ON REGULARITY OF TRANSITION PROBABILITIES AND INVARIANT MEASURES OF SINGULAR DIFFUSIONS UNDER MINIMAL CONDITIONS

VLADIMIR I. BOGACHEV, NICOLAI V. KRYLOV,
AND MICHAEL RÖCKNER

Abstract. Let $A = (a^{ij})$ be a matrix-valued Borel mapping on a domain $\Omega \subset \mathbb{R}^d$, let $b = (b^i)$ be a vector field on Ω, and let $L_{A,b}\varphi = a^{ij}\partial_{x_i}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi$. We study Borel measures μ on Ω that satisfy the elliptic equation $L^*_{A,b}\mu = 0$ in the weak sense: $\int L_{A,b}\varphi\, d\mu = 0$ for all $\varphi \in C_0^\infty(\Omega)$. We prove that, under mild conditions, μ has a density. If A is locally uniformly nondegenerate, $A \in H^{p,1}_{loc}$ and $b \in L^p_{loc}$ for some $p > d$, then this density belongs to $H^{p,1}_{loc}$. Actually, we prove Sobolev regularity for solutions of certain generalized nonlinear elliptic inequalities. Analogous results are obtained in the parabolic case. These results are applied to transition probabilities and invariant measures of diffusion processes.

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1. Introduction and notation

Though the results and methods of this paper are purely analytic, we first want to describe our original motivation in the direction of probabilistic applications (which also explains the title of this paper). A

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fundamental problem in probability theory is the study of equilibrium distributions (i.e., invariant measures) of diffusion processes governed by the stochastic differential equation on $\mathbb{R}^d$

$$d\xi_t = \sigma(\xi_t)dw_t + b(\xi_t)dt \tag{1.1}$$

under minimal conditions on the coefficients σ and b. The main issues are, of course, existence and uniqueness of such invariant measures, but also their regularity properties. In order to avoid a priori conditions that ensure the existence of a solution to (1.1) or a corresponding semigroup describing its transition probabilities, we shall use the purely analytic reformulation of the notion of an invariant measure in terms of the underlying generator

$$L = \frac{1}{2}(\sigma\sigma^T)^{ij}\partial_i\partial_j + b^i\partial_i,$$

i.e., we consider solutions μ to the equation

$$L^*\mu = 0.$$

In the terminology of [4], we are thus concerned with the adjoint solutions for L (see also [13] where additional references can be found). Existence and uniqueness results for probability measures solving this equation have been proved in [2], [10], again mainly based on analytic tools. In particular, the corresponding uniqueness results depend on the third issue, namely, the regularity of solutions in an essential way. In the present paper, we, therefore, address the latter problem again. Our results widely generalize our previous ones in the elliptic case [8], since the assumptions on the diffusion coefficients are much more general and we also include the parabolic case.

To be more specific, let Ω be an open set in $\mathbb{R}^d$, let $A = (a^{ij})$ be a Borel mapping on Ω with values in the space of the nonnegative matrices on $\mathbb{R}^d$, and let $b = (b^i)\colon\ \Omega \to \mathbb{R}^d$ be a Borel vector field. Let us set

$$L_{A,b}\varphi = a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi, \qquad \varphi \in C_0^\infty(\Omega), \tag{1.2}$$

where we use the standard summation rule for repeated indices. Suppose that μ is a locally finite Borel measure on Ω such that a^{ij}, $|b| \in L^1_{loc}(\mu)$ and

$$L^*_{A,b}\mu = 0 \tag{1.3}$$

in the following sense:

$$\int_\Omega L_{A,b}\varphi\, d\mu = 0, \qquad \forall\, \varphi \in C_0^\infty(\Omega). \tag{1.4}$$

Our first main result states that if μ is nonnegative, then the measure $(\det A)^{1/d}\mu$ has a density in $L_{loc}^{\frac{d}{d-1}}(\Omega, dx)$ and if A is Hölder continuous and nondegenerate, then the same is true for μ itself even if μ is only a signed measure. In fact, we prove such a result in the case of certain nonlinear elliptic inequalities. This extends a result from [23], where b was assumed to be locally bounded. Then we prove that if the a^{ij}'s belong to the Sobolev class $H_{loc}^{p,1}(\Omega)$, $\det A \geq c > 0$, and $|b| \in L_{loc}^{p}(\Omega)$, where $p > d$, then μ has a density from $H_{loc}^{p,1}(\Omega)$. The latter improves our result from [8], where the a^{ij}'s were assumed to be infinitely differentiable. In comparison with classical generalized solutions of second order elliptic equations (treated, e.g., in [15], [16], [27], [32], [37], [40]), a novelty of the problem addressed in this work is that solutions of (1.3) are not a priori in the first Sobolev class; the fact that they are indeed in $H_{loc}^{p,1}$ is one of the main results that enables one to apply to such solutions the classical theory, in particular, Harnack's inequality, etc. Sobolev regularity results and Harnack's inequality play an important role in the proof of an existence result in [10], which generalizes a well known theorem of Hasminskii to the case of non-Lipschitzian coefficients. Namely, the existence of a probability measure μ satisfying (1.3) has been proved in [10] under the following conditions: A is locally uniformly nondegenerate, $a^{ij} \in H_{loc}^{p,1}(\mathbb{R}^d)$, $b^i \in L_{loc}^{p}(\mathbb{R}^d)$, $p > d$, and there exists a positive function $V \in C^2(\mathbb{R}^d)$ such that $V(x) \to +\infty$ and $L_{A,b}V(x) \to -\infty$ as $|x| \to +\infty$. Finally, regularity results are important also for the uniqueness problem and have been used in [2] for solving Varadhan's uniqueness problem (see Section 4).

As a consequence, we obtain by results in [38] that an extension of $\big(L_{A,b}, C_0^\infty(\Omega)\big)$ generates a sub-Markovian C_0-semigroup $(T_t)_{t\geq 0}$ on $L^1(\Omega,\mu)$ and that μ is a sub-invariant measure with respect to $(T_t)_{t\geq 0}$. Furthermore, if $\Omega = \mathbb{R}^d$, then μ is invariant with respect to $(T_t)_{t\geq 0}$ if and only if this extension coincides with the closure of $\big(L_{A,b}, C_0^\infty(\Omega)\big)$ on $L^1(\Omega,\mu)$ (cf. Section 4). We also prove that $(T_t)_{t\geq 0}$ is strongly Feller (see Theorem 4.1).

In Section 3, we prove analogous results in the parabolic case, i.e., we consider a locally finite Borel measure μ on $\Omega_T := \Omega \times (0,T)$ such that

$$\int_{\Omega_T} [\partial_t\varphi + L_{A,b}\varphi]\, d\mu = 0, \qquad \forall\, \varphi \in C_0^\infty\big(\Omega \times (0,T)\big), \tag{1.5}$$

where now a^{ij} and b depend on t and belong to $L_{loc}^1(\Omega_T,\mu)$. We prove that μ is absolutely continuous provided A is Hölder continuous in x uniformly with respect to t and nondegenerate. If, in addition, A is locally uniformly nondegenerate, the $H^{p,1}$-norm of A in x is uniformly bounded in t and b is in $L^p(\Omega_T)$, where $p > d+2$, then μ has a Hölder continuous density which belongs to the space $\mathbb{H}^{p,1}\big((0,T),\Omega)\big)$ of L^p-mappings with values in $H^{p,1}(\Omega)$. This result applies to the transition probabilities of singular diffusion processes, e.g., diffusions in the sense of Dirichlet forms (see [31]). Typically, such transition probabilities are constructed by means of certain Markovian semigroups, and their analytic properties are not given in advance.

In Section 4, the results obtained in Section 3 are applied to the study of the above mentioned semigroup $(T_t)_{t\geq 0}$ associated with a probability measure μ satisfying the elliptic equation (1.3). We show that it is strongly Feller and is given by continuous densities $p(t,x,y)$ which satisfy the weak parabolic equation. This yields Sobolev regularity of $p(t,x,y)$. In addition, if μ is invariant for $(T_t)_{t\geq 0}$, then it is a unique invariant measure for this semigroup.

Let us introduce some notation. For any number $p \neq 1$ we denote $p' = p/(p-1)$. Let $\Omega \subset \mathbb{R}^d$ be an open set. If μ is a locally finite (not necessarily nonnegative) Borel measure on Ω, we denote by $L^1_{loc}(\Omega,\mu)$ the class of all functions f such that $\zeta f \in L^1(\Omega,\mu)$ for every $\zeta \in C_0^\infty(\Omega)$. Here $L^p(\Omega,\mu) = L^p(\Omega,|\mu|)$, where $|\mu|$ is the variation of μ. Lebesgue measure is denoted by dx, and as usual $L^p_{loc}(\Omega) := L^p_{loc}(\Omega, dx)$. The same notation is used for spaces of vector-valued mappings on Ω. We also ocasionally identify μ with its Radon–Nikodym density with respect to Lebesgue measure if this density exists. The Borel σ-field of a space X is denoted by $\mathcal{B}(X)$.

Let $C_0^\infty(\Omega)$ be the class of all infinitely differentiable functions with compact support in Ω. We set $\partial_i\varphi = \frac{\partial\varphi}{\partial x_i}$, $\partial_t\varphi = \frac{\partial\varphi}{\partial t}$.

Let $H^{p,r}(\Omega)$, $p \geq 1$, $r \geq 0$, be the standard Sobolev space of functions on Ω whose generalized derivatives up to order r are in L^p equipped with its natural norm. In the case where r is an integer, one of the standard equivalent norms is defined as the sum of the L^p-norms of all derivatives up to order r (see, e.g., [1]). In particular,

$$\|f\|_{H^{p,1}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

The closure of $C_0^\infty(\Omega)$ in $H^{p,r}(\Omega)$, $r \geq 0$, is denoted by $H_0^{p,r}(\Omega)$. Note that if B is an open ball, then $H_0^{p,1}(B)$ is the class of all $f \in H^{p,1}(B)$ whose extensions to $\mathbb{R}^d$ by zero outside B belong to $H^{p,1}(\mathbb{R}^d)$. Finally, $H^{p',-r}(\Omega)$ is the dual of $H_0^{p,r}(\Omega)$. We denote by $H^{p,r}_{loc}(\Omega)$ the class of all functions f on Ω such that $\zeta f \in H^{p,r}(\mathbb{R}^d)$ for every $\zeta \in C_0^\infty(\Omega)$.

We denote by $B_R(x_0)$ the open ball of radius R centered at x_0.

Let $A=(a^{ij})_{i,j=1}^d$ be a matrix. We write $A\geq cI$ if $a^{ij}x_ix_j\geq c|x|^2$ for each $x\in\mathbb{R}^d$.

Let J be an interval and let U be an open set in $\mathbb{R}^d$. Let $\mathbb{H}^{p,s}(U,J)$ denote the space of all measurable functions u on $U\times J$ such that $u(\,\cdot\,,t)\in H^{p,s}(U)$ with finite norm

$$\|u\|_{\mathbb{H}^{p,s}(U,J)}=\biggl(\int_J\|u(\,\cdot\,,t)\|^p_{H^{p,s}(U)}\,dt\biggr)^{1/p}.$$

The space $\mathbb{H}^{p,s}_0(U,J)$ is defined analogously with $H^{p,s}_0(U)$ instead of $H^{p,s}(U)$ and $\mathbb{H}^{p',-s}(U,J)$ denotes its dual.

In relation to parabolic equations, it is useful to introduce also the following spaces. Let $\mathcal{H}^{p,1}(U,J)$ be the space of all functions $u\in\mathbb{H}^{p,1}(U,J)$ with $\partial_tu\in\mathbb{H}^{p,-1}(U,J)$ and finite norm

$$\|u\|_{\mathcal{H}^{p,1}(U,J)}=\|\partial_tu\|_{\mathbb{H}^{p,-1}(U,J)}+\|u\|_{\mathbb{H}^{p,1}(U,J)}.$$

$H^{p,2;1}_0(B_R,[0,T])$ denotes the closure of the space of smooth functions u on $B_R\times[0,T]$ that vanish on $\partial B_R\times[0,T]\bigcup B_R\times\{0\}$ (i.e., have zero limits on this part of the boundary) with respect to the norm

$$\begin{aligned}\|u\|_{H^{p,2;1}_0(B_R,[0,T])}&=\|u\|_{L^p(B_R\times[0,T])}+\|\partial_tu\|_{L^p(B_R\times[0,T])}\\&\quad+\|\nabla_xu\|_{L^p(B_R\times[0,T])}+\sum_{i,j=1}^d\|\partial_i\partial_ju\|_{L^p(B_R\times[0,T])}.\end{aligned}$$

2. Elliptic case

Suppose that $A=(a^{ij})_{i,j=1}^d$ is a Borel measurable mapping on a domain $\Omega\subset\mathbb{R}^d$ with values in the space of the nonnegative symmetric operators on $\mathbb{R}^d$.

Theorem 2.1. *Let μ be a locally finite Borel measure on Ω such that $a^{ij}\in L^1_{loc}(\mu)$, and for some $C>0$, one has*

$$\int_\Omega a^{ij}\partial_i\partial_j\varphi\,d\mu\leq C(\sup_\Omega|\varphi|+\sup_\Omega|\nabla\varphi|) \tag{2.1}$$

for all nonnegative $\varphi\in C_0^\infty(\Omega)$. Then:

(i) *If μ is nonnegative, then* $(\det A)^{1/d}\mu$ *has a density which belongs to $L^{d'}_{loc}(\Omega,dx)$;*

(ii) *If A is locally Hölder continuous and nondegenerate, then μ has a density which belongs to $L^r_{loc}(\Omega,dx)$ for every $r\in[1,d')$.*

Proof. We shall start with case (ii) which is simpler. Let D_0 be a ball with compact closure in Ω and let $\zeta\in C_0^\infty(\Omega)$ be such that $0\leq\zeta\leq1$, $\zeta=1$ on D_0 and the support of ζ belongs to a ball $D\subset\Omega$. Let us consider the measure $\nu = \zeta\,\mu$. By substituting $\zeta\psi$ in place of φ in (2.1), for every nonnegative smooth function ψ on Ω, we obtain

$$\int_D a^{ij}\partial_i\partial_j\psi\,d\nu \le C_1(\sup_\Omega|\psi| + \sup_\Omega|\nabla\psi|), \tag{2.2}$$

where

$$C_1 = C + \big(C + 2d^2\sup_{i,j}\|a^{ij}\|_{L^1(D,\mu)}\big)\sup_D|\nabla\zeta| + \|a^{ij}\|_{L^1(D,\mu)}\sup_D|\partial_i\partial_j\zeta| \tag{2.3}$$

is independent of ψ. It is easily seen that (2.2) remains valid for every nonnegative $\psi \in C^2(\overline{D})$. By considering the function $\psi + \sup|\psi|$, we arrive at the estimate

$$\Big|\int_D a^{ij}\partial_i\partial_j\psi\,d\nu\Big| \le C_1(\sup_D|\psi| + \sup_D|\nabla\psi|), \qquad \forall\,\psi \in C^2(\overline{D}). \tag{2.4}$$

Now let $r > d$. It is known (see, e.g., [16, Theorem 6.14]) that, given $f \in C_0^\infty(D)$, there exists a function $u \in C^2(\overline{D})$ such that

$$a^{ij}\partial_i\partial_j u = f$$

on D and $u = 0$ on ∂D. Moreover, by [16, Lemma 9.17], there exists a constant C_2 independent of f such that

$$\|u\|_{H^{2,r}(D)} \le C_2\|f\|_{L^r(D)}.$$

By the Sobolev embedding theorem, we obtain

$$\sup_D|\nabla u| + \sup_D|u| \le C_3\|f\|_{L^r(D)}.$$

Together with (2.4) this yields

$$\int_D f\,d\nu \le C_1C_3\|f\|_{L^r(D)}, \qquad \forall\,f \in C_0^\infty(D). \tag{2.5}$$

Hence ν is absolutely continuous with $\nu = g\,dx$, $g \in L^{r'}(D)$.

Let us now consider case (i). The above reasoning does not work in this case even for bounded uniformly nondegenerate A, since the equation $a^{ij}\partial_i\partial_j u = f$ need not be solvable; for continuous A, the solution u of this equation is only in $H^{r,2}$ and not in C^2, hence one cannot pass from C_0^∞-functions to u in (2.4). In order to overcome this difficulty, we need the assumption that μ is nonnegative and we are going to proceed similarly to [23]. As above, by considering a suitable function ζ, we arrive at estimate (2.4) for the measure $\nu = \zeta\mu$ on the ball $B_{R_0}(x_0)$. Note that the support of the measure ν is contained in a ball $B_R(x_0)$ with radius $R = R_0 - r$, where $r > 0$. In that case, instead of solving the elliptic equation, we shall employ a result from [19, p. 442] (a more general result used below in the parabolic case was established in [20]; see also [22, Ch. III, §2]) according to which, for every nonnegative continuous function f on $\mathbb{R}^d$ vanishing outside $\overline{B_R}(x_0)$, there exists a nonnegative continuous concave function z on $\overline{B_R}(x_0)$ with the following property:

$$-\alpha^{ij}\partial_i\partial_j z \geq |\det(\alpha^{ij})|^{1/d} f$$

in the sense of distributions on $B_{R_0}(x_0)$ for every nonnegative matrix (α^{ij}) and

$$\sup_{B_{R_0}(x_0)} z \leq N||f||_{L^d(B_{R_0}(x_0))},$$

where N is independent of f. Let g be a fixed smooth probability density on $\mathbb{R}^d$ whose support is contained in the centered ball of radius 1 and let

$$v_\varepsilon = v * g_\varepsilon, \qquad g_\varepsilon(x) = \varepsilon^{-d} g(\varepsilon^{-1}x).$$

Then, for every nonnegative matrix (α^{ij}) and every $\varepsilon \in (0, r)$, one has the estimates

$$-\alpha^{ij}\partial_i\partial_j z_\varepsilon(x) \geq |\det(\alpha^{ij})|^{1/d} f_\varepsilon(x), \tag{2.6}$$

$$\sup_{B_R(x_0)} |z_\varepsilon| \leq N\|f\|_{L^d(B_{R_0}(x_0))} \tag{2.7}$$

on $B_R(x_0)$, where N is independent of f, α^{ij}, and ε. Clearly, the functions z_ε are smooth, nonnegative and concave on $B_{R+r/2}(x_0)$ if $\varepsilon < r/2$. We observe that, for every nonnegative continuously differentiable concave function w on $B_{R+r/2}(x_0)$, one has

$$|\nabla w(x)| \leq 2r^{-1} \sup_{y \in B_R(x_0)} w(y), \qquad \forall x \in B_R(x_0).$$

This estimate follows by considering the one dimensional case. Together with (2.4), (2.6) applied to $\alpha^{ij} = a^{ij}(x)$ and (2.7), this yields the estimate

$$\int |\det(a^{ij})|^{1/d} f_\varepsilon \, d\nu \leq \left| \int a^{ij}\partial_i\partial_j z_\varepsilon \, dx \right| \leq C_1 \sup_{B_R(x_0)} \Big(|\nabla z_\varepsilon| + |z_\varepsilon| \Big)$$
$$\leq C_1 N(1 + 2r^{-1})\|f\|_{L^d(B_{R_0}(x_0))}.$$

As in case (ii), we complete the proof. □

Corollary 2.2. *Let μ be a locally finite signed Borel measure on Ω and let a^{ij}, b^i, $c \in L^1_{loc}(\Omega, \mu)$. Assume that*

$$\int_\Omega (L_{A,b}\varphi + c\varphi)\, d\mu \leq 0 \qquad \textit{for all nonnegative } \ \varphi \in C_0^\infty(\Omega). \tag{2.8}$$

Then:

(i) *If μ is nonnegative, then* $(\det A)^{1/d}\mu$ *has a density which belongs to* $L^{d'}_{loc}(\Omega, dx)$*;*

(ii) *If A is locally Hölder continuous and nondegenerate, then μ has a density which belongs to* $L^{r}_{loc}(\Omega, dx)$ *for every* $r \in [1, d')$.

In particular, the above statements are true if (1.4) *holds.*

Proof. It suffices to note that, for every bounded open $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$, one has

$$\int\limits_{\Omega_0} (b^i \partial_i \varphi + c\varphi)\, d\mu \le \sup_{\Omega_0} |\nabla \varphi| \int\limits_{\Omega_0} |b|\, d|\mu| + \sup_{\Omega_0} |\varphi| \int\limits_{\Omega_0} |c|\, d|\mu|$$

for every smooth function φ with support in Ω_0. □

In a similar manner, one proves the following more general result.

Corollary 2.3. *Let μ and ν be two locally finite signed Borel measures on Ω and let* a^{ij}, b^i, $c \in L^1_{loc}(\Omega, \mu)$. *Assume that*

$$\int\limits_{\Omega} \Big[L_{A,b}\varphi + c\varphi \Big]\, d\mu = \int \varphi\, d\nu \qquad \text{for all nonnegative } \ \varphi \in C_0^\infty(\Omega). \tag{2.9}$$

Then:

(i) *If μ is nonnegative, then* $(\det A)^{1/d}\mu$ *has a density which belongs to* $L^{d'}_{loc}(\Omega, dx)$*;*

(ii) *If A is locally Hölder continuous and nondegenerate, then μ has a density which belongs to* $L^{r}_{loc}(\Omega, dx)$ *for every* $r \in [1, d')$.

Remark 2.4. Assume that the hypotheses of Corollary 2.2 (ii) are fulfilled. Let $B_{R_1}(x_0) \subset \Omega$ be a ball of radius $R_1 > 0$ centered at x_0. Then, for every $R < R_1$ and $r < d'$, there exists N depending only on $R_1, R, r, d, \inf\limits_{B_{R_1}} \det A, \sup\limits_{i,j} \sup\limits_{B_{R_1}} |a^{ij}|$, and the Hölder norm of A on B_{R_1} such that for the density ϱ of μ one has

$$\|\varrho\|_{L^r(B_R)} \le N \big\| 1 + |b| + |c| \big\|_{L^1(B_{R_1}, \mu)}.$$

In addition, for d fixed, N is a locally bounded function of the above mentioned quantities. This is readily seen from (2.3) and the proof of Theorem 2.1.

We now proceed to the regularity results. Throughout the rest of this section we assume that $A(x)$ *is continuous in* x.

Let B_R be an open ball of radius R. First we want to recall some simple properties of the space $H^{p,-1}(B_R)$, which is the dual of $H_0^{p',1}(B_R)$

for $p \in (1, \infty)$. It is known (see, e.g., [1, Ch. III, Theorem 3.10]) that every $u \in H^{p,-1}(B_R)$ can be written as

$$u = \partial_i f^i, \qquad f^i \in L^p(B_R),\ i = 1, \dots, d, \tag{2.10}$$

and, for all representations (2.10), one has

$$\|u\|_{H^{p,-1}(B_R)} \le \|f\|_{L^p(B_R)}. \tag{2.11}$$

If $pq \ge p + q$, $|b| \in L^p(B_R)$, and $\mu \in L^q(B_R)$, then, by Hölder's inequality, $|b|\mu \in L^r(B_R)$ and

$$||b\mu||_{L^r(B_R)} \le ||b||_{L^p(B_R)}||\mu||_{L^q(B_R)} \quad \text{with } r = pq/(p+q). \tag{2.12}$$

The Sobolev embedding theorem says that $H_0^{r,1}(B_R) \subset L^{rd/(d-r)}(B_R)$ if $1 \le r < d$. In addition, $H_0^{d,1}(B_R) \subset L^s(B_R)$ for every $s \in [1, +\infty)$, and $H_0^{r,1}(B_R) \subset C(B_R)$ if $r > d$. By turning to dual spaces and using scaling to control the norms of the embeddings, we arrive at the following well known lemma (see, e.g., [16, Theorem 7.10]).

Lemma 2.5. (i) *Let $d' < r < \infty$ and $R > 0$. Then $L^{rd/(r+d)}(B_R) \subset H^{r,-1}(B_R)$ and the embedding operator is bounded. In addition, there exists N independent of R such that*

$$||u||_{H^{r,-1}(B_R)} \le N||u||_{L^{rd/(r+d)}(B_R)} \tag{2.13}$$

for all $u \in L^{rd/(r+d)}(B_R)$ and all $R > 0$.

(ii) *Let $1 < r < d'$ and $R > 0$. Then $L^1(B_R) \subset H^{r,-1}(B_R)$ and the embedding operator is bounded. In addition, there exists N independent of R such that*

$$||u||_{H^{r,-1}(B_R)} \le NR^{1-d/r'}||u||_{L^1(B_R)} \tag{2.14}$$

for all $u \in L^1(B_R)$ and all $R > 0$.

(iii) *Let $r = d'$, $s > 1$, and $R > 0$. Then $L^s(B_R) \subset H^{r,-1}(B_R)$. In addition, there exists N independent of R such that*

$$\|u\|_{H^{r,-1}(B_R)} \le NR^{\frac{2s+d}{s}}\|u\|_{L^s(B_R)} \tag{2.15}$$

for all $u \in L^s(B_R)$ and all $R > 0$.

Lemma 2.6. *Let p and q be two numbers satisfying $p \ge d$, $q \ge p'$ but not such that $p = d = q'$. Let $R_1 > 0$. Assume that the functions $a^{ij} \in H^{p,1}(B_{R_1})$ are continuous and $A \ge \lambda I$ for some $\lambda > 0$. Then, there exist $N > 0$ and $R_0 > 0$ depending only on p, q, d, λ, R_1, the modulus of continuity of A, $\|a^{ij}\|_{H^{p,1}(B_{R_1})}$, and the rate of decreasing to zero of $\|\nabla a^{ij}\|_{L^d(B_R)}$ as $R \to 0$, such that for all $R < R_0$ and $\varphi \in H_0^{q,1}(B_R)$, one has*

$$f := a^{ij}\partial_i\partial_j\varphi \in H^{q,-1}(B_R)$$

and

$$||\nabla\varphi||_{L^q(B_R)} \le N||f||_{H^{q,-1}(B_R)}. \tag{2.16}$$

Proof. We may assume that $R_1 = 1$. Note $f \in H^{q,-1}(B_R)$. This follows from the fact that, for every bounded function $\zeta \in H^{p,1}(B_R)$, the operator $\psi \mapsto \zeta\psi$ is continuous on $H_0^{q',1}(B_R)$ by the estimate $\|\nabla(\zeta\psi)\|_{L^{q'}(B_R)} \le C\|\nabla\psi\|_{L^{q'}(B_R)}$. Indeed, if $q' < d$, we have $|\nabla\zeta| \in L^p(B_R)$ and $\psi \in L^{q'd/(d-q')}(B_R)$. Hence $|\psi\nabla\zeta| \in L^s(B_R)$, where

$$s = \frac{pq'd/(d-q')}{p+q'd/(d-q')} = \frac{pq'd}{pd-pq'+q'd} = q'\frac{pd}{pd-pq'+q'd} \ge q'.$$

In addition,

$$\begin{aligned}\|\psi\nabla\zeta\|_{L^s(B_R)} &\le \|\psi\|_{L^{q'd/(d-q')}(B_R)}\|\nabla\zeta\|_{L^p(B_R)}\\ &\le C\|\nabla\psi\|_{L^{q'}(B_R)}\|\nabla\zeta\|_{L^p(B_R)}.\end{aligned}$$

The case $q' > d$, where ψ is bounded and $q' \le p$, and the case $q' = d$, where $q' < p$ and ψ is in all $L^r(B_R)$, $r < \infty$, are similar.

We note that

$$\partial_i(a^{ij}\partial_j\varphi) = f + \partial_i a^{ij}\partial_j\varphi =: f + g.$$

It follows from [32, Section 5.5] (see also [15, Section 4.3] or [36, Theorem 1]) that

$$||\nabla\varphi||_{L^q(B_R)} \le N_1\Big(||g||_{H^{q,-1}(B_R)} + ||f||_{H^{q,-1}(B_R)}\Big), \tag{2.17}$$

where N_1 is independent of $R \in (0,1]$ and φ. Let $a = (a^j)$, $a^j = \partial_i a^{ij}$.

Now we consider three cases.

Case $q > d'$. By Lemma 2.5(i) and (2.12) we have

$$||g||_{H^{q,-1}(B_R)} \le N_2||a||_{L^d(B_R)}||\nabla\varphi||_{L^q(B_R)},$$

which along with (2.17) yields

$$||\nabla\varphi||_{L^q(B_R)} \le N_1N_2||a||_{L^d(B_R)}||\nabla\varphi||_{L^q(B_R)} + N_1||f||_{H^{q,-1}(B_R)}. \tag{2.18}$$

We emphasize that N_1 and N_2 are independent of R and f and note that, since $|\nabla a^{ij}| \in L^p(B_R)$ and $p \ge d$, we can choose R so small that

$$N_1N_2||a||_{L^d(B_R)} \le 1/2.$$

For such an R, (2.18) implies (2.16).

Case $p' < q \le d'$. In that case it follows from $q > p'$ that, for r defined by $rd/(r+d) = pq/(p+q)$, we have $r > d' \ge q$. Therefore, for $R \in (0,1)$, we obtain

$$||g||_{H^{q,-1}(B_R)} \le N_3R^{d(r-q)/(rq)}||g||_{H^{r,-1}(B_R)} \le N||a||_{L^p(B_R)}||\nabla\varphi||_{L^q(B_R)},$$

and we can finish the proof as above.

Case $q = p' < d'$. As is easy to see, this is the only remaining case. By Lemma 2.5 and (2.12), for $R \in (0,1)$, we have

$$||g||_{H^{q,-1}(B_R)} \le N_4 R^{1-d/p} ||g||_{L^1(B_R)} \le N_4 ||a||_{L^p(B_R)} ||\nabla \varphi||_{L^q(B_R)},$$

and the argument from the first case applies again to complete the proof. □

Theorem 2.7. *Let $d \ge 2$, $p \ge d$, $q \in (1,\infty)$, $R_1 > 0$, let $a^{ij} \in H^{p,1}(B_{R_1})$ be continuous, and let $A \ge \lambda I$, where $\lambda > 0$. Then there exist $R_0 > 0$ and $N_0 > 0$ with the following properties. Let $R < R_0$ and let μ be a measure of finite total variation on B_R such that for any $\varphi \in C_0^2(B_R) := C^2(\overline{B_R}) \cap \{u\colon u|_{\partial B_R} = 0\}$ we have*

$$\left| \int_{B_R} a^{ij} \partial_i \partial_j \varphi \, d\mu \right| \le N ||\nabla \phi||_{L^q(B_R)} \tag{2.19}$$

with N independent of φ. Furthermore, assume one of the following:
a) *$p > d$ or*
b) *$p = d > q'$ and $\mu \in \bigcup_{r>1} L^r(B_R)$.*

Then, $\mu \in H_0^{q' \wedge p,1}(B_R)$ (where we identify μ with its density) and

$$||\mu||_{H_0^{q'\wedge p,1}(B_R)} \le N_0. \tag{2.20}$$

In addition, R_0 depends only on p, q, d, λ, R_1, $\|a^{ij}\|_{H^{p,1}(B_{R_1})}$, and the rate of decreasing of $\|\nabla a^{ij}\|_{L^d(B_R)}$ as $R \to 0$, and N_0 depends on the same quantities and N.

Proof. We break the proof into three cases.

Case $q \ge p'$ *and* $q \ne d'$. Take $f = (f^1, \dots, f^d) \in C^2(\overline{B_R})$ and solve the equation

$$a^{ij} \partial_i \partial_j \varphi = \partial_i f^i$$

in B_R with zero boundary conditions. If a) holds, then $p > d$ and A is Hölder continuous in B_R and, by Hölder space theory (see, e.g., [24, Ch. 10]), there exists a unique solution $\varphi \in C_0^2(B_R)$ of our problem, which we can substitute into (2.19). If b) holds, then, since A is continuous, $\partial_i \partial_j \varphi$ are summable to any power by L^p-theory (see, e.g., [32, Section 5.6], [15, Section 4.3]), and, owing to $\mu \in \bigcup_{r>1} L^r(B_R)$, we again can substitute φ into (2.19). By Lemma 2.6 and (2.19) we have

$$\left| \int_{B_R} \partial_i f^i \, \mu(dx) \right| \le N ||\partial_i f^i||_{H^{q,-1}(B_R)},$$

which implies our claim.

Case $q = d' \ge p'$. In this case by our assumptions we must have that $p > d$, so that Lemma 2.6 is still applicable.

Case $1 < q < p' < d'$. As is easy to see, this is the only remaining case. Observe that, of course, (2.19) is satisfied with $r = (p' + d')/2$ in place of q and, by the first case, we have $\mu \in H_0^{r',1}(B_R)$ if R is sufficiently small. Since $r' > d$, by the Sobolev embedding theorem, μ is bounded in B_R. Furthermore, we note that (2.19) means that

$$\varphi \mapsto \int_{B_R} a^{ij}\partial_i\partial_j\varphi\mu\,dx$$

is a linear functional defined on a dense subspace $C_0^2(B_R)$ of $H_0^{q,1}(B_R)$ and bounded in the $H_0^{q,1}(B_R)$-norm. By the duality between $H_0^{q,1}(B_R)$ and $H^{q',-1}(B_R)$, we have that

$$\int_{B_R} a^{ij}\partial_i\partial_j\varphi\mu\,dx = \int_{B_R} f\varphi\,dx,$$

where $f \in H^{q',-1}(B_R) \subset H^{p,-1}(B_R)$. Thus, μ is a generalized solution of the equation

$$\partial_j(a^{ij}\partial_i\mu) = f - \partial_j(\partial_i a^{ij}\mu) =: g.$$

Here $\partial_i a^{ij}\mu \in L^p(B_R)$, since μ is bounded, so that $g \in H^{p,-1}(B_R)$. Since μ belongs to $H_0^{r',1}(B_R)$ and $r' > d \geq 2$, by [32, Theorem 5.5.3], we conclude that $\mu \in H_0^{p,1}(B_R)$. $\square$

Theorem 2.8. *Let $p > d$, $r \in (p', \infty)$, $\mu \in L^r_{loc}(\Omega, dx)$, $a^{ij} \in H^{p,1}_{loc}(\Omega)$, and let either $\beta \in L^p_{loc}(\Omega, dx)$ or $\beta \in L^p_{loc}(\Omega, \mu)$. Suppose that A is locally uniformly nondegenerate. Assume that, for every $\varphi \in C_0^\infty(\Omega)$, we have*

$$\left|\int_\Omega a^{ij}\partial_i\partial_j\varphi\mu\,dx\right| \leq \int_\Omega (|\varphi| + |\nabla\varphi|)|\beta\mu|\,dx. \tag{2.21}$$

Then, $\mu \in H^{p,1}_{loc}(\Omega)$.

Proof. Since $r > p'$, we have $pr > p + r$. Then $q = pr/(pr - p - r) > 1$ and $q' = pr/(p + r) > 1$. According to (2.12), $\beta\mu \in L^{q'}_{loc}(\Omega)$ provided $\beta \in L^p_{loc}(\Omega)$. The same is true if $\beta \in L^p_{loc}(\Omega, \mu)$, since

$$|\beta|^{q'}|\mu|^{q'} = |\beta|^{pr/(p+r)}|\mu|^{r/(p+r)}|\mu|^{(pr-r)/(p+r)},$$

where $|\beta|^{pr/(p+r)}|\mu|^{r/(p+r)} \in L^s_{loc}(\Omega)$ and $|\mu|^{(pr-r)/(p+r)} \in L^{s'}_{loc}(\Omega)$ with $s = (p+r)/r$. Observe that, for every $R > 0$ such that $B_R := B_R(x_0) \subset$

Ω, $\eta \in C_0^\infty(B_R)$, and $\varphi \in C_0^2(B_R)$, one has

$$\begin{aligned}
&\left|\int_{B_R} a^{ij}\partial_i\partial_j\varphi(\eta\mu)\,dx\right| \le \left|\int_{B_R} a^{ij}\partial_i\partial_j(\varphi\eta)\mu\,dx\right| \\
&+\left|\int_{B_R} a^{ij}\partial_i\partial_j\eta(\varphi\mu)\,dx\right| + 2\int_{B_R}\|A\||\nabla\eta||\nabla\varphi||\mu|\,dx \\
&\le N_1\int_{B_R}(|\varphi|+|\nabla\varphi|)|\beta\mu|\,dx \qquad (2.22)\\
&\le N_2||\beta\mu||_{L^{q'}(B_R)}\,||\nabla\varphi||_{L^q(B_R)} = N_3||\nabla\varphi||_{L^q(B_R)},
\end{aligned}$$

where the constants N_1, N_2, and N_3 are independent of φ. The last inequality above is due to the estimate $\|\varphi\|_{L^q(B_R)} \le N\|\nabla\varphi\|_{L^q(B_R)}$ with some N. It follows by Theorem 2.7 that $\eta\mu \in H_0^{q'\wedge p,1}(B_R)$ if R is small enough, and, since we can take any point as x_0 and $q' < p$, we have

$$\mu \in H_{loc}^{q',1}(\Omega). \qquad (2.23)$$

Moreover, if $q' < d$, then by Sobolev embedding $\mu \in L_{loc}^{r_1}(\Omega)$ with

$$r_1 = q'd/(d-q') = \frac{prd}{(p+r)d-pr}.$$

The inequality $q' < d$ is equivalent to $r < pd/(p-d)$. Thus on the interval

$$\Big(p/(p-1), pd/(p-d)\Big)$$

we have a mapping $T\colon\, r \to r_1$ with the property that if $\mu \in L_{loc}^r(\Omega)$, then $\mu \in L_{loc}^{r_1}(\Omega)$. It is easy to see that

$$\frac{r_1}{r} = \frac{pd}{pd-r(p-d)} \ge \frac{pd}{pd-p'(p-d)} = \frac{d'}{p'} > 1,$$

where the first inequality is due to $pr > p + r$ and the second one to $p > d$. Therefore, after finitely many applications of T to the given r we will come to a point

$$s \in \Big(p/(p-1), pd/(p-d)\Big)$$

such that $t = T(s) \ge pd/(p-d)$ and $\mu \in L^t(B_R)$. Actually, without loss of generality we may assume that $t > pd/(p-d)$ since otherwise we could just slightly decrease the initial point r.

This shows that we could assume from the very beginning that $r > pd/(p-d)$ that is $q' > d$. In that case (2.23) implies that the function μ is locally bounded which shows that (2.22) is true with $q' = p$. Now it only remains to apply again Theorem 2.7, and the theorem is proved. $\square$

It should be noted that the assertion of Theorem 2.8 is valid under the following alternative assumptions on a^{ij}, β, μ: μ is a locally bounded Borel measure on Ω such that $a^{ij} \in H^{p,1}_{loc}(\Omega)$ are continuous, A is locally uniformly nondegenerate, $\beta \in L^p_{loc}(\Omega, \mu)$ (or $\beta \in L^1_{loc}(\Omega, \mu) \bigcap L^p_{loc}(\Omega, dx)$). This follows by Theorem 2.1.

Corollary 2.9. *Let μ be a locally finite Borel measure on Ω. Let A be locally uniformly nondegenerate in Ω with $a^{ij} \in H^{p,1}_{loc}(\Omega)$, where $p > d$, and let either b^i, $c \in L^p_{loc}(\Omega, dx)$ or b^i, $c \in L^p_{loc}(\Omega, \mu)$. Assume that, for every $\varphi \in C_0^\infty(\Omega)$, one has*

$$\int \Big[a^{ij} \partial_i \partial_j \varphi + b^i \partial_i \varphi + c\varphi \Big] \, d\mu = 0, \tag{2.24}$$

where we assume that b^i, c are locally μ-integrable. Then, μ has a density in $H^{p,1}_{loc}(\Omega)$ that is locally Hölder continuous.

Proof. It suffices to take $\beta = |b| + |c|$ and apply Theorem 2.8. □

It should be noted that, in the situation of the above corollary, the integrability of b^i with respect to μ and not with respect to Lebesgue measure appears naturally in many applications. For example, let μ be the projection to $\mathbb{R}^d$ of a probability measure μ_0 on $\mathbb{R}^\infty$ that is invariant for the diffusion $\xi_t = (\xi_t^n)_{n=1}^\infty$ on $\mathbb{R}^\infty$ governed by the infinite system of stochastic differentiable equations $d\xi_t^n = dw_t^n + b_0^n(\xi_t)dt$ with independent Brownian motions w_t^n and $b_0^n \in L^p(\mu_0)$. Then μ satisfies (1.3) with $A = I/2$ and b^i given by the conditional expectations of b_0^i, $i = 1, \ldots, d$, with respect to the σ-field generated by the first d coordinates (see [9, proof of Proposition 3.3]). Then $b^i \in L^p(\mu)$, but there is no information about local Lebesgue integrability of b^i. It should also be noted that the condition $b^i \in L^p_{loc}(\Omega, \mu)$ can be fulfilled for b^i that are very singular with respect to Lebesgue measure. For example, let ϱ be a smooth probability density on $\mathbb{R}^d$ and let $b = \nabla\varrho/\varrho$. Then the measure $\mu = \varrho \, dx$ satisfies the equation $L^*_{I,b}\mu = 0$. It is easy to find ϱ such that $|b| \in L^p(\mathbb{R}^d, \mu)$, but b is not locally Lebesgue integrable (say, take $d = 1$ and $\varrho(x) = \exp\big(-|x|^{-2}\big)$ near zero).

For applications to optimal control, see [5, Ch. II], where, in particular, the $H^{p,1}_{loc}$-regularity was proved for probability measures on $\mathbb{R}^d$ satisfying (1.3) with $A = I$ and $b = b_1 + b_2$, where b_1 is Lipschitzian and b_2 is bounded, under the additional assumption of the existence of a certain Lyapunov function (see [5, Ch. II, Theorem 5.5]).

Corollary 2.10. *In the situation of Corollary 2.9, let $b^i \in L^p_{loc}(\Omega, dx)$, let μ be nonnegative and let ϱ be its continuous density. Then, for every compact set K contained in a connected open set U with compact* *closure in* Ω*, one has*

$$\sup_K \varrho \le C \inf_K \varrho,$$

where the constant C *depends only on* $\|a^{ij}\|_{H^{p,1}(U)}$, $\|b\|_{L^p(U)}$, $\inf_U \det A$, *and* K *(if* K *is a ball, then dependence on* K *is through the distance from* K *to* ∂U*). In particular,* ϱ *does not vanish in* U *if it is not identically zero in* U*.*

Proof. This follows from Harnack's inequality for the solutions of the equation $\partial_i(a^{ij}\partial_j u+\partial_j a^{ij}u-b^i u)+cu=0$ (see [42, Corollary 5.3]), which is satisfied by the density of μ by the integration by parts formula. $\square$

Corollary 2.11. *Suppose that in the situation of Corollary* 2.9 *one has* $b^i \in L^p_{loc}(\Omega, dx)$ *and that* $c \le 0$*. Assume also that the continuous density* ϱ *of* μ *is strictly positive on the boundary of a bounded open set* $U \subset \overline{U} \subset \Omega$*. Then* ϱ *is strictly positive on* U*.*

Proof. This follows from [43, Theorem 7], which applies with $\gamma = 0$. $\square$

Clearly, the assumption that $b^i \in L^p_{loc}(\Omega, dx)$ in Corollary 2.10 and Corollary 2.11 cannot be replaced by the alternative assumption $b^i \in L^p_{loc}(\Omega, \mu)$ from Corollary 2.9. Indeed, it suffices to take $b = \nabla\varrho/\varrho$ as noted before Corollary 2.10. Corollary 2.9 can be generalized as follows.

Corollary 2.12. *Let* $p > d$*, let* $a^{ij} \in H^{p,1}_{loc}(\Omega)$*,* b^i*,* f^i*,* $c \in L^p_{loc}(\Omega)$*,* $i,j = 1,\dots,d$*, and let* A *be locally uniformly nondegenerate in* Ω*. Assume that* μ *is a locally finite Borel measure on* Ω *such that* b^i*,* $c \in L^1_{loc}(\Omega, \mu)$ *and, for every* $\varphi \in C_0^\infty(\Omega)$*, one has*

$$\int \Big[a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi + c\varphi\Big]\, d\mu = \int f^i \partial_i\varphi\, dx. \tag{2.25}$$

Then, $\mu \in H^{p,1}_{loc}(\Omega)$*.*

Proof. Let $B := B_R(x_0) \subset \Omega$. In order to reduce our claim to Corollary 2.9, it suffices to note that, for sufficiently small $R > 0$, the boundary value problem

$$\partial_i(a^{ij}\partial_j u + \partial_j a^{ij} u - b^i u) + cu = \partial_i f^i, \qquad u|_{\partial B} = 0,$$

has a solution $u \in H^{p,1}_0(B)$, since $c, b^i, f^i \in L^p(B)$, $a^{ij} \in H^{p,1}(B)$, and $p > d$ (the existence results in the class $H^{2,1}_0(B)$ are found in [27], [37], [42]; the $H^{p,1}$-regularity is established in [32, §5.5] and follows also from Theorem 2.8). Now, the measure $\mu - u dx$ on B satisfies the hypotheses of Corollary 2.9. $\square$

Remark 2.13. It is easily seen that in Corollary 2.9 one cannot omit the hypotheses that A is locally uniformly nondegenerate and $a^{ij} \in H^{p,1}_{loc}$. Indeed, if A and b vanish at a point x_0, then Dirac's measure at x_0 satisfies our elliptic equation. In particular, if it is not given in advance that μ is absolutely continuous, then one cannot take an arbitrary Lebesgue version of A. In order to see that μ cannot be more regular than A, let us take a probability measure μ with a smooth density that satisfies $L^*_{I,b_0}\mu = 0$, e.g., let μ be the standard Gaussian measure and $b_0(x) = -x$. Now, if g is any Borel function with $1 \le g \le 2$, then the measure $g \cdot \mu$ satisfies our elliptic equation with $A = g^{-1}I$ and $b = g^{-1}b_0$. In particular, we obtain an example, where A and b are Hölder continuous and A is uniformly nondegenerate, but the density of μ is not weakly differentiable. Also, the condition $p > d$ is essential for the membership of μ in a Sobolev class even if $A = I$ (see a simple example in [8]). However, if μ is a probability measure on $\mathbb{R}^d$, then the condition $|b| \in L^2(\mathbb{R}^d, \mu)$ implies that $\mu = \varrho\, dx$ with $\varrho \in H^{1,1}(\mathbb{R}^d)$ and $|\nabla\varrho|^2/\varrho \in L^1(\mathbb{R}^d)$ (see [9] and [7]).

Remark 2.14. Let μ be a Borel measure on a ball B_{R_1} of radius $R_1 > 0$ in Ω satisfying $L^*_{A,b}\mu + c\mu = f\, dx$, where $a^{ij}, b^i, c \in L^1(B_{R_1}, \mu)$, $f \in L^p(B_{R_1})$, and

$$A \ge \lambda_1 I, \quad \|a^{ij}\|_{H^{p,1}(B_{R_1})} \le \lambda_2, \quad \|b^i\|_{L^p(B_{R_1})} + \|c\|_{L^p(B_{R_1})} \le \lambda_2,\ p > d.$$

Then it follows from the above results that, for any $R < R_1$, μ has a continuous density $u \in H^{p,1}(B_R)$ that satisfies the equation

$$\partial_i(a^{ij}\partial_j u + \partial_j a^{ij} u - b^i u) + cu = f.$$

Therefore, by results in [32, Sect. 5.5] (see, e.g., [32, Theorem 5.5.5′]), one has

$$\begin{aligned}\|u\|_{H^{p,1}(B_R)} &\le \Lambda_{d,R_1}(\lambda_1,\lambda_2,R)\Big[\|u\|_{L^1(B_R)} + \|f\|_{L^p(B_R)}\Big]\\ &\le \Lambda_{d,R_1}(\lambda_1,\lambda_2,R)\Big[\mathrm{mes}(B_R)\sup_{B_R}|u| + \|f\|_{L^p(B_R)}\Big], \quad (2.26)\end{aligned}$$

where $\Lambda_{d,R_1}(\lambda_1,\lambda_2,R)$ is a locally bounded function on $(0,+\infty)^3$ and is independent of u.

Proposition 2.15. *Let $A_k = (a^{ij}_k)$ be a sequence of continuous mappings on $\mathbb{R}^d$ with values in the symmetric matrices and let $b_k = (b^i_k)$ be a sequence of Borel vector fields on $\mathbb{R}^d$. Suppose that for every ball $B_r \subset \mathbb{R}^d$, there exist numbers $c_r > 0$, $\alpha_r > 0$, and $p = p_r > d$ such that*

$$A_k \ge c_r I, \qquad \|a^{ij}_k\|_{H^{p,1}(B_r)} + \|b^i_k\|_{L^p(B_r)} \le \alpha_r$$

for all i, j, k. *Assume that there exist probability measures* μ_k *on* $\mathbb{R}^d$ *such that* $L^*_{A_k,b_k}\mu_k = 0$ *in the same sense as above. Then the measures* μ_k *have continuous strictly positive densities that are uniformly Hölder continuous on every ball. If, in addition, the sequence* μ_k *is uniformly tight, then it is compact with respect to the variation norm, and every measure* μ *from its closure has a continuous strictly positive density that belongs to* $H^{p,1}(B_r)$ *for every* $r > 0$.

Proof. It follows from our hypotheses and Theorem 2.8 that the measures μ_k have continuous densities f_k. Suppose that $\sup_k \sup_{B_r} f_k = \infty$. By Harnack's inequality (see Corollary 2.10), for every $r > 0$, one has $\sup_{B_r} f_k \leq M(r, c_1, c_2) \inf_{B_r} f_k$, where $M(r, c_1, c_2)$ is independent of k. Then $\sup_k \inf_{B_r} f_k = \infty$. This is impossible, since the f_k's are probability densities. By (2.26), the sequence $\{f_k\}$ is bounded in $H^{p,1}(B_r)$. By the Sobolev embedding theorem, $\{f_k\}$ is uniformly Hölder continuous on B_r, in particular, is compact with respect to the sup-norm. If $\{\mu_k\}$ is uniformly tight, then a subsequence $\{\mu_{k_i}\}$ converges weakly to some probability measure μ. Passing to a subsequence once again we may assume that the functions f_{k_i} converge uniformly on compacts and are uniformly bounded in $H^{p,1}(B_r)$ for each $r > 0$. Hence μ has a density $f \in H^{p,1}(B_r)$. Then we obtain a continuous and strictly positive version of f. By Scheffé's theorem, the probability measures μ_{k_i} converge to μ in the variation norm. This reasoning applies to any subsequence in $\{\mu_k\}$, whence we obtain the desired conclusion. □

Remark 2.16. The above proposition can be extended as follows. Let Ω be an open set in $\mathbb{R}^d$ and let Ω be a union of increasing open sets Ω_k such that the closure of Ω_k is compact and contained in Ω_{k+1}. Let μ_k be probability measures on Ω_k satisfying the equations $L^*_{A_k,b_k}\mu_k = 0$ on Ω_k, where A_k is a sequence of continuous mappings on Ω_k with values in the nonnegative matrices that are locally uniformly nondegenerate and bounded in the $H^{p,1}$-norm, and let b_k be a sequence of Borel mappings that are locally uniformly bounded in the L^p-norm. Then the analogue of the statement of the previous proposition is true. Moreover, the same is true for Riemannian manifolds of dimension d.

Remark 2.17. It should be noted that the problems discussed above are considerably simplified in the one dimensional case. For example, equation (1.3) is equivalent to the first order equation $(a\mu)' - b\mu = const$ in the sense of distributions, whence it follows that the measure $a\mu$ is absolutely continuous and, if a is locally absolutely continuous and does not vanish, μ admits a locally absolutely continuous density. Now the above equation can be explicitly solved.

3. Parabolic case

In this section, we extend the results from the previous section to the parabolic case. Our strategy is essentially the same as in the elliptic case, but due to some additional technicalities we include complete proofs. Let us start again with the existence of densities.

Let $\Omega_T = \Omega \times (0,T)$, $T > 0$, and let $A(\,\cdot\,,\,\cdot\,) = \big(a^{ij}(\,\cdot\,,\,\cdot\,)\big)_{i,j=1}^d$ be a Borel mapping on Ω_T with values in the nonnegative matrices on $\mathbb{R}^d$.

Let B_R be an open ball of radius R and center x_0 with compact closure in Ω. Let us consider the following boundary value problem in the cylinder $B_{R,T} = B_R \times (0,T)$:

$$\partial_t u + a^{ij}\partial_i\partial_j u = f, \qquad u(x,0) = 0,\ u|_{\partial B_R \times [0,T]} = 0. \tag{3.1}$$

We shall need the following estimate for the solutions of (3.1). Assume that A is uniformly continuous in x uniformly in t, i.e., for all i,j, one has

$$\lim_{\delta\to 0} \sup_t \sup_{|x-y|\le\delta} |a^{ij}(x,t) - a^{ij}(y,t)| = 0. \tag{3.2}$$

Let $c_1 I \le A \le c_2 I$, where $c_1, c_2 > 0$. Then there exists two numbers $R(d,c_1,c_2), N_0(d,c_1,c_2) > 0$, depending only on d, c_1, c_2, and the modulus of continuity of A in x, such that, for every $R < R(d,c_1,c_2)$ and every $f \in C_0^\infty(B_{R,T})$, there exists a function $u \in H_0^{p,2;1}(B_R,[0,T])$ that is a solution of (3.1) and satisfies the estimate

$$\|u\|_{H_0^{p,2;1}(B_R,[0,T])} \le N_0(d,c_1,c_2)\|f\|_{L^p(B_{R,T})}. \tag{3.3}$$

In addition, $R(d,c_1,c_2)$ and $N_0(d,c_1,c_2)$ are locally bounded functions of $c_1, c_2 > 0$.

It is known (see, e.g., [24, Ch. 10], [29, Ch. IV], [28, Ch. IV]) that if A is Hölder continuous with respect to (x,t), then u has two derivatives in x and one derivative in t that are continuous on the closure of $B_{R,T}$. Moreover, according to [18] or [29] (see, in particular, [29, Corollary 15.4, Theorem 15.6]), if A is Hölder continuous in x uniformly with respect to t, then, for every compact set $K \subset B_R$, the second order derivatives in x and the first order derivative in t are continuous in $x \in K$ uniformly with respect to t.

Estimate (3.3) for all $R > 0$ is obtained in [28, Ch. IV, Theorem 9.2] in the case where A is continuous in (x,t). Let us explain how to prove (3.3) and the local existence result under condition (3.2). In fact, the existence result is obtained in [29, Section 16], but we need also (3.3). To this end, note that (3.3) holds if A is independent of x (see, e.g., [25, Theorem 4.10]). Suppose first that there is a solution u. Let

$a_0^{ij}(t) = a^{ij}(t, x_0)$ and $b^{ij} = a_0^{ij} - a^{ij}$. Let us rewrite the equation as

$$\partial_t u + a_0^{ij}\partial_i\partial_j u = f + b^{ij}\partial_i\partial_j u.$$

By (3.2), we can choose $R > 0$ such that

$$\|b^{ij}\partial_i\partial_j u\|_{L^p(B_{R,T})} \le \frac{1}{2N_0(d, c_1, c_2)}\|u\|_{H_0^{p,2;1}(B_R,[0,T])}.$$

Then we obtain (3.3) with $2N_0(d, c_1, c_2)$ in place of $N_0(d, c_1, c_2)$. This a priori estimate implies the existence result. Indeed, we can find a sequence of Hölder continuous mappings A_k such that $A_k(x, t) \to A(x, t)$ almost everywhere, $c_1 I \le A_k \le c_2 I$, and (3.2) is fulfilled uniformly in k. Then the sequence of the corresponding solutions u_k is bounded in the space $H_0^{p,2;1}(B_R, [0, T])$, hence a subsequence of $\{u_k\}$ converges weakly in this space to some u, which is a solution corresponding to A (note that, for every $f \in C_0^\infty(B_{R,T})$, this is the solution constructed in [29, Section 16], i.e., the functions $\partial_i\partial_j u$ and $\partial_t u$ are locally Hölder continuous in x uniformly with respect to t).

Now let $p > d+2$. Then (3.3) combined with the embedding theorem (see [6, Ch. III, Theorem 10.4] or [25, Theorem 7.2]) yields the existence of a constant $N(d, c_1, c_2)$ such that u has a modification which is continuously differentiable in x such that

$$\sup_{B_{R,T}} |\nabla_x u| \le N(d, c_1, c_2)\|f\|_{L^p(B_{R,T})}. \tag{3.4}$$

Theorem 3.1. *Let μ be a locally finite Borel measure on Ω_T such that $a^{ij} \in L^1_{loc}(\Omega_T, \mu)$ and*

$$\int_{\Omega_T} \Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi\Big]\, d\mu \le C(\sup_{\Omega_T}|\varphi| + \sup_{\Omega_T}|\nabla_x\varphi|) \tag{3.5}$$

for all nonnegative $\varphi \in C_0^\infty(\Omega_T)$. Then:

(i) *If μ is nonnegative, then $(\det A)^{1/(d+1)}\mu = \varrho\, dx\, dt$, where $\varrho \in L_{loc}^{(d+1)'}(\Omega_T)$;*

(ii) *If, on every compact set in Ω_T, A is uniformly bounded, uniformly nondegenerate, and Hölder continuous in x uniformly with respect to t, then $\mu = \varrho\, dx\, dt$, where $\varrho \in L^r_{loc}(\Omega_T)$ for every $r \in [1, (d+2)')$.*

Proof. In order to prove statement (i) we shall modify the reasoning used in the elliptic case as follows. As in the elliptic case, by choosing an apropriate cut-off function ζ with compact support in D, where $D = B \times J$, J is a subinterval with compact closure in $(0, T)$ and $B = B_R(x_0)$ is a ball with compact closure in Ω, and letting $\nu = \zeta\mu$, we arrive at the estimate

$$\int_D \Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi\Big]\,d\nu \le N\sup_D|\nabla_x\varphi| + N\sup_D|\varphi| \tag{3.6}$$

for all φ that are continuously differentiable in t and twice continuously differentiable in x on $\overline{D}$. Now we need [20, Theorem 2], which states the following. There exist two constants $\alpha, k(R,d) > 0$ such that, for every nonnegative function $f \in C_0^\infty(D)$, there exists a bounded nonpositive function z on $B_{2R}(x_0)\times\mathbb{R}^1$ that is convex on $B_{2R}(x_0)$ for each fixed t and has the following two properties: for any smooth probability density g with support in the centered unit ball of $\mathbb{R}^{d+1}$, $g_\varepsilon(y,t) = \varepsilon^{-d-1}g(y/\varepsilon, t/\varepsilon)$, and $\psi_\varepsilon = \psi * g_\varepsilon$, one has

$$\begin{aligned} &\alpha\big(\det(\alpha^{ij})\big)^{1/(d+1)} f_\varepsilon \le \partial_t z_\varepsilon + \alpha^{ij}\partial_i\partial_j z_\varepsilon, \\ &|z(x,t)| \le k(R,d)\|f\|_{L^{d+1}(D)}, \qquad (x,t)\in D \end{aligned} \tag{3.7}$$

for each $\varepsilon\in(0,R)$ and each nonnegative-definite symmetric matrix (α^{ij}). Note that in [20, Theorem 2], there is a minus in front of $\partial_t z_\varepsilon$. However, the same is true with plus, since one can replace t by $-t$, which does not influence all other terms. Clearly, for all sufficiently small ε, we obtain

$$|z_\varepsilon(x,t)| \le 2k(R,d)\|f\|_{L^{d+1}(D)}.$$

We observe that z_ε is nonpositive and convex on $B_{3R/2}(x_0)$ for $\varepsilon < R/2$. Therefore, as in the elliptic case, one has

$$\sup_{x\in B_R(x_0)}|\nabla_x z_\varepsilon(x,t)| \le 2R^{-1}\sup_{x\in B_R(x_0)}|z_\varepsilon(x,t)|.$$

Hence, from (3.7) applied to $\alpha^{ij} = a^{ij}(x,t)$, for all sufficiently small ε, taking into account that ν is nonnegative we obtain

$$\begin{aligned} &\int_D |\det A|^{1/(d+1)} f_\varepsilon\,d\nu \le \alpha^{-1}\int_D\Big[\partial_t z_\varepsilon + a^{ij}\partial_i\partial_j z_\varepsilon\Big]\,d\nu \\ &\le \alpha^{-1}N\Big(\sup_D|z_\varepsilon| + \sup_D|\nabla_x z_\varepsilon|\Big) \le \alpha^{-1}N(1+2R^{-1})\sup_D|z_\varepsilon| \\ &\le 2\alpha^{-1}k(R,d)N(1+2R^{-1})\|f\|_{L^{d+1}(D)}. \end{aligned}$$

Letting ε go to zero, we obtain the same estimate for f, which yields that $\sqrt[d+1]{\det A}\,\nu = \varrho\,dxdt$ with $\varrho\in L^{(d+1)'}(D)$.

Statement (ii) is also proved similarly to the elliptic case. Let $p > d+2$. As above, we arrive at the estimate (3.6). Let $f\in C_0^\infty(D)$. We shall first assume that A is Hölder continuous in both variables. This case is simpler, but exhibits the main idea of the general case. Let u be the solution of the equation $\partial_t u + a^{ij}\partial_i\partial_j u = f$ on D with zero data. We recall that u is twice continuously differentiable in x and continuously differentiable in t. Let $R > 0$ be small enough. By (3.4), there exists a constant N_0 independent of f such that

$$\sup_D |\nabla_x u| \le N_0 \|f\|_{L^p(D)}. \tag{3.8}$$

Therefore, estimating $\sup_D |u|$ by $R \sup_D |\nabla_x u|$, we obtain

$$\begin{aligned}\int_D f\, d\nu &= \int_D \Big[\partial_t u + a^{ij}\partial_i\partial_j u\Big]\, d\nu \\ &\le N(1+R)\sup_D |\nabla_x u| \le N_0 N(1+R)\|f\|_{L^p(D)},\end{aligned}$$

whence it follows that $\nu = \varrho_\nu\, dx\, dt$ with $\varrho_\nu \in L^{p'}(D)$.

Now let us explain how to handle the general case where A is Hölder continuous in x uniformly with respect to t. We can extend a^{ij} from D to $\mathbb{R}^{d+1}$ such that $\lambda_1 I \le A \le \lambda_2 I$ for some $\lambda_1, \lambda_2 > 0$, and condition (3.2) is fulfilled on $\mathbb{R}^{d+1}$. As noted above, for every $f \in C_0^\infty(D)$, there exists a function $\varphi \in H_0^{p,2;1}\big(B_{3R}(x_0), [0,T]\big)$ satisfying $\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi = f$ in $B_{3R}(x_0) \times (0,T)$ such that the second order derivatives in x and the first order derivative in t are uniformly bounded and continuous in $x \in B_{2R}(x_0)$ uniformly with respect to $t \in [0,T]$. Moreover, if $R > 0$ is sufficiently small, then, by (3.4), we have

$$\begin{aligned}&\|\varphi\|_{H_0^{p,2;1}(B_R(x_0),[0,T])} \le \|\varphi\|_{H_0^{p,2;1}(B_{3R}(x_0),[0,T])} \\ &\le N_0(d,c_1,c_2)\|f\|_{L^p(B_{3R}(x_0)\times(0,T))} = N_0(d,c_1,c_2)\|f\|_{L^p(D)}.\end{aligned}$$

Let us verify (3.6) for such φ.

Step 1. Let $\nu(dxdt) = \xi_t(dx)\eta(dt)$ be a factorization of ν such that, for *every* $t \in \mathbb{R}^1$, ξ_t is a non zero measure on $B_R(x_0) \times \{t\}$. First, we prove that η is absolutely continuous.

By substituting $\varphi(x,t) = f(t)h(x)$ in (3.6), for any smooth compactly supported f and h, we find

$$\left|\int_{\mathbb{R}} f'\psi\, \eta(dt)\right| \le N \sup |f|,$$

where N is independent of f and

$$\psi(t) = \int_{\mathbb{R}^d} h(x)\, \xi_t(dx).$$

It follows that the measure $\psi\,\eta$ is absolutely continuous and the measure η is absolutely continuous on the set $\Gamma(h) := \Big\{t\colon\ |\psi(t)| \ne 0\Big\}$. Since the space $C(\overline{B})$ is separable and ξ_t is non zero, we can find $h_k \in C_0^\infty(\mathbb{R}^d)$ such that $\mathbb{R}^1 = \cup_i \Gamma(h_i)$, which proves our claim.

Step 2. We observe that (3.6) also holds for any φ such that its second order derivatives in x and the first order derivative in t are bounded and continuous in x uniformly with respect to t. Indeed, let $\varphi_\varepsilon = \varphi * g_\varepsilon$, where g_ε is the same as above with $g(y,t) = g_1(y)g_2(t)$, where g_1 and g_2 are smooth probability densities. Then the functions $\partial_t\varphi_\varepsilon$, $\partial_i\varphi_\varepsilon$, and $\partial_i\partial_j\varphi_\varepsilon$ are bounded uniformly in ε and converge ν-a.e. to the corresponding derivatives of φ. To see this, let us observe that if a function ψ is bounded and continuous in x uniformly with respect to t, then $\psi_\varepsilon \to \psi$ ν-a.e. Indeed,

$$\lim_{\varepsilon\to 0}\sup_x\left|\int \psi(x+\varepsilon y, t+\varepsilon s)g_1(y)g_2(s)\,dy\,ds - \int \psi(x, t+\varepsilon s)g_2(s)\,ds\right| = 0.$$

On the other hand, for every fixed x and almost all t, one has

$$\int \psi(x, t+\varepsilon s)g_2(s)\,ds \to \psi(x,t).$$

Since η is absolutely continuous, we obtain the almost sure convergence with respect to ν, which yields (3.6) by the dominated convergence theorem. Now the rest of the proof is the same as in the special case considered above. □

Corollary 3.2. *Let μ be a locally finite Borel measure on Ω_T such that a^{ij}, b^i, $c \in L^1_{loc}(\Omega_T, \mu)$, and*

$$\int_{\Omega_T}\Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi + c\varphi\Big]\,d\mu = 0 \qquad \forall\,\varphi \in C_0^\infty(\Omega_T). \tag{3.9}$$

Then statements (i) *and* (ii) *of Theorem* 3.1 *are true. In addition, in case* (ii)*, if $J = [T_0, T_1] \subset (0,T)$, B is a ball with compact closure in Ω, and W is a neighborhood of $\overline{B}\times J$ with compact closure, then, for each $r < (d+2)'$, one has*

$$\|\varrho\|_{L^r(B\times J)} \le C(d,r,A,W)\Big(|\mu|(W) + \|c\|_{L^1(W,\mu)} + \|b\|_{L^1(W,\mu)}\Big),$$

where $C(d,r,A,W)$ depends only on d, r, the Hölder norms of a^{ij} with respect to x on W, $\inf_W \det A$, $\sup_{i,j}\sup_W |a^{ij}|$, and the distance from $B\times J$ to ∂W. An analogous statement is true in case (i).

Proof. Let us consider the measure $\nu = \zeta\,\mu$, where $\zeta \in C_0^\infty(W)$ is such that $0 \le \zeta \le 1$ and $\zeta|_{B\times J} = 1$. Then ν satisfies (3.6) with W instead of D and some constant $N = N_0\Big(|\mu|(W) + \|c\|_{L^1(W,\mu)} + \|b\|_{L^1(W,\mu)}\Big)$, where N_0 depends on $\sup_{i,j}\sup_W\big[|\partial_t\zeta| + |\partial_i\zeta| + |\partial_i\partial_j\zeta| + |a^{ij}|\big]$. Now the claim is readily seen from the proof of the above theorem. □

Remark 3.3. Assume that in the situation of Corollary 3.2, one has, in addition, that $|b|+|c|\in L^p_{loc}(\Omega_T)$, where $p>r'$. Then one has

$$\|\varrho\|_{L^r(B\times J)}\leq C(d,r,A,W)\Big(|\mu|(W)+(\|c\|_{L^p(W)}\|+\|b\|_{L^p(W)})\|\varrho\|_{L^{p'}(W)}\Big).$$

Remark 3.4. (i) Let us note that if there exists a diffusion process $\xi=(\xi_t^{s,x})$ governed by the stochastic differentiable equation

$$d\xi_t^{s,x}=\sqrt{2A(\xi_t^{s,x},t)}dW_t+b(\xi_t^{s,x},t)dt,\quad \xi_s^{s,x}=x,$$

then the above results apply to the transition probabilities $P(s,x;t,dy)$ of ξ. Namely, for any fixed (s,x), the measure $\mu=P(s,x;t,dy)\,dt$ satisfies (3.9) with $c=0$. Hence, for almost every t, the measure $P(s,x;t,dy)$ is absolutely continuous. This fact is well known for locally bounded b (see [21, Ch. II, §2], [39, Ch. 7, Ch. 9]). However, the measure $P(s,x;t,dy)$ may fail to be absolutely continuous for all t. In [14], an example is constructed such that $b=0$ and $A(x,t)$ is uniformly continuous, uniformly bounded and uniformly positive, but, for some fixed t, the measures $P(s,x;t,dy)$ are purely singular with respect to Lebesgue measure for all $s\in(0,1)$ and all x (a similar example is constructed in [35]).

(ii) It is worth noting that N.I. Portenko [34] employed analogous assumptions on A and b in the study of generalized diffusion processes. In particular, it is shown in [34, Chapter II] that if A is uniformly bounded, uniformly positive, and uniformly Hölder continuous, and b is a measurable vector field on $\mathbb{R}^n$ such that $|b|\in L^p(\mathbb{R}^d)$ with $p>d+2$, then there exists a continuous function $G(s,x,t,y)$, $0\leq s<t\leq T$, $x,y\in\mathbb{R}^d$, that is the transition probability density for a continuous Markov process $\Big(x(t),\mathcal{M}_t^s,P_{s,x}\Big)$, where $P_{s,x}$ is a probability measure on the σ-field $\mathcal{M}_t^s$ on the space Ω of continuous paths $x(\,\cdot\,)\colon\ [0,+\infty)\to\mathbb{R}^d$ generated by the evaluation mappings $x(\,\cdot\,)\mapsto x(u)$ with $u\in[s,t]$, $P_{s,x}\big\{x(s)=x\big\}=1$, and $P_{s,x}$-almost surely

$$x(t)-x(s)=\int_s^t b\big(x(\tau),\tau\big)\,d\tau+\int_s^t\sqrt{A\big(x(\tau),\tau\big)}\,dw_s(\tau)$$

with a certain Wiener process $\big(w_s(t),\mathcal{M}_t^s,P_{s,x}\big)$. Some additional comments on associated Markov processes are made in the last section.

(iii) We also note that the generalized parabolic equations considered in Corollary 3.2 have been used in [12] for the study of certain flows of probability measures. The above results yield the absolute continuity of such measures. Stronger regularity properties will be established for them below under certain additional assumptions on the coefficients.

Let $B_R \subset \Omega$ be a ball and $B_{R,T} = B_R \times (0,T)$. We set

$$S_{R,T} = (\partial B_R \times [0,T]) \cup (B_R \times \{0\}).$$

We shall now consider the following more general equation:

$$\partial_t u + a^{ij}\partial_i\partial_j u = \partial_i f^i, \qquad f_i \in L^q(B_{R,T}), \tag{3.10}$$

where $a^{ij} \in \mathbb{H}^{p,1}(B_R,[0,T])$. We recall that $u \in \mathbb{H}_0^{q,1}(B_R,[0,T])$, where $q \geq 1$, is called a solution of (3.10) with zero boundary conditions on $S_{R,T}$ if, for all $\varphi \in C_0^\infty(B_{R,T})$, one has

$$\int \Big[u\partial_t\varphi + a^{ij}\partial_i u \partial_j\varphi + u\partial_i a^{ij}\partial_j\varphi \Big]\, dx\, dt = \int f^i \partial_i\varphi \, dx\, dt.$$

If $f^i \in C_0^\infty(B_{R,T})$ and $u \in H^{q,2;1}(B_R,[0,T])$, then (3.10) is fulfilled almost everywhere.

We shall assume that the functions $a^{ij}(x,t)$ are continuous in x uniformly in t, i.e. (3.2) is fulfilled. Note that a^{ij} has a modification with such a property provided that

$$\sup_t \|a^{ij}(\,\cdot\,,t)\|_{H^{p,1}(B_R)} < \infty, \qquad \text{where } p > d. \tag{3.11}$$

Lemma 3.5. *Suppose that $A \geq \lambda I$ for some $\lambda > 0$ and that* (3.11) *is fulfilled. Let $q \geq p'$. If $R > 0$ is sufficiently small, then there exist $M > 0$ such that*

$$\|u\|_{\mathbb{H}^{q,1}(B_R,[0,T])} \leq M\|\partial_i f^i\|_{\mathbb{H}^{q,-1}(B_R,[0,T])} \leq M\|(f^i)\|_{L^q(B_{R,T})}$$

for every $u \in H_0^{q,1}(B_R,[0,T])$ that satisfies (3.10) *in the above sense.*

Proof. We set $a_0^{ij}(t) = a^{ij}(x_0,t)$. Let us write our equation in the following form:

$$\partial_t u + a_0^{ij}\partial_i\partial_j u = \partial_i f^i + \partial_i(b^{ij}\partial_j u) + c^i\partial_i u,$$

$$b^{ij}(x,t) = a_0^{ij}(t) - a^{ij}(x,t), \quad c^i(x,t) = \partial_j a^{ij}(x,t).$$

Let $c(x,t) = \big(c^i(x,t)\big)_{i=1}^d$. According to [26, Theorem 4.10], there exists N_0 independent of u and $R < 1$ such that

$$\|u\|_{\mathbb{H}^{q,1}(B_R,[0,T])} \leq N_0\|\partial_i f^i + \partial_i(b^{ij}\partial_j u) + c^i\partial_i u\|_{\mathbb{H}^{q,-1}(B_R,[0,T])}.$$

It should be noted that although the theorem cited has been proved for $\mathbb{R}^d$ instead of a ball, the same statement is true for B_R. In fact, this can be deduced from the case of $\mathbb{R}^d$, since the extension of a function in $H_0^{q,1}(B_R)$ by zero outside B_R belongs to $H_0^{q,1}(\mathbb{R}^d)$ with the same norm.

Let $q > d'$. By Lemma 2.5(i), for each fixed t, one has

$$\|c^i\partial_i u(\,\cdot\,,t)\|_{H^{q,-1}(B_R)} \leq N\|c(\,\cdot\,,t)\|_{L^d(B_R)}\|\nabla_x u(\,\cdot\,,t)\|_{L^q(B_R)},$$

where N is independent of t, u and R. By (3.11), we can choose $R > 0$ sufficiently small such that

$$\sup_t \|c(\,\cdot\,,t)\|_{L^d(B_R)} \le \frac{1}{4NN_0}.$$

Hence

$$\|c^i\partial_i u\|_{\mathbb{H}^{q,-1}(B_R,[0,T])} \le \frac{1}{4N_0}\|\nabla_x u\|_{L^q(B_{R,T})} \le \frac{1}{4N_0}\|u\|_{\mathbb{H}^{q,1}(B_R,[0,T])}.$$

In addition, letting $v = (b^{1j}\partial_j u, \dots, b^{dj}\partial_j u)$, by (2.11), one has

$$\|\partial_i(b^{ij}\partial_j u)\|_{\mathbb{H}^{q,-1}(B_R,[0,T])} \le \|v\|_{L^q(B_{R,T})} \le \sup_{B_{R,T}} |b^{ij}|\, \|\nabla_x u\|_{L^q(B_{R,T})}.$$

Since A is uniformly continuous in x uniformly in t, we can make R smaller in such a way that

$$\sup_t \sup_{x\in B_R} |b^{ij}(x,t)| \le \frac{1}{4N_0}.$$

Then we obtain

$$\|u\|_{\mathbb{H}^{q,1}(B_R,[0,T])} \le N_0\|\partial_i f^i\|_{\mathbb{H}^{q,-1}(B_R,[0,T])} + \frac{1}{2}\|u\|_{\mathbb{H}^{q,1}(B_R,[0,T])}.$$

The two remaining cases $q < d'$ and $q = d'$ are considered in a similar manner by using statements (ii) and (iii) of Lemma 2.5 for estimating $\|c^i\partial_i u\|_{\mathbb{H}^{q,-1}(B_R,[0,T])}$. Thus, we obtain the first inequality in the lemma, and the second follows by (2.11). □

Corollary 3.6. *In the situation of Lemma* 3.5*, for every collection $f^i \in L^q(B_{R,T})$, $i = 1, \dots, d$, there exists $u \in \mathbb{H}_0^{q,1}(B_R,[0,T])$ satisfying equation* (3.10) *in the above sense.*

Proof. It suffices to approximate a^{ij} and f^i by smooth functions a_k^{ij} and f_k^i in $\mathbb{H}^{p,1}(B_R,[0,T])$ and $L^q(B_{R,T})$, respectively, in such a way that (3.11) holds uniformly in k and the A_k are uniformly nondegenerate. Then, by Lemma 3.5, the corresponding solutions u_k are uniformly bounded in $\mathbb{H}_0^{q,1}(B_R,[0,T])$, hence have a weak cluster point $u \in \mathbb{H}_0^{q,1}(B_R,[0,T])$, which is a solution corresponding to a^{ij} and f^i. □

Theorem 3.7. *Let $d \ge 2$, $p > d$, $q \in [p', +\infty)$. Let A be uniformly bounded and uniformly nondegenerate and let* (3.11) *hold. Suppose that μ is a finite measure on $B_{R,T}$ such that, for some $N > 0$, one has*

$$\left|\int \Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi\Big]\,d\mu\right| \le N\|\nabla_x\varphi\|_{L^q(B_{R,T})}, \qquad \forall\, \varphi \in C_0^\infty(B_{R,T}). \tag{3.12}$$

Then $\mu \in \mathbb{H}^{q',1}(B_{R'},[t_0,t_1])$ and $\mu \in \mathcal{H}^{q',1}(B_{R'},[t_0,t_1])$ for every $R' < R$ and $[t_0,t_1] \subset (0,T)$.

Proof. The reasoning is similar to the elliptic case. Passing to smaller R and a time interval $[t_0, t_1] \subset (0, T)$, replacing μ be the measure $\zeta\,\mu$, where $\zeta \in C_0^\infty\big(B_{R'} \times (t_0, t_1)\big)$, and arguing as in the proof of Theorem 3.1 (see Step 2 in the proof of assertion (ii)), we may assume that (3.12) is fulfilled for all functions φ such that $\partial_t\varphi$ and $\partial_i\partial_j\varphi$ are uniformly bounded and continuous in x uniformly with respect to t. Let $f_j \in C_0^\infty\big(B_{R'} \times (t_0, t_1)\big)$, $j = 1, \dots, d$. Let us solve the equation

$$\partial_t\varphi + a^{ij}\varphi = \partial_i f^i$$

with zero boundary conditions on $B_{R_0,T}$. By our hypotheses, φ has two derivatives in x and one derivative in t that are bounded and continuous in $x \in B_{R'}$ uniformly with respect to $t \in [t_0, t_1]$. Letting $\psi = \partial_i f^i$, we have, by Lemma 3.5,

$$\begin{aligned}\left|\int \psi\, d\mu\right| &\le N\|\nabla_x\varphi\|_{L^q(B_{R,T})} \le NM\|\psi\|_{\mathbb{H}^{q,-1}(B_R,[0,T])} \\ &= NM\|\psi\|_{\mathbb{H}^{q,-1}(B_{R'},[t_0,t_1])}.\end{aligned}$$

Hence $\mu \in \mathbb{H}_0^{q',1}(B_{R'}, [t_0, t_1])$. Moreover, we have by the above estimate

$$-\partial_t\mu + \partial_i\partial_j(a^{ij}\mu) \in \mathbb{H}^{q',-1}(B_{R'}, [t_0, t_1]).$$

Therefore, in order to show that $\partial_t\mu \in \mathbb{H}^{q',-1}(B_{R'}, [t_0, t_1])$, it remains to check that $\partial_i\partial_j(a^{ij}\mu) \in \mathbb{H}^{q',-1}(B_{R'}, [t_0, t_1])$. For this it suffices to prove that $a^{ij}\mu \in \mathbb{H}_0^{q',1}(B_{R'}, [t_0, t_1])$. This is proved exactly as in the proof of Lemma 2.6. Let us recall that both inclusions have been established, of course, for $\zeta\mu$ in place of μ, so that for the initial measure μ we obtain the inclusions announced in the theorem for yet smaller R' and $[t_0, t_1]$. □

Theorem 3.8. *Let A be locally bounded and locally uniformly non-degenerate on $B_{R,T}$ and let* (3.11) *be fulfilled, where now we assume that $p > d + 2$. Let μ be a finite Borel measure on $B_{R,T}$ such that $\mu \in L^r(B_{R,T})$ with some $r > p'$. Let $\beta \in L^p(B_{R,T})$. Suppose that for all $\varphi \in C_0^\infty(B_{R,T})$, one has*

$$\left|\int \Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi\Big]\, d\mu\right| \le \int \Big(|\varphi| + |\nabla_x\varphi|\Big)|\beta\mu|\, dx. \tag{3.13}$$

Then μ has a density that is locally Hölder continuous on $B_R \times (0, T)$ and belongs to $\mathbb{H}^{p,1}(B_{R'}, [T_0, T_1])$ and $\mathcal{H}^{p,1}(B_{R'}, [T_0, T_1])$ for all $R' < R$ and $[T_0, T_1] \subset (0, T)$.

Proof. Let us fix $R' < R$ and a closed interval $[T_0, T_1] \subset (0, T)$. Since $\mu \in L^r(B_{R,T})$ with some $r > p(p-1)^{-1} = p'$, it follows by (2.12) that $|b|\mu \in L^s(B_{R,T})$ with $s = pr/(p + r) < p$. Hence (3.12) holds with

$q = s' > p'$. Then, by Theorem 3.7, $\mu \in \mathbb{H}^{s,1}(B_{R_1}, [t_0, t_1])$ and $\mu \in \mathcal{H}^{s,1}(B_{R_1}, [t_0, t_1])$ for every $[t_0, t_1] \subset (0, T)$ and $R_1 < R$. In particular, we can take any $t_0 \in (0, T_0)$, $t_1 \in (T_1, T)$, $R_1 \in (R', R)$.

Let us recall an embedding theorem for the spaces $\mathcal{H}^{p,1}(\mathbb{R}^d, [0, T])$. A proof can be found, e.g., in [26, Corollary 7.6] or in [25, Theorem 7.2] (where the restriction $p > 2$ was only needed in the case of stochastic Sobolev spaces). Let $q > p > 1$ and $(d+2)(1/p - 1/q) < 1$. Then there is a constant $N(d, p, q, T)$ such that, for each $u \in \mathcal{H}^{p,1}(\mathbb{R}^d, [0, T])$, one has

$$\|u\|_{L^q(\mathbb{R}^d \times [0,T])} \leq N(d, p, q, T)\|u\|_{\mathcal{H}^{p,1}(\mathbb{R}^d,[0,T])}. \tag{3.14}$$

It is easily seen that the same is true for a ball instead of $\mathbb{R}^d$. According to (3.14), we obtain that $\mu \in L^{r_1}(B_{R_1} \times [t_0, t_1])$ for each r_1 with

$$\frac{1}{r_1} > \frac{1}{s} - \frac{1}{d+2}.$$

This is equivalent to the following relation:

$$\frac{1}{r_1} > \frac{p+r}{pr} - \frac{1}{d+2} = \frac{(d+2)(p+r) - pr}{pr(d+2)},$$

which can be written as

$$r_1 < r\frac{p(d+2)}{(d+2)(p+r) - pr},$$

provided $(d+2)(p+r) - pr > 0$ (otherwise one can take for r_1 any number $r_1 > 1$). We observe that one can choose r_1 sufficiently close to the right-hand side of the above relation in such a way that

$$\begin{aligned} \frac{r_1}{r} = \frac{p(d+2)}{(d+2)(p+r) - pr} &= \frac{p}{p + r - \frac{pr}{d+2}} \\ &= \frac{p}{p - r(\frac{p}{d+2} - 1)} \geq \frac{p}{p - \frac{p}{d+2} + 1} > 1. \end{aligned}$$

Therefore, repeating the above procedure finitely many times we arrive at the situation with $(d+2)(p+r) - pr \leq 0$, when $\mu \in L^r(B_{R_0} \times [\tau_0, \tau_1])$ for all $r \in (1, +\infty)$, where $\tau_0 < T_0$, $\tau_1 > T_1$, $R_0 > R'$. Hence $\mu \in \mathbb{H}^{s,1}(B_{R_0}, [\tau_0, \tau_1])$ and $\mu \in \mathcal{H}^{s,1}(B_{R_0}, [\tau_0, \tau_1])$ for each $s < p$. Let us choose $s > d + 2$, $\alpha > 0$ and $\beta > 0$ such that $\beta > \alpha > 1/s$ and $s(1 - 2\beta) > d$, which is possible, since $p > d + 2$. By [26, Theorem 7.2] (see also [6, Ch. III]), we have $\mu \in C^{\alpha - 1/s}\Big([\tau_0, \tau_1], H^{s,1-2\beta}(B_{R_0})\Big)$, which is the space of $(\alpha - 1/s)$-Hölder continuous mappings on $[\tau_0, \tau_1]$ with values in $H^{s,1-2\beta}(B_{R_0})$ and is contained in $C^{\alpha - 1/s}\Big([\tau_0, \tau_1], C^\gamma(B_{R_0})\Big)$ for some $\gamma > 0$ (where $C^\gamma(B_{R_0})$ is the space of γ-Hölder functions on B_{R_0})

by the Sobolev embedding theorem, since $s(1-2\beta) > d$. Thus, μ has a Hölder continuous density. Then (3.12) holds with $q = p'$, whence our statement follows by Theorem 3.7. $\square$

Corollary 3.9. *Let $p > d+2$ and let A be locally uniformly bounded and locally uniformly nondegenerate on Ω_T and let* (3.11) *be fulfilled* (*with $p > d+2$*) *for every ball B_R with compact closure in Ω. Let b^i, $c \in L^p_{loc}(\Omega_T)$. Assume that μ is a locally finite signed Borel measure on Ω_T such that b^i, $c \in L^1_{loc}(\Omega_T, \mu)$ and*

$$\int_{\Omega_T} \Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi + c\varphi\Big]\, d\mu = 0, \qquad \forall\, \varphi \in C_0^\infty(\Omega_T). \quad (3.15)$$

Then μ has a locally Hölder continuous density that belongs to the spaces $\mathbb{H}^{p,1}(U,J)$ and $\mathcal{H}^{p,1}(U,J)$ for every interval J and open set U such that $U \times J$ has compact closure in Ω_T.

Proof. It suffices to note that, by Theorem 3.1, μ has a density in $L^r_{loc}(\Omega_T)$ with any $r < (d+2)'$, in particular, with some $r > p'$. Hence Theorem 3.8 applies with $\beta = |b| + |c|$. $\square$

It is worth noting that if μ is nonnegative, then, by Harnack's inequality (see, e.g., [3], [17], [41]) the continuous version of its density is strictly positive in every component of Ω in which it is not identically zero.

Corollary 3.10. *Assume that, for all $\varphi \in C_0^\infty(\Omega_T)$, one has*

$$\int_{\Omega_T} \Big[\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi + c\varphi\Big]\, d\mu = \int f^i\partial_i\varphi\, dx\, dt, \quad (3.16)$$

where $f^i \in L^p_{loc}(\Omega_T)$ and a^{ij}, b^i, c satisfy the same hypotheses as in Corollary 3.9*. Then the assertion of Corollary* 3.9 *with* (3.15) *replaced by* (3.16) *is still true.*

Proof. As in the elliptic case, our assertion reduces to the case where we have (3.15). Indeed, for sufficiently small $R > 0$ and any $[T_0, T_1] \subset (0, T)$, there exists $u \in \mathbb{H}_0^{p,1}(B_R, [T_0, T_1])$ satisfying the equation

$$-\partial_t u + \partial_i(a^{ij}\partial_j u + \partial_j a^{ij} u - b^i u) + cu = -\partial_i f^i.$$

This fact is proved in [28, Ch. III] for the space $\mathbb{H}_0^{2,1}(B_R, [T_0, T_1])$, but the same is true for $\mathbb{H}_0^{p,1}(B_R, [T_0, T_1])$, which follows from Lemma 3.5. Then the measure $\mu - u dx dt$ satisfies (3.15) with $B_R \times (T_0, T_1)$ instead of Ω_T. $\square$

A closer look at the above proof leads to the following result.

Corollary 3.11. *Let μ satisfy the hypotheses of Corollary* 3.9*, let B_{R_0} be an open ball with compact closure in Ω, and let $[t_1, t_2] \subset (0, T)$. Then, for every closed interval $[\tau_1, \tau_2] \subset (t_1, t_2)$ and $R < R_0$, there exists a constant N, depending on t_1, t_2, τ_1, τ_2, R_0, R, $\|c\|_{L^p(B_{R_0}\times[t_1,t_2])}$, $\inf\limits_{B_{R_0}\times[t_1,t_2]} \det A$, $\sup\limits_{t\in[t_1,t_2]} \|a^{ij}(\,\cdot\,,t)\|_{H^{p,1}(B_{R_0})}$, and $\|b^i\|_{L^p(B_{R_0}\times[t_1,t_2])}$, such that N is a locally bounded function of the above mentioned quantities and one has*

$$\|\mu\|_{\mathbb{H}^{p,1}(B_R,[\tau_1,\tau_2])} \le N\|\mu\|_{L^{p'}(B_{R_0}\times[t_1,t_2])}.$$

Moreover, one can choose N in such a way that in the case where μ is nonnegative we have

$$\|\mu\|_{\mathbb{H}^{p,1}(B_R,[\tau_1,\tau_2])} \le N\|\mu\|_{L^{1}(B_{R_0}\times[t_1,t_2])}. \tag{3.17}$$

Proof. By Remark 3.3 we obtain an a priori estimate for the L^r-norm of any solution with any $r \in \big(p', (d+2)'\big)$. Now the first claim follows by the reasoning in the proof of Theorem 3.8. Finally, estimate (3.17) for nonnegative solutions follows from Harnack's inequality. Indeed, suppose that $s_1 < s_2 < s_3 < s_4$ and $W_1 = B_{R_1} \times [s_1, s_2]$, $W_2 = B_{R_1} \times [s_3, s_4]$ have compact closures in $B_R \times (t_1, t_2)$. Letting ϱ be the continuous version of the density of a nonnegative solution μ, according to Harnack's inequality (see [3, Theorem 3] or [41, Theorem 5.1]), we have

$$\max_{W_1} \varrho \le C \min_{W_2} \varrho,$$

where C depends (as a locally bounded function) on s_i, t_i, R_0, R, R_1, $\|c\|_{L^p(B_{R_0}\times[t_1,t_2])}$, $\|b^i\|_{L^p(B_{R_0}\times[t_1,t_2])}$, $\sup\limits_{t\in[t_1,t_2]} \|a^{ij}(\,\cdot\,,t)\|_{H^{p,1}(B_{R_0})}$, and $\inf\limits_{B_{R_0}\times[t_1,t_2]} \det A$. Therefore, we have

$$\max_{W_1} \varrho \le C|W_2|^{-1} \int_{B_{R_0}\times[t_1,t_2]} \varrho(x,t)\,dx\,dt,$$

where $|W_2|$ is Lebesgue measure of W_2. Now (3.17) follows by the first claim (we take s_i such that $[\tau_1, \tau_2] \subset (s_1, s_2)$). □

Remark 3.12. We observe that the assumption $p > d + 2$ was used in the proof of Theorem 3.8 in order to improve the initial integrability of a solution and also to guarantee its Hölder continuity (by an appropriate embedding theorem). It is clear from the first step of that proof (or from Theorem 3.7) that if we assume only that $p > d$, then we obtain that $\mu \in \mathbb{H}^{s,1}(B_{R'}, [T_0, T_1])$ and $\mu \in \mathcal{H}^{s,1}(B_{R'}, [T_0, T_1])$ with

$$s = pr/(p + r) > d$$

provided that $r > pd/(p - d)$.

4. Applications to transition probabilities

Let us return to the stochastic equation (1.1). It is known that if A has two bounded derivatives and is uniformly nondegenerate on $\mathbb{R}^d$ and b has a bounded derivative, then there exists a diffusion process ξ_t^x that satisfies (1.1) with $\xi_0^x = x$, $x \in \mathbb{R}^d$. Its transition probabilities $P(t,x,dy)$ have densities $p(t,x,y)$ satisfying the following Kolmogorov equations:

$$\frac{\partial}{\partial t} p(t,x,y) = a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} p(t,x,y) + b^i \frac{\partial}{\partial x_i} p(t,x,y) \tag{4.1}$$

$$\frac{\partial}{\partial t} p(t,x,y) = \frac{\partial^2}{\partial y_i \partial y_j}\Big(a^{ij} p(t,x,y)\Big) - \frac{\partial}{\partial y_i}\Big(b^i p(t,x,y)\Big) \tag{4.2}$$

for all $t > 0$, $x, y \in \mathbb{R}^d$; in addition, $P(0,x,dy) = \delta_x$. This means that the measures $P(t,x,dy)\,dt$ satisfy (1.5). The transition semigroup $(T_t)_{t\geq 0}$ of ξ_t^x is given on $C_b(\mathbb{R}^d)$ by

$$T_t f(x) = \int f(y)\, P(t,x,dy).$$

If a probability measure μ is invariant for $(T_t)_{t\geq 0}$, i.e.,

$$\int T_t f\, d\mu = \int f\, d\mu, \qquad \forall\, f \in C_b(\mathbb{R}^d), \tag{4.3}$$

then μ satisfies the elliptic equation (1.3), and, conversely, if μ satisfies (1.3), then it is invariant for $(T_t)_{t\geq 0}$.

Now let us assume that a probability measure μ on Ω satisfies the elliptic equation (1.3), where A is continuous and nondegenerate, $a^{ij} \in H^{p,1}_{loc}(\Omega)$, $b^i \in L^p_{loc}(\Omega)$ with some $p > d$. Then our results imply (at least if $p \geq 2$) that [38, Theorem 1.5] applies. Hence there exists a closed extension $\big(\widehat{L}_{A,b}, D(\widehat{L}_{A,b})\big)$ of $\big(L_{A,b}, C_0^\infty(\Omega)\big)$ on $L^1(\Omega,\mu)$ that generates a sub-Markovian C_0-semigroup $(T_t)_{t\geq 0}$ of linear contractions on $L^1(\Omega,\mu)$. In general, μ is only $(T_t)_{t\geq 0}$-subinvariant (see [38, Example 1.12]). But, if $\Omega = \mathbb{R}^d$, by [38, Corollary 2.2], it follows that μ is invariant with respect to $(T_t)_{t\geq 0}$ if and only if $\big(\widehat{L}_{A,b}, D(\widehat{L}_{A,b})\big)$ is the closure of $\big(L_{A,b}, C_0^\infty(\Omega)\big)$ on $L^1(\Omega,\mu)$. For a necessary and sufficient condition for the latter, we refer to [38, Proposition 1.9] and, for easy to verify sufficient conditions, to [38, Proposition 1.10]. In addition, as shown in [38], there exists a diffusion associated with $(T_t)_{t\geq 0}$. We shall now prove that in the case $p > d+2$ the semigroup $(T_t)_{t\geq 0}$ is strongly Feller and that the dual semigroup satisfies the parabolic equation (4.2) in the weak sense. Then we deduce that μ is the unique invariant probability measure for $(T_t)_{t\geq 0}$.

Theorem 4.1. *Let $(T_t)_{t\geq 0}$ be the sub-Markovian semigroup on $L^1(\Omega,\mu)$ generated by the closed extension $\big(\widehat{L}_{A,b}, D(\widehat{L}_{A,b})\big)$ of $\big(L_{A,b}, C_0^\infty(\Omega)\big)$ specified above. Suppose that $p > d+2$. Then there exist unique sub-probability kernels $K_t(\,\cdot\,, dy)$, $t>0$, on Ω such that*

$$K_t(x,dy) = p(t,x,y)\,dy,$$

where $p(t,x,y)$ is a locally Hölder continuous nonnegative function on $(0,+\infty)\times\Omega\times\Omega$, and for every $f\in L^1(\Omega,\mu)$, the function

$$x \mapsto K_t f(x) := \int_\Omega f(y)\,p(t,x,y)\,dy$$

is a μ-version of $T_t f$ such that $(t,x)\mapsto K_t f(x)$ is continuous on $(0,+\infty)\times\Omega$. In addition, if μ_0 is a bounded measure on Ω and

$$K_t^*\mu_0(dy) := \int_\Omega K_t(x,dy)\,\mu_0(dx),$$

then the measure $K_t^*\mu_0(dy)\,dt$ satisfies (1.5) for all $T>0$. In particular, $K_t^*\mu_0$, $t>0$, has a density in $H^{p,1}_{loc}(\Omega)$.

Proof. Step 1. Consider the operator L_{A,b_0} on $C_0^\infty(\Omega)$ with the first-order term $b_0 = (b_0^1,\dots,b_0^d)$ given by

$$b_0^j = 2\partial_i a^{ij} + 2a^{ij}\frac{\partial_i\varrho}{\varrho} - b^j,$$

where ϱ is the continuous version of $d\mu/dx$, which according to Corollary 2.9 and Corollary 2.10 is in $H^{p,1}_{loc}(\Omega)$ and strictly positive in every connected component of Ω where it is not identically zero. We may assume that ϱ does not vanish in Ω. Note that by our assumption, $b_0^j\in L^p_{loc}(\Omega)$ and that

$$\int L_{A,b}\varphi\,\psi\,d\mu = \int \varphi L_{A,b_0}\psi\,d\mu, \qquad \forall\,\varphi,\,\psi\in C_0^\infty(\Omega). \tag{4.4}$$

Indeed, by the integration by parts formula, the left-hand side of (4.4) is equal to

$$\int\Big[-\psi\partial_i a^{ij}\partial_j\varphi\,\varrho - a^{ij}\partial_i\psi\partial_j\varphi\,\varrho - \psi a^{ij}\partial_j\varphi\partial_i\varrho + \psi b^i\partial_i\varphi\,\varrho\Big]\,dx.$$

In a similar manner, the left-hand side can be written as

$$\int\Big[\varphi\partial_i a^{ij}\partial_j\psi\,\varrho - a^{ij}\partial_i\varphi\partial_j\psi\,\varrho + \varphi a^{ij}\partial_j\psi\partial_i\varrho - \varphi b^i\partial_i\psi\,\varrho\Big]\,dx.$$

Therefore, (4.4) is equivalent to the equality

$$\int\Big[-\partial_i a^{ij}\partial_j(\psi\varphi)\,\varrho - a^{ij}\partial_j(\psi\varphi)\partial_i\varrho + b^i\partial_i(\psi\varphi)\,\varrho\Big]\,dx = 0.$$

It remains to note that the latter can be written as

$$\int \Big[a^{ij}\partial_i\partial_j(\psi\varphi) + b^i\partial_i(\psi\varphi)\Big]\varrho\,dx = 0.$$

Hence (4.4) is proved. In particular, if $\psi = 1$ on the support of φ, we obtain

$$L^*_{A,b_0}\mu = 0.$$

Now [38, Theorem 1.5] also applies to $\big(L_{A,b_0}, C_0^\infty(\Omega)\big)$, so it has a closed extension $\big(\widehat{L}_{A,b_0}, D(\widehat{L}_{A,b_0})\big)$ which generates a sub-Markovian C_0-semigroup $(T_t')_{t\geq 0}$ on $L^1(\Omega,\mu)$. Moreover, as follows from the construction in [38] (see, in particular, [38, Remark 1.7(ii)]), we have

$$\int T_t f\, g\,d\mu = \int f\, T_t' g\,d\mu, \qquad \forall\, f,\, g \in L^\infty(\Omega,\mu),\ t\geq 0. \tag{4.5}$$

Step 2. Let us show that for all $g\in L^1(\Omega,\mu)$, $t>0$, there exists a μ-version $\widetilde{T_t g}$ of $T_t g$ such that

$$(x,t)\mapsto \widetilde{T_t g}(x) \text{ is continuous on } \Omega\times(0,+\infty).$$

First, by the continuity of $t\mapsto T_t g$ in $L^1(\Omega,\mu)$, for each fixed $t\geq 0$, one can choose a version of $T_t g$ (denoted by the same symbol) such that $(x,t)\mapsto T_t g(x)$ is jointly measurable (see [33, Section III.4]).

Assume that $g\in L^\infty(\Omega,\mu)$. Define a measure ν on $\Omega\times(0,+\infty)$ by

$$\nu(dx\,dt) := T_t g(x)\,\mu(dx)\,dt. \tag{4.6}$$

Then

$$\int_{\Omega_T}\Big(\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi + b_0^i\partial_i\varphi\Big)\,d\nu = 0, \qquad \forall\,\varphi\in C_0^\infty(\Omega_T). \tag{4.7}$$

Indeed, since $(T_t)_{t\geq 0}$ is continuous on $L^{p'}(\Omega,\mu)$, it suffices to verify (4.7) for $g\in C_0^\infty(\Omega)$. Moreover, it suffices to do this for φ of the form $\varphi=\varphi_1\varphi_2$, where $\varphi_1\in C_0^\infty(\Omega)$, $\varphi_2\in C_0^\infty\big((0,T)\big)$. It follows by (4.5) that

$$\int_\Omega L_{A,b_0}\varphi_1\,T_t g\,d\mu = \int_\Omega \varphi_1 T_t L_{A,b} g\,d\mu.$$

By integrating by parts with respect to t we obtain

$$\begin{aligned}\int_{\Omega_T}\Big(\partial_t\varphi + a^{ij}\partial_i\partial_j\varphi + b_0^i\partial_i\varphi\Big)\,d\nu &= \int\!\!\int \varphi_1(x)\varphi_2(t)T_t L_{A,b}g(x)\,\mu(dx)\,dt\\ &\quad - \int \varphi_2(t)\frac{d}{dt}\int \varphi_1(x)\,T_t g(x)\,\mu(dx)\,dt.\end{aligned}$$

Since

$$\frac{d}{dt}\int \varphi_1(x)T_t g(x)\,\mu(dx) = \int \varphi_1(x)T_t L_{A,b}g(x)\,\mu(dx),$$

we obtain (4.7). Because of the continuity and strict positivity of ϱ, (4.7) and Corollary 3.9 imply that there exists a continuous function $u\colon \Omega\times(0,+\infty)\to\mathbb{R}^1$ such that for dt-a.e. $t\in(0,+\infty)$ we have that $u(\,\cdot\,,t)$ is a version of $T_t g$. However, we observe that the latter is true for every $t>0$. Indeed, for any $t>0$, there exists a sequence $t_n\to t$ and a Borel set $S_t\subset\Omega$ such that $\mu(S_t)=1$ and $T_{t_n}g(x)=u(x,t_n)$ for all $x\in S_t$. It remains to note that the right-hand side of this equality converges to $u(x,t)$ for each x and that $T_{t_n}g\to T_t g$ in $L^1(\Omega,\mu)$, hence in measure. Therefore, we can set $\widetilde{T_t g}(x)=u(x,t)$.

Now let $g\in L^1(\Omega,\mu)$. We may assume that $g\geq 0$. Let $g_k=\min(g,k)$. According to Corollary 3.11 (see (3.17)), equality (4.7) implies that, for every ball B_R with compact closure in Ω and every $[T_1,T_2]\subset(0,+\infty)$, the sequence of functions $(x,t)\mapsto\widetilde{T_t g_k}(x)$ is bounded in $\mathbb{H}^{p,1}(B_R,[T_1,T_2])$. For every fixed t, the functions $T_t g_k$ converge to $T_t g$ in $L^1(\Omega,\mu)$. Hence the function $(x,t)\mapsto\widetilde{T_t g}(x)$ is in $\mathbb{H}^{p,1}(B_R,[T_1,T_2])$. Moreover, the functions $\widetilde{T_t g_k}(x)$ converge to $\widetilde{T_t g}(x)$ in $L^p(B_R\times[T_1,T_2])$. Therefore, since the measure $T_t g_k\,\mu(dx)\,dt$ satisfies (4.7), the measure $T_t g\,\mu(dx)\,dt$ also does. As above, this yields our claim by Corollary 3.9.

We observe that if a sequence $\{g_j\}$ converges in $L^1(\Omega,\mu)$, then, for every compact set $Q\subset\Omega$, the continuous functions $\widetilde{T_t g_j}$ converge uniformly on Q for every $t>0$. This follows by the closed graph theorem and the above reasoning (note that the positive and negative parts of g_j converge in $L^1(\Omega,\mu)$), since the natural embedding $C(Q)\to L^1(Q,\mu)$ is continuous (certainly, this is also seen from the above reasoning). In fact, this shows that there exists a constant $C(Q,t)$ such that

$$\sup_{x\in Q}|\widetilde{T_t g}(x)|\leq C(Q,t)\|g\|_{L^1(\Omega,\mu)},\qquad \forall\, g\in L^1(\Omega,\mu). \tag{4.8}$$

Moreover, there is $\alpha>0$ and a constant $C_1(Q,T)$ such that

$$\sup_{x,y\in Q}\frac{|\widetilde{T_t g}(x)-\widetilde{T_t g}(y)|}{|x-y|^\alpha}\leq C_1(Q,t)\|g\|_{L^1(\Omega,\mu)},\qquad \forall\, g\in L^1(\Omega,\mu). \tag{4.9}$$

which is proved by applying the same reasoning to the Banach space of α-Hölder continuous functions.

Step 3. By (4.8) we obtain that there exist sub-probability measures $K_t(x,dy)$ such that, for every $x\in\Omega$ and $t>0$, one has

$$K_t(x,dy)=p_t(x,y)\mu(dy),$$

where $y\mapsto p_t(x,y)$ is a bounded μ-measurable function, and

$$K_t g(x) := \int g(y)\, K_t(x, dy) = \widetilde{T_t} g(x) \tag{4.10}$$

for all $g \in L^1(\Omega, \mu)$, $x \in \Omega$, and $t > 0$.

Let $K'_t(\,\cdot\,, dy)$ be defined analogously to $K_t(\,\cdot\,, dy)$ above, but with T'_t in place of T_t. Then $K'_t(x, dy) = p'_t(x, y)\mu(dy)$, where $p'_t(x, \cdot\,) \in L^\infty(\Omega, \mu)$.

Step 4. Now we apply (4.5) and Fubini's theorem to (4.7) which yields for all $g \in L^\infty(\Omega, \mu)$ and all $\varphi \in C_0^\infty(\Omega_T)$

$$\int g(y) \int_{\Omega_T} \Big(\partial_t \varphi + a^{ij}\partial_i\partial_j\varphi + b_0^i \partial_i \varphi\Big) K'_t(y, dx)\, dt\, \mu(dy) = 0. \tag{4.11}$$

Interchanging the roles of b_0 and b as well as T_t and T'_t in our reasoning so far we also have for all $g \in L^\infty(\Omega, \mu)$ and all $\varphi \in C_0^\infty(\Omega_T)$

$$\int g(y) \int_{\Omega_T} \Big(\partial_t \varphi + a^{ij}\partial_i\partial_j\varphi + b^i \partial_i \varphi\Big) K_t(y, dx)\, dt\, \mu(dy) = 0. \tag{4.12}$$

We observe that the function

$$\Phi\colon\ (y, t) \mapsto \int_\Omega \Big(\partial_t \varphi + a^{ij}\partial_i\partial_j\varphi + b^i \partial_i \varphi\Big) K_t(y, dx)$$

is continuous on $\Omega \times (0, +\infty)$. Indeed, if we denote the integrand by $g(t, x)$, then $|g(t, x) - g(s, x)| \le \theta(x)|t - s|$, where θ is μ-integrable. Now our claim readily follows from the joint continuity of the function $(x, t) \mapsto \widetilde{T_t} g(x)$ for any $g \in L^1(\Omega, \mu)$. Integrating Φ with respect to t yields a continuous function. Since (4.12) holds for any g, one has for all $y \in \Omega$

$$\int_{\Omega_T} \Big(\partial_t \varphi + a^{ij}\partial_i\partial_j\varphi + b^i \partial_i \varphi\Big) K_t(y, dx)\, dt = 0.$$

This means that, for every $y \in \Omega$, the measure $\nu = K_t(y, dx)\, dt$ satisfies (4.7). Therefore, there exist locally Hölder continuous functions $(x, t) \mapsto p_t(y, x)$ (a version of the above defined $p_t(y, x)$) on $\Omega \times (0, +\infty)$ such that

$$K_t(y, dx) = p_t(y, x)\, \mu(dx), \qquad y \in \Omega, t > 0.$$

According to Corollary 3.11 (estimate (3.17)) and the embedding theorem used in the proof of Theorem 3.8, the functions $(x, t) \mapsto p_t(y, x)$, $y \in \Omega$, have locally uniformly bounded Hölder and supremum norms. In addition, (4.9) implies that, for every $t > 0$, we have

$$\sup_{x, y \in Q}\, \sup_{z \in \Omega} \frac{|p_t(x, z) - p_t(y, z)|}{|x - y|^\alpha} \le C_1(Q, t).$$

Note also that (4.8) yields that

$$\sup_{x\in Q}\sup_{z\in\Omega} p_t(x,z) \le C(Q,t). \tag{4.13}$$

It is worth noting that, for a fixed compact set Q, we can choose quantities $C(Q,t)$ and $C_1(Q,t)$ such that they are bounded in t in every closed interval in $(0,+\infty)$. By the same reasoning we obtain that one can choose densities $p'_t(y,x)$ of the measures $K'_t(y,dx)$, $t>0$, $y\in\Omega$, with respect to μ such that $p'_t(y,x)$ is locally Hölder continuous in (t,x) and y. It is readily seen from (4.5) that $p_t(x,y)=p'_t(y,x)$ for all $x,y\in\Omega$, $t>0$. Setting $p(t,x,y)=p_t(x,y)\varrho(y)$ for $x,y\in\Omega$, $t>0$, we complete the proof of the first part of the theorem.

Step 5. Since, as shown in the previous step, the measure $K_t(y,dx)\,dt$ satisfies (4.7) for all $y\in\Omega$, by integrating over y with respect to μ_0 we obtain that the same holds for the measure $K_t^*\mu_0\,dt$. Now the last part of the assertion follows by Corollary 3.9. In order to apply this corollary, we have to verify that $|b|^p$ is locally integrable with respect to the measure $K_t^*\mu_0\,dt$. This measure has the density

$$f(x,t)=\int_\Omega p_t(y,x)\,\mu_0(dy)$$

with respect to $\mu\,dt$. It suffices to note that $f(x,t)$ is bounded on compact sets in $\Omega\times(0,+\infty)$. Indeed, the function $p'_t(x,y)$ satisfies an estimate of the type (4.13). By the equality $p'_t(x,y)=p_t(y,x)$ we obtain that $\sup_{x\in Q}\sup_{y\in\Omega} p_t(y,x)\le C(Q,t)$ for every compact set $Q\subset\Omega$, where $C(Q,t)$ is locally bounded in t. □

Remark 4.2. Let $f\in L^1(\Omega,\mu)$. It has been shown above that for every compact set $E\subset\Omega\times(0,+\infty)$, there exists a constant $C(E)$ such that

$$\sup_{(x,t)\in E}\sup_{y\in\Omega} p_t(x,y)\le C(E),$$

where $p_t(x,y)$ is the jointly continuous version.

Corollary 4.3. *Suppose that in the situation of Theorem* 4.1 *the set* Ω *is connected and that* μ *is invariant with respect to* $(T_t)_{t\ge0}$ *(or equivalently* $(K_t)_{t\ge0}$ *from Theorem* 4.1*). Let* ν *be a Borel probability measure on* Ω *invariant with respect to* $(K_t)_{t\ge0}$*. Then* $\nu=\mu$*.*

Proof. We know that μ has a continuous strictly positive density ϱ_1. By Theorem 4.1, $K_t^*\nu\,dt=\nu\,dt$ satisfies the equation

$$\int_{\Omega\times(0,+\infty)}\Big(\partial_t\varphi+a^{ij}\partial_i\partial_j\varphi+b^i\partial_i\varphi\Big)\,\nu(dx)\,dt=0$$

for all $\varphi(x,t) = \varphi_1(x)\varphi_2(t)$, $\varphi_1 \in C_0^\infty(\Omega)$, $\varphi_2 \in C_0^\infty(0,+\infty)$. Hence ν satisfies the elliptic equation (1.3). Therefore, ν has a continuous strictly positive density ϱ_2. According to [2, Proposition 2.2], the measure $\mu_0 = |\mu - \nu|$ with the continuous density $\varrho = |\varrho_1 - \varrho_2|$ is also invariant with respect to $(T_t)_{t\geq 0}$. The above reasoning applies also to μ_0, hence μ_0 satisfies the elliptic equation. Thus, ϱ is either identically zero or does not vanish in Ω. In the latter case by continuity $\varrho_1 > \varrho_2$ or $\varrho_2 > \varrho_1$, since Ω is connected. But this is impossible, because ϱ_1 and ϱ_2 have equal integrals. □

Proposition 4.4. *Suppose that the hypotheses of Theorem* 4.1 *are fulfilled except that now we have only $p > d$. Let $\psi \in C_0^\infty(\Omega)$. Then, for every $t \geq 0$, $T_t\psi$ has a continuous version $\widetilde{T_t\psi}$ and, for every compact set $K \subset \Omega$, one has $\lim_{t\to 0} \widetilde{T_t\psi}(x) = \psi(x)$ uniformly in $x \in K$.*

Proof. Let B be a closed ball in Ω. Let $f_t(x) = T_t L_{A,b}\psi(x)$. For every $t > 0$, the measure $\nu_t = T_t\psi\,\mu = T_t\psi\varrho\,dx$ on Ω satisfies the equation $L^*_{A,b_0}\nu_t = f_t\,\mu$. Indeed, for any $\varphi \in C_0^\infty(\Omega)$, it follows by (4.5) that

$$\int L_{A,b_0}\varphi\,d\nu_t = \int \varphi T_t L_{A,b}\psi\,d\mu = \int \varphi f_t\,d\mu.$$

Note that, since $\varrho \leq C$ on B, we have

$$\|f_t\varrho\|_{L^p(B)} \leq C^{(p-1)/p}\|f_t\|_{L^p(\mu)} \leq C^{(p-1)/p}\|L_{A,b}\psi\|_{L^p(\mu)}.$$

In addition, $\sup_t \|T_t\psi\varrho\|_{L^1(B)} < \infty$. According to Remark 2.14, we have $\nu_t = \varrho_t\,dx$, where $\varrho_t \in H^{p,1}_{loc}(\Omega)$ and $\sup_t \|\varrho_t\|_{H^{p,1}(B)} < \infty$. Therefore, the continuous versions of ϱ_t are Hölder continuous on B uniformly in $t > 0$. Hence the same is true for the functions $u(x,t) = \varrho_t(x)/\varrho(x)$. If $t_n \to 0$, then $u(x,t_n) \to \psi(x)$ in measure, since $T_{t_n}\psi \to \psi$ in $L^1(\Omega,\mu)$ and $T_t\psi = u(\,\cdot\,,t)$ μ-a.e. for each t. Setting $\widetilde{T_t\psi}(x) = u(t,x)$, it remains to note that if a sequence of functions is uniformly Hölder continuous and uniformly bounded on K and converges in measure, then it converges uniformly on K. □

Remark 4.5. Theorem 4.1 and Proposition 4.4 together with results in [38] imply that $\big(L_{A,b}, C_0^\infty(\Omega)\big)$ generates a strong Markov diffusion process on Ω, which can start from every point $x \in \Omega$ in a natural way. This will be the subject of a forthcoming paper.

Let us briefly comment the uniqueness problem for (1.3) in the case $\Omega = \mathbb{R}^d$. W. Stannat has recently shown by simple examples that even if $A = I$ and b is infinitely differentiable on $\mathbb{R}^d$ with $d \geq 2$, there might be many probability measures satisfying (1.3) (cf. [11]). On the other hand, answering a question posed by Varadhan [44], it has been shown in [2] that the uniqueness statement is true for probability measures under certain additional assumptions. The proof in [2] is essentially based on the regularity result from [8] and Harnack's inequality. The results from [2] can be extended to the non-constant diffusion term case by using the regularity results from this work. In particular, Corollary 4.3 is one of such extensions. We note, however, that the uniqueness of an invariant probability in that corollary is a priori not the same as the uniqueness of a probability satisfying the elliptic equation (1.3). We address the uniqueness problem for the elliptic equation (1.3) in [11], where, in particular, we clarify these two notions of uniqueness completely.

References

[1] Adams, R.A., Sobolev Spaces. Academic, New York, 1975.

[2] Albeverio, S., Bogachev, V. and Röckner, M., On uniqueness of invariant measures for finite- and infinite-dimensional diffusions. *Comm. Pure Appl. Math.*, 52 (1999), 325–362.

[3] Aronson, D.G., and J. Serrin, J., Local behavior of solutions of quasilinear parabolic equations. *Arch. Rational Mech. Anal.*, 25 (1967), 81–122.

[4] Bauman, P., Positive solutions of elliptic equations in nondivergence form and their adjoints. *Ark. Mat.*, 22 (1984), 153–173.

[5] Bensoussan, A., Perturbation Methods in Optimal Control. Wiley/Gauthier-Villars, 1988.

[6] Besov, O.V., Il'in, V.P. and Nikol'skii, S.M., Integral Representations of Functions and Imbedding Theorems. Moscow, Nauka, 1975 (in Russian); English transl.: V.H. Winston and Sons, Washington, D.C., vol. I, 1978; vol. II, 1979.

[7] Bogachev, V.I., Krylov, N.V., andRöckner, M., Regularity of invariant measures: the case of non-constant diffusion part. *J. Funct. Anal.*, 138 (1996), 223–242.

[8] Bogachev, V.I., Krylov, N.V. and Röckner, M., Elliptic regularity and essential self-adjointness of Dirichlet operators. *Annali Scuola Norm. Super. di Pisa Cl. Sci.*, (4) 24 (1997), no. 3, 451–461.

[9] Bogachev, V.I. and Röckner, M., Regularity of invariant measures on finite and infinite dimensional spaces and applications. *J. Funct. Anal.*, 133 (1995), 168–223.

[10] Bogachev, V.I. and Röckner, M., A generalization of Hasminskii's theorem on existence of invariant measures for locally integrable drifts. SFB 343 Preprint, Bielefeld Univ., N 98–072 (1998); to appear in *Theory Probab. Appl.*

[11] Bogachev, V.I., Röckner, M. and Stannat, W., Uniqueness of invariant measures and maximal dissipativity of diffusion operators on L^1. Preprint (1999).

[12] Cattiaux, P. and Léonard, C., Minimization of the Kullback information of diffusion processes. *Ann. Inst. H. Poincaré*, 30 (1994), no. 1, 83–132; Correction: ibid., 31 (1995), no. 4, 705–707.

[13] Escauriaza, L., Bounds for the fundamental solution of elliptic and parabolic equations in nondivergence form. *Comm. Partial Diff. Equations* (to appear).

[14] Fabes, E.B. and Kenig, C.E., Examples of singular parabolic measures and singular transition probability densities. *Duke Math. J.*, 48 (1981), 845–856.
[15] Giaquinta, M., Introduction to Regularity Theory for Nonlinear Elliptic Systems. Birkhäuser Verlag, Basel – Boston – Berlin, 1993.
[16] Gilbarg, D., and Trudinger, N.S., Elliptic Partial Differential Equations of Second Order. Springer, Berlin – New York, 1977.
[17] Ivanov, A.V., The Harnack inequality for generalized solutions of second order quasilinear parabolic equations. *Trudy Mat. Inst. Steklov*, 102 (1967), 51–84 (in Russian).
[18] Knerr, B., Parabolic interior Schauder estimates by the maximum principle. *Arch. Rational Mech. Anal.*, 75 (1980), 51–58.
[19] Krylov, N.V., An inequality in the theory of stochastic integrals. *Theory Probab. Appl.*, 16 (1971), no. 3, 438–448.
[20] Krylov, N.V., Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation. *Sibirskii Mat. Zhurn.*, 17 (1976), no. 2, 290–303 (in Russian); English transl.: *Siberian J. Math.*, 17 (1976), n 2, 226–236.
[21] Krylov, N.V., Controlled Diffusion Processes. Moscow, Nauka, 1977 (in Russian); English transl.: Springer–Verlag, 1980.
[22] Krylov, N.V., Nonlinear Elliptic and Parabolic Equations of Second Order. Nauka, Moscow, 1985 (in Russian); English transl.: Reidel, Dordrecht, 1987.
[23] Krylov, N.V., An approach to controlled diffusion processes. *Theory Probab. Appl.*, 31 (1985), no. 4, 605–626.
[24] Krylov, N.V., Lectures on Elliptic and Parabolic Equations in Hölder Spaces. Amer. Math. Soc., Rhode Island, 1996.
[25] Krylov, N.V., An analytic approach to SPDEs, in: *"Stochastic partial differential equations: six perspectives"*. R. Carmona and B. Rozovskii, eds., pp. 185–241; Amer. Math. Soc., Rhode Island, Providence, 1999.
[26] Krylov, N.V., Some properties of traces for stochastic and deterministic parabolic weighted Sobolev spaces. Preprint 99–091 SFB 343 (http://www.uni-bielefeld.de/homedir/index.html); submitted to *J. Funct. Anal.*
[27] Ladyz'enskaya, O.A. and Ural'tseva, N.N., Linear and Quasilinear Elliptic Equations. New York, Academic Press, 1968.
[28] Ladyz'enskaya, O.A., Solonnikov, V.A. and Ural'tseva, N.N., Linear and Quasilinear Equations of Parabolic Type. Amer. Math. Soc., Rhode Island, 1968.
[29] Lieberman, G.M., Intermediate Schauder theory for second order parabolic equations IV: time irregularity and regularity. *Diff. and Integr. Equations*, 5 (1992), no. 6, 1219–1236.
[30] Lieberman, G.M., Second Order Parabolic Differential Equations. World Sci., 1996.
[31] Ma, Z.-M. and Röckner, M., Introduction to the Theory of (Non-symmetric) Dirichlet Forms. Springer, Berlin – New York, 1992.
[32] Morrey, C.B., Multiple Integrals in the Calculus of Variations. Springer–Verlag, Berlin – Heidelberg – New York, 1966.
[33] Neveu, J., Bases Mathématiques du Calcul des Probabilités. Masson et Cie, Paris, 1964.

[34] Portenko, N.I., Generalized Diffusion Processes. Moscow, Nauka, 1982 (in Russian); English tranls.: American Mathematical Society, Providence, Rhode Island, 1990.

[35] Safonov, M.V., An example of a diffusion process with the singular distribution at a fixed moment, in: *Abstracts Comm. Third Intern. Vilnius Conf. on Probab. Theory and Math. Statistics*, 1981, v. II, pp. 133–134.

[36] Semenov, Yu.A., Smoothness of generalized solutions of the equation $(\lambda - \sum_{i,j} \nabla_i a_{ij} \nabla_j)u = f$ with continuous coefficients. *Mat. Sbornik*, 118 (1982), no. 3, 399–410 (in Russian); English transl.: *Math. USSR Sbornik*, 46 (1983), no. 3, 403–415.

[37] Stampacchia, G., Équations Elliptiques du Second Ordre à Coefficients Discontinus. Les Presses de l'Université de Montréal, 1966.

[38] Stannat, W., (Nonsymmetric) Dirichlet operators on L^1: existence, uniqueness and associated Markov processes. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, (4) 28 (1999), no. 1, 99–140.

[39] Stroock, D.W. and Varadhan, S.R.S., Multidimensional Diffusion Processes. Springer-Verlag, Berlin – New York, 1979.

[40] Troianiello, G.M., Elliptic Differential Equations and Obstacle Problems. Plenum Press, New York, 1987.

[41] Trudinger, N.S., Pointwise estimates and quasilinear parabolic equations. *Comm. Pure Appl. Math.*, 21 (1968), 205–226.

[42] Trudinger, N.S., Linear elliptic operators with measurable coefficients. *Ann. Scuola Normale Super. Pisa*, (3) 27 (1973), 265–308.

[43] Trudinger, N.S., Maximum principles for linear, non-uniformly elliptic operators with measurable coefficients. *Math. Z.*, 156 (1977), 291–301.

[44] Varadhan, S.R.S., Lectures on Diffusion Problems and Partial Differential Equations. Tata Institute of Fundamental Research, Bombay, 1980.

Department of Mechanics and Mathematics, Moscow State University, 119899 Moscow, Russia
E-mail address: `vbogach@mech.math.msu.su`

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: `krylov@math.umn.edu`

Fakultät für Mathematik, Universität Bielefeld, D–33501 Bielefeld, Germany
E-mail address: `roeckner@mathematik.uni-bielefeld.de`