Reduction of Shimura varieties at parahoric levels

by

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(December 15, 1999)

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Introduction

One aim of this paper is to give some evidence for a generalization of the conjecture of Langlands and Rapoport [4] (§5) concerning the reduction of a Shimura variety Sh_G attached to a connected reductive group G over $\mathbb{Q}$ with simply connected derived group and a $G(\mathbb{R})$−conjugacy class of homomorphisms $h\colon \mathbb{S} = \mathbb{C}^\times \to G_\mathbb{R}$ satisfying the usual conditions (cf. [3], 2.1.1.1–3). Assuming that G splits over an unramified extension of $\mathbb{Q}_p$ for a rational prime p, we give a conjectural description of a suitable reduction of the varieties $Sh_{G,C}$ at a place $\wp$ of the reflex field dividing p, where $C = C_p \times C^p$ for a fixed subgroup $C_p \subset G(\mathbb{Q}_p)$ and every sufficiently small open subgroup $C^p \subset G(\mathbb{A}_f^p)$. In the original conjecture, C_p was assumed to be either hyperspecial or the only maximal compact subgroup of $G(\mathbb{Q}_p)$. In the present paper C_p may be any parahoric subgroup. The generalized conjecture is explained in §1. In the remainder of the paper we consider the special case of a quaternionic Shimura variety. Since our second aim is to provide the geometric information which is necessary to generalize the results of [8] concerning the zeta function of quaternionic Shimura varieties, we extend most of the results of Part I in [7] to the case of an arbitrary parahoric subgroup C_p. We will often give rather short arguments and refer to [7] for details. In particular, we will see, that there are cases of good reduction where C_p is not maximal compact, and we conjecture that, in general, the existence of a good reduction can be characterized by a simple condition for the data G, h, C_p, and the place $\wp$ which is stated in §1.

§1. The general conjecture

Before we fix the settings for the present section, we recall some facts about parahoric subgroups.

Remark 1.1 Let G be a connected reductive group over a field K which is henselian with respect to a non-trivial discrete valuation with perfect residue class field. For every facet $\mathfrak{F}$ in the Bruhat–Tits building $\mathcal{B}(G,K)$, let $\mathcal{P}_\mathfrak{F} \subseteq G(K)$ denote the corresponding parahoric subgroup (cf. [2], 5.2). Then, $\mathfrak{F}$ is contained in the closure of another facet $\mathfrak{F}'$ if and only if $\mathcal{P}_{\mathfrak{F}'} \subseteq \mathcal{P}_\mathfrak{F}$ (cf. [2], 4.6.27 and proof of 5.2.8). In particular, there is a bijection between the set of facets of $\mathcal{B}(G,K)$ and the set of parahoric subgroups of $G(K)$. Moreover, the subgroup $G' \subseteq G(K)$ generated by all parahoric subgroups acts transitively on the set of chambers in $\mathcal{B}(G,K)$, and for every chamber $\mathfrak{C}$ with $\mathfrak{F} \subseteq \overline{\mathfrak{C}}$, the set $\{\, g \in G' \mid g\mathfrak{F} \subseteq \overline{\mathfrak{C}} \,\}$ coincides with $\mathcal{P}_\mathfrak{F}$ (cf. [2], 5.2.12 and [1], 2.1.5 and 2.1.6).

Lemma 1.2 *Let K and G be as above, and assume that the derived group G^{der} is simply connected and that G splits over an unramified extension of K. Let G^{ab} denote the maximal abelian quotient of G and $G^1 \subseteq G(K)$ the pre-image of the maximal compact subgroup of $G^{\mathrm{ab}}(K)$. Then, for every facet $\mathfrak{F}$ in $\mathcal{B}(G,K)$ and every point $x \in \mathfrak{F}$,*

$$\mathcal{P}_{\mathfrak{F}} = \mathrm{Stab}_{G(K)}(x) = \mathrm{Stab}_{G^1}(\mathfrak{F}) .$$

Proof. It is clear from the definition of a parahoric subgroup that we may replace K with its maximal unramified extension and $\mathfrak{F}$ with the unique facet containing it in the corresponding larger building. Let $S \subseteq G$ be a split maximal torus. Then we have an exact sequence of split tori

$$1 \longrightarrow S^{\mathrm{der}} \longrightarrow S \longrightarrow G^{\mathrm{ab}} \longrightarrow 1 .$$

It gives rise to a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & S^{\mathrm{der}}(K) \arrow[r] \arrow[d, "\mathrm{ord}_{S^{\mathrm{der}}}"] & S(K) \arrow[r] \arrow[d, "\mathrm{ord}_S"] & G^{\mathrm{ab}}(K) \arrow[r] \arrow[d, "\mathrm{ord}_{G^{\mathrm{ab}}}"] & 1 \\ 0 \arrow[r] & X_*(S^{\mathrm{der}}) \arrow[r] & X_*(S) \arrow[r] & X_*(G^{\mathrm{ab}}) \arrow[r] & 0 \end{tikzcd}$$

with exact rows and surjective columns. Let $H \subseteq S(K)$ and $H^{\mathrm{ab}} \subseteq G^{\mathrm{ab}}(K)$ denote the maximal compact subgroups. Applying the snake lemma to the diagram above, we see that the map $H \to H^{\mathrm{ab}}$ is surjective. This implies that $G^1 = G^{\mathrm{der}}(K) \cdot H$. Since G^{der} is simply connected, $G^{\mathrm{der}}(K)$ is contained in the subgroup $G' \subseteq G(K)$ defined above (cf. [2], 5.2.11), hence $G^1 = G'$. Together with the inclusions

$$\mathcal{P}_{\mathfrak{F}} \subseteq \mathrm{Stab}_{G(K)}(x) \subseteq \mathrm{Stab}_{G^1}(\mathfrak{F})$$

(cf. [2], 4.6.28) and the preceding remark this implies the claim. ∎

Let now Sh_G be a Shimura variety associated with a connected reductive group G over $\mathbb{Q}$ with simply connected derived group G^{der} and a homomorphism $h\colon \mathbb{S} \to G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$. Let $\mathfrak{G}_G$ be the neutral $\overline{\mathbb{Q}}|\mathbb{Q}$−groupoid defined by G and $\mathfrak{G}_\infty$ the $\mathbb{C}|\mathbb{R}$−groupoid corresponding to $(\mathbb{G}_{m,\mathbb{R}}\, , g_{\iota,\iota} = -1)$ where ι denotes complex conjugation (cf. [7], §B1). For every place ℓ of $\mathbb{Q}$, we fix an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_\ell$ and denote the pull-back of a $\overline{\mathbb{Q}}|\mathbb{Q}$−groupoid $\mathfrak{G}$ to $\overline{\mathbb{Q}}_\ell|\mathbb{Q}_\ell$ by $\mathfrak{G}(\ell)$. The image of the canonical cocharacter of $\mathbb{S}$ with respect to h is a homomorphism $\mu_h\colon \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ of algebraic groups over $\mathbb{C}$, and restriction of h^{-1} to $\mathbb{G}_{m,\mathbb{R}} \subset \mathbb{S}$ determines a cocharacter w_h of the maximal $\mathbb{R}$−split torus in the center of G. The pair $(w_h, f_\iota = \mu_h(-1))$ corresponds to a morphism of $\mathbb{C}|\mathbb{R}$−groupoids

$$\xi_\infty\colon \mathfrak{G}_\infty \longrightarrow \mathfrak{G}_G(\infty) .$$

The equivalence class of ξ_∞ is uniquely determined by the $G(\mathbb{R})$−conjugacy class of the homomorphism h.

We fix a rational prime p such that G splits over an unramified extension of $\mathbb{Q}_p$ and denote the completion of the maximal unramified extension $\mathbb{Q}_p^{\mathrm{nr}} \subset \overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ by $\mathcal{K}$. If S and S' are split maximal tori in $G_{\mathcal{K}}$ and W_S and $W_{S'}$ denote their Weyl groups, there is an element $g \in G(\mathcal{K})$ such that $S' = gSg^{-1}$, and the bijection

$$(X_*(S) \otimes_{\mathbb{Z}} \mathbb{R})/W_S \longrightarrow (X_*(S') \otimes_{\mathbb{Z}} \mathbb{R})/W_{S'}$$

induced by $\mathrm{ad}(g)$ does not depend on the choice of g. Let

$$\widetilde{X}(G,\mathcal{K}) = \varprojlim_S (X_*(S) \otimes_{\mathbb{Z}} \mathbb{R})/W_S\,,$$

and for every field K of characteristic 0, let $\mathcal{C}(G,K)$ be the set of $G(K)$−conjugacy classes of homomorphisms $\mathbb{G}_{m,K} \to G_K$. There are natural bijections (cf. [6], 1.7)

$$\varprojlim_S X_*(S)/W_S \longrightarrow \mathcal{C}(G,\mathcal{K}) \longleftarrow \mathcal{C}(G,\mathbb{Q}_p^{\mathrm{nr}}) \longrightarrow \mathcal{C}(G,\overline{\mathbb{Q}}_p) \longleftarrow \mathcal{C}(G,\overline{\mathbb{Q}}) \longrightarrow \mathcal{C}(G,\mathbb{C})$$

hence a canonical injection $\mathcal{C}(G,\mathbb{C}) \subset \widetilde{X}(G,\mathcal{K})$, and we denote the image of a homomorphism $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ in $\widetilde{X}(G,\mathcal{K})$ by $[\mu]$. Moreover, if $x, y \in \mathcal{B}(G,\mathcal{K})$ are any two points and S a split maximal torus in $G_{\mathcal{K}}$ such that x and y are contained in the corresponding apartment, then there is a unique element $v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $x = y + v$, and the image $\mathrm{inv}(x,y) \in \widetilde{X}(G,\mathcal{K})$ of v is uniquely determined by the $G(\mathcal{K})$−orbit of the pair (x,y).

Consider now a chamber $\mathfrak{C} \subseteq \mathcal{B}(G,\mathbb{Q}_p)$. Let $\mathfrak{F}_1,\ldots,\mathfrak{F}_r$ be all facets of minimal dimension in the closure of the unique chamber in $\mathcal{B}(G,\mathcal{K})$ containing $\mathfrak{C}$ (cf. [9], 1.10.3), and let $x_i \in \mathfrak{F}_i$ be any points. The set

$$Y_h = \{\, g \in G(\mathcal{K}) \mid \mathrm{inv}(gx_i,x_i) = [\mu_h] \text{ for all } i \in \{1,\ldots,r\}\,\}$$

depends only on the $G(\mathbb{R})$−conjugacy class of h and the chamber $\mathfrak{C}$, and if $\mathfrak{C}$ is replaced with $g\mathfrak{C}$ for a $g \in G(\mathbb{Q}_p)$, Y_h is replaced with gY_hg^{-1}. Moreover, if σ denotes the Frobenius automorphism of $\mathbb{Q}_p^{\mathrm{nr}}$ and $e = [\,E : \mathbb{Q}_p\,]$ where E denotes the subfield of $\mathbb{Q}_p^{\mathrm{nr}}$ such that $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}|E)$ is the stabilizer of $[\mu_h]$ for the natural action of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}|\mathbb{Q}_p)$ on $\widetilde{X}(G,\mathcal{K})$, then Y_h is invariant under the action of σ^e on $G(\mathcal{K})$.

Let $\mathfrak{G}_p$ be the pull-back of the Dieudonné groupoid $\mathfrak{D}$ to $\overline{\mathbb{Q}}_p|\mathbb{Q}_p$ and d_σ the distinguished element of $\mathfrak{D}(\mathbb{Q}_p^{\mathrm{nr}})$ where $\mathbb{Q}_p^{\mathrm{nr}}$ is considered as a $\mathbb{Q}_p^{\mathrm{nr}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}}$−algebra via the map $x \otimes y \mapsto x\sigma(y)$ (cf. [7], §B1). For every morphism $\theta\colon \mathfrak{G}_p \to \mathfrak{G}_G(p)$, there is a morphism θ_0 from $\mathfrak{D}$ to the neutral $\mathbb{Q}_p^{\mathrm{nr}}|\mathbb{Q}_p$−groupoid defined by $G_{\mathbb{Q}_p}$ and an element $g_0 \in \mathrm{Isom}(\theta,\bar{\theta}_0)(\mathbb{Q}_p)$ where $\bar{\theta}_0$ denotes the pull-back of θ_0 (cf. [6], Proposition 3.35). Let $b = \theta_0(d_\sigma) \in G(\mathbb{Q}_p^{\mathrm{nr}})$ and

$$\begin{aligned} \iota\colon\ \mathrm{Aut}(\theta)(\mathbb{Q}_p) &\longrightarrow \mathrm{Aut}(\theta_0)(\mathbb{Q}_p) \subset G(\mathbb{Q}_p^{\mathrm{nr}}) \\ \alpha &\longmapsto g_0\alpha g_0^{-1}\,, \end{aligned}$$

and consider the map

$$\mathcal{F}\colon G(\mathcal{K}) \longrightarrow G(\mathcal{K})$$
$$g \longmapsto b\sigma(g)\,.$$

The σ–conjugacy class $\mathrm{cls}(\theta) \subset G(\mathcal{K})$ of b and the isomorphism class of the set

$$X(\theta) = \{\, g \in G(\mathcal{K}) \mid \mathcal{F}(g) \in gY_h \,\}$$

together with the automorphism Φ induced by $\mathcal{F}^e$, the action of $\mathrm{Aut}(\theta)(\mathbb{Q}_p)$ by left multiplication via ι, and the obvious action of $Z_G(\mathbb{Q}_p) \subset G(\mathcal{K})$ are independent of the choice of θ_0 and g_0 (cf. [7], §B3). Moreover, for every $g \in G(\mathbb{Q}_p)$, right multiplication by g^{-1} induces an isomorphism between $X(\theta)$ together with these actions and the analogous set which is obtained using the chamber $g\mathfrak{C}$ instead of $\mathfrak{C}$.

For every prime $\ell \neq p$, let $\mathfrak{G}_\ell$ denote the trivial $\overline{\mathbb{Q}}_\ell|\mathbb{Q}_\ell$–groupoid and

$$\xi_\ell\colon \mathfrak{G}_\ell \longrightarrow \mathfrak{G}_G(\ell)$$

the canonical section. Let $(\,\mathfrak{Q},\, \zeta_\ell\colon \mathfrak{G}_\ell \to \mathfrak{Q}(\ell) \mid \ell \text{ place of } \mathbb{Q}\,)$ be a quasi-motivic groupoid (cf. [7], Definition B2.7) and

$$\psi_h\colon \mathfrak{Q} \longrightarrow \mathfrak{G}_{G^{\mathrm{ab}}}$$

the morphism of $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoids corresponding to the cocharacter of G^{ab} induced by μ_h (cf. [7], Definition B2.10).

Definition 1.3 A morphism $\varphi\colon \mathfrak{Q} \to \mathfrak{G}_G$ of $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoids is called *admissible* if it satisfies the following conditions.

(i) The morphism $\varphi^{\mathrm{ab}}\colon \mathfrak{Q} \to \mathfrak{G}_{G^{\mathrm{ab}}}$ induced by φ is equivalent to ψ_h.

(ii) For every place $\ell \neq p$ of $\mathbb{Q}$, $\varphi(\ell) \circ \zeta_\ell$ is equivalent to ξ_ℓ.

(iii) The σ–conjugacy class $\mathrm{cls}(\varphi(p) \circ \zeta_p) \subset G(\mathcal{K})$ contains an element of Y_h.

This definition is independent of the choice of the chamber $\mathfrak{C}$. For every admissible morphism $\varphi\colon \mathfrak{Q} \to \mathfrak{G}_G$, we consider the set

$$X^p(\varphi) = \{\, g \in G(\overline{\mathbb{A}_f^p}) \mid \text{for all } \ell \neq \infty, p,\; g_\ell \in \mathrm{Isom}(\xi_\ell\,, \varphi(\ell) \circ \zeta_\ell)(\mathbb{Q}_\ell) \,\}$$

where $\overline{\mathbb{A}_f^p}$ denotes the image of $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ in $\prod_{\ell \neq \infty, p} \overline{\mathbb{Q}}_\ell$, and g_ℓ the image of g in $G(\overline{\mathbb{Q}}_\ell)$. The group $\mathrm{Aut}(\varphi)(\mathbb{Q}) \subset G(\overline{\mathbb{Q}})$ acts on $X^p(\varphi)$ by left multiplication and on

$$X_p(\varphi) = X(\varphi(p) \circ \zeta_p)$$

via the canonical embedding into $\mathrm{Aut}(\varphi(p) \circ \zeta_p)(\mathbb{Q}_p)$.

Fix now a parahoric subgroup $C_p \subset G(\mathbb{Q}_p)$. The corresponding facet $\mathfrak{F}$ of $\mathcal{B}(G,\mathbb{Q}_p)$ is contained in a facet of $\mathcal{B}(G,\mathcal{K})$ which determines a $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}|\mathbb{Q}_p)$–invariant parahoric subgroup $C_p(\mathcal{K}) \subset G(\mathcal{K})$ such that

$$C_p(\mathcal{K}) \cap G(\mathbb{Q}_p) = C_p\,.$$

For an admissible morphism $\varphi\colon \mathfrak{Q} \to \mathfrak{G}_G$, let $X_p(\varphi) \subseteq G(\mathcal{K})$ be defined as above with respect to any chamber $\mathfrak{C}$ in $\mathcal{B}(G,\mathbb{Q}_p)$ with $\mathfrak{F} \subseteq \overline{\mathfrak{C}}$. Since by Remark 1.1 every other chamber having this property is of the form $g\mathfrak{C}$ for a $g \in C_p$, the image $X_p(\varphi)/C_p(\mathcal{K})$ of this set in $G(\mathcal{K})/C_p(\mathcal{K})$ is independent of the choice of $\mathfrak{C}$. Let

$$S(\varphi) = \varprojlim_{C^p} \mathrm{Aut}(\varphi)(\mathbb{Q})\backslash\big(X_p(\varphi)/C_p(\mathcal{K}) \times X^p(\varphi)/C^p\big)\,,$$

where the limit is extended over all open subgroups $C^p \subset G(\mathbb{A}_f^p)$. The isomorphism class of the set $S(\varphi)$ together with the obvious right action of $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ and the automorphism Φ is determined by the equivalence class of φ.

Via the fixed embeddings of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$, the field E is identified with the completion of the reflex field of the Shimura variety Sh_G at a certain place above p. For every open compact subgroup $C^p \subset G(\mathbb{A}_f^p)$, we consider $Sh_{G,C_p\times C^p}$ as a variety over E and put

$$Sh_{G,C_p} = \varprojlim_{C^p} Sh_{G,C_p\times C^p}\,.$$

This is a scheme over E with a continuous right action of $G(\mathbb{A}_f^p)$ and an action of $Z_G(\mathbb{Q}_p)$. Let $\mathbb{F}$ denote an algebraic closure of the residue class field $\kappa(E)$ of O_E.

Definition 1.4 An *admissible model* for Sh_{G,C_p} is a projective system of schemes $\widetilde{Sh}_{G,C_p\times C^p}$ over O_E with respect to the set of all open compact subgroups C^p of $G(\mathbb{A}_f^p)$ together with a continuous action of $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ on

$$\widetilde{Sh}_{G,C_p} = \varprojlim_{C^p} \widetilde{Sh}_{G,C_p\times C^p}$$

and a $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$–equivariant isomorphism of E–schemes

$$\widetilde{Sh}_{G,C_p} \otimes_{O_E} E \cong Sh_{G,C_p}$$

such that the following conditions hold.

(i) There is a $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$–equivariant bijection

$$\widetilde{Sh}_{G,C_p}(\mathbb{F}) \cong \bigsqcup_{\varphi} S(\varphi)\,,$$

where the disjoint union is taken over a set of representatives for the equivalence classes of admissible morphisms $\varphi\colon \mathfrak{Q} \to \mathfrak{G}_G$, which carries the action of the Frobenius automorphism over $\kappa(E)$ on the left to the action of Φ on the right.

(ii) For every sufficiently small C^p, $\widetilde{Sh}_{G,C_p\times C^p}$ can be embedded as a dense open subscheme into a normal projective flat O_E–scheme such that the complement (with its reduced structure) is flat over O_E.

(iii) If $C_1^p \subset C_2^p$ are sufficiently small, then the morphism

$$\widetilde{Sh}_{G,C_p\times C_1^p} \longrightarrow \widetilde{Sh}_{G,C_p\times C_2^p}$$

is étale.

Definition 1.5 The subgroup $C_p \subset G(\mathbb{Q}_p)$ is called *h–hyperspecial* if there is a connected normal algebraic subgroup $H \subseteq G_{\mathbb{Q}_p}$ such that:

(i) the class of μ_h in $\mathcal{C}(G,\overline{\mathbb{Q}}_p)$ is contained in the image of $\mathcal{C}(H,\overline{\mathbb{Q}}_p)$;

(ii) $C_p(\mathcal{K})\cap H(\mathcal{K})$ is a special subgroup of $H(\mathcal{K})$.

Conjecture 1.6 *There exists an admissible model for Sh_{G,C_p} over O_E. There exists a smooth admissible model (i.e. for every sufficiently small C^p, $\widetilde{Sh}_{G,C_p\times C^p}$ is smooth over O_E), if and only if C_p is h–hyperspecial.*

Note that in the situation considered in §B3 of [7], the set on the right hand side of the bijection in Definition 1.4 above is a subset of the corresponding set in Definition B3.11 of [7] (where Y_h depends on the group C_p). In the case considered below these sets coincide, but I was not able to see this in general. Moreover, the condition for the existence of a smooth admissible model stated above coincides with the condition considered in [7] (which does not involve the group C_p) in the special case, but may, in general, be stronger. Note also that condition (i) in Definition 1.5 remains unchanged when the embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$ are replaced by others which determine the same place of the reflex field.

§2. Quaternionic Shimura varieties and related moduli problems

Let $D^\times$ denote the algebraic group over $\mathbb{Q}$ associated with a quaternion division algebra D over a totally real number field F, let $\mathcal{Z}$ and $\mathcal{V}$ denote the sets of infinite places of F where D is split or ramified, respectively, and assume that $\mathcal{Z}$ is not empty. There is a natural conjugacy class of homomorphisms $h_D\colon \mathbb{S}\to(D\otimes_{\mathbb{Q}}\mathbb{R})^\times$ of algebraic groups over $\mathbb{R}$ defining a Shimura variety Sh_D over a totally real number field $E(D)\subset\overline{\mathbb{Q}}$. We fix a totally imaginary quadratic extension $K|F$ with complex conjugation $^-$ and a subset $\mathcal{V}_K\subset \mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})$ such that restriction to F induces a bijection $\mathcal{V}_K \to \mathcal{V}$. It determines a homomorphism $h_K\colon \mathbb{S} \to (K \otimes_{\mathbb{Q}} \mathbb{R})^\times$. Let $E(K) \subset \overline{\mathbb{Q}}$ be the extension of $E(D)$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}|E(K))$ is the stabilizer of $\mathcal{V}_K$ in $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$, and let G denote the algebraic group over $\mathbb{Q}$ given by the exact sequence

$$
\begin{aligned}
1 \longrightarrow F^\times \longrightarrow D^\times &\times K^\times \xrightarrow{\tau} G \longrightarrow 1 \\
f \longmapsto &(f, f^{-1}).
\end{aligned}
\tag{2.1}
$$

The conjugacy class of

$$h_G = \tau_{\mathbb{R}} \circ (h_D \times h_K)\colon \mathbb{S} \longrightarrow G_{\mathbb{R}}$$

defines a Shimura variety Sh_G over $E(K)$ which is closely related to Sh_D. For more details see [7], §1. For every open compact subgroup $C \subset G(\mathbb{A}_f)$, $Sh_{G,C} = Sh_G/C$ is a coarse moduli scheme for a contravariant functor $\underline{M}_{G,C}$ on the category of $E(K)$–schemes. Before we explain this functor, we fix some notation.

In the sequel, all schemes which are not explicitly specified are assumed to be locally noetherian and abelian schemes are always projective. If p is a prime number, S a $\mathbb{Z}_{(p)}$–scheme, s a geometric point of S, and A an abelian scheme over S, $\widehat{\mathrm{T}}^p(A_s)$ denotes the product of the Tate modules $\mathrm{T}_\ell(A_s)$ of the fibre A_s for all primes $\ell \neq p$, and $\widehat{\mathrm{V}}^p(A_s) = \widehat{\mathrm{T}}^p(A_s) \otimes_{\mathbb{Z}} \mathbb{Q}$. For every polarization $\lambda\colon A \to \widehat{A}$,

$$E_s^\lambda\colon \widehat{\mathrm{T}}^p(A_s) \times \widehat{\mathrm{T}}^p(A_s) \longrightarrow \widehat{\mathbb{Z}}^p(1)_s$$

denotes the Riemann form of λ at the geometric point s. If S is connected and T is any $\widehat{\mathbb{Z}}^p$–module equipped with an action of a group C, a C–class of isomorphisms $T \to \widehat{\mathrm{T}}^p(A)$ is an orbit of the action of C on the set of isomorphisms $T \to \widehat{\mathrm{T}}^p(A_s)$ which is invariant under the action of the fundamental group $\pi_1(S,s)$ for one (hence every) geometric point s of S. Similarly, we shall speak of a C–class of isomorphisms $V \to \widehat{\mathrm{V}}^p(A)$ if V is an $\mathbb{A}_f^p$–module on which C acts. If S is a $\mathbb{Q}$–scheme, we can include the Tate module for $\ell = p$ and omit the superscript p.

Let $L \subset D$ be a quadratic extension of F, $\mathcal{R} = O_{E(K)}[\Delta^{-1}]$ where Δ denotes the product of all rational primes which are ramified in K or L, S an $\mathcal{R}$–scheme, and $\mathfrak{F}$ a locally free $\mathcal{O}_S$–module together with a homomorphism

$$O_L \otimes_{O_F} O_K \longrightarrow \mathrm{End}_{\mathcal{O}_S} \mathfrak{F}.$$

Since $\mathrm{Gal}(\overline{\mathbb{Q}}|E(K))$ stabilizes $\mathcal{V}_K$, there is a natural decomposition

$$O_K \otimes_{\mathbb{Z}} \mathcal{R} = O(\mathcal{Z}) \times O(\mathcal{V}_K) \times \overline{O}(\mathcal{V}_K),$$

where

$$
\begin{aligned}
O(\mathcal{Z}) &= \{\, x \in O_K \otimes_{\mathbb{Z}} \mathcal{R} \mid (\varphi \otimes \mathrm{id})(x) = 0 \text{ if } \varphi \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C}),\ \varphi|_F \in \mathcal{V} \,\} \\
O(\mathcal{V}_K) &= \{\, x \in O_K \otimes_{\mathbb{Z}} \mathcal{R} \mid (\varphi \otimes \mathrm{id})(x) = 0 \text{ if } \varphi \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C}) \smallsetminus \mathcal{V}_K \,\} \\
\overline{O}(\mathcal{V}_K) &= \{\, x \in O_K \otimes_{\mathbb{Z}} \mathcal{R} \mid (\overline{\varphi} \otimes \mathrm{id})(x) = 0 \text{ if } \varphi \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C}) \smallsetminus \mathcal{V}_K \,\}.
\end{aligned}
$$

It induces a corresponding decomposition

$$\mathfrak{F} = \mathfrak{F}(\mathcal{Z}) \times \mathfrak{F}(\mathcal{V}_K) \times \overline{\mathfrak{F}}(\mathcal{V}_K)\,,$$

and $\mathfrak{F}$ is said to be of type $(L, \mathcal{V}_K)$ if locally with respect to the étale topology on S the following conditions are satisfied.

- ▷ $\mathfrak{F}$ is a free $O_L \otimes_{\mathbb{Z}} \mathcal{O}_S$−module.
- ▷ $\mathfrak{F}(\mathcal{Z})$ is a free $O_L \otimes_{O_F} O(\mathcal{Z}) \otimes_{\mathcal{R}} \mathcal{O}_S$−module.
- ▷ $\overline{\mathfrak{F}}(\mathcal{V}_K) = 0$.

Note that, if S is connected and s is a geometric point of S, then $\mathfrak{F}$ is of type $(L, \mathcal{V}_K)$ if and only if $\mathfrak{F}_s$ is of type $(L, \mathcal{V}_K)$.

Let $'$ be a positive involution on $B = D \otimes_F K$ and

$$\psi\colon B \times B \longrightarrow \mathbb{Q}$$

a non-degenerate alternating bilinear form such that, for all $d \in D$ and $b, x, y \in B$,

$$\psi(bxd, y) = \psi(x, b'yd^*)\,,$$

where * denotes the main involution on D. The action of $D^\times \times K^\times$ on B via $(d\,, k) \circ x = kxd^*$ for all $d \in D^\times$, $k \in K^\times$, and $x \in B$ induces an isomorphism between G and the algebraic group of B−module automorphisms of B which preserve the set $\{\, \psi \circ (f \times \mathrm{id}) \mid f \in F^\times \,\}$.

Definition 2.2 For every open compact subgroup $C \subset G(\mathbb{A}_f)$ and every connected $E(K)$−scheme S, $\underline{M}_{G,C}(S)$ is the set of isomorphism classes of triples $(A, \iota, \overline{\eta})$ where

- ▷ A is an abelian scheme over S up to isogeny;
- ▷ $\iota\colon B \to \mathrm{End}^\circ A = \mathrm{End}\, A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a homomorphism;
- ▷ $\overline{\eta}$ is a C−class of $B \otimes_{\mathbb{Q}} \mathbb{A}_f$−module isomorphisms $B \otimes_{\mathbb{Q}} \mathbb{A}_f \to \widehat{\mathrm{V}}(A)$;

such that the following conditions are satisfied.

- ▷ $\mathrm{Lie}\, A$ is of type $(L, \mathcal{V}_K)$.
- ▷ There is a connected finite étale covering $T \to S$ and a polarization λ of A_T such that, for a geometric point t of T, there is an element $f \in F \otimes_{\mathbb{Q}} \mathbb{A}_f(-1)_t$ satisfying $E_t^\lambda \circ (f\eta \times \eta) = \psi$.

For an arbitrary $E(K)$−scheme S, $\underline{M}_{G,C}(S)$ is the product of the sets corresponding to the connected components of S.

Note that the functor in Definition 2.3 of [7] can be identified with a subfunctor of $\underline{M}_{G,C}$ defined above, and that every morphism from the former to a sheaf on the category of $E(K)$–schemes with respect to the étale topology extends uniquely to the latter. On a simply connected $E(K)$–scheme S the functors agree, and $Sh_{G,C}$ is a coarse moduli scheme for both of them.

Consider now a rational prime p which is unramified in F, a prime $\wp$ of $E(D)$ dividing it, and a commutative diagram

(2.3)
$$\begin{tikzcd} E(D) \arrow[r, dotted] \arrow[d, dotted] & \overline{\mathbb{Q}} \subset \mathbb{C} \arrow[d] \\ E = E(D)_\wp \arrow[r] & \mathbb{Q}_p^{\mathrm{nr}} \subset \overline{\mathbb{Q}}_p \end{tikzcd}$$

where $\mathbb{Q}_p^{\mathrm{nr}}$ denotes the maximal unramified extension of $\mathbb{Q}_p$ and the dotted arrows indicate the natural inclusions. In particular, this defines a bijection

$$\mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R}) \longleftrightarrow \bigsqcup_{i=1}^{m} \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \mathbb{Q}_p^{\mathrm{nr}})$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are all primes of F dividing p. Let I_0 denote the set of all $i \in \{1, \ldots, m\}$ such that D splits at $\mathfrak{p}_i$, and let $O_D \subset D$ be an order which is maximal at $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ and contains O_L. For every $i \in \{1, \ldots, m\}$, we fix a prime $\mathfrak{q}_i$ of K dividing $\mathfrak{p}_i$, and we assume that the following conditions are satisfied (cf. [7], §2).

(2.4) All $\mathfrak{p}_i$ split in K, and diagram (2.3) identifies $\mathcal{V}_K \subset \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$ with a subset of $\bigsqcup_{i=1}^{m} \mathrm{Hom}_{\mathbb{Q}_p}(K_{\mathfrak{q}_i}, \mathbb{Q}_p^{\mathrm{nr}})$.

(2.5) If $\mathcal{V} = \varnothing$, then K does not split D.

(2.6) L is totally imaginary and all $\mathfrak{p}_i$ stay prime in L.

(2.7) ψ restricts to a perfect pairing

$$\psi\colon O_B \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \times O_B \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \longrightarrow \mathbb{Z}_{(p)}$$

where $O_B = O_D \otimes_{O_F} O_K$.

(2.8) There are elements $\Pi \in O_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $b_0 \in (O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ such that

- ▷ $(b_0\Pi)' = -b_0\Pi$;
- ▷ for all $l \in L$, $\Pi l = l^*\Pi$ and $(b_0 l)' = b_0 l$;
- ▷ $\Pi^2 \in F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ has component 1 in $F_{\mathfrak{p}_i}$ if $i \in I_0$ and component p otherwise.

In particular, (2.3) defines an embedding $E(K) \subset E$. For every $i \in I_0$, we fix a maximal proper left ideal $X_i \subset O_{D_{\mathfrak{p}_i}} = O_D \otimes_{O_F} O_{F_{\mathfrak{p}_i}}$. For any subset $I \subseteq I_0$,

$$X_I = \prod_{i \in I} X_i \times \prod_{i \notin I} O_{D_{\mathfrak{p}_i}} \subset O_D \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \text{and} \quad \overline{X}_I = X_I \otimes_{O_F} O_K \subset O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

are left ideals. Let $f_I \in F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ be the element which has component $p \in F_{\mathfrak{p}_i}$ for all $i \in I$ and component 1 otherwise, and consider $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$−modules $T \subseteq T'$ which are free of rank 4 over $O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and a perfect alternating $\mathbb{Z}_p$−bilinear form $\Phi\colon T \times T \to \mathbb{Z}_p$ such that:

- ▹ $f_I T' \subseteq T$;
- ▹ for all $i \in I$, $T \not\subseteq \mathfrak{q}_i T'$ and $f_I T' \not\subseteq \mathfrak{q}_i T$;
- ▹ for all $x, y \in T$ and $b \in O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $\Phi(bx, y) = \Phi(x, b'y)$;
- ▹ $\Phi \circ (f_I \times \mathrm{id})$ extends to a perfect $\mathbb{Z}_p$−bilinear form $\Phi'\colon T' \times T' \to \mathbb{Z}_p$.

Then there is an isomorphism $\eta\colon O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \to T$ of $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$−modules such that $\eta(\overline{X}_I) = f_I T'$ and $\Phi \circ (\eta \times \eta)$ is the form induced by ψ.

Fix now a subset $I \subseteq I_0$ and assume that the subgroup $C \subset G(\mathbb{A}_f)$ satisfies the following condition.

(2.9) $C = \{ g \in G(\mathbb{Q}_p) \mid g(O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p) = O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p,\ g\overline{X}_I = \overline{X}_I \} \times C^p$ for an open subgroup $C^p \subset G(\mathbb{A}_f^p)$, and there is an integer $N > 2$ such that, for all $b \in O_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $g \in C$, $g(b) - b \in NO_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

Using the preceding observation, we can eliminate the rational Tate module $V_p(A)$ in Definition 2.2 and define a functor $\underline{\mathcal{M}}_{G,C}$ on the category of O_E−schemes such that the restrictions of $\underline{M}_{G,C}$ and $\underline{\mathcal{M}}_{G,C}$ to the category of E−schemes are canonically isomorphic.

Definition 2.10 For every connected O_E−scheme S, $\underline{\mathcal{M}}_{G,C}(S)$ is the set of isomorphism classes of 5−tuples $(A, A', \alpha, \iota, \overline{\eta})$ where

- ▹ A and A' are abelian schemes over S up to p−principal isogeny;
- ▹ $\alpha \in \mathrm{Hom}^p(A, A') = \mathrm{Hom}(A, A') \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$;
- ▹ $\iota\colon O_B \to \mathrm{End}^p A = \mathrm{End}\, A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a ring homomorphism;
- ▹ $\overline{\eta}$ is a C^p−class of $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$−module isomorphisms $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to \widehat{\mathrm{V}}^p(A)$;

such that there are $\alpha' \in \mathrm{Hom}^p(A', A)$, a homomorphism $\iota'\colon O_B \to \mathrm{End}^p A'$, and p−principal polarizations λ of A_T and λ' of A'_T for a certain connected finite étale covering $T \to S$ satisfying the following conditions.

(i) $\operatorname{Lie} A$ and $\operatorname{Lie} A'$ are of type $(L, \mathcal{V}_K)$.

(ii) For all $b \in O_B$, $\iota'(b) \circ \alpha = \alpha \circ \iota(b)$.

(iii) $\alpha' \circ \alpha = \iota(f_0)$ for an $f_0 \in F^\times \cap f_I (O_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$.

(iv) For every $i \in I$, $\alpha \notin \mathfrak{q}_i \operatorname{Hom}^p(A\,, A')$ and $\alpha' \notin \mathfrak{q}_i \operatorname{Hom}^p(A'\,, A)$, where the O_K–module structures are defined by ι' and ι, respectively.

(v) $\widehat{\alpha}_T \circ \lambda' = \lambda \circ \alpha'_T$ in $\operatorname{Hom}^p(A'_T\,, \widehat{A}_T)$.

(vi) For a geometric point t of T, there is an element $f \in F \otimes_{\mathbb{Q}} \mathbb{A}_f^p(-1)_t$ such that $E_t^\lambda \circ (f\eta \times \eta) = \psi$.

For an arbitrary O_E–scheme S, $\underline{\mathcal{M}}_{G,C}(S)$ is the product of the sets corresponding to the connected components of S.

Remark 2.11 If in the preceding definition condition (vi) is satisfied for one geometric point t, then it is satisfied for every geometric point of T. In this case, we call λ an admissible polarization of $(A, \iota, \overline{\eta})$ over T. It is unique up to a factor in $(O_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})^\times$, and λ' is an admissible polarization of $(A', \iota', \widehat{\mathrm{V}}^p(\alpha) \circ \overline{\eta})$ over T. Moreover, for every $b \in O_B$, b' is contained in $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and $\lambda \circ \iota(b')_T = \widehat{\iota(b)}_T \circ \lambda$.

Lemma 2.12 *Let $\alpha\colon X_1 \to X_2$ be an isogeny of p–divisible groups over a connected $\mathbb{Z}_{(p)}$–scheme S and $L|\mathbb{Q}_p$ a finite unramified extension. Assume that there are $\mathbb{Z}_p$–algebra homomorphisms $O_L \to \operatorname{End} X_i$ ($i = 1, 2$) such that α is O_L–equivariant and the Lie algebras of X_1 and X_2 are locally with respect to the étale topology on S isomorphic as $O_L \otimes_{\mathbb{Z}} \mathcal{O}_S$–modules. Then the rank of the locally free finite group scheme $\mathcal{N} = \ker(\alpha)$ is a power of $(O_L : pO_L)$.*

Proof. Since S is connected, the rank of $\mathcal{N}$ is constant, and we may assume that $S = \operatorname{Spec}(k)$ for an algebraically closed field k. If k has characteristic 0, then $\mathcal{N}$ is étale and the set of k–valued points is an O_L–module M of finite length l. Since O_L/pO_L is the only simple O_L–module, we obtain

$$\operatorname{rank}(\mathcal{N}) = \#M = (O_L : pO_L)^l\,.$$

If k has characteristic p, let $\mathrm{W}(k)$ denote the Witt ring of k. The action of O_L gives rise to a decomposition

$$M_i = \bigoplus_{\varphi} M_{i,\varphi}$$

of the covariant Dieudonné module M_i of X_i ($i = 1, 2$) where φ runs through the set of $\mathbb{Z}_p$–algebra homomorphisms $O_L \to \mathrm{W}(k)$. The assumption concerning the Lie algebras implies that M_1/VM_1 and M_2/VM_2 are isomorphic as $O_L \otimes_{\mathbb{Z}} k$−modules, i.e. for every φ,

$$M_{1,\varphi}/VM_{1,\sigma\varphi} \cong M_{2,\varphi}/VM_{2,\sigma\varphi} \tag{2.13}$$

as k−vector spaces where σ denotes the Frobenius automorphism of $\mathrm{W}(k)$. The isogeny α induces an O_L−equivariant embedding $M_1 \to M_2$ with finite cokernel. Let l_φ denote the length of the $\mathrm{W}(k)$−module $M_{2,\varphi}/M_{1,\varphi}$. The isomorphism (2.13) implies that $l_\varphi = l_{\sigma\varphi}$, i.e. there is an l such that $l_\varphi = l$ for all $\varphi\colon O_L \to \mathrm{W}(k)$. But then

$$\mathrm{rank}(\mathcal{N}) = p^{[\,M_2:M_1\,]} = (O_L : pO_L)^l$$

where $[\,M_2 : M_1\,]$ denotes the length of the $\mathrm{W}(k)$−module M_2/M_1. ∎

Proposition 2.14 *$\underline{\mathcal{M}}_{G,C}$ is representable by a projective O_E−scheme $\widetilde{Sh}_{G,C}$, and if $C' \subseteq C$ is another subgroup satisfying condition* (2.9), *then the canonical morphism $\widetilde{Sh}_{G,C'} \to \widetilde{Sh}_{G,C}$ is étale.*

Proof. Consider first the case $I = \varnothing$. Since α in Definition 2.10 is then always p−principal, $\underline{\mathcal{M}}_{G,C}$ is isomorphic to the functor referred to in Remark 2.17 of [7]. This implies the first assertion in this special case. For the general case, we consider an abelian scheme A_0 over a connected O_E−scheme S_0, a homomorphism $\iota_0\colon O_B \to \mathrm{End}^p A_0$ such that $\mathrm{Lie}\, A_0$ has type $(L, \mathcal{V}_K)$, and a C^p−class $\overline{\eta}_0$ of $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$−module isomorphisms $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to \widehat{\mathrm{V}}^p(A_0)$ such that $(A_0, \iota_0, \overline{\eta}_0)$ admits an admissible polarization over a certain connected finite étale covering of S_0, and we show that the following functor $\mathcal{F}_0$ is representable by a projective S_0−scheme. For a connected S_0−scheme S, $\mathcal{F}_0(S)$ is the set of all $(A, A', \alpha, \iota, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(S)$ such that $(A, \iota, \overline{\eta})$ is the pull-back of $(A_0, \iota_0, \overline{\eta}_0)$ to S, and in general, $\mathcal{F}_0(S)$ is the product of the sets corresponding to the connected components of S. To start with, we fix any $(A, A', \alpha, \iota, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(S)$ for a connected O_E−scheme S and α', ι', λ, λ' as in the definition. The natural isomorphism $O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_{i=1}^m (O_{K_{\mathfrak{q}_i}} \times O_{K_{\overline{\mathfrak{q}}_i}})$ induces a decomposition $P = \prod_{i=1}^m (P_i \times \overline{P}_i)$ of the p−divisible group of A. P_i and $\overline{P}_i$ are p−divisible groups of height $4f_i$ where $f_i = [\,F_{\mathfrak{p}_i} : \mathbb{Q}_p\,]$, and ι induces an action of O_D on P_i and $\overline{P}_i$. The p−divisible group P' of A' decomposes in a similar way, and α induces isogenies $\alpha_i\colon P_i \to P_i'$ and $\overline{\alpha}_i\colon \overline{P}_i \to \overline{P}_i'$ which commute with the actions of O_D. For $i \notin I$, α_i and $\overline{\alpha}_i$ are isomorphisms, and for $i \in I$, α' induces isogenies $\alpha_i'\colon P_i' \to P_i$ and $\overline{\alpha}_i'\colon \overline{P}_i' \to \overline{P}_i$ such that $\alpha_i' \circ \alpha_i \in p\,\mathrm{Aut}\, P_i$ and $\overline{\alpha}_i' \circ \overline{\alpha}_i \in p\,\mathrm{Aut}\, \overline{P}_i$. Since $\mathrm{Lie}\, P$ and $\mathrm{Lie}\, P'$ are both of type $(L, \mathcal{V}_K)$, the preceding lemma implies that the rank of the locally free group scheme $\mathcal{N}_i = \ker(\alpha_i)$ is a power of p^{2f_i}. By condition (iv) in Definition 2.10 it follows that, for every $i \in I$, $\mathcal{N}_i$ has rank p^{2f_i}.

Let now A_i, $\overline{A}_i$, A'_i, and $\overline{A}'_i$ denote the kernels of multiplication by p in P_i, $\overline{P}_i$, P'_i, and $\overline{P}'_i$, respectively. Then, for all $i \in I$, $\alpha'_i(A'_i) = \mathcal{N}_i \subset A_i$. Hence, if $\mathcal{D}(\mathcal{N})$ denotes the Cartier dual of a finite locally free group scheme $\mathcal{N}$, we have a natural surjection $\mathcal{D}(A_i) \to \mathcal{D}(\mathcal{N}_i)$, and $\mathcal{D}(\alpha'_i)$ induces an injection $\mathcal{D}(\mathcal{N}_i) \to \mathcal{D}(A'_i)$. Moreover, by condition (v) in Definition 2.10, λ and λ' define isomorphisms λ_i and λ'_i such that the diagram

$$\begin{tikzcd}
(\overline{A}_i)_T \arrow[r, "\lambda_i"] \arrow[d, "(\overline{\alpha}_i)_T"'] & \mathcal{D}(A_i)_T \arrow[d, "\mathcal{D}(\alpha'_i)_T"] \\
(\overline{A}'_i)_T \arrow[r, "\lambda'_i"'] & \mathcal{D}(A'_i)_T
\end{tikzcd}$$

commutes. In particular, we obtain an exact sequence

$$0 \longrightarrow (\overline{\mathcal{N}}_i)_T \xrightarrow{\lambda_i} \mathcal{D}(A_i)_T \longrightarrow \mathcal{D}(\mathcal{N}_i)_T \longrightarrow 0$$

where $\overline{\mathcal{N}}_i = \ker(\overline{\alpha}_i) \subset \overline{A}_i$.

Consider, on the other hand, a connected S_0–scheme S and an admissible polarization λ of $(A, \iota, \overline{\eta}) = (A_0, \iota_0, \overline{\eta}_0)_S$ over a connected finite étale covering T of S. Let, for every $i \in I$, the isomorphism $\lambda_i \colon (\overline{A}_i)_T \to \mathcal{D}(A_i)_T$ be defined as above, and fix locally free closed subgroup schemes $\mathcal{N}_i \subset A_i$ which are invariant with respect to the action of O_D. Then $\mathcal{M}_i = \lambda_i^{-1}\Big(\ker\big(\mathcal{D}(A_i)_T \to \mathcal{D}(\mathcal{N}_i)_T\big)\Big)$ is O_D–invariant, too. Moreover, if $T' \to S$ is another connected finite étale covering and $\beta_1, \beta_2 \colon T' \to T$ are two morphisms of S–schemes, then the polarizations of $A_{T'}$ induced by λ via β_1 and β_2, respectively, differ by a factor in $(O_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})^\times$. Hence, the subgroup schemes of $(\overline{A}_i)_{T'}$ induced by $\mathcal{M}_i$ via β_1 and β_2, respectively, coincide. Therefore, there is a unique O_D–invariant subgroup scheme $\overline{\mathcal{N}}_i \subset \overline{A}_i$ such that $(\overline{\mathcal{N}}_i)_T = \mathcal{M}_i$. We consider $\mathcal{N} = \prod_{i \in I} \mathcal{N}_i \times \overline{\mathcal{N}}_i$ as an O_B–invariant subgroup scheme of A and denote the Lie algebra of $A' = A/\mathcal{N}$ by $\mathcal{L}(\,\mathcal{N}_i \mid i \in I\,)$.

Let now, for every connected S_0–scheme S, $\mathcal{F}(S)$ be the set of all $(\,\mathcal{N}_i \mid i \in I\,)$ where $\mathcal{N}_i$ is an O_D–invariant subgroup scheme of A_i which is locally free of rank p^{2f_i} and such that $\mathcal{L}(\,\mathcal{N}_i \mid i \in I\,)$ has type $(L, \mathcal{V}_K)$. For an arbitrary S_0–scheme S, let $\mathcal{F}(S)$ be the product of the sets corresponding to the connected components of S. The arguments above show that $\mathcal{F}_0$ is isomorphic to this functor $\mathcal{F}$. Since $\mathcal{F}$ is obviously representable by a projective S_0–scheme, the first assertion of the proposition is proved in the general case.

For the second assertion, it suffices to show that, for every surjective homomorphism $R \to R_0$ of strictly henselian local O_E–algebras, the natural map

$$\underline{\mathcal{M}}_{G,C'}(R) \longrightarrow \underline{\mathcal{M}}_{G,C}(R) \underset{\underline{\mathcal{M}}_{G,C}(R_0)}{\times} \underline{\mathcal{M}}_{G,C'}(R_0)$$

is bijective. Let $\mathfrak{a}_0 = (A_0, A_0', \alpha_0, \iota_0, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(R_0)$ be the image of a fixed element $\mathfrak{a} = (A, A', \alpha, \iota, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(R)$, and consider any $\mathfrak{a}_0' \in \underline{\mathcal{M}}_{G,C'}(R_0)$ lifting $\mathfrak{a}_0$. It is represented by $(A_0, A_0', \alpha_0, \iota_0, \overline{\eta}')$ with $\overline{\eta}' \subseteq \overline{\eta}$, and $\mathfrak{a}' = (A, A', \alpha, \iota, \overline{\eta}')$ is an element of $\underline{\mathcal{M}}_{G,C'}(R)$ which is mapped to the pair $(\mathfrak{a}, \mathfrak{a}_0')$. Any $\mathfrak{a}'' \in \underline{\mathcal{M}}_{G,C'}(R)$ which has this property is represented by $(A, A', \alpha, \iota, \overline{\eta}'')$ for some $\overline{\eta}'' \subseteq \overline{\eta}$, and there is a $\beta_0 \in \operatorname{Aut} \mathfrak{a}_0$ such that $\widehat{\mathrm{V}}^p(\beta_0) \circ \overline{\eta}' = \overline{\eta}''$. The proof of Lemma 2.13 in [7] shows that β_0 is given by the action of an element in $O_F^\times$. Hence it can be lifted to an isomorphism $\beta\colon \mathfrak{a}' \to \mathfrak{a}''$. This completes the proof of the proposition. ∎

Corollary 2.15 *There is a natural projective system of isomorphisms*

$$\widetilde{Sh}_{G,C} \otimes_{O_E} E \cong Sh_{G,C} \otimes_{E(K)} E$$

with respect to all C which satisfy (2.9)*. Its limit is a $K^\times(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$–equivariant isomorphism of E–schemes.*

Proof. This follows from the uniqueness of a coarse moduli scheme. ∎

§3. The local structure of $\widetilde{Sh}_G$

Let $\mathbb{F}$ denote the residue class field of $\mathbb{Q}_p^{\mathrm{nr}}$ and σ the Frobenius automorphism of the Witt ring $\mathrm{W}(\mathbb{F})$ of $\mathbb{F}$. Via diagram (2.3) $\mathrm{W}(\mathbb{F})$ becomes an O_E–algebra. Let $\widehat{\mathcal{O}}_x$ denote the completion of the local ring at a geometric point $x \in \widetilde{Sh}_{G,C}(\mathbb{F})$ with respect to the étale topology on $\widetilde{Sh}_{G,C}$. We fix a finite field $\kappa \subset \mathbb{F}$ containing the residue class field of E and an element $k \in K^\times(\mathbb{A}_f)$ such that $\gamma(x) = k(x)$ where γ denotes the Frobenius automorphism of $\mathbb{F}|\kappa$. The action of k on $\widetilde{Sh}_{G,C}$ defines a $\mathrm{W}(\mathbb{F})$–algebra isomorphism $k_x\colon \widehat{\mathcal{O}}_{\gamma(x)} \to \widehat{\mathcal{O}}_x$. On the other hand, for every $\varrho \in \mathrm{Gal}(\mathbb{F}|\kappa)$, there is a canonical isomorphism $\varrho_x\colon \widehat{\mathcal{O}}_x \to \widehat{\mathcal{O}}_{\varrho(x)}$ extending the action of ϱ on $\mathrm{W}(\mathbb{F})$. In this section we describe $\widehat{\mathcal{O}}_x$ together with the unique twisted action of $\mathrm{Gal}(\mathbb{F}|\kappa)$ where γ acts as $k_x \circ \gamma_x$ and every ϱ in a sufficiently small open subgroup of $\mathrm{Gal}(\mathbb{F}|\kappa)$ acts as ϱ_x.

For any number field Ω in which p is unramified, (2.3) induces a natural bijection

$$\mathrm{Hom}(\Omega, \mathbb{C}) \longleftrightarrow \mathrm{Hom}(O_\Omega, \mathrm{W}(\mathbb{F}))\,.$$

In particular, this defines an isomorphism of $\mathrm{W}(\mathbb{F})$–algebras

$$\prod_\varphi w(\varphi) \times \overline{w}(\varphi):\ O_{L\otimes_F K} \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}) \longrightarrow \prod_{\varphi \in \mathrm{Hom}(L, \mathbb{C})} \mathrm{W}(\mathbb{F}) \times \mathrm{W}(\mathbb{F})$$

where $w(\varphi)|_{O_L} = \overline{w}(\varphi)|_{O_L}$ corresponds to φ and, for a certain $i \in \{1, \dots, m\}$, $w(\varphi)(\mathfrak{q}_i) = \overline{w}(\varphi)(\bar{\mathfrak{q}}_i) \subset p\mathrm{W}(\mathbb{F})$. If $\mathfrak{F}$ is a sheaf of $O_{L\otimes_F K} \otimes_{\mathbb{Z}} \mathcal{O}_S$-modules on a $\mathrm{W}(\mathbb{F})$-scheme S, we obtain a decomposition

$$\mathfrak{F} = \prod_{\varphi \in \mathrm{Hom}(L, \mathbb{C})} (\mathfrak{F}_\varphi \times \overline{\mathfrak{F}}_\varphi)\,. \tag{3.1}$$

Definition 3.2 For a triple $Z = (Z_1, Z_1', Z_2)$ consisting of $Z_1, Z_1' \subseteq \mathrm{Hom}(F, \mathbb{R})$ and $Z_2 \subseteq \mathrm{Hom}(L, \mathbb{C})$ and another such triple $\widetilde{Z} = (\widetilde{Z}_1, \widetilde{Z}_1', \widetilde{Z}_2)$, let

$$Z \cup \widetilde{Z} = (Z_1 \cup \widetilde{Z}_1, Z_1' \cup \widetilde{Z}_1', Z_2 \cup \widetilde{Z}_2)\,,$$

and write $\widetilde{Z} \leqslant Z$ if $\widetilde{Z}_1 \subseteq Z_1$, $\widetilde{Z}_1' \subseteq Z_1'$, and $\widetilde{Z}_2 \subseteq Z_2$. Z is called *admissible* if $Z_1 \cup Z_1' = \mathcal{Z}_1$ and $Z_2 \cup Z_2^* = \mathcal{Z}_2$ where

$$\begin{aligned}
\mathcal{Z}_0 &= \mathcal{Z} \cap \bigsqcup_{i \in I_0 \smallsetminus I} \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \mathbb{Q}_p^{\mathrm{nr}}) \\
\mathcal{Z}_1 &= \mathcal{Z} \cap \bigsqcup_{i \in I} \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \mathbb{Q}_p^{\mathrm{nr}}) \\
\mathcal{Z}_2 &= \{\, \varphi \in \mathrm{Hom}(L, \mathbb{C}) \mid \varphi|_F \in \mathcal{Z} \smallsetminus (\mathcal{Z}_0 \cup \mathcal{Z}_1) \,\}\,.
\end{aligned}$$

For every admissible triple Z, let

$$d(Z) = \#\{\, v \in \mathcal{Z} \smallsetminus (Z_1 \cap Z_1') \mid v = \varphi|_F \text{ for a } \varphi \in \mathrm{Hom}(L, \mathbb{C}) \smallsetminus Z_2 \,\}\,.$$

Z is called *minimal* if $Z_1 \cap Z_1' = \varnothing = Z_2 \cap Z_2^*$, i.e. $d(Z) = \#\mathcal{Z}$.

Definition 3.3 For a connected $\mathbb{F}$-scheme S and $\mathfrak{a} = (A, A', \alpha, \iota, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(S)$, let $\mathcal{Z}(\mathfrak{a}) = (\mathcal{Z}_1(\mathfrak{a}), \mathcal{Z}_1'(\mathfrak{a}), \mathcal{Z}_2(\mathfrak{a}))$ with

$$\begin{aligned}
\mathcal{Z}_1(\mathfrak{a}) &= \{\, \varphi|_F \mid \varphi \in \mathrm{Hom}(L, \mathbb{C}) \text{ such that } \alpha(\mathrm{Lie}\, A)_\varphi = 0 \,\} \\
\mathcal{Z}_1'(\mathfrak{a}) &= \{\, \varphi|_F \mid \varphi \in \mathrm{Hom}(L, \mathbb{C}) \text{ such that } \alpha'(\mathrm{Lie}\, A')_\varphi = 0 \,\} \\
\mathcal{Z}_2(\mathfrak{a}) &= \{\, \varphi \in \mathrm{Hom}(L, \mathbb{C}) \mid \Pi(\mathrm{Lie}\, A)_\varphi = 0 \,\}
\end{aligned}$$

where α' is chosen as in Definition 2.10 and Π as in (2.8), and set

$$\begin{aligned}
\mathcal{Z}_1^+(\mathfrak{a}) &= \mathcal{Z}_1(\mathfrak{a}) \cap \mathcal{Z}_1'(\mathfrak{a}) \\
\mathcal{Z}_2^+(\mathfrak{a}) &= \mathcal{Z}_2(\mathfrak{a}) \cap \mathcal{Z}_2(\mathfrak{a})^*
\end{aligned}$$

where * denotes the action of the non-trivial automorphism of $L|F$.

Remark 3.4 If in the preceding definition $\mathfrak{a}$ is the pull-back of $\mathfrak{a}_0$ with respect to a morphism $S \to S_0$ of connected $\mathbb{F}$-schemes, then $\mathcal{Z}(\mathfrak{a}_0) \leqslant \mathcal{Z}(\mathfrak{a})$. Moreover, if S is integral, then $\mathcal{Z}(\mathfrak{a})$ is an admissible triple.

Let now $\mathfrak{a} = (A, A', \alpha, \iota, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(\mathbb{F})$ be the element which corresponds to the geometric point x considered above. The main result of the present section is the following generalization of Proposition 3.5 in [7].

Proposition 3.5 *There is an isomorphism of* $\mathrm{W}(\mathbb{F})$*–algebras*

$$\widehat{\mathcal{O}}_x \cong \mathcal{O}(\mathfrak{a}) = \mathrm{W}(\mathbb{F})[\![\, X_v,\, X'_{v'},\, X_\varphi \mid v \in \mathcal{Z}_0 \cup \mathcal{Z}_1(\mathfrak{a}),\, v' \in \mathcal{Z}'_1(\mathfrak{a}),\, \varphi \in \mathcal{Z}_2(\mathfrak{a}) \,]\!]/\mathcal{I}$$

where $\mathcal{I} = (\, X_v X'_v - p,\, X_\varphi X_{\varphi^*} - p \mid v \in \mathcal{Z}_1^+(\mathfrak{a}),\, \varphi \in \mathcal{Z}_2^+(\mathfrak{a})\,)$ *such that the twisted action of* $\varrho \in \mathrm{Gal}(\mathbb{F}\,|\,\kappa)$ *on* $\widehat{\mathcal{O}}_x$ *corresponds to the extension of the natural action on* $\mathrm{W}(\mathbb{F})$ *given by* $\varrho(X_v) = X_{\varrho v}$, $\varrho(X'_{v'}) = X'_{\varrho v'}$, *and* $\varrho(X_\varphi) = X_{\varrho\varphi}$ *for all* $v \in \mathcal{Z}_0 \cup \mathcal{Z}_1(\mathfrak{a})$, $v' \in \mathcal{Z}'_1(\mathfrak{a})$, *and* $\varphi \in \mathcal{Z}_2(\mathfrak{a})$. *For every local artinian* $\mathbb{F}$*–algebra* R_0 *with residue class field* $\mathbb{F}$ *and every* $\mathrm{W}(\mathbb{F})$*–algebra homomorphism* $h_0\colon \widehat{\mathcal{O}}_x \to R_0$, *the corresponding element* $\mathfrak{a}_0 \in \underline{\mathcal{M}}_{G,C}(R_0)$ *satisfies*

$$\begin{aligned}
\mathcal{Z}_1(\mathfrak{a}_0) &= \mathcal{Z}_1(\mathfrak{a}) \smallsetminus \{\, v \in \mathcal{Z}_1^+(\mathfrak{a}) \mid h_0(X_v) \neq 0 \,\}\,; \\
\mathcal{Z}'_1(\mathfrak{a}_0) &= \mathcal{Z}'_1(\mathfrak{a}) \smallsetminus \{\, v \in \mathcal{Z}_1^+(\mathfrak{a}) \mid h_0(X'_v) \neq 0 \,\}\,; \\
\mathcal{Z}_2(\mathfrak{a}_0) &= \mathcal{Z}_2(\mathfrak{a}) \smallsetminus \{\, \varphi \in \mathcal{Z}_2^+(\mathfrak{a}) \mid h_0(X_\varphi) \neq 0 \,\}\,.
\end{aligned}$$

Proof. Let $\Gamma \subseteq \mathcal{O}(\mathfrak{a})$ denote the Galois invariant $\mathrm{W}(\mathbb{F})$–subalgebra generated by $\{\, X_v,\, X'_v,\, X_\varphi \mid v \in \mathcal{Z}_1^+(\mathfrak{a}),\, \varphi \in \mathcal{Z}_2^+(\mathfrak{a})\,\}$. As in [7] we construct a formally smooth $\mathrm{W}(\mathbb{F})$–algebra homomorphism $\Gamma \to \widehat{\mathcal{O}}_x$ and show that the relative tangent space together with the twisted action of γ has the required structure. To start with, we consider the element $\widetilde{\mathfrak{a}} = (\widetilde{A}, \widetilde{A}', \widetilde{\alpha}, \widetilde{\iota}, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(\widehat{\mathcal{O}}_x)$ which corresponds to the natural map $\mathrm{Spec}(\widehat{\mathcal{O}}_x) \to \widetilde{Sh}_{G,C}$. It has image $\mathfrak{a}$ in $\underline{\mathcal{M}}_{G,C}(\mathbb{F})$, and there is an isomorphism $\theta\colon \gamma(\widetilde{\mathfrak{a}}) \to k(\widetilde{\mathfrak{a}})$. We fix elements $\Pi \in O_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $b_0 \in (O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ as in (2.8) and $\widetilde{\alpha}'$, $\widetilde{\iota}'$, $\widetilde{\lambda}$, $\widetilde{\lambda}'$, and f_0 as in Definition 2.10 (with respect to $\widetilde{\mathfrak{a}}$). Let $f = f_0^{-1} f_I \in (O_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$, and let $\mathcal{Z}^\circ = \{\, \varphi_v \mid v \in \mathcal{Z}_0 \cup \mathcal{Z}_1 \,\}$ where $\varphi_v \in \mathrm{Hom}(L\,, \mathbb{C})$ is chosen such that $\varphi_v|_F = v$.

For every $\varphi \in \mathcal{Z}_2^+(\mathfrak{a})$, $(\mathrm{Lie}\,\widetilde{A})_\varphi$ is a free $\widehat{\mathcal{O}}_x$–module of rank 1, and θ induces a γ–linear isomorphism $(\mathrm{Lie}\,\widetilde{A})_\varphi \to (\mathrm{Lie}\,\widetilde{A})_{\gamma\varphi}$. We choose generators $t_\varphi \in (\mathrm{Lie}\,\widetilde{A})_\varphi$ such that $\theta(t_\varphi) = t_{\gamma\varphi}$, and obtain elements x_φ in the maximal ideal $\widehat{\mathfrak{m}}_x \subset \widehat{\mathcal{O}}_x$ such that $\Pi t_\varphi = x_\varphi t_{\varphi^*}$. These elements satisfy the relations $\gamma(x_\varphi) = x_{\gamma\varphi}$ and $x_\varphi x_{\varphi^*} = p$. Similarly, we choose generators $t_v \in (\mathrm{Lie}\,\widetilde{A})_{\varphi_v}$ and $t'_v \in (\mathrm{Lie}\,\widetilde{A}')_{\varphi_v}$, for all $v \in \mathcal{Z}_1^+(\mathfrak{a})$, such that $\theta(t_v) \in \{t_{\gamma v}, \Pi t_{\gamma v}\}$ and $\theta(t'_v) \in \{t'_{\gamma v}, \Pi t'_{\gamma v}\}$. This determines $x_v, x'_v \in \widehat{\mathfrak{m}}_x$ such that $\widetilde{\alpha}(t_v) = x_v t'_v$ and $\widetilde{\alpha}'(f t'_v) = x'_v t_v$, and these elements satisfy the equations $\gamma(x_v) = x_{\gamma v}$, $\gamma(x'_v) = x'_{\gamma v}$, and $x_v x'_v = p$. Thus we obtain a γ–equivariant $\mathrm{W}(\mathbb{F})$–algebra homomorphism

$$\begin{gathered}
\Gamma \longrightarrow \widehat{\mathcal{O}}_x \\
X_v \longmapsto x_v\ ,\ X'_v \longmapsto x'_v\ ,\ X_\varphi \longmapsto x_\varphi\,.
\end{gathered}$$

Consider now, for every positive integer n, the Lie algebra of the universal extension of the pull-back of $\widetilde{A}$ to $\operatorname{Spec}(\widehat{\mathcal{O}}_x/\widehat{\mathfrak{m}}_x^n)$, and let $\widetilde{M}$ denote the projective limit over all n and $\widetilde{V}$ the kernel of the natural surjective $O_B \otimes_{\mathbb{Z}} \widehat{\mathcal{O}}_x$−module homomorphism $\widetilde{M} \to \operatorname{Lie} \widetilde{A}$. The isomorphism θ induces a γ−linear automorphism of $\widetilde{M}$ which respects $\widetilde{V}$, and $\widetilde{\lambda} \circ \widetilde{\iota}(b_0)$ gives rise to a perfect alternating bilinear form

$$\widetilde{\Phi} \colon \widetilde{M} \times \widetilde{M} \longrightarrow \widehat{\mathcal{O}}_x$$

such that $\widetilde{V}$ is a maximal isotropic subspace. It has the following properties:

$$\begin{aligned} \widetilde{\Phi} \circ (k \times \mathrm{id}) &= \widetilde{\Phi} \circ (\mathrm{id} \times \overline{k}) \quad \text{for all } k \in O_K\,; \\ \widetilde{\Phi} \circ (l \times \mathrm{id}) &= \widetilde{\Phi} \circ (\mathrm{id} \times l) \quad \text{for all } l \in O_L\,; \\ \widetilde{\Phi} \circ (\Pi \times \mathrm{id}) &= -\widetilde{\Phi} \circ (\mathrm{id} \times \Pi)\ . \end{aligned}$$

Recall that $\widetilde{M}$ and $\widetilde{V}$ decompose as described in (3.1). For $\varphi \in \operatorname{Hom}(L, \mathbb{C})$, $\widetilde{M}_\varphi$ is a free $\widehat{\mathcal{O}}_x$−module of rank 2, and $\widetilde{V}_\varphi$ is a direct summand of rank 1 if $\varphi|_F \in \mathcal{Z}$ and vanishes if $\varphi|_F \in \mathcal{V}$. Π maps $\widetilde{M}_\varphi$ into $\widetilde{M}_{\varphi^*}$ and $\widetilde{V}_\varphi$ into $\widetilde{V}_{\varphi^*}$. If $\varphi|_F$ is contained in $\operatorname{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \mathbb{Q}_p^{\mathrm{nr}})$ for some $i \in I_0$, these maps are isomorphisms. In addition, the bilinear form $\widetilde{\Phi}$ induces perfect pairings

$$\widetilde{\Phi} \colon \widetilde{M}_\varphi \times \overline{\widetilde{M}}_\varphi \longrightarrow \widehat{\mathcal{O}}_x$$

such that $\widetilde{\Phi}(\widetilde{V}_\varphi \times \overline{\widetilde{V}}_\varphi) = 0$. Applying a similar construction to $\widetilde{A}'$, $\widetilde{\iota}'$, and $\widetilde{\lambda}'$, we obtain modules $\widetilde{V}' \subset \widetilde{M}'$, a bilinear form $\widetilde{\Phi}'$, and decompositions as above. $\widetilde{\alpha}$ defines homomorphisms $\widetilde{M}_\varphi \to \widetilde{M}'_\varphi$ which map $\widetilde{V}_\varphi$ into $\widetilde{V}'_\varphi$, and $\widetilde{\alpha}'$ defines maps in the opposite direction. If $\varphi|_F$ is contained in $\operatorname{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \mathbb{Q}_p^{\mathrm{nr}})$ for an $i \in \{1, \ldots, m\} \smallsetminus I$, these maps are isomorphisms. For every $v \in \mathcal{Z}_1^+(\mathfrak{a})$, we lift t_v and t'_v to elements $e_v \in \widetilde{M}_{\varphi_v}$ and $e'_v \in \widetilde{M}'_{\varphi_v}$, respectively, and for every $\varphi \in \mathcal{Z}_2^+(\mathfrak{a})$, we lift t_φ to an element $e_\varphi \in \widetilde{M}_\varphi$.

Lemma 3.6 *The following relations hold.*

(i) *For* $v \in \mathcal{Z}_1^+(\mathfrak{a})$, $\widetilde{V}_{\varphi_v} = \widehat{\mathcal{O}}_x\big(\widetilde{\alpha}'(f e'_v) - x'_v e_v\big)$ *and* $\widetilde{V}'_{\varphi_v} = \widehat{\mathcal{O}}_x\big(\widetilde{\alpha}(e_v) - x_v e'_v\big)$.

(ii) *For* $v \in \mathcal{Z}_1 \smallsetminus \mathcal{Z}'_1(\mathfrak{a})$, $\widetilde{\alpha}\widetilde{V}_{\varphi_v} = \widetilde{V}'_{\varphi_v}$.

(iii) *For* $v \in \mathcal{Z}_1 \smallsetminus \mathcal{Z}_1(\mathfrak{a})$, $\widetilde{\alpha}'\widetilde{V}'_{\varphi_v} = \widetilde{V}_{\varphi_v}$.

(iv) *For* $\varphi \in \mathcal{Z}_2^+(\mathfrak{a})$, $\widetilde{V}_\varphi = \widehat{\mathcal{O}}_x\big(\Pi e_{\varphi^*} - x_{\varphi^*} e_\varphi\big)$.

(v) *For* $\varphi \in \mathcal{Z}_2 \smallsetminus \mathcal{Z}_2(\mathfrak{a})$, $\Pi\widetilde{V}_{\varphi^*} = \widetilde{V}_\varphi$.

Proof. Let N and N' denote the Dieudonné modules of the Barsotti–Tate groups of A and A', respectively, in the sense of Zink [10] (Part II, page 51). The action of $O_{L\otimes_F K}$ gives rise to decompositions as in (3.1), and for all $\varphi\in \mathrm{Hom}(L,\mathbb{C})$, α defines an injection of N_φ into N'_φ. The map α' such that $\alpha'\circ\alpha=\iota(f_0)$ defines an injection in the other direction, and condition (iv) in Definition 2.10 implies that, for every $i\in I$, there is a φ such that $\varphi|_F\in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i},\mathbb{Q}_p^{\mathrm{nr}})$ and $\alpha' N'_\varphi\neq pN_\varphi$, hence $\alpha N_\varphi\neq N'_\varphi$. Since both $\mathrm{Lie}\,A$ and $\mathrm{Lie}\,A'$ are of type $(L,\mathcal{V}_K)$, $[\,N_\varphi : VN_{\sigma\varphi}\,]=[\,N'_\varphi : VN'_{\sigma\varphi}\,]$, hence $[\,N'_\varphi : \alpha N_\varphi\,]=[\,N'_{\sigma\varphi} : \alpha N_{\sigma\varphi}\,]$ for all φ. It follows that $\alpha N_\varphi\neq N'_\varphi$ and, similarly, $\alpha' N'_\varphi\neq N_\varphi$ for all φ such that $\varphi|_F\in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i},\mathbb{Q}_p^{\mathrm{nr}})$ for an $i\in I$. In particular, if $v=\varphi|_F\in\mathcal{Z}_1$, then

$$[\,N'_\varphi:\alpha N_\varphi\,]=[\,N_\varphi:\alpha' N'_\varphi\,]=[\,N_\varphi:VN_{\sigma\varphi}\,]=[\,N'_\varphi:VN'_{\sigma\varphi}\,]=1\,.$$

Since, on the other hand, $v\in\mathcal{Z}_1(\mathfrak{a})$ if and only if $\alpha N_\varphi\subseteq VN'_{\sigma\varphi}$, and $v\in\mathcal{Z}'_1(\mathfrak{a})$ if and only if $\alpha' N'_\varphi\subseteq VN_{\sigma\varphi}$, we conclude that $\alpha N_\varphi=VN'_{\sigma\varphi}$ and $\alpha' N'_\varphi=VN_{\sigma\varphi}$ if $v\in\mathcal{Z}_1^+(\mathfrak{a})$, $\alpha VN_{\sigma\varphi}\nsubseteq pN'_\varphi$ if $v\notin\mathcal{Z}'_1(\mathfrak{a})$, and $\alpha' VN'_{\sigma\varphi}\nsubseteq pN_\varphi$ if $v\notin\mathcal{Z}_1(\mathfrak{a})$.

Recall now that there are canonical isomorphisms

$$\begin{array}{ccccccc}
\widetilde{V}\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F} & \cong & VN/pN & & \widetilde{V}'\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F} & \cong & VN'/pN' \\
\bigcap & & \bigcap & \text{and} & \bigcap & & \bigcap \\
\widetilde{M}\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F} & \cong & N/pN & & \widetilde{M}'\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F} & \cong & N'/pN'
\end{array}$$

which respect the actions of O_B. Since the homomorphisms induced by $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ on the left hand side of these diagrams correspond to the homomorphisms induced by α and α', respectively, on the right hand side, we can restate the results obtained above as follows. For any φ such that $v=\varphi|_F\in\mathcal{Z}_1$, $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ induce surjective homomorphisms $(\mathrm{Lie}\,\widetilde{A})_\varphi\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F}\to\widetilde{V}'_\varphi\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F}$ and $(\mathrm{Lie}\,\widetilde{A}')_\varphi\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F}\to\widetilde{V}_\varphi\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F}$ if $v\in\mathcal{Z}_1^+(\mathfrak{a})$, $\widetilde{\alpha}$ induces a surjective homomorphism $\widetilde{V}_\varphi\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F}\to\widetilde{V}'_\varphi\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F}$ if $v\notin\mathcal{Z}'_1(\mathfrak{a})$, and $\widetilde{\alpha}'$ induces a surjective homomorphism $\widetilde{V}'_\varphi\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F}\to\widetilde{V}_\varphi\otimes_{\widehat{\mathcal{O}}_x}\mathbb{F}$ if $v\notin\mathcal{Z}_1(\mathfrak{a})$. By the Lemma of Nakayama this implies (i), (ii), and (iii). The proof of (iv) and (v) is similar and can be found in [7] (Lemma 3.6). ∎

Using the preceding lemma we can now show that the map $\Gamma\to\widehat{\mathcal{O}}_x$ is formally smooth, and we can describe its relative tangent space

$$T_x=\widehat{\mathfrak{m}}_x/(\mathfrak{m}_\Gamma\widehat{\mathcal{O}}_x+\widehat{\mathfrak{m}}_x^2)$$

where $\mathfrak{m}_\Gamma$ denotes the maximal ideal of Γ.

Lemma 3.7 *For every surjective homomorphism $\beta\colon R\to R_0$ of local artinian Γ–algebras with residue class field $\mathbb{F}$ such that $\mathfrak{m}\mathfrak{c}=0$, where $\mathfrak{c}$ denotes the kernel of β and $\mathfrak{m}$ the maximal ideal of R, the induced map*

$$\mathrm{Hom}_\Gamma(\widehat{\mathcal{O}}_x,R)\longrightarrow\mathrm{Hom}_\Gamma(\widehat{\mathcal{O}}_x,R_0)$$

is surjective. Moreover, there is a basis

$$\{\, \xi_v \mid v \in \mathcal{Z}_0 \cup \mathcal{Z}_1 \smallsetminus \mathcal{Z}_1'(\mathfrak{a}) \,\} \cup \{\, \xi_v' \mid v \in \mathcal{Z}_1 \smallsetminus \mathcal{Z}_1(\mathfrak{a}) \,\} \cup \{\, \xi_\varphi \mid \varphi \in \mathcal{Z}_2 \smallsetminus \mathcal{Z}_2(\mathfrak{a})^* \,\}$$

of the $\mathbb{F}$*-vector space* T_x *such that* $\gamma(\xi_v) = \xi_{\gamma v}$ *for all* $v \in \mathcal{Z}_0 \cup \mathcal{Z}_1 \smallsetminus \mathcal{Z}_1'(\mathfrak{a})$, $\gamma(\xi_v') = \xi_{\gamma v}'$ *for all* $v \in \mathcal{Z}_1 \smallsetminus \mathcal{Z}_1(\mathfrak{a})$, *and* $\gamma(\xi_\varphi) = \xi_{\gamma\varphi}$ *for all* $\varphi \in \mathcal{Z}_2 \smallsetminus \mathcal{Z}_2(\mathfrak{a})^*$.

Proof. We consider the pull-backs $\mathfrak{a}_0 = (A_0, A_0', \alpha_0, \iota_0, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(R_0)$ of $\widetilde{\mathfrak{a}}$ and $(\alpha_0', \iota_0', \lambda_0, \lambda_0')$ of $(\widetilde{\alpha}', \widetilde{\iota}', \widetilde{\lambda}, \widetilde{\lambda}')$ with respect to a homomorphism $h_0 \in \mathrm{Hom}_\Gamma(\widehat{\mathcal{O}}_x, R_0)$. $\lambda_0 \circ \iota_0(b_0)$ defines a perfect alternating bilinear form Φ on the Lie algebra M of the universal extension of A_0 over R, and since M is an $O_B \otimes_{\mathbb{Z}} R$-module, it admits a natural decomposition (3.1). Let $V_0 \subset M_0 = M \otimes_R R_0$ denote the kernel of the natural surjection onto $\mathrm{Lie}\, A_0$ and Φ_0 the form on M_0 induced by Φ. Similarly, we obtain M', Φ', $V_0' \subset M_0'$, and Φ_0' using A_0' and $\lambda_0' \circ \iota_0'(b_0)$. α_0 and α_0' induce $O_B \otimes_{\mathbb{Z}} R$-module homomorphisms $\alpha\colon M \to M'$ and $\alpha'\colon M' \to M$, and there is a canonical isomorphism

$$(\widetilde{V} \subset \widetilde{M}, \widetilde{\Phi}, \widetilde{V}' \subset \widetilde{M}', \widetilde{\Phi}') \otimes_{\widehat{\mathcal{O}}_x} R_0 \cong (V_0 \subset M_0, \Phi_0, V_0' \subset M_0', \Phi_0').$$

For every $v \in \mathcal{Z}_1^+(\mathfrak{a})$, we lift $e_v \otimes 1 \in M_{0,\varphi_v}$ and $e_v' \otimes 1 \in M_{0,\varphi_v}'$ to elements $m_v \in M_{\varphi_v}$ and $m_v' \in M_{\varphi_v}'$ and define $V_{\varphi_v} = R(\alpha'(fm_v') - X_v' m_v) \subset M_{\varphi_v}$ and $V_{\varphi_v}' = R(\alpha(m_v) - X_v m_v') \subset M_{\varphi_v}'$. For every $\varphi \in \mathcal{Z}_2^+(\mathfrak{a})$, we lift $e_\varphi \otimes 1 \in M_{0,\varphi}$ to an element $m_\varphi \in M_\varphi$ and define $V_\varphi = R(\Pi m_{\varphi^*} - X_{\varphi^*} m_\varphi) \subset M_\varphi$. Lemma 3.6 and the condition $\mathfrak{mc} = 0$ imply that these submodules are independent of the choice of m_v, m_v', and m_φ, and lift the corresponding factors of V_0 and V_0'.

By the criterion of Messing [5] (Chapter V, Theorem 1.10), there is a canonical bijective correspondence between the set of homomorphisms $h \in \mathrm{Hom}_\Gamma(\widehat{\mathcal{O}}_x, R)$ such that $\beta h = h_0$ and the set of pairs (V, V') of O_B-invariant direct summands of M and M', respectively, which lift (V_0, V_0') and satisfy the following conditions.

- ▷ $\Phi(V \times V) = \Phi'(V' \times V') = 0$.
- ▷ For every $\varphi \in \{\, \varphi_v \mid v \in \mathcal{Z}_1^+(\mathfrak{a}) \,\} \cup \mathcal{Z}_2^+(\mathfrak{a})$, V_φ defined above is the component corresponding to φ in the decomposition (3.1) of V, and for every $v \in \mathcal{Z}_1^+(\mathfrak{a})$, V_{φ_v}' defined above is the component corresponding to φ_v in the decomposition (3.1) of V'.
- ▷ $\alpha V \subseteq V'$ and $\alpha' V' \subseteq V$.

Such a pair is determined by $(\, V_{\varphi_v} \mid v \in \mathcal{Z}_0 \cup \mathcal{Z}_1 \smallsetminus \mathcal{Z}_1'(\mathfrak{a}) \,)$, $(\, V_{\varphi_v}' \mid v \in \mathcal{Z}_1 \smallsetminus \mathcal{Z}_1(\mathfrak{a}) \,)$, and $(\, V_\varphi \mid \varphi \in \mathcal{Z}_2 \smallsetminus \mathcal{Z}_2(\mathfrak{a})^* \,)$ since Lemma 3.6 and the Lemma of Nakayama imply that, for all $v \in \mathcal{Z}_1 \smallsetminus \mathcal{Z}_1(\mathfrak{a})$, $V_{\varphi_v} = \alpha' V_{\varphi_v}'$, for all $v \in \mathcal{Z}_0 \cup \mathcal{Z}_1 \smallsetminus \mathcal{Z}_1'(\mathfrak{a})$, $V_{\varphi_v}' = \alpha V_{\varphi_v}$, for all $\varphi \in \mathcal{Z}_2(\mathfrak{a})$, $V'_\varphi = \alpha V_\varphi$, for all $\varphi \in \mathrm{Hom}(L, \mathbb{C}) \smallsetminus (\mathcal{Z}^\circ \cup \mathcal{Z}_2(\mathfrak{a}))$,

$$(V_\varphi, V'_\varphi) = \begin{cases} (0,0) & \text{if } \varphi|_F \in \mathcal{V}\,; \\ (\Pi V_{\varphi^*}, \Pi V'_{\varphi^*}) & \text{if } \varphi|_F \in \mathcal{Z}\,; \end{cases}$$

and for all $\varphi \in \mathrm{Hom}(L, \mathbb{C})$,

$$\begin{aligned} \overline{V}_\varphi &= \{\, m \in \overline{M}_\varphi \mid \Phi(m, V_\varphi) = 0 \,\} \\ \overline{V}'_\varphi &= \{\, m \in \overline{M}'_\varphi \mid \Phi'(m, V'_\varphi) = 0 \,\}\,. \end{aligned}$$

Since every direct summand of $N \otimes_R R_0$ for a free R−module N of finite rank can be lifted to a direct summand of N, this proves the first assertion of the lemma, and the second assertion follows as in [7] (Lemma 3.8). ∎

As in the proof of Proposition 3.5 in [7] the preceding lemma implies that the homomorphism $\Gamma \to \widehat{\mathcal{O}}_x$ can be extended to a Galois equivariant isomorphism $\mathcal{O}(\mathfrak{a}) \to \widehat{\mathcal{O}}_x$. Thus it remains to determine $\mathcal{Z}_1(\mathfrak{a}_0)$, $\mathcal{Z}'_1(\mathfrak{a}_0)$, and $\mathcal{Z}_2(\mathfrak{a}_0)$ for the pull-back $\mathfrak{a}_0 = (A_0, A'_0, \alpha_0, \iota_0, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(R_0)$ of $\widetilde{\mathfrak{a}}$ with respect to a homomorphism $h_0 \in \mathrm{Hom}_{\mathrm{W}(\mathbb{F})}(\widehat{\mathcal{O}}_x, R_0)$ for a local artinian $\mathbb{F}$−algebra R_0 with residue class field $\mathbb{F}$. Let α'_0 denote the pull-back of $\widetilde{\alpha}'$, and consider a $v \in \mathcal{Z}_1$. If $v \notin \mathcal{Z}'_1(\mathfrak{a})$, then α'_0 induces a surjective map $(\mathrm{Lie}\, A'_0)_{\varphi_v} \to (\mathrm{Lie}\, A_0)_{\varphi_v}$ since this holds after base change to the residue class field. Hence $\alpha(\mathrm{Lie}\, A_0)_{\varphi_v} = \alpha\alpha'(\mathrm{Lie}\, A'_0)_{\varphi_v} = p(\mathrm{Lie}\, A'_0)_{\varphi_v} = 0$, i.e. $v \in \mathcal{Z}_1(\mathfrak{a}_0)$. If, on the other hand, $v \in \mathcal{Z}_1^+(\mathfrak{a})$, then $v \in \mathcal{Z}_1(\mathfrak{a}_0)$ if and only if $\widetilde{\alpha}(e_v) \otimes 1 \in \widetilde{M}'_{\varphi_v} \otimes_{\widehat{\mathcal{O}}_x} R_0$ is contained in the submodule $\widetilde{V}'_{\varphi_v} \otimes_{\widehat{\mathcal{O}}_x} R_0$ which, by definition of x_v already contains $\widetilde{\alpha}(e_v) \otimes 1 - e'_v \otimes h_0(x_v)$. Hence $v \in \mathcal{Z}_1(\mathfrak{a}_0)$ if and only if $t'_v \otimes h_0(x_v) = 0$ in $(\mathrm{Lie}\, A'_0)_{\varphi_v} = (\mathrm{Lie}\, \widetilde{A}')_{\varphi_v} \otimes_{\widehat{\mathcal{O}}_x} R_0$. Since $t'_v \otimes 1$ is a generator of the free R_0−module $(\mathrm{Lie}\, A'_0)_{\varphi_v}$, this is equivalent to $h_0(x_v) = 0$, and we obtain our description of $\mathcal{Z}_1(\mathfrak{a}_0)$. Similarly, we can determine $\mathcal{Z}'_1(\mathfrak{a}_0)$. Since $\mathcal{Z}_2(\mathfrak{a}_0)$ has already been dealt with in [7] (Proposition 3.5), this completes the proof of the proposition. ∎

Definition 3.8 For every admissible triple Z, $Sh_{G,C,Z}$ denotes the closed subscheme of $\widetilde{Sh}_{G,C} \otimes_{O_E} \mathbb{F}$ which represents the functor $\underline{M}_{G,C,Z}$ such that, for every connected $\mathbb{F}$−scheme S,

$$\underline{M}_{G,C,Z}(S) = \{\, \mathfrak{a} \in \underline{\mathcal{M}}_{G,C}(S) \mid Z \leqslant \mathcal{Z}(\mathfrak{a}) \,\}$$

and for an arbitrary $\mathbb{F}$−scheme S, $\underline{M}_{G,C,Z}(S)$ is the product of the sets corresponding to the connected components of S.

Proposition 3.9 *The geometric special fibre of* $\widetilde{Sh}_{G,C}$ *is the union of the closed subschemes* $Sh_{G,C,Z}$ *where* Z *runs through the set of all minimal admissible triples. For every admissible triple* Z, $Sh_{G,C,Z}$ *is a non-singular projective variety of pure dimension* $d(Z)$ *over* $\mathbb{F}$, *and if* $\widetilde{Z}$ *is another admissible triple,*

$$Sh_{G,C,Z} \cap Sh_{G,C,\widetilde{Z}} = Sh_{G,C,Z\cup\widetilde{Z}}\,.$$

Moreover, if x *is a closed point of* $Sh_{G,C,Z}$ *corresponding to* $\mathfrak{a} \in \underline{M}_{G,C,Z}(\mathbb{F})$ *and* $\widehat{\mathcal{O}}_x$ *denotes the completion of the local ring of* x *in* $\widetilde{Sh}_{G,C} \otimes_{O_E} \mathbb{F}$, *then there is an* $\mathbb{F}$*–algebra isomorphism*

$$\widehat{\mathcal{O}}_x \cong \mathbb{F}[\![\, X_v,\, X'_{v'},\, X_\varphi \mid v \in \mathcal{Z}_0 \cup \mathcal{Z}_1(\mathfrak{a}),\, v' \in \mathcal{Z}'_1(\mathfrak{a}),\, \varphi \in \mathcal{Z}_2(\mathfrak{a}) \,]\!]/\mathcal{I}$$

where $\mathcal{I} = (\, X_v X'_v,\, X_\varphi X_{\varphi^*} \mid v \in \mathcal{Z}_1^+(\mathfrak{a}),\, \varphi \in \mathcal{Z}_2^+(\mathfrak{a})\,)$ *such that the ideal of* $Sh_{G,C,Z}$ *in* $\widehat{\mathcal{O}}_x$ *is generated by*

$$\{\, X_v \mid v \in Z_1 \cap \mathcal{Z}'_1(\mathfrak{a}) \,\} \cup \{\, X'_v \mid v \in Z'_1 \cap \mathcal{Z}_1(\mathfrak{a}) \,\} \cup \{\, X_\varphi \mid \varphi \in Z_2 \cap \mathcal{Z}_2(\mathfrak{a})^* \,\}\,.$$

Proof. In §4 (Corollary 4.5) we will see that there is an element $\mathfrak{a} \in \underline{\mathcal{M}}_{G,C}(\mathbb{F})$ such that $\mathcal{Z}_1(\mathfrak{a}) = \mathcal{Z}'_1(\mathfrak{a}) = \mathcal{Z}_1$ and $\mathcal{Z}_2(\mathfrak{a}) = \mathcal{Z}_2$. In particular, the subschemes defined above are not empty. The assertions concerning the structure of $\widehat{\mathcal{O}}_x$ and the ideal of $Sh_{G,C,Z}$ in $\widehat{\mathcal{O}}_x$ are immediate consequences of Proposition 3.5, and the remaining assertions follow since $\mathcal{Z}_1 \smallsetminus (Z_1 \cap Z'_1)$ is the disjoint union of $\mathcal{Z}_1(\mathfrak{a}) \smallsetminus (Z_1 \cap \mathcal{Z}'_1(\mathfrak{a}))$ and $\mathcal{Z}'_1(\mathfrak{a}) \smallsetminus (Z'_1 \cap \mathcal{Z}_1(\mathfrak{a}))$, and $\mathcal{Z}_2 \smallsetminus (Z_2 \cap Z_2^*)$ is the disjoint union of $\mathcal{Z}_2 \smallsetminus Z_2$ and $(\mathcal{Z}_2 \smallsetminus Z_2)^*$ and also the disjoint union of $\mathcal{Z}_2(\mathfrak{a}) \smallsetminus (Z_2 \cap \mathcal{Z}_2(\mathfrak{a})^*)$ and $\big(\mathcal{Z}_2(\mathfrak{a}) \smallsetminus (Z_2 \cap \mathcal{Z}_2(\mathfrak{a})^*)\big)^*$. ∎

§4. $\mathbb{F}$–valued points

In this section we describe the set $\underline{\mathcal{M}}_{G,C}(\mathbb{F})$ together with the action of the Frobenius automorphism of $\mathbb{F}$ over the residue class field $\kappa(E)$ of O_E. As in [7] we split this set into various isogeny classes. To this end, we introduce the following notation. For every $i \in \{1, \ldots, m\}$, let

$$f_i = [\, F_{\mathfrak{p}_i} : \mathbb{Q}_p \,]\,, \quad d_i = \#\big(\mathcal{Z} \cap \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \mathbb{Q}_p^{\mathrm{nr}})\big) \quad \text{and} \quad \delta_i = \begin{cases} 0 & \text{if } i \in I_0\,; \\ 1 & \text{otherwise.} \end{cases}$$

The quaternion algebra over F which is ramified at all infinite places, has the same invariants as D at all finite places not dividing p, and invariant $(d_i + \delta_i)/2$ at $\mathfrak{p}_i$ for every $i \in \{1, \ldots, m\}$ is denoted by D', and the set of isomorphism classes of pairs (L', ε) where $L'|F$ is a totally imaginary quadratic extension which splits D at all places not dividing p and ε is a map from the set of all primes of L' dividing p into the set of non-negative integers such that $\varepsilon(\mathfrak{p}) \neq \varepsilon(\overline{\mathfrak{p}})$ for at least one $\mathfrak{p}$, and

$$\sum_{\mathfrak{p} \mid \mathfrak{p}_i} (\varepsilon(\mathfrak{p}) + \delta_i) = d_i + \delta_i$$

for all i, is denoted by $\mathfrak{I}_D$. For every $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$, $Z_\mathfrak{t}$ denotes the centre of the algebraic group

$$H_\mathfrak{t} = \begin{cases} D'^\times & \text{if } \mathfrak{t} = D'\,; \\ L'^\times & \text{if } \mathfrak{t} = (L', \varepsilon)\,; \end{cases}$$

over $\mathbb{Q}$, and $A_\mathfrak{t}$ an abelian variety of dimension $2[\,K : \mathbb{Q}\,]$ over $\mathbb{F}$ together with an embedding $\iota_\mathfrak{t}\colon B \to \operatorname{End}^\circ A_\mathfrak{t}$ such that the commutant B' of B in $\operatorname{End}^\circ A_\mathfrak{t}$ and the Frobenius endomorphism π of $A_\mathfrak{t}$ over a finite field of definition $\mathbb{F}_q \subset \mathbb{F}$ for $A_\mathfrak{t}$ and all of its endomorphisms satisfy the following conditions.

- ▷ If $\mathfrak{t} = D'$, then $B' \cong D' \otimes_F K$ and $\pi \in K$ with

  $$\operatorname{ord}_{\overline{\mathfrak{q}}_i}(\pi) = d_i \operatorname{ord}_p(q)/2f_i$$

  for all $i \in \{1, \ldots, m\}$.

- ▷ If $\mathfrak{t} = (L', \varepsilon)$, then $B' \cong L' \otimes_F K$ and $\pi \in B'$ with

  $$\operatorname{ord}_\mathfrak{q}(\pi) = (d_i + \varepsilon(\mathfrak{p}) - \varepsilon(\overline{\mathfrak{p}})) \operatorname{ord}_\mathfrak{q}(q)/2f_i$$

  for all $i \in \{1, \ldots, m\}$ and all primes $\mathfrak{p}$ in L' and $\mathfrak{q}$ in B' such that $\mathfrak{q}$ divides $\mathfrak{p}$ and $\overline{\mathfrak{q}}_i$.

The pair $(A_\mathfrak{t}, \iota_\mathfrak{t})$ is uniquely determined up to isogeny. Let $\mathfrak{T}_D$ denote the set of all $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$ for which there are a polarization $\lambda_\mathfrak{t}$ of $A_\mathfrak{t}$, an isomorphism $\eta_\mathfrak{t}\colon B \otimes_\mathbb{Q} \mathbb{A}_f^p \to \widehat{\mathrm{V}}^p(A_\mathfrak{t})$ of $B \otimes_\mathbb{Q} \mathbb{A}_f^p$–modules, and an $f \in F \otimes_\mathbb{Q} \mathbb{A}_f^p(-1)$ such that $E^{\lambda_\mathfrak{t}} \circ (f\eta_\mathfrak{t} \times \eta_\mathfrak{t}) = \psi$. Then

$$\{D'\} \cup \{\, (L', \varepsilon) \in \mathfrak{I}_D \mid L' \text{ splits } D \,\} \subseteq \mathfrak{T}_D\,.$$

Let $d \in D$ act on B as right multiplication by d^*. We fix an $F \otimes_\mathbb{Q} \mathbb{A}_f^p$–algebra isomorphism $D' \otimes_\mathbb{Q} \mathbb{A}_f^p \cong D \otimes_\mathbb{Q} \mathbb{A}_f^p$ and, for every $(L', \varepsilon) \in \mathfrak{I}_D$, an $F \otimes_\mathbb{Q} \mathbb{A}_f^p$–algebra homomorphism $L' \otimes_\mathbb{Q} \mathbb{A}_f^p \to D \otimes_\mathbb{Q} \mathbb{A}_f^p$. For every $\mathfrak{t} \in \mathfrak{T}_D$, this induces an action of $H_\mathfrak{t}(\mathbb{A}_f^p)$ on $B \otimes_\mathbb{Q} \mathbb{A}_f^p$, and we may identify the commutant of B in $\operatorname{End}^\circ A_\mathfrak{t}$ with $D' \otimes_F K$ or $L' \otimes_F K$, respectively, and choose $\lambda_\mathfrak{t}$ and $\eta_\mathfrak{t}$ such that the latter respects the actions of $H_\mathfrak{t}(\mathbb{A}_f^p)$. Let now $G_\mathfrak{t}$ denote the group of all $\gamma \in \operatorname{End}^\circ A_\mathfrak{t}$ with $\eta_\mathfrak{t}^{-1} \circ \widehat{\mathrm{V}}^p(\gamma) \circ \eta_\mathfrak{t} \in G(\mathbb{A}_f^p)$ considered as an algebraic group over $\mathbb{Q}$, and let $\underline{\mathcal{M}}_{G,C}(\mathfrak{t})$

be the set of all $(A, A', \alpha, \iota, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(\mathbb{F})$ such that there is an isogeny $\beta\colon A \to A_{\mathfrak{t}}$ with $\eta_{\mathfrak{t}}^{-1} \circ \widehat{\mathrm{V}}^p(\beta) \circ \eta \in G(\mathbb{A}_f^p)$. Then there are an exact sequence

$$\begin{aligned} 1 \longrightarrow F^\times \longrightarrow H_{\mathfrak{t}} \times K^\times &\longrightarrow G_{\mathfrak{t}} \longrightarrow 1 \\ f \longmapsto (f\,, f^{-1}) & \\ (h\,, k) &\longmapsto h \otimes k \end{aligned}$$

and a decomposition

$$\underline{\mathcal{M}}_{G,C}(\mathbb{F}) = \coprod_{\mathfrak{t} \in \mathfrak{I}_D} \underline{\mathcal{M}}_{G,C}(\mathfrak{t})\,.$$

For proofs of the facts mentioned above we refer to [7] (§4).

Our next aim is to describe the sets $\underline{\mathcal{M}}_{G,C}(\mathfrak{t})$. This generalizes the results of §5 in [7]. We fix $\Pi \in O_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $b_0 \in (O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ as in (2.8), and for each $i \in \{1, \ldots, m\}$, a homomorphism $\varphi_i\colon L_{\mathfrak{p}_i} \to \mathcal{K}$ where $\mathcal{K}$ denotes the completion of $\mathbb{Q}_p^{\mathrm{nr}}$, i.e. the quotient field of $W(\mathbb{F})$. Let n denote the least common multiple of the integers $2f_i$ and consider any $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$. If $\mathfrak{t} = D'$, let $z_{\mathfrak{t}} \in Z_{\mathfrak{t}}(\mathbb{Q}_p)$ be the element with components $p^{-d_i n/2f_i} \in F_{\mathfrak{p}_i}$ and fix an $F \otimes_{\mathbb{Q}} \mathcal{K}$−algebra isomorphism $j_{\mathfrak{t}}\colon D' \otimes_{\mathbb{Q}} \mathcal{K} \to D \otimes_{\mathbb{Q}} \mathcal{K}$ which commutes with the actions of σ^n. If $\mathfrak{t} = (L', \varepsilon)$, let $z_{\mathfrak{t}} \in Z_{\mathfrak{t}}(\mathbb{Q}_p)$ be the element with components $p^{-n(\mathfrak{p})} \in L'_{\mathfrak{p}}$, where $n(\mathfrak{p}) = (d_i + \varepsilon(\mathfrak{p}) - \varepsilon(\overline{\mathfrak{p}}))n/2f_i$ if $\mathfrak{p}$ divides $\mathfrak{p}_i$, and fix an $F \otimes_{\mathbb{Q}} \mathcal{K}$−algebra homomorphism $j_{\mathfrak{t}}\colon L' \otimes_{\mathbb{Q}} \mathcal{K} \to D \otimes_{\mathbb{Q}} \mathcal{K}$ which commutes with the actions of σ^n. Let Y_I denote the set of all $d \in D^\times(\mathcal{K})$ such that

- ▷ $O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}) \subseteq (O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))\,d$;
- ▷ $X_I \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}) \subseteq (X_I \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\,d$;
- ▷ for every $v \in \mathrm{Hom}(F\,, \mathcal{K}) \cong \mathrm{Hom}(F\,, \mathbb{R})$,

$$\mathrm{ord}_p\big((v \otimes \mathrm{id}_{\mathcal{K}})(\mathrm{Nm}_{D|F}(d))\big) = \begin{cases} -1 & \text{if } v \in \mathcal{Z}\,; \\ \ \ 0 & \text{if } v \in \mathcal{V}\,. \end{cases}$$

To every $d \in Y_I$, we attach the admissible triple $\mathcal{Z}(d) = (\mathcal{Z}_1(d), \mathcal{Z}'_1(d), \mathcal{Z}_2(d))$ with

$$\begin{aligned} \mathcal{Z}_1(d) &= \{\, v \in \mathcal{Z}_1 \mid f_v(X_I \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\,d = f_v(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))\,\} \\ \mathcal{Z}'_1(d) &= \{\, v \in \mathcal{Z}_1 \mid f_v(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))\,pd = f_v(X_I \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\,\} \\ \mathcal{Z}_2(d) &= \{\, \varphi \in \mathcal{Z}_2 \mid l_\varphi(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))\Pi d = l_\varphi(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))\,\} \end{aligned}$$

where $f_v \in O_F \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})$ and $l_\varphi \in O_L \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})$ denote the idempotents corresponding to v and φ, respectively. For any two lattices M_1 and M_2 in a finite-dimensional $\mathcal{K}$−vector space, let

$$[\,M_1 : M_2\,] = \mathrm{length}_{\mathrm{W}(\mathbb{F})}\big(M_1/(M_1 \cap M_2)\big) - \mathrm{length}_{\mathrm{W}(\mathbb{F})}\big(M_2/(M_1 \cap M_2)\big)\,.$$

Lemma 4.1 *For every* $\mathfrak{t}$, *there is an element* $d_\mathfrak{t} \in D^\times(\mathcal{K})$ *such that*

$$j_\mathfrak{t}(\sigma(h)) = d_\mathfrak{t}\sigma(j_\mathfrak{t}(h))d_\mathfrak{t}^{-1}$$

for all $h \in H_\mathfrak{t}(\mathcal{K})$ *and*

$$d_\mathfrak{t}\sigma(d_\mathfrak{t})\cdots\sigma^{n-1}(d_\mathfrak{t}) = j_\mathfrak{t}(z_\mathfrak{t})\,.$$

The σ–*conjugacy class of* $d_\mathfrak{t}$ *is uniquely determined and contains an element* $d \in Y_I$. *If* $\mathfrak{t} = D'$, *this element can be chosen such that* $\mathcal{Z}(d) = (\mathcal{Z}_1, \mathcal{Z}_1, \mathcal{Z}_2)$.

Proof. In the proof of Lemma 5.2 in [7] we have explicitly defined an element $y_\mathfrak{t} \in Y_\varnothing$ which is σ–conjugate to an element $d_\mathfrak{t}$ satisfying the equations in our claim. Moreover, we have shown that the σ–conjugacy class of $d_\mathfrak{t}$ is uniquely determined. If $\mathfrak{t} = D'$ then $\mathcal{Z}_2(y_\mathfrak{t}) = \mathcal{Z}_2$. We fix now an $i \in I$ and denote the image of $y_\mathfrak{t}$ in $(D_{\mathfrak{p}_i} \otimes_{\mathbb{Q}_p} \mathcal{K})^\times$ by y. The second part of our claim will follow if we show that there is an element $\delta \in (D_{\mathfrak{p}_i} \otimes_{\mathbb{Q}_p} \mathcal{K})^\times$ such that $\mathrm{Nm}_{D|F}(\delta) \in F_{\mathfrak{p}_i}(O_{F_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))^\times$ and such that $d = \delta y \sigma(\delta)^{-1}$ satisfies

- ▷ $O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}) \subseteq (O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))d$ and $X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}) \subseteq (X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))d$;
- ▷ $(X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))d \subseteq O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F})$ and $(O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))pd \subseteq X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F})$ if $\mathfrak{t} = D'$.

Consider the $\mathcal{K}$–algebra isomorphism

$$\prod_{j=0}^{f_i-1}\omega_j \colon D_{\mathfrak{p}_i} \otimes_{\mathbb{Q}_p} \mathcal{K} \longrightarrow \prod_{j=0}^{f_i-1} \mathrm{M}_2(\mathcal{K})$$

such that

$$\omega_j(l) = \begin{pmatrix} \sigma^{-j}\varphi_i(l) & 0 \\ 0 & \sigma^{f_i-j}\varphi_i(l) \end{pmatrix} \qquad \text{for all } l \in L \subset D \text{ and}$$
$$\omega_j(\Pi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let $y_j = \omega_j(y)$. We define $O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F})$–submodules $M' \subset M \subset D_{\mathfrak{p}_i} \otimes_{\mathbb{Q}_p} \mathcal{K}$ as follows. If $\mathfrak{t} = (L', \varepsilon)$ and $\varepsilon(\mathfrak{p}) < \varepsilon(\overline{\mathfrak{p}})$ for a prime $\mathfrak{p}$ of L' above $\mathfrak{p}_i$, then there is a unique $j_0 \in \{0, \ldots, f_i - 1\}$ such that $\sigma^{-j_0}\varphi_i|_F \in \mathcal{Z}$ and

$$\#\{\, j \in \{j_0+1, \ldots, f_i-1\} \mid \sigma^{-j}\varphi_i|_F \in \mathcal{Z}\,\} = \varepsilon(\mathfrak{p})\,,$$

and $y_{\mathfrak{t}}$ was defined such that, for all $j \in \{0, \dots, f_i - 1\}$,

$$y_j = \begin{cases} 1 & \text{if } \sigma^{-j}\varphi_i|_F \in \mathcal{V}\,; \\ \begin{pmatrix} 0 & 1 \\ p^{-1} & 0 \end{pmatrix} & \text{if } j = j_0\,; \\ \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} & \text{otherwise}\,. \end{cases}$$

Let $M_j = \mathrm{M}_2(\mathrm{W}(\mathbb{F}))$ and

$$M'_j = \begin{cases} \mathrm{M}_2(\mathrm{W}(\mathbb{F}))\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} & \text{if } j \leqslant j_0\,; \\ \mathrm{M}_2(\mathrm{W}(\mathbb{F}))\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} & \text{if } j > j_0\,. \end{cases}$$

If $\mathfrak{t}$ does not satisfy the condition above then, for all $j \in \{0, \dots, f_i - 1\}$,

$$y_j = \begin{cases} 1 & \text{if } \sigma^{-j}\varphi_i|_F \in \mathcal{V}\,; \\ \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} & \text{otherwise}\,; \end{cases}$$

and we choose a matrix $g \in \mathrm{Gl}_2(\mathcal{K})$ such that

$$\sigma^{f_i}(g)\begin{pmatrix} 0 & p^{d_i} \\ 1 & 0 \end{pmatrix} g^{-1} = g_0^{d_i} \quad \text{where} \quad g_0 = \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix}.$$

Let $M_0 = \mathrm{M}_2(\mathrm{W}(\mathbb{F}))\,g$, $M'_0 = \mathrm{M}_2(\mathrm{W}(\mathbb{F}))\,g_0 g$, and for all $j \in \{0, \dots, f_i - 2\}$,

$$\begin{aligned} M_{j+1} &= \sigma^{-1}(M_j) &\quad \text{and} \quad M'_{j+1} &= \sigma^{-1}(M'_j) &\quad \text{if} \quad \sigma^{-j}\varphi_i|_F \in \mathcal{V}\,; \\ M_{j+1} &= \sigma^{-1}(M'_j y_j) &\quad \text{and} \quad M'_{j+1} &= \sigma^{-1}(M_j p y_j) &\quad \text{if} \quad \sigma^{-j}\varphi_i|_F \in \mathcal{Z}\,. \end{aligned}$$

For all $\mathfrak{t}$, we take M and M' such that, for all $j \in \{0, \dots, f_i - 1\}$, $\omega_j(M) = M_j$ and $\omega_j(M') = M'_j$. A simple computation shows that, for these $O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F})$–modules,

- ▷ $M' \subset M$;
- ▷ $\sigma(M) \subseteq My$ and $\sigma(M') \subseteq M'y$;
- ▷ $M'y \subseteq \sigma(M)$ and $Mpy \subseteq \sigma(M')$ if $\mathfrak{t} = D'$.

Since for all j, $[\,M_j : M'_j\,] = 1 = [\,\omega_j(O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F})) : \omega_j(X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\,]$, there is a $\delta \in (D_{\mathfrak{p}_i} \otimes_{\mathbb{Q}_p} \mathcal{K})^\times$ such that $M = (O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\delta$ and $M' = (X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\delta$, and since for all j, $[\,\mathrm{M}_2(\mathrm{W}(\mathbb{F})) : M_j\,] = [\,\mathrm{M}_2(\mathrm{W}(\mathbb{F})) : M_0\,]$, this element has all the required properties. ∎

For all $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$, we fix $d_\mathfrak{t}$ as in Lemma 4.1 and consider the non-empty set

$$X_{\mathfrak{t},I} = \{\, d \in D^\times(\mathcal{K}) \mid \mathcal{F}_\mathfrak{t}(d) \in dY_I \,\}$$

where $\mathcal{F}_\mathfrak{t}(d) = d_\mathfrak{t}\sigma(d)$. For $e = [\,E : \mathbb{Q}_p\,]$, $\mathcal{F}_\mathfrak{t}^e$ induces an automorphism $\mathrm{Fr}_\mathfrak{t}$ of $X_{\mathfrak{t},I}$, the group

$$C_I = \{\, d \in (O_D \otimes_\mathbb{Z} \mathrm{W}(\mathbb{F}))^\times \mid (X_I \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\, d = X_I \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}) \,\}$$

acts by right multiplication, and $H_\mathfrak{t}(\mathbb{Q}_p)$ acts by left multiplication via the embedding

$$H_\mathfrak{t}(\mathbb{Q}_p) \subset H_\mathfrak{t}(\mathcal{K}) \longrightarrow D^\times(\mathcal{K})$$

induced by $j_\mathfrak{t}$. The set $X_{\mathfrak{t},I}$ together with these actions is, up to isomorphism, independent of the choice of $j_\mathfrak{t}$ and $d_\mathfrak{t}$. For every $d \in X_{\mathfrak{t},I}$, let

$$\mathcal{Z}_\mathfrak{t}(d) = \mathcal{Z}(d^{-1}\mathcal{F}_\mathfrak{t}(d))\,.$$

Lemma 4.2 *For every $i \in I_0$ and every $x \in O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F})$, there is an element $c \in (O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))^\times$ such that $(X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\sigma(c)xc^{-1} \subseteq X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F})$.*

Proof. We may assume that $x \notin (O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))^\times$ since, otherwise, there is an element $c \in (O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))^\times$ such that $x^{-1} = c^{-1}\sigma(c)$. For every $j \in \{0, \ldots, f_i - 1\}$, let $x_j = \omega_j(x) \in \mathrm{M}_2(\mathrm{W}(\mathbb{F}))$ where ω_j denotes the homomorphism introduced in the proof of Lemma 4.1 and let $y_j = \sigma^j(x_j)\sigma^{j-1}(x_{j-1}) \cdots x_0$. We choose a maximal proper left ideal $\mathfrak{m}_0 \subset \mathrm{M}_2(\mathrm{W}(\mathbb{F}))$ containing y_{f_i-1}. For every $j \in \{1, \ldots, f_i\}$, $\mathfrak{c}_j = \sigma^{-j}\big(\mathfrak{m}_0 y_{j-1}^{-1} \cap \mathrm{M}_2(\mathrm{W}(\mathbb{F}))\big)$ is either equal to or a maximal proper left ideal in $\mathrm{M}_2(\mathrm{W}(\mathbb{F}))$, and we choose a maximal proper left ideal $\mathfrak{m}_j \subset \mathrm{M}_2(\mathrm{W}(\mathbb{F}))$ which is contained in $\mathfrak{c}_j$ as follows. Let

$$\mathfrak{m}_{f_i} = \mathfrak{m}_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \subset \mathrm{M}_2(\mathrm{W}(\mathbb{F})) = \mathfrak{c}_{f_i}$$

and assume that, for $j \in \{1, \ldots, f_i - 1\}$, $\mathfrak{m}_{j+1}$ has already been chosen. If $\mathfrak{c}_j$ is not equal to $\mathrm{M}_2(\mathrm{W}(\mathbb{F}))$, $\mathfrak{m}_j = \mathfrak{c}_j$ satisfies

$$\sigma(\mathfrak{m}_{j+1})x_j \subseteq \sigma(\mathfrak{c}_{j+1})x_j \subseteq \mathfrak{c}_j = \mathfrak{m}_j\,.$$

Otherwise,

$$\sigma(\mathfrak{m}_{j+1})x_j \subset \mathrm{M}_2(\mathrm{W}(\mathbb{F}))x_j \subseteq \mathrm{M}_2(\mathrm{W}(\mathbb{F}))$$

and we fix a maximal proper left ideal $\mathfrak{m}_j \subset \mathrm{M}_2(\mathrm{W}(\mathbb{F}))$ with $\sigma(\mathfrak{m}_{j+1})x_j \subseteq \mathfrak{m}_j$. Note that $\sigma(\mathfrak{m}_1)x_0 \subseteq \sigma(\mathfrak{c}_1)x_0 \subseteq \mathfrak{m}_0$. Consider now the element $c \in (O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))^\times$ such that, for all $j \in \{0, \ldots, f_i - 1\}$, $c_j = \omega_j(c) \in \mathrm{Gl}_2(\mathrm{W}(\mathbb{F}))$ satisfies

$$\omega_j(X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\, c_j = \mathfrak{m}_j\,.$$

Since, for all $j \in \{0, \dots, f_i - 1\}$,

$$\begin{aligned}\omega_j\big((X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\sigma(c)xc^{-1}\big) &= (\omega_j \circ \sigma)\big((X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))c\big)\, x_j c_j^{-1} \\ &= \sigma(\mathfrak{m}_{j+1}) x_j c_j^{-1} \subseteq \mathfrak{m}_j c_j^{-1} = \omega_j(X_i \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\,,\end{aligned}$$

this element has the required property. ∎

Corollary 4.3 *The natural map $X_{\mathfrak{t},I_0} \to X_{\mathfrak{t},I}/C_I$ is surjective.*

Proof. Consider an element $d \in X_{\mathfrak{t},I}$ and apply, for all $i \in I_0 \smallsetminus I$, the preceding lemma to the component of $\mathcal{F}_{\mathfrak{t}}(d)^{-1}d$ in $O_{D_{\mathfrak{p}_i}} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F})$. ∎

From now on we fix $\mathfrak{t} \in \mathfrak{T}_D$. Then $\eta_{\mathfrak{t}}$ gives rise to a commutative diagram

$$\begin{array}{ccc} H_{\mathfrak{t}}(\mathbb{A}_f^p) \times K^\times(\mathbb{A}_f^p) & \longrightarrow & D^\times(\mathbb{A}_f^p) \times K^\times(\mathbb{A}_f^p) \\ \downarrow & & \downarrow \\ G_{\mathfrak{t}}(\mathbb{A}_f^p) & \longrightarrow & G(\mathbb{A}_f^p) \end{array}$$

and $\lambda_{\mathfrak{t}} \circ \iota_{\mathfrak{t}}(b_0)$ induces a perfect pairing Φ on the rational covariant Dieudonné module $(N, \mathcal{F})$ of $A_{\mathfrak{t}}$. The natural isomorphism $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{i=1}^m (K_{\mathfrak{q}_i} \times K_{\overline{\mathfrak{q}}_i})$ induces a decomposition $N = \bigoplus_{i=1}^m (N_i \oplus \overline{N}_i)$ such that Φ identifies $\overline{N}_i$ with the dual of N_i. For every $i \in \{1, \dots, m\}$, we fix a lattice $M_i \subset l_{\varphi_i} N_i$. For every O_D−invariant lattice $M \subset \bigoplus_{i=1}^m N_i$, let $f_M \in F^\times(\mathbb{Q}_p)$ be the element which has components $p^{n_i} \in F_{\mathfrak{p}_i}$ where $n_i = \big[\, l_{\varphi_i} M : M_i \,\big]$, and let

$$M^\# = f_M M \oplus \overline{M} \subset N$$

where $\overline{M} \subset \bigoplus_{i=1}^m \overline{N}_i$ is the lattice which is dual to M with respect to Φ. Let $k_D \in K^\times(\mathbb{Q}_p)$ be the element which, for every $i \in \{1, \dots, m\}$, has component 1 in $K_{\overline{\mathfrak{q}}_i}^\times$ and component p^{e_i} in $K_{\mathfrak{q}_i}^\times$ where

$$e_i = \#\{\, j \in \{0, \dots, e-1\} \mid \sigma^{-j}\varphi_i|_F \in \mathcal{V} \,\}\,.$$

We fix an isomorphism

$$\eta_p \colon D \otimes_{\mathbb{Q}} \mathcal{K} \longrightarrow \bigoplus_{i=1}^m N_i$$

of $D \otimes_{\mathbb{Q}} \mathcal{K}$−modules such that the action of an element $h \in H_{\mathfrak{t}}(\mathcal{K})$ on $\bigoplus_{i=1}^m N_i$ (via $H_{\mathfrak{t}} \to G_{\mathfrak{t}}$) corresponds to right multiplication by $j_{\mathfrak{t}}(h)^* \in D^\times(\mathcal{K})$, and $\mathcal{F}$ corresponds to the map $x \mapsto \sigma(x) d_{\mathfrak{t}}^{-1}$ (cf. [7], Lemma 5.5). For every $(A, A', \alpha, \iota, \overline{\eta})$ in $\underline{\mathcal{M}}_{G,C}(\mathfrak{t})$ we choose an isogeny $\beta \colon A \to A_{\mathfrak{t}}$ with $\eta_{\mathfrak{t}}^{-1} \circ \widehat{\mathrm{V}}^p(\beta) \circ \eta \in G(\mathbb{A}_f^p)$ and use α and β

to identify the covariant Dieudonné modules of A and A' with lattices in N. This determines a bijection

$$\underline{\mathcal{M}}_{G,C}(\mathfrak{t}) \cong G_{\mathfrak{t}}(\mathbb{Q})\big\backslash\big(Y_{\mathfrak{t},I} \times G(\mathbb{A}_f^p)/C^p\big)$$

where $Y_{\mathfrak{t},I}$ denotes the set of all pairs (M, M') of O_B-invariant lattices in N which satisfy the following conditions.

- ▹ $pM \subseteq \mathcal{F}M \subseteq M$ and $pM' \subseteq \mathcal{F}M' \subseteq M'$.
- ▹ $\mathcal{F}M/pM$ and $\mathcal{F}M'/pM'$ are $O_L \otimes_{O_F} O_K \otimes_{\mathbb{Z}} \mathbb{F}-$modules of type $(L, \mathcal{V}_K)$.
- ▹ $f_I M' \subseteq M \subseteq M'$.
- ▹ For all $i \in I$, $M \cap N_i \nsubseteq pM'$ and $M' \cap N_i \nsubseteq M$.
- ▹ There is an $f \in F^{\times}(\mathbb{Q}_p)$ such that the forms $\Phi \circ (f \times \mathrm{id})\colon M \times M \to \mathrm{W}(\mathbb{F})$ and $\Phi \circ (f_I f \times \mathrm{id})\colon M' \times M' \to \mathrm{W}(\mathbb{F})$ are perfect.

If $\mathfrak{a} \in \underline{\mathcal{M}}_{G,C}(\mathfrak{t})$ corresponds to the element determined by $(M, M') \in Y_{\mathfrak{t},I}$ and $g \in G(\mathbb{A}_f^p)$, then $v \in \mathcal{Z}_1$ is contained in $\mathcal{Z}_1(\mathfrak{a})$ if and only if $f_v\mathcal{F}M = pf_vM'$, $v \in \mathcal{Z}_1$ is contained in $\mathcal{Z}_1'(\mathfrak{a})$ if and only if $f_v\mathcal{F}M' = f_vM$, and $\varphi \in \mathcal{Z}_2$ is contained in $\mathcal{Z}_2(\mathfrak{a})$ if and only if $l_\varphi\mathcal{F}M = l_\varphi\Pi M$. One can show as in [7] (Lemmas 5.9 and 5.10) that the map which sends every $(d, k) \in X_{\mathfrak{t},I} \times K^{\times}(\mathbb{Q}_p)$ to the pair of lattices $(kM(d)^{\#}, kM_I(d)^{\#})$ where

$$\begin{aligned} M(d) &= \eta_p\big((O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))\, d^{-1}\big) \\ M_I(d) &= \eta_p\big((X_I \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))\, d^{-1}\big) \end{aligned}$$

induces an isomorphism

$$F^{\times}(\mathbb{Q}_p)\big\backslash\big(X_{\mathfrak{t},I}/C_I \times K^{\times}(\mathbb{Q}_p)/(O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\times}\big) \longrightarrow Y_{\mathfrak{t},I}$$

where an element $f \in F^{\times}(\mathbb{Q}_p)$ acts on $X_{\mathfrak{t},I}$ as multiplication by f and on $K^{\times}(\mathbb{Q}_p)$ as multiplication by f^{-1}. If $M = M(d)^{\#}$ and $M' = M_I(d)^{\#}$ for a $d \in X_{\mathfrak{t},I}$, then $v \in \mathcal{Z}_1$ is contained in $\mathcal{Z}_1(d^{-1}\mathcal{F}_{\mathfrak{t}}(d))$ if and only if $f_v\mathcal{F}M = pf_vM'$, $v \in \mathcal{Z}_1$ is contained in $\mathcal{Z}_1'(d^{-1}\mathcal{F}_{\mathfrak{t}}(d))$ if and only if $f_v\mathcal{F}M' = f_vM$, and $\varphi \in \mathcal{Z}_2$ is contained in $\mathcal{Z}_2(d^{-1}\mathcal{F}_{\mathfrak{t}}(d))$ if and only if $l_\varphi\mathcal{F}M = l_\varphi\Pi M$. Hence we obtain the following generalization of Proposition 5.11 and Corollary 5.13 (loc. cit.).

Proposition 4.4 *There is a projective system of bijections*

$$\begin{array}{c} \underline{\mathcal{M}}_{G,C}(\mathfrak{t}) \\ \big\downarrow \\ (H_{\mathfrak{t}}(\mathbb{Q}) \times K^{\times})\Big\backslash\Big(F^{\times}(\mathbb{Q}_p)\big\backslash\big(X_{\mathfrak{t},I}/C_I \times K^{\times}(\mathbb{Q}_p)/(O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\times}\big) \times G(\mathbb{A}_f^p)/C^p\Big) \end{array}$$

such that the action of $K^\times(\mathbb{Q}_p)\times G(\mathbb{A}_f^p)$ on the projective limit with respect to all C which satisfy (2.9) corresponds to the obvious right multiplication, and the Frobenius automorphism of $\mathbb{F}\,|\,\kappa(E)$ acts via the endomorphism $\mathrm{Fr}_{\mathfrak{t}}\times k_D^{-1}$ of $X_{\mathfrak{t},I}\times K^\times(\mathbb{Q}_p)$. If $\mathfrak{a}\in\underline{\mathcal{M}}_{G,C}(\mathfrak{t})$ corresponds to the element determined by $d\in X_{\mathfrak{t},I}$, $k\in K^\times(\mathbb{Q}_p)$, and $g\in G(\mathbb{A}_f^p)$, then $\mathcal{Z}(\mathfrak{a})=\mathcal{Z}_{\mathfrak{t}}(d)$.

Corollary 4.5 *There is an element $\mathfrak{a}\in\underline{\mathcal{M}}_{G,C}(\mathbb{F})$ such that $\mathcal{Z}(\mathfrak{a})=(\mathcal{Z}_1,\mathcal{Z}_1,\mathcal{Z}_2)$.*

Proof. Let $\mathfrak{t}=D'$. The last assertion of Lemma 4.1 implies that there is an element $d\in X_{\mathfrak{t},I}$ such that $\mathcal{Z}_{\mathfrak{t}}(d)=(\mathcal{Z}_1,\mathcal{Z}_1,\mathcal{Z}_2)$, and by Proposition 4.4 there is an element $\mathfrak{a}\in\underline{\mathcal{M}}_{G,C}(\mathfrak{t})$ such that $\mathcal{Z}(\mathfrak{a})=\mathcal{Z}_{\mathfrak{t}}(d)$. ∎

§5. A reduction of Sh_D

We fix a parahoric subgroup $C_p\subset D^\times(\mathbb{Q}_p)$ and choose an order $O_D\subset D$, for every $i\in I_0$, a maximal proper left ideal $X_i\subset O_{D_{\mathfrak{p}_i}}$, and a subset $I\subseteq I_0$ such that

$$C_p=\{\,d\in(O_D\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times\mid X_I d=X_I\,\}\quad\text{where}\quad X_I=\prod_{i\in I}X_i\times\prod_{i\notin I}O_{D_{\mathfrak{p}_i}}.$$

Moreover, we choose quadratic extensions $L|F$ and $K|F$, an embedding $O_L\subset O_D$, a subset $\mathcal{V}_K\subset\mathrm{Hom}_{\mathbb{Q}}(K,\mathbb{C})$, an alternating form ψ on $B=D\otimes_F K$, and for every $i\in\{1,\dots,m\}$, a prime $\mathfrak{q}_i$ of K dividing $\mathfrak{p}_i$. All these choices are made as in §2. In particular, we impose conditions (2.4), … , (2.8). For every subgroup $C\subset D^\times(\mathbb{A}_f)$ which satisfies the following condition, we construct a reduction of $Sh_{D,C}$ over O_E.

(5.1) $C=C_p\times C^p$ where C^p is an open subgroup of $D^\times(\mathbb{A}_f^p)$, and there is an integer $N>2$ such that $C\subseteq(1+NO_D\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}})^\times$.

For every such C, let

$$\begin{aligned}N_C&=\min\{\,N>0\mid(1+NO_F\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}})^\times\subset C\,\}\\ C_K&=(F^\times(\mathbb{A}_f)\cap C)(1+N_CO_K\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}})^\times\subset K^\times(\mathbb{A}_f)\\ \overline{C}&=\tau_{\mathbb{A}_f}(C\times C_K)\subset G(\mathbb{A}_f)\end{aligned}$$

where G and τ are defined by the exact sequence (2.1). Then $N_C>2$, $\overline{C}$ satisfies condition (2.9), and there are a projective O_E–scheme $\mathcal{M}_{\overline{C}}$ and an isomorphism ϕ_C of schemes over $\mathbb{Z}_p^{\mathrm{nr}}=\mathrm{W}(\mathbb{F})\cap\mathbb{Q}_p^{\mathrm{nr}}$ such that the diagram

(5.2)
$$\begin{tikzcd}\mathcal{M}_{\overline{C}}\otimes_{O_E}\mathbb{Z}_p^{\mathrm{nr}} \arrow[r, "\phi_C"] \arrow[d, "\mathrm{id}\times\sigma^e"'] & \widetilde{Sh}_{G,\overline{C}}\otimes_{O_E}\mathbb{Z}_p^{\mathrm{nr}} \arrow[d, "k_D^{-1}\times\sigma^e"] \\ \mathcal{M}_{\overline{C}}\otimes_{O_E}\mathbb{Z}_p^{\mathrm{nr}} \arrow[r, "\phi_C"'] & \widetilde{Sh}_{G,\overline{C}}\otimes_{O_E}\mathbb{Z}_p^{\mathrm{nr}}\end{tikzcd}$$

commutes. These schemes and isomorphisms form projective systems with respect to the directed set of all C which satisfy condition (5.1), and there is a unique action of $K^\times(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ on the projective limit of the schemes $\mathcal{M}_{\overline{C}}$ with respect to which the limit of the isomorphisms ϕ_C is equivariant. Moreover, there is a natural projective system of $K^\times(\mathbb{A}_f)$−equivariant morphisms

$$\pi_C\colon \mathcal{M}_{\overline{C}} \otimes_{O_E} E \longrightarrow K^\times(\mathbb{A}_f)/F^\times(\mathbb{A}_f)C_K K^\times$$

where the right hand side denotes the constant E−scheme. Its limit is invariant with respect to the action of $D^\times(\mathbb{A}_f^p)$ considered as a subgroup of $G(\mathbb{A}_f^p)$ via τ, and there is a projective system of isomorphisms

$$Sh_{D,C} \otimes_{E(D)} E \longrightarrow \pi_C^{-1}(1)$$

the limit of which is equivariant with respect to the actions of $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$ (cf. [7], §2). Lemma 6.5 in [7] and Proposition 3.5 imply that the local ring at every closed point of $\mathcal{M}_{\overline{C}}$ is a normal domain of characteristic 0, hence $\mathcal{M}_{\overline{C}}$ is normal and flat over O_E, and π_C extends uniquely to a $K^\times(\mathbb{A}_f)$−equivariant morphism of O_E−schemes

$$\pi_C\colon \mathcal{M}_{\overline{C}} \longrightarrow K^\times(\mathbb{A}_f)/F^\times(\mathbb{A}_f)C_K K^\times$$

where now the right hand side denotes the constant O_E−scheme. The limit is again $D^\times(\mathbb{A}_f^p)$−invariant, and we define

$$\widetilde{Sh}_{D,C} = \pi_C^{-1}(1) \subseteq \mathcal{M}_{\overline{C}}\ .$$

Their limit $\widetilde{Sh}_{D,C_p}$ carries a natural action of $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$. For a fixed prime $\ell \neq p$ and every non-negative integer k, let

$$\Psi_C^k = \bigsqcup_{x \in \widetilde{Sh}_{D,C}(\mathbb{F})} \Psi_x^k$$

where Ψ_x^k denotes the stalk of the k−th $\mathbb{Q}_\ell$−sheaf of vanishing cycles at x.

The statement of the main result of this section requires the following notation. For every $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$, let

$$S(\mathfrak{t}) = \varprojlim_{C^p} H_{\mathfrak{t}}(\mathbb{Q})\Big\backslash\big(X_{\mathfrak{t},I}/C_I \times D^\times(\mathbb{A}_f^p)/C^p\big)$$

where the limit is taken with respect to all open compact subgroups $C^p \subset D^\times(\mathbb{A}_f^p)$ and where $H_{\mathfrak{t}}(\mathbb{Q})$ acts via its embeddings into $H_{\mathfrak{t}}(\mathbb{Q}_p)$ and $H_{\mathfrak{t}}(\mathbb{A}_f^p) \subseteq D^\times(\mathbb{A}_f^p)$. Let $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$ act on $S(\mathfrak{t})$ as the obvious right multiplication. For every $d \in X_{\mathfrak{t},I}$ and every $k \geqslant 0$, let

$$R_{\mathfrak{t},d}^k = \bigwedge\nolimits^k R_{\mathfrak{t},d}$$

where $\mathcal{Z}_{\mathfrak{t}}(d) = (Z_1, Z_1', Z_2)$ and $R_{\mathfrak{t},d}$ is the cokernel of the injection

$$\mathbb{Q}_\ell^{\mathcal{Z} \smallsetminus \mathcal{Z}_0} \longrightarrow \mathbb{Q}_\ell^{Z_1} \oplus \mathbb{Q}_\ell^{Z_1'} \oplus \mathbb{Q}_\ell^{Z_2}$$

defined by the inclusion maps $Z_1 \to \mathcal{Z}$, $Z_1' \to \mathcal{Z}$, and the restriction map $Z_2 \to \mathcal{Z}$. Since $\mathcal{Z}_{\mathfrak{t}}$ is invariant with respect to the actions of $H_{\mathfrak{t}}(\mathbb{Q}_p)$ and C_I on $X_{\mathfrak{t},I}$, there is an obvious action of $H_{\mathfrak{t}}(\mathbb{Q}_p)$ on

$$R^k_{\mathfrak{t},I} = \bigsqcup_{d \in X_{\mathfrak{t},I}/C_I} R^k_{\mathfrak{t},d}$$

such that the natural map $R^k_{\mathfrak{t},I} \to X_{\mathfrak{t},I}/C_I$ is equivariant. The action of σ^e on $\mathcal{Z}_1$ and $\mathcal{Z}_2$ induces isomorphisms $R_{\mathfrak{t},\mathrm{Fr}_{\mathfrak{t}}(d)} \to R_{\mathfrak{t},d}$ of $\mathbb{Q}_\ell$–vector spaces. We compose them with multiplication by p^e and then take the k–th exterior power to define an automorphism $\mathrm{Fr}_{\mathfrak{t}}$ of $R^k_{\mathfrak{t},I}$ for every $k \geqslant 0$. Using the commutative diagrams (5.2) and the morphisms π_C, we can now translate the results of the preceding sections into statements concerning $\widetilde{Sh}_{D,C_p}$.

Theorem 5.3 *The O_E–scheme $\widetilde{Sh}_{D,C_p}$ has the following properties.*

(i) *There is a natural $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$–equivariant isomorphism of E–schemes*

$$\widetilde{Sh}_{D,C_p} \otimes_{O_E} E \cong Sh_{D,C_p}.$$

(ii) *There is an $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$–equivariant bijection*

$$\widetilde{Sh}_{D,C_p}(\mathbb{F}) \cong \bigsqcup_{\mathfrak{t} \in \mathfrak{T}_D} S(\mathfrak{t})$$

such that the action of the Frobenius automorphism over $\kappa(E)$ on $\widetilde{Sh}_{D,C_p}(\mathbb{F})$ corresponds to the automorphism induced by $\mathrm{Fr}_{\mathfrak{t}}$ on $X_{\mathfrak{t},I}$ for all $\mathfrak{t}$.

(iii) *For every C which satisfies (5.1), $\widetilde{Sh}_{D,C_p}/C^p = \widetilde{Sh}_{D,C}$ is a normal projective flat O_E–scheme, and for every subgroup $C' \subset C$ which satisfies the same condition, the natural map $\widetilde{Sh}_{D,C'} \to \widetilde{Sh}_{D,C}$ is a finite étale covering.*

(iv) *$\widetilde{Sh}_{D,C} \otimes_{O_E} \mathbb{F}$ is the union of closed subschemes $Sh_{D,C,Z}$ where Z runs through the set of all minimal admissible triples. For every admissible triple Z, there is a non-singular projective subvariety $Sh_{D,C,Z}$ of pure dimension $d(Z)$ over $\mathbb{F}$, and if $\widetilde{Z}$ is another admissible triple, then*

$$Sh_{D,C,Z} \cap Sh_{D,C,\widetilde{Z}} = Sh_{D,C,Z \cup \widetilde{Z}}.$$

(v) *If $\widehat{\mathcal{O}}_x$ denotes the strict completion of the local ring at a closed point $x \in \widetilde{Sh}_{D,C}$ with respect to a fixed embedding $\kappa(x) \subset \mathbb{F}$, $d \in X_{\mathfrak{t},I}$ is the first component of a representative of this $\mathbb{F}$–valued point under the bijection in* (ii), *and $\mathcal{Z}_{\mathfrak{t}}(d) = (Z_1, Z_1', Z_2)$, then there is an isomorphism of* $\mathrm{W}(\mathbb{F})$*–algebras*

$$\widehat{\mathcal{O}}_x \cong \mathrm{W}(\mathbb{F})[\![X_v, X_{v'}', X_\varphi \mid v \in \mathcal{Z}_0 \cup Z_1, v' \in Z_1', \varphi \in Z_2]\!]/\mathcal{I}$$

where $\mathcal{I} = (X_v X_v' - p, X_\varphi X_{\varphi^*} - p \mid v \in Z_1 \cap Z_1', \varphi \in Z_2 \cap Z_2^*)$ such that the action of $\varrho \in \mathrm{Gal}(\mathbb{F}\,|\,\kappa(x))$ on $\widehat{\mathcal{O}}_x$ corresponds to the extension of the natural action on $\mathrm{W}(\mathbb{F})$ given by $\varrho(X_v) = X_{\varrho v}$, $\varrho(X_{v'}') = X_{\varrho v'}'$, and $\varrho(X_\varphi) = X_{\varrho\varphi}$ for all $v \in \mathcal{Z}_0 \cup Z_1$, $v' \in Z_1'$, and $\varphi \in Z_2$. Moreover, x is contained in $Sh_{D,C,\widetilde{Z}}$ for an admissible triple $\widetilde{Z} = (\widetilde{Z}_1, \widetilde{Z}_1', \widetilde{Z}_2)$ if and only if $\widetilde{Z} \leqslant \mathcal{Z}_{\mathfrak{t}}(d)$. In this case the ideal of $Sh_{D,C,\widetilde{Z}}$ in $\widehat{\mathcal{O}}_x$ is generated by p and

$$\{ X_v \mid v \in \widetilde{Z}_1 \cap Z_1' \} \cup \{ X_v' \mid v \in \widetilde{Z}_1' \cap Z_1 \} \cup \{ X_\varphi \mid \varphi \in \widetilde{Z}_2 \cap Z_2^* \}.$$

(vi) *The action of the Galois group of E on the sheaves of vanishing cycles on $\widetilde{Sh}_{D,C} \otimes_{O_E} \mathbb{F}$ factors through the Galois group of $\kappa(E)$. Moreover, for every admissible triple Z, these sheaves are constant on $Sh_{D,C,Z} \smallsetminus \bigcup_{Z<\widetilde{Z}} Sh_{D,C,\widetilde{Z}}$, and there are commutative diagrams*

$$\begin{tikzcd}
\Psi_C^k \arrow[r, "\mu_C^k"] \arrow[d] & \bigsqcup_{\mathfrak{t}\in\mathfrak{T}_D} H_{\mathfrak{t}}(\mathbb{Q})\backslash\big(R_{\mathfrak{t},I}^k \times D^\times(\mathbb{A}_f^p)/C^p\big) \arrow[d] \\
\widetilde{Sh}_{D,C}(\mathbb{F}) \arrow[r, "\mu_C"] & \bigsqcup_{\mathfrak{t}\in\mathfrak{T}_D} H_{\mathfrak{t}}(\mathbb{Q})\backslash\big(X_{\mathfrak{t},I}/C_I \times D^\times(\mathbb{A}_f^p)/C^p\big)
\end{tikzcd}$$

where μ_C denotes the bijection induced by (ii), *the vertical maps are the natural projections, and μ_C^k is a bijection which identifies the action of the Frobenius automorphism over $\kappa(E)$ on Ψ_C^k with the automorphism induced by the actions of $\mathrm{Fr}_{\mathfrak{t}}$ on $R_{\mathfrak{t},I}^k$ for all $\mathfrak{t}$.*

Proof. This follows from the results in the preceding sections in the same way as in the proof of Theorem 6.6 in [7] with the correction given at the end of [8] and with the additional input provided by Proposition 2.14. ∎

Let now $z = \#\mathcal{Z}$ and $z_0 = \#\mathcal{Z}_0$. For every subset $C \subset D^\times(\mathbb{A}_f)$ which satisfies (5.1) and every non-negative integer $j \leqslant z - z_0$, the subschemes $Sh_{D,C,Z}$ where Z is an admissible triple with $d(Z) = z - j$ are permuted by the action of the Frobenius automorphism over $\kappa(E)$ on $\widetilde{Sh}_{D,C} \otimes_{O_E} \mathbb{F}$. This defines a $\kappa(E)$–scheme $W_{j,C}$, an isomorphism

$$W_{j,C} \otimes_{\kappa(E)} \mathbb{F} \cong \bigsqcup_{d(Z)=z-j} Sh_{D,C,Z},$$

and a morphism $\nu_{j,C}\colon W_{j,C} \to \widetilde{Sh}_{D,C} \otimes_{O_E} \kappa(E)$. The schemes $W_{j,C}$ form a projective system with respect to C, and on the limit we have a continuous action of $D^\times(\mathbb{A}_f^p)$ such that the limit of the projective system of morphisms $\nu_{j,C}$ is equivariant.

Corollary 5.4 *For a subgroup $C \subset D^\times(\mathbb{A}_f)$ which satisfies* (5.1), *$\nu_{0,C}$ is surjective, and $\widetilde{Sh}_{D,C}$ is smooth if and only if $\mathcal{Z} = \mathcal{Z}_0$. Moreover, for every $j \in \{0, \ldots, z - z_0\}$, $W_{j,C}$ is a smooth projective $\kappa(E)$–scheme of pure dimension $z - j$. Let Y_j denote the scheme theoretic image of $\nu_{j,C}$. Then Y_{j+1} (with $Y_{z-z_0+1} = \varnothing$) is a closed subscheme of Y_j. Let $D_j = \nu_{j,C}^{-1}(Y_{j+1})$ and $U_j = Y_j \smallsetminus Y_{j+1}$. Then $D_j \otimes_{\kappa(E)} \mathbb{F}$ is a union of smooth divisors with, at most, normal crossings in $W_{j,C} \otimes_{\kappa(E)} \mathbb{F}$, $\nu_{j,C}\colon W_{j,C} \smallsetminus D_j \to U_j$ is an isomorphism, and the restrictions of the sheaves of vanishing cycles on $\widetilde{Sh}_{D,C} \otimes_{O_E} \mathbb{F}$ to $U_j \otimes_{\kappa(E)} \mathbb{F}$ are constant.*

§6. Comparison with the general conjecture

We retain the notation of the preceding section. As in §1, C_p determines a parahoric subgroup $C_p(\mathcal{K}) \subset D^\times(\mathcal{K})$. Obviously, we have $C_p(\mathcal{K}) = C_I$. On the other hand, $\{\, d \in C_p \mid X_{I_0} d = X_{I_0} \,\}$ is an Iwahori subgroup of $D^\times(\mathbb{Q}_p)$, and we use the corresponding chamber $\mathfrak{C}$ to define the set $Y_{h_D} \subset D^\times(\mathcal{K})$ as in §1.

Lemma 6.1 $Y_{h_D} = Y_{I_0}$.

Proof. We fix an isomorphism of $F \otimes_{\mathbb{Q}} \mathcal{K}$–algebras

$$\omega\colon D \otimes_{\mathbb{Q}} \mathcal{K} \longrightarrow F \otimes_{\mathbb{Q}} \mathrm{M}_2(\mathcal{K})$$

such that $\omega(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})) \subseteq O_F \otimes_{\mathbb{Z}} \mathrm{M}_2(\mathrm{W}(\mathbb{F}))$ and $1 \otimes \gamma \in \omega(X_{I_0} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))$ where

$$\gamma = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{M}_2(\mathcal{K}).$$

Then we have

$$\text{(6.2)}\quad \omega\big(\{\, d \in O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}) \mid (X_{I_0} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}))d \subseteq X_{I_0} \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}) \,\}\big) \\ = O_F \otimes_{\mathbb{Z}} \big(\mathrm{M}_2(\mathrm{W}(\mathbb{F})) \cap \gamma^{-1}\, \mathrm{M}_2(\mathrm{W}(\mathbb{F}))\gamma\big).$$

Moreover, ω induces isomorphisms

$$\mathcal{B}(D^\times, \mathcal{K}) \cong \prod_v \mathcal{B}(\mathrm{Gl}_2, \mathcal{K}) \quad\text{and}\quad \widetilde{X}(D^\times, \mathcal{K}) \cong \prod_v \widetilde{X}(\mathrm{Gl}_2, \mathcal{K})$$

where v runs through the set of all embeddings of F into $\mathcal{K}$. Let $x_0 \in \mathcal{B}(\mathrm{Gl}_2, \mathcal{K})$ be any point which is fixed by $\mathrm{Gl}_2(\mathrm{W}(\mathbb{F}))$. Then x_0 and $\gamma^{-1}(x_0)$ are contained in the closure of a unique chamber $\mathfrak{C}_0 \subset \mathcal{B}(\mathrm{Gl}_2, \mathcal{K})$, and, under the above isomorphism, $\mathfrak{C}$ is contained in $\prod_v \mathfrak{C}_0$. Let, for $n \in \{0, 1\}$,

$$Y_n = \{\, g^{-1} \mid g \in \mathrm{M}_2(\mathrm{W}(\mathbb{F})),\ \det(g) \in p^n \mathrm{W}(\mathbb{F})^\times \,\}$$
$$Y'_n = \{\, g \in \mathrm{Gl}_2(\mathcal{K}) \mid \mathrm{inv}(g x_0, x_0) = [\mu_n] \,\}$$

where $\mu_n \colon \mathbb{G}_{m,\mathcal{K}} \to \mathrm{Gl}_{2,\mathcal{K}}$ denotes the homomorphism of algebraic groups over $\mathcal{K}$ with $\mu_n(p) = \gamma^n$. Then, (6.2) implies that

$$\omega(Y_{I_0}) = \prod_{v \in \mathcal{Z}} (Y_1 \cap \gamma^{-1} Y_1 \gamma) \times \prod_{v \in \mathcal{V}} (Y_0 \cap \gamma^{-1} Y_0 \gamma)\,,$$

and since, under the above isomorphism, the component of $[\mu_{h_D}] \in \widetilde{X}(D^\times, \mathcal{K})$ corresponding to v is $[\mu_n] \in \widetilde{X}(\mathrm{Gl}_2, \mathcal{K})$ where $n = 1$ if $v \in \mathcal{Z}$ and $n = 0$ if $v \in \mathcal{V}$, we have

$$\omega(Y_{h_D}) = \prod_{v \in \mathcal{Z}} (Y'_1 \cap \gamma^{-1} Y'_1 \gamma) \times \prod_{v \in \mathcal{V}} (Y'_0 \cap \gamma^{-1} Y'_0 \gamma)\,.$$

On the other hand, it is easy to see that, for $n \in \{0, 1\}$,

$$Y'_n = \mathrm{Gl}_2(\mathrm{W}(\mathbb{F}))\, \gamma^{-n}\, \mathrm{Gl}_2(\mathrm{W}(\mathbb{F})) = Y_n\,.$$

This proves our claim ∎

Note that, in general, the set Y_h occurring in Definition 1.3 differs from the corresponding set in Definition B3.5 of [7]. In the special case we are considering in the present paper, the preceding lemma and Lemma 6.8 in [7] imply that these two sets are Y_{I_0} and $Y_\varnothing$, respectively. By Lemma 4.2, a σ−conjugacy class in $D^\times(\mathcal{K})$ contains an element of Y_{I_0} if and only if it contains an element of $Y_\varnothing$. Therefore, a morphism of $\overline{\mathbb{Q}}|\mathbb{Q}$−groupoids $\varphi \colon \mathfrak{Q} \to \mathfrak{G}_{D^\times}$ is admissible if and only it is admissible in the sense of Definition B3.5 in [7]. Recall now the following facts which have been obtained in the proof of Proposition 6.10 in [7].

Lemma 6.3 *There is a bijection from the set $\{D'\} \cup \mathfrak{I}_D$ onto the set of all equivalence classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_{D^\times}$ such that, if $\mathfrak{t}$ is mapped to the equivalence class of φ, then there are an isomorphism $\varrho \colon H_\mathfrak{t} \to \mathrm{Aut}(\varphi)$ of algebraic groups over $\mathbb{Q}$, an element $g_1 \in X^p(\varphi) \subset D^\times(\overline{\mathbb{A}^p_f})$, θ_0 and g_0 as in § 1, and $j_\mathfrak{t}$ and $d_\mathfrak{t}$ as in § 4 which satisfy the following conditions.*

- ▷ *For every $h \in H_\mathfrak{t}(\mathbb{A}^p_f) \subseteq D^\times(\mathbb{A}^p_f)$, $\varrho(h) \circ g_1 = g_1 h$.*
- ▷ $d_\mathfrak{t} = \theta_0(d_\sigma) \in D^\times(\mathbb{Q}_p^{\mathrm{nr}})$.

▷ *The diagram*

$$\begin{tikzcd}
h \arrow[d, mapsto] & H_{\mathfrak{t}}(\mathbb{Q}_p) \arrow[r] \arrow[d] & H_{\mathfrak{t}}(\mathcal{K}) \arrow[d] & h \arrow[d, mapsto] \\
g_0\,\varrho(h)\,g_0^{-1} & \mathrm{Aut}(\theta_0)(\mathbb{Q}_p) \arrow[r] & D^\times(\mathcal{K}) & j_{\mathfrak{t}}(h)\ ,
\end{tikzcd}$$

in which the horizontal arrows indicate the natural inclusions, commutes.

Proposition 6.4 *For every parahoric subgroup $C_p \subset D^\times(\mathbb{Q}_p)$, the following assertions hold.*

(i) *$\widetilde{Sh}_{D,C_p}$ provides an admissible model for Sh_{D,C_p} if and only if $\mathfrak{I}_D \subset \mathfrak{T}_D$. In particular, this condition is satisfied if either D is totally indefinite or, for all $i \in \{1,\ldots,m\} \smallsetminus I_0$, $d_i = 0$.*

(ii) *$\widetilde{Sh}_{D,C_p}$ provides a smooth admissible model for Sh_{D,C_p} if and only if C_p is h_D–hyperspecial.*

Proof. Let $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$ correspond to the equivalence class of $\varphi\colon \mathfrak{Q} \to \mathfrak{G}_{D^\times}$, and choose ϱ, g_1, θ_0, g_0, $j_{\mathfrak{t}}$, and $d_{\mathfrak{t}}$ as in Lemma 6.3. Then, Lemma 6.1 implies that $X_p(\varphi) = X_{\mathfrak{t},I_0} \subset D^\times(\mathcal{K})$. The commutative diagram in Lemma 6.3 shows that the action of an element $h \in H_{\mathfrak{t}}(\mathbb{Q})$ on $X_{\mathfrak{t},I_0}$ coincides with the action of $\varrho(h) \in \mathrm{Aut}(\varphi)(\mathbb{Q})$ on $X_p(\varphi)$, and the condition on the choice of $d_{\mathfrak{t}}$ ensures that the action of $\mathrm{Fr}_{\mathfrak{t}}$ on $X_{\mathfrak{t},I_0}$ coincides with the action of Φ on $X_p(\varphi)$. On the other hand, as noted above, we have $C_p(\mathcal{K}) = C_I$. Together with Corollary 4.3 this implies that $X_{\mathfrak{t},I}/C_I$ is the image of $X_p(\varphi)$ in $D^\times(\mathcal{K})/C_p(\mathcal{K})$. Moreover, by the choice of g_1, the map which sends d to $g_1 d$ defines a bijection between $D^\times(\mathbb{A}_f^p)$ and $X^p(\varphi)$ such that the action of an element $h \in H_{\mathfrak{t}}(\mathbb{Q})$ on $D^\times(\mathbb{A}_f^p)$ corresponds to the action of $\varrho(h)$ on $X^p(\varphi)$. Hence, there is an $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$–equivariant bijection between $S(\varphi)$ and $S(\mathfrak{t})$ which identifies the actions of Φ and $\mathrm{Fr}_{\mathfrak{t}}$, and (i) is a consequence of Theorem 5.3 above and of Lemma 4.8 in [7].

Concerning assertion (ii) we remark that, in general, if there is a subgroup $H \subseteq G_{\mathbb{Q}_p}$ which satisfies the conditions in Definition 1.5, then the group generated by H and the connected component of 1 in the center of G satisfies these conditions, too. Now every connected normal algebraic subgroup of $(D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ which contains the torus $(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ has the form

$$H = \prod_{i \in J} D_{\mathfrak{p}_i}^\times \times \prod_{i \notin J} F_{\mathfrak{p}_i}^\times$$

for a subset $J \subseteq \{1,\ldots,m\}$, and this group satisfies the conditions in Definition 1.5 if and only if

$$\mathcal{Z} \subseteq \bigsqcup_{i \in J} \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \mathbb{Q}_p^{\mathrm{nr}}) \quad \text{and} \quad J \subseteq I_0 \smallsetminus I\,.$$

Obviously, a subset J with these properties exists if and only if $\mathcal{Z} = \mathcal{Z}_0$. Together with (i) and Corollary 5.4 this completes the proof. ∎

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