The coinvariant algebra and representation types of blocks of category $\mathcal{O}$

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Abstract

Let $\mathfrak{G}$ be a finite dimensional semisimple Lie algebra over the complex numbers. Let A be the finite dimensional algebra of a (regular or singular) block of the BGG–category $\mathcal{O}$. By results of Soergel [S], A has a combinatorial description in terms of a subalgebra C_0 of the coinvariant algebra C. In [KM], an embeddding has been constructed from C_0–mod into the category $\mathcal{F}(\Delta)$ of A–modules having a Verma flag. This is the main tool for our classification of $\mathcal{F}(\Delta)$ into finite, tame and wild representation type. As a consequence we also obtain a classification of A–mod into finite, tame and wild representation type, thus reproving a recent result of Futorny, Nakano and Pollack [FNP].

1 Introduction and statement of results

Let $\mathfrak{G}$ be a finite dimensional semisimple Lie algebra over the complex numbers. Fix a triangular decomposition, $\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+$. Then the BGG-category $\mathcal{O}$, which has been defined in [BGG], decomposes into a direct sum of indecomposable subcategories, called *blocks*. By a result of Soergel ([S]) we can restrict our attention from now on to blocks with integral support, since any other block is equivalent to one with integral support (note, however, that applying such an equivalence may involve changing $\mathfrak{G}$ and thus also changing the root system). Each block is equivalent to the module category of a finite dimensional associative algebra A which is unique up to Morita equivalence. The full subcategory of A–mod consisting of modules having a Verma flag is denoted by $\mathcal{F}(\Delta)$. Associated with a block having integral support are the following combinatorial data: an antidominant integral weight λ, the Weyl group W and the stabilizer subgroup W_0 of W fixing λ. Denote the root systems corresponding to W and to W_0 by Φ and Φ_0, respectively. Our classifications will depend on these root systems only.

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Theorem 1. *Fix a block and its finite dimensional algebra A. Then the representation type of the category $\mathcal{F}(\Delta)$ is given by the following table. All cases not listed are wild.*

representation type	Φ	Φ_0
finite	*any* Φ	$\Phi_0 = \Phi$
	A_1	$\emptyset$
	A_2	A_1
	A_3	A_2
	B_2	A_1
	A_4	A_3
tame	$A_1 \times A_1$	$\emptyset$
	A_5	A_4
	B_3	B_2

Of course, an equality $\Phi = \Phi_0$ occurs precisely for A a simple algebra.

As a consequence of the proof of Theorem 1 we also obtain:

Theorem 2. *Fix a block and its finite dimensional algebra A. Then the representation type of the category A–mod is given by the following table. All cases not listed are wild.*

representation type	Φ	Φ_0
finite	*any* Φ	$\Phi_0 = \Phi$
	A_1	$\emptyset$
	A_2	A_1
tame	A_3	A_2
	B_2	A_1

Theorem 2 has been proved in [FNP] by completely different methods. The formulation there is for simple Lie algebras. However, our more general formulation follows easily from the results in [FNP].

Our method is based on Soergel's [S] description of blocks of $\mathcal{O}$ as endomorphism rings over certain subalgebras C_0 of coinvariant algebras and on a result in [KM] which states that the module category of C_0 embeds into $\mathcal{F}(\Delta)$ (we reprove this result here for the sake of completeness). The algebra C_0 is local and commutative and explicit computations therein can be easily performed using a basis coming from Schubert calculus. Therefore, the category C_0–mod is seen to be wild unless we are in one of a small number of limited situations where we really have to study the categories A-mod and $\mathcal{F}(\Delta)$. These situations have to be dealt with case by case. (A full classification of C_0-mod into finite, tame and wild type has been obtained by I.Gordon and A.Premet [GoP].)

In Section 2 we collect some information on coinvariant algebras and their relation to blocks of $\mathcal{O}$. In Section 3 we consider some special cases, starting with the regular block for $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. In Section 4 a proof of Theorem 1 and of Theorem 2 is given.

2 Coinvariant algebras

We fix $\mathfrak{G}$, $\mathcal{O}$, the root system Φ and the Weyl group W as before. The Weyl group W is a Coxeter group with length function l. Moreover, we fix a subset, Θ, of the set of simple roots and denote by W_0 the "parabolic" subgroup of W generated by the corresponding simple reflections. Its root system is called Φ_0. The Weyl group acts as a reflection group on the complex vector space V and thus on the symmetric algebra $S(V)$. By definition, the *coinvariant algebra* C is the quotient of $S(V)$ modulo the two-sided ideal generated by homogenous W-invariants of degree at least one. By C_0 we denote the (graded) subalgebra of C consisting of W_0-invariant elements.

It is well-known that C is isomorphic to the cohomology algebra of the flag manifold $\mathcal{G}/\mathcal{B}$. Therefore, Schubert calculus provides us with a special basis of C, which contains a basis of C_0 as well. This has been worked out in Hiller's book [H], from which we take the following information (Chapter IV, in particular Sections 3 and 4).

There is a complete set of left coset representatives of W_0 in W consisting of elements w such that each w has minimal length in its coset. These are precisely those w which satisfy $l(ws) = l(w) + 1$ for all simple reflections $s \in \Theta$. The algebra C_0 has a basis consisting of elements X_w indexed by these coset representatives, such that the following properties are satisfied: The element X_w is homogenous of degree l. For a simple reflection, $s = s_\alpha$, and any element, $w \in W$, the product $X_s X_w$ in C is given by the *Pieri formula*:

$$X_s X_w = \sum_{\beta \in \Phi^+, l(ws_\beta) = l(w)+1} (\beta^\vee, \omega_\alpha) X_{ws_\beta}. \tag{1}$$

If s and w are coset representatives, then this is the product formula in C_0. In particular, if w is chosen in such a way that the product ws is longer than w, then the product $X_s X_w$ is not zero. In fact, in this case X_{ws} occurs with a non–zero coefficient in the right hand term.

Let us now recall the role of C and C_0 in describing blocks of $\mathcal{O}$. Fix a block by choosing an antidominant integral weight λ. Its stabilizer is a subgroup, W_0. Then, by Soergel's *Struktursatz 2* and *Endomorphismensatz 3* from [S], the finite dimensional algebra A associated with this block is the endomorphism ring of a certain module, P, (the "big projective module") over C_0:

Theorem 3 (Soergel, [S]). *$A \simeq End_{C_0}(P)$.*

We remark that C_0 is a symmetric algebra. In fact, by Soergel's result it is the endomorphism ring over A of the unique indecomposable projective–injective module P. Hence C_0 is self–injective. As a subalgebra of C, the algebra C_0 is commutative, thus it is symmetric.

Clearly, the module category of C_0 can be embedded into this block. From [KM] it follows that even the following is true:

Theorem 4. *The category C_0-mod embeds into $\mathcal{F}(\Delta)$.*

Proof. This follows by combining Theorem 4 in [KM] with Theorem 1 in [KM]. Here, we give another - more direct - proof which is built up from arguments already used in [KM].

By Soergel's result, the algebra C_0 is the endomorphism ring of the projective–injective A–module P which we can write as Ae for some primitive idempotent $e \in A$. Thus we can identify C_0 with eAe. It is well–known (see e.g. [A]) that the category eAe–mod is equivalent to the full subcategory of A–mod consisting of modules M which are P–copresented, i.e. there is an exact sequence $0 \to M \to P_1 \xrightarrow{\varphi} P_2$ with $P_1, P_2 \in add(P)$ direct sums of copies of P. Fix such a module M and a copresentation. It will be enough to show that M has a Verma flag.

Let F be a minimal submodule of P_1 such that it has the following properties: F contains M; there exists a Δ-filtration $F_0 \subset F_1 \subset \cdots \subset P_1$ of P_1 such that F equals some F_i. We are going to show that F equals M.

Let us recall some well–known properties of blocks of $\mathcal{O}$ (see e.g. [D, BGG]). The projective module P has a Verma flag. There exists a unique simple Verma module, L, in this block which is the socle of each Verma module. Moreover, L is both the socle and the top of P.

In particular, the image of $\varphi : P_1 \to P_2$ has a socle which is a direct sum of copies of L. By the minimality of F, the quotient F/M cannot have a composition factor isomorphic to L. Since M is sent to zero by φ, F must be in the kernel as well. This implies equality $F = M$. □

The embedding preserves isomorphisms and indecomposability. As observed in [KM], we therefore have the following lower bound for representation types:

Corollary 1. *If C_0–mod has infinite representation type, then $\mathcal{F}(\Delta)$ and A–mod also have infinite type. If C_0–mod is wild, then $\mathcal{F}(\Delta)$ and A–mod are wild as well.*

3 Some explicit examples

In this section we consider some algebras which will appear in special cases where the lower bound from Corollary 1 cannot be used.

3.1 The case $A_1 \times A_1$

If the root system is of type $A_1 \times A_1$, then C_0 has two generators and $\mathbb{C}$–dimension four. In fact, $C_0 \simeq \mathbb{C}[x, y]/(x^2, y^2)$. Then C_0 is known to be tame [Ba]; in fact, it is a string algebra ([GeP, R]) whose Auslander–Reiten quiver is well–known.

The dimension of C_0 coincides with the number of cosets of W_0 in W. Therefore, this case occurs only for the root system $A_1 \times A_1$ with trivial W_0. Picking $(\Phi, \Phi_0) = (B_2, A_1)$ yields the same dimension, but in this case C_0 is generated by one element.

The algebra A by definition is a tensor product of two copies of the principal block B of $\mathfrak{sl}(2)$, i.e. of type A_1. By quiver and relations the last one is given as follows:

$$B: \qquad \begin{tikzcd} a \bullet \arrow[r, shift left, "\alpha"] & \bullet\, b \arrow[l, shift left, "\beta"] \end{tikzcd}$$ modulo the ideal generated by the relation: $\beta \cdot \alpha = 0$.

Therefore, the algebra A has the following quiver and relations:

$$A: \qquad \begin{tikzcd} a \bullet \arrow[r, shift left, "\alpha"] \arrow[d, shift right, "\gamma"'] & \bullet\, b \arrow[l, shift left, "\beta"] \arrow[d, shift right, "\gamma"'] \\ c \bullet \arrow[r, shift left, "\alpha"] \arrow[u, shift right, "\delta"'] & \bullet\, d \arrow[l, shift left, "\beta"] \arrow[u, shift right, "\delta"'] \end{tikzcd}$$

modulo the ideal generated by the following relations:

$\beta \cdot \alpha = 0, \quad \delta \cdot \gamma = 0,$

$\alpha \cdot \gamma = \gamma \cdot \alpha, \quad \alpha \cdot \delta = \delta \cdot \alpha,$

$\beta \cdot \gamma = \gamma \cdot \beta, \quad \beta \cdot \delta = \delta \cdot \beta.$

Here are the composition series of the indecomposable projective modules:

$$P(a) = \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array}, \quad P(b) = \begin{array}{ccc} & b & \\ d & & a \\ b & & c \\ & d & \end{array}, \quad P(c) = \begin{array}{ccc} & c & \\ a & & d \\ b & & c \\ & d & \end{array}, \quad P(d) = \begin{array}{ccccc} & & d & & \\ & b & & c & \\ d & & a & & d \\ & b & & c & \\ & & d & & \end{array}.$$

The projective resolutions of the standard modules look as follows:

$$0 \to P(a) \to \Delta(a) = \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array} \to 0$$

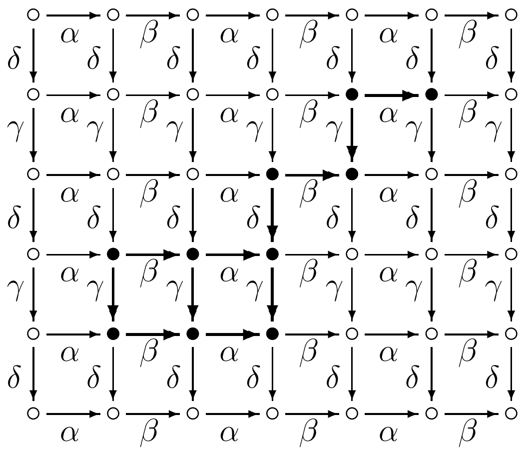
$$0 \to P(a) \to P(b) \to \Delta(b) = \begin{array}{c} b \\ d \end{array} \to 0$$

$$0 \to P(a) \to P(c) \to \Delta(c) = \begin{array}{c} c \\ d \end{array} \to 0$$

$$0 \to P(a) \to P(b) \oplus P(c) \to P(d) \to \Delta(d) = \ d \ \to 0$$

Lemma 1. *The algebra A is wild.*

Proof. The universal covering $\widetilde{A}$ of the algebra A is given by a quiver, $\widetilde{Q}$, which is a two-dimensional lattice with arrows labeled as in the figure below, and satisfying the same relations as for the algebra A. We also marked in the figure a convex subcategory, E, (supported by the 10 thick vertices) of the covering $\widetilde{A}$. As E is contained in Unger's list of wild algebras [U], we conclude that $\widetilde{A}$ and therefore also A is wild. $\square$



We now turn to investigate the category $\mathcal{F}(\Delta)$. Unfortunately, it is sandwiched between the tame category C_0–mod and the wild module category A–mod, hence we have to use different methods to determine the representation type of $\mathcal{F}(\Delta)$ in this case. We therefore recall from [K] some facts on A_∞–categories: Let $E = \oplus_{i\geq 0} \operatorname{Ext}^i_A(\oplus\Delta(x), \oplus\Delta(x))$ denote the (graded) Ext–algebra of $\mathcal{F}(\Delta)$. It is endowed with an A_∞–structure, given by maps $m_n : E^{\otimes n} \to E$ for $n \geq 2$, where $m_2 : E \otimes E \to E$ is the Yoneda product. The data $(E, (m_n)_{n\geq 2})$ define a category $\operatorname{filt}(E)$, which we refer to as the category of representations of the A_∞–algebra E. In case the maps m_n vanish for $n \geq 3$, the algebra (E, m_2) is a graded associative algebra and the category $\operatorname{filt}(E)$ is what is known as the category of representations of the differential graded algebra E (with zero differential, of course).

Proposition 1 (Keller [K],Theorem 7.7). *The category $\mathcal{F}(\Delta)$ is equivalent to filt(E).*

Thus we can determine the representation type of $\mathcal{F}(\Delta)$ by studying the category $\operatorname{filt}(E)$.

Proposition 2. *The category $\mathcal{F}(\Delta)$ is tame.*

Proof. Of course, we do not verify tameness directly; instead, we reduce this problem to some problem which is known to be tame. First compute the Yoneda product of the Ext–algebra E of $\mathcal{F}(\Delta)$ (e.g. by representing elements in E as shifted morphisms between complexes which are given by the above projective resolutions). It is given by the following (bi–)quiver Q with relations, where the arrows labeled by greek letters describe the degree 0 part of E and the latin letters stand for the degree 1 part.

$$E:\quad \begin{tikzcd} \Delta(d) \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "a_1"'] \arrow[d, shift right, "\beta_1"'] \arrow[d, shift left, "b_1"] \arrow[dr, shift left, "c_1"] \arrow[dr, shift right, "c_2"'] & \Delta(b) \arrow[d, shift right, "a_2"'] \arrow[d, shift left, "\alpha_2"] \\ \Delta(c) \arrow[r, shift left, "b_2"] \arrow[r, shift right, "\beta_2"'] & \Delta(a) \end{tikzcd}$$

relations:

$\alpha_2 \cdot \alpha_1 = \beta_2 \cdot \beta_1$

$a_2 \cdot \alpha_1 = c_1 = \beta_2 \cdot b_1$

$\alpha_2 \cdot a_1 = c_2 = b_2 \cdot \beta_1$

$a_2 \cdot a_1 = b_2 \cdot b_1.$

Since A is quasi-hereditary, the algebra E is directed. In fact, there are no paths in the quiver Q of length > 2, hence the maps m_n vanish for $n \geq 3$ (c.f. [K], section 3.4). Thus, the category $\mathrm{filt}(E)$ can be described as follows: A representation $V \in \mathrm{filt}(E)$ is given by linear maps V_x for each latin letter x of Q such that the relation $V_{a_2}V_{a_1} = V_{b_2}V_{b_1}$ holds. In other words, the objects of the category $\mathrm{filt}(E)$ are the representations of the quiver (with relations) given by the latin letters.

The arrows indexed by greek letters come into play when we define morphisms: A morphism in $\mathrm{filt}(E)$ from V to W is given by a family of linear maps

$$H = (H_{\Delta(a)}, H_{\Delta(b)}, H_{\Delta(c)}, H_{\Delta(d)}, H_{\alpha_1}, H_{\alpha_2}, H_{\beta_1}, H_{\beta_2})$$

satisfying $H_{\alpha_2}H_{\alpha_1} = H_{\beta_2}H_{\beta_1}$ such that the following equations hold:

$$H_{\Delta(b)}V_{a_1} = W_{a_1}H_{\Delta(d)}, \quad H_{\Delta(a)}V_{a_2} = W_{a_2}H_{\Delta(b)},$$

$$H_{\Delta(c)}V_{b_1} = W_{b_1}H_{\Delta(d)}, \quad H_{\Delta(a)}V_{b_2} = W_{b_2}H_{\Delta(c)},$$

$$H_{\beta_2}V_{b_1} + H_{\Delta(a)}V_{c_1} = W_{c_1}H_{\Delta(d)} + W_{a_2}H_{\alpha_1},$$

$$H_{\alpha_2}V_{a_1} + H_{\Delta(a)}V_{c_2} = W_{c_2}H_{\Delta(d)} + W_{b_2}H_{\beta_1}.$$

In order to show finally that $\mathrm{filt}(E)$ is tame, we use the classical reduction method for differential graded categories ([KR1, KR2]): Consider the (graded) subalgebra G of E generated by the arrows c_1 and c_2. The reduction of E with respect to G leads to a new differential graded category, K, whose category of representations is equivalent to $\mathrm{filt}(E)$.

In our case, the quotient E/G is given by a commutative square, a well-known representation-finite algebra. It is therefore easy to compute the category K and it turns out that K is given by a bundle of chains in the sense of [Bo], namely by the following two pairs of posets

$$(x \to \bullet\,, \quad y \to \bullet) \quad \text{and} \quad (\tilde{x} \to \bullet\,, \quad \tilde{y} \to \bullet)$$

with involution $x \mapsto \tilde{x}$ and $y \mapsto \tilde{y}$. Thus, by [Bo] or equivalently [C], the category of representations of K is tame and its indecomposable objects have been completely classified.

$\square$

3.2 An enlargement of $A_1 \times A_1$

When proving the theorems we have to consider some enlargements of the case $A_1 \times A_1$. We therefore collect in the next two subsections auxiliary statements about quasi-hereditary algebras that are given by composition series of projective modules and by projective resolutions of standard modules. In the proofs later on, quotients and subalgebras of blocks of category $\mathcal{O}$ will appear which have precisely these properties (this in particular will ensure the existence of these algebras). It will turn out that for proving tameness or wildness of the corresponding categories $\mathcal{F}(\Delta)$ the partial information we have on these algebras is sufficient.

The application we are aiming at allows us to make some assumptions which simplify notation. In particular, we only consider algebras with a duality. In category $\mathcal{O}$ terminology the algebras appearing in this context are multiplicity free. Thus all indecomposable projective modules are contained in a unique 'big' projective module. The quotients or subalgebras to be studied now inherit this property which explains another assumption made below.

Thus, let A be a quasi-hereditary algebra (admitting a duality fixing simples) which has five primitive idempotents $\{a, b, c, d, e\}$ such that the standard modules have the following projective resolutions:

$$0 \to P(a) \to \Delta(a) = \begin{array}{ccc} & a & \\ b & & c \\ & d & \\ & e & \end{array} \to 0$$

$$0 \to P(a) \to P(b) \to \Delta(b) = \begin{array}{c} b \\ d \\ e \end{array} \to 0$$

$$0 \to P(a) \to P(c) \to \Delta(c) = \begin{array}{c} c \\ d \\ e \end{array} \to 0$$

$$0 \to P(a) \to P(b) \oplus P(c) \to P(d) \to \Delta(d) = \begin{array}{c} d \\ e \end{array} \to 0$$

$$0 \to P(d) \to P(e) \to \Delta(e) = \ e \ \to 0$$

The composition series of the indecomposable projective modules are:

$$P(a) = \begin{array}{ccc} & a & \\ b & & c \\ & d & \\ & e & \end{array}, \quad P(b) = \begin{array}{cccc} & b & & \\ d & & a & \\ e & b & & c \\ & & d & \\ & & e & \end{array}, \quad P(c) = \begin{array}{cccc} & & c & \\ & a & & d \\ b & & c & e \\ & d & & \\ & e & & \end{array}, \quad P(d) = \begin{array}{ccccc} & & d & & \\ & b & e & c & \\ d & & a & & d \\ e & b & & c & e \\ & & d & & \\ & & e & & \end{array},$$

$$P(e) = \begin{array}{ccccc} & & e & & \\ & & d & & \\ & b & e & c & \\ d & & a & & d \\ e & b & & c & e \\ & & d & & \\ & & e & & \end{array}$$

We further suppose that the projectives admit inclusions $P(a) \subset P(b) \subset P(d)$, $P(a) \subset P(c) \subset P(d)$ and $P(d) \subset P(e)$.

Proposition 3. *The category $\mathcal{F}(\Delta)$ is wild.*

Proof. As in the proof of Proposition 2, we consider $E = \oplus_{i \geq 0} \operatorname{Ext}^i_A(\oplus\Delta(x), \oplus\Delta(x))$, the Ext–algebra of the standard modules. Let F be the (graded) subalgebra of E generated by extensions of $\Delta(a) \oplus \Delta(d) \oplus \Delta(e)$, i.e.

$$F = \oplus_{i \geq 0} \operatorname{Ext}^i_A(\Delta(a) \oplus \Delta(d) \oplus \Delta(e), \Delta(a) \oplus \Delta(d) \oplus \Delta(e)).$$

We know that E is endowed with an A_∞–structure such that $\mathcal{F}(\Delta)$ and $\operatorname{filt}(E)$ are equivalent. Our strategy is the following: we show that the A_∞–structure restricted to F is just given by the Yoneda-product (i.e. $m_n : E^{\otimes n} \to E$ vanishes for $n \geq 3$ when restricted to $F^{\otimes n}$). Thus, $\operatorname{filt}(F)$ can be viewed as the category of representations of the (differential) graded algebra F and it is a full subcategory of $\operatorname{filt}(E)$. Therefore it suffices to show that $\operatorname{filt}(F)$ is wild. Note that this proof cannot be obtained on the level of quasi-hereditary algebras: the category $\operatorname{filt}(F)$ consists of those objects in $\mathcal{F}(\Delta)$ whose multiplicity of $\Delta(b)$ and $\Delta(c)$ is zero. The corresponding algebra fAf with $f = a + d + e$ however is not quasi-hereditary.

From the projective resolutions of the standard modules we see that the quiver Q of the graded algebra F contains the following wild subquiver Γ with all arrows of degree 1:

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \rightrightarrows & \bullet \\ \Delta(e) & & \Delta(d) & & \Delta(a) \end{array}$$

The quiver Q of F is directed since A is quasi-hereditary. As a directed quiver with three vertices, Q has no paths of length more than two, hence the maps $m_n : E^{\otimes n} \to E$ vanish when restricted to $F^{\otimes n}$ for $n \geq 3$. From the definition of morphisms in $\operatorname{filt}(F)$ it follows that the obvious functor $\operatorname{rep} \Gamma \to \operatorname{filt}(F)$ preserves isomorphism classes and indecomposability, hence the category $\operatorname{filt}(F)$ is wild. $\square$

3.3 Another enlargement of $A_1 \times A_1$

We consider finally a second enlargement of the case $A_1 \times A_1$: Let A be a quasi-hereditary algebra (admitting a duality fixing simples) with primitive idempotents $\{a, a', b, c, d\}$ whose standard modules have the following projective resolutions:

$$0 \to P(a) \to \Delta(a) = \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array} \to 0$$

$$0 \to P(a') \to \Delta(a') = \begin{array}{ccc} & a' & \\ b & & c \\ & d & \end{array} \to 0$$

$$0 \to P(a) \oplus P(a') \to P(b) \to \Delta(b) = \begin{array}{c} b \\ d \end{array} \to 0$$

$$0 \to P(a) \oplus P(a') \to P(c) \to \Delta(c) = \begin{array}{c} c \\ d \end{array} \to 0$$

$$0 \to P(a) \oplus P(a') \to P(b) \oplus P(c) \to P(d) \to \Delta(d) = d \to 0$$

The composition series of the indecomposable projective modules are:

$$P(b) = \begin{array}{cccc} & b & & \\ d & a & a' & \\ b & c & c & b \\ d & & & d \end{array}, \quad P(c) = \begin{array}{cccc} & & c & \\ & a & a' & d \\ b & c & c & b \\ d & & & d \end{array}, \quad P(d) = \begin{array}{cccc} & \multicolumn{2}{c}{d} & \\ & b & c & \\ d & a & a' & d \\ b & c & c & b \\ d & & & d \end{array}.$$

Proposition 4. *The category $\mathcal{F}(\Delta)$ is wild in this case.*

Proof. This works completely analogous to the case we solved before: From the projective resolutions of the standard modules we obtain the following wild subquiver Γ of E with all arrows of degree 1:

$$\begin{tikzcd} \underset{\Delta(a)}{\bullet} & \underset{\Delta(d)}{\bullet} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] & \underset{\Delta(a')}{\bullet} \end{tikzcd}$$

It is also clear that $\mathrm{Hom}_A(\Delta(a), \Delta(a'))$ and $\mathrm{Hom}_A(\Delta(a'), \Delta(a))$ are zero, thus the wildness of the subquiver Γ implies that $\mathcal{F}(\Delta)$ is wild. $\square$

4 Proof of Theorems

We fix A and C_0 as before. Recall that C_0 is a graded symmetric algebra. We will go through several cases depending on Φ_0. In each case, the first step of the proof consists of determining the representation type of C_0. If C_0 is wild, then this case is finished. Only few situations remain where C_0 is not wild. In a second step we have to deal with these situations.

In order to decide about the representation type of C_0 we will use the following well–known lemma.

Lemma 2. *A local algebra is wild if its minimal number of generators is at least three.*

Proof. This follows from [HR]; see also [E], I.10.10 (a). □

Now let us distinguish four cases according to the number of simple reflections not contained in Θ, i.e. the corank of Φ_0 in Φ. If this number is zero, then $W = W_0$ and the algebra A is simple. If the corank is three or bigger, then the minimal number of generators of C_0 is at least three, hence by Lemma 2 the algebra C_0 is wild. Thus it remains to consider the two cases of the corank being one or two.

Let us first suppose that Φ_0 has corank two. There are two simple reflections, s_α and s_β, outside of Θ. Denote by w the product $s_\alpha s_\beta$. The four elements 1, s_α, s_β and w are minimal length representatives of four different cosets of W_0 in W, and each of them defines a basis element in C_0. If the dimension of C_0 is four, then we are in the case $A_1 \times A_1$ which has been discussed in subsection 3.1. Otherwise the dimension of C_0 must be at least six since it is symmetric. Passing to quotients or subalgebras of A if necessary we arrive at an algebra A as in subsection 3.3 or at an algebra A' which differs from this A by having some multiplicities (of simple modules in Verma modules) bigger than one, i.e. by some arrows occuring with multiplicities. Clearly, the case of A' also falls into wild type. Note that in this case our lower bound may be too weak, since C_0 may be tame (e.g. in type A_2, see [GoP]).

It remains to consider the case of Φ_0 having corank one. Denote by s the simple reflection which is not in Θ. In degree one, the graded algebra C_0 consists of the scalar multiples of X_s. If all other degrees are one–dimensional as well, then C_0 is isomorphic to $\mathbb{C}[x]/(x^m)$ for some m. It is obviously of finite representation type, thus our lower bound is trivial.

In this case, we can determine A itself. In fact, by Soergel's double centralizer property (Theorem 3), the algebra A then is the Auslander algebra of C_0, i.e. the endomorphism ring of the sum of all indecomposable C_0–modules. By [DR], Section 7, the category $\mathcal{F}(\Delta)$ has finite representation type if and only if $m \leq 5$; it is tame for $m = 6$ and wild otherwise. The full category A–mod is of finite type for $m \leq 3$ only, tame for $m = 4$ and wild otherwise.

The number m is the $\mathbb{C}$–dimension of C_0, hence it equals the number of cosets of W_0 in W. Consequently, we have the following choices for the root systems (Φ, Φ_0): For $m = 1$, of course, $\Phi = \Phi_0$ and no further restriction. For $m = 2$ we have $(A_1, \emptyset)$. For $m = 3$ we have (A_2, A_1); $m = 4$ means (A_3, A_2) or (B_2, A_1); $m = 5$ occurs for (A_4, A_3) and $m = 6$ happens precisely for (A_5, A_4) and for (B_3, B_2).

From now on, we may assume that we are not in the situation of Lemma 2, that Φ_0 has corank one and that some graded pieces of C_0 have $\mathbb{C}$-dimension bigger than one. Denote the smallest degree where this happens by $l + 1$. Note that the dimension of C_0 must be bigger than four in this case. As before, we have a generator X_s associated with a simple reflection, $s \notin \Theta$. If the dimension in degree $l + 1$ is bigger than two, then we are in the situation of Lemma 2. Thus we may assume that the dimension is two. Denote the two standard basis elements by X_{w_1} and X_{w_2}.

If $w_1 s$ is longer than w_1, then the Pieri formula ((1) in Section 2) implies that the product $X_{w_1} X_s$ is not zero. If also $w_2 s$ is longer than w_2, then one of the elements X_{w_1} and X_{w_2} is an additional generator and (passing to quotients or subalgebras if necessary and up to forgetting higher multiplicities) we are in the situation of subsection 3.2.

Suppose now that $w_1 s$ is longer than w_1, but $w_2 s$ is shorter than w_2. Then the element $w_2 s$ must be the minimal length representative of its W_0–coset (if not, write $w_2 s = w_0 v$ for some $w_0 \in W_0$ and v of length at most $l - 1$; then $w_2 = w_0 v s$ cannot be minimal, since its length $l + 1$ is bigger than that of vs). Therefore $X_{w_2 s}$ generates the degree l–piece of C_0. The Pieri formula now tells us that X_{w_2} occurs with a non–zero coefficient in the expansion of the product $X_{w_2 s} X_s$. It follows that X_{w_1} is an additional generator and we are again in the situation of subsection 3.2.

Finally, it cannot happen that both $w_1 s$ and $w_2 s$ are shorter than w_1 and w_2. If so, by the same arguments as above, the basis elements $X_{w_1 s}$ and $X_{w_2 s}$ would be linearly dependent. Hence $w_1 s$ and $w_2 s$ would have to coincide, a contradiction.

This finishes the proofs of Theorems 1 and 2.

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