $j = 1, \ldots, n$. Hence

$$A = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} + w_{0,1} e e^T.$$

In both cases we then proceed by induction taking the neighbours of the root as roots for the branches.

The other implication can be proved by induction also. If i and j are in different branches connected with the root then

$$a(i,j) = 0 = \sum_{\{r,s\} \in P_{i,0} \cap P_{j,0}} w_{r,s}.$$

If there is just one neighbour of the root, vertex 1, then for all j

$$a(1,j) = a(j,1) = \tau_1 = w_{0,1} = \sum_{\{r,s\} \in P_{1,0} \cap P_{j,0}} w_{r,s}.$$

$\square$

Note that matrices of tree structure can have negative entries. Thus the τ_i of (2.7) can be negative. But if all τ_i in (2.7) are nonnegative we obtain a subclass of certain so-called pre-ultrametric matrices (see [F] and [VN]). If all τ_i are positive, (2.7) gives certain so-called strictly ultrametric matrices defined by [MNS] and characterized in [NV1]. Pre-ultrametric matrices as well as strictly ultrametric matrices are nonnegative inverse M-matrices which can be decomposed as the sum of $2n-1$ rank one matrices similar to (2.7). For more details on ultrametric matrices see [NV1], [VN], [NV2].

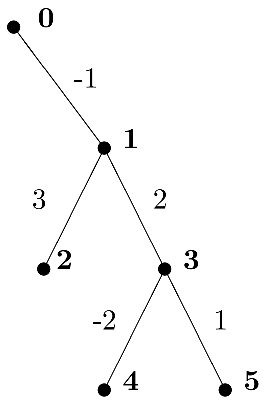
Example 2.6 For illustration consider the weighted rooted tree in Figure 1. The 5×5 matrix of tree structure is then given by

$$A = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 2 \end{bmatrix}.$$

Moreover A can be decomposed as

$$A = \sum_{i=1}^{5} \tau_i \mathbf{u}_i \mathbf{u}_i^T,$$

where the u_i are given as in Figure 1 and the $\{\tau_1, \ldots, \tau_5\}$ in (2.7) are $\{-1, 3, 2, -2, 1\}$.



$u_1 := (1,1,1,1,1)^T$

$u_2 := (1,0,0,0,0)^T$; $u_3 := (0,0,1,1,1)^T$

$u_4 := (0,0,0,1,0)^T$; $u_5 := (0,0,0,0,1)^T$

FIGURE 1

3 Main Results

We start this section with a simple observation

Observation 3.1 *Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ be nonsingular, irreducible and acyclic. Assume that the diagonal entries of $C = [c_{i,j}] := A^{-1}$ are nonzero. Then $c_{i,j} \neq 0$ for all $i, j \in \{1, \ldots, n\}$.*

Proof. With Theorem 2.3, C satisfies the treeangle property. Now let $P_{i,j} = \{(k_1, k_2), (k_2, k_3), \ldots (k_{r-1}, k_r)\}$, where $k_1 = i$ and $k_r = j$, be the unique path from i to j. As mention in [Kl] the treeangle property implies

$$c_{i,j} = c_{k_1,k_2} \prod_{s=2}^{r-1} \frac{c_{k_s,k_s+1}}{c_{k_s,k_s}}. \tag{3.8}$$

Thus, if there is one $c_{i,i+1} = 0$ then C is reducible. Hence all $c_{i,i+1}$ are nonzero which imply that all other entries are nonzero also. □

Lemma 3.2 *Let Γ be a tree and let $C = [c_{i,j}] \in \mathbb{R}^{n,n}$ be irreducible. Then the following are equivalent:*

(1) C satisfies the treeangle property with respect to Γ with $Ce_1 = \alpha e$ and $e_1^T C = \alpha e^T$ for some $\alpha \in \mathbb{R}$.

(2) C is symmetric and C is of tree structure with respect to $\Gamma_{(0)}$ where $w_{0,1} = \alpha$.

Proof. (1) $\Rightarrow$ (2): First, observe that the first row and column of C are the same. We then proceed by induction on n. For $n = 2$ the statement is true.

Now delete the first row and column of C and let $C_{[1]}$ the remaining matrix. Obviously, $C_{[1]}$ satisfies the treeangle property with respect to $\Gamma_{[1]}$, i.e. the graph obtained from Γ by deleting vertex 1 and all edges connecting 1 with other vertices. First assume that $\Gamma_{[1]}$ is connected. This implies that there was just one edge from vertex 1 to another vertex, say w.l.o.g vertex 2. Since the first row and column of C are the same we can write C as

$$C = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} + \alpha e e^T,$$

for some $B \in \mathbb{R}^{n-1,n-1}$ and $e = (1, \ldots, 1)^T \in \mathbb{R}^{n,n}$. The treeangle property for C says

$$\begin{aligned} c_{1,j} c_{2,2} &= c_{1,2} c_{2,j} \quad \text{for} \quad j > 2 \\ c_{i,1} c_{2,2} &= c_{i,2} c_{2,1} \quad \text{for} \quad i > 2. \end{aligned}$$

Thus

$$c_{2,2} = c_{2,j} = c_{i,2} \quad \text{for} \quad j > 2,\ i > 2.$$

Hence, $C_{[1]}e_1 = \tilde{c}e$ and $e_1^T C_{[1]} = \tilde{c}e^T$. With the induction hypothesis and Theorem 2.5 we obtain the result.

Now assume that $C_{[1]}$ is reducible, i.e. there are more then one vertices connected with 1. Thus we have different branches connected with vertex 1. Now consider two vertices i and j of different branches. The treeangle property implies

$$c_{i,j}c_{1,1} = c_{i,1}c_{1,j}.$$

Thus

$$c_{i,j} = \alpha.$$

Hence C is of the form

$$C = \begin{bmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_s \end{bmatrix} + \alpha ee^T,$$

for some square matrices $C_1, \ldots, C_s$. Now consider the submatrices of C corresponding to the different branches. Obviously they satisfy the treeangle property with respect to the subgraphs. The first submatrix, say A_{11}, which include vertex 1 obviously satisfies that the first row and column are the same. Now consider a submatrix consisting of vertices $k, k+1, \ldots, t$ where k is connected with 1. Then for $j \in \{k+1, \ldots, t\}$

$$\begin{aligned} c_{1,j}c_{k,k} &= c_{1,k}c_{k,j} \\ c_{j,1}c_{k,k} &= c_{j,k}c_{k,1}. \end{aligned}$$

Hence

$$c_{k,k} = c_{k,j} = c_{j,k}.$$

Thus the submatrices satisfy the inductive assumptions. Again, with the induction hypothesis and Theorem 2.5 we get (2).

(2) $\Rightarrow$ (1): Since C is of ultrametric structure it is clear that $Ce_1 = \alpha e$ and $e_1^T C = \alpha e^T$. We then prove that C satisfies the treeangle property with respect to Γ. Let $i, j, k \in \{1, \ldots, n\}$ with $P_{i,k} \subseteq P_{i,j}$. If $P_{0,k} \subseteq P_{0,i}$ then $c_{i,k} = c_{k,k}$ and $c_{i,j} = c_{k,j}$. If $P_{0,k} \subseteq P_{0,j}$ then $c_{j,k} = c_{k,j} = c_{k,k}$ and $c_{i,j} = c_{k,i} = c_{i,k}$. In both cases the treeangle property is fulfilled. $\square$

The next theorem gives a characterization of acyclic matrices which satisfies $Ae = \gamma e_1$ and $e^T A = \gamma e_1^T$. Moreover, we give the exact formula for the inverse.

Theorem 3.3 *Let Γ be a tree and let $A = [a_{i,j}] \in \mathrm{IR}^{\mathrm{n,n}}$ be nonsingular. Then the following are equivalent:*

(1) A is acyclic with $G(A) = \Gamma$ and $Ae = \gamma e_1$ and $e^T A = \gamma e_1^T$.

(2) A is symmetric and A^{-1} is of tree structure with respect to the weighted rooted tree $\Gamma_{(0)}$ where the weights are $w_{i,j} = -1/a_{i,j}$ and $w_{0,1} = 1/\gamma$.

Proof. (1) $\Rightarrow$ (2): Obviously the first row and column of A^{-1} are $\gamma^{-1}e$ and $\gamma^{-1}e^T$. Therefore we have

$$A^{-1} = \begin{bmatrix} \gamma^{-1} & \dots & \gamma^{-1} \\ \vdots & B & \\ \gamma^{-1} & & \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \tilde{B} & \\ 0 & & \end{bmatrix} + \frac{1}{\gamma} ee^T,$$

for some $B, \tilde{B} \in \mathrm{IR}^{\mathrm{n-1,n-1}}$. Similarly we partition A:

$$A = \begin{bmatrix} a_{11} & b \\ c^T & A_{22} \end{bmatrix}. \tag{3.9}$$

Since A is nonsingular we obtain that the Schur complement $A^{-1}/\gamma^{-1} = B - \gamma^{-1} e_{n-1} e_{n-1}^T$ is also nonsingular. Using the formula for the inverse of 2×2 block matrices we obtain

$$A_{22} = (B - \gamma^{-1} e_{n-1} e_{n-1}^T)^{-1} = \tilde{B}^{-1}.$$

Now first let A_{22} be irreducible, i.e. vertex 1 is only connected with vertex 2. Then since $Ae = \gamma e_1$ and A is acyclic we have

$$\begin{aligned} A_{22} e &= -a_{21} e_1 \qquad \text{or} \quad -\tfrac{1}{a_{21}} e = A_{22}^{-1} e_1 \\ e^T A_{22} &= -a_{21} e_1^T \qquad \text{or} \quad -\tfrac{1}{a_{21}} e^T = e_1^T A_{22}^{-1}. \end{aligned}$$

Hence, $a_{21} = a_{12}$ and

$$A^{-1} = \frac{1}{\gamma} ee^T - \frac{1}{a_{12}} (0, e_{n-1}^T)^T (0, e_{n-1}^T) + \begin{bmatrix} O & O \\ O & \hat{B} \end{bmatrix}$$

for some $\hat{B} \in \mathrm{IR}^{\mathrm{n-2\times n-2}}$.

If A_{22} is reducible, i.e. there are more than one branches connected with vertex 1, we can apply the above proof for each branch or each irreducible component of A_{22}. We then proceed by induction.
(2) $\Rightarrow$ (1): Since A^{-1} is of tree structure we have as above

$$A^{-1} = \begin{bmatrix} \gamma^{-1} & \dots & \gamma^{-1} \\ \vdots & B & \\ \gamma^{-1} & & \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \tilde{B} & \\ 0 & & \end{bmatrix} + \frac{1}{\gamma} ee^T,$$

for some $B, \tilde{B} \in \mathbb{R}^{n-1,n-1}$. Hence $Ae = \gamma e_1$ and $e^T A = \gamma e_1^T$. We partition A as in (3.9) and get $A_{22} = \tilde{B}^{-1}$. Then frist assume that $\tilde{B}$ and therefore A_{22} is irreducible. Then $\tilde{B} e_1 = w_{12} e_{n-1}$ and $e_1^T \tilde{B} = w_{12} e_{n-1}^T$ and $\tilde{B}$ is of tree structure with respect to the branch starting at vertex 2. Moreover since

$$Ae = \gamma e_1 \quad \text{and} \quad A_{22} e_{n-1} = \tilde{B}^{-1} e_{n-1} = w_{12}^{-1} e_1$$

we obtain

$$a_{21} = -\frac{1}{w_{12}} \quad \text{and} \quad a_{i,1} = 0 \text{ for } i > 2.$$

If $\tilde{B}$ is reducible we apply the above steps for each irreducible component, i.e. for each branch connected with vertex 1. By induction we obtain the result. □

Note that in Theorem 3.3 there is no restriction on the signs of the entries of the matrix. Moreover, Theorem 3.3 also gives formulas for the entries of the inverse of a matrix of tree structure. The diagonal entries of the inverse can be obtained using $Ae = \gamma e_1$.

Theorem 3.3 extends some results by Kirkland, Neumann, and Shader [KNS] as well as by Fiedler [F]. There inverses of $n-1 \times n-1$ principal submatrices of combinatorically symmetric ([KNS]) or symmetric ([F]) singular M–matrices whose graphs are trees, are considered.

Next we compare Theorem 3.3 with Klein's Theorems 2.2 and 2.3. First of course here we give formulas for the inverse in boths ways. But on the other hand the assumption for A^{-1} being of special tree structure which implies $A^{-1} e_1 = w_{01} e$ seems to be more restrictive then the assumption of nonvanishing diagonal entries. But the following example shows that Theorem 3.3 is not included in Theorem 2.2.

Example 3.4 Let A be as follows

$$A = \begin{bmatrix} 8 & 8 & 8 & 8 \\ 8 & 0 & 0 & 0 \\ 8 & 0 & 2 & 0 \\ 8 & 0 & 0 & 1 \end{bmatrix}.$$

A is of tree structure with respect to the tree given in Figure 2.

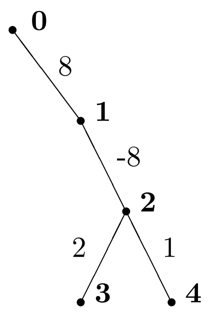


FIGURE 2

The inverse of A is

$$A^{-1} = \frac{1}{8}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 11 & -4 & -8 \\ 0 & -4 & 4 & 0 \\ 0 & -8 & 0 & 8 \end{bmatrix},$$

which is acyclic and $G(A^{-1})$ is the graph given in Figure 2.

We then obtain our generalization of the Gantmacher and Krein result.

Theorem 3.5 *Let $A \in \mathbb{R}^{n,n}$ be nonsingular, irreducible and acyclic with $G(A) = \Gamma$. Assume that the diagonal entries of A^{-1} are nonzero. Then there exist matrices T and R of the form*

$$T = \begin{bmatrix} d_1 & d_1 & \cdots & \cdots & d_1 \\ f_1 & d_2 & \cdots & \cdots & d_2 \\ f_1 & f_2 & d_3 & \cdots & d_3 \\ \vdots & \vdots & & \ddots & \vdots \\ f_1 & f_2 & \cdots & \cdots & d_n \end{bmatrix} \qquad R = \begin{bmatrix} f_1 & f_2 & \cdots & \cdots & f_n \\ d_2 & f_2 & \cdots & \cdots & f_n \\ d_3 & d_3 & f_3 & \cdots & f_n \\ \vdots & \vdots & & \ddots & \vdots \\ d_n & d_n & \cdots & \cdots & f_n \end{bmatrix}. \tag{3.10}$$

and a matrix U of tree structure with respect to the weighted tree $\Gamma_{(0)}$ such that

$$A^{-1} = T \circ U \circ R.$$

Conversely, let $U = [u_{i,j}] \in \mathbb{R}^{n,n}$ be nonsingular. If U is of tree structure with respect to a given rooted tree $\Gamma_{(0)}$. Then for any matrices T and R of the form (3.10) with $f_i d_i \neq 0$ for all i, the matrix $(T \circ U \circ R)^{-1}$ is acyclic and $G((T \circ U \circ R)^{-1}) = \Gamma$.

Proof. Let $F = diag(f_1, \ldots, f_n) = diag(e_1^T A^{-1})$ and $D = diag(d_1, \ldots, d_n) = diag(A^{-1} e_1)$. With Observation 3.1 D and F are nonsingular. Thus FAD is nonsingular and $G(FAD) = \Gamma$. Moreover $(FAD)^{-1} e_1 = ce$ and $e_1^T (FAD)^{-1} = ce^T$ for some $c \in \mathbb{R}$. Thus with Lemma 3.2

$$(FAD)^{-1} = U$$

for a matrix U of tree structure with respect to $\Gamma_{(0)}$. Hence

$$A^{-1} = DUF.$$

But we then observe that

$$A^{-1} = DUF = T \circ U \circ R.$$

For the converse we obtain with Theorem 3.3 that U^{-1} is acyclic with $G(U^{-1}) = \Gamma$. Since, $T \circ U \circ R = DUF$ with $D = diag(d_1, \ldots, d_n)$ and $F = diag(f_1, \ldots, f_n)$, we obtain the desired result. $\square$

For the symmetric case we obtain

Corollary 3.6 *Let $A \in \mathbb{R}^{n,n}$ be symmetric nonsingular, irreducible and acyclic with $G(A) = \Gamma$. Assume that the diagonal entries of A^{-1} are nonzero. Then there exist matrices T and R of the form*

$$(3.11) \qquad T = \begin{bmatrix} d_1 & d_1 & \cdots & \cdots & d_1 \\ d_1 & d_2 & \cdots & \cdots & d_2 \\ d_1 & d_2 & d_3 & \cdots & d_3 \\ \vdots & \vdots & & \ddots & \vdots \\ d_1 & d_2 & \cdots & \cdots & d_n \end{bmatrix} \qquad R = \begin{bmatrix} d_1 & d_2 & \cdots & \cdots & d_n \\ d_2 & d_2 & \cdots & \cdots & d_n \\ d_3 & d_3 & d_3 & \cdots & d_n \\ \vdots & \vdots & & \ddots & \vdots \\ d_n & d_n & \cdots & \cdots & d_n \end{bmatrix}.$$

and a matrix U of tree structure with respect to the weighted tree $\Gamma_{(0)}$ such that

$$A^{-1} = T \circ U \circ R.$$

Conversely, let $U = [u_{i,j}] \in \mathbb{R}^{n,n}$ be nonsingular. If U is of tree structure with respect to a given rooted tree $\Gamma_{(0)}$. Then for any matrices T and R of the form (3.11) *with $d_i \neq 0$ for all i, the matrix $(T \circ U \circ R)^{-1}$ is acyclic and $G((T \circ U \circ R)^{-1}) = \Gamma$.*

Theorem 3.5 implies several well-known results as corollaries.

Corollary 3.7 *Let $A \in \mathbb{R}^{n,n}$ be nonsingular, irreducible and tridiagonal. Assume that the diagonal entries of A^{-1} are nonzero. Then there exist four vectors $u, v, x, y \in \mathbb{R}^n$ with $u_i v_i = x_i y_i$ for all i, such that $A^{-1} =: C = [c_{i,j}]$ is given by*

$$(3.12) \qquad c_{i,j} = \begin{cases} u_i v_j & i \le j, \\ x_i y_j & i \ge j \end{cases}$$

i.e.

$$(3.13) \qquad C = \begin{bmatrix} u_1 & u_1 & \cdots & \cdots & u_1 \\ x_1 & u_2 & \cdots & \cdots & u_2 \\ x_1 & x_2 & u_3 & \cdots & u_3 \\ \vdots & \vdots & & \ddots & \vdots \\ x_1 & x_2 & \cdots & \cdots & u_n \end{bmatrix} \circ \begin{bmatrix} v_1 & v_2 & \cdots & \cdots & v_n \\ y_2 & v_2 & \cdots & \cdots & v_n \\ y_3 & y_3 & v_3 & \cdots & v_n \\ \vdots & \vdots & & \ddots & \vdots \\ y_n & y_n & \cdots & \cdots & v_n \end{bmatrix}.$$

Conversely, if C is nonsingular and of the form (3.12) *with $u_i v_i = x_i y_i \neq 0$ for all i then C^{-1} is tridiagonal.*

Proof. Consider the Hadamard product form of A given in Theorem 3.5. The matrix U is of tree structure with respect to $\Gamma_{(0)}$ where $\Gamma = G(A)$. But $G(A)$ is just a path. Hence U is a type D matrix as in (1.2). Therefore

$$C = \begin{bmatrix} d_1 & d_1 & \cdots & \cdots & d_1 \\ f_1 & d_2 & \cdots & \cdots & d_2 \\ f_1 & f_2 & d_3 & \cdots & d_3 \\ \vdots & \vdots & & \ddots & \vdots \\ f_1 & f_2 & \cdots & \cdots & d_n \end{bmatrix} \circ \begin{bmatrix} g_1 & g_1 & \cdots & \cdots & g_1 \\ g_1 & g_2 & \cdots & \cdots & g_2 \\ g_1 & g_2 & g_3 & \cdots & g_3 \\ \vdots & \vdots & & \ddots & \vdots \\ g_1 & g_2 & \cdots & \cdots & g_n \end{bmatrix} \circ \begin{bmatrix} f_1 & f_2 & \cdots & \cdots & f_n \\ d_2 & f_2 & \cdots & \cdots & f_n \\ d_3 & d_3 & f_3 & \cdots & f_n \\ \vdots & \vdots & & \ddots & \vdots \\ d_n & d_n & \cdots & \cdots & f_n \end{bmatrix}.$$

Thus with $u_i = d_i g_i, x_i = f_i g_i$ and $v_i = f_i, y_i = d_i$. We get the required form of C. Moreover, $u_i v_i = x_i y_i$ for all i. For the converse we set $f_i = v_i, d_i = y_i$ and $g_i = u_i/d_i$. With Theorem 3.5 we obtain that C^{-1} is tridiagonal. □

For the symmetric case we obtain the well-known result:

Corollary 3.8 *Let $A \in \mathbb{R}^{n,n}$ be nonsingular, symmetric, irreducible and tridiagonal. Assume that the diagonal entries of A^{-1} are nonzero. Then there exist two vectors $u, v \in \mathbb{R}^n$ such that $A^{-1} =: C = [c_{i,j}]$ is given by*

$$C = \begin{bmatrix} u_1 & u_1 & \cdots & u_1 \\ u_1 & u_2 & \cdots & u_2 \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & \cdots & u_n \end{bmatrix} \circ \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ v_2 & v_2 & \cdots & v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n & v_n & \cdots & v_n \end{bmatrix}. \tag{3.14}$$

Conversely, if C is nonsingular and of the form (3.14) *with $u_i v_i \neq 0$, then C^{-1} is tridiagonal.*

Proof. The result follows immediately from Corollary 3.7 and the fact that $d_i = f_i$ for all i. □

In the above corollaries tridiagonal matrices are considered. In some sense the counter part of these matrices are matrices whose graphs are stars. For these matrices we obtain:

Corollary 3.9 *Let $A \in \mathbb{R}^{n,n}$ be nonsingular and irreducible and let $G(A)$ be a star, i.e. w.l.o.g. $a_{i,j} = 0$ whenever i or j is not 1, $i \neq j$. Assume that diagonal entries of A^{-1} are nonzero. Then there exist diagonal matrices D_1, D_2, D_3 and a constant α such that*

$$A^{-1} = D_1(\alpha e e^t + D_3)D_2. \tag{3.15}$$

Moreover there exist matrices T and R as in (3.10) *such that*

$$A^{-1} = T \circ (\alpha e e^T + D_3) \circ R. \tag{3.16}$$

Proof. We apply Theorem 3.5. Since the center of the star is vertex 1 the related matrix of tree structure is $\alpha e e^T + D_3$. □

Example 3.10

$$A = \begin{bmatrix} 10 & -2 & -4 & -4 & -4 \\ -1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 4 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & 0 & 4 \end{bmatrix}$$

We have

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 4 & 2 & 4 & 2 \\ 2 & 8 & 2 & 4 & 2 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 1 & 4 & 1 \\ 1 & 2 & 1 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 2 & \\ & & & & 1 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} * \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

Example 3.11

$$B = \begin{bmatrix} -1 & -2 & -3 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ -3 & 0 & 6 & 3 & -6 \\ 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & -6 & 0 & 6 \end{bmatrix}.$$

The graph $G(B)$ is just the graph given in Figure 1. The inverse of B is $\frac{1}{6} * A$, with A as in Example 2.6:

$$A = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 2 \end{bmatrix}.$$

Note that A is of special tree structure with respect to $G(B)$. Moreover we have $D = F = -I$, where I denotes the identity matrix.

Theorem 3.5 describes a simple way to construct matrices whose inverses are acyclic. However, one have to guarantee that the matrices of tree structure are nonsingular. The next theorem gives an useful characterization of nonsingularity. These characterization was already observed in [MNST] for so-called generalized ultrametric matrices.

Theorem 3.12 *Let $C = [c_{i,j}] \in \mathbb{R}^{n,n}$ be of ultrametric structure with respect to a given rooted tree. Then C is nonsingular if and only if C does not contain a row or column of zeros, and no two rows or two columns are the same.*

Proof. It is clear that C is singular if C does contain a row or column of zeros, or two rows or two columns are the same. We prove the other implication with an induction on the dimension of C. For $n = 2$ this is obviously true.
So assume that C does not contain a row or column of zeros, and no two rows or two columns are the same. Moreover C has the structure

$$C = \begin{bmatrix} w_{0,1} & \dots & w_{0,1} \\ \vdots & B & \\ w_{0,1} & & \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \tilde{B} & \\ 0 & & \end{bmatrix} + w_{0,1} e e^T,$$

for some $B, \tilde{B} \in \mathbb{R}^{n-1,n-1}$ and some $w_{0,1} \neq 0$. For the Schur complement $C/w_{0,1}$ we obtain

$$C/w_{0,1} = B - w_{0,1} e_{n-1} e_{n-1}^T = \tilde{B}.$$

Obviously, $\tilde{B}$ does not contain a row or column of zeros, and no two rows or two columns are the same. Otherwise C would do so. On the other $\tilde{B}$ is of tree structure or the direct sum of matrices of tree structure. Thus by the induction hyphotesis we get the result. □

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