STOCHASTIC CALCULUS OF GENERALIZED DIRICHLET FORMS AND APPLICATIONS

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Contents

Introduction

One of the main purposes of this work is to analyze stochastic processes which are associated to a certain class of (in general non coercive) bilinear forms, also called generalized Dirichlet forms. The bilinear forms themselves are associated to differential operators of at most second order with measurable coefficients in finite or infinitely many variables. In general they are not elliptic and not symmetric. As in the classical case functions (in the domain of the form) observed along the paths of the associated processes will be characterized as a sum of a martingale and a zero quadratic variation part or as a martingale and a zero energy term and are often called Dirichlet processes. The theory of symmetric Dirichlet forms (cf. [8]) has proved to be quite effective for the study of such processes and so robust that even non symmetric cases (cf. [16], [15], [9]) and infinite dimensional ones (cf. [15], [3]) can be covered. An inconvenience of these generalizations of [8] is that a sector condition for the bilinear form has to be assumed. Extending the work [23] on generalized Dirichlet forms, which contains as a main result the construction of an associated strong Markov process with nice sample path properties, we develop here a detailed analysis for the study of these processes without the rather restrictive sector condition. This analysis goes far beyond what had been achieved in [23] and reveals in addition the probabilistic counterpart of generalized Dirichlet forms as far as the calculus of the associated process is concerned. We give, in particular, purely analytic conditions in terms of the bilinear form to see when the associated processes are stable (in the class of Dirichlet processes) under composition with C^1-functions and develop a change of variables rule for the martingale part leading to an extension of Itô's formula. We also present a new localization procedure which seems also to be new for the classical theories. This localization is independent of the topology induced by the sample paths and only depends on an analytic property of the form.

To give now a detailed overview about this work and about how we proceed let us first explain (following [23]) what a generalized Dirichlet form is. Right after this we first describe the theoretical part of our results and then present applications.

Let $(E, \mathcal{B}, m)$ be a σ-finite measure space and let $(\mathcal{A}, \mathcal{V})$ be a coercive closed form on the corresponding L^2-space $\mathcal{H} := L^2(E; m)$. Let $(\Lambda, D(\Lambda))$ be a linear operator on $\mathcal{H}$ such that

(i) $(\Lambda, D(\Lambda))$ generates a C_0-semigroup of contractions $(U_t)_{t\geq 0}$.
(ii) $(U_t)_{t\geq 0}$ can be restricted to a C_0-semigroup on $\mathcal{V}$.

Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we obtain that $\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}' \hookrightarrow \mathcal{V}'$ densely and continuously. Let $(\Lambda, \mathcal{F})$ be the closure of $\Lambda : D(\Lambda) \cap \mathcal{V} \to \mathcal{V}'$ and $(\widehat{\Lambda}, \widehat{\mathcal{F}})$ the dual operator. Then define

$$\mathcal{E}(u, v) := \begin{cases} \mathcal{A}(u, v) - \langle \Lambda u, v \rangle & \text{for } u \in \mathcal{F},\ v \in \mathcal{V} \\ \mathcal{A}(u, v) - \langle \widehat{\Lambda} v, u \rangle & \text{for } u \in \mathcal{V},\ v \in \widehat{\mathcal{F}} \end{cases}$$

and $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_\mathcal{H}$ for $\alpha > 0$. Here, $\langle\cdot,\cdot\rangle$ denotes the dualization between $\mathcal{V}'$ and $\mathcal{V}$ and $\langle\cdot,\cdot\rangle$ coincides with the inner product $(\cdot,\cdot)_\mathcal{H}$ in $\mathcal{H}$ when restricted to $\mathcal{H}\times\mathcal{V}$. $\mathcal{E}$ is called a generalized Dirichlet form if the associated resolvent $(G_\alpha)_{\alpha>0}$ is sub-Markovian. Hence the class of generalized Dirichlet forms contains in particular symmetric and coercive Dirichlet forms (choose $\Lambda = 0$) (cf. [8], [15]) and also time dependent Dirichlet forms (choose $\Lambda = \frac{\partial}{\partial t}$ cf. section 4.1 **a)**) as in [17]. In contrast to the classical theory (i.e. $\Lambda = 0$) it is not known whether regularity or quasi-regularity alone implies the existence of an associated process. Here quasi-regularity of a generalized Dirichlet form is defined similarly to [15] (cf. [23] or Definition 1.3 below). An additional structural assumption on $\mathcal{F}$ is made in [23, $IV.2, D3$] (i.e. the existence of a nice intermediate space $\mathcal{Y}$ has to be assumed) in order to construct explicitly an associated m-tight special standard process $\mathbf{M}$. Since we do not make use of this technical assumption and since it may be subject to some further progress, we instead prefer to assume merely the existence of $\mathbf{M}$ whenever this is necessary in the theoretical part of this work.

Our first problem is to obtain an appropriate analytic description of $\mathcal{E}$-exceptional sets, i.e. sets which are not hit by the associated process. For $\mathcal{C},\mathcal{D}\subset\mathcal{H}$ let $\mathcal{C}_\mathcal{D} := \{u\in\mathcal{C} \,|\, \exists v\in \mathcal{D},\ u\le v\}$ and let $\mathcal{P}$ (resp. $\widehat{\mathcal{P}}$) denote the 1-excessive (resp. 1-coexcessive) elements in $\mathcal{V}$. Using the integration theory of Daniell and Stone we first show (cf. Theorem 1.4 below) that any $\hat{u}\in\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$ is associated to a unique positive measure $\mu_{\hat{u}}$ on $(E,\mathcal{B})$ charging no $\mathcal{E}$-exceptional set, via the relation

$$\int \widetilde{f}\, d\mu_{\hat{u}} = \lim_{\alpha\to\infty} \mathcal{E}_1(f, \alpha\widehat{G}_{\alpha+1}\hat{u}) \quad \forall \widetilde{f}\in\widetilde{\mathcal{P}}_\mathcal{F} - \widetilde{\mathcal{P}}_\mathcal{F}. \tag{1}$$

Here $\widetilde{\mathcal{D}}$ for $\mathcal{D}\subset\mathcal{H}$ stands for the totality of $\mathcal{E}$-quasi-continuous (:=$\mathcal{E}$-q.c. cf. the paragraph right before Definition 1.3 for the meaning of this) m-versions of elements in $\mathcal{D}$ (conversely $\mathcal{D}$ for $\widetilde{\mathcal{D}}\subset\widetilde{\mathcal{H}}$ stands for the totality of m-classes represented by elements in $\widetilde{\mathcal{D}}$) and $(\widehat{G}_\alpha)_{\alpha>0}$ is the coresolvent associated to $\mathcal{E}$. Note that we can not assume that $\mathcal{F}$ is stable under normal contractions (e.g. let $\mathcal{A}$ be zero and $\mathcal{F}$ be the domain of a second order differential operator). Hence there is no naturally given vector lattice structure in $\mathcal{F}$ in contrast to the classical case where $\mathcal{F}$ is a Sobolev space of order one or an abstract analogue. We compensate this lack of structure by introducing the vector lattice $\widetilde{\mathcal{P}}_\mathcal{F} - \widetilde{\mathcal{P}}_\mathcal{F}$ which goes slightly out of $\mathcal{F}$. Therefore we have to consider a limit in (1). The $\mathcal{E}$-exceptional sets are then exactly those sets which are annihilated by every element in the following class of measures

$$\widehat{S}_{00} := \{\mu_{\hat{u}} \mid\ \hat{u}\in\widehat{\mathcal{P}}_{\widehat{G}_1\mathcal{H}_b^+} \text{and } \mu_{\hat{u}}(E) < \infty\} \tag{2}$$

where for a linear operator G on $\mathcal{H}$, $\mathcal{D}\subset\mathcal{H}$, $G\mathcal{D} := \{Gh \mid h\in\mathcal{D}\}$ and $\mathcal{H}_b^+$ denotes the positive and bounded elements in $\mathcal{H}$. Note that although our strategy to obtain a description of $\mathcal{E}$-exceptional sets is similar to the one in [8], the proofs (cf. section 1.2) of the statements corresponding to (1) and (2) turn out to have nothing more in common with the related ones in [8]. In the symmetric case our class of measures $\widehat{S}_{00}$ is smaller than the corresponding one in [8, *p*.78]. A consequence of this is that the uniform convergence in Lemma 2.7 can be determined (cf. Remark 2.9) w.r.t. a weaker semi-norm than in [8, *Lemma* 5.1.2.].

In section 1.3 we analyze the structure of the space S of smooth measures, i.e. positive measures μ on $(E,\mathcal{B})$ which do not charge $\mathcal{E}$-exceptional sets and for which there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact subsets of E such that $\mu(F_k)<\infty$ for any k. Our main theorem here is the following: given $\mu\in S$, then we can show the existence of an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ such that $1_{F_k^{reg}}\cdot\mu\in\widehat{S}_{00}$ for any $k\in\mathbb{N}$ (cf. Theorem 1.21 below). Here $1_{F_k^{reg}}\cdot\mu$ denotes the restriction of μ to the regular points F_k^{reg} of F_k. The proof of Theorem 1.21 is completely different from the corresponding one in [8]. Our method is, in particular, based on properties of the associated strong Markov process. Since $F_k\setminus F_k^{reg}$ is semi-polar, we have to emphasize here that from the potential theoretic point of view there is an important difference between our situation and the symmetric case: semi-polar sets are not polar in general. Semi-polar sets are also not $\mathcal{E}$-exceptional as we can see from Remark 1.19. But since $\bigcap_{k\ge1}(E\setminus F_k^{reg})$ is $\mathcal{E}$-exceptional by Lemma 1.18 we can see that any $\mu\in S$ is "approximated" by $1_{F_k^{reg}}\cdot\mu$, $k\in\mathbb{N}$. This is what we really need for later purposes as e.g. the still open proof of the Revuz correspondence (cf. the next paragraph where a coversion of Theorem 1.21 is used).

In section 2.1 following [20] we associate to every positive continuous additive functional (PCAF) A of the associated process $\mathbf{M}=(\Omega,(\mathcal{F}_t)_{t\ge0},(Y_t)_{t\ge0},(P_z)_{z\in E_\Delta})$ (cf. section 2.1, section 5, for the exact definition) its corresponding Revuz measure μ_A (by abuse of notation the expressions μ_A, $\mu_{\hat{u}}$ (cf. (1)), are used simultaneously). Theorem 2.1 characterizes μ_A as the unique positive measure on $(E,\mathcal{B})$, charging no $\mathcal{E}$-exceptional set such that

$$\int_E f\,d\mu_A \;=\; \lim_{\alpha\to\infty}\alpha E_m\Big[\int_0^\infty e^{-\alpha t}f(Y_t)\,dA_t\Big]\quad\text{for all f }\in\mathcal{B}^+. \tag{3}$$

Here E_m (resp. E_z) denotes the expectation w.r.t. $P_m(\cdot):=\int_E P_z(\cdot)\,m(dz)$ (resp. P_z). In particular we obtain that μ_A is a smooth measure. In Theorem 2.5 we show similarly to [8] that the correspondence $A\mapsto\mu_A$ is injective. The bijectivity of this mapping, also known as the Revuz correspondence, is still an open question but we note that according to our results in section 1.3 it only remains to show the existence of a PCAF $A^{(k)}$ such that $\overline{U_1(1_{F_k^{\widehat{reg}}}\cdot\mu)}(z)=E_z\Big[\int_0^\infty e^{-s}dA_s^{(k)}\Big]$ for $\mathcal{E}$-quasi-every (:=$\mathcal{E}$-q.e. which means up to an $\mathcal{E}$-exceptional set and $F_k^{\widehat{reg}}$ are the coregular points of F_k) $z\in E$ and any $k\in\mathbb{N}$. Here $\overline{U_1(1_{F_k^{\widehat{reg}}}\cdot\mu)}$ is some $\mathcal{E}$-quasi-lowersemicontinuous m-version (cf. the paragraph right before Lemma 1.7 for the meaning of this) of the 1-excessive function $U_1(1_{F_k^{\widehat{reg}}}\cdot\mu)$ corresponding to the measure $1_{F_k^{\widehat{reg}}}\cdot\mu\in S_{00}$ and S_{00} is similarly defined as $\widehat{S}_{00}$, but in terms of the coresolvent.

Section 2.2 is devoted to the Fukushima-decomposition of additive functionals (AF's cf.

section 2.1 for the definition) and its extended version to functions not necessarily in the domain $\mathcal{F}$ of the generalized Dirichlet form. As usual in non-symmetric cases the energy of an AF A is defined by

$$e(A) \;=\; \frac{1}{2}\lim_{\alpha\to\infty}\alpha^2 E_m\Big[\int_0^\infty e^{-\alpha t}A_t^2\,dt\Big],$$

whenever this limit exists in $[0,\infty]$. We set $\overline{e}(A)$ for the same expression, but with $\overline{\lim}$ instead of lim. Let $\widetilde{u}\in\widetilde{\mathcal{H}}$ with m-version u (in the sequel $\widetilde{u}$, if it exists, will always denote an $\mathcal{E}$-q.c. m-version of a function u, converseley u will always denote the m-class represented by $\widetilde{u}$). The additive functional $A^{[u]}:=(\widetilde{u}(Y_t)-\widetilde{u}(Y_0))_{t\geq 0}$ is independent of the choice of $\widetilde{u}$ (i.e. defines the same equivalence class of AF's for any $\mathcal{E}$-q.c. m-version $\widetilde{u}$ of u). Let $(\widehat{G}_\alpha)_{\alpha>0}$ be sub-Markovian. We then have

$$\overline{e}(A^{[u]})\leq\overline{\lim_{\alpha\to\infty}}\;\alpha(u-\alpha G_\alpha u,u)_{\mathcal{H}},$$

and by Lemma 2.6 we know that the last term is dominated by $(K+1)^2\|u\|_{\mathcal{F}}^2$ whenever $u\in\mathcal{F}$. Here $\|\cdot\|_{\mathcal{F}}$ denotes the graph norm corresponding to Λ and K is the sector constant of $\mathcal{A}$. The martingale additive functionals of finite energy $\overset{\circ}{\mathcal{M}}$ and the continuous additive functionals of zero energy $\mathcal{N}_c$ are defined as usual (cf. section 2.2). In Theorem 2.11(i) we show that the AF $A^{[u]}$, $u\in\mathcal{F}$, can uniquely be decomposed as

$$A^{[u]} \;=\; M^{[u]}+N^{[u]},\quad M^{[u]}\in\overset{\circ}{\mathcal{M}},\quad N^{[u]}\in\mathcal{N}_c. \tag{4}$$

The identity (4) means that both sides are equivalent as additive functionals and reduces to the well-known Fukushima decomposition in the case $\Lambda=0$. After all our preparations the proof of (4) is similar to the corresponding one in [8]. Note that for the proof we only assume the quasi-regularity of $\mathcal{E}$, the existence of an associated process and the sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$. No dual process is needed. Let us define the following linear space

$$\widetilde{\mathcal{H}}^{dec}:=\{\widetilde{u}\in\widetilde{\mathcal{H}}\mid\exists M^{[u]}\in\overset{\circ}{\mathcal{M}},N^{[u]}\in\mathcal{N}_c\text{ such that }A^{[u]}=M^{[u]}+N^{[u]}\}.$$

If $\widetilde{u}_n$, $\widetilde{u}\in\widetilde{\mathcal{H}}$, $n\in\mathbb{N}$, satisfy the conditions of Lemma 2.7 and $\widetilde{u}_n\in\widetilde{\mathcal{H}}^{dec}$, $n\in\mathbb{N}$, are such that $\overline{e}(A^{[u_n-u]})\underset{n\to\infty}{\longrightarrow}0$, we show in Theorem 2.11(ii) that then $\widetilde{u}\in\widetilde{\mathcal{H}}^{dec}$. This procedure to check whether for a given $\widetilde{u}\in\widetilde{\mathcal{H}}$ decomposition (4) extends to $A^{[u]}$ shows to be very practical and will be used quite often in the sequel and especially in the examples.

In section 2.3 we assume that the coform $\widehat{\mathcal{E}}$ (cf. the paragraph before Lemma 1.20 for the definition) is also quasi-regular and that there exists a coassociated process $\widehat{\mathbf{M}}$. We also assume that every $M^{[G_1h]}$, $h\in\mathcal{H}$, is continuous as an additive functional. We then show the following: if $\widetilde{v}=(\widetilde{v}_1,...,\widetilde{v}_n)$ is a fixed n-tuple of m-essentially bounded elements in $\widetilde{\mathcal{H}}^{dec}$ such that $\Phi(\widetilde{v})\in\widetilde{\mathcal{H}}^{dec}$ for all $\Phi\in C^1(\mathbb{R}^n)$ with $\Phi(0)=0$ and such that there exists $(v_{ik})_{k\in\mathbb{N}} \subset \mathcal{H}$ with $\overline{e}(A^{[v_i-G_1v_{ik}]}) \to 0$, as $k\to\infty$, $1\le i\le n$, then we have the chain rule for the energy measure $\mu_{<M^{[\Phi(v)]}>}$ associated via (3) to the quadratic variation $< M^{[\Phi(v)]} >$ of $M^{[\Phi(v)]}$. This is equivalent to the martingale transformation

$$M^{[\Phi(v_1,\dots,v_n)]} = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(\widetilde{v}_1,\dots,\widetilde{v}_n) \bullet M^{[v_i]} \tag{5}$$

where $\frac{\partial\Phi}{\partial x_i}(\widetilde{v}) \bullet M^{[v_i]}$, $1\le i\le n$, is a version of the usual stochastic integral of the continuous process $\frac{\partial\Phi}{\partial x_i}(\widetilde{v}(Y))$ w.r.t. to the continuous martingale $M^{[v_i]}$ (cf. Lemma 2.15(i)). As a simple example consider sectorial Dirichlet forms as in [8], [16], [15], [3], where (5) is satisfied for any n-tuple $\widetilde{v}=(\widetilde{v}_1,...,\widetilde{v}_n)$ of $\mathcal{E}$-q.c. m-essentially bounded elements in the domain of the form. More complex examples are given in section 4 (cf. e.g. p.75).

In section 3 **a)** as a new input we give a condition easy to check in examples (as an application cf. e.g. Proposition 4.2), whether the associated process is a diffusion (i.e. has continuous sample paths) up to his life time. According to Theorem 3.2 this is the case whenever for any $U\subset E$, U open, there exists $(u_n)_{n\in\mathbb{N}}\subset\mathcal{F}^+$, $u_n\le u_{n+1}$, $supp(u_n)\subset U$, $0\le\sup_{n\in\mathbb{N}} u_n\le 1_U$, $\sup_{n\in\mathbb{N}}\widetilde{u}_n>0$ $\mathcal{E}$-q.e. on U and

$$\mathcal{E}(u_n,v)=0 \text{ for all } v\in\{\widehat{U}_1\mu-(\widehat{U}_1\mu)_U \mid \mu\in\widehat{S}_{00}\} \text{ and } n\in\mathbb{N}. \tag{6}$$

Here $\widehat{U}_1\mu$ is the 1-coexcessive function associated to $\mu\in\widehat{S}_{00}$ by (1) and $(\widehat{U}_1\mu)_U$ is the smallest 1-coexcessive function dominating $\widehat{U}_1\mu\cdot 1_U$. We remark that in the case of quasi-regular Dirichlet forms in the sense of [15] the existence of a sequence $(u_n)_{n\in\mathbb{N}}$ with all the above mentioned properties except (6) is always guaranteed (cf. [15, Proposition V.1.7.]). One only has to check (6) for given U, $(u_n)_{n\in\mathbb{N}}$ in this case.
In part **b)** of section 3 supplementary to quasi-regularity, existence of an associated process and sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ we assume that $\mathcal{E}$ satisfies the following assumptions

Alg There exists a linear space $\mathcal{Y}\subset L^\infty(E;m)\cap\mathcal{V}^{\mathcal{F}}\cap\mathcal{H}^{dec}$ of bounded functions such that $u\cdot v\in\mathcal{Y}$ for $u,v\in\mathcal{Y}$.

Diag There exists constants $C,\gamma\ge 0$ such that $\overline{e}(A^{[u]})\le C\mathcal{A}_\gamma(u,u)$ for any $u\in\mathcal{Y}$.

Diag' There exists constants $C,\gamma\ge0$, and a Dirichlet form $(A,D(A))$ on $\mathcal{H}$ such that $\mathcal{Y}\subset D(A)\cap\mathcal{F}$, $\overline{e}(A^{[u]})\le CA_\gamma(u,u)$, $A(u,u)\le\mathcal{E}(u,u)$, for any $u\in\mathcal{Y}$.

Here $\mathcal{V}^{\mathcal{F}} := \{h\in\mathcal{H}\mid \sup_{\alpha>0}\alpha(h-\alpha G_\alpha h,h)_{\mathcal{H}}<\infty\}$. Note that **Alg** and **Diag** (or **Alg** and **Diag'**) are satisfied in nearly all examples below and in particular for sectorial Dirichlet forms where $\mathcal{Y}$ coincides with the m-essentially bounded elements in the domain of the form (cf. Remark 3.7). We then show that (4) automatically extends to $A^{[\Phi(v_1,\dots,v_n)]}$ where Φ is as above and $v_1,\dots,v_n\in\mathcal{Y}$. In particular, if $\mathcal{Y}=C_0^k(V)$ is a dense subset of $\mathcal{F}$ (here $E=V$ is an open subset of $\mathbb{R}^d$ and $C_0^k(V)$ denotes the space of all k times continuously differentiable functions on V with compact support) for some $k\in\mathbb{N}\cup\{\infty\}$, by

Lemma 3.4(ii) (4) extends to $A^{[\Phi(v_1,\dots,v_n)]}$ where now $v_1,\dots,v_n$ are m-essentially bounded elements of $\mathcal{F}$.
In part **c)** of section 3 we present a new localization procedure. This localization is based on a property of the energy measure (cf. Lemma 3.5(iii)) which is shown here for the first time and is given as follows: if $\widetilde{u}\in\widetilde{\mathcal{L}}:=\{\widetilde{u}\in\widetilde{\mathcal{V}}^{\mathcal{F}}\mid\exists(h_n)_{n\in\mathbb{N}}\subset\mathcal{H}$ with $\overline{e}(A^{[u-G_1h_n]})\to 0$ as $n\to\infty$, and $\exists C,\gamma\geq 0$ such that $\forall\xi\in C_b^\infty(\mathbb{R})$ with $\xi(0)=0$ we have $\widetilde{u},\xi(\widetilde{u})\in\widetilde{\mathcal{H}}^{dec}$ and $e(M^{[\xi(u)]})\leq C\mathcal{A}_\gamma(\xi(u),\xi(u))\}$ or $\widetilde{u}\in\widetilde{\mathcal{L}}'$ (see Lemma 3.5(iii) for the definition) is constant $\mu_{<M^{[u]}>}$-a.s. on $B\in\mathcal{B}$ then

$$\mu_{<M^{[u]}>}(B)=0. \tag{7}$$

Here $C_b^\infty(\mathbb{R})$ denotes the space of all infinitely many continuously differentiable functions on $\mathbb{R}$ with bounded derivatives of any order. Note that "constant $\mathcal{E}$-q.e. on B" implies "constant $\mu_{<M^{[u]}>}$-a.s on B". In particular, if $\mu_{<M^{[u]}>}$ is absolutely continuous w.r.t. the reference measure m "constant m-a.s on B" implies "constant $\mu_{<M^{[u]}>}$-a.s. on B". Note furthermore that in the case of sectorial Dirichlet forms as in [8],[16], [15], the m-classes of $\widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{L}}'$) contain the domain of the Dirichlet form. Since also in our situation (cf. Lemma 3.5(iv), Theorem 3.6(ii) and (iii)) $\widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{L}}'$) is large enough, (7) provides the possibility to localize w.r.t. closed subsets of E, especially w.r.t. some fixed $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact subsets of E. Concerning the localization our assumptions correspond to the assumption which is known to be equivalent to the strong local property in the finite dimensional symmetric case (cf. assumptions **Dif**, $\mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}$, **Plt**, of section 3 **c)**).

In addition to the new theoretical results described above we also present new applications. For example, we present here for the first time a martingale transformation for time dependent Dirichlet forms assuming only that $C_0^k(V)$ is a dense subset of $\mathcal{F}$ for some $k\geq 1$ (provided that the usual assumptions hold, e.g. that the associated process is a conservative diffusion cf. section 4.1 **a)**).
In section 4.1 **c)** we show that the Fukushima decomposition holds for the process determined by an extension of the differential operator

$$Lu(t,x):=\sum_{i,j=1}^d\frac{\partial}{\partial x_j}\frac{\partial u}{\partial x_i}(t,x)+\sum_{i=1}^d b_i(t,x)\frac{\partial u}{\partial x_i}(t,x)+\frac{\partial u}{\partial t}(t,x);u\in C_0^\infty(\mathbb{R}^{d+1}).$$

Here $d\geq 1$, $b_i\in L^1_{\text{loc}}(\mathbb{R}\times\mathbb{R}^d,dt\otimes dx)$, $1\leq i\leq d$, $b_i(t,\cdot)\in L^1_{\text{loc}}(\mathbb{R}^d,dx)$ for all $t\in\mathbb{R}$, $1\leq i\leq d$, $B:=(b_1,\dots,b_d)$ satisfies $\int_{\mathbf{R}^{d+1}}\langle B,\nabla u\rangle\,dtdx\leq 0$ for all $u\in C_0^\infty(\mathbb{R}^{d+1}),u\geq 0$, and we assume that there exists a function $L=L_2+L_\infty$ with $L_2\in\mathcal{B}^+(\mathbb{R})\cap L^2(\mathbb{R},dt)$ and $L_\infty\in\mathcal{B}^+(\mathbb{R})\cap L^\infty(\mathbb{R},dt)$ and a constant $\mathbf{M}$ such that

$$|B(t,x)-B(t,y)|\leq L(t)|x-y|_{\mathbf{R}^d}\text{ for all } x,y\in\mathbb{R}^d,t\in\mathbb{R}$$

$$\text{and }\ |B(t,x)|\leq M(|(t,x)|_{\mathbf{R}^{d+1}}+1),\ x\in\mathbb{R}^d,\ t\in\mathbb{R}.$$

In this case since **Alg** is satisfied with $\mathcal{Y}=C_0^\infty(\mathbb{R}^{d+1})$ for the bilinear form corresponding to L we also know that Fukushima's decomposition extends to $A^{[\Phi(v_1,\dots,v_n)]}$ where

$v_1, \ldots, v_n$ are in $\mathcal{Y}$. But unless we do not make some supplementary assumption on the divergence of the vector field B we even cannot show the existence of a coprocess.
In section 4.1 **d)** we consider a positive measure $\mu = \varphi^2\, dx$ on $\mathcal{B}(\mathbb{R}^d)$ such that $\varphi \in H^{1,2}_{loc}(\mathbb{R}^d)$, i.e. $\varphi \cdot \chi \in H^{1,2}_0(\mathbb{R}^d)$ for all $\chi \in C_0^\infty(\mathbb{R}^d)$ and such that $supp(\mu) \equiv \mathbb{R}^d$. Then the bilinear form given by

$$\mathcal{E}^0(u,v) = \frac{1}{2}\int \langle \nabla u, \nabla v\rangle\, d\mu \ ; \ u, v \in C_0^\infty(\mathbb{R}^d)$$

is closable on $L^2(\mathbb{R}^d, \mu)$ (cf. [15] for the definition of closability). Let $B \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^d, \mu)$, i.e. $B = (B_1, \ldots, B_d) : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and $\int_V \langle B, B\rangle\, d\mu < \infty$ for all V relatively compact in $\mathbb{R}^d$, and such that

$$\int \langle \frac{1}{2}B - \frac{\nabla\varphi}{\varphi}, \nabla u\rangle\, d\mu = 0 \text{ for all } u \in C_0^\infty(\mathbb{R}^d).$$

Let us further assume that the semi-group corresponding to a suitable extension of $Lu := \frac{1}{2}\Delta u + \langle \frac{1}{2}B, \nabla u\rangle$, $u \in C_0^\infty(\mathbb{R}^d)$, is conservative. This is for instance the case if there exists a positive constant C such that $\langle B(x), x\rangle \le C(|x|^2_{\mathbb{R}^d} ln(|x|^2_{\mathbb{R}^d} + 1) + 1)$ for all x in $\mathbb{R}^d$. Then using our localization procedure and results of [24] on the existence of an associated process we can contruct explicitly a conservative diffusion $\mathbf{M} = \Big(\Omega, \mathcal{F}_\infty, (Y_t)_{t\ge 0}, (P_z)_{z\in\mathbf{R}^d_\Delta}\Big)$ which satisfies

$$Y_t - Y_0 = W_t + \int_0^t \frac{1}{2}B(Y_s)\, ds \tag{8}$$

where $W_t = (W_t^1, ..., W_t^d)$ is the standard Brownian motion on $\mathbb{R}^d$ and (8) means that both sides are equivalent as additive functionals. This generalizes a result on the distorted Brownian motion obtained by Fukushima (cf. [7] where (8) was only shown for $B = 2\frac{\nabla\varphi}{\varphi}$). Using again our localization procedure we obtain the following extension of Itô's formula

$$\Phi(X_t) - \Phi(X_0) = \sum_{i=1}^d \int_0^t \Phi_{x_i}(X_s)\, dM_s^{[u_i]} + \int_0^t \langle \frac{1}{2}B, \nabla\Phi\rangle(X_s)\, ds + \lim_{n\to\infty}\int_0^t \frac{1}{2}\Delta p_n^k(X_s)\, ds,$$

for all $t \le \sigma_k$, $k \in \mathbb{N}$. Here $(p_n^k)_{k,n\in\mathbf{N}}$ are polynomials specified as at the end of section 4.1 **d)**, $\Phi \in C^1(\mathbb{R}^d)$, and σ_k is the first hitting time of the complement of the closed ball in $\mathbb{R}^d$ with radius k.
We also consider the infinite dimensional analogon of the preceding example, i.e. in section 4.2 we construct weak solutions to stochastic differential equations in infinite dimensions of the type

$$dY_t \ = \ dW_t + \frac{1}{2}\beta_H^\mu(Y_t)dt + \overline{\beta}(Y_t)dt \ , \ Y_0 = z. \tag{9}$$

Here $(Y_t)_{t\ge 0}$ takes values in some real separable Banach space E, $z \in E$, $(W_t)_{t\ge 0}$ is an E-valued Brownian motion, $\overline{\beta}$ is some square integrable vector field on E of divergence zero

taking values in a real separable Hilbert space $H \subset E$ and $\beta_H^\mu : E \to E$ is the logarithmic derivative of μ associated with H (cf. section 4.2). In the symmetric case, when $\overline{\beta} = 0$, equation (9) has been studied intensively in [1].
We then give a first application of (9) for more explicit maps β_H^μ and $\overline{\beta}$ using existence results of [2] on invariant probability measures for some given linear operator L, i.e. measures μ solving the equation $\int Lu\, d\mu = 0$ for all finitely based smooth functions u. We also use results of [24] on the existence of diffusions associated to extensions of such operators. More precisely, in this case we assume E also to be a Hilbert space and that $H \subset E$ densely by a Hilbert-Schmidt map. We then apply (9) with $\overline{\beta} = \frac{1}{2}(B - \beta_H^\mu)$ where $B : E \to E$ is a Borel measurable vector field of the form $B = -id_E + v$, $v : E \to H$, satisfying *(B.1)-(B.3)* of section 4.2. Under these assumptions on B there exists an invariant probability measure μ such that the stochastic differential equation

$$dY_t \;=\; dW_t - \frac{1}{2}Y_t dt + \frac{1}{2}v(Y_t)dt\ ,\ Y_0 = z \tag{10}$$

admits a weak solution $\mathbf{M} = (\Omega, (\mathcal{F})_{t\geq 0}, (Y_t)_{t\geq 0}, (P_z)_{z\in E})$ for μ-a.e. (even (quasi-)every) $z \in E$. In particular, μ is absolutely continuous w.r.t. the Gaussian measure γ with Cameron-Martin space H on E and with Radon-Nikodym derivative φ^2 where φ is in $H^{1,2}(E;\gamma)$, i.e., the Sobolev space over (E, H, γ). Moreover $\beta_H^\mu = -id_E + 2\frac{\nabla\varphi}{\varphi}$. It is known (see [2, Theorem 3.10]) that the generator of $\mathbf{M}$ restricted to the finitely based smooth functions $L = \frac{1}{2}\Delta_H + \frac{1}{2}B \cdot \nabla$ is μ-symmetric if and only if $v = 2\frac{\nabla\varphi}{\varphi}$ or equivalently $B = \beta_H^\mu$. In our general (i.e. non-symmetric) situation, $2\frac{\nabla\varphi}{\varphi}$ is the orthogonal projection of v onto the closure of the set $\{\nabla u \mid u \in \mathcal{F}\mathcal{C}_b^\infty\}$ in $L^2(E, H; \mu)$. The diffusion $\widehat{\mathbf{M}} = (\widehat{\Omega}, (\widehat{\mathcal{F}})_{t\geq 0}, (\widehat{Y}_t)_{t\geq 0}, (\widehat{P}_z)_{z\in E})$, which is in duality to $\mathbf{M}$ w.r.t. μ, weakly solves

$$d\widehat{Y}_t \;=\; d\widehat{W}_t - \frac{1}{2}\widehat{Y}_t dt + 2\frac{\nabla\varphi}{\varphi}(\widehat{Y}_t)dt - \frac{1}{2}v(\widehat{Y}_t)dt\ ,\ \widehat{Y}_0 = z \tag{11}$$

for μ-a.e. (even (quasi-)every) $z \in E$, where $(\widehat{W}_t)_{t\geq 0}$ is an E-valued $(\widehat{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion starting at $0 \in E$ with covariance given by the inner product of H. Thus (cf. e.g. [6]), adding the drifts of (10) and (11) we obtain $2\beta_H^\mu$ as in the symmetric case.
At the end of section 4.2 we show that $\mathbf{M}$ also satisfies an Itô-type formula, i.e. if $\Psi \in C^1(\mathbb{R}^n)$ and $\widetilde{f}_1, \dots, \widetilde{f}_n \in \widetilde{D(\overline{L})}_b$ then

$$\Psi(\widetilde{f}_1, \dots, \widetilde{f}_n)(Y_t) - \Psi(\widetilde{f}_1, \dots, \widetilde{f}_n)(Y_0) = \sum_{i=1}^n \frac{\partial\Psi}{\partial x_i}(\widetilde{f}_1, \dots, \widetilde{f}_n) \bullet M_t^{[f_i]} + N_t^{[\Psi(f_1,\dots,f_n)]}.$$

In particular, $N_t^{[\Psi(f_1,\dots,f_n)]} = \lim_{n\to\infty} N_t^{[p_n(f_1,\dots,f_n)]}$ where $(p_n)_{n\in\mathbb{N}}$ are polynomials as specified at the end of section 4.2 and the martingale part is a version of the usual stochastic integral (cf. Lemma 2.15(i)).

Finally, I would like to thank Professor Michael Röckner, who led me to the study of Dirichlet forms. I am very grateful for his strong interest and steady encouragement. I am also grateful to Wilhelm Stannat for numerous discussions and valuable comments.

1 Framework and supplementary Potential Theory of generalized Dirichlet forms

1.1 Framework

Let E be a Hausdorff space such that its Borel σ-algebra $\mathcal{B}(E)$ is generated by the set $\mathcal{C}(E)$ of all continuous functions on E. Let m be a σ-finite measure on $(E, \mathcal{B}(E))$ such that $\mathcal{H} = L^2(E, m)$ is a separable (real) Hilbert space with inner product $(\cdot, \cdot)_{\mathcal{H}}$. Let $(\mathcal{A}, \mathcal{V})$ be a real valued coercive closed form on $\mathcal{H}$, i.e. $\mathcal{V}$ is a dense linear subspace of $\mathcal{H}$, $\mathcal{A} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is a positive definite bilinear map, $\mathcal{V}$ is a Hilbert space with inner product $\widetilde{\mathcal{A}}_1(u, v) := \frac{1}{2}(\mathcal{A}(u, v) + \mathcal{A}(v, u)) + (u, v)_{\mathcal{H}}$, and $\mathcal{A}$ satisfies the *weak sector condition*

$$|\mathcal{A}_1(u, v)| \leq K \mathcal{A}_1(u, u)^{1/2} \mathcal{A}_1(v, v)^{1/2},$$

$u, v \in \mathcal{V}$, with *sector constant* K. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ densely and continuously. Since $\mathcal{V}$ is a dense linear subspace of $\mathcal{H}$, $(\mathcal{V}, \widetilde{\mathcal{A}}_1(\cdot, \cdot)^{1/2})$ is again a separable real Hilbert space. Let $\| \cdot \|_{\mathcal{V}}$ be the corresponding norm.
For a linear operator Λ defined on a linear subspace D of one of the Hilbert spaces $\mathcal{V}$, $\mathcal{H}$ or $\mathcal{V}'$ we will use from now on the notation (Λ, D). Let $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator on $\mathcal{H}$ satisfying the following conditions:

D1 (i) $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a C_0-semigroup of contractions $(U_t)_{t \geq 0}$.
(ii) $(U_t)_{t \geq 0}$ can be restricted to a C_0-semigroup on $\mathcal{V}$.

Denote by $(\Lambda, D(\Lambda, \mathcal{V}))$ the generator corresponding to the restricted semigroup. From $[23, \textit{Lemma}\, I.2.3., p.12]$ we have that if $(\Lambda, D(\Lambda, \mathcal{H}))$ satisfies **D1** then $\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}'$ is closable as an operator from $\mathcal{V}$ into $\mathcal{V}'$. Let $(\Lambda, \mathcal{F})$ denote its closure, then $\mathcal{F}$ is a real Hilbert space with corresponding norm

$$\|u\|_{\mathcal{F}}^2 := \|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}'}^2.$$

By $[23, \textit{Lemma}\, I.2.4., p.13]$ the adjoint semigroup $(\widehat{U}_t)_{t \geq 0}$ of $(U_t)_{t \geq 0}$ can be extended to a C_0-semigroup on $\mathcal{V}'$ and the corresponding generator $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{V}'))$ is the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$. Let $\widehat{\mathcal{F}} := D(\widehat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$. Then $\widehat{\mathcal{F}}$ is a real Hilbert space with corresponding norm

$$\|u\|_{\widehat{\mathcal{F}}}^2 := \|u\|_{\mathcal{V}}^2 + \|\widehat{\Lambda} u\|_{\mathcal{V}'}^2.$$

Let the form $\mathcal{E}$ be given by

$$\mathcal{E}(u, v) := \begin{cases} \mathcal{A}(u, v) - \langle \Lambda u, v \rangle & \text{for } u \in \mathcal{F},\ v \in \mathcal{V} \\ \mathcal{A}(u, v) - \langle \widehat{\Lambda} v, u \rangle & \text{for } u \in \mathcal{V},\ v \in \widehat{\mathcal{F}} \end{cases}$$

and $\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha (u, v)_{\mathcal{H}}$ for $\alpha > 0$. $\mathcal{E}$ is called the *bilinear form associated with* $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$.

Here, $\langle \cdot, \cdot \rangle$ denotes the dualization between $\mathcal{V}'$ and $\mathcal{V}$. Note that $\langle \cdot, \cdot \rangle$ restricted to $\mathcal{H} \times \mathcal{V}$ coincides with $(\cdot, \cdot)_{\mathcal{H}}$ and that $\mathcal{E}$ is well-defined. It follows, from [23, *Proposition I.3.4., p.19*], that for all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \to \mathcal{F}$ and $\widehat{W}_\alpha : \mathcal{V}' \to \widehat{\mathcal{F}}$ such that $\mathcal{E}_\alpha(W_\alpha f, u) = \langle f, u \rangle = \mathcal{E}_\alpha(u, \widehat{W}_\alpha f)$, $\forall f \in \mathcal{V}'$, $u \in \mathcal{V}$. Furthermore $(W_\alpha)_{\alpha>0}$ and $(\widehat{W}_\alpha)_{\alpha>0}$ satisfy the resolvent equation

$$W_\alpha - W_\beta = (\beta - \alpha) W_\alpha W_\beta \quad \textit{and} \quad \widehat{W}_\alpha - \widehat{W}_\beta = (\beta - \alpha) \widehat{W}_\alpha \widehat{W}_\beta.$$

Restricting W_α to $\mathcal{H}$ we get a strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ on $\mathcal{H}$ satisfying $\lim_{\alpha\to\infty} \alpha G_\alpha f = f$ in $\mathcal{V}$ for all $f \in \mathcal{V}$. The resolvent $(G_\alpha)_{\alpha>0}$ is called the *resolvent associated with* $\mathcal{E}$. Let $(\widehat{G}_\alpha)_{\alpha>0}$ be the adjoint of $(G_\alpha)_{\alpha>0}$ in $\mathcal{H}$. $(\widehat{G}_\alpha)_{\alpha>0}$ is called the *coresolvent associated with* $\mathcal{E}$.
A bounded linear operator $G : \mathcal{H} \to \mathcal{H}$ is called *sub-Markovian* if $0 \le Gf \le 1$ for all $f \in \mathcal{H}$ with $0 \le f \le 1$. By [23, *Proposition I.4.6., p.24*] we have that $(G_\alpha)_{\alpha>0}$ is sub-Markovian if and only if

$$\textbf{D2} \qquad u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V} \quad \textit{and} \quad \mathcal{E}(u, u - u^+ \wedge 1) \ge 0$$

is satisfied.

Definition 1.1 *The bilinear form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ is called a* **generalized Dirichlet form** *if* **D2** *holds.*

Example 1.2 *(i) Let $(\mathcal{A}, \mathcal{V})$ be a Dirichlet form (cf. e.g. [15]) and $\Lambda = 0$. Then $\mathcal{F} = \mathcal{V} = \widehat{\mathcal{F}}$. And $\mathcal{E} = \mathcal{A}$ is a generalized Dirichlet form since the resolvent of $\mathcal{A}$ is sub-Markovian and therefore* **D2** *is satisfied.*
(ii) Let $\mathcal{A} = 0$ on $\mathcal{V} := \mathcal{H}$ and $(\Lambda, D(\Lambda))$ be a Dirichlet operator (cf. e.g. [15]) generating a C_0-semigroup of contractions on $\mathcal{H}$. In this case $\mathcal{F} = D(\Lambda)$, $\widehat{\mathcal{F}} = D(\widehat{\Lambda})$ and the corresponding bilinear form $\mathcal{E}(u, v) = (-\Lambda u, v)_{\mathcal{H}}$ if $u \in D(\Lambda)$, $v \in \mathcal{H}$, and $\mathcal{E}(u, v) = (u, -\widehat{\Lambda} v)_{\mathcal{H}}$ if $u \in \mathcal{H}$, $v \in D(\widehat{\Lambda})$, is a generalized Dirichlet form.

An element u of $\mathcal{H}$ is called 1-excessive (resp. 1-coexcessive) if $\beta G_{\beta+1} u \le u$ (resp. $\beta \widehat{G}_{\beta+1} u \le u$) for all $\beta \ge 0$. Let $\mathcal{P}$ (resp. $\widehat{\mathcal{P}}$) denote the 1-excessive (resp. 1-coexcessive) elements of $\mathcal{V}$. Let $\mathcal{C}, \mathcal{D} \subset \mathcal{H}$. We define $\mathcal{D}_{\mathcal{C}} := \{u \in \mathcal{D} \mid \exists f \in \mathcal{C},\ u \le f\}$. For an arbitrary Borel set $B \in \mathcal{B}(E)$ and an element $u \in \mathcal{H}$ such that $\{v \in \mathcal{H} \mid v \ge u \cdot 1_B\} \cap \mathcal{F} \ne \emptyset$ (resp. $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$) let $u_B := e_{u \cdot 1_B}$ be the 1-*reduced function* (resp. $\hat{u}_B := \hat{e}_{\hat{u} \cdot 1_B}$ be the 1-*coreduced function*) of $u \cdot 1_B$ (resp. $\hat{u} \cdot 1_B$) as defined in [23, *Definition III.1.8., p.65*]. Here we use the notation 1_B for the characteristic function of B. Note that in general only if B is open our definition of reduced function coincides with the one of [8, p.92], [15, Exercise III.3.10(ii), p.84]. In particular, if $B \in \mathcal{B}(E)$ is such that $m(B) = 0$, then $u_B = 0$. We will use the following quite often in the sequel (cf. [23, Proposition III.1.6. and proof of Proposition III.1.7.]): for $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$, $B \in \mathcal{B}(E)$ there exists $\hat{u}_B^\alpha \in \widehat{\mathcal{F}} \cap \widehat{\mathcal{P}}$ such that $\hat{u}_B^\alpha \le \hat{u}_B^\beta$, $0 < \alpha \le \beta$, $\hat{u}_B^\alpha \to \hat{u}$, $\alpha \to \infty$, strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$, and

$$\mathcal{E}_1(v, \hat{u}_B^\alpha) = \alpha((\hat{u}_B^\alpha - \hat{u} \cdot 1_B)^-, v)_{\mathcal{H}} \text{ for any } v \in \mathcal{V} \tag{12}$$

where f^- denotes the negative part of f. Similarly for $u \in \mathcal{P}_{\mathcal{F}}$ there exists $u_B^\alpha \in \mathcal{F} \cap \mathcal{P}$ such that $u_B^\alpha \le u_B^\beta$, $0 < \alpha \le \beta$, $u_B^\alpha \to u$, $\alpha \to \infty$, strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$ and

$$\mathcal{E}_1(u_B^\alpha, v) = \alpha((u_B^\alpha - u \cdot 1_B)^-, v)_{\mathcal{H}} \text{ for any } v \in \mathcal{V}.$$

Since by $[23, Proposition III.1.7.(ii)]$ $\hat{u}_B \cdot 1_B = \hat{u} \cdot 1_B$, $u_B \cdot 1_B = u \cdot 1_B$ we then have for any $\alpha > 0$

$$\mathcal{E}_1(u_B^\alpha, \hat{u}) = \mathcal{E}_1(u, \hat{u}_B^\alpha).$$

Note that then (by our definition of reduced functions for not necessarily open sets) [23, Lemma III.2.1.(ii)] extends to general Borel sets, i.e. $\mathcal{E}_1(f_B, \hat{f}) = \mathcal{E}_1(f, \hat{f}_B)$ for any $f \in \mathcal{F} \cap \mathcal{P}$, $\hat{f} \in \widehat{\mathcal{F}} \cap \widehat{\mathcal{P}}$, $B \in \mathcal{B}(E)$.
If $B = E$ we rather use the notation e_u instead of u_E.
Let $A \subset E$. We set $A^c := E \setminus A$, i.e. the complement of A in E. An increasing sequence of closed subsets $(F_k)_{k\ge1}$ is called an $\mathcal{E}$-nest, if for every function $u \in \mathcal{P} \cap \mathcal{F}$ it follows that $u_{F_k^c} \to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$. A subset $N \subset E$ is called $\mathcal{E}$-exceptional if there is an $\mathcal{E}$-nest $(F_k)_{k\ge1}$ such that $N \subset \cap_{k\ge1} E \setminus F_k$. A property of points in E holds $\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.) if the property holds outside some $\mathcal{E}$-exceptional set. A function f defined up to some $\mathcal{E}$-exceptional set $N \subset E$ is called $\mathcal{E}$-quasi-continuous ($\mathcal{E}$-q.c.)(resp. $\mathcal{E}$-quasi-lower-semicontinuous ($\mathcal{E}$-q.l.s.c.)) if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$, such that $\bigcup_{k\ge1} F_k \subset E \setminus N$ and $f_{|F_k}$ is continuous (resp. lower-semicontinuous) for all k.
For an $\mathcal{E}$-nest $(F_k)_{k\ge1}$ let

$$C(\{F_k\}) = \{f : A \to \mathbb{R} \mid \bigcup_{k\ge1} F_k \subset A \subset E,\ f_{|F_k} \textit{ is continuous } \forall k\}$$

$$C_l(\{F_k\}) = \{f : A \to \mathbb{R} \mid \bigcup_{k\ge1} F_k \subset A \subset E,\ f_{|F_k} \textit{ is lower-semicontinuous } \forall k\}$$

We denote by $\widetilde{f}$ an $\mathcal{E}$-q.c. m-version of f, conversely f denotes the m-class represented by an $\mathcal{E}$-q.c. m-version $\widetilde{f}$ of f.

Definition 1.3 *The generalized Dirichlet form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ is called quasi-regular if:*
(i) There exists an $\mathcal{E}$-nest $(E_k)_{k\ge1}$ consisting of compact sets.
(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-q.c. m-versions.
(iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having $\mathcal{E}$-q.c. m-versions $\widetilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\widetilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

1.2 Measures associated to coexcessive functions

Let us first make a remark about a notational convention: in the sequel before each statement we will name the assumptions on the generalized Dirichlet form which we need to show the statement. We do this in the following way: we define abbreviations for these assumptions and put the abbreviations in brackets just before the statement (cf e.g. Theorem 1.4 below).

From now on we assume that we are given a quasi-regular generalized Dirichlet form. We write **QR** as an abbreviation for this assumption. We remark that **QR** will not be subdivided in weaker assumptions, i.e. **QR** will be the weakest assumption on the generalized Dirichlet form which we will make.
By quasi-regularity every element in $\mathcal{F}$ admits an $\mathcal{E}$-q.c. m-version (cf. [23, Proposition IV.1.8.]). For a subset $\mathcal{G} \subset \mathcal{H}$ denote by $\widetilde{\mathcal{G}}$ all the $\mathcal{E}$-q.c. m-versions of elements in $\mathcal{G}$. In particular $\widetilde{\mathcal{P}}_{\mathcal{F}}$ denotes the set of all $\mathcal{E}$-q.c. m-versions of 1-excessive elements in $\mathcal{V}$ which are dominated by elements of $\mathcal{F}$. Note that $\widetilde{\mathcal{F} \cap \mathcal{P}} \subset \widetilde{\mathcal{P}}_{\mathcal{F}}$ and that $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ is a linear lattice, that is $\widetilde{u} \wedge \alpha \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ for all $\alpha \geq 0$ and all $\widetilde{u} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$. We emphasize that an element in $\mathcal{P}_{\mathcal{F}}$ not necessarily admits an $\mathcal{E}$-q.c. m-version.
We denote by $\mathcal{B}$ the $\mathcal{B}(E)$-measurable functions on E and by $\mathcal{B}_b$, $\mathcal{B}^+$ the bounded respectively positive elements in $\mathcal{B}$. We also set $\mathcal{B}_b^+ := \mathcal{B}_b \cap \mathcal{B}^+$. Let $\mathcal{D} \subset \mathcal{H}$. We denote by $\mathcal{D}_b$, $\mathcal{D}^+$ the bounded respectively positive elements of $\mathcal{D}$. As above we set $\mathcal{D}_b^+ := \mathcal{D}_b \cap \mathcal{D}^+$. We are now in the situation to state an integral representation theorem for elements in $\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$.

Theorem 1.4 *(**QR**) Let $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$. Then there exists a unique σ-finite and positive measure $\mu_{\hat{u}}$ on $(E, \mathcal{B}(E))$ charging no $\mathcal{E}$-exceptional set, such that*

$$\int \widetilde{f}\, d\mu_{\hat{u}} = \lim_{\alpha \to \infty} \mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u}) \quad \forall \widetilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}} \ .$$

Proof Set $I_{\hat{u}}(\widetilde{f}) = \lim_{\alpha \to \infty} \mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u})$, $\widetilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$. The limit exists since $\mathcal{E}_1(f, \alpha \widehat{G}_{\alpha+1} \hat{u})$ splits into two parts which are both increasing and bounded. Then $I_{\hat{u}}$ is a nonnegative linear functional on $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$. Let $(\widetilde{f}_n)_{n \in \mathbb{N}} \subset \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ such that $\widetilde{f}_n \downarrow 0$ pointwise on E for $n \to \infty$. Similar to the proof of Theorem 1 in [5] we will show that

$$I_{\hat{u}}(\widetilde{f}_n) \downarrow 0 \ \text{ as } \ n \to \infty. \tag{13}$$

Fix $\varphi \in L^1(E; m) \cap \mathcal{B}$ such that $0 < \varphi \leq 1$. By [23, *Lemma III*.3.10., *p*.73] there exists an $\mathcal{E}$-nest $(F_k)_{k \in \mathbb{N}}$, such that $\widetilde{G_1 \varphi} \geq \frac{1}{k}$ everywhere on F_k for all $k \in \mathbb{N}$. Since $\mathcal{E}$ is quasi-regular we may assume that F_k, $k \in \mathbb{N}$, is compact. We may further assume by [23, *Lemma* 3.5., *p*.71] that $(\widetilde{f}_n)_{n \in \mathbb{N}} \subset C(\{F_k\})$. From Dini's Theorem we know that given $k_0 \in \mathbb{N}$ there exists $n(k_0) \in \mathbb{N}$, such that for all $n \geq n(k_0)$

$$f_n \leq \frac{1}{k_0} G_1 \varphi \quad \text{ m-a.e. on } F_{k_0}.$$

Since $f_n \leq f_1 \in \mathcal{P}_{\mathcal{F}} - \mathcal{P}_{\mathcal{F}}$ there exists $f \in \mathcal{F}$ such that $f_n \leq f$ and therefore we have for all $n \in \mathbb{N}$

$$f_n \leq \frac{1}{k_0} G_1 \varphi + f_{F_{k_0}^c} \quad \text{ m-a.e..}$$

Let $\hat{f} \in \widehat{\mathcal{F}}$ such that $\hat{u} \leq \hat{f}$. Then

$$\begin{aligned} I_{\hat{u}}(\widetilde{f}_n) &= \lim_{\alpha \to \infty} \mathcal{E}_1(f_n, \alpha \widehat{G}_{\alpha+1} \hat{u}) \\ &\leq \limsup_{\alpha \to \infty} \mathcal{E}_1(\frac{1}{k_0} G_1 \varphi + f_{F_{k_0}^c}, \alpha \widehat{G}_{\alpha+1} \hat{u}) \\ &\leq \mathcal{E}_1(\frac{1}{k_0} G_1 \varphi + f_{F_{k_0}^c}, \hat{f}). \end{aligned}$$

Since $\lim_{k\to\infty} \frac{1}{k} G_1\varphi + f_{F_k^c} = 0$ weakly in $\mathcal{V}$ we conclude that $\lim_{n\to\infty} I_{\hat{u}}(\widetilde{f}_n) = 0$ and (13) is shown. By the Theorem of Daniell-Stone there exists a unique measure say $\mu_{\hat{u}}$ on $\sigma(\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}})$ (i.e. the σ-Algebra generated by $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$) such that $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}} \subset L^1(\mu_{\hat{u}})$. By [23, *Proposition* IV.1.9., *p.*77] we know that $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ separates the points of $E \setminus N$ where N is an $\mathcal{E}$-exceptional set and consequently as in [15, Remark IV.3.2.(iv), p.102] we have $\sigma(\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}) \supset \mathcal{B}(E \setminus N)$ where $\mathcal{B}(E \setminus N)$ is the Borel σ-Algebra of $E \setminus N$. Since $\mu_{\hat{u}}(\widetilde{N}) = \lim_{\alpha\to\infty} \mathcal{E}_1(1_{\widetilde{N}}, \alpha\widehat{G}_{\alpha+1}\hat{u}) = 0$ for every $\mathcal{E}$-exceptional set $\widetilde{N}$ we may assume that $\mu_{\hat{u}}$ is a Borel-measure. Finally $\int_E \widetilde{G_1\varphi}\, d\mu_{\hat{u}} \le \mathcal{E}_1(G_1\varphi\,,\, \hat{f}) < \infty$ implies that $\mu_{\hat{u}}$ is σ-finite.

□

Let $\mathcal{D} \subset \mathcal{H}$. For a linear operator G on $\mathcal{H}$ with domain $D(G) \supset \mathcal{D}$ we set $G\mathcal{D} := \{Gh \mid h \in \mathcal{D}\}$.

Remark 1.5 *In the time dependent case (cf. section 4.1 below), whereas in the case of classical Dirichlet forms we have $\mathcal{P}_{\mathcal{F}} = \mathcal{P}$ and $\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}} = \widehat{\mathcal{P}}$. More generally this holds for any generalized Dirichlet form with $\mathcal{F} = \widehat{\mathcal{F}}$ and $-\Lambda f = \widehat{\Lambda} f$ for any $f \in G_1\mathcal{H}_b \cup \widehat{G}_1\mathcal{H}_b$. We only show that $\mathcal{P}_{\mathcal{F}} = \mathcal{P}$. Indeed let $u \in \mathcal{P}$, $h \in \mathcal{H}_b^+$. Since $(G_\alpha)_{\alpha>0}$ is positivity preserving by the assumption $\mathcal{F} = \widehat{\mathcal{F}}$ we have $f := \alpha G_{\alpha+1}h \in \widehat{\mathcal{F}}^+ \cap G_1\mathcal{H}_b$ hence $0 \le \mathcal{E}_1(u,f)$ by [23, Proposition III.1.4.]. Now*

$$\begin{aligned} 0 \le \mathcal{E}_1(u,f) &= 2\widetilde{\mathcal{A}}_1(u,f) - \mathcal{E}_1(f,u) \\ &= {}_{\mathcal{V}'}< 2\widetilde{\mathcal{A}}_1(u,\cdot), f >_{\mathcal{V}} - \mathcal{E}_1(f,u) \\ &= \mathcal{E}_1(f, \widehat{W}_1(2\widetilde{\mathcal{A}}_1(u,\cdot)) - u) \\ &= (h, \{\widehat{W}_1(2\widetilde{\mathcal{A}}_1(u,\cdot)) - u\} - \alpha\widehat{G}_{\alpha+1}\{\widehat{W}_1(2\widetilde{\mathcal{A}}_1(u,\cdot)) - u\})_{\mathcal{H}}. \end{aligned}$$

implies that $\widehat{W}_1(2\widetilde{\mathcal{A}}_1(u,\cdot)) - u$ is 1-coexcessive. In particular we have $u \le \widehat{W}_1(2\widetilde{\mathcal{A}}_1(u,\cdot)) \in \mathcal{F}$ and therefore $u \in \mathcal{P}_{\mathcal{F}}$. The converse inclusion is trivial and $\widehat{\mathcal{P}}_{\widehat{\mathcal{F}}} = \widehat{\mathcal{P}}$ can be shown similarly.

From now on we fix an m-tight special standard process $\mathbf{M} = (\Omega, (\mathcal{F}_t)_{t\ge0}, (Y_t)_{t\ge0}, (P_z)_{z\in E_\Delta})$ with lifetime ζ and shift operator $(\theta_t)_{t\ge0}$ such that the resolvent $R_\alpha f$ of $\mathbf{M}$ is an $\mathcal{E}$-q.c. m-version of $G_\alpha f$ for all $\alpha > 0$, $f \in \mathcal{H} \cap \mathcal{B}_b$. $\mathbf{M}$ is then said to be properly associated in the resolvent sense with $\mathcal{E}$. The exact definition of such a process $\mathbf{M}$ is given in the Appendix. Recall that we always assume that $(\mathcal{F}_t)_{t\ge0}$ is the (universally completed) natural filtration of $(Y_t)_{t\ge0}$ and that any real-valued function u on E is extended to E_Δ by setting $u(\Delta) = 0$. We use the abbreviation $\mathbf{M^{ex}}$ for the assumption that such a process exists.
In addition to quasi-regularity a structural condition on the domain $\mathcal{F}$ of the generalized Dirichlet form is imposed in [23, *IV*.2, *D*3] in order to construct explicitly an associated m-tight special standard process. Since we make no use of this technical assumption and since it may be subject to some further progress we instead prefer to assume the existence of $\mathbf{M}$. We will use the resolvent of $\mathbf{M}$ in the proofs of Lemma 1.6, Lemma 1.7 and Theorem 1.9 below but we remark that the statement of our main result Theorem 1.9 is

independent of $\mathbf{M}$ and only depends on the generalized Dirichlet form.
Let P be a probability measure on $(\Omega, \mathcal{F}_\infty)$. Let $A, B \in \mathcal{F}_\infty$ be two events. We say that A holds P-a.s. on B, if $P(A; B) := P(A \cap B) = P(B)$. An $(\mathcal{F}_t)$-stopping time τ is called a *terminal time* provided $t + \tau \circ \theta_t = \tau$ P_z-a.s. on $\{\tau > t\}$ for any $z \in E$. Define for $A \subset E_\Delta$

$$\sigma_A := \inf\{t > 0 \mid Y_t \in A\}, \qquad D_A := \inf\{t \geq 0 \mid Y_t \in A\}.$$

A terminal time τ is called *exact* provided $t_n \downarrow 0$ implies that $t_n + \tau \circ \theta_{t_n} \downarrow \tau$ P_z-a.s. for every $z \in E$. Note that if $A \subset E_\Delta$ is such that σ_A, D_A are $(\mathcal{F}_t)$-stopping times, then σ_A is an exact terminal time, whereas D_A is in general only a terminal time and may fail to be exact since $\lim_{t \downarrow 0} t + D_A \circ \theta_t \downarrow \sigma_A$ P_z-a.s. for every $z \in E$. For $(\mathcal{F}_t)$-stopping times σ, τ define

$$R_\alpha^{\sigma,\tau} f(z) := E_z\Big[\int_\sigma^\tau e^{-\alpha s} f(Y_s) ds\Big], \quad \alpha > 0, \quad z \in E, \quad f \in \mathcal{B}(E)^+$$

and

$$p_t^\tau f(z) := E_z\Big[f(Y_t) 1_{\{t<\tau\}}\Big], \quad t > 0, \quad z \in E, \quad f \in \mathcal{B}(E)^+.$$

In particular $(p_t^\zeta)_{t>0} := (p_t)_{t>0}$ is then the transition semigroup of $\mathbf{M}$ and $(R_\alpha^{0,\zeta})_{\alpha>0}$ is the resolvent of $\mathbf{M}$. Here we rather use the notation p_t instead of p_t^ζ (resp. R_α instead of $R_\alpha^{0,\zeta}$). We will show explicitly in Lemma 1.7(i) below that the terminal time property of an $(\mathcal{F}_t)$-stopping time τ implies the semigroup property of $(p_t^\tau)_{t>0}$ hence the Resolvent equation for $(R_\alpha^{0,\tau})_{\alpha>0}$.
Let $B \in \mathcal{B}(E)$. Then $\{\sigma_B = 0\} \in \mathcal{F}_0$ and according to Blumenthal's 0-1 law we know that $P_z(\sigma_B = 0) = 0$ or 1. Let us denote the regular points for B by

$$B^{reg} := \{z \in E \mid P_z(\sigma_B = 0) = 1\}.$$

From its definition we see that B^{reg} is universally measurable. Also obviously by right-continuity of the associated process we have $B^{reg} \subset \overline{B}$ where $\overline{B}$ denotes the closure of B in E.

Lemma 1.6 *(**QR**, $\mathbf{M}^{\mathbf{ex}}$) Let $B \in \mathcal{B}(E)$. Then $m(B \setminus B^{reg}) = 0$ and $P_m(D_B = \sigma_B) = 1$.*

Proof Let $\varphi \in L^2(E; m) \cap \mathcal{B}$, $0 < \varphi \leq 1$. Then $0 \leq R_1^{0,D_F}\varphi \leq R_1\varphi$ and therefore $R_1^{0,D_F}\varphi \in L^2(E; m) \cap \mathcal{B}_b^+$. By strong continuity of $(U_t)_{t>0}$ we may subtract a decreasing sequence $(t_n)_{n\in\mathbb{N}} \subset (0, \infty)$ converging to zero such that $\lim_{n\to\infty} U_{t_n} R_1^{0,D_F}\varphi(z) = R_1^{0,D_F}\varphi(z)$ for m-a.e. $z \in E$. Since $p_{t_n} R_1^{0,D_F}\varphi$ is an m-version of $U_{t_n} R_1^{0,D_F}\varphi$ for every n we have also $\lim_{n\to\infty} p_{t_n} R_1^{0,D_F}\varphi(z) = R_1^{0,D_F}\varphi(z)$ for m-a.e. $z \in E$. Note that $\lim_{t\downarrow 0} D_B \circ \theta_t + t = \sigma_B$. Now, using the strong Markov property and Lebesgue's Theorem we have for any $z \in E$

$$\begin{aligned}
\lim_{n\to\infty} p_{t_n} E_\cdot\Big[\int_0^{D_B} e^{-s}\varphi(Y_s)\, ds\Big](z) &= \lim_{n\to\infty} E_z\Big[E_{Y_{t_n}}\Big[\int_0^{D_B} e^{-s}\varphi(Y_s)\, ds\Big]\Big] \\
&= \lim_{n\to\infty} e^{t_n} E_z\Big[\int_{t_n}^{D_B\circ\theta_{t_n}+t_n} e^{-s}\varphi(Y_s)\, ds\Big] \\
&= E_z\Big[\int_0^{\sigma_B} e^{-s}\varphi(Y_s)\, ds\Big].
\end{aligned}$$

It follows that $E_z\Big[\int_{D_B}^{\sigma_B} e^{-s}\varphi(Y_s)\,ds\Big]=0$ for m-a.e. $z\in E$. But

$$E_z\Big[\int_{D_B}^{\sigma_B} e^{-s}\varphi(Y_s)\,ds\Big] \text{ is } \begin{cases} =0 & \text{for } z\in B^{reg}\cup B^c \\ >0 & \text{for } z\in B\setminus B^{reg} \end{cases}$$

and therefore $m(B\setminus B^{reg})=0$. Clearly $P_z(D_B=\sigma_B)=1$ for all $z\in B^{reg}\cup B^c$ hence $P_m(D_B=\sigma_B)=1$.

□

Given a finite measure μ on measurable space $(G,\mathcal{G})$. The completion of $\mathcal{G}$ w.r.t. μ is denoted by $\mathcal{G}^\mu$. An element of $\mathcal{B}^*(E):=\bigcap_{P\in\mathcal{P}(E)}\mathcal{B}(E)^P$ where $\mathcal{P}(E)$ denotes the family of all probability measures on $(E,\mathcal{B}(E))$ is called a universally measurable set. Let $\mathcal{B}^*$ denote the $\mathcal{B}^*(E)$-measurable functions on E.
Let $\gamma\geq 0$. A function $f\in\mathcal{H}\cap\mathcal{B}^{*+}$ is called *γ-supermedian for $(R_\alpha)_{\alpha>0}$* if $\alpha R_{\alpha+\gamma}f\leq f$, $\alpha>0$. In particular γ-supermedian functions $f\in\mathcal{H}\cap\mathcal{B}^{*+}$ are m-versions of γ-excessive elements in $\mathcal{H}$. $f\in\mathcal{H}\cap\mathcal{B}^{*+}$ is called *γ-excessive for $(R_\alpha)_{\alpha>0}$* if f is γ-supermedian for $(R_\alpha)_{\alpha>0}$ and if $\lim_{\alpha\to\infty}\alpha R_{\alpha+\gamma}f=f$.
We already remarked that $u\in\mathcal{P}$ not necessarily admits an $\mathcal{E}$-q.c. m-version. By quasi-regularity however we know that there exists an an $\mathcal{E}$-q.c. m-version $\widetilde{\alpha G_{\alpha+1}u}$ of $\alpha G_{\alpha+1}u$. Since $\alpha G_{\alpha+1}u$ increases m-a.s. if α increases we know from [23, Corollary III.3.3.] that $\widetilde{\alpha G_{\alpha+1}u}$ increases $\mathcal{E}$-q.e. if α increases. Hence we may define an $\mathcal{E}$-q.l.s.c. m-version of u by

$$\overline{u}:=\sup_{\alpha>0}\widetilde{\alpha G_{\alpha+1}u}$$

$\overline{u}$ is called an $\mathcal{E}$-q.l.s.c. regularization of $u\in\mathcal{P}$. Surely any two $\mathcal{E}$-q.l.s.c. regularizations of $u\in\mathcal{P}$ coincide $\mathcal{E}$-q.e. hence any $\mathcal{E}$-q.l.s.c. regularization of $u\in\mathcal{P}$ coincides $\mathcal{E}$-q.e. with the "canonical" regularization $\overline{u}=\sup_{\alpha>0}\alpha R_{\alpha+1}u$. If not otherwise stated we will always choose the canonical regularization for $u\in\mathcal{P}$.
Let μ be a positive measure on $(E,\mathcal{B}(E))$ charging no $\mathcal{E}$-exceptional set. Since by assumption there exists an $\mathcal{E}$-nest consisting of compact sets support of μ *supp*(μ) is defined.
Let us remark here that Lemma 1.7(i) will not be used until later in Lemma 2.4 and there only in terms of the coassociated process.

Lemma 1.7 *(***QR**, **M**$^{\mathbf{ex}}$*)*
(i) Let τ be a terminal time. Then $(R_\alpha^{0,\tau})_{\alpha>0}$ satisfies the resolvent equation. If f is γ-supermedian for $(R_\alpha)_{\alpha>0}$ ($\gamma\geq 0$) then f is γ-supermedian for $(R_\alpha^{0,\tau})_{\alpha>0}$.
(ii) Let τ be a terminal time. Let $f\in L^2(E;m)^+\cap\mathcal{B}_b^{*+}$. Then $R_1^{\tau,\infty}f$ is 1-supermedian for $(R_\alpha)_{\alpha>0}$ and $R_1^{\tau,\infty}f\in\mathcal{P}\cap\mathcal{B}_b^*$. If in addition τ is exact then $R_1^{\tau,\infty}f$ is 1-excessive for $(R_\alpha)_{\alpha>0}$. In this case we have in particular that $\overline{R_1^{\tau,\infty}f}(z)=R_1^{\tau,\infty}f(z)$ for every $z\in E$.
(iii) Let $g\in L^2(E;m)^+$, $F\subset E$ be closed. Then $\mu_{(\widehat{G}_1g)_F}(E\setminus F^{reg})=0$. In particular supp$(\mu_{(\widehat{G}_1g)_F})\subset F$.

Proof (i) Since $(R_\alpha^{0,\tau})_{\alpha>0}$ is the Laplace transform of $(p_t^\tau)_{t>0}$ it is enough to show the semigroup property for $(p_t^\tau)_{t>0}$. Let $s,\,t>0$. The terminal time property of τ implies

$\{s+t<\tau\} = \{s<\tau\circ\theta_t\}\cap\{t<\tau\}$ P_z-a.s. for any $z\in E$. Hence, using the strong Markov property of $\mathbf{M}$ and the monotone convergence theorem, for any $f\in\mathcal{B}^+$, $z\in E$

$$\begin{aligned} p^\tau_{t+s}f(z) &= E_z\Big[\, f(Y_{t+s})1_{\{t+s<\tau\}}\,\Big] \\ &= E_z\Big[\, f(Y_{t+s})1_{\{s<\tau\circ\theta_t\}}1_{\{t<\tau\}}\,\Big] \\ &= E_z\Big[\, E_{Y_t}\Big[\, f(Y_s)1_{\{t<\tau\}}\,\Big]\,\Big] = p^\tau_t p^\tau_s f(z). \end{aligned}$$

Let f be γ-supermedian for $(R_\alpha)_{\alpha>0}$ then

$$\alpha R^{0,\tau}_{\alpha+\gamma} f \le \alpha R_{\alpha+\gamma} f \le f.$$

(ii) Since τ is a terminal time we have $\tau\circ\theta_t + t \ge \tau$ P_z-a.s. for any $z\in E$. Hence the strong Markov property of $\mathbf{M}$ implies

$$\begin{aligned} e^{-t}p_t R^{\tau,\infty}_1 f(z) &= E_z\Big[\, e^{-t}E_{Y_t}\Big[\, \int_\tau^\infty e^{-s}f(Y_s)ds\,\Big]\,\Big] \\ &= E_z\Big[\, \int_{\tau\circ\theta_t+t}^\infty e^{-s}f(Y_s)ds\,\Big] \le R^{\tau,\infty}_1 f(z). \end{aligned}$$

It follows that $R^{\tau,\infty}_1 f$ is 1-supermedian for $(R_\alpha)_{\alpha>0}$ because

$$\alpha R_{\alpha+1}R^{\tau,\infty}_1 f(z) = \int_0^\infty \alpha e^{-\alpha t}E_z\Big[\, \int_{\tau\circ\theta_t+t}^\infty e^{-s}f(Y_s)ds\,\Big]dt \le R^{\tau,\infty}_1 f(z).$$

Furthermore $R^{\tau,\infty}_1 f \le R_1 f$ implies $R^{\tau,\infty}_1 f\in\mathcal{V}\cap\mathcal{B}^*$ by [23, Lemma III.2.1.(i)]. Note that $R^{\tau,\infty}_1 f$ is finite $\mathcal{E}$-q.e.. Then, using the exactness of τ and Lebesgue's Theorem we have

$$\begin{aligned} \lim_{\alpha\to\infty}\alpha R_{\alpha+1}R^{\tau,\infty}_1 f(z) &= \lim_{\alpha\to\infty}\int_0^\infty \alpha e^{-(\alpha+1)t}p_t R^{\tau,\infty}_1 f(z)dt \\ &= \lim_{\alpha\to\infty}\{\int_0^\infty e^{-t}p_{\frac{t}{\alpha+1}}R^{\tau,\infty}_1 f(z)dt - \int_0^\infty e^{-(\alpha+1)t}p_t R^{\tau,\infty}_1 f(z)dt\} \\ &= R^{\tau,\infty}_1 f(z) \quad \text{for every} \quad z\in E. \end{aligned}$$

Clearly $\lim_{\alpha\to\infty}\alpha R_{\alpha+1}R^{\tau,\infty}_1 f(z) = \sup_{\alpha>0}\alpha R_{\alpha+1}R^{\tau,\infty}_1 f(z)$ for every $z\in E$ hence $\overline{R^{\tau,\infty}_1 f} = R^{\tau,\infty}_1 f$.

(iii) Fix $\varphi\in L^1(E;m)\cap\mathcal{B}$ such that $0<\varphi\le 1$. Since σ_F is an exact terminal time we know from (ii) that $R^{\sigma_F,\infty}_1\varphi$ is 1-excessive for $(R_\alpha)_{\alpha>0}$ and $R^{\sigma_F,\infty}_1\varphi\in\mathcal{V}\cap\mathcal{B}^*$. Furthermore by Lemma 1.6 $R^{\sigma_F,\infty}_1\varphi = R_1\varphi$ m-a.s. on F and therefore by [23, Proposition III.1.7.(ii)] $R^{\sigma_F,\infty}_1\varphi \ge (G_1\varphi)_F$ m-a.e.. Hence

$$\overline{(G_1\varphi)_F} = \sup_{n\ge1} nR_{n+1}(G_1\varphi)_F \le \sup_{n\ge1} nR_{n+1}R^{\sigma_F,\infty}_1\varphi = R^{\sigma_F,\infty}_1\varphi \quad \mathcal{E}\text{-q.e..} \tag{14}$$

Furthermore

$$R_1\varphi - R_1^{\sigma_F,\infty}\varphi \text{ is } \begin{cases} > 0 & \mathcal{E}-q.e. \text{ on } E\setminus F^{reg} \\ = 0 & \text{on } F^{reg}. \end{cases}$$

Since $\mu_{(\widehat{G}_1\varphi)_F}$ does not charge $\mathcal{E}$-exceptional sets it follows from (14) that

$$\int R_1\varphi - R_1^{\sigma_F,\infty}\varphi \, d\mu_{(\widehat{G}_1 g)_F} \leq \int R_1\varphi - \overline{(G_1\varphi)}_F \, d\mu_{(\widehat{G}_1 g)_F}$$

but the expression on the right hand side is equal to zero since by (12)

$$\begin{aligned} \int R_1\varphi \, d\mu_{(\widehat{G}_1 g)_F} &= \lim_{\alpha\to\infty} \alpha(G_1\varphi, ((\widehat{G}_1 g)_F^\alpha - \widehat{G}_1 g\cdot 1_F)^-)_{\mathcal{H}} \\ &= \lim_{\alpha\to\infty} \alpha((G_1\varphi)_F, ((\widehat{G}_1 g)_F^\alpha - \widehat{G}_1 g\cdot 1_F)^-)_{\mathcal{H}} \\ &= \lim_{\alpha\to\infty}\lim_{\beta\to\infty} \mathcal{E}_1(\beta\, R_{\beta+1}(G_1\varphi)_F, (\widehat{G}_1 g)_F^\alpha) \\ &\leq \sup_{\beta\geq 1} \int \beta\, R_{\beta+1}(G_1\varphi)_F \, d\mu_{(\widehat{G}_1 g)_F} = \int \overline{(G_1\varphi)}_F \, d\mu_{(\hat{G}_1 g)_F}. \end{aligned}$$

Now $\mu_{(\widehat{G}_1 g)_F}(E\setminus F^{reg}) = 0$ follows by a standard argument, because $\mu_{(\widehat{G}_1 g)_F}$ is σ-finite.

□

We will need the following remark in the proof of the following theorem.

Remark 1.8 *(analogous to [15, Remark IV.3.2, p.101]) By quasi-regularity we may construct an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ of compact sets and a sequence $(\widetilde{u}_n)_{n\in\mathbb{N}} \subset C(\{F_k\})$ such that*

$$\rho(x,y) := \sum_{n=1}^{\infty} 2^{-n}(|\widetilde{u}_n(x) - \widetilde{u}_n(y)| \wedge 1); x, y \in F_k,$$

defines a separating metric on F_k for each k which is compatible to the trace topology on F_k inherited from E.

To show the equivalence of (i) and (ii) in Theorem 1.9 below we will use another equivalent description for $\mathcal{E}$-exceptional sets via a finite Choquet capacity called the φ-capacity. To explain this let $\varphi \in L^2(E;m)\cap\mathcal{B}$, $0<\varphi\leq 1$. For $U\subset E$, U open let $\mathrm{cap}_\varphi(U) := ((G_1\varphi)_U, \varphi)_{\mathcal{H}}$ and for arbitrary $A\subset E$ let $\mathrm{cap}_\varphi(A) := \inf\{((G_1\varphi)_U,\varphi)_{\mathcal{H}} \mid U\supset A,\, U \text{ open}\}$. It is shown in [23, *Proposition III*.2.10.] that an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is an $\mathcal{E}$-nest if and only if $\lim_{k\to\infty}\mathrm{cap}_\varphi(F_k^c) = 0$. Hence the $\mathcal{E}$-exceptional sets are exactly the zero sets of the set function cap_φ restricted to $\mathcal{B}(E)$.
As a generalization of [8, *p*.78] we introduce the following class of measures

$$\widehat{S}_{00} := \{\mu_{\hat{u}} \mid \ \widehat{u}\in\widehat{\mathcal{P}}_{\widehat{G}_1\mathcal{H}_b^+} \text{ and } \mu_{\hat{u}}(E)<\infty\}$$

where $\widehat{G}_1\mathcal{H}_b^+ := \{\widehat{G}_1 h \mid h\in\mathcal{H}_b^+\}$.

Theorem 1.9 *(**QR**, $\mathbf{M^{ex}}$) For $B \in B(E)$ the following conditions are equivalent:*

(i) B is $\mathcal{E}$-exceptional

(ii) $\mu(B) = 0\ \forall \mu \in \hat{S}_{00}$

Proof (i) $\Rightarrow$ (ii) is clear. We show $\neg$(i) $\Rightarrow \neg$(ii). Fix $\varphi \in L^1(E;m) \cap \mathcal{B}$ such that $0 < \varphi \le 1$. If B is not $\mathcal{E}$-exceptional then $\mathrm{cap}_\varphi(B) = \inf\{((\widehat{G}_1\varphi)_U, \varphi)_{\mathcal{H}} \mid U \supset B, U \text{ open}\} > 0$ where we used that by the discussion below (12) $((G_1\varphi)_U, \varphi)_{\mathcal{H}} = ((\widehat{G}_1\varphi)_U, \varphi)_{\mathcal{H}}$. Since cap_φ is regular by Choquet's capacibility theorem there exists a compact $K \subset B$ with $\mathrm{cap}_\varphi(K) > 0$.

Let $D_0^+ - D_0^+ = \{f_n; n \in \mathbb{N}\}$ be a countable dense subset of bounded functions in $\mathcal{F}$ with $\mathcal{E}$-q.c. m-versions $\overline{D}_0^+ - \overline{D}_0^+ = \{\widetilde{f}_n; n \in \mathbb{N}\} \subset \widetilde{D}_0^+ - \widetilde{D}_0^+ \subset \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ which separate the points of $E \setminus N$ where N is an $\mathcal{E}$-exceptional set (cf. [23, *Proposition IV*.1.9.(*ii*), *p*.77] for the existence). There exists further (cf. [23, *Lemma IV*.1.10., *p*.77]) an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact metrizable sets such that $\{R_1\varphi, \widetilde{f}_n; n \in \mathbb{N}\} \subset C(\{F_k\})$ and such that $R_1\varphi \ge \frac{1}{k}$ $\mathcal{E}$-q.e. on F_k for all $k \ge 1$. We may assume, that $N \subset \bigcap_{k\ge1} F_k^c$. Since $\mathrm{cap}_\varphi(K) = \lim_{k\to\infty} \mathrm{cap}_\varphi(K \cap F_k)$ we may choose $n_0 \in \mathbb{N}$ such that $\mathrm{cap}_\varphi(K \cap F_k) > 0$ for all $k \ge n_0$. Since $\mathrm{cap}_\varphi(F_k^c) \underset{k\to\infty}{\longrightarrow} 0$, there exists $k_0 \ge n_0$ with

$$\mathrm{cap}_\varphi(K \cap F_{k_0}) - \mathrm{cap}_\varphi(F_{k_0}^c) \quad > \quad 0. \tag{15}$$

Let $\rho_{F_{k_0}}$ be a metric on F_{k_0} which is compatible with the relative topology on F_{k_0} inherited from E (cf. Remark 1.8). Define for $n \in \mathbb{N}$

$$B_n := \{z \in F_{k_0} \mid \rho_{F_{k_0}}(z, K \cap F_{k_0}) < \frac{1}{n}\}, \quad \overline{B}_n := \{z \in F_{k_0} \mid \rho_{F_{k_0}}(z, K \cap F_{k_0}) \le \frac{1}{n}\}.$$

Then $B_n \cup F_{k_0}^c \subset E$ is open for all n and thus

$$\begin{aligned}
\mathrm{cap}_\varphi(K \cap F_{k_0}) &= \inf_{\substack{K\cap F_{k_0}\subset U\\ U \text{ open}}} \mathrm{cap}_\varphi(U)\\
&\le \inf_{n\ge1} \mathrm{cap}_\varphi(B_n \cup F_{k_0}^c)\\
&= \inf_{n\ge1} \mathcal{E}_1(G_1\varphi, (\widehat{G}_1\varphi)_{B_n\cup F_{k_0}^c})\\
&\le \inf_{n\ge1} \int R_1\varphi \, d\mu_{(\widehat{G}_1\varphi)_{\overline{B}_n}} + \mathrm{cap}_\varphi(F_{k_0}^c)
\end{aligned}$$

since $(\widehat{G}_1\varphi)_{B_n\cup F_{k_0}^c} \le (\widehat{G}_1\varphi)_{\overline{B}_n} + (\widehat{G}_1\varphi)_{F_{k_0}^c}$. It now follows from (15) that

$$0 \quad < \quad \int R_1\varphi \, d\mu_{(\widehat{G}_1\varphi)_\infty} \tag{16}$$

where $(\widehat{G}_1\varphi)_\infty$ is defined to be the weak limit of $((\widehat{G}_1\varphi)_{\overline{B}_n})_{n\in\mathbb{N}}$ in $\mathcal{V}$. Note that $(\widehat{G}_1\varphi)_\infty \in \widehat{\mathcal{P}}_{\widehat{G}_1\mathcal{H}_b^+}$ and thus there exists a unique $\mu_{(\widehat{G}_1\varphi)_\infty}$ by Theorem 1.4. For convenience we set

$\hat\mu_n := \mu_{(\widehat{G}_1\varphi)_{\overline{B}_n}}$, $n \in \mathbb{N}$, and $\hat\mu_\infty := \mu_{(\widehat{G}_1\varphi)_\infty}$. Note that by Lemma 1.7(iii) we have that $supp(\hat\mu_n) \subset \overline{B}_n$ for any n. We have to show $supp(\hat\mu_\infty) \subset K \cap F_{k_0}$, because then by (16)

$$0 < \int R_1\varphi \, d\hat\mu_\infty = \int R_1\varphi \, 1_{K\cap F_{k_0}} \, d\hat\mu_\infty \leq \hat\mu_\infty(K \cap F_{k_0}) \leq \hat\mu_\infty(B).$$

Clearly $supp(\hat\mu_\infty) \subset K \cap F_{k_0}$ implies $\hat\mu_\infty(E) < \infty$ and thus $\hat\mu_\infty \in \hat{S}_{00}$. We will proceed in several steps.

1. Step : There exists a subsequence such that $\hat\mu_{n_k}$ converges weakly to some μ:
By Lemma 1.7 we know that $supp(\hat\mu_n) \subset F_{k_0}$ for all $n \in \mathbb{N}$ and thus we have for all $f \in C_0(F_{k_0})$

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \Big| \int f \, d\hat\mu_n \Big| &\leq \|f\|_\infty \sup_{n\in\mathbb{N}} \hat\mu_n(F_{k_0}) \\
&\leq \|f\|_\infty \sup_{n\in\mathbb{N}} \mathcal{E}_1(k_0 \, G_1\varphi, (\widehat{G}_1\varphi)_{\overline{B}_n}) \\
&\leq \|f\|_\infty \mathcal{E}_1(k_0 \, G_1\varphi, \widehat{G}_1\varphi) \\
&< \infty.
\end{aligned}$$

It follows that $\{\hat\mu_n; n \in \mathbb{N}\}$ is relatively compact for the vague topology. Let us choose a subsequence $(\hat\mu_{n_k})_{k\in\mathbb{N}}$ which is convergent to some μ with respect to the vague topology. Since F_{k_0} is compact it follows that $(\hat\mu_{n_k})_{k\in\mathbb{N}}$ is weakly convergent to μ.

2. Step : μ is finite and $supp(\mu) \subset K \cap F_{k_0}$:
Since $1_{F_{k_0}} \in C_b(F_{k_0})$ it follows that

$$\mu(F_{k_0}) = \lim_{k\to\infty} \hat\mu_{n_k}(F_{k_0}) \leq k_0 \, \mathcal{E}_1(G_1\varphi, \widehat{G}_1\varphi) < \infty.$$

Further, since $\overline{B}_j^{\,c} \uparrow (K \cap F_{k_0})^c$ as $j \to \infty$ (the complements are taken in F_{k_0}) we conclude by the Porte-Manteau-Theorem and Lemma 1.7 that

$$\mu((K \cap F_{k_0})^c) = \lim_{j\to\infty} \mu(\overline{B}_j^{\,c}) \leq \lim_{j\to\infty} \liminf_{n_k \geq j} \hat\mu_{n_k}(\overline{B}_j^{\,c}) = 0.$$

3. Step : μ does not charge $\mathcal{E}$-exceptional sets:
Setting $\hat\mu(A) = \mu(A \cap F_{k_0})$ for $A \in \mathcal{B}(E)$ we may interpret μ as a Borel measure on E. We will make no distinction between μ and $\hat\mu$ in the following. Let $(E_k)_{k\in\mathbb{N}}$ be an arbitrary $\mathcal{E}$-nest. Then (with the complements in E)

$$\begin{aligned}
\mu(\cap_{k\geq 1} E_k^c) &= \lim_{k\to\infty} \mu(E_k^c) \\
&\leq \lim_{k\to\infty} \liminf_{j\to\infty} \hat\mu_{n_j}(E_k^c) \\
&\leq \lim_{k\to\infty} \liminf_{j\to\infty} \int_{E_k^c} k_0 \, R_1\varphi \, d\hat\mu_{n_j} \\
&\leq k_0 \lim_{k\to\infty} \liminf_{j\to\infty} \int E_\cdot\Big[\int_{\sigma_{E_k^c}}^\infty e^{-t}\varphi(Y_t) \, dt \Big] d\hat\mu_{n_j} \\
&\leq k_0 \lim_{k\to\infty} \mathrm{cap}_\varphi(E_k^c) \\
&= 0
\end{aligned}$$

implies that $\mu(N)=0$ for every $\mathcal{E}$-exceptional set $N\in\mathcal{B}(E)$.
4. Step : $\mu=\hat{\mu}_\infty$:
Let $f\in\mathcal{F}$. There exists $(f_{m_k})_{k\in\mathbb{N}}\subset D_0^+-D_0^+$ such that $\lim_{k\to\infty} f_{m_k}=f$ in $\mathcal{F}$. By $[23, Corollary\, III.3.8., p.73]$ we may assume that $(\widetilde{f}_{m_k})_{k\in\mathbb{N}}\subset\overline{D}_0^+-\overline{D}_0^+$ converges $\mathcal{E}$-q.e. to some $\mathcal{E}$-q.c m-version $\widetilde{f}$ of f. We will show that $(\widetilde{f}_{m_k})_{k\in\mathbb{N}}$ is $L^1(\mu)$-Cauchy. Since $\mid\widetilde{f}_{m_k}-\widetilde{f}_{m_l}\mid\in C_b(F_{k_0})$ and $\mid\widetilde{f}_{m_k}-\widetilde{f}_{m_l}\mid\le\overline{e}_{f_{m_k}-f_{m_l}}+\overline{e}_{f_{m_l}-f_{m_k}}$ $\mathcal{E}$-q.e. (cf. the proof Lemma 2.8 for this estimation) where $\overline{e}_{f_{m_k}-f_{m_l}},\overline{e}_{f_{m_l}-f_{m_k}}$ are canonical regularizations we have

$$\begin{aligned}\int\mid\widetilde{f}_{m_k}-\widetilde{f}_{m_l}\mid\,d\mu &\le \lim_{j\to\infty}\int\overline{e}_{f_{m_k}-f_{m_l}}+\overline{e}_{f_{m_l}-f_{m_k}}\,d\hat{\mu}_{n_j}\\ &\le \mathcal{E}_1(e_{f_{m_k}-f_{m_l}}+e_{f_{m_l}-f_{m_k}},\widehat{G}_1\varphi)\\ &\le \|e_{f_{m_k}-f_{m_l}}+e_{f_{m_l}-f_{m_k}}\|_{\mathcal{H}}\|\varphi\|_{\mathcal{H}}\end{aligned}$$

and we conclude by $[23, Lemma\, III.2.2.(i), p.66]$. Then for a new subsequence eventually

$$\begin{aligned}\int\widetilde{f}\,d\mu &= \lim_{k\to\infty}\lim_{j\to\infty}\int\widetilde{f}_{m_k}\,d\hat{\mu}_{n_j}\\ &= \lim_{k\to\infty}\lim_{j\to\infty}\mathcal{E}_1(f_{m_k},(\widehat{G}_1\varphi)_{\overline{B}_j})\\ &= \mathcal{E}_1(f,(\widehat{G}_1\varphi)_\infty),\end{aligned}$$

but since μ does not charge $\mathcal{E}$-exceptional sets by **3.Step** the equality holds for every $\mathcal{E}$-q.c. m-version $\widetilde{f}$ of f. Now let $\widetilde{f}\in\widetilde{\mathcal{P}}_{\mathcal{F}}$. Then since $\lim_{\alpha\to\infty}\alpha R_{\alpha+1}u(z)=u(z)$ for $\mathcal{E}$-q.e. $z\in E$ if u is $\mathcal{E}$-q.c. and bounded we have

$$\begin{aligned}\int\widetilde{f}\,d\mu &= \sup_{n\ge1}\int\widetilde{f}\wedge n\,d\mu\\ &= \sup_{n\ge1}\lim_{\alpha\to\infty}\int\alpha R_{\alpha+1}(\widetilde{f}\wedge n)\,d\mu\\ &= \sup_{n\ge1}\lim_{\alpha\to\infty}\mathcal{E}_1(\alpha G_{\alpha+1}(f\wedge n),(\widehat{G}_1\varphi)_\infty)\\ &= \lim_{\alpha\to\infty}\mathcal{E}_1(f,\alpha\hat{G}_{\alpha+1}(\widehat{G}_1\varphi)_\infty).\end{aligned}$$

Hence $\mu=\hat{\mu}_\infty$ by Theorem 1.4.

□

Remark 1.10 *(i) $A\subset E$ is called nearly Borel if there exists $B_1,B_2\in\mathcal{B}(E)$ such that $B_1\subset A\subset B_2$ and $B_2\setminus B_1$ is $\mathcal{E}$-exceptional. Then Theorem 1.9 extends to nearly Borel sets. Indeed, we have $A\subset B_1\cup(B_2\setminus B_1)$ and*

$$cap_\varphi(A)=cap_\varphi(B_1)=cap_\varphi(B_2)$$

hence if $cap_\varphi(A)>0$ by Theorem 1.9 there exists $\mu\in\widehat{\mathcal{S}}_{00}$ with $\mu(B_1)>0$ but then $\mu(A)=\mu(B_1)>0$. The fact that A is in general not $\mathcal{B}(E)$-measurable doesn't matter

since for convenience only we restricted ourselves to $(E,\mathcal{B}(E))$ in Theorem 1.4. Actually $\mu \in \widehat{\mathcal{S}}_{00}$ is defined on $\bigcap_{\mu\in\widehat{\mathcal{S}}_{00}} \sigma(\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}})^{\mu}$ (cf. the paragraph before Lemma 1.7 for the meaning of this) which contains any nearly Borel measurable set. Finally, we can call the nearly Borel set A $\mathcal{E}$-exceptional if $cap_{\varphi}(B_1) = 0$.
(ii) Since we may divide each $\mu \in \widehat{S}_{00} \setminus \{0\}$ by its total mass the assertion of Theorem 1.9 *remains true if we replace $\widehat{S}_{00}$ by $\{\mu \in \widehat{S}_{00} \mid \mu(E) = 1\}$. Note also that if $(\widehat{G}_{\alpha})_{\alpha>0}$ is sub-Markovian we may replace $\widehat{S}_{00}$ by the larger class $\{\mu_{\hat{u}} \mid \ \|\hat{u}\|_{\infty} < \infty$ and $\mu_{\hat{u}}(E) < \infty\}$ and then our definition coincides with the one of [8, p.78].*
(iii) Note that if every element in $\mathcal{P}_{G_1\mathcal{H}_b^+}$ admits an $\mathcal{E}$-q.c. m-version (e.g. in the case where $\mathcal{E}$ is a quasi-regular (semi-)Dirichlet form in the sense of [15],[14]) then before **1.Step** : *in the proof of Theorem 1.9 one can show directly supp$(\hat{\mu}_{\infty}) \subset K$. Indeed we may assume that $G_1\varphi$, $(G_1\varphi)_{\overline{B}_j}$, $\alpha\, R_{\alpha+1}(G_1\varphi)_{\overline{B}_j}$ are continuous on $\overline{B}_j$ for every j, $\alpha \in \mathbb{N}$. We may also assume that $G_1\varphi \geq \frac{1}{k_0}$ on each $\overline{B}_j$. We then have for each $j \in \mathbb{N}$*

$$\begin{aligned}\int G_1\varphi - (G_1\varphi)_{\overline{B}_j}\, d\hat{\mu}_{\infty} &= \lim_{\alpha\to\infty} \lim_{n\geq j} \lim_{\beta\to\infty} \int (G_1\varphi)_{\overline{B}_j} - \alpha\, R_{\alpha+1}(G_1\varphi)_{\overline{B}_j}\, d\mu_{(\widehat{G}_1\varphi)^{\beta}_{\overline{B}_n}} \\ &\leq \lim_{\alpha\to\infty} \|(G_1\varphi)_{\overline{B}_j} - \alpha\, R_{\alpha+1}(G_1\varphi)_{\overline{B}_j}\|_{\infty,\overline{B}_j} \int k_0 G_1\varphi\, d\hat{\mu}_{\infty}\end{aligned}$$

and the last expression is zero by Dini's Theorem (here $\|\cdot\|_{\infty,\overline{B}_j}$ denotes the sup norm on the compact space $\overline{B}_j$). We then conclude as in the proof of Lemma 1.7.

1.3 Smooth measures

In this subsection similar to [8], [18] we will define smooth measures and measures of finite (co-)energy integral and show that these measures have properties similar to those in [8], [18]. Throughout the whole section we assume that we are given a quasi-regular generalized Dirichlet form (**QR**) and an m-tight special standard process $\mathbf{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$ ($\mathbf{M}^{\mathbf{ex}}$).

Definition 1.11 *A positive measure μ on $(E,\mathcal{B}(E))$ is said to be of finite 1-order co-energy integral if there exists $\widehat{U}_1\mu \in \mathcal{V}$, such that*

$$\int_E \widetilde{G_1 h}\, d\mu = \mathcal{E}_1(G_1 h, \widehat{U}_1\mu) \tag{17}$$

for all $h \in \mathcal{H}$ and for all $\mathcal{E}$-q.c. m-versions $\widetilde{G_1 h}$ of $G_1 h$. The measures of finite 1-order co-energy integral are denoted by $\widehat{S}_0$.

Let $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$ and $\mu_{\hat{u}}$ be the associated measure of Theorem 1.4. Then $\widehat{U}_1\mu_{\hat{u}} = \hat{u}$. Hence obviously $\widehat{S}_{00} \subset \widehat{S}_0$. Clearly $\mu \in \widehat{S}_0$ does not charge $\mathcal{E}$-exceptional sets. Furthermore

$\widehat{U}_1\mu \in \widehat{\mathcal{P}}$ because for any $f \in \mathcal{H}^+$ we have

$$\begin{aligned}(f, \widehat{U}_1\mu - \alpha\widehat{G}_{\alpha+1}\widehat{U}_1\mu)_{\mathcal{H}} &= \int_E \widetilde{G_1 f}\, d\mu - (\alpha G_{\alpha+1} f, \widehat{U}_1\mu)_{\mathcal{H}} \\ &= \int_E \widetilde{G_1 f} - \alpha\widetilde{G_{\alpha+1} G_1} f\, d\mu \geq 0\end{aligned}$$

since $\widetilde{G_1 f} - \alpha\widetilde{G_{\alpha+1} G_1} f \geq 0$ $\mathcal{E}$-q.e. hence μ-a.e..
Let $\overline{\mathcal{P}}_{G_1\mathcal{H}_b^+}$ denote the totality of $\mathcal{E}$-q.l.s.c. regularizations of the elements in $\mathcal{P}_{G_1\mathcal{H}_b^+}$. Let $\overline{u} \in \overline{\mathcal{P}}_{G_1\mathcal{H}_b^+}$. Then $\int_E \overline{u}\, d\mu = \sup_{\alpha>0} \int_E \alpha R_{\alpha+1} u\, d\mu = \lim_{\alpha\to\infty} \mathcal{E}_1(u, \alpha\widehat{G}_{\alpha+1}\widehat{U}_1\mu)$ exists as a bounded and increasing limit for all $\overline{u} \in \overline{\mathcal{P}}_{G_1\mathcal{H}_b^+}$. Now let $\widetilde{w} \in \widetilde{\mathcal{P}}$. Since $\lim_{\alpha\to\infty} \alpha R_{\alpha+1}\widetilde{v} = \widetilde{v}$ $\mathcal{E}$-q.e. for any $\widetilde{v} \in C(\{F_k\}) \cap \mathcal{B}_b$ we have $\mathcal{E}$-q.e.

$$\begin{aligned}\widetilde{w} = \sup_{n\geq 1} \widetilde{w} \wedge n &= \sup_{n\geq 1} \lim_{\alpha\to\infty} \alpha R_{\alpha+1}(\widetilde{w} \wedge n) \\ &= \sup_{n\geq 1} \sup_{\alpha>0} \alpha R_{\alpha+1}(\widetilde{w} \wedge n) \\ &= \sup_{\alpha>0} \alpha R_{\alpha+1}\widetilde{w}.\end{aligned}$$

Hence if a function $u \in \mathcal{P}_{G_1\mathcal{H}_b^+}$ admits an $\mathcal{E}$-q.c. m-version then this m-version coincides $\mathcal{E}$-q.e. with its canonical regularization. Thus $\int_E \widetilde{u}\, d\mu = \int_E \overline{u}\, d\mu$ and therefore (17) extends to all $\widetilde{f} \in \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+} - \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+}$ in the sense of Theorem 1.4. Note that $\widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+} - \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+}$ is also a vector lattice which separates the points of $E \setminus N$ and hence could have also been used as a space of test functions in Theorem 1.4.
On the other hand only if $\widehat{U}_1\mu \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$ similarly to **4.Step** of the proof of Theorem 1.9 we can show that (17) extends to all $\widetilde{f} \in \widetilde{\mathcal{F}}$. Also only if $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$ (and not for all $\hat{u} \in \widehat{\mathcal{P}}$!) by Theorem 1.4 we can show the existence of $\mu_{\hat{u}} \in \widehat{S}_0$.
In the following proof $(ii) \Rightarrow (i)$ of Lemma 1.12 we shall see that $\mu \in \widehat{S}_0$ can be identified with some $\overline{L}_\mu \in (\mathcal{V}')'$, i.e. the bidual of $\mathcal{V}$.

Lemma 1.12 *(**QR**) The following statements are equivalent for a positive measure μ on $(E, \mathcal{B}(E))$:*
(i) μ is of finite 1-order co-energy integral.
(ii) There exists $C > 0$, such that

$$|\int_E \widetilde{G_1 h}\, d\mu| \leq C\|G_1 h\|_{\mathcal{F}}$$

for all $h \in \mathcal{H}$ and for all $\mathcal{E}$-q.c. m-versions $\widetilde{G_1 h}$ of $G_1 h$.

Proof (cf. [18]) Let us assume that (ii) holds. Clearly μ then does not charge $\mathcal{E}$-exceptional sets. Define $L_\mu(h) = \int_E \widetilde{G_1 h}\, d\mu$, $h \in \mathcal{H}$. Since $|L_\mu(h)| \leq C\|G_1 h\|_{\mathcal{F}} \leq C\|W_1\|_{L(\mathcal{V}')}\|h\|_{\mathcal{V}'}$ where $\|W_1\|_{L(\mathcal{V}')}$ denotes the operator norm of $W_1 : \mathcal{V}' \to \mathcal{F}$. Since $\mathcal{H} \subset \mathcal{V}'$ dense we may extend L_μ to a continuous linear functional $\overline{L}_\mu$ on $\mathcal{V}'$. But then by [28, IV.8.Theorem 1] there exists a unique $\widehat{U}_1\mu \in \mathcal{V}$, such that $\overline{L}_\mu(f) = _{\mathcal{V}'}< f, \widehat{U}_1\mu >_{\mathcal{V}}$ for all f in $\mathcal{V}'$ and (i) holds. (i) $\Rightarrow$ (ii) is clear.

□

Definition 1.13 *A positive measure μ on $(E, \mathcal{B}(E))$ charging no $\mathcal{E}$-exceptional set is called smooth if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ of compact subsets of E, such that*

$$\mu(F_k) < \infty \quad \textit{for all} \quad k \in \mathbb{N}.$$

The smooth measures are denoted by S.

Until the end of the section we assume that the coresolvent $(\widehat{G}_\alpha)_{\alpha>0}$ is sub-Markovian. We abbreviate this assumption by $\widehat{\mathbf{SUB}}$. The following lemma will be needed as a preparation for Lemma 1.15 below.

Lemma 1.14 *($\mathbf{QR}$, $\widehat{\mathbf{SUB}}$) Let $\widetilde{u} \in \widetilde{\mathcal{F}}$, $\varphi \in \mathcal{H}$, $0 < \varphi \leq 1$. Then $cap_\varphi(|\widetilde{u}| > \lambda) \leq 2\frac{(K+1)^2}{\lambda^2}\|u\|^2_{\mathcal{F}}$.*

Proof Let $U := \{\widetilde{u} > 0\}$, $V := \{-\widetilde{u} > 0\}$. Then, since $G_1\varphi \leq \frac{u}{\lambda}$ m-a.e. on U, $G_1\varphi \leq -\frac{u}{\lambda}$ m-a.e. on V

$$\begin{aligned}
\mathrm{cap}_\varphi(\{|\,\widetilde{u}\,| > \lambda\}) &\leq \mathrm{cap}_\varphi(\{\widetilde{u} > \lambda\}) + \mathrm{cap}_\varphi(\{-\widetilde{u} > \lambda\}) \\
&= \mathcal{E}_1(G_1\varphi, (\widehat{G}_1\varphi)_U) + \mathcal{E}_1(G_1\varphi, (\widehat{G}_1\varphi)_V) \\
&\leq \mathcal{E}_1(\frac{u}{\lambda}, (\widehat{G}_1\varphi)_U) + \mathcal{E}_1(-\frac{u}{\lambda}, (\widehat{G}_1\varphi)_V) \\
&\leq \frac{(K+1)}{\lambda}\|u\|_{\mathcal{F}}(\|(\hat{G}_1\varphi)_U\|_{\mathcal{V}} + \|(\widehat{G}_1\varphi)_V\|_{\mathcal{V}}).
\end{aligned}$$

By sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ we have in particular that $(\widehat{G}_1\varphi)^\alpha_U \leq 1$ m-a.e. on U, hence

$$\begin{aligned}
\|(\widehat{G}_1\varphi)_U\|^2_{\mathcal{V}} &\leq \varlimsup_{\alpha\to\infty} \mathcal{E}_1((\widehat{G}_1\varphi)^\alpha_U, (\widehat{G}_1\varphi)^\alpha_U) \\
&\leq \varlimsup_{\alpha\to\infty} \mathcal{E}_1(\frac{u}{\lambda}, (\widehat{G}_1\varphi)^\alpha_U) \\
&\leq \frac{(K+1)}{\lambda}\|u\|_{\mathcal{F}}\|(\widehat{G}_1\varphi)_U\|_{\mathcal{V}}.
\end{aligned}$$

Therefore $\|(\widehat{G}_1\varphi)_U\|_{\mathcal{V}} \leq \frac{(K+1)}{\lambda}\|u\|_{\mathcal{F}}$. Similarly we get $\|(\widehat{G}_1\varphi)_V\|_{\mathcal{V}} \leq \frac{(K+1)}{\lambda}\|u\|_{\mathcal{F}}$ and the assertion follows.

□

Using the preceding Lemma 1.14 and Lemma 1.12(ii) the following lemma can be shown exactly as in [8, Lemma 2.2.8.,p.81].

Lemma 1.15 *($\mathbf{QR}$, $\widehat{\mathbf{SUB}}$) Let $\varphi \in \mathcal{H}$, $0 < \varphi \leq 1$. Let ν be a finite positive measure on $(E, \mathcal{B}(E))$ such that there exists $C > 0$ with*

$$\nu(B) \leq C\, cap_\varphi(B) \textit{ for all } B \in \mathcal{B}(E).$$

Then $\nu \in \widehat{S}_0$.

Since cap_φ, $\varphi \in \mathcal{H}$, $0 < \varphi \le 1$ is a Choquet capacity and the proof of [8, Lemma 2.2.9.,p.81] only uses general properties of Choquet capacities the following lemma can be shown exactly as [8, Lemma 2.2.9.,p.81]. Note that for the proof we need much less then **QR**.

Lemma 1.16 *(**QR**) Let ν be a finite positive measure on $(E, \mathcal{B}(E))$ charging no $\mathcal{E}$-exceptional set (i.e. a finite smooth measure). Let $\varphi \in \mathcal{H}$, $0 < \varphi \le 1$. Then there exists an $\mathcal{E}$-nest $(F_k)_{k \in \mathbb{N}}$, such that*

$$\nu(A) \le 2^k \, cap_\varphi(A) \quad \textit{for any Borel set } A \subset F_k.$$

Theorem 1.17 *(**QR**, $\widehat{\mathbf{SUB}}$) Let μ be a positive measure on $(E, \mathcal{B}(E))$. Then the following statements are equivalent:*
(i) $\mu \in S$.
(ii) There exits an $\mathcal{E}$-nest $(F_k)_{k \in \mathbb{N}}$ consisting of compact subsets of E, such that

$$1_{F_k} \cdot \mu \in \widehat{S}_0 \textit{ for each } k$$

where $1_A \cdot \mu(B) := \mu(A \cap B)$ for $A \subset E$, $B \in \mathcal{B}(E)$.

Proof Let us assume (i). Then there exists an $\mathcal{E}$-nest $(E_k)_{k \in \mathbb{N}}$ consisting of compact subsets of E, such that $1_{E_k} \cdot \mu$ is a finite positive measure charging no $\mathcal{E}$-exceptional set for any k. Let $\varphi \in \mathcal{H}$, $0 < \varphi \le 1$. By Lemma 1.16 we can find an $\mathcal{E}$-nest $(\widetilde{E}_k)_{k \in \mathbb{N}}$ such that $1_{E_k \cap \tilde{E}_k} \cdot \mu(A) \le 2^k \text{cap}_\varphi(A)$ for any k and for any Borel set $A \subset \tilde{E}_k$ but then also for for any k and for any $A \in \mathcal{B}(E)$. Therefore (ii) follows by Lemma 1.15 with $F_k := E_k \cap \tilde{E}_k$.
Let us assume (ii). Let $\varphi \in \mathcal{H}$, $0 < \varphi \le 1$. There exits an $\mathcal{E}$-nest $(E_k)_{k \in \mathbb{N}}$ consisting of compact subsets of E and an $\mathcal{E}$-q.c. m-version of $\widetilde{G_1\varphi}$ of $G_1\varphi$, such that $1_{E_k} \cdot \mu \in \hat{S}_0$ and such that $k\widetilde{G_1\varphi} \ge 1$ on E_k for each k. Therefore $\mu(E_k) \le k \int_E \widetilde{G_1\varphi} \, 1_{E_k} \, d\mu = \mathcal{E}_1(G_1\varphi, \widehat{U}_1(1_{E_k} \cdot \mu)) < \infty$.

□

In the following we will need some preparations in order establish a relation between the classes $\widehat{S}_{00}$ (which we defined in section 1.2) and S. The methods in [8], [16] to develop such a relation rely heavily on the symmetry of the domain of the form, the sector condition and the invariance of the Dirichlet space under truncation. Since in general none of the above mentioned properties are available for generalized Dirichlet forms we have to develop a different procedure. We remark that this procedure takes advantage of the behaviour of the associated process in an essential way.

For $B \in \mathcal{B}(E)$ let

$$B^0 := \{z \in E \mid P_z(\sigma_{B^c} > 0) = 1\}.$$

If $F \subset E$ is closed then F^0 is called the fine interior of F.

In the following Lemma 1.18 we shall not make use of the sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha > 0}$.

Lemma 1.18 *(**QR**, $\mathbf{M^{ex}}$) Let $(F_k)_{k\in\mathbb{N}}$ be an $\mathcal{E}$-nest. Then*

$$\bigcap_{k\geq 1}(E\setminus F_k^0)\quad \textit{and}\quad \bigcap_{k\geq 1}(E\setminus F_k^{reg})\quad \textit{is}\ \ \mathcal{E}\textit{-exceptional.}$$

Proof Let $B\in\mathcal{B}(E)$. We first remark that B^0, B^{reg} is nearly Borel. To show this let $\varphi\in L^2(E;m)\cap\mathcal{B}$, $0<\varphi\leq 1$. Then

$$\{R_1\varphi - R_1^{\sigma_{B^c},\infty}\varphi>0\}=\bigcup_{n\in\mathbb{N}}\{R_1\varphi - R_1^{\sigma_{B^c},\infty}\varphi\geq\frac{1}{n}\}.$$

Thus, since by Lemma 1.7(ii) $R_1^{\sigma_{B^c},\infty}\varphi$ is $\mathcal{E}$-q.l.s.c. it is easy to see that $\{R_1\varphi - R_1^{\sigma_{B^c},\infty}\varphi>0\}$ is nearly Borel. Since $\{R_1\varphi - R_1^{\sigma_{B^c},\infty}\varphi>0\}=B^0$ up to an $\mathcal{E}$-exceptional set B^0 is nearly Borel too. The same is also true for B^{reg} since $B^{reg}=((B^c)^0)^c$. Now let $\mu\in\widehat{S}_{00}$. Then

$$\begin{aligned}\int_E R_1^{\sigma_{F_k^c},\infty}\varphi\,d\mu &\geq \int_{E\setminus F_k^0}E_z\Big[\int_{\sigma_{F_k^c}}^{\infty}e^{-s}\varphi(Y_s)\,ds\Big]\,\mu(dz)\\ &= \int_{E\setminus F_k^0}R_1\varphi\,d\mu.\end{aligned}$$

By [23, *Lemma IV*.3.9.] $R_1^{\sigma_{F_k^c},\infty}\varphi$ is an $\mathcal{E}$-q.l.s.c. m-version of $(G_1\varphi)_{F_k^c}$ and since $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest we have $\lim_{k\to\infty}(G_1\varphi)_{F_k^c}=0$ weakly in $\mathcal{V}$. Therefore

$$0=\lim_{k\to\infty}\int R_1^{\sigma_{F_k^c},\infty}\varphi\,d\mu\geq\int_{\cap_{k\geq 1}(E\setminus F_k^0)}R_1\varphi\,d\mu$$

which implies $\mu(\cap_{k\geq1}(E\setminus F_k^0))=0$. By Remark 1.10(i) we then have that $\cap_{k\geq1}(E\setminus F_k^0)$ is $\mathcal{E}$-exceptional. Since $B^0\subset B^{reg}$ for any $B\subset E$ we have $\cap_{k\geq1}(E\setminus F_k^{reg})\subset\cap_{k\geq1}(E\setminus F_k^0)$ and then $\cap_{k\geq1}(E\setminus F_k^{reg})$ is $\mathcal{E}$-exceptional too.

□

Remark 1.19 *Note that the assertion of Lemma 1.18 is not trivial as one might suspect because in contrast to the case of regular symmetric Dirichlet forms (cf. [8, Theorem 4.1.3., p.139]) $F_k\setminus F_k^{reg}$ is not polar. In our framework, as it is well known from the parabolic case, semi-polar sets are not polar in general. This is from the potential theoretic point of view an important difference to the case of classical Dirichlet forms in the sense of [8], [16], [15]. As an example consider the uniform motion to the right on the real line, i.e.* $\mathcal{H}=\mathcal{V}=L^2(\mathbb{R},dx)$, $\mathcal{F}=\widehat{\mathcal{F}}=H^{1,2}(\mathbb{R})$, $p_tf(x)=f(x+t)$, $x\in\mathbb{R},t\geq0$. *Let $[a,b]$ be the closed interval from a to b. Then $[a,b]\setminus[a,b]^{reg}=\{b\}$ is semi-polar but surely hit if we start at $c<b$. Thus $[a,b]\setminus[a,b]^{reg}$ is not polar. Furthermore, since the Dirac measure δ_x is in S_0 for any $x\in\mathbb{R}$ we have also that $[a,b]\setminus[a,b]^{reg}$ is not $\mathcal{E}$-exceptional.*

For the rest of the section let us assume that in **D1** (ii) the adjoint semigroup $(\widehat{U}_t)_{t\geq0}$ of $(U_t)_{t\geq0}$ can also be restricted to a C_0-semigroup on $\mathcal{V}$. Let $(\widehat{\Lambda},D(\widehat{\Lambda},\mathcal{H}))$ denote the

generator of $(\widehat{U}_t)_{t\geq 0}$ on $\mathcal{H}$, $\widehat{\mathcal{A}}(u,v) := \mathcal{A}(v,u)$, $u,v \in \mathcal{V}$ and let the coform $\widehat{\mathcal{E}}$ be defined as the bilinear form associated with $(\widehat{\mathcal{A}}, \mathcal{V})$ and $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{H}))$. Note that since $(\widehat{G}_\alpha)_{\alpha>0}$ was assumed to be sub-Markovian the corresponding statement of **D2** holds for the coform. The coform is hence a generalized Dirichlet form too. Let us further assume up to the end of this section that the coform $\widehat{\mathcal{E}}$ is quasi-regular too. We will abbreviate the assumption that $\widehat{\mathcal{E}}$ is a quasi-regular generalized Dirichlet form by $\widehat{\mathbf{QR}}$.
We fix an m-tight special standard process $\widehat{\mathbf{M}} = (\widehat{\Omega}, (\widehat{\mathcal{F}}_t)_{t\geq 0}, (\widehat{Y}_t)_{t\geq 0}, (\widehat{P}_z)_{z\in E_\Delta})$ with lifetime $\widehat{\zeta}$ and shift operator $(\widehat{\theta}_t)_{t\geq 0}$ such that the resolvent $\widehat{R}_\alpha f = \widehat{E}_\cdot[\int_0^\infty e^{-\alpha t} f(\widehat{Y}_t)\,dt]$ is an $\widehat{\mathcal{E}}$-q.c. m-version of $\widehat{G}_\alpha f$ for all $f \in \mathcal{H} \cap \mathcal{B}_b$. $\widehat{\mathbf{M}}$ is then said to be properly coassociated in the resolvent sense with $\widehat{\mathcal{E}}$. Recall that we always assume that $(\widehat{\mathcal{F}}_t)_{t\geq 0}$ denotes the (universally completed) natural filtration. Necessary and sufficient conditions for the existence of such a process are given in [23]. $\widehat{\mathbf{M}}$ is then in duality to $\mathbf{M}$ w.r.t. m. We will use the abbreviation $\widehat{\mathbf{M}^{\mathbf{ex}}}$ to express our assumption that such a process exists. Symbols with a superposed hat as

$$\widehat{E}_\cdot[...], \widehat{\sigma}_B, \widehat{D}_B, B^{\widehat{0}}, B^{\widehat{reg}}, \widehat{\mathcal{E}}\text{-nest}, \widehat{\mathcal{E}}\text{-exceptional}, \widehat{\mathcal{E}}\text{-}q.c., ...\ etc.$$

correspond to the coassociated process or the coform and are defined analoguous to the corresponding objects in terms of the associated process $\mathbf{M}$.
We remark that by the discussion right below (12) we have for any open set U that

$$\text{cap}_\varphi(U) = \mathcal{E}_1((G_1\varphi)_U, \widehat{G}_1\varphi) = \mathcal{E}_1(G_1\varphi, (\widehat{G}_1\varphi)_U) =: \widehat{\text{cap}}_\varphi(U).$$

But since analoguous to the corresponding statement for $\mathcal{E}$ (cf. paragraph before Theorem 1.9) we have that an increasing sequence of closed sets $(F_k)_{k\in\mathbf{N}}$ is an $\widehat{\mathcal{E}}$-nest if and only if $\lim_{k\to\infty} \widehat{\text{cap}}_\varphi(F_k^c) = 0$ we can see that $\widehat{\mathcal{E}}$-nests and $\mathcal{E}$-nests coincide hence $\widehat{\mathcal{E}}$-exceptional sets and $\mathcal{E}$-exceptional sets coincide.

Lemma 1.20 *(i) (**QR**, $\mathbf{M^{ex}}$) Let $g \in L^2(E;m) \cap \mathcal{B}_b^+$. Let $F \subset E$, F closed. Then there exists relatively compact subsets $(B_{n,k})_{n,k\geq 1}$ of E (resp. compact subsets $(\overline{B}_{n,k})_{n,k\geq 1}$ of E) such that $B_{n+1,k} \subset \overline{B}_{n+1,k} \subset B_{n,k} \subset B_{n,k+1}$, $P_\mu(\bigcup_{k\geq 1}\bigcap_{n\geq 1} B_{n,k}) = P_\mu(F)$ for any $\mu \in \widehat{S}_{00}$ and*

$$R_1^{D_F,\infty} g(z) = \lim_{k\to\infty}\lim_{n\to\infty} R_1^{D_{B_{n,k}},\infty} g(z) = \lim_{k\to\infty}\lim_{n\to\infty} R_1^{\sigma_{B_{n,k}},\infty} g(z) = \lim_{k\to\infty}\lim_{n\to\infty} \overline{(G_1 g)}_{\overline{B}_{n,k}}(z)$$

for $\mathcal{E}$-q.e. $z \in E$. In particular there exists open subsets $(U_{n,k})_{n,k\geq 1}$ of E such that

$$R_1^{D_F,\infty} g(z) = \lim_{k\to\infty}\lim_{n\to\infty} R_1^{\sigma_{U_{n,k}},\infty} g(z)$$

for $\mathcal{E}$-q.e. $z \in E$.
*(ii) (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}^{\mathbf{ex}}}$) Let $F \subset E$, F closed. Then*

$$supp(\mu_{E_\cdot\left[\int_{D_F}^\infty e^{-s} g(Y_s)ds\right]}) \subset F$$

for any $g \in L^2(E;m) \cap \mathcal{B}^+$.

Proof (i) By quasi-regularity there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ of compact sets. Let ρ_k, $k \geq 1$ be a metric on F_k compatible with the relative topology on F_k inherited from E (cf. Remark 1.8). Define for $n, k \geq 1$

$$B_{n,k} := \{z \in F_k \mid \rho_k(F \cap F_k, z) < \frac{1}{n}\}, \quad \overline{B}_{n,k} := \{z \in F_k \mid \rho_k(F \cap F_k, z) \leq \frac{1}{n}\}.$$

Obviously $\lim_{n\to\infty} D_{B_{n,k}} \leq D_{F\cap F_k}$. Also note that since $D_{B_{n,k}}$ is increasing in n and $B_{n,k} \supset F \cap F_k$ for all n we have $\{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\} = \bigcap_{n\geq 1}\{D_{B_{n,k}} < \zeta\} \supset \{D_{F\cap F_k} < \zeta\}$. Fix $z \in E$. Since $\mathbf{M}$ is special standard by quasi-left continuity up to ζ we have

$$\lim_{n\to\infty} Y_{D_{B_{n,k}}} = Y_{\lim_{n\to\infty} D_{B_{n,k}}} \quad P_z\text{-a.s. on } \{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}.$$

But on $\{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}$ we have P_z-a.s. $Y_{D_{B_{n,k}}} \in \overline{B}_{n,k}$ and hence $\lim_{n\to\infty} Y_{D_{B_{n,k}}} = Y_{\lim_{n\to\infty} D_{B_{n,k}}} \in \bigcap_{n\geq 1} \overline{B}_{n,k} = F \cap F_k$. It follows that

$$\lim_{n\to\infty} D_{B_{n,k}} = D_{F\cap F_k} \quad P_z\text{-a.s. on } \{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}.$$

Since $z \in E$ was arbitrary this holds for every $z \in E$. For $A \in \mathcal{F}_\infty$, f $\mathcal{F}_\infty$-measurable, let $E_z\Big[f; A\Big] := E_z\Big[f 1_A\Big]$. Now using that $\lim_{k\to\infty} R_1^{\sigma_{F_k^c},\infty} g = 0$ $\mathcal{E}$-q.e. and $\{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\} \supset \{D_{F\cap F_k} < \zeta\}$ we obtain for $\mathcal{E}$-q.e. $z \in E$

$$\begin{aligned}
R_1^{D_F,\infty} g(z) &= \lim_{k\to\infty} R_1^{D_{F\cap F_k},\infty} g(z) \\
&= \lim_{k\to\infty} E_z\Big[\int_{D_{F\cap F_k}}^\infty e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}\Big] \\
&= \lim_{k\to\infty} E_z\Big[\int_{D_{F\cap F_k}\wedge\sigma_{F_k^c}}^\infty e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}\Big] \\
&= \lim_{k\to\infty}\lim_{n\to\infty} E_z\Big[\int_{D_{B_{n,k}}\wedge\sigma_{F_k^c}}^\infty e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} < \zeta\}\Big] \\
&= \lim_{k\to\infty}\lim_{n\to\infty} R_1^{D_{B_{n,k}}\wedge\sigma_{F_k^c},\infty} g(z)
\end{aligned}$$

where the last identity followed since by Lebesgue's Theorem for $\mathcal{E}$-q.e. $z \in E$

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty} E_z\Big[\int_{D_{B_{n,k}}\wedge\sigma_{F_k^c}}^\infty e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} \geq \zeta\}\Big] \\
= &\lim_{k\to\infty} E_z\Big[\int_{\lim_{n\to\infty} D_{B_{n,k}}\wedge\sigma_{F_k^c}}^\infty e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} \geq \zeta\}\Big] \\
\leq &\lim_{k\to\infty} E_z\Big[\int_{\sigma_{F_k^c}}^\infty e^{-s} g(Y_s)ds; \{\lim_{n\to\infty} D_{B_{n,k}} \geq \zeta\}\Big] = 0.
\end{aligned}$$

Observe that $U_{n,k} := B_{n,k} \cup F_k^c$ is open in E hence $D_{B_{n,k}} \wedge \sigma_{F_k^c} = \sigma_{B_{n,k}} \wedge \sigma_{F_k^c} = \sigma_{U_{n,k}}$. Therefore

$$\begin{aligned}\lim_{k\to\infty}\lim_{n\to\infty} R_1^{D_{B_{n,k}}\wedge\sigma_{F_k^c},\infty} g(z) &= \lim_{k\to\infty}\lim_{n\to\infty} R_1^{D_{B_{n,k}},\infty} g(z)\\ &= \lim_{k\to\infty}\lim_{n\to\infty} R_1^{\sigma_{B_{n,k}},\infty} g(z)\end{aligned}$$

for $\mathcal{E}$-q.e. $z \in E$. Finally, we also have (recall that the bar over an 1-excessive function denotes when not otherwise stated the "canonical" $\mathcal{E}$-q.l.s.c. regularization) $\overline{(G_1g)_{\overline{B}_{n+1,k}}}(z) \le \overline{(G_1g)_{B_{n,k}}}(z) \le \overline{(G_1g)_{\overline{B}_{n,k}}}(z)$ and $\overline{(G_1g)_{B_{n,k}\cup F_k^c}}(z) \le \overline{(G_1g)_{B_{n,k}}}(z) + \overline{(G_1g)_{F_k^c}}(z)$ for $\mathcal{E}$-q.e. $z \in E$. Hence

$$\begin{aligned}\lim_{k\to\infty}\lim_{n\to\infty} R_1^{D_{B_{n,k}}\wedge\sigma_{F_k^c},\infty} g(z) &= \lim_{k\to\infty}\lim_{n\to\infty} \overline{(G_1g)_{B_{n,k}\cup F_k^c}}(z)\\ &= \lim_{k\to\infty}\lim_{n\to\infty} \overline{(G_1g)_{B_{n,k}}}(z)\\ &= \lim_{k\to\infty}\lim_{n\to\infty} \overline{(G_1g)_{\overline{B}_{n,k}}}(z)\end{aligned}$$

for $\mathcal{E}$-q.e. $z \in E$ and (i) follows.
(ii) Let $(B_{n,k})_{n,k\ge1}$, be as in (i). Let $\varphi \in L^2(E;m)\cap\mathcal{B}$, $0<\varphi\le1$. By (i) but in terms of the coassociated process we have

$$\widehat{E}_\cdot\Big(\int_{\widehat{D}_F}^\infty e^{-s} g(\widehat{Y}_s)ds\Big) = \lim_{k\to\infty}\lim_{n\to\infty} (\widehat{G}_1g)_{\overline{B}_{n,k}} \quad \text{m-a.s..}$$

Now similar to the proof of Theorem 1.9 $(\widehat{G}_1g)_{\overline{B}_{n,k}}$ converges weakly in $\mathcal{V}$ (as $n\to\infty$) to some $(\widehat{G}_1g)_{\overline{B}_{\infty,k}}$ such that $supp(\mu_{(\widehat{G}_1g)_{\overline{B}_{\infty,k}}}) \subset F\cap F_k$. Hence

$$\begin{aligned}\int R_1\varphi\, d\mu_{\widehat{E}_\cdot(\int_{\widehat{D}_F}^\infty e^{-s}g(\widehat{Y}_s)ds)} &= \lim_{k\to\infty}\int R_1\varphi\, d\mu_{(\widehat{G}_1g)_{\overline{B}_{\infty,k}}}\\ &= \lim_{k\to\infty}\int R_1^{D_{F\cap F_k},\infty}\varphi\, d\mu_{(\widehat{G}_1g)_{\overline{B}_{\infty,k}}}.\end{aligned}$$

Define for $k\ge1$

$$B'_{l,m} := \{z\in F_m \mid \rho_m(F\cap F_k, z) < \frac{1}{l}\}.$$

Then by (i) and since $\sigma_{B'_{l,m}}$, l, $m\ge1$, is exact we have

$$\begin{aligned}&\lim_{k\to\infty}\int R_1^{D_{F\cap F_k},\infty}\varphi\, d\mu_{(\widehat{G}_1g)_{\overline{B}_{\infty,k}}}\\ &= \lim_{k\to\infty}\lim_{m\to\infty}\lim_{l\to\infty}\lim_{\alpha\to\infty}\int \alpha R_{\alpha+1}R_1^{\sigma_{B'_{l,m}},\infty}\varphi\, d\mu_{(\widehat{G}_1g)_{\overline{B}_{\infty,k}}}\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{k\to\infty}\lim_{m\to\infty}\lim_{l\to\infty}\lim_{\alpha\to\infty}\int \alpha R_{\alpha+1}R_1^{\sigma_{B'_{l,m}},\infty}\varphi\, d\mu_{\widehat{E}.(\int_{\widehat{D}_F}^{\infty} e^{-s}g(\widehat{Y}_s)ds)}\\
&= \int R_1^{D_F,\infty}\varphi\, d\mu_{\widehat{E}.(\int_{\widehat{D}_F}^{\infty} e^{-s}g(\widehat{Y}_s)ds)}
\end{aligned}$$

and the assertion follows.

□

We are now in the situation to formulate the main theorem of this section.

Theorem 1.21 *(**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$) Let $\mu \in S$. Then there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact subsets of E such that*

$$1_{F_k^{reg}}\cdot\mu \in \widehat{S}_{00} \quad \textit{for each} \quad k\geq 1.$$

Proof By Theorem 1.17 we know that there exists an $\mathcal{E}$-nest $(E_k)_{k\geq 1}$ consisting of compact subsets of E, such that $1_{E_k}\cdot\mu\in\widehat{S}_0$ for each k. Choose $k_0\geq 1$. Let $\varphi\in\mathcal{H}$, $0<\varphi\leq 1$. Let $\widetilde{\widehat{G}_1\varphi}$ be an $\mathcal{E}$-q.c. m-version of $\widehat{G}_1\varphi$, let $\overline{\widehat{U}_1(1_{E_{k_0}}\cdot\mu)}$ be an $\mathcal{E}$-q.l.s.c. regularization of $\widehat{U}_1(1_{E_{k_0}}\cdot\mu)$. We may assume that both m-versions are chosen w.r.t. $(E_k)_{k\geq 1}$ and that $\widetilde{\widehat{G}_1\varphi}\geq\frac{1}{k}$ $\mathcal{E}$-q.e. on E_k for each k. Observe that $\overline{\widehat{U}_1(1_{E_{k_0}}\cdot\mu)}$, $\widetilde{\widehat{G}_1\varphi}$ are finite $\mathcal{E}$-q.e. and that $\widetilde{\widehat{G}_1\varphi}>0$ $\mathcal{E}$-q.e.. Define

$$F_k := \{z\in E_k \mid \overline{\widehat{U}_1(1_{E_{k_0}}\cdot\mu)} \leq k^2\widetilde{\widehat{G}_1\varphi}\}.$$

Obviously $(F_k)_{k\in\mathbb{N}}$ is an increasing sequence of compact subsets of E. We first show that $(F_k)_{k\in\mathbb{N}}$ is an $\mathcal{E}$-nest. Indeed

$$\begin{aligned}
\lim_{k\to\infty}\mathrm{cap}_\varphi(F_k^c) &= \lim_{k\to\infty}\mathrm{cap}_\varphi(E_k^c\cup\{z\in E_k \mid \overline{\widehat{U}_1(1_{E_{k_0}}\cdot\mu)} > k^2\widetilde{\widehat{G}_1\varphi}\})\\
&\leq \lim_{k\to\infty}\mathrm{cap}_\varphi(E_k^c) + \lim_{k\to\infty}\mathrm{cap}_\varphi(\{z\in E_k \mid \overline{\widehat{U}_1(1_{E_{k_0}}\cdot\mu)} > k\})\\
&\leq \lim_{k\to\infty}\mathrm{cap}_\varphi(\{\overline{\widehat{U}_1(1_{E_{k_0}}\cdot\mu)} > k\})\\
&\leq \lim_{k\to\infty}\frac{1}{k}\|\varphi\|_{\mathcal{H}}\|\widehat{U}_1(1_{E_{k_0}}\cdot\mu)\|_{\mathcal{H}} = 0
\end{aligned}$$

where the last inequality followed form [23, Proposition 3.6.,p.71].
Since $F_k^{reg}\subset E_k$ implies $1_{F_k^{reg}}\cdot\mu(B)\leq 1_{E_k}\cdot\mu(B)$ for any $B\in\mathcal{B}(E)$ we know further from Lemma 1.16, Lemma 1.15 that $1_{F_k^{reg}}\cdot\mu\in\widehat{S}_0$ for each k. Lemma 1.6 implies that $R_1^{\sigma_{F_k},\infty}\varphi(z) = R_1^{D_{F_k},\infty}\varphi(z)$ for m-a.e. $z\in E$. By Lemma 1.20(ii) we have

$$(\varphi, \widehat{U}_1(1_{F_k^{reg}}\cdot\mu))_{\mathcal{H}} = \int R_1\varphi\, 1_{F_k^{reg}}\, d\mu$$

$$
\begin{aligned}
&= \int E_z\Big[\int_{\sigma_{F_k}}^{\infty} e^{-s}\varphi(Y_s)\,ds\Big]\,1_{F_k^{reg}}(z)\,\mu(dz)\\
&= \lim_{\alpha\to\infty}\int \alpha R_{\alpha+1}E_{\cdot}\Big[\int_{\sigma_{F_k}}^{\infty} e^{-s}\varphi(Y_s)\,ds\Big](z)\,1_{F_k^{reg}}(z)\,\mu(dz)\\
&= \lim_{\alpha\to\infty}\int \alpha\widehat{R}_{\alpha+1}\widehat{U}_1(1_{F_k^{reg}}\cdot\mu))\,d\mu_{E_{\cdot}\Big[\int_{D_{F_k}}^{\infty} e^{-s}\varphi(Y_s)\,ds\Big]}\\
&= \int \overline{\widehat{U}_1(1_{F_k^{reg}}\cdot\mu)}\wedge k^2\widetilde{\widehat{G}_1\varphi}\,d\mu_{E_{\cdot}\Big[\int_{D_{F_k}}^{\infty} e^{-s}\varphi(Y_s)\,ds\Big]}\\
&\le (\varphi,\widehat{U}_1(1_{F_k^{reg}}\cdot\mu)\wedge k^2\widehat{G}_1\varphi)_{\mathcal{H}}.
\end{aligned}
$$

Therefore $\widehat{U}_1(1_{F_k^{reg}}\cdot\mu)\le\widehat{U}_1(1_{F_k^{reg}}\cdot\mu)\wedge k^2\widehat{G}_1\varphi$ which implies the assertion.

$\square$

2 Stochastic analysis by additive functionals

2.1 Positive continuous additive functionals and Revuz measure

We assume that we are given a generalized Dirichlet form $\mathcal{E}$ which is quasi-regular and an m-tight special standard process $\mathbf{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$.

A family $(A_t)_{t\geq 0}$ of extended real valued functions on Ω is called an *additive functional* (abbreviated AF) of $\mathbf{M} = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$, if:

(i) $A_t(\cdot)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.

(ii) There exists a *defining* set $\Lambda \in \mathcal{F}_\infty$ and an $\mathcal{E}$-exceptional set $N \subset E$, such that $P_z[\Lambda] = 1$ for all $z \in E\backslash N$, $\theta_t(\Lambda) \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right continuous on $[0,\infty)$ and has left limits on $(0,\zeta(\omega))$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta(\omega)$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s,t \geq 0$

Two additive functionals $A := (A_t)_{t\geq 0}$, $B := (B_t)_{t\geq 0}$ are called equivalent (notation $A = B$) if for each $t > 0$ $P_z(A_t = B_t) = 1$ for $\mathcal{E}$-q.e. $z \in E$. We can then find a common defining set Λ and a common $\mathcal{E}$-exceptional set N, such that $A_t(\omega) = B_t(\omega)$ for all $\omega \in \Lambda$ and $t \geq 0$. An AF A is called a *continuous additive functional* (abbreviated CAF), if $t \mapsto A_t(\omega)$ is continuous on $[0,\infty)$, a *positive continuous additive functional* (abbreviated PCAF) if it is a CAF and $A_t(\omega) \geq 0$ and a finite AF, if $\mid A_t(\omega) \mid < \infty$ for all $t \geq 0, \omega \in \Lambda$. For a PCAF A and a function $f \in \mathcal{B}^+$ we set $(f\cdot A)_t = \int_0^t f(X_s)\, dA_s$. For a Borel measure ν on E and $B \in \mathcal{B}(E)$ let $P_\nu(B) := \int P_z(B)\nu(dz)$ and let E_ν be the expectation w.r.t. P_ν. The energy of an AF A of $\mathbf{M}$ is then defined by

$$e(A) \;=\; \frac{1}{2}\lim_{\alpha\to\infty} \alpha^2 E_m\Big[\int_0^\infty e^{-\alpha t} A_t^2\, dt\Big], \tag{18}$$

whenever this limit exists in $[0,\infty]$. We will set $\overline{e}(A)$ for the same expression but with $\overline{\lim}$ instead of lim. As usual we set

$$U_A^\alpha f(z) := E_z\Big[\int_0^\infty e^{-\alpha t} f(Y_t)\, dA_t\Big]$$

for a PCAF A of $\mathbf{M}$ and $f \in \mathcal{B}^+$. Then we have the following resolvent equations (cf. e.g. [22, 36.16]) for $0 \leq \alpha < \beta$ and $f \in \mathcal{B}^+$

$$U_A^\alpha f = U_A^\beta f + (\beta-\alpha)R_\alpha U_A^\beta f = U_A^\beta f + (\beta-\alpha)R_\beta U_A^\alpha f, \tag{19}$$

but one has to be careful not to subtract when no finiteness assumptions on $U_A^\beta f$ are made.

From now on up to the end of this section let us assume that the coresolvent $(\widehat{G}_\alpha)_{\alpha>0}$ associated with $\mathcal{E}$ is sub-Markovian. As before we use the abbreviation $\widehat{\mathbf{SUB}}$ for this assumption.

Theorem 2.1 *(**QR**, $\mathbf{M}^{\mathbf{ex}}$, $\widehat{\mathbf{SUB}}$) Let A be a* PCAF *of* $\mathbf{M}$*. Then there exists a unique positive measure μ_A on $(E, \mathcal{B}(E))$, charging no $\mathcal{E}$-exceptional set and called the Revuz measure of A, such that*

$$\int_E f\, d\mu_A \;=\; \lim_{\alpha\to\infty} \alpha E_m\Big[\int_0^\infty e^{-\alpha t} f(Y_t)\, dA_t\Big] \quad \text{for all } f \in \mathcal{B}^+. \tag{20}$$

Furthermore, there exists an $\mathcal{E}$-nest of compact sets $(F_n)_{n\geq 1}$ such that $\mu_A(F_n) < \infty$ and such that $U_A^1 1_{F_n}$ is an $\mathcal{E}$-q.l.s.c. m-version of some element in $\mathcal{P}_{G_1\mathcal{H}_b^+}$ for each n.

Proof By sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ and (19) we have

$$\int_E \alpha U_A^\alpha f\, dm = \int_E \alpha U_A^{\alpha+1} f + \alpha R_\alpha U_A^{\alpha+1} f\, dm \leq \int_E (\alpha+1) U_A^{\alpha+1} f\, dm.$$

Hence μ_A exists as an increasing limit. Clearly μ_A does not charge $\mathcal{E}$-exceptional sets. Fix $\varphi \in L^1(E;m)\cap\mathcal{B}$ such that $0<\varphi\leq 1$. For $z\in E$ set

$$\Phi(z) := E_z\Big[\int_0^\zeta e^{-s}\varphi(Y_s) e^{-A_s}\, ds\Big].$$

We have for $\mathcal{E}$-q.e. $z \in E$

$$\begin{aligned} e^{-t} p_t(R_1\varphi - \Phi)(z) &= E_z\Big[\int_t^\zeta e^{-s}\varphi(Y_s)(1 - e^{-A_s} e^{A_t})\, ds\Big] \\ &\leq E_z\Big[\int_t^\zeta e^{-s}\varphi(Y_s)(1 - e^{-A_s})\, ds\Big] \\ &\leq (R_1\varphi - \Phi)(z) \end{aligned} \tag{21}$$

and

$$\lim_{t\downarrow 0} e^{-t} p_t(R_1\varphi - \Phi)(z) \;=\; (R_1\varphi - \Phi)(z). \tag{22}$$

By (21) we know that $R_1\varphi - \Phi$ is 1-supermedian for $(R_\alpha)_{\alpha>0}$. Since $R_1\varphi - \Phi \leq R_1\varphi \in L^2(E;m)$ it follows from [23, *Lemma III*.2.1., *p*.65] that $R_1\varphi - \Phi$ is an m-version of some 1-excessive element in $\mathcal{P}_{G_1\mathcal{H}_b^+}$. It is easy to see that (21) together with (22) imply

$$\sup_{\alpha\geq 1} \alpha R_{\alpha+1}(R_1\varphi - \Phi)(z) = (R_1\varphi - \Phi)(z)$$

for $\mathcal{E}$-q.e. $z \in E$ and therefore $R_1\varphi - \Phi$ is $\mathcal{E}$-q.l.s.c.. A simple calculation (cf. the original proof [20, *p*.509]) gives

$$U_A^1\Phi = R_1\varphi - \Phi \quad \mathcal{E}\text{-q.e.}$$

hence $-\Phi$ is $\mathcal{E}$-q.l.s.c.. Let $(E_n)_{n\geq 1}$ be an $\mathcal{E}$-nest of compact sets such that $-\Phi \in C_l(\{E_n\})$. It follows that

$$F_n := E_n \cap \{\Phi \geq \frac{1}{n}\},\ n\geq 1,$$

is closed in E_n hence compact. We will show that $(F_n)_{n\geq 1}$ is an $\mathcal{E}$-nest. To see this set

$$B_n = E_n \cap \{\Phi < \frac{1}{n}\}, \ n \geq 1,$$

Then $F_n^c = B_n \cup E_n^c$. Let $\{t_i\}_{i\in\mathbb{N}} \subset (0,\infty)$ dense and define for $A \subset \mathcal{B}(E)$ stopping times $\sigma_A^k := \min\{t_i \mid 1 \leq i \leq k, Y_{t_i} \in A\}$, $k \geq 1$, with the convention $\min \emptyset = \infty$. Also set $e^{-\infty} f(Y_\infty) = 0$ for $f \in \mathcal{B}_b^+$. Observe that for $U \subset E$, U open $\sigma_U = \lim_{k\to\infty} \sigma_U^k$. Then, using Lebesgue's Theorem

$$\begin{aligned} E_z\Big[\int_{\sigma_{F_n^c}}^{\zeta} e^{-s}\varphi(Y_s)e^{-A_s}\,ds\Big] &= \lim_{k\to\infty} E_z\Big[\int_{\sigma^k_{B_n\cup E_n^c}}^{\zeta} e^{-s}\varphi(Y_s)e^{-A_s}\,ds\Big] \\ &= \lim_{k\to\infty} E_z\Big[\int_{\sigma^k_{B_n}}^{\zeta} e^{-s}\varphi(Y_s)e^{-A_s}\,ds\Big] \\ &= \lim_{k\to\infty} E_z\Big[e^{-\sigma^k_{B_n} - A_{\sigma^k_{B_n}}}\,\Phi(Y_{\sigma^k_{B_n}})\Big] \\ &\leq \frac{1}{n} \end{aligned}$$

for $\mathcal{E}$-q.e. $z \in E$. Using again Lebesgue's Theorem and that $e^{-s}\varphi(Y_s)e^{-A_s} > 0$ on $s < \zeta$ $P_z - a.s$ for $\mathcal{E}$-q.e. $z \in E$ we obtain $P_m(\lim_{n\to\infty} \sigma_{F_n^c} < \zeta) = 0$ and consequently $(F_n)_{n\geq 1}$ is an $\mathcal{E}$-nest by [23, *Remark IV.3.6., p.91*].
Finally (as in [20, *Lemme II.2, p.508*]) $U_A^1 1_{F_n} \leq n\, U_A^1 \Phi \leq n\, R_1\varphi$ $\mathcal{E}$-q.e. implies that $\mu_A(F_n) < \infty$, $n \geq 1$. Indeed, by the resolvent equation (19) and the sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ we have for all $\beta \geq 1$, $n \geq 1$

$$\begin{aligned} \int \beta\,(R_\beta n\,\varphi - U_A^\beta 1_{F_n})\,dm &= \int \beta\,(R_1 n\,\varphi - U_A^1 1_{F_n}) - (\beta-1)\beta\, R_\beta(R_1 n\,\varphi - U_A^1 1_{F_n})\,dm \\ &\geq \int R_1 n\,\varphi - U_A^1 1_{F_n}\,dm \ \geq \ 0, \end{aligned}$$

hence $\int \beta\, R_\beta n\,\varphi\,dm \geq \int \beta\, U_A^\beta 1_{F_n}\,dm$ for all $\beta \geq 1$, and therefore $\mu_A(F_n) \leq \int n\,\varphi\,dm < \infty$, $n \geq 1$. Clearly $U_A^1 1_{F_n}$ is $\mathcal{E}$-q.l.s.c. and by [23, Lemma III.2.1.(i),p.65] $U_A^1 1_{F_n}$ is an m-version of some element in $\mathcal{P}_{G_1\mathcal{H}_b^+}$ for each n.

□

Remark 2.2 *(i) (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{SUB}}$) Let A be a* PCAF *and let μ_A be the associated Revuz measure of Theorem 2.1. We know that there exists an $\mathcal{E}$-nest of compact sets $(F_n)_{n\in\mathbb{N}}$ such that $\mu_A(F_n) < \infty$ and such that $U_A^1 1_{F_n}$ is an $\mathcal{E}$-q.l.s.c. m-version of some element in*

$\mathcal{P}_{G_1\mathcal{H}_b^+}$. *Hence by (19) for any $\mu \in \widehat{S}_{00}$, $t > 0$*

$$
\begin{aligned}
E_\mu\Big[\,A_t\,\Big] &= \lim_{n\to\infty}\int E_\cdot\Big[\int_0^t 1_{F_n}(Y_s)\,dA_s\Big]\,d\mu \\
&\le e^t\lim_{n\to\infty}\int E_\cdot\Big[\int_0^\infty e^{-s}1_{F_n}(Y_s)\,dA_s\Big]\,d\mu \\
&= e^t\lim_{n\to\infty}\lim_{\alpha\to\infty}\mathcal{E}_1(\alpha R_{\alpha+1}\,U_A^1 1_{F_n},\widehat{U}_1\mu) \\
&= e^t\lim_{n\to\infty}\lim_{\alpha\to\infty}\mathcal{E}_1(R_1(\alpha U_A^{\alpha+1}1_{F_n}),\widehat{U}_1\mu) = e^t|\widehat{U}_1\mu|_\infty\mu_A(E).
\end{aligned}
$$

where $|\cdot|_\infty$ denotes the $L^\infty(E;m)$-norm.
*(ii) (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{SUB}}$) Since μ_A is σ-finite there exists $g \in L^1(E;\mu_A)$ with $g > 0$. Let $f \in \mathcal{B}^+$. Then*

$$
\begin{aligned}
\int f\,d\mu_A &= \sup_{n\ge 1}\int ng\wedge f\,d\mu_A \\
&= \sup_{n\ge 1}\lim_{\alpha\to\infty}\alpha(U_A^\alpha(ng\wedge f),1) \\
&= \sup_{n\ge 1}\sup_{t>0}\frac{1}{t}E_m\Big[\int_0^t (ng\wedge f)(Y_s)\,dA_s\Big] \\
&= \sup_{t>0}\frac{1}{t}E_m\Big[\int_0^t f(Y_s)\,dA_s\Big].
\end{aligned}
$$

The third equality follows from a theorem of Tauber (cf. [29, V 4.3, p.192]) and since $t \mapsto E_m\Big[\int_0^t f(Y_s)dA_s\Big]$ is subadditive in t.

Let us from now on up to the end of this section assume that the coform $\widehat{\mathcal{E}}$ is a quasi-regular generalized Dirichlet form and that there exists an m-tight special standard process $\widehat{\mathbf{M}}$ which is properly coassociated in the resolvent sense with $\widehat{\mathcal{E}}$ (cf. paragraph just before Lemma 1.20). As before we use the abbreviations $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$ for these assumptions. Note that the assumption $\widehat{\mathbf{QR}}$ includes the assumption $\widehat{\mathbf{SUB}}$.

Lemma 2.3 *(**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$) Let μ_A be the measure defined in Theorem 2.1. Then we have for every $f \in \mathcal{B}^+$, $g \in L^2(E;m)\cap\mathcal{B}_b(E)^+$, $\beta > 0$*

$$
\int f\widehat{R}_\beta g\,d\mu_A = \lim_{\alpha\to\infty}\alpha(U_A^{\alpha+\beta}f,\widehat{R}_\beta g). \tag{23}
$$

Proof (cf. proof of [16, *Lemma* 4.1.7., *p*.91]) Let $f \in \mathcal{B}^+$, $g \in L^2(E;m)\cap\mathcal{B}_b^+$, $\beta > 0$. Similar to the proof of 2.1 the measure $\mu_A^{\widehat{R}_\beta g}(f) := \lim_{\alpha\to\infty}\alpha(U_A^{\alpha+\beta}f,\widehat{R}_\beta g)$ exists and is σ-finite. Furthermore since $t \mapsto E_{\widehat{R}_\beta g\,m}\Big[\int_0^t e^{-\beta s}d(f\cdot A)_s\Big]$ is subadditive in t, it follows

similar to Remark 2.2 that $\mu_A^{\widehat{R}_\beta g}(f) = \lim_{t\downarrow 0}\frac{1}{t}E_{\widehat{R}_\beta g\, m}\Big[\int_0^t e^{-\beta s} f(Y_s)\,dA_s\Big]$.
Now let $g \in L^2(E;m)\cap \mathcal{B}_b(E)^+$ such that $0\le g\le 1$ and $h := \beta\widehat{R}_\beta g$, $\beta>0$. Let further $f\in\mathcal{B}^+\cap L^1(E;\mu_A)$. By Remark 2.2 we have

$$\begin{aligned}\int hf\,d\mu_A &= \lim_{t\downarrow 0}\frac{1}{t}E_m\Big[\int_0^t h(Y_s)f(Y_s)\,dA_s\Big]\\ &= \lim_{t\downarrow 0}\frac{1}{t}E_m\Big[\int_0^t h(Y_s)\,d(f\cdot A)_s\Big].\end{aligned}$$

Since h is $\mathcal{E}$-q.c. ($=\widehat{\mathcal{E}}$-q.c.) it follows that the integrand above is an integral in the sense of Lebesgue-Stieltjès. Therefore we have

$$\begin{aligned}\int hf\,d\mu_A &= \lim_{t\downarrow 0}\frac{1}{t}E_m\Big[\lim_{n\to\infty}\sum_{k=0}^{n-1}h(Y_{kd})((f\cdot A)_{(k+1)d}-(f\cdot A)_{kd})\Big]\\ &= \lim_{t\downarrow 0}\lim_{n\to\infty}\frac{1}{t}\sum_{k=0}^{n-1}E_m\Big[h(Y_{kd})((f\cdot A)_d\circ\theta_{kd})\Big]\end{aligned}$$

where $d := \frac{t}{n}$. Then by the strong Markov property

$$\int hf\,d\mu_A = \lim_{t\downarrow 0}\lim_{n\to\infty}\frac{1}{t}\sum_{k=0}^{n-1}\int_E E_z\Big[h(Y_{kd})E_{Y_{kd}}\Big[(f\cdot A)_d\Big]\Big]\,m(dz).$$

Set $u(z) := h(z)E_z\Big[(f\cdot A)_d\Big]$. The sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ implies that m is p_t-supermedian and then

$$\begin{aligned}\int hf\,d\mu_A &= \lim_{t\downarrow 0}\lim_{n\to\infty}\frac{1}{t}\sum_{k=0}^{n-1}\int_E p_{kd}u(z)\,m(dz)\\ &\le \lim_{t\downarrow 0}\lim_{n\to\infty}\frac{n}{t}\int_E h(z)E_z\Big[(f\cdot A)_d\Big]\,m(dz)\\ &= \lim_{d\downarrow 0}\frac{1}{d}E_{hm}\Big[(f\cdot A)_d\Big]\\ &= \lim_{d\downarrow 0}\frac{1}{d}E_{hm}\Big[\int_0^d e^{-\beta s}d(f\cdot A)_s\Big] \;=\; \mu_A^h(f).\end{aligned}$$

h is β-coexcessive, so is $1-h$. Therefore we will get as above $\int(1-h)f\,d\mu_A\le\mu_A(f)-\mu_A^h(f)$, which further implies (23) for $h=\alpha\widehat{R}_\alpha g$, $g\in L^2(E;m)\cap\mathcal{B}_b^+$, $f\in\mathcal{B}^+\cap L^1(E;\mu_A)$. Since μ_A is σ-finite the statement of the Lemma follows by a standard argument.

□

Before we state and prove Lemma 2.4 let us remark the following. Although a lot of statements in [23, Chapter III.,IV.3] are only formulated for 1-excessive functions (resp. 1-coexcessive functions) they readily extend to γ-excessive functions (resp. γ-coexcessive functions) for any $\gamma > 0$. One only has to observe that the solutions $h_\alpha \in \mathcal{F}$ and $\hat{h}_\alpha \in \widehat{\mathcal{F}}$ in [23, *Proposition III*.1.6.] are given for $\mathcal{E}_1$ hence for $\gamma = 1$. Actually these solutions exist for any $\gamma > 0$. We will make use of this fact in the proof of Lemma 2.4 and shall here present briefly the modified statements of [23] if we allow γ to vary over $(0, \infty)$.

For any $\gamma > 0$, $f \in \mathcal{B}_b^{*+}$, $E_z\Big[e^{-\gamma\sigma_U} R_\gamma f(Y_{\sigma_U})\Big]$ is an $\mathcal{E}$-q.l.s.c. m-version of the γ-reduced function $(R_\gamma f)_{U,\gamma}$ of $R_\gamma f$ on U (cf. [23, IV.3.4.] where this is proved for $\gamma = 1$).
If $f \in \mathcal{F}$ is γ-excessive, $\hat{f} \in \widehat{\mathcal{F}}$ is γ-coexcessive and $U \subset E$ is open then

$$\mathcal{E}_\gamma(f_{U,\gamma}, \hat{f}) = \mathcal{E}_\gamma(f, \hat{f}_{U,\gamma})$$

where $f_{U,\gamma}$ (resp. $\hat{f}_{U,\gamma}$) is the γ-reduced function of f on U (resp. the γ-coreduced function of $\hat{f}$ on U).

Let $\hat{\tau}$ be an exact $(\widehat{\mathcal{F}}_t)$-terminal time. Let us define $(\widehat{R}_\alpha^{0,\hat{\tau}})_{\alpha>0}$ exactly as $(R_\alpha^{0,\tau})_{\alpha>0}$ (cf. paragraph before Lemma 1.6) but in terms of the coassociated process. Note that $(\widehat{R}_\alpha^{0,\hat{\tau}})_{\alpha>0}$ is a resolvent as was already shown for $(R_\alpha^{0,\tau})_{\alpha>0}$ in Lemma 1.7(i). A function $f \in \mathcal{H} \cap \mathcal{B}^{*+}$ is called *1-cosupermedian for* $(\widehat{R}_\alpha^{0,\hat{\tau}})_{\alpha>0}$ if $\alpha \widehat{R}_{\alpha+1}^{0,\hat{\tau}} f \le f$, $\alpha > 0$. $f \in \mathcal{H} \cap \mathcal{B}^{*+}$ is called *1-coexcessive for* $(\widehat{R}_\alpha^{0,\hat{\tau}})_{\alpha>0}$ if f is 1-supermedian for $(\widehat{R}_\alpha^{0,\hat{\tau}})_{\alpha>0}$ and if $\lim_{\alpha\to\infty} \alpha \widehat{R}_{\alpha+1}^{0,\hat{\tau}} f = f$.

Lemma 2.4 *(*$\mathbf{QR}$*,* $\mathbf{M^{ex}}$*,* $\widehat{\mathbf{QR}}$*,* $\widehat{\mathbf{M}}^{\mathbf{ex}}$*) Let $F \subset E$, F closed. Let A be a* PCAF *and let μ_A be the corresponding Revuz measure of Theorem 2.1. Then*

$$\lim_{\alpha\to\infty} \alpha E_{\hat{h}m}\Big[\int_0^{\sigma_{F^c}} e^{-(\alpha+1)t} f(Y_t)\, dA_t\Big] = \int_{E\setminus \widehat{F^{reg}}} \hat{h}\, f\, d\mu_A$$

for any γ-coexcessive function $\hat{h}$ w.r.t. $(\widehat{R}_\alpha^{0,\hat{\sigma}_{F^c}})_{\alpha>0}$ and any $f \in \mathcal{B}^+$. Furthermore the limit is increasing.

Proof It is enough to show the statement when $f \in \mathcal{B}_b^+$ and $\hat{h}$ is 1-coexcessive w.r.t. $(\widehat{R}_\alpha^{0,\hat{\sigma}_{F^c}})_{\alpha>0}$. By Theorem 2.1 there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets such that $U_A^1 1_{F_k} \in \mathcal{P}_{G_1\mathcal{H}_b^+}$ for any k. It is easy to see that the strong Markov property of $\mathbf{M}$ implies $E_z\Big[e^{-\beta\sigma} U_A^\beta g(Y_\sigma)\Big] = E_z\Big[\int_\sigma^\infty e^{-\beta s} g(Y_s) dA_s\Big]$, for any $\mathcal{F}_t$-stopping time σ, $\beta > 0$, $g \in \mathcal{B}^+$. Then by (19) and the remark in the paragraph just before the statement of this lemma

$$\begin{aligned}
&\alpha E_{\hat{h}m}\Big[\int_0^{\sigma_{F^c}} e^{-(\alpha+1)t} f(Y_t)\, d(1_{F_k}\cdot A)_t\Big] \\
=\ & \alpha(\hat{h}, U^{\alpha+1}_{(1_{F_k}\cdot A)} f - E_\cdot\Big[e^{-(\alpha+1)\sigma_{F^c}} U^{\alpha+1}_{(1_{F_k}\cdot A)} f(Y_{\sigma_{F^c}})\Big])_{\mathcal{H}} \\
=\ & \lim_{\beta\to\infty} \alpha(\hat{h}, \beta R_{\beta+\alpha+1} U^{\alpha+1}_{(1_{F_k}\cdot A)} f - E_\cdot\Big[e^{-(\alpha+1)\sigma_{F^c}} \beta R_{\beta+\alpha+1} U^{\alpha+1}_{(1_{F_k}\cdot A)} f(Y_{\sigma_{F^c}})\Big])_{\mathcal{H}}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\beta\to\infty} \alpha\mathcal{E}_{\alpha+1}(R_{\alpha+1}\beta U^{\beta+\alpha+1}_{(1_{F_k}\cdot A)}f - (R_{\alpha+1}\beta U^{\beta+\alpha+1}_{(1_{F_k}\cdot A)}f)_{F^c,\alpha+1}, \widehat{R}_{\alpha+1}\hat{h}) \\
&= \lim_{\beta\to\infty} \alpha\mathcal{E}_{\alpha+1}(R_{\alpha+1}\beta U^{\beta+\alpha+1}_{(1_{F_k}\cdot A)}f, \widehat{R}_{\alpha+1}\hat{h} - (\widehat{R}_{\alpha+1}\hat{h})_{F^c,\alpha+1}) \\
&= \int_{F_k} f\alpha \widehat{R}^{0,\hat{\sigma}_{F^c}}_{\alpha+1}\hat{h}\, d\mu_A.
\end{aligned}
$$

Now the assertion follows if we let first k and then α tend to infinity.

□

Theorem 2.5 *(*$\mathbf{QR}$*,* $\mathbf{M^{ex}}$*,* $\widehat{\mathbf{QR}}$*,* $\widehat{\mathbf{M}}^{\mathbf{ex}}$*) Let A, B be two* PCAF*'s. If $\mu_A = \mu_B$ then $A = B$ (i.e. $A \sim B$).*

Proof Fix $\varphi \in L^1(E;m) \cap \mathcal{B}$, $0 < \varphi \le 1$. We may choose (cf. the proof of Theorem 2.1) an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$ consisting of compact sets such that $U^\beta_A 1_{F_k}$, $U^\beta_B 1_{F_k} \le kR_1\varphi$ for every $k \in \mathbb{N}$. It follows from (19) and (23) that for any $g \in L^2(E;m) \cap \mathcal{B}^+_b$, $k \ge 1$

$$
\begin{aligned}
(U^\beta_A 1_{F_k}, g)_{\mathcal{H}} &= \lim_{\alpha\to\infty} (\alpha R_{\alpha+\beta} U^\beta_A 1_{F_k}, g)_{\mathcal{H}} \\
&= \lim_{\alpha\to\infty} (\alpha U^{\alpha+\beta}_A 1_{F_k}, \widehat{R}_\beta g)_{\mathcal{H}} \\
&= \int \widehat{R}_\beta g 1_{F_k}\, d\mu_A \\
&= \int \widehat{R}_\beta g 1_{F_k}\, d\mu_B \\
&= \lim_{\alpha\to\infty} (\alpha U^{\alpha+\beta}_B 1_{F_k}, \widehat{R}_\beta g)_{\mathcal{H}} \\
&= \lim_{\alpha\to\infty} (\alpha R_{\alpha+\beta} U^\beta_B 1_{F_k}, g)_{\mathcal{H}} = (U^\beta_B 1_{F_k}, g)_{\mathcal{H}}.
\end{aligned}
$$

Hence $U^\beta_A 1_{F_k} = U^\beta_B 1_{F_k}$ m-a.e. and therefore $R_{\alpha+\beta}U^\beta_A 1_{F_k} = R_{\alpha+\beta}U^\beta_B 1_{F_k}$ $\mathcal{E}$-q.e.. It follows

$$
U^\beta_A 1_{F_k} = \lim_{\alpha\to\infty} \alpha R_{\alpha+\beta} U^\beta_A 1_{F_k} = \lim_{\alpha\to\infty} \alpha R_{\alpha+\beta} U^\beta_B 1_{F_k} = U^\beta_B 1_{F_k} \quad \mathcal{E}\text{-q.e..}
$$

Now fix k_0. Let $N \subset E$ be such that $\overline{u}(z) := U^1_A 1_{F_{k_0}}(z) = U^1_B 1_{F_{k_0}}(z)$ for all $z \in E \setminus N$. Let $A^{(1)} := \int_0^\cdot 1_{F_{k_0}}(Y_s)dA_s$, $A^{(2)} := \int_0^\cdot 1_{F_{k_0}}(Y_s)dB_s$. Let

$$
v_{ij}(z) := E_z\Big[\int_0^\infty e^{-t} dA^{(i)}_t \int_0^\infty e^{-t} dA^{(j)}_t\Big], \quad 1 \le i,j \le 2.
$$

A simple calculation (cf. [8, proof of Theorem 5.1.2.]) leads to

$$
v_{ij}(z) = \Sigma_{k=i,j} E_z\Big[\int_0^\infty e^{-2t}\overline{u}(Y_t) dA^{(k)}_t\Big], \quad z \in E \setminus N.
$$

But since $\overline{u}(z) \le k_0 R_1\varphi(z) \le k_0$ for $\mathcal{E}$-q.e. $z \in E$ we have

$$
\begin{aligned}
v_{ij}(z) &\le k_0 \Sigma_{k=i,j} E_z\Big[\int_0^\infty e^{-2t} dA_t^{(k)}\Big] \\
&\le 2k_0 \overline{u}(z) \\
&\le 2k_0^2 R_1\varphi(z).
\end{aligned}
$$

and hence $\int v_{ij}\, d\mu < \infty$ for any $\mu \in \widehat{S}_{00}$. It follows

$$
E_\mu\Big[\{\int_0^\infty e^{-t} dA_t^{(1)} - \int_0^\infty e^{-t} dA_t^{(2)}\}^2\Big] = \int v_{11} - 2v_{12} + v_{22}\, d\mu = 0.
$$

From this it is easy to see that $(1_{F_{k_0}} \cdot A)$ is equivalent to $(1_{F_{k_0}} \cdot B)$. Since k_0 was arbitrary we get $A = B$.

$\square$

2.2 Fukushima's decomposition of AF's and its extension

From the beginning of this section we assume that we are given a generalized Dirichlet form $\mathcal{E}$ which is quasi-regular and an m-tight special standard process $\mathbf{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$. We do further assume that the coresolvent $(\widehat{G}_\alpha)_{\alpha>0}$ associated with $\mathcal{E}$ is sub-Markovian. As before we use the abbreviations **QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{SUB}}$ for these assumptions.
For the proof of the main theorem of this section namely 2.11 below we follow the same strategy as in [8, Chapter 5].
Let $\widetilde{u}$ be an $\mathcal{E}$-q.c. function. Then by quasi-regularity $(\widetilde{u}(Y_t) - \widetilde{u}(Y_0))_{t\ge 0}$ is an AF of $\mathbf{M}$, and independent (up to equivalence) of the special choice $\widetilde{u}$. We then set

$$
A^{[u]} = (\widetilde{u}(Y_t) - \widetilde{u}(Y_0))_{t\ge 0}. \tag{24}
$$

It follows from the sub-Markovianity of $(\widehat{G}_\alpha)_{\alpha>0}$ that for $\widetilde{u} \in \widetilde{\mathcal{H}}$

$$
\begin{aligned}
\overline{e}(A^{[u]}) &= \frac{1}{2}\overline{\lim_{\alpha\to\infty}}\, \alpha^2 E_m\Big[\int_0^\infty e^{-\alpha t}(\widetilde{u}(Y_t) - \widetilde{u}(Y_0))^2\, dt\Big] \\
&= \frac{1}{2}\overline{\lim_{\alpha\to\infty}}\, \alpha \int \alpha R_\alpha u^2 - 2\alpha u R_\alpha u + 2u^2 - u^2\, dm \\
&= \overline{\lim_{\alpha\to\infty}}\, \alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}} - \frac{1}{2}\alpha \int u^2(1 - \alpha\widehat{G}_\alpha 1)\, dm \\
&\le \overline{\lim_{\alpha\to\infty}}\, \alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}}.
\end{aligned} \tag{25}
$$

Note that $\alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}} = \mathcal{E}(\alpha G_\alpha u, u) = \mathcal{E}(u, \alpha\widehat{G}_\alpha u)$ for $u \in \mathcal{V}$. Hence in the case where $(G_\alpha)_{\alpha>0}$ is strongly continuous on $\mathcal{F}$ or $(\widehat{G}_\alpha)_{\alpha>0}$ is strongly continuous on $\mathcal{V}$ we know that $\overline{e}(A^{[u]})$ is dominated by $\mathcal{E}(u,u)$ for all $u \in \mathcal{F}$. But in general we have the following.

Lemma 2.6 *(**QR**, $\mathbf{M^{ex}}$) Let $u \in \mathcal{F}$. Then*

$$\overline{\lim_{\alpha\to\infty}}\ \alpha(u-\alpha G_\alpha u,u)_{\mathcal{H}} \le (K+1)^2\|u\|_{\mathcal{F}}^2 \tag{26}$$

In particular it holds that $\overline{e}(A^{[u]}) \le (K+1)^2\|u\|_{\mathcal{F}}^2$ *if in addition* $(\widehat{G}_\alpha)_{\alpha>0}$ *is sub-Markovian.*

Proof Since

$$\begin{aligned}\overline{\lim_{\alpha\to\infty}}\ \alpha(u-\alpha G_\alpha u,u)_{\mathcal{H}} &= \overline{\lim_{\alpha\to\infty}}\,\mathcal{E}(u,\alpha\widehat{G}_\alpha u)\\ &\le \overline{\lim_{\alpha\to\infty}}\,\mathcal{E}_1(u,\alpha\widehat{G}_\alpha u)\\ &\le \overline{\lim_{\alpha\to\infty}}(K+1)\|u\|_{\mathcal{F}}\|\alpha\widehat{G}_\alpha u\|_{\mathcal{V}}\end{aligned}$$

and

$$\begin{aligned}\|\alpha\widehat{G}_\alpha u\|_{\mathcal{V}}^2 &\le \mathcal{E}_1(\alpha\widehat{G}_\alpha u,\alpha\widehat{G}_\alpha u)\\ &\le \mathcal{E}_1(u,\alpha\widehat{G}_\alpha u)\\ &\le (K+1)\|u\|_{\mathcal{F}}\|\alpha\widehat{G}_\alpha u\|_{\mathcal{V}}\end{aligned}$$

the first assertion follows. The final assertion now follows from (25).

□

We will now introduce the spaces which will be relevant for our further investigations. Define

$$\begin{aligned}\mathcal{M} &:= \{M|M \text{ is a finite AF}, E_z\Big[\,M_t^2\,\Big]<\infty, E_z\Big[\,M_t\,\Big]=0\\ &\qquad \text{for } \mathcal{E}\text{-q.e } z\in E \text{ and all } t\ge 0\}.\end{aligned}$$

$M \in \mathcal{M}$ is called a *martingale additive functional* (MAF). Furthermore define

$$\overset{\circ}{\mathcal{M}} = \{M\in\mathcal{M}|\ e(M)<\infty\}. \tag{27}$$

The elements of $\overset{\circ}{\mathcal{M}}$ are called MAF's of finite energy.
Let $M \in \mathcal{M}$. There exists an $\mathcal{E}$-exceptional set N, such that $(M_t,\mathcal{F}_t,P_z)_{t\ge0}$ is a square integrable martingale for all $z \in E\setminus N$. Now the following will be used quite often in the sequel: there exists a unique (up to equivalence) PCAF $< M >$, called the sharp bracket of M, such that $(M_t^2 - < M >_t, \mathcal{F}_t, P_z)_{t\ge0}$ is a martingale for all $z\in E\setminus N$.
We comment this. Formulations in probability theory concern a single probability space in general. On the other hand the MAF's M of $\mathcal{M}$ are defined w.r.t. a possibly uncountable family $(P_z)_{z\in E\setminus N}$ of probability measures. Hence the classical Doob-Meyer decompostion of the submartingale M^2 would only provide us a process $< M >^z$ such that $M^2 - < M >^z$ would be a P_z-martingale for any $z\in E\setminus N$. The sharp bracket independent of the paramenter $z \in E\setminus N$ was first constructed in [13, III.Théorème 3]. Following the lines

of argument in $[8, A.3.]$ we will briefly illustrate how this can be done. We remark that the construction has nothing to do with topological specialities of the state space E since only processes from abstract probabilty spaces into $\mathbb{R}$ are involved. We also remark that the continuity of $< M >$ (as in the classical case) results from the quasi-left continuity of the (universally completed) natural filtration $(\mathcal{F}_t)_{t\geq 0}$.
Since M is a P_z-martingale for any $z \in E \setminus N$ we can consider its usual P_z-square bracket $[M]^z$. It is well known that

$$\sum_{0\leq i\leq 2^n} (M_{t_{i+1}} - M_{t_i})^2 \ ; \ t_i = it2^{-n}$$

converges in $L^1(P_z)$ and in probability P_z to $[M]^z$. Hence by $[22, Theorem\,(51.17)]$ or $[8, Lemma\,A.3.3]$ there exists a càdlàg $\mathcal{F}_t$-adapted process $[M]$ which is P_z-inditinguishable from $[M]^z$ for any $z \in E \setminus N$. Then $< M >:= [M]^p$, i.e. the dual predictable projection of $[M]$ is the desired process. It is positive and an AF by $[8, Theorem\,A.3.17]$ and a method which has been called a perfection. It remains to show the continuity. Since $\mathbf{M}$ is special it follows from $[22, Theorem\,(47.6)]$ $\mathcal{F}_T = \mathcal{F}_{T-}$ for any predictable stopping time T hence any $\mathcal{F}_t$-martingale is quasi-left continuous which further implies that $[M]^p$ is continuous. Since $< M >$ is P_z-indistinguishable from the unique increasing predictable process $< M >^z$ in the Doob-Meyer decomposition of M^2 w.r.t. P_z we know further that $M^2 - < M >$ is a P_z-martingale.
It now follows that one half of the total mass of the Revuz measure $\mu_{<M>}$ associated to the sharp bracket of $M \in \mathcal{M}$ is equal to the energy of M, i.e.

$$e(M) = \frac{1}{2}\int_E d\mu_{<M>}. \tag{28}$$

For $M,\, L \in \overset{\circ}{\mathcal{M}}$ let

$$< M, L >_t = \frac{1}{2}\,(< M + L >_t - < M >_t - < L >_t).$$

Then $(< M, L >_t)_{t\geq 0}$ is a CAF of bounded variation on each finite interval of t and satisfies

$$E_z(M_t\, L_t) = E_z(< M, L >_t) \quad \forall t \geq 0,\ \mathcal{E}\text{-q.e. } z \in E.$$

Furthermore the finite signed measure $\mu_{<M,L>}$ defined by $\mu_{<M,L>} = \frac{1}{2}(\mu_{<M+L>} - \mu_{<M>} - \mu_{<L>})$ is related to $< M, L >$ in the sense of relation (20). If $f \in \mathcal{B}_b^+$, then $f \cdot \mu_{<\cdot,\cdot>}$ is symmetric, bilinear and positive on $\overset{\circ}{\mathcal{M}} \times \overset{\circ}{\mathcal{M}}$, where $f \cdot \mu_{<M,L>}(A) := \int_A f\, d\mu_{<M,L>}$ for every $A \in \mathcal{B}(E)$ and every pair $(M, L) \in \overset{\circ}{\mathcal{M}} \times \overset{\circ}{\mathcal{M}}$. Hence by (28) and Cauchy-Schwarz's inequality we have for any $f \in \mathcal{B}_b^+$

$$\left| \int_E f\, d\mu_{<M,L>} \right| \leq \|f\|_\infty e(M)^{1/2} e(L)^{1/2}.$$

Define

$$\begin{aligned} \mathcal{N}_c \ &= \ \{N | N \text{ is a finite CAF}, e(N) = 0, E_z\Big[\,|N_t|\,\Big] < \infty \\ &\qquad \text{for } \mathcal{E}\text{-q.e.} z \in E \text{ and all } t \geq 0\}. \end{aligned}$$

The "isometry" (28) and the continuity statement (26) are fundamental for the stochastic calculus related to $\mathcal{E}$.
We set $\mathcal{C} = \overset{\circ}{\mathcal{M}} \bigoplus \mathcal{N}_c$. Namely $\mathcal{C}$ consists of AF's such that

$$A_t = M_t + N_t \quad M \in \overset{\circ}{\mathcal{M}},\ N \in \mathcal{N}_c.$$

$\mathcal{C}$ is a linear space of AF's of finite energy. Furthermore by Remark 2.2(i) this decomposition is unique. We define the mutual energy of $A, B \in \mathcal{C}$ by

$$e(A,B) = \frac{1}{2} \lim_{\alpha\to\infty} \alpha^2 E_m\Big[\int_0^\infty e^{-\alpha t} A_t\, B_t\, dt\Big].$$

By the Cauchy-Schwarz inequality we know that $e(A,B) = 0$ when either A or B is in $\mathcal{N}_c$. Therefore

$$e(A) = e(M) \ \text{ if } \ A = M + N,\ M \in \overset{\circ}{\mathcal{M}},\ N \in \mathcal{N}_c. \tag{29}$$

Using Theorem 1.9 and the Lemma of Borel-Cantelli the proof of the following lemma is similar to the proof of [8, Lemma 5.1.2.(i),p.182]

Lemma 2.7 *(**QR**, $\mathbf{M^{ex}}$) Let $(F_k)_{k\ge1}$ be an $\mathcal{E}$-nest. Let $\widetilde{u}$, $\widetilde{u}_n \in C(\{F_k\})$, $n \in \mathbb{N}$. Let $(S_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, such that $\lim_{n\to\infty} S_n = 0$. Suppose that there exists for each $\mu \in \widehat{S}_{00}$ and $T > 0$ a constant $C^{T,\mu}$, such that*

$$P_\mu(\sup_{0\le t\le T} |\widetilde{u}(Y_t) - \widetilde{u}_n(Y_t)| > \varepsilon) \ \le \ \frac{C^{T,\mu}}{\varepsilon} S_n.$$

Then there exists a subsequence $(\widetilde{u}_{n_k})_{k\in\mathbb{N}}$, such that for $\mathcal{E}$-q.e. $z \in E$

$P_z(\widetilde{u}_{n_k}(Y_t)$ converges to $\widetilde{u}(Y_t)$ uniformly in t on each compact interval of $[0,\infty)) = 1$.

In contrast to [8], [18] in the following lemma we determine convergence w.r.t. a weaker semi-norm (cf. Remark 2.9 below).

Lemma 2.8 *(**QR**, $\mathbf{M^{ex}}$) Let $\widetilde{u} \in \widetilde{\mathcal{H}}_{\mathcal{F}^+_-}$ where $\mathcal{H}_{\mathcal{F}^+_-} := \{u \in \mathcal{H} \mid u, -u \in \mathcal{H}_\mathcal{F}\}$ and let $\varepsilon > 0$. Then we have for any $\mu \in \widehat{S}_{00}$ and $T > 0$*

$$P_\mu(\sup_{0\le t\le T} |\widetilde{u}(Y_t)| > \varepsilon) \ \le \ \frac{e^T}{\varepsilon} \|h\|_\mathcal{H} \|e_u + e_{-u}\|_\mathcal{H},$$

where h is in $\mathcal{H}_b^+$ such that $\widehat{U}_1\mu \le \widehat{G}_1 h$.

Proof Set $U = \{|\widetilde{u}| > \varepsilon\}$. Since $\{\sup_{0\le t\le T} |\widetilde{u}(Y_t)| > \varepsilon\} = \{\exists t \in [0,T] \mid Y_t \in U; t < \zeta\} \subset \{\sigma_U \le T; \sigma_U < \zeta\}$ we have P_z-a.s. for $\mathcal{E}$-q.e. $z \in E$

$$\frac{e^{T-\sigma_U}}{\varepsilon} |\widetilde{u}(Y_{\sigma_U})| \ \begin{cases} \ge 1 & \text{on } \ \{\sigma_U \le T; \sigma_U < \zeta\} \\ \ge 0 & \text{elsewhere} \end{cases}$$

because it holds P_z-a.s. for $\mathcal{E}$-q.e. $z \in E$ that $t \mapsto \widetilde{u}(Y_t)$ is right continuous on $[0,\zeta)$ and that $\widetilde{u}(Y_t) = 0$ for $t \geq \zeta$. Let $\overline{e}_u$, $\overline{e}_{-u}$ be $\mathcal{E}$-q.l.s.c. regularizations of e_u, e_{-u}. Since $|\widetilde{u}| = \sup_{n>1} |\widetilde{u}|\wedge n = \sup_{n>1} \lim_{\alpha\to\infty} \alpha R_{\alpha+1}(|\widetilde{u}|\wedge n) \leq \lim_{\alpha\to\infty} \alpha R_{\alpha+1}(e_u+e_{-u}) \leq \overline{e}_u+\overline{e}_{-u}$ $\mathcal{E}$-q.e. it follows that

$$
\begin{aligned}
P_\mu(\sup_{0\leq t\leq T} |\widetilde{u}(Y_t)| > \varepsilon) &\leq \frac{e^T}{\varepsilon} E_\mu\Big[\, e^{-\sigma_U} |\widetilde{u}(Y_{\sigma_U})| \,\Big] \\
&\leq \frac{e^T}{\varepsilon} \int \overline{e}_u + \overline{e}_{-u}\, d\mu \\
&\leq \frac{e^T}{\varepsilon} \parallel h \parallel_{\mathcal{H}} \parallel e_u + e_{-u} \parallel_{\mathcal{H}} .
\end{aligned}
$$

□

Remark 2.9 *Let us define a semi-norm on $\widetilde{\mathcal{H}}_{\mathcal{F}^+_-}$ by $\|v\|_e := \|e_v + e_{-v}\|_{\mathcal{H}}$. Let $(\widetilde{u}_n)_{n\in\mathbb{N}} \subset \widetilde{\mathcal{H}}_{\mathcal{F}^+_-}$ be $\|\cdot\|_e$-convergent to $\widetilde{u} \in \widetilde{\mathcal{H}}_{\mathcal{F}^+_-}$. Then, using Lemma 2.8 we see that Lemma 2.7 applies. Since for $f \in \mathcal{F}$, $\|f\|_e \leq 6K\|f\|_{\mathcal{F}}$ we have in particular, that if $u, u_n \in \mathcal{F}$, $n \in \mathbb{N}$ such that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $\mathcal{F}$ then $u_n \underset{n\to\infty}{\longrightarrow} u$ w.r.t. $\|\cdot\|_e$.*

Using Theorem 1.9 and Remark 2.2(i) the proof of the following theorem is similar to [8, p.203].

Theorem 2.10 *(**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{SUB}}$) Let $(M^n)_{n\in\mathbb{N}} \subset \overset{\circ}{\mathcal{M}}$ be e-Cauchy. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a unique $M \in \overset{\circ}{\mathcal{M}}$, such that $\lim_{n\to\infty} e(M^n - M) = 0$ and for $\mathcal{E}$-q.e. $z \in E$*

$$
P_z(\lim_{k\to\infty} M_t^{n_k} = M_t \text{ uniformly in } t \text{ on each compact interval of } [0,\infty)) \;=\; 1.
$$

Theorem 2.11 *(i) (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{SUB}}$) Let $u \in \mathcal{F}$. There exists a unique $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and a unique $N^{[u]} \in \mathcal{N}_c$ such that*

$$
A^{[u]} \;=\; M^{[u]} + N^{[u]}. \tag{30}
$$

*(ii) (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{SUB}}$) Let $(F_k)_{k\geq 1}$ be an $\mathcal{E}$-nest. Let $\widetilde{u}$, $\widetilde{u}_n \in C(\{F_k\})$, $n \in \mathbb{N}$, such that we have (30) for $A^{[u_n]}$, $n \in \mathbb{N}$ and such that $\overline{e}(A^{[u_n-u]}) \underset{n\to\infty}{\longrightarrow} 0$. Assume furthermore that the conditions of Lemma 2.7 are satisfied for $\widetilde{u}$, $\widetilde{u}_n$, $n \in \mathbb{N}$. Then (30) extends to $A^{[u]}$.*

Proof (of Theorem 2.11) After all preparations (among others Theorem 1.4, Theorem 1.9, Theorem 2.1, Remark 2.2(i) for the uniqueness of the decomposition, Lemma 2.6, Lemma 2.8) we can finally show (i) similar to the proof of the corresponding statement in [8, Theorem 5.2.2., p.203ff]. Therefore we omit the proof of (i) and only show (ii).
Let (30) be valid for $\widetilde{u}_n$, $n \in \mathbb{N}$. By the uniqueness of the decomposition we know that $M^{[u_n]} - M^{[u_m]} = M^{[u_n-u_m]}$. Hence by (29) we have

$$
e(M^{[u_n-u_m]}) \;=\; e(A^{[u_n-u_m]}) \;\leq\; 2\,\overline{e}(A^{[u-u_n]}) + 2\,\overline{e}(A^{[u-u_m]}).
$$

It follows that $(M^{[u_n]})_{n\in\mathbf{N}} \subset \overset{\circ}{\mathcal{M}}$ is e-Cauchy. Hence by Theorem 2.10 it makes sense to set

$$\begin{aligned} M^{[u]} &= \lim_{n\to\infty} M^{[u_n]} \ \text{ in } \ (\overset{\circ}{\mathcal{M}}, e) \\ N^{[u]} &= A^{[u]} - M^{[u]}. \end{aligned}$$

It only remains to show $N^{[u]} \in \mathcal{N}_c$. Note that there exists a subsequence n_k such that

$$P_z(N_t^{[u_{n_k}]} \text{ converges uniformly in } t \text{ on each compact interval of } [0,\infty)) \;=\; 1$$

for $\mathcal{E}$-q.e. $z \in E$ because by Lemma 2.7 and Theorem 2.10 the same is true for $A^{[u]}$ and $M^{[u]}$. Therefore $N^{[u]}$ is a CAF. Finally

$$\begin{aligned} \overline{e}(N^{[u]}) &= \overline{e}(A^{[u-u_n]} - (M^{[u]} - M^{[u_n]}) + N^{[u_n]}) \\ &\le 3\,\overline{e}(A^{[u-u_n]}) + 3\,e(M^{[u]} - M^{[u_n]}) \end{aligned}$$

implies that $N^{[u]}$ is of zero energy.

□

2.3 An Itô-type formula

From the beginning of this section we assume that we are given a generalized Dirichlet form $\mathcal{E}$ which is quasi-regular and an m-tight special standard process $\mathbf{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$. We also assume that the coresolvent $(\widehat{G}_\alpha)_{\alpha>0}$ associated with $\mathcal{E}$ is sub-Markovian. As before we use the abbreviations **QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{SUB}}$ for these assumptions.

Lemma 2.12 *(*__QR__*,* $\mathbf{M^{ex}}$*,* $\widehat{\mathbf{SUB}}$*) Let $f \in \mathcal{B}_b(E)$ and $M \in \overset{\circ}{\mathcal{M}}$. Then there exists a unique element denoted by $f \bullet M \in \overset{\circ}{\mathcal{M}}$, such that*

$$\frac{1}{2}\int_E f\,d\mu_{<M,L>} \;=\; e(f\bullet M, L) \quad \textit{for all } \ L \in \overset{\circ}{\mathcal{M}}, \tag{31}$$

Proof Let $f \in \mathcal{B}_b$ and $f = f^+ - f^-$ be its decomposition in positive and negative part. Since for any $M, L \in \overset{\circ}{\mathcal{M}}$

$$\begin{aligned} \Big|\int_E f\,d\mu_{<M,L>}\Big| &\le \Big(\int_E f^+\,d\mu_{<M>}\Big)^{1/2}\Big(\int_E f^+\,d\mu_{<L>}\Big)^{1/2} \\ &\quad + \Big(\int_E f^-\,d\mu_{<M>}\Big)^{1/2}\Big(\int_E f^-\,d\mu_{<L>}\Big)^{1/2} \le 4\|f\|_\infty e(M)^{1/2} e(L)^{1/2} \end{aligned}$$

.

the map $L \mapsto \frac{1}{2}\int_E f\,d\mu_{<M,L>}$, $L \in \overset{\circ}{\mathcal{M}}$ is a continuous linear functional on $\overset{\circ}{\mathcal{M}}$. The assertion then follows from the Riesz lemma.

□

$f \bullet M$ in Lemma 2.12 is called the *stochastic integral* of $f(Y)$ w.r.t. M. Later on in Lemma 2.15(i) we will give the justification for this definition.

Lemma 2.13 *(*$\mathbf{QR}$*,* $\mathbf{M^{ex}}$*) Let* μ *be a finite smooth measure. Then*

$$\widetilde{G_1\mathcal{H}_b} \subset L^p(E;\mu) \ \ dense\ for\ any \ \ p \geq 1.$$

Proof Since $\widetilde{G_1\mathcal{H}_b}$ separates the points of $E \setminus N$ where N is an $\mathcal{E}$-exceptional set $\mathcal{L} := \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+} - \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+}$ also does. Note that $\sigma(\mathcal{L}) \supset \mathcal{B}(E \setminus N)$ and that $\mathcal{L}$ is closed under the operation $f \wedge g$. The closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ in $L^p(E;\mu)$ is a monotone vector space in the sense of [22, *Appendices* A0] and also closed under the operation $f \wedge g$. Hence noting that smooth measures do not charge $\mathcal{E}$-exceptional sets an argument as in the proof of [22, *Appendices* A0.8] then implies that $\overline{\mathcal{L}} \supset \mathcal{B}_b$ and thus $\overline{\mathcal{L}}$ is dense in any $L^p(E;\mu)$, $p \geq 1$. It remains to show that any $u \in \mathcal{L}$ is an $L^p(E;\mu)$-limit of elements in $\widetilde{G_1\mathcal{H}_b}$. But this is clear since for any $\widetilde{u} \in \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+}$

$$\int |\widetilde{u} - \alpha R_{\alpha+1}\widetilde{u}|^p d\mu \to 0 \ as\ \alpha \to \infty.$$

□

Let us from now on up to the end of this section assume that the coform $\widehat{\mathcal{E}}$ is a quasi-regular generalized Dirichlet form and that there exists an m-tight special standard process $\widehat{\mathbf{M}}$ which is properly coassociated in the resolvent sense with $\widehat{\mathcal{E}}$ (cf. paragraph just before Lemma 1.20). As before we use the abbreviations $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$ for these assumptions. Note that the assumption $\widehat{\mathbf{QR}}$ includes the assumption $\widehat{\mathbf{SUB}}$.

Remark 2.14 *(*$\mathbf{QR}$*,* $\mathbf{M^{ex}}$*,* $\widehat{\mathbf{QR}}$*,* $\widehat{\mathbf{M}}^{\mathbf{ex}}$*) By quasi-regularity of* $\widehat{\mathcal{E}}$ *we know that* $\widetilde{\widetilde{\widehat{\mathcal{P}}}}_{\widehat{G}_1\mathcal{H}_b^+} - \widetilde{\widetilde{\widehat{\mathcal{P}}}}_{\widehat{G}_1\mathcal{H}_b^+}$ *separates the points of* $E \setminus N$*, where* N *is an* $\widehat{\mathcal{E}}$*-exceptional set (cf. proof of Lemma 2.13). Let* μ*,* ν *be finite measures on* $(E, \mathcal{B}(E))$ *charging no* $\widehat{\mathcal{E}}$*-exceptional set. Then by an argument similar to the one in the proof of Lemma 2.13 it follows that* $\mu = \nu$ *if* $\int \widehat{R}_\alpha \widetilde{f}\, d\mu = \int \widehat{R}_\alpha \widetilde{f}\, d\nu$ *for all* $\widetilde{f} \in \widetilde{\widetilde{\widehat{\mathcal{P}}}}_{\widehat{G}_1\mathcal{H}_b^+}$*.*

Let us assume from now on up to the end of this section that the martingale part $M^{[u]}$ of the decomposition of Theorem 2.11 is continuous for all u in $G_1\mathcal{H}_b$. We will use the abbreviation $\mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}$ for this assumption. Note that by $\mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}$ similar to [8, *Theorem* A.3.20., *Lemma* 5.5.1.(*ii*)] any $M \in \mathcal{M}$ is continuous. In fact we only have to replace $g \in C_0(X)$ in [8, *Theorem* A.3.20.] by $g \in \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+} - \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+}$ and to use monotone class arguments as in the proof of Lemma 2.13.

Lemma 2.15 *(i) (**QR**, $\mathbf{M^{ex}}$) Let $M \in \overset{\circ}{\mathcal{M}}$. If $f \in \mathcal{B}_b(E)$ and $t \mapsto f(Y_t)$, $t \in [0,\infty)$ is a continuous process then the abstract stochastic integral $f \bullet M$ of Lemma 2.12 is a P_z-version of the usual stochastic integral of $f(Y_t)$ w.r.t. M for $\mathcal{E}$-q.e. $z \in E$, i.e. for any $t > 0$ and $\mathcal{E}$-q.e. $z \in E$ we have*

$$\lim_{|\Delta|\to 0} E_z\Big[\, ((f \bullet M)_t^{(\Delta)} - (f \bullet M)_t)^2 \Big] = 0$$

where

$$(f \bullet M)_t^{(\Delta)} := \sum_{i=0}^{n-1} f(Y_{t_i})(M_{t_{i+1}} - M_{t_i})$$

and Δ denotes the partition $0 = t_0 < t_1 < ... < t_n = t$, $|\Delta| = \max_{1 \le i \le n} |t_i - t_{i-1}|$.
*(ii) (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$, $\mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}$) The notion of stochastic integral extends uniquely to $f \in L^2(E; \mu_{<M>})$, $M \in \overset{\circ}{\mathcal{M}}$ in the following sense: if $(\widetilde{g}_n)_{n\in\mathbb{N}} \subset \widetilde{G_1\mathcal{H}_b}$ such that $\widetilde{g}_n \to f$ in $L^2(E; \mu_{<M>})$ then $f \bullet M := \lim_{n\to\infty} \widetilde{g}_n \bullet M$ in $\overset{\circ}{\mathcal{M}}$. Furthermore for $M, L \in \overset{\circ}{\mathcal{M}}$ and $f, g \in L^2(\mu_{<M>}) \cap L^2(\mu_{<L>})$ we have*

$$\mu_{<f\bullet M, L>} \;=\; f \cdot \mu_{<M,L>}$$

hence in particular the following representation

$$e(f \bullet M, g \bullet L) = \frac{1}{2} \int fg \, d\mu_{<M,L>}.$$

Proof (i) Let N be a common exceptional set for M and $< M >$ such that $M^2 - < M >$ is a martingale w.r.t. P_z for any $z \in E \setminus N$. By Theorem [8, A.3.19.] with $S = E \setminus N$ and the discussion right before the quoted theorem there exists uniquely $\widetilde{M} \in \mathcal{M}$ such that for any $z \in E \setminus N$ we have that $\widetilde{M}^2 - \int_0^\cdot f^2(Y_s) d < M >_s$ is a P_z-martingale and

$$\lim_{|\Delta|\to 0} E_z\Big[\, ((f \bullet M)_t^{(\Delta)} - \widetilde{M}_t)^2 \Big] = 0 \; for\ any\ t > 0.$$

Note that the process $\widetilde{M}$ results from a procedure which made the family of usual P_z-stochastic integrals $((f\bullet M)^z)_{z\in E\setminus N}$ of the square integrable predictable process $f(Y)$ w.r.t. the continuous martingale M independent of the parameter z in the sense that $P_z(\widetilde{M}_t = (f \bullet M)_t^z \, \forall t \ge 0) = 1$ for any $z \in E \setminus N$ (cf. [22, (51.17) Lemma], [8, Lemma A.3.2.]). Since $f \in L^2(E; \mu_{<M>})$ we have by Remark 2.2(ii) that $e(\widetilde{M}) = \frac{1}{2} \int f^2 d\mu_{<M>} < \infty$. hence $M \in \overset{\circ}{\mathcal{M}}$. By polarization we have for any $L \in \overset{\circ}{\mathcal{M}}$

$$E_z\Big[\, < \widetilde{M}, L >_t \, \Big] = E_z\Big[\, \int_0^t f(Y_s) d < M, L >_s \, \Big]$$

hence for all $g \in L^2(E; m) \cap \mathcal{B}_b$, $\beta > 0$,

$$\int_E \widehat{R}_\beta g \, d\mu_{<\widetilde{M},L>} \;=\; \int_E (\widehat{R}_\beta g) f \, d\mu_{<M,L>}$$

and consequently $d\mu_{<\widetilde{M},L>} = f \cdot d\mu_{<M,L>}$ by Remark 2.14. It follows in particular that $e(\widetilde{M}, L) = \frac{1}{2}\int f d\mu_{<M,L>}$ hence $\widetilde{M} = f \bullet M$ in view of Lemma 2.12.
(ii) Since $e(\widetilde{g}_n \bullet M - \widetilde{g}_m \bullet M) = e((\widetilde{g}_n - \widetilde{g}_m) \bullet M) = \frac{1}{2}\int (\widetilde{g}_n - \widetilde{g}_m)^2 d\mu_{<M>}$ we see that there exists a unique $f \bullet M \in \overset{\circ}{\mathcal{M}}$ such that $f \bullet M = \lim_{n\to\infty} \widetilde{g}_n \bullet M$ in $\overset{\circ}{\mathcal{M}}$.
By our assumption the martingale part $M^{[u]}$ of the decomposition of Theorem 2.11 is continuous for all u in $G_1\mathcal{H}_b$ hence $t \mapsto \widetilde{u}(Y_t)$, $t \in [0, \infty)$ is a continuous process for any $\widetilde{u} \in \widetilde{G_1\mathcal{H}_b}$. Obviously $\widetilde{u} \in L^2(E; \mu_{<M>})$ for any $M \in \overset{\circ}{\mathcal{M}}$. Hence by (i) we know that

$$d\mu_{<\widetilde{u}\bullet M,L>} = \widetilde{u} \cdot d\mu_{<M,L>} \tag{32}$$

for any $M, L \in \overset{\circ}{\mathcal{M}}$. Since the pointwise limit of $\alpha R_{\alpha+1}\widetilde{u}$, $\widetilde{u} \in \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+}$ is $\mathcal{E}$-q.e. monotone and all the measures in question are bounded and do not charge $\mathcal{E}$-exceptional sets it is easy to see that (32) extends to $\widetilde{u} \in \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+}$. Hence (32) holds for any $\widetilde{u} \in \mathcal{L} := \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+} - \widetilde{\mathcal{P}}_{G_1\mathcal{H}_b^+}$. The closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ in $L^2(E; \mu_{<M+L>} + \mu_{<M>} + \mu_{<L>})$ is a monotone vector space in the sense of [22, *Appendices* A0] and closed under the operation $f \wedge g$ since this was already true for $\mathcal{L}$. Hence an argument as in the proof of Lemma 2.13 implies that (32) extends to all $\widetilde{u} \in \mathcal{B}_b$. Finally let $f \in L^2(\mu_{<M>}) \cap L^2(\mu_{<L>})$. Set $f_n := (-n) \vee f \wedge n$, $n \geq 1$. Then $f_n \to f$ in $L^2(\mu_{<M>}) \cap L^2(\mu_{<L>})$ hence in $L^2(\mu_{<M+L>})$ because

$$\begin{aligned}\int (f_n - f_m)^2 d\mu_{<M+L>} &= 4e((f_n - f_m) \bullet M, (f_n - f_m) \bullet L) \\ &\quad + \int (f_n - f_m)^2 d\mu_{<M>} + \int (f_n - f_m)^2 d\mu_{<L>}\end{aligned}$$

and then it is easy to see that (32) also holds for $\widetilde{u} \in L^2(\mu_{<M>}) \cap L^2(\mu_{<L>})$. In particular we then have $d\mu_{<f\bullet M, g\bullet L>} = fg \cdot d\mu_{<M,L>}$.

□

Let us now consider the following linear space

$$\widetilde{\mathcal{H}}^{dec} := \{\widetilde{u} \in \widetilde{\mathcal{H}} \mid \exists M^{[u]} \in \overset{\circ}{\mathcal{M}}, N^{[u]} \in \mathcal{N}_c \textit{ such that } A^{[u]} = M^{[u]} + N^{[u]}\}$$

and let $\mathcal{H}^{dec}$ denote the totality of m-versions of elements in $\widetilde{\mathcal{H}}^{dec}$.

The proof of the next Theorem is based on the proof of Theorem 5.3.2. in [16, p.160]. For convenience we write $\mu_{<u,v>}$ instead of $\mu_{<M^{[u]},M^{[v]}>}$ and $\mu_{<u>}$ instead of $\mu_{<M^{[u]}>}$. For $V \subset \mathbb{R}^d$, V open, $d \geq 1$, let $C^k(V)$, $k \in \mathbb{N} \cup \{\infty\}$, denote the space of k times continuously differentiable functions on V.

Theorem 2.16 *[Product rule] (*$\mathbf{QR}$, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$, $\mathbf{M_{cont}^{[G_1\mathcal{H}_b]}}$*) Let* $\widetilde{f} = (\widetilde{f}_1, ..., \widetilde{f}_n)$ *be an* n*-tuple of* $\mathcal{E}$*-q.c.* m*-versions of elements in* $\mathcal{H}_b^{dec}$ *such that* $\Phi(\widetilde{f}) \in \widetilde{\mathcal{H}}^{dec}$ *for all* $\Phi \in C^1(\mathbb{R}^n)$ *with* $\Phi(0) = 0$ *and assume that there exists* $(h_{ik})_{k\in\mathbb{N}} \subset \mathcal{H}$ *with* $\overline{e}(A^{[f_i - G_1 h_{ik}]}) \to 0$, *as* $k \to \infty$, $1 \leq i \leq n$. *Let* Φ, $\Psi \in C^1(\mathbb{R}^n)$, $\Phi(0) = \Psi(0) = 0$ *and* $w \in \mathcal{H}_b^{dec}$, *then:*

$$\mu_{<\Phi(f)\cdot\Psi(f),w>} = \Phi(\widetilde{f}_1, ..., \widetilde{f}_n) \cdot \mu_{<\Psi(f),w>} + \Psi(\widetilde{f}_1, ..., \widetilde{f}_n) \cdot \mu_{<\Phi(f),w>}. \tag{33}$$

Proof It is enough to show

$$\int_E h\,d\mu_{<\Phi^2(f),w>} \;=\; 2\int_E h\,\Phi(\widetilde{f}_1,...,\widetilde{f}_n)\,d\mu_{<\Phi(f),w>}$$

for $h=\widehat{R}_\beta g$, $g\in\mathcal{B}_b^+\cap L^1(E;m)$, $\beta>0$, because then we will consider $\mu_{<(\Phi(f)+\Psi(f))^2,w>}$. We may furthermore assume that $\int h\,dm=1$. Then by (23)

$$\begin{aligned}
&\int_E h\,d\mu_{<\Phi^2(f),w>}\\
=\;& \lim_{\alpha\to\infty}\alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}M_t^{[\Phi^2(f)]}M_t^{[w]}\,dt\Big]\\
=\;& \lim_{\alpha\to\infty}\alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(\Phi^2(\widetilde{f}(Y_t))-\Phi^2(\widetilde{f}(Y_0))(\widetilde{w}(Y_t)-\widetilde{w}(Y_0))\,dt\Big]\\
=\;& 2\lim_{\alpha\to\infty}\alpha\,(\alpha+\beta)E_{h\Phi(\widetilde{f})\,m}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(\Phi(\widetilde{f}(Y_t))-\Phi(\widetilde{f}(Y_0)))(\widetilde{w}(Y_t)-\widetilde{w}(Y_0))\,dt\Big]\\
&+\lim_{\alpha\to\infty}\alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(\Phi(\widetilde{f}(Y_t))-\Phi(\widetilde{f}(Y_0)))^2(\widetilde{w}(Y_t)-\widetilde{w}(Y_0))\,dt\Big]\\
=\;& 2\lim_{\alpha\to\infty}I_\alpha+\lim_{\alpha\to\infty}II_\alpha.
\end{aligned}$$

By (23) and Lebesgue's Theorem we have

$$\begin{aligned}
\lim_{\alpha\to\infty}I_\alpha \;&=\; \lim_{\alpha\to\infty}\alpha(U^{\alpha+\beta}_{<\Phi(f),w>}1,h\Phi(\widetilde{f}))\\
&=\; \lim_{\alpha\to\infty}\alpha\lim_{\gamma\to\infty}(\gamma\,R_{\gamma+\alpha+\beta}\,U^{\alpha+\beta}_{<\Phi(f),w>}1,h\Phi(\widetilde{f}))\\
&=\; \lim_{\alpha\to\infty}\alpha\lim_{\gamma\to\infty}(\gamma\,U^{\gamma+\alpha+\beta}_{<\Phi(f),w>}1,\widehat{R}_{\alpha+\beta}(h\Phi(\widetilde{f})))\\
&=\; \lim_{\alpha\to\infty}\alpha\int_E \widehat{R}_{\alpha+\beta}(h\Phi(\widetilde{f}))\,d\mu_{<\Phi(f),w>}\\
&=\; \int_E h\Phi(\widetilde{f})\,d\mu_{<\Phi(f),w>}
\end{aligned}$$

and for some constant $L>0$ $\lim_{\alpha\to\infty}II_\alpha$ is dominated by

$$nL\sum_{i=1}^n\overline{\lim_{\alpha\to\infty}}\,\alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(\widetilde{f}_i(Y_t)-\widetilde{f}_i(Y_0))^2\mid\widetilde{w}(Y_t)-\widetilde{w}(Y_0)\mid\,dt\Big].$$

For $1\le i\le n$, $k\in\mathbb{N}$, we set $f_{ki}=R_1h_{ik}$. Since we assumed that $\overline{e}(A^{[f_i-f_{ki}]})\to 0$, as $k\to\infty$, $1\le i\le n$, it is enough to show that for $1\le i\le n$

$$\overline{\lim_{\alpha\to\infty}}\,\alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(f_{ki}(Y_t)-f_{ki}(Y_0))^2\mid\widetilde{w}(Y_t)-\widetilde{w}(Y_0)\mid\,dt\Big]$$

is 0 for any fixed k, i. We may even assume that h_{ik}, $1 \le i \le n$, $k \in \mathbb{N}$, is bounded (cf. proof of Proposition 2.19). By our assumption $(M_t^{[f_{ki}]}, \mathcal{F}_t, P_{hm})_{t\ge 0}$ is a continuous square integrable martingale and consequently by the Burkholder-Davis-Gundy inequality

$$E_{hm}\Big[(M_t^{[f_{ki}]})^4 \Big] \;\; \le \;\; CE_{hm}\Big[< M^{[f_{ki}]} >_t^2 \Big]$$

where $C > 0$ is a constant independent of f_{ki}. Then

$$\begin{aligned}
& \overline{\lim_{\alpha\to\infty}}\, \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(f_{ki}(Y_t) - f_{ki}(Y_0))^2 \mid \widetilde{w}(Y_t) - \widetilde{w}(Y_0) \mid \, dt \Big] \\
\le \;\; & 2\, \overline{\lim_{\alpha\to\infty}}\, (\alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(M_t^{[f_{ki}]})^4\, dt \Big])^{\frac{1}{2}}(\int h\, d\mu_{<w>})^{\frac{1}{2}} \\
\le \;\; & 2\, \overline{\lim_{\alpha\to\infty}}\, (\alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t} < M^{[f_{ki}]} >_t^2 \; dt \Big])^{\frac{1}{2}}(\int h\, d\mu_{<w>})^{\frac{1}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}
& \overline{\lim_{\alpha\to\infty}}\, \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(< M^{[f_{ki}]} >_t)^2\, dt \Big] \\
= \;\; & \overline{\lim_{\alpha\to\infty}}\, \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}\, 2\int_0^t < M^{[f_{ki}]} >_t - < M^{[f_{ki}]} >_s \; d < M^{[f_{ki}]} >_s \; dt \Big] \\
= \;\; & 2\, \overline{\lim_{\alpha\to\infty}}\, \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty \int_s^\infty e^{-(\alpha+\beta)\,t} < M^{[f_{ki}]} >_{t-s} \circ\vartheta_s \, dtd < M^{[f_{ki}]} >_s \; \Big] \\
= \;\; & 2\, \overline{\lim_{\alpha\to\infty}}\, \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,s}E_{Y_s}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t} < M^{[f_{ki}]} >_t \; dt \Big] d < M^{[f_{ki}]} >_s \; \Big] \\
= \;\; & 2\, \overline{\lim_{\alpha\to\infty}}\, \alpha \int h\, U^{\alpha+\beta}_{<M^{[f_{ki}]}>} U^{\alpha+\beta}_{<M^{[f_{ki}]}>} 1\, dm \\
\le \;\; & 2 \int h\, U^{\gamma}_{<M^{[f_{ki}]}>} 1\, d\mu_{<M^{[f_{ki}]}>}
\end{aligned}$$

for every $\gamma > 0$. Now since $U^1_{<M^{[f_{ki}]}>}1$ is bounded $\mathcal{E}$-q.e. for any fixed k, $1 \le i \le n$, and since

$$U^{\gamma+1}_{<M^{[f_{ki}]}>}1(z) \;\; = \;\; U^1_{<M^{[f_{ki}]}>}1(z) - \gamma R_{\gamma+1} U^1_{<M^{[f_{ki}]}>}1(z) \;\downarrow\; 0$$

for $\mathcal{E}$-q.e. z in E as $\gamma \to \infty$ we conclude by Lebesgue's Theorem.

□

Theorem 2.17 *[Chain rule] (*$\mathbf{QR}$, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$, $\mathbf{M}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}_{\mathbf{cont}}$*) Let* $\widetilde{f} = (\widetilde{f}_1, ..., \widetilde{f}_n)$ *be an* n*-tuple of* $\mathcal{E}$*-q.c. m-versions of elements in* $\mathcal{H}_b^{dec}$ *such that* $\Phi(\widetilde{f}) \in \widetilde{\mathcal{H}}^{dec}$ *for all* $\Phi \in C^1(\mathbb{R}^n)$ *with* $\Phi(0) = 0$ *and assume that there exists* $(f_{ik})_{k\in\mathbb{N}} \subset \mathcal{H}$ *with* $\overline{e}(A^{[f_i - G_1 f_{ik}]}) \to 0$ *, as* $k \to \infty$, $1 \le i \le n$*. Let* $\Phi \in C^1(\mathbb{R}^n)$, $\Phi(0) = 0$ *and* $w \in \mathcal{H}_b^{dec}$*, then*

$$\mu_{<\Phi(f),w>} \;\; = \;\; \sum_{i=1}^n \frac{\partial\Phi}{\partial x_i}(\widetilde{f}) \cdot \mu_{<f_i,w>}. \tag{34}$$

Proof Any powers of the coordinate functions satisfy (34) by the product rule and then by the product rule again all polynomials of n variables vanishing at the origin. Let $K \subset \mathbb{R}^n$ be a compact set such that $\widetilde{f}(z) \in K$ for $\mathcal{E}$-q.e. $z \in E$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ on $K \subset\subset K' = supp(\chi)$. There exists (cf. $[4, II.4.2, 4.3, p.57]$) a sequence of polynomials $(p_j)_{j\in\mathbf{N}}$, vanishing at the origin, such that $p_j \longrightarrow \Phi$, $\frac{\partial p_j}{\partial x_i} \longrightarrow \frac{\partial \Phi}{\partial x_i}$, $1 \le i \le n$, uniformly on K' as $j \to \infty$. Note that $[(\Phi - p_j)\chi](\widetilde{f})(z) = (\Phi - p_j)(\widetilde{f})(z)$ for $\mathcal{E}$-q.e. $z \in E$. Then we have for all $g \in L^2(E;m) \cap \mathcal{B}_b$, $\alpha > 0$

$$\begin{aligned}\sum_{i=1}^n \int \widehat{R}_\alpha\, g\, \frac{\partial \Phi}{\partial x_i}(\widetilde{f})\, d\mu_{<f_i,w>} &= \sum_{i=1}^n \lim_{j\to\infty} \int \widehat{R}_\alpha\, g\, \frac{\partial p_j}{\partial x_i}(\widetilde{f})\, d\mu_{<f_i,w>} \\ &= \lim_{j\to\infty} \int \widehat{R}_\alpha\, g\, d\mu_{<p_j(f),w>} \\ &= \int \widehat{R}_\alpha\, g\, d\mu_{<\Phi(f),w>}\end{aligned}$$

where the second identity followed from the product rule. Because of (28) the third identity follows since we assumed to have decomposition (30) for all $\Phi(f)$ like above, hence

$$\begin{aligned}e(M^{[\Phi(f)]} - M^{[p_j(f)]}) &= e(A^{[((\Phi - p_j)\chi)(f)]}) \\ &\le n \sum_{i=1}^n \left\| \frac{\partial((\Phi - p_j)\chi)}{\partial x_i} \right\|_\infty^2 \overline{e}(A^{[f_i]}).\end{aligned}$$

The last expression tends to zero as $j \to \infty$.

$\square$

Summarizing we get the following

Theorem 2.18 *[Itô's formula]* (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$, $\mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}$) *Let $\widetilde{f} = (\widetilde{f}_1, ..., \widetilde{f}_n)$ be an n-tuple of $\mathcal{E}$-q.c. m-versions of elements in $\mathcal{H}_b^{dec}$ such that $\Phi(\widetilde{f}) \in \widetilde{\mathcal{H}}^{dec}$ for all $\Phi \in C^1(\mathbb{R}^n)$ with $\Phi(0) = 0$ and assume that there exists $(f_{ik})_{k\in\mathbf{N}} \subset \mathcal{H}$ with $\overline{e}(A^{[f_i - G_1 f_{ik}]}) \to 0$, as $k \to \infty$, $1 \le i \le n$. Then we have*

$$\Phi(\widetilde{f}_1, \ldots, \widetilde{f}_n)(Y_t) - \Phi(\widetilde{f}_1, \ldots, \widetilde{f}_n)(Y_0) = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}(\widetilde{f}_1, \ldots, \widetilde{f}_n) \bullet M_t^{[f_i]} + N_t^{[\Phi(f_1,\ldots,f_n)]}$$

for all Φ like above and this decomposition is orthogonal w.r.t. $e(\cdot,\cdot)$. In particular Lemma 2.15(i) applies.

Proof The assertion follows by Lemma 2.12, Lemma 2.15(ii) and Theorem 2.17 because

$$e(M^{[\Phi(f)]} - \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}(\widetilde{f}) \bullet M^{[f_i]}) = \frac{1}{2} \int d\mu_{<M^{[\Phi(f)]} - \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}(\widetilde{f}) \bullet M^{[f_i]}>} = 0.$$

$\square$

If we strengthen the assumptions on Φ we can show directly the chain rule in the next proposition. This will be used in the proof of Lemma 3.5(iii) below. Let $C_b^k(\mathbb{R}^n)$, $k \in \mathbb{N} \cup \{\infty\}$, denote the space of all k-times continuously differentiable functions on $\mathbb{R}^n$ with all partial derivatives up to order k bounded.

Proposition 2.19 *(* $\mathbf{QR}$, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$, $\mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}$*) Let $\widetilde{f} = (\widetilde{f}_1, ..., \widetilde{f}_n)$ be an n-tuple of $\mathcal{E}$-q.c. m-versions of elements in $\mathcal{H}^{dec}$. Let $\Phi \in C_b^2(\mathbb{R}^n)$ such that $\Phi(\widetilde{f}) \in \widetilde{\mathcal{H}}^{dec}$. Assume that there exists $(f_{ik})_{k\in\mathbb{N}} \subset \mathcal{H}$ with $\overline{e}(A^{[f_i - G_1 f_{ik}]}) \to 0$, as $k \to \infty$, $1 \le i \le n$. Let $\widetilde{w} \in \widetilde{\mathcal{H}}^{dec}$ such that there exists $(h_l)_{l\in\mathbb{N}} \subset \mathcal{H}$ with $\overline{e}(A^{[w - G_1 h_l]}) \to 0$, as $l \to \infty$. Then*

$$\mu_{<\Phi(f),w>} \;=\; \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(\widetilde{f}) \cdot \mu_{<f_i,w>}. \tag{35}$$

Proof We first show the statement for w replaced by $v := R_1(h_l \wedge m)$ where l, m arbitrary but fixed. Let h be as in the proof of Theorem 2.16. Using Taylor's formula similar to the just mentioned proof we have

$$\begin{aligned}
&\int_E h\, d\mu_{<\Phi(f),v>} \\
=\; & \lim_{\alpha\to\infty} \alpha\,(\alpha+\beta) E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(\Phi(\widetilde{f}(Y_t)) - \Phi(\widetilde{f}(Y_0)))A_t^{[v]}\,dt\Big] \\
=\; & \lim_{\alpha\to\infty} \alpha\,(\alpha+\beta) E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}\sum_{i=1}^{n}\frac{\partial \Phi}{\partial x_i}(\widetilde{f}(Y_0))(\widetilde{f}_i(Y_t) - \widetilde{f}_i(Y_0))A_t^{[v]}\,dt\Big] \\
+\; & \lim_{\alpha\to\infty} \alpha\,(\alpha+\beta) E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}C_t^{ij}(\widetilde{f}_i(Y_t) - \widetilde{f}_i(Y_0))(\widetilde{f}_j(Y_t) - \widetilde{f}_j(Y_0))A_t^{[v]}\,dt\Big] \\
=\; & \lim_{\alpha\to\infty} I_\alpha + \lim_{\alpha\to\infty} II_\alpha.
\end{aligned}$$

where $C_t^{ij}(\omega) := \frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Phi(\widetilde{f}(Y_0(\omega)) + \theta_\omega(\widetilde{f}(Y_t(\omega)) - \widetilde{f}(Y_0(\omega)))$ and $0 \le \theta_\omega \le 1$. We remark that θ needs not to be measurable but that the composite function with the θ in it is.
Similar to the proof of Theorem 2.16 (now since $\frac{\partial\Phi}{\partial x_i}$ is uniformly bounded $1 \le i \le n$) it follows from Lebesgue's theorem that

$$\begin{aligned}
\lim_{\alpha\to\infty} I_\alpha \;&=\; \sum_{i=1}^{n} \lim_{\alpha\to\infty} \alpha(U^{\alpha+\beta}_{<f_i,v>}1, h\frac{\partial\Phi}{\partial x_i}(\widetilde{f})) \\
&=\; \sum_{i=1}^{n} \lim_{\alpha\to\infty} \alpha \int_E \widehat{R}_{\alpha+\beta}(h\frac{\partial\Phi}{\partial x_i}(\widetilde{f}))\, d\mu_{<f_i,v>} \\
&=\; \sum_{i=1}^{n} \int_E h\frac{\partial\Phi}{\partial x_i}(\widetilde{f})\, d\mu_{<f_i,v>}.
\end{aligned}$$

Since $\frac{1}{2}\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Phi$ is uniformly bounded it follows that $|\lim_{\alpha\to\infty} II_\alpha|$ is dominated by

$$L\sum_{i=1}^{n} \overline{\lim_{\alpha\to\infty}}\, \alpha\,(\alpha+\beta)E_{hm}\Big[\int_0^\infty e^{-(\alpha+\beta)\,t}(\widetilde{f}_i(Y_t) - \widetilde{f}_i(Y_0))^2 \mid A_t^{[v]} \mid\, dt\Big] \tag{36}$$

for some constant $L > 0$. To show that the expression (36) is 0 it is enough to show that the same expression but with f_i replaced by $R_1 f_{ik}$ is 0 for any k. This is because of our assumption that $\overline{e}(A^{[f_i - R_1 f_{ik}]}) \to 0$, as $k \to \infty$, $1 \le i \le n$. Since by (25) $\overline{e}(A^{[R_1 f_{ik} - R_1(f_{ik} \wedge m)]}) \le \mathcal{E}(R_1 f_{ik} - R_1(f_{ik} \wedge m), R_1 f_{ik} - R_1(f_{ik} \wedge m))$ which tends to 0 as $m \to \infty$ it is even enough to show that expression (36) with $\widetilde{f}_i$ replaced by $R_1(f_{ik} \wedge m)$ is 0 for any m, k. But this has already been done in the proof of Theorem 2.16.
Finally since h, and $\frac{\partial \Phi}{\partial x_i}(\widetilde{f})$, $1 \le i \le n$ are uniformly bounded and $\overline{e}(A^{[w - G_1 h_l]}) \to 0$, as $l \to \infty$ we have (by continuity of the measures in question with respect to the energy)

$$\begin{aligned}
\int h \, d\mu_{<\Phi(f),w>} &= \lim_{l\to\infty} \lim_{m\to\infty} \int h \, d\mu_{<\Phi(f),R_1(h_l \wedge m)>} \\
&= \lim_{l\to\infty} \lim_{m\to\infty} \sum_{i=1}^{n} \int h \frac{\partial \Phi}{\partial x_i}(\widetilde{f}) \, d\mu_{<f_i,R_1(h_l \wedge m)>} \\
&= \sum_{i=1}^{n} \int h \frac{\partial \Phi}{\partial x_i}(\widetilde{f}) \, d\mu_{<f_i,w>}.
\end{aligned}$$

$\square$

3 Local property, extension of Fukushima's decomposition and localization for an important class of generalized Dirichlet forms

The following section will serve as a preparation for our examples below. For an important class of generalized Dirchlet forms (i.e. generalized Dirchlet forms satisfying in particular **Alg** and **Diag** or **Alg** and **Diag'** below) we will develop a localization procedure, a martingale transformation as well as a simple condition to check whether the associated process is a diffusion up to his life time. As usual from the beginning on we assume that we are given a generalized Dirichlet form $\mathcal{E}$ which is quasi-regular and an m-tight special standard process $\mathbf{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$. As before we use the abbreviations **QR**, $\mathbf{M^{ex}}$ for these assumptions.

a) Local property

We first start with a lemma reflecting the relation between analytic and probabilistic aspects of a generalized Dirichlet form. For $u \in L^2(E;m)$ let $supp[u] := supp(|u| \cdot m)$ be the support of u.

Lemma 3.1 *(**QR**, $\mathbf{M^{ex}}$) Let $U \subset E$, U open. Let $u \in \mathcal{F}^+$, $supp[u] \subset U$. Then the following assertions are equivalent:*
(i) $\mathcal{E}(u,v) = 0$ for all $v \in \{\widehat{U}_1\mu - (\widehat{U}_1\mu)_U \mid \mu \in \widehat{S}_{00}\}$.
(ii) $E_z\Big[e^{-\sigma_U}\widetilde{u}(Y_{\sigma_U})\Big] = 0$ for $\mathcal{E}$-q.e. $z \in U^c$.

Proof Let $\mu \in \widehat{S}_{00}$. By [23, Lemma III.2.2.(ii)] we have that $e_{f-\alpha G_{\alpha+1}f} \to 0$ in $\mathcal{H}$ as $\alpha \to \infty$ for any $f \in \mathcal{F}$. Since $P_\mu(\sigma_N < \infty) = 0$ if N is $\mathcal{E}$-exceptional and $|\widetilde{u}| \le \overline{e}_u + \overline{e}_{-u}$ $\mathcal{E}$-q.e. for any $u \in \mathcal{F}$ we then have

$$\begin{aligned}
E_\mu\Big[|e^{-\sigma_U}(\widetilde{u} - \alpha R_{\alpha+1}\widetilde{u})(Y_{\sigma_U})|\Big] &\le E_\mu\Big[e^{-\sigma_U}(\overline{e}_{u-\alpha G_{\alpha+1}u} + \overline{e}_{\alpha G_{\alpha+1}u-u})(Y_{\sigma_U})\Big] \\
&\le \int_E \overline{e}_{u-\alpha G_{\alpha+1}u} + \overline{e}_{\alpha G_{\alpha+1}u-u}\, d\mu \\
&\le \|h\|_{\mathcal{H}}\|u - \alpha G_{\alpha+1}u\|_e
\end{aligned}$$

where $h \in \mathcal{H}_b^+$ is such that $\widehat{U}_1\mu \le \widehat{G}_1 h$ and (cf. Remark 2.9)$\|f\|_e = \|e_f + e_{-f}\|_{\mathcal{H}}$. It follows that $e^{-\sigma_U}\alpha R_{\alpha+1}\widetilde{u}(Y_{\sigma_U})$ converges to $e^{-\sigma_U}\widetilde{u}(Y_{\sigma_U})$ in $L^1(P_\mu)$. Hence

$$\begin{aligned}
\int_E E_\cdot\Big[e^{-\sigma_U}\widetilde{u}(Y_{\sigma_U})\Big]\,d\mu &= \lim_{\alpha\to\infty}\int_E E_\cdot\Big[e^{-\sigma_U}\alpha R_{\alpha+1}\widetilde{u}(Y_{\sigma_U})\Big]\,d\mu \\
&= \lim_{\alpha\to\infty}\int_E E_\cdot\Big[\int_{\sigma_U}^\infty e^{-s}\alpha(\widetilde{u} - \alpha R_{\alpha+1}\widetilde{u})(Y_s)ds\Big]\,d\mu \\
&= \lim_{\alpha\to\infty}\mathcal{E}_1((G_1(\alpha u)_U - (G_1(\alpha^2 G_{\alpha+1}u))_U, \widehat{U}_1\mu)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\alpha\to\infty} \mathcal{E}_1(G_1(\alpha(u-\alpha G_{\alpha+1}u)), (\widehat{U}_1\mu)_U) \\
&= \mathcal{E}_1(u, (\widehat{U}_1\mu)_U)).
\end{aligned}
$$

and therefore
$$
\int_E \widetilde{u} - E_\cdot\Big[e^{-\sigma_U}\widetilde{u}(Y_{\sigma_U})\Big]\, d\mu = \mathcal{E}_1(u, \widehat{U}_1\mu - (\widehat{U}_1\mu)_U)).
$$
By right-continuity every point is regular in U. Hence if we assume (i) we can easily see that (ii) holds. Clearly the converse is also true.

□

Let $V \subset \mathbb{R}^d$, $d \geq 1$, V open and $k \in \mathbb{N} \cup \{\infty\}$. Let $C_0^k(V)$ denote the space of k times continuously differentiable functions on V with compact support.

Theorem 3.2 *(**QR**, $\mathbf{M^{ex}}$) Assume that for any $U \subset E$, U open, there exists $(u_n)_{n\in\mathbb{N}} \subset \mathcal{F}^+$ such that $u_n \leq u_{n+1}$, supp$(u_n) \subset U$, $0 \leq \sup_{n\in\mathbb{N}} u_n \leq 1_U$ and $\sup_{n\in\mathbb{N}} \widetilde{u}_n > 0$ $\mathcal{E}$-q.e. on U. Suppose that Lemma 3.1(i) holds for u_n, $n \in \mathbb{N}$, and any $U \subset E$, U open. Then*

$$
P_z(t \mapsto Y_t \text{ is continuous on } [0,\zeta)) = 1 \text{ for } \mathcal{E}\text{-q.e. } z \in E. \tag{37}
$$

Proof Let $N := \{x \in U \mid \sup_{n\in\mathbb{N}} \widetilde{u}_n \leq 0\}$. Then N is $\mathcal{E}$-exceptional (we may assume that $N \in \mathcal{B}(E)$) by our assumption on $(u_n)_{n\in\mathbb{N}}$. Then by Lemma 3.1(ii) $0 = E_z\Big[e^{-\sigma_U}\widetilde{u}_n(Y_{\sigma_U})\Big] = E_\cdot\Big[e^{-\sigma_U}(\widetilde{u}_n + 1_N)(Y_{\sigma_U})\Big]$ for $\mathcal{E}$-q.e. $z \in U^c$. Therefore, using the monotone convergence theorem we see that also $E_z\Big[e^{-\sigma_U}1_U(Y_{\sigma_U})\Big] = 0$ for $\mathcal{E}$-q.e. $z \in U^c$. Then, exactly as in [15, p.153, 1.9(ii) $\Rightarrow$ (1.8.)] we show the assertion.

□

Any generalized Dirichlet form satisfying the conditions of Theorem 3.2 is here said to have the *local property*. **M** is then said to be a *diffuson up to* ζ.

Example 3.3 *Let $E = V$ and let $C_0^k(V) \subset \mathcal{F}$. Since obviously $C_0^k(V)$ has the first property of Theorem 3.2 we know that **M** is a diffusion up to ζ whenever Lemma 3.1(i) is satisfied for any $u \in C_0^k(V)$ with supp$(u) \subset U$, and all $U \subset V$, U open.*

Up to the end of this section we assume that the coresolvent $(\widehat{G}_\alpha)_{\alpha>0}$ associated to $\mathcal{E}$ is sub-Markovian. As before we use the abbreviation $\widehat{\mathbf{SUB}}$ for this assumption.

b) Extension of Fukushima's decomposition

Let us now explain some contraction principles related to $\mathcal{E}$ which we shall use below as a technical tool in order to show the extension of Fukushima's decomposition. We remark that we could have assumed much less than **QR** and $\mathbf{M}^{ex}$ but that the sub-Markovianity

of $(G_\alpha)_{\alpha>0}$ and $(\widehat{G}_\alpha)_{\alpha>0}$ is really indispensible to show the following contraction properties.
Set $\widetilde{G}_\alpha := \frac{1}{2}(G_\alpha + \widehat{G}_\alpha)$, $\alpha > 0$. Then $\widetilde{G}_\alpha$ is a symmetric operator which is sub-Markovian since $(G_\alpha)_{\alpha>0}$ and $(\widehat{G}_\alpha)_{\alpha>0}$ are sub-Markovian. Now let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$|T(x)| \le \sum_{k=1}^n |x_k| \quad \text{and} \quad |T(x) - T(y)| \le \sum_{k=1}^n |x_k - y_k|$$

for all $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$. We call such a T a normal contraction on $\mathbb{R}^n$. From [15, proof of Theorem 4.12.] we know that

$$(T(u_1, ..., u_n) - \alpha\widetilde{G}_\alpha T(u_1, ..., u_n), T(u_1, ..., u_n))_{\mathcal{H}}^{1/2} \le \sum_{k=1}^n (u_k - \alpha\widetilde{G}_\alpha u_k, u_k)_{\mathcal{H}}^{1/2} \tag{38}$$

for any n-tuple $u = (u_1, ..., u_n)$ of elements in $\mathcal{H}$, $\alpha > 0$. Note that $\alpha(h - \alpha G_\alpha h, h)_{\mathcal{H}} = \alpha(h - \alpha\widetilde{G}_\alpha h, h)_{\mathcal{H}}$ for any $h \in \mathcal{H}$, $\alpha > 0$ hence (38) holds also if we replace $\widetilde{G}_\alpha$ by G_α.
Let $\Phi \in C_b^1(\mathbb{R}^n)$ such that $\Phi(0) = 0$. Set $\| \frac{\partial \Phi}{\partial x_i} \|_\infty := C_i$. We may assume that $C_i \neq 0$, $1 \le i \le n$. Then $T(x_1, ..., x_n) := \Phi(C_1^{-1}x_1, ..., C_n^{-1}x_n)$ is a normal contraction. Hence by (38)

$$(\Phi(u) - \alpha G_\alpha \Phi(u), \Phi(u))_{\mathcal{H}}^{1/2} \le \sum_{k=1}^n C_k (u_k - \alpha G_\alpha u_k, u_k)_{\mathcal{H}}^{1/2}. \tag{39}$$

Let us now consider the following intermediate linear space

$$\mathcal{V}^{\mathcal{F}} := \{h \in \mathcal{H} \mid \sup_{\alpha>0} \alpha(h - \alpha G_\alpha h, h)_{\mathcal{H}} < \infty\}.$$

We have $\mathcal{F} \subset \mathcal{V}^{\mathcal{F}} \subset \mathcal{V}$. Indeed, since $\sup_{\alpha>0} \mathcal{A}(\alpha G_\alpha u, \alpha G_\alpha u) \le \sup_{\alpha>0} \alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}}$ for any $u \in \mathcal{H}$ and $(G_\alpha)_{\alpha>0}$ is strongly continuous on $\mathcal{H}$ it follows from [15, *Lemma I*.2.12.] that $\mathcal{V}^{\mathcal{F}} \subset \mathcal{V}$. As in the proof of Lemma 2.6 we can see that $\sup_{\alpha>0} \alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}} \le (K+1)^2 \|u\|_{\mathcal{F}}$ for any $u \in \mathcal{F}$ hence $\mathcal{F} \subset \mathcal{V}^{\mathcal{F}}$.
Let $u = (u_1, ..., u_n)$ be an n-tuple of elements in $\mathcal{V}^{\mathcal{F}}$, Φ as above. Since by (39)

$$\alpha(\Phi(u) - \alpha G_\alpha \Phi(u), \Phi(u))_{\mathcal{H}} \quad \le \quad n \sup_{1\le k\le n} C_k^2 \sum_{k=1}^n \alpha(u_k - \alpha G_\alpha u_k, u_k)_{\mathcal{H}}$$

it follows that $\Phi(u_1, ..., u_n) \in \mathcal{V}^{\mathcal{F}}$. But $\Phi(u_1, ..., u_n) \in \mathcal{F}$ even if $u_1, ..., u_n \in \mathcal{F}$ is in general not true.

Recall the following definition:

$$\widetilde{\mathcal{H}}^{dec} := \{\widetilde{u} \in \widetilde{\mathcal{H}} \mid \exists M^{[u]} \in \overset{\circ}{\mathcal{M}}, N^{[u]} \in \mathcal{N}_c \text{ such that } A^{[u]} = M^{[u]} + N^{[u]}\}$$

and let $\mathcal{H}^{dec}$ denote the totality of m-versions of elements in $\widetilde{\mathcal{H}}^{dec}$.

For the rest of this section we will consider the following conditions:

Alg There exists a linear space $\mathcal{Y} \subset L^\infty(E;m) \cap \mathcal{V}^{\mathcal{F}} \cap \mathcal{H}^{dec}$ of bounded functions such that $u \cdot v \in \mathcal{Y}$ for $u, v \in \mathcal{Y}$.

Diag There exists constants $C, \gamma \geq 0$ such that $\overline{e}(A^{[u]}) \leq C\mathcal{A}_\gamma(u,u)$ for any $u \in \mathcal{Y}$.

Diag' There exists constants $C, \gamma \geq 0$, and a Dirichlet form $(A, D(A))$ on $\mathcal{H}$ such that $\mathcal{Y} \subset D(A) \cap \mathcal{F}$, $\overline{e}(A^{[u]}) \leq CA_\gamma(u,u)$, $A(u,u) \leq \mathcal{E}(u,u)$, for any $u \in \mathcal{Y}$.

Note that there is a qualitative difference between **Diag** (resp. **Diag'**)and (26).

Proposition 3.4 *(i) (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{SUB}}$, **Alg**) Let $\Phi \in C^1(\mathbb{R}^n)$, $\Phi(0) = 0$. Let $f = (f_1, ..., f_n)$ be an n-tuple of elements in $\mathcal{Y}$. Then $\Phi(f) \in \mathcal{H}^{dec}$. In particular if* **Diag** *(resp.* **Diag'***) holds then*

$$e(M^{[\Phi(f)]}) \leq C\mathcal{A}_\gamma(\Phi(f), \Phi(f))$$

(resp. $e(M^{[\Phi(f)]}) \leq CA_\gamma(\Phi(f), \Phi(f))$).
*(ii) (**QR**, $\widehat{\mathbf{SUB}}$, **Alg**, **Diag** or **Diag'**) Let $E = V$ and assume that $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ for some $k \in \mathbb{N} \cup \{\infty\}$ or assume that $\mathcal{Y} = \mathcal{F}\mathcal{C}_b^\infty \subset \mathcal{F}$ (for the definition of $\mathcal{F}\mathcal{C}_b^\infty$ cf. section 4.2). Then (i) holds with $f_1, ..., f_n \in \overline{\mathcal{Y}} \cap \mathcal{F}_b$ where $\overline{\mathcal{Y}}$ denotes the closure of $\mathcal{Y}$ w.r.t. $\|\cdot\|_{\mathcal{F}}$.*

Proof (i) We may assume that $n = 1$ since all estimations we will use hold in a similar form for $n > 1$. We then will show that the conditions of Theorem 2.11(ii) are satisfied. Let $K \subset \mathbb{R}$ be a compact set such that $\widetilde{f}(z) \in K$ for $\mathcal{E}$-q.e. z. Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi = 1$ on $K \subset\subset K' = supp(\chi)$. Let $(p_n)_{n \in \mathbb{N}}$ be polynomials (cf. proof of Theorem 2.17) vanishing at the origin and such that $p_n \longrightarrow \Phi$, $\frac{\partial p_n}{\partial x} \longrightarrow \frac{\partial \Phi}{\partial x}$, uniformly on K' as $n \to \infty$. By **Alg** we have $p_n(\widetilde{f}) \in \widetilde{\mathcal{H}}^{dec}$ for any $n \in \mathbb{N}$. Then

$$\overline{e}(A^{[\Phi(f)-p_n(f)]}) \leq \left\| \frac{\partial((\Phi - p_n) \cdot \chi)}{\partial x} \right\|_\infty^2 \overline{e}(A^{[f]}).$$

Similarly to Lemma 2.8 we have for any $\mu \in \widehat{S}_{00}$

$$P_\mu(\sup_{0 \leq t \leq T} |\Phi(\widetilde{f}) - p_n(\widetilde{f})|(Y_t) > \varepsilon) \leq \frac{e^T}{\varepsilon} C'\mu(E) \left\| \frac{\partial((\Phi - p_n) \cdot \chi)}{\partial x} \right\|_\infty$$

where C' is a constant such that $\widetilde{f} \leq C'$ $\mathcal{E}$-q.e. on E. Since $\left\| \frac{\partial((\Phi - p_n) \cdot \chi)}{\partial x} \right\|_\infty \to 0$ as $n \to \infty$ Theorem 2.11(ii) now implies that $\Phi(\widetilde{f}) \in \widetilde{\mathcal{H}}^{dec}$. Since also $p_n(f) \to \Phi(f)$ w.r.t. $\widetilde{\mathcal{A}}_1^{1/2}$ (resp. $\widetilde{A}_1^{1/2}$ for any Dirichlet form $(A, D(A))$ with $f \in D(A)$ where $\widetilde{A}_1(u,v) :=$

$\frac{1}{2}(A(u,v)+A(v,u))+(u,v)_{\mathcal{H}},\ u,v\in D(A))$ as $n\to\infty$ it follows finally from (29) and **Diag** (resp. **Diag'**) that

$$e(M^{[\Phi(f)]})=\lim_{n\to\infty}e(A^{[p_n(f)]})\leq C\mathcal{A}_\gamma(\Phi(f),\Phi(f))$$

(resp. $e(M^{[\Phi(f)]})\leq CA_\gamma(\Phi(f),\Phi(f))$).
(ii) Note first that $\mathcal{Y}=C_0^k(V)\subset\mathcal{F}$ dense for some $k\in\mathbb{N}\cup\{\infty\}$ implies $\mathbf{M^{ex}}$ by [23, *Proposition IV*.2.1.]. As in (i) it is enough to show the statement when $n=1$ and as in (i) we will show that the conditions of Theorem 2.11(ii) are satisfied. We will also only show the statement when $k=1$ and when **Diag** holds. The assertion for the case **Diag'** follows similarly. We will proceed in two steps.
a) Let $f\in\mathcal{F}_b$. We first show that there exists $(v_n)_{n\in\mathbb{N}}\subset C_0^1(V)$ such that $\sup_{n\in\mathbb{N}}\|v_n\|_\infty<\infty$ and $v_n\to f$ w.r.t. $\widetilde{\mathcal{A}}_1^{1/2}$ as $n\to\infty$.
Let $(u_n)_{n\in\mathbb{N}}\subset C_0^1(V)$ such that $u_n\to f$ in $\mathcal{F}$ as $n\to\infty$. Let $D\subset\mathbb{R}$ be a compact set such that $\widetilde{f}(z)\in D$ for $\mathcal{E}$-q.e. $z\in V$. Let $\xi\in C_0^\infty(\mathbb{R})$, $\xi(x)=x$ on $D\cup\{0\}$. Since $\xi(f)=f$ and $|f-\xi(u_n)|\leq\|\frac{\partial\xi}{\partial x}\|_\infty|f-u_n|$ m-a.e. it follows that $\xi(u_n)\to f$ in $L^2(V;m)$ as $n\to\infty$. Using (39) and Lemma 2.6 we have $\sup_{n\in\mathbb{N}}\mathcal{A}(\xi(u_n),\xi(u_n))\leq(K+1)^2\left\|\frac{\partial\xi}{\partial x}\right\|_\infty^2\sup_{n\in\mathbb{N}}\|u_n\|_{\mathcal{F}}^2<\infty$. Hence there exists a subsequence such that $\frac{1}{n}\sum_{k=1}^n\xi(u_{n_k})\to f$ w.r.t. $\widetilde{\mathcal{A}}_1^{1/2}$ as $n\to\infty$. It then suffices to set $v_n:=\frac{1}{n}\sum_{k=1}^n\xi(u_{n_k})$.
b) Let ξ, $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, f be as in **a)**. Let $K\subset\mathbb{R}$ be a compact set such that $\widetilde{v}_n(z),\widetilde{f}(z)\in K$ for $\mathcal{E}$-q.e. z. Let $\chi\in C_0^\infty(\mathbb{R})$, $\chi=1$ on $K\subset\subset K'=supp(\chi)$. Let $(p_n)_{n\in\mathbb{N}}$ be polynomials as in (i). Note that by (i) we have that $p_n(\widetilde{v}_n)\in\widetilde{\mathcal{H}}^{dec}$ for any n. Then

$$\begin{aligned}
\overline{e}(A^{[\Phi(f)-p_n(v_n)]}) &\leq 2\overline{e}(A^{[\Phi\cdot\chi(f)-p_n\cdot\chi(f)]})+2\overline{e}(A^{[p_n\cdot\chi(f)-p_n\cdot\chi(v_n)]})\\
&\leq \left\|\frac{\partial((\Phi-p_n)\cdot\chi)}{\partial x}\right\|_\infty^2 2\overline{e}(A^{[f]})+\left\|\frac{\partial(p_n\cdot\chi)}{\partial x}\right\|_\infty^2 2\overline{e}(A^{[f-v_n]})\\
&= \left\|\frac{\partial((\Phi-p_n)\cdot\chi)}{\partial x}\right\|_\infty^2 2e(M^{[f]})+\left\|\frac{\partial(p_n\cdot\chi)}{\partial x}\right\|_\infty^2 2\lim_{k\to\infty}\overline{e}(A^{[u_k-v_n]})\\
&\leq \left\|\frac{\partial((\Phi-p_n)\cdot\chi)}{\partial x}\right\|_\infty^2 2e(M^{[f]})+\left\|\frac{\partial(p_n\cdot\chi)}{\partial x}\right\|_\infty^2 2C\mathcal{A}_\gamma(f-v_n,f-v_n).
\end{aligned}$$

But since $\|\frac{\partial(p_n\cdot\chi)}{\partial x}\|_\infty^2$ is uniformily bounded in n the r.h.s. above tends to zero as $n\to\infty$. Noting that $|\Phi(\widetilde{f})-p_n(\widetilde{v}_n)|\leq\|\frac{\partial((\Phi-p_n)\cdot\chi)}{\partial x}\|_\infty|\widetilde{f}|+\|\frac{\partial(p_n\cdot\chi)}{\partial x}\|_\infty|\widetilde{f}-\widetilde{v}_n|$ and $|\widetilde{f}-\widetilde{v}_n|\leq\|\frac{\partial\xi}{\partial x}\|_\infty\frac{1}{n}\sum_{k=1}^n|\widetilde{f}-\widetilde{u}_{n_k}|$ $\mathcal{E}$-q.e. and that $\frac{1}{n}\sum_{k=1}^n\|f-u_{n_k}\|_e$ tends to zero as $n\to\infty$ we can easily see that the conditions of Lemma 2.7 are also satisfied. Surely we have also $e(M^{[\Phi(f)]})=\lim_{n\to\infty}e(M^{[p_n(v_n)]})\leq C\mathcal{A}_\gamma(\Phi(f),\Phi(f))$.

□

c) Localization

A is called a *local additive functional* (abbreviated local AF) if A satisfies all properties of an AF except that the additivity $A_{t+s}(\omega)=A_s(\omega)+A_t(\theta_s(\omega))$ for $\omega\in\Lambda$ is only

required for those $t, s \geq 0$ with $t + s < \zeta(\omega)$. Thus, local AF's and AF's are the same if $\mathbf{M}$ is conservative (i.e. $P_z(\zeta = \infty) = 1$ for $\mathcal{E}$-q.e. $z \in E$).
For any two local AF's $A^{(1)}, A^{(2)}$ and an $\mathcal{F}_t$-stopping time σ we write

$$A_t^{(1)} = A_t^{(2)} \ \forall t \leq \sigma \quad \textit{or} \quad A_t^{(1)} = A_t^{(2)} \ \forall t < \sigma$$

if the probability of these events w.r.t. P_z is equal to one for $\mathcal{E}$-q.e. $z \in E$. $A^{(1)}$ is called *equivalent* to $A^{(2)}$ if

$$A_t^{(1)} = A_t^{(2)} \ \forall t < \zeta.$$

Up to the end of this section let us consider the following additional conditions

$$\begin{array}{c} \textbf{Dif} \ P_z(t \mapsto Y_t \text{ is continuous on } [0, \zeta)) = 1 \text{ for } \mathcal{E}\text{-q.e. } z \in E \\ \mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]} \ M^{[G_1 h]} \text{ is a continuous MAF for any } h \in \mathcal{H}_b^+ \\ \textbf{Plt} \text{ There exists an } \mathcal{E}\text{-nest } (F_k)_{k\in\mathbb{N}} \text{ such that } P_z(t \wedge \sigma_{F_k^c} < \zeta) = 1 \\ \text{for any } t \geq 0,\ k \in \mathbb{N} \text{ and } \mathcal{E}\text{-q.e. } z \in E. \end{array}$$

Note that $\mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}$ implies that any $M \in \overset{\circ}{\mathcal{M}}$ is continuous. Note also that under suitable conditions $\mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}$ implies **Dif** and **Plt** but we do not investigate this question. By quasi-regularity we may assume that an $\mathcal{E}$-nest which satisfies **Plt** consists of compact subsets of E. Up to the end of this section we will fix such an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$.

Let $\widetilde{\mathcal{D}} \subset \widetilde{\mathcal{H}}^{dec}$. An $\mathcal{E}$-q.e. defined function u on E is called *locally* in $\widetilde{\mathcal{D}}$ w.r.t. $(F_k)_{k\in\mathbb{N}}$ (abbreviated $u \in \widetilde{\mathcal{D}}_{loc,(F_k)_{k\in\mathbb{N}}}$) if there exists $(\widetilde{u}_k)_{k\in\mathbb{N}} \subset \widetilde{\mathcal{D}}$ such that for any k

$$u = \widetilde{u}_k \quad \mathcal{E}\text{-q.e.} \ \ \textit{on} \ \ F_k.$$

Any $(\widetilde{u}_k)_{k\in\mathbb{N}}$ as above is called a localizing sequence for u.
Now let us define the space of functions which we will use for our localization procedure:

$$\begin{aligned} [\widetilde{\mathcal{H}}^{dec}]_{loc,(F_k)_{k\in\mathbb{N}}} := \{u \in \widetilde{\mathcal{H}}^{dec}_{loc,(F_k)_{k\in\mathbb{N}}} \mid\ & u \text{ admits a localizing sequence } (\widetilde{u}_k)_{k\in\mathbb{N}} \subset \widetilde{\mathcal{H}}^{dec} \\ & \text{such that for any } l \geq k \text{ we have that } M^{[u_k - u_l]} = 0 \ \forall t \leq \sigma_{F_k^c}\}. \end{aligned}$$

A local AF M is called *locally* in $\overset{\circ}{\mathcal{M}}$ w.r.t. $(F_k)_{k\in\mathbb{N}}$ (abbreviated $M \in \overset{\circ}{\mathcal{M}}_{loc,(F_k)_{k\in\mathbb{N}}}$) if there exists $(M^k)_{k\in\mathbb{N}} \subset \overset{\circ}{\mathcal{M}}$ such that for any k

$$M_t = M_t^k \ \ \forall t \leq \sigma_{F_k^c}.$$

A local AF N is called *locally* in $\mathcal{N}_c$ w.r.t. $(F_k)_{k\in\mathbb{N}}$ (abbreviated $N \in \mathcal{N}_{c,loc,(F_k)_{k\in\mathbb{N}}}$) if there exists $(N^k)_{k\in\mathbb{N}} \subset \mathcal{N}_c$ such that for any k

$$N_t = N_t^k \ \ \forall t \leq \sigma_{F_k^c}.$$

Lemma 3.5 *(i) (*$\mathbf{QR}$, $\mathbf{M^{ex}}$*) Any $u \in [\widetilde{\mathcal{H}}^{dec}]_{loc,(F_k)_{k\in\mathbb{N}}}$ is $\mathcal{E}$-quasi-continuous. If $u, v \in [\widetilde{\mathcal{H}}^{dec}]_{loc,(F_k)_{k\in\mathbb{N}}}$ are such that $u - v = const.$ $\mathcal{E}$-q.e. then $A^{[u]} = A^{[v]}$ (i.e. equivalent as*

AF's). For any localizing sequence $(\widetilde{u}_k)_{k\in\mathbb{N}}$ for u the AF $A^{[u]}$ is equivalent (in the sense of local AF's) to the local AF A where $A_t := \lim_{k\to\infty} A_t^{[u_k]}$ if $t<\zeta$, $A_t := 0$ elsewhere.

*(ii) (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$, **Dif**, **Plt**) Let $M \in \overset{\circ}{\mathcal{M}}$. Let $F \subset F_k$ for some k, F closed. If $1_F \cdot \mu_{<M>} = 0$ then $M_t = 0$ $\forall t \le \sigma_{F^c}$.*

*(iii) (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$, $\mathbf{M^{[G_1\mathcal{H}_b]}_{cont}}$) Let $\widetilde{u} \in \widetilde{\mathcal{L}} := \{\widetilde{u} \in \widetilde{\mathcal{V}}^{\mathcal{F}} \mid \exists C,\gamma \ge 0$ such that $\forall \xi \in C_b^\infty(\mathbb{R})$ with $\xi(0)=0$ we have $\widetilde{u}, \xi(\widetilde{u}) \in \widetilde{\mathcal{H}}^{dec}, e(M^{[\xi(u)]}) \le C\mathcal{A}_\gamma(\xi(u),\xi(u)), \exists (h_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ with $\overline{e}(A^{[u-G_1h_n]}) \to 0$ as $n\to\infty\}$ or let $\widetilde{u} \in \widetilde{\mathcal{L}}' := \{\widetilde{u} \in \widetilde{\mathcal{V}}^{\mathcal{F}} \mid \exists C,\gamma \ge 0$ and a Dirichlet form $(A, D(A))$ on $\mathcal{H}$ such that $\forall \xi \in C_b^\infty(\mathbb{R})$ with $\xi(0)=0$ we have $\widetilde{u}, \xi(\widetilde{u}) \in \widetilde{\mathcal{H}}^{dec} \cap \widetilde{D(A)}, e(M^{[\xi(u)]}) \le CA_\gamma(\xi(u),\xi(u)), \exists (h_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ with $\overline{e}(A^{[u-G_1h_n]}) \to 0$ as $n\to\infty\}$. Let $\widetilde{u} = const$ $\mu_{<u>}$-a.s. on $B \in \mathcal{B}(E)$. Then*

$$1_B \cdot \mu_{<u>} = 0.$$

Here $C_b^\infty(\mathbb{R})$ denotes the space of all infinitely many continuously differentiable functions on $\mathbb{R}$ with bounded derivative of any order.

*(iv) (**QR**, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$, $\mathbf{M^{[G_1\mathcal{H}_b]}_{cont}}$, **Alg**, **Diag** or **Diag'**) Assume that for any $u \in \mathcal{Y}$ there exists $(h_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ with $\overline{e}(A^{[u-G_1h_n]}) \to 0$ as $n\to\infty$. If **Diag** holds then $\widetilde{\mathcal{Y}} \subset \widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{F}}_b \subset \widetilde{\mathcal{L}}$ if in addition $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ dense for some $k \in \mathbb{N}\cup\{\infty\}$). If **Diag'** holds then $\widetilde{\mathcal{Y}} \subset \widetilde{\mathcal{L}}'$ (resp. $\widetilde{\mathcal{F}}_b \subset \widetilde{\mathcal{L}}'$ if in addition $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ dense for some $k \in \mathbb{N}\cup\{\infty\}$). If **Diag** or **Diag'** hold then $1_B \bullet \mu_{<\Phi(u_1,\dots,u_n)>} = 0$ for any $\Phi \in C^1(\mathbb{R}^n)$ with $\Phi(0)=0$ and $\widetilde{u}_1,\dots,\widetilde{u}_n \in \widetilde{\mathcal{Y}}$ such that $\widetilde{u}_i = const$ $\mu_{<u_i>}$-a.s. on B, $1 \le i \le n$. If $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ dense for some $k \in \mathbb{N}\cup\{\infty\}$ and **Diag** or **Diag'** hold then $1_B \bullet \mu_{<\Phi(u_1,\dots,u_n)>} = 0$ remains true for any $\widetilde{u}_1,\dots,\widetilde{u}_n \in \widetilde{\mathcal{F}}_b$ as above.*

Proof (i) is obvious.
(ii) By **Dif** and **Plt** we have for $\mathcal{E}$-q.e $z \in E$ and any $t \ge 0$

$$\begin{aligned} E_z\Big[< M >_{t\wedge\sigma_{F^c}} \Big] &= E_z\Big[\int_0^{t\wedge\sigma_{F^c}} 1_F(Y_s)\, d < M >_s \Big] \\ &\le E_z\Big[\int_0^t 1_F(Y_s)\, d < M >_s \Big]. \end{aligned}$$

From Theorem 2.5 we know that $E_\cdot\Big[\int_0^t 1_F(Y_s)\, d < M >_s \Big] = 0$ $\mathcal{E}$-q.e. hence the same is true for $E_\cdot\Big[< M >_{t\wedge\sigma_{F^c}} \Big]$. Now $((M_t)^2 - < M >_t)_{t\ge 0}$ is a martingale w.r.t. P_z for $\mathcal{E}$-q.e. $z \in E$. The optional sampling theorem then implies

$$E_z\Big[(M_{t\wedge\sigma_{F^c}})^2 \Big] = E_z\Big[< M >_{t\wedge\sigma_{F^c}} \Big] = 0$$

and the assertion follows.
(iii) It is enough to assume that $\widetilde{u} = 0$ or $\widetilde{u} = 2\pi$ $\mu_{<u>}$-a.s. on B. Let $\widetilde{u} \in \widetilde{\mathcal{L}}$. By Proposition 2.19 we have for any $k \in \mathbb{N}$

$$\int 1_B\, d\mu_{<u>} = \int \cos(k\widetilde{u}) 1_B\, d\mu_{<u>} = \int 1_B\, d\mu_{<\frac{1}{k}\sin(ku),u>}. \tag{40}$$

Obviously $\frac{1}{k}\sin(ku) \to 0$ in $L^2(E;m)$ as $k \to \infty$ and $\frac{1}{k}\sin(ku) \in \mathcal{V}^{\mathcal{F}}(\subset \mathcal{V})$ since $u \in \mathcal{V}^{\mathcal{F}}$. By (39) we have

$$\begin{aligned}\mathcal{A}(\frac{1}{k}\sin(ku), \frac{1}{k}\sin(ku)) &\le \overline{\lim_{\alpha\to\infty}}\,\mathcal{E}(\alpha G_\alpha(\frac{1}{k}\sin(ku)), \alpha G_\alpha(\frac{1}{k}\sin(ku)))\\ &\le \overline{\lim_{\alpha\to\infty}}\,\alpha(\frac{1}{k}\sin(ku) - \alpha G_\alpha(\frac{1}{k}\sin(ku)), \frac{1}{k}\sin(ku))_{\mathcal{H}}\\ &\le \overline{\lim_{\alpha\to\infty}}\,\alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}} < \infty.\end{aligned}$$

Hence by [15, *Lemma I*.2.12.] there exists a subsequence such that $\frac{1}{N}\sum_{l=1}^N \frac{1}{k_l}\sin(k_l u) \to 0$ w.r.t. $\widetilde{\mathcal{A}}_1^{1/2}$ as $N \to \infty$. Therefore by (40) we have

$$\begin{aligned}\int 1_B\, d\mu_{<u>} &= \lim_{N\to\infty}\int 1_B\, d\mu_{<\frac{1}{N}\sum_{l=1}^N \frac{1}{k_l}\sin(k_l u), u>}\\ &\le \lim_{N\to\infty} 2e(M^{[\frac{1}{N}\sum_{l=1}^N \frac{1}{k_l}\sin(k_l u)]})^{1/2} e(M^{[u]})^{1/2}\\ &\le 2e(M^{[u]})^{1/2}\lim_{N\to\infty} C^{1/2}\mathcal{A}_\gamma(\frac{1}{N}\sum_{l=1}^N \frac{1}{k_l}\sin(k_l u), \frac{1}{N}\sum_{l=1}^N \frac{1}{k_l}\sin(k_l u))^{1/2} = 0.\end{aligned}$$

In the same manner we show $\int 1_B\, d\mu_{<u>} = 0$ for $\widetilde{u} \in \widetilde{\mathcal{L}}'$.
(iv) Let $\widetilde{u}_1, ..., \widetilde{u}_n \in \widetilde{\mathcal{Y}}$ (resp. $\widetilde{u}_1, ..., \widetilde{u}_n \in \widetilde{\mathcal{F}}_b$). By Proposition 3.4(i) (resp. Proposition 3.4(ii) if in addition $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ dense for some $k \in \mathbb{N}\cup\{\infty\}$) we have $\Phi(\widetilde{u}_1, ..., \widetilde{u}_n) \in \widetilde{\mathcal{H}}^{dec}$. It then follows by Theorem 2.17 that

$$\int 1_B\, d\mu_{<\Phi(u_1,...,u_n)>} = \sum_{i=1}^n \int \frac{\partial\Phi}{\partial x_i}(\widetilde{u}_1, ..., \widetilde{u}_n)\, 1_B\, d\mu_{<u_i,\Phi(u_1,...,u_n)>}.$$

Let $\widetilde{u}_i = const$ $\mu_{<u_i>}$-a.s. on B, $1 \le i \le n$. If **Diag** holds the expression on the r.h.s. is zero by (iii) (and Cauchy-Schwarz inequality) because by Proposition 3.4(i) (resp. Proposition 3.4(ii)) $\widetilde{\mathcal{Y}} \subset \widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{F}}_b \subset \widetilde{\mathcal{L}}$). If **Diag'** holds instead of **Diag** then also by Proposition 3.4(i) (resp. Proposition 3.4(ii)) $\widetilde{\mathcal{Y}} \subset \widetilde{\mathcal{L}}'$ (resp. $\widetilde{\mathcal{F}}_b \subset \widetilde{\mathcal{L}}'$) and $\int 1_B\, d\mu_{<\Phi(u_1,...,u_n)>} = 0$ as above.

□

For an appropriate defining set and exceptional set the quadratic variation $< M >$ of $M \in \overset{\circ}{\mathcal{M}}_{loc,(F_k)_{k\in\mathbb{N}}}$ is well defined as a PCAF by

$$< M >_t = < M^k >_t \quad \forall t \le \sigma_{F_k^c}, \quad k \ge 1. \tag{41}$$

We may then consider its Revuz measure $\mu_{<M>}$. Now we have the following:

Theorem 3.6 *(i) (**QR**, $\mathbf{M^{ex}}$, $\mathbf{M_{cont}^{[G_1\mathcal{H}_b]}}$) Let $u \in [\widetilde{\mathcal{H}}^{dec}]_{loc,(F_k)_{k\in\mathbb{N}}}$. Then there exists $M^{[u]} \in \overset{\circ}{\mathcal{M}}_{loc,(F_k)_{k\in\mathbb{N}}}$ and $N^{[u]} \in \mathcal{N}_{c,loc,(F_k)_{k\in\mathbb{N}}}$ such that*

$$A^{[u]} = M^{[u]} + N^{[u]}.$$

Such a decomposition is unique up to equivalence of local AF*'s.*
(ii) ($\mathbf{QR}$, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$, $\mathbf{M_{cont}^{[G_1\mathcal{H}_b]}}$, $\mathbf{Dif}$, $\mathbf{Plt}$*) Let* $\widetilde{\mathcal{L}}$,$\widetilde{\mathcal{L}'}$*, be as in Lemma 3.5(iii). Then* $\widetilde{\mathcal{L}}_{loc,(F_k)_{k\in\mathbb{N}}}, \widetilde{\mathcal{L}'}_{loc,(F_k)_{k\in\mathbb{N}}} \subset [\widetilde{\mathcal{H}}^{dec}]_{loc,(F_k)_{k\in\mathbb{N}}}$.
(iii) ($\mathbf{QR}$, $\mathbf{M^{ex}}$, $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$, $\mathbf{M_{cont}^{[G_1\mathcal{H}_b]}}$, $\mathbf{Dif}$, $\mathbf{Plt}$, $\mathbf{Alg}$, $\mathbf{Diag}$ *or* $\mathbf{Diag'}$*) Assume that for any* $u \in \mathcal{Y}$ *there exists* $(h_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ *with* $\overline{e}(A^{[u-G_1h_n]}) \to 0$ *as* $n \to \infty$*. If* $\mathbf{Diag}$ *holds then* $\widetilde{\mathcal{Y}}_{loc,(F_k)_{k\in\mathbb{N}}} \subset \widetilde{\mathcal{L}}_{loc,(F_k)_{k\in\mathbb{N}}}$ *(resp.* $\widetilde{\mathcal{F}}_{b\,loc,(F_k)_{k\in\mathbb{N}}} \subset \widetilde{\mathcal{L}}_{loc,(F_k)_{k\in\mathbb{N}}}$ *if in addition* $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ *dense for some* $k \in \mathbb{N}\cup\{\infty\}$*). If* $\mathbf{Diag'}$ *holds then* $\widetilde{\mathcal{Y}}_{loc,(F_k)_{k\in\mathbb{N}}} \subset \widetilde{\mathcal{L}'}_{loc,(F_k)_{k\in\mathbb{N}}}$ *(resp.* $\widetilde{\mathcal{F}}_{b\,loc,(F_k)_{k\in\mathbb{N}}} \subset \widetilde{\mathcal{L}'}_{loc,(F_k)_{k\in\mathbb{N}}}$ *if in addition* $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ *dense for some* $k \in \mathbb{N}\cup\{\infty\}$*). If* $f_1,...,f_n \in \widetilde{\mathcal{Y}}_{loc,(F_k)_{k\in\mathbb{N}}}$ *and* $\Phi \in C^1(\mathbb{R}^n)$ *then* $A^{[\Phi(f_1,...,f_n)]}$ *decomposes as in (i) and*

$$< M^{[\Phi(f)]} >_t = \sum_{i,j=1}^{n} \int_0^t \Phi_{x_i}(f(Y_s))\Phi_{x_j}(f(Y_s))d < M^{[f_i]}, M^{[f_j]} >_s .$$

Here $f = (f_1,...,f_n)$ *and* $\Phi_{x_i} = \frac{\partial\Phi}{\partial x_i}$, $1 \le i \le n$*. The assertion remains true for any* $f_1,...,f_n \in \widetilde{\mathcal{F}}_{b\,loc,(F_k)_{k\in\mathbb{N}}}$ *provided that in addition* $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ *dense for some* $k \in \mathbb{N}\cup\{\infty\}$*.*

Proof (i) **Existence:** Let $u \in [\widetilde{\mathcal{H}}^{dec}]_{loc,(F_k)_{k\in\mathbb{N}}}$ and let $(\widetilde{u}_k)_{k\in\mathbb{N}} \subset \widetilde{\mathcal{H}}^{dec}$ be an associated localizing sequence. For any $l \ge k$ we have $M_t^{[u_k]} = M_t^{[u_l]}$ $\forall t \le \sigma_{F_k^c}$. Hence $M_t^{[u]} := \lim_{k\to\infty} M_t^{[u_k]}$ if $t < \zeta$, $M_t^{[u]} := 0$ elsewhere, is well defined as an element of $\overset{\circ}{\mathcal{M}}_{loc,(F_k)_{k\in\mathbb{N}}}$. It then suffices to set $N^{[u]} := A^{[u]} - M^{[u]}$.
Uniqueness: By $\mathbf{M_{cont}^{[G_1\mathcal{H}_b]}}$ we have that any $M \in \overset{\circ}{\mathcal{M}}$ is continuous. Hence for any stopping time σ

$$< M >_{t\wedge\sigma} = \lim_{k\to\infty} \sum_{l=1}^{[kt]} (M_{\frac{l+1}{k}\wedge\sigma} - M_{\frac{l}{k}\wedge\sigma})^2 \ in\ L^1(P_m).$$

On the other hand (cf. [8, (5.2.14), p.201]) $\lim_{k\to\infty} \sum_{l=1}^{[kt]} (N_{\frac{l+1}{k}\wedge\sigma} - N_{\frac{l}{k}\wedge\sigma})^2$ in $L^1(P_m)$ for any $N \in \mathcal{N}_c$.
Now let $M \in \overset{\circ}{\mathcal{M}}_{loc,(F_k)_{k\in\mathbb{N}}} \bigcap \mathcal{N}_{c,loc,(F_k)_{k\in\mathbb{N}}}$ with $(M^k)_{k\in\mathbb{N}} \subset \overset{\circ}{\mathcal{M}}$, $(N^k)_{k\in\mathbb{N}} \subset \mathcal{N}_{c,loc,(F_k)_{k\in\mathbb{N}}}$ as associated sequences . The preceding observation implies $P_m(< M >_{t\wedge\sigma_{F_k^c}} = < M^k >_{t\wedge\sigma_{F_k^c}} = 0\ \forall t \le \sigma_{F_k^c}) = 1$ which means $P_m(< M >_t = < M^k >_t = 0\ \forall t \le \sigma_{F_k^c}) = 1$.
Now consider the Revuz measure $\mu_{<M>}$ of M. Since $P_m(< M >_t = < M^k >_t = 0\ \forall t \le \sigma_{F_k^c}) = 1$ for any k we know from Theorem 2.4 that

$$\int_{E\setminus F_k^{\widehat{0}}} d\mu_{<M>} \quad = \quad \lim_{\alpha\to\infty} \alpha E_m \Big[\int_0^{\sigma_{F_k^c}} e^{-(\alpha+1)t}\, d < M^k >_t \ \Big] = 0.$$

for any k. But since $\bigcap_{k\ge1}(E \setminus F_k^{\widehat{0}})$ is $\widehat{\mathcal{E}}$-exceptional by Lemma 1.18 applied to the coassociated process and since any $\widehat{\mathcal{E}}$-exceptional set is $\mathcal{E}$-exceptional we have $\mu_{<M>} = 0$ hence $< M >= 0$.
(ii) Let $(\widetilde{f}_k)_{k\in\mathbb{N}}$ be a localizing sequence for $f \in \widetilde{\mathcal{L}}_{loc,(F_k)_{k\in\mathbb{N}}}$ (resp. $f \in \widetilde{\mathcal{L}'}_{loc,(F_k)_{k\in\mathbb{N}}}$). Let

$l \geq k$. Since $\widetilde{f}_k - \widetilde{f}_l = 0$ $\mathcal{E}$-q.e. on F_k we have $1_{F_k} \cdot d\mu_{<M^{[f_k - f_l]}>} = 0$ by Lemma 3.5(iii). Then Lemma 3.5(ii) implies $M_t^{[f_k - f_l]} = 0 \; \forall t \leq \sigma_{F_k^c}$ for any $l \geq k$ hence $f \in [\widetilde{\mathcal{H}}^{dec}]_{loc,(F_k)_{k\in\mathbb{N}}}$.
(iii) We only consider the case where **Diag** holds. Since by Lemma 3.5(iv) $\widetilde{\mathcal{Y}} \subset \widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{F}}_b \subset \widetilde{\mathcal{L}}$ if $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ dense for some $k \in \mathbb{N} \cup \{\infty\}$) it follows that $\widetilde{\mathcal{Y}}_{loc,(F_k)_{k\in\mathbb{N}}} \subset \widetilde{\mathcal{L}}_{loc,(F_k)_{k\in\mathbb{N}}}$ (resp. $\widetilde{\mathcal{F}}_{b\,loc,(F_k)_{k\in\mathbb{N}}} \subset \widetilde{\mathcal{L}}_{loc,(F_k)_{k\in\mathbb{N}}}$ if $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ dense for some $k \in \mathbb{N} \cup \{\infty\}$). For the second assertion let $f_1, ..., f_n \in \widetilde{\mathcal{Y}}_{loc,(F_k)_{k\in\mathbb{N}}}$ and $\Phi \in C^1(\mathbb{R}^n)$. Set $\Psi(x) = \Phi(x) - \Phi(0)$ then $\Psi(f_1, ..., f_n) \in \widetilde{\mathcal{L}}_{loc,(F_k)_{k\in\mathbb{N}}}$ by Proposition 3.4(i). Since $A^{[\Phi(f)]} = A^{[\Psi(f)]}$ there exists $M^{[\Phi(f)]} \in \overset{\circ}{\mathcal{M}}_{loc,(F_k)_{k\in\mathbb{N}}}$, $N^{[\Phi(f)]} \in \mathcal{N}_{c,loc,(F_k)_{k\in\mathbb{N}}}$, such that $A^{[\Phi(f)]} = M^{[\Phi(f)]} + N^{[\Phi(f)]}$. Indeed, if $(\widetilde{f}_{ik})_{k\in\mathbb{N}}$, $1 \leq i \leq n$, are localizing sequences for f_i, $1 \leq i \leq n$, it suffices to set $M_t^{[\Phi(f)]} := \lim_{k\to\infty} M_t^{[\Psi(f_{1k},...,f_{nk})]}$ if $t < \zeta$, $M_t^{[\Phi(f)]} := 0$ elsewhere, $N^{[\Phi(f)]} := A^{[\Phi(f)]} - M^{[\Phi(f)]}$. Since for any $k \in \mathbb{N}$ by Lemma 2.4 and (41)

$$1_{F_k^{\hat{0}}} \cdot d\mu_{<M^{[\Phi(f)]}>} = 1_{F_k^{\hat{0}}} \sum_{i,j=1}^{n} \Phi_{x_i}(f)\Phi_{x_j}(f) \cdot d\mu_{<M^{[f_i]},M^{[f_j]}>}$$

the assertion follows from Theorem 2.5. The assertion for $f_1, ..., f_n \in \widetilde{\mathcal{F}}_{b\,loc,(F_k)_{k\in\mathbb{N}}}$ provided that $\mathcal{Y} = C_0^k(V) \subset \mathcal{F}$ dense for some $k \in \mathbb{N} \cup \{\infty\}$ follows analoguous by Proposition 3.4(ii).

$\square$

Remark 3.7 *Given a Dirichlet form $(\mathcal{A}, \mathcal{V})$ on $\mathcal{H}$ (cf. [15, Definition I.4.5.]). We can construct an associated generalized Dirichlet form $\mathcal{E} = \mathcal{A}$ setting $\Lambda = 0$, $\mathcal{F} = \mathcal{V} = \widehat{\mathcal{F}}$ (cf. Example 1.2(i)). If $\mathcal{A}$ is quasi-regular in the sense of [14] it follows that the correponding generalized Dirichlet form satisfies* **QR** *(cf. [23, Remark III.2.6.(ii)]). Moreover by [23, IV.4(a)] we have* $\mathbf{M^{ex}}$*. By [15, Theorem I.4.4.]* $\widehat{\mathbf{SUB}}$ *holds. Then clearly* $\widehat{\mathbf{QR}}$, $\widehat{\mathbf{M}}^{\mathbf{ex}}$ *are satisfied. Also* **Alg** *and* **Diag** *hold automatically where $\mathcal{Y} = \mathcal{V} \cap L^\infty(E; m)$. Now assume that $\mathcal{E}$ has the local property in the sense of Lemma 3.1(i). Then, using the equivalence Lemma 3.1(i) $\Leftrightarrow$ (ii) and [15, Proposition V.1.7.] we show as in Theorem 3.2 that* $\mathbf{M}$ *is a diffusion up to ζ hence* **Dif** *holds. Finally, if we have that $p_t 1(x) = 1$ for $\mathcal{E}$-q.e. x, $t > 0$ then both* $\mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}$ *and* **Plt** *are satisfied and we can apply Theorem 2.11(i), Theorem 2.17, Theorem 2.18, Proposition 3.4(i), Theorem 3.6(ii) and (iii) to obtain the full results well known from classical Dirichlet form theory. In particular we can apply Lemma 3.5(iii) in its whole expanse, i.e. $\widetilde{\mathcal{L}} = \widetilde{\mathcal{V}}$.*

4 Examples

4.1 Time dependent Dirichlet forms and finite dimensional examples

a) Time dependent Dirichlet forms

Let $(\mathcal{E}^{(t)}, V)_{t\in\mathbb{R}}$ be a family of Dirichlet forms with sector constant K independent of t and common domain V in some separable L^2-space $H = L^2(X,\mu)$ with norm $\|\cdot\|_H$-where V is a Hilbert space with corresponding norm $\|\cdot\|_V$-satisfying the following conditions:

(a) $t \mapsto \mathcal{E}^{(t)}(u,v)$ is $\mathcal{B}(\mathbb{R})$-measurable for all $u, v \in V$.
(b) There exists a constant c such that

$$c\|u\|_V^2 \leq \mathcal{E}_1^{(t)}(u,u) \leq c^{-1}\|u\|_V^2 \ \text{ for all } u \in V,\ t \in \mathbb{R}.$$

For $W = H, V$ or V', (i.e. the dual of V with operatornorm $\|\cdot\|_{V'}^2$) let $L^2(\mathbb{R}, W) := \{u : \mathbb{R} \to W \text{ measurable} \mid \int_{\mathbb{R}} \|u(t)\|_W^2\, dt < \infty\}$. For $u, v \in \overline{\mathcal{V}} := L^2(\mathbb{R}; V)$ let $\overline{\mathcal{A}}(u,v) := \int_{\mathbb{R}} \mathcal{E}^{(t)}(u(t), v(t))\, dt$. Then $(\overline{\mathcal{A}}, \overline{\mathcal{V}})$ is a coercive closed form on $\overline{\mathcal{H}} := L^2(\mathbb{R}; H)$ with sector constant K. For $h \in \overline{\mathcal{H}}$ and $t \geq 0$ let

$$\overline{U}_t h(s) := h(s+t)\ .$$

The family of operators $(\overline{U}_t)_{t\geq 0}$ defines a C_0-semigroup of contractions on $\overline{\mathcal{H}}$ such that both $(\overline{U}_t)_{t\geq 0}$ and the adjoint semigroup $(\widehat{\overline{U}}_t)_{t\geq 0}$ can be extended to C_0-semigroups of contractions on $\overline{\mathcal{V}}' = L^2(\mathbb{R}; V')$. Denote by $(\frac{d}{dt}, D(\frac{d}{dt}, \overline{\mathcal{V}}'))$ and $(\frac{\widehat{d}}{dt}, D(\frac{\widehat{d}}{dt}, \overline{\mathcal{V}}'))$ the generators corresponding to the extended semigroups. It can be shown that

$$D(\frac{\widehat{d}}{dt}, \overline{\mathcal{V}}') = D(\frac{d}{dt}, \overline{\mathcal{V}}') = \{u | u \in \overline{\mathcal{V}}', \frac{du}{dt} \in \overline{\mathcal{V}}'\} =: H^{1,2}(\mathbb{R}, V')$$

and $\frac{\widehat{d}}{dt} = -\frac{d}{dt}$ (cf. [12, Subsection 3.4.3]), where $\frac{du}{dt}$ is to be understood in the sense of distributions taking their values in V'.
Let $\overline{\mathcal{F}} := D(\frac{d}{dt}, \overline{\mathcal{V}}') \cap \overline{\mathcal{V}}$ and $\widehat{\overline{\mathcal{F}}} := D(\frac{\widehat{d}}{dt}, \overline{\mathcal{F}}') \cap \overline{\mathcal{V}}$. The time dependent Dirichlet form corresponding to $(\mathcal{E}^{(t)}, V)_{t\in\mathbb{R}}$ is now given as follows:

$$\overline{\mathcal{E}}(u,v) := \begin{cases} \overline{\mathcal{A}}(u,v) - \langle \frac{du}{dt}, v\rangle & \text{if } u \in \overline{\mathcal{F}}, v \in \overline{\mathcal{V}} \\ \overline{\mathcal{A}}(u,v) + \langle \frac{dv}{dt}, u\rangle & \text{if } u \in \overline{\mathcal{V}}, v \in \widehat{\overline{\mathcal{F}}}\ , \end{cases}$$

(cf. [17, (2.8)]).
$\overline{\mathcal{E}}$ can be identified with a generalized Dirichlet form as follows. Let $\mathcal{H} := L^2(\mathbb{R} \times X,\, dt \otimes d\mu)$ and denote by $T : \mathcal{H} \to \overline{\mathcal{H}}$, $h(t,x) \mapsto (t \mapsto h(t,\cdot))$, the unique isomorphism between the two spaces (cf. [19, Theorem II.10]). $(\overline{\mathcal{A}}, \overline{\mathcal{V}})$ can then be identified with a Dirichlet form $(\mathcal{A}, \mathcal{V})$ given by $\mathcal{V} := T^{-1}(\overline{\mathcal{V}})$ and $\mathcal{A}(u,v) := \overline{\mathcal{A}}(T(u), T(v))$. Similarly, $(\overline{U}_t)_{t\geq 0}$ induces a

C_0-semigroup $(U_t)_{t\geq 0}$ of contractions on $\mathcal{H}$ if we define $U_t := T^{-1}\overline{U}_t T$, $t\geq 0$. We denote the corresponding generator by $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t},\mathcal{H}))$. It is easy to see that both $(U_t)_{t\geq 0}$ can be restricted to a C_0-semigroup on $\mathcal{V}$ and at the same time be extended to a C_0-semigroup on $\mathcal{V}'$. In particular, $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t},\mathcal{H}))$ satisfies assumption D1. Since $(U_t)_{t\geq 0}$ is sub-Markovian the bilinear form $\mathcal{E}$ associated with $(\mathcal{A},\mathcal{V})$ and $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t},\mathcal{H}))$ is a generalized Dirichlet form. By the *time dependent Dirichlet form* corresponding to $(\mathcal{E}^{(t)},V)_{t\in\mathbf{R}}$ we mean the generalized Dirichlet form $\mathcal{E}$ just defined.
According to [23, IV.4(a)(iii)] $\mathcal{E}$ is quasi-regular and there exists an associated process $\mathbf{M}$. Similarly the coform $\widehat{\mathcal{E}}$ is also quasi-regular and there exists a coassociated process $\widehat{\mathbf{M}}$. Let us assume that $E=W\subset\mathbb{R}^d$, W open, and that $C_0^k(W)\subset\mathcal{F}$ dense for some $k\in\mathbb{N}\cup\{\infty\}$. Then using Theorem 3.2 we obtain immediately that $\mathbf{M}$ and $\widehat{\mathbf{M}}$ are diffusions up to their life times if all the $\mathcal{E}^{(t)}$, $\widehat{\mathcal{E}}^{(t)}$ are local in the classical sense. Obviously **Alg** and **Diag** are satisfied with $\mathcal{Y}=C_0^k(W)$ hence Proposition 3.4(ii) applies and Fukushima's decomposition extends to $A^{[\Phi(f_1,\dots,f_n)]}$, $\Phi\in C^1(\mathbb{R}^n)$, $\Phi(0)=0$, $f_1,\dots,f_n\in\mathcal{F}_b$. Now, if $\mathbf{M}$ is conservative then by Theorem 2.18 we have a change of variables rule for $M^{[\Phi(f_1,\dots,f_n)]}$. The generalized Dirichlet form in the next example **b)** has all the above properties if the time-dependent potential V is zero.

b) Time dependent potentials

Let $d\geq 1$, and $V:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, $V\in L^1_{\text{loc}}(\mathbb{R}\times\mathbb{R}^d, dt\otimes dx)$, $V\geq 0$, be a time dependent potential. Let $(\mathcal{E}, C_0^\infty(\mathbb{R}^{d+1}))$ be the linear form

$$\begin{aligned}\mathcal{E}(u,v) \;&:=\; \int_{\mathbf{R}}\int_{\mathbf{R}^d}\langle\nabla u,\nabla v\rangle\,dx\,dt+\int_{\mathbf{R}}\int_{\mathbf{R}^d}uvV\,dx\,dt \\ &\;-\;\int_{\mathbf{R}}\int_{\mathbf{R}^d}\frac{\partial u}{\partial t}v\,dx\,dt\ ;\ u,v\in C_0^\infty(\mathbb{R}^{d+1}),\end{aligned}\tag{42}$$

where ∇u means gradient w.r.t. x, i.e. $\nabla u(t,x)=\left(\frac{\partial u}{\partial x_1}(t,x),\dots,\frac{\partial u}{\partial x_d}(t,x)\right)$. There is a generalized Dirichlet form in $\mathcal{H}:=L^2(\mathbb{R}\times\mathbb{R}^d,\ dt\otimes dx)$ extending the bilinear form $\mathcal{E}$. By [23, Lemma II.1.1] the bilinear form

$$\mathcal{A}(u,v):=\int_{\mathbf{R}}\int_{\mathbf{R}^d}\langle\nabla u,\nabla v\rangle\,dx\,dt+\int_{\mathbf{R}}\int_{\mathbf{R}^d}uvV\,dx\,dt\,;\ u,v\in C_0^\infty(\mathbb{R}^{d+1})$$

is closable in $\mathcal{H}$ and the closure $(\mathcal{A},\mathcal{V})$ is a symmetric Dirichlet form. Consider now the following assumption on V:

$$\int_{\mathbf{R}}\int_{\mathbf{R}^d}\frac{\partial u}{\partial t}uV\,dx\,dt \;\leq\; c\,\mathcal{A}_1(u,u)\text{ for all }u\in C_0^\infty(\mathbb{R}^{d+1}),\tag{43}$$

for some constant $c\geq 0$. Let V satisfy (43). The bilinear form associated with $(\mathcal{A},\mathcal{V})$ and $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t},\mathcal{H}))$ extends the bilinear form $\mathcal{E}$ (cf. (42)), and is a generalized Dirichlet form by [23, Proposition I.4.7] since $(U_t)_{t\geq 0}$ is sub-Markovian, hence $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t},\mathcal{H}))$ a Dirichlet

operator, and $(\mathcal{A}, \mathcal{V})$ a Dirichlet form.
By [23, IV.4(b)] we know that $\mathcal{E}$ is quasi-regular and there exists an associated process. Furthermore, since $\mathcal{A}$ is a symmetric Dirichlet form and since the coresolvent associated to the time derivative is sub-Markovian we know (cf. [23, Proposition I.4.7.]) that the coresolvent associated to $\mathcal{E}$ is also sub-Markovian. Hence Fukushima's decomposition holds for $A^{[f]}$, $f \in \mathcal{F}$, where $\mathcal{F} = \{u \in \mathcal{V} \mid \frac{\partial u}{\partial t} \in \mathcal{V}'\}$. Moreover, by [23, IV.4(b)] we know that $C_0^\infty(\mathbb{R}^{d+1}) \subset \mathcal{F}$ dense. Hence **Alg** and **Diag** hold with $\mathcal{Y} = C_0^\infty(\mathbb{R}^{d+1})$, $C = 2c$, $\gamma = 1$, and by Proposition 3.4(ii) Fukushima's decomposition extends to $A^{[\Phi(f_1,\ldots,f_n)]}$, $\Phi \in C^1(\mathbb{R}^n)$, $\Phi(0) = 0$, $f_1, \ldots, f_n \in \mathcal{F}_b$. By Example 3.3 the associated process is obviously a diffusion up to his life time.

c) First order perturbations of time dependent Dirichlet forms on $\mathbb{R}^d$

Let $d \geq 1$ and $b_i \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d, dt \otimes dx)$, $1 \leq i \leq d$, and $b_i(t, \cdot) \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$ for all $t \in \mathbb{R}$, $1 \leq i \leq d$. Consider the bilinear form

$$\begin{aligned} \mathcal{E}(u,v) &= \sum_{i=1}^d \int_{\mathbf{R}} \int_{\mathbf{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx\, dt - \sum_{i=1}^d \int_{\mathbf{R}} \int_{\mathbf{R}^d} b_i \frac{\partial u}{\partial x_i} v\, dx\, dt \\ &- \int_{\mathbf{R}} \int_{\mathbf{R}^d} \frac{\partial u}{\partial t} v\, dx\, dt\ ; u, v \in C_0^\infty(\mathbb{R}^{d+1}). \end{aligned}$$

Let $B := (b_1, \ldots, b_d)$ and assume that

$$\int_{\mathbf{R}^{d+1}} \langle B, \nabla u \rangle\, dxdt \quad \leq \quad 0 \text{ for all } u \in C_0^\infty(\mathbb{R}^{d+1}), u \geq 0.$$

We suppose furthermore that there exist functions $L = L_2 + L_\infty$ with $L_2 \in \mathcal{B}^+(\mathbb{R}) \cap L^2(\mathbb{R}, dt)$ and $L_\infty \in \mathcal{B}^+(\mathbb{R}) \cap L^\infty(\mathbb{R}, dt)$ and a constant $\mathbf{M}$ such that

$$|B(t,x) - B(t,y)| \leq L(t)|x-y|_{\mathbf{R}^d} \text{ for all } x, y \in \mathbb{R}^d, t \in \mathbb{R}$$

$$\text{and } |B(t,x)| \leq M(|(t,x)|_{\mathbf{R}^{d+1}} + 1),\ x \in \mathbb{R}^d,\ t \in \mathbb{R}.$$

Here $|\cdot|_{\mathbf{R}^k}$ is the Euclidean norm on $\mathbb{R}^k$, $k \geq 1$. Let $(\mathcal{A}, \mathcal{V})$ be the symmetric Dirichlet form given by the closure of

$$\mathcal{A}(u,v) := \int_{\mathbf{R}} \int_{\mathbf{R}^d} \langle \nabla u, \nabla v \rangle\, dx\, dt\ ;\ u, v \in C_0^\infty(\mathbb{R}^{d+1})\ ,$$

in $\mathcal{H}$. ∇u means gradient w.r.t. x, i.e., $\nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d})$. Let $\Lambda u := \langle B, \nabla u \rangle + \frac{\partial u}{\partial t}$, $u \in C_0^\infty(\mathbb{R}^{d+1})$. By [23, Proposition II.2.8.] $(\Lambda, C_0^\infty(\mathbb{R}^{d+1}))$ is closable on $\mathcal{H}$ and the closure $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a C_0-semigroup $(U_t)_{t\geq 0}$ of contractions which can be restricted to C_0-semigroup on $\mathcal{V}$. Since $(U_t)_{t\geq 0}$ is sub-Markovian the bilinear form $\mathcal{E}'$ associated with

$(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ is a generalized Dirichlet form. By [23, IV.4(d)] we have that $\mathcal{E}'$ is quasi-regular and that $\mathcal{Y} = C_0^\infty(\mathbb{R}^{d+1}) \subset \mathcal{F}$ dense which in particular implies the existence of an associated process (cf. [23, Proposition IV.2.1.]). From [23, II.(2.3.1)] we can see that $(U_t)_{t\geq 0}$ is also an L^1-contraction. This implies that the adjoint semigroup $(\widehat{U}_t)_{t\geq 0}$ of $(U_t)_{t\geq 0}$ is sub-Markovian and then applying [23, Proposition I.4.7.] we have that the coresolvent assoicated to $\mathcal{E}'$ is also sub-Markovian. Therefore Fukushima's decomposition holds for $A^{[f]}$, $f \in \mathcal{F}$, and extends to $A^{[\Phi(f_1,\dots,f_n)]}$, $\Phi \in C^1(\mathbb{R}^{d+1})$, $\Phi(0) = 0$, $f_1, \dots, f_n \in \mathcal{Y}$ by Proposition 3.4(i) because **Alg** holds. However, unless we do not make some supplementary assumption on the divergence of the vector field B we can even not show the existence of a coprocess.

d) Symmetric Dirichlet forms on $\mathbb{R}^d$ perturbed by divergence zero vector fields

Let μ be a positive measure on $\mathcal{B}(\mathbb{R}^d)$ with $supp(\mu) \equiv \mathbb{R}^d$. Suppose that $d\mu \ll dx$ and that the density admits a representation φ^2, where $\varphi \in H^{1,2}_{loc}(\mathbb{R}^d)$, i.e. $\varphi \cdot \chi \in H^{1,2}_0(\mathbb{R}^d)$ for all $u \in C_0^\infty(\mathbb{R}^d)$ and where $H^{1,2}_0(\mathbb{R}^d)$ is given as the closure of $C_0^\infty(\mathbb{R}^d)$ w.r.t. $\int \langle \nabla u, \nabla v\rangle\, dx$. Consider the closure of

$$\mathcal{E}^0(u,v) = \frac{1}{2}\int \langle \nabla u, \nabla v\rangle\, d\mu\ ;\ u, v \in C_0^\infty(\mathbb{R}^d)$$

on $L^2(\mathbb{R}^d, \mu)$ (cf. [15, II.2b] for the closability) which we denote by $(\mathcal{E}^0, H^{1,2}_0(\mathbb{R}^d, \mu))$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$.
In particular, the measure μ can be infinite. Let $(L^0, D(L^0))$ by the generator associated to $(\mathcal{E}^0, H^{1,2}_0(\mathbb{R}^d, \mu))$. By the construction we have that $C_0^\infty(\mathbb{R}^d) \subset D(L^0)$ and $L^0 u = \frac{1}{2}\Delta u + \langle \frac{\nabla \varphi}{\varphi}, \nabla u\rangle$ for $u \in C_0^\infty(\mathbb{R}^d)$. Let $B \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^d, \mu)$, i.e. $B = (B_1, \dots, B_d) : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and $\int_V \langle B, B\rangle\, d\mu < \infty$ for all V relatively compact in $\mathbb{R}^d$. Suppose that

$$\int \Delta u + \langle B, \nabla u\rangle\, d\mu = 0 \text{ for all } u \in C_0^\infty(\mathbb{R}^d)\ .$$

Note that there is no symmetric bilinear form associated with $\Delta + B \cdot \nabla$ but that $\frac{1}{2}\Delta u + \frac{1}{2}\langle B, \nabla u\rangle = L^0 u + \langle \overline{\beta}, \nabla u\rangle$ where $\overline{\beta} = \frac{1}{2}\left(B - 2\frac{\nabla\varphi}{\varphi}\right)$, $\overline{\beta} \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^d, \mu)$ and

$$\int \langle \overline{\beta}, \nabla u\rangle\, d\mu = 0 \text{ for all } u \in C_0^\infty(\mathbb{R}^d)$$

which induces some kind of symmetry for the first order part of $L^0 + \overline{\beta} \cdot \nabla$ because $-\overline{\beta}$ satisfies the same conditions as $\overline{\beta}$ (i.e. we can also consider the operator $L^0 - \overline{\beta} \cdot \nabla$ cf. below). For a subspace $W \subset L^2(\mathbb{R}^d, \mu)$ let W_0 denote the space of all $u \in W$ such that $supp(|u|\mu)$ is compact and let W_b as usual denote the bounded elements in W. Set $W_{0,b} = W_0 \cap W_b$. Then $Lu := L^0 u + \langle \overline{\beta}, \nabla u\rangle$, $u \in D(L^0)_{0,b}$ is an extension of $\frac{1}{2}\Delta u + \frac{1}{2}\langle B, \nabla u\rangle$, $u \in C_0^\infty(\mathbb{R}^d)$. By [24, Theorem 1.5] the exists a closed extension $(\overline{L}, D(\overline{L}))$ of $(L, D(L^0)_{0,b})$ on $L^1(\mathbb{R}^d, \mu)$ generating a strongly continuous resolvent $(\overline{G}_\alpha)_{\alpha>0}$ which is sub-Markovian. Furthermore we have $D(\overline{L})_b \subset H_0^{1,2}(\mathbb{R}^d,\mu)$ and

$$\mathcal{E}^0(u,v) - \int \langle \overline{\beta}, \nabla u\rangle v\, d\mu = -\int \overline{L}u\ v\, d\mu\ ;\ u \in D(\overline{L})_b,\ v \in H_0^{1,2}(\mathbb{R}^d,\mu)_{0,b}$$

and

$$\mathcal{E}^0(u,u) \le -\int \overline{L}u\ u\, d\mu\ ;\ u \in D(\overline{L})_b\ .$$

Remark 4.1 *(i) Since $-\overline{\beta}$ satisfies the same assumptions as $\overline{\beta}$ similarly to $(\overline{L}, D(\overline{L}))$ we can construct a closed extension $(\overline{L}', D(\overline{L}'))$ of $L^0u - \langle\overline{\beta},\nabla u\rangle$ generating a strongly continious resolvent $(\overline{G}'_\alpha)_{\alpha>0}$ which is sub-Markovian. It follows that*

$$\int \overline{G}_\alpha u\, v\, d\mu = \int u\overline{G}'_\alpha v\, d\mu \textit{ for all } u,v \in L^1(\mathbb{R}^d,\mu)_b\ .$$

Note also that similarly to the case of symmetric Dirichlet operators which admit a carré du champ cf. [3, I.4] $D(\overline{L})_b$ is an algebra.

Let $(\overline{T}_t)_{t\ge0}$ (resp. $(\overline{T}'_t)_{t\ge0}$) be the semigroup corresponding to $(\overline{L}, D(\overline{L}))$ (resp. $(\overline{L}', D(\overline{L}'))$). Since $(\overline{T}_t)_{t\ge0}$ (resp. $(\overline{T}'_t)_{t\ge0})$) is a sub-Markovian semigroup of contractions it determines uniquely a semigroup $(T_t)_{t\ge0}$ (resp. $(T'_t)_{t\ge0}$) of contractions on $L^2(\mathbb{R}^d,\mu)$ by the Riesz-Thorin Interpolation Theorem (cf. [19, Theorem IX.17]). $(T_t)_{t\ge0}$ (resp. $(T'_t)_{t>0}$) is strongly continuous again. Let $(L, D(L))$ (resp. $(L', D(L'))$) be the associated generator and $(G_\alpha)_{\alpha>0}$ (resp. $(G'_\alpha)_{\alpha>0}$) be the associated resolvent. Note that T_t (resp. $G_\alpha, T'_t, G'_\alpha$) coincides with $\overline{T}_t$ (resp. $\overline{G}_\alpha, \overline{T}'_t, \overline{G}'_\alpha$) on $L^1(\mathbb{R}^d,\mu)\cap L^2(\mathbb{R}^d,\mu)$. Let $(\cdot,\cdot)$ be the inner product in $L^2(\mathbb{R}^d,\mu)$. According to Example 1.2(ii) $(L, D(L))$ is associated to a generalized Dirichlet form on $D(L)\times L^2(\mathbb{R}^d,\mu) \cup L^2(\mathbb{R}^d,\mu)\times D(L')$ by

$$\mathcal{E}(u,v) := \begin{cases} (-Lu, v) & \text{for } u \in D(L),\ v \in L^2(\mathbb{R}^d,\mu) \\ (u, -L'v) & \text{for } u \in L^2(\mathbb{R}^d,\mu),\ v \in D(L') \end{cases}$$

Furthermore $(L, D(L))$ is quasi-regular in terms of Definition 1.3 and by [24, Theorem 3.5] there exists a μ-tight special standard process $\mathbf{M} = \left(\Omega, \mathcal{F}_\infty, (X_t)_{t\ge0}, (P_z)_{z\in\ R^d_\Delta}\right)$ with life time ζ which is associated with $(L, D(L))$ in the sense that $E.(\int_0^\infty e^{-\alpha t} f(X_t)\, dt)$ is an $\mathcal{E}^0$-q.c. m-version of $G_\alpha f$ for all $f \in L^2(\mathbb{R}^d,\mu)\cap B_b$. Note that by [24, Lemma 3.4] $\mathcal{E}$-exceptional and $\mathcal{E}^0$-exceptional sets coincide because $\mathcal{E}$-nests an $\mathcal{E}^0$-nests coincide.
Let us give here a quite shorter hence more transparent proof of [24, Proposition 3.6] which states the following:

Proposition 4.2 *It holds that*

$$P_z(t \mapsto X_t \textit{ is continuous on } [0,\zeta)) = 1 \textit{ for } \mathcal{E}\textit{-q.e. } z \in E.$$

Proof Let $u \in C_0^\infty(\mathbb{R}^d)$. Since for any $v \in L^2(\mathbb{R}^d, \mu)$

$$\mathcal{E}(u,v) = (-\frac{1}{2}\Delta u, v) - \int \langle \frac{1}{2}B, \nabla u\rangle \; v \, d\mu$$

the assertion follows by Example 3.3.

$\square$

Let us now summarize what we have achieved so far. Since we have **QR** and $\mathbf{M^{ex}}$ by Remark 4.1 we have also $\widehat{\mathbf{QR}}$ and $\widehat{\mathbf{M}}^{\mathbf{ex}}$. Clearly, **SUB** and $\widehat{\mathbf{SUB}}$ hold. By Lemma 4.2 we have **Dif**. It is clear that the coassociated process is also a diffusion up to $\widehat{\zeta}$.

Let us from now on assume that the semigroup $(T_t)_{t\geq 0}$ is conservative, i.e. $T_t 1 = 1$ for any $t > 0$. Adapting arguments from [24] we can see that this is the case if there exists a positive constant C such that $\langle B(x), x\rangle \leq C(|x|^2_{\mathbb{R}^d} ln(|x|^2_{\mathbb{R}^d} + 1) + 1)$ for all x in $\mathbb{R}^d$. Here $|\cdot|_{\mathbb{R}^d}$ denotes the Euclidean norm on $\mathbb{R}^d$. When $(T_t)_{t\geq 0}$ is conservative it is easy to see that $\int 1 - \alpha R_\alpha 1 \, d\mu = 0$ for any $\mu \in \widehat{S}_{00}$. Hence $P_z(\zeta = \infty) = 1$ for $\mathcal{E}$-q.e. $z \in \mathbb{R}^d$ which further implies that $\mathbf{M}_{\mathbf{cont}}^{[\mathbf{G_1}\mathcal{H}_\mathbf{b}]}$ and **Plt** hold. Note that also **Alg** and **Diag'** are satisfied with $\mathcal{Y} = C_0^\infty(\mathbb{R}^d), \gamma = 0, C = 1$, and $(A, D(A)) = (\mathcal{E}^0, H_0^{1,2}(\mathbb{R}^d, \mu))$.
Let $D_k = \{x \in \mathbb{R}^d \mid |x|_{\mathbb{R}^d} \leq k\}$. Then $(D_k)_{k\geq 1}$ is an $\mathcal{E}$-nest since it is an $\mathcal{E}^0$-nest.
Let $(u_i^k)_{k\geq 1} \subset C_0^\infty(\mathbb{R}^d)$, $1 \leq i \leq d$, be such that $u_i^k(x) = x_i, x = (x_1, \ldots, x_d) \in D_k$, $u_i^k(x) = 0, x \notin D_{k+1}$. Then $(u_i^k)_{k\geq 1}$ is a localizing sequence for the coordinate projections $u_i(x) = x_i, x \in \mathbb{R}^d, 1 \leq i \leq d$. Hence by Theorem 3.6(iii) with $\widetilde{\mathcal{Y}} \supset C_0^\infty(\mathbb{R}^d)$ we have

$$u_i(X_t) - u_i(X_0) = M_t^{[u_i]} + N_t^{[u_i]} \; .$$

Let us show that $M^{[u_i]}$ is a 1-dimensional Brownian motion. To do so let us first calculate the energy measure related to $\langle M^{[u]}\rangle$, $u \in C_0^\infty(\mathbb{R}^d)$. Let $g \in L^1(\mathbb{R}^d, \mu)_b$ and $(R'_\alpha)_{\alpha>0}$ be the resolvent of the coassociated process. Then

$$\begin{aligned}
\int R'_\gamma g \, d\mu_{\langle M^{[u]}\rangle} &= \lim_{\alpha\to\infty} \alpha(u - \alpha G_\alpha u, uG'_\gamma g) - \alpha(u^2, G'_\gamma g - \alpha G'_\alpha G'_\alpha g) \\
&= 2(-Lu, uG'_\gamma g) - (-Lu^2, G'_\gamma g) \\
&= 2(-L^0 u, uG'_\gamma g) - (-L^0 u^2, G'_\gamma g) \\
&+ 2\int \langle \overline{\beta}, \nabla u\rangle u G'_\gamma g \, d\mu - \int \langle \overline{\beta}, \nabla u^2\rangle G'_\gamma g \, d\mu \\
&= 2\int \langle \nabla u, \nabla(uG'_\gamma g)\rangle \, d\mu - \int \langle \nabla u^2, G'_\gamma g\rangle \, d\mu \\
&= \int \langle \nabla u, \nabla u\rangle G'_\gamma g \, d\mu \; .
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int R'_\gamma g \, d\mu_{\int_0^\cdot \langle \nabla u(X_s), \nabla u(X_s)\rangle \, ds} &= \lim_{\alpha\to\infty} (G'_\gamma g, \alpha G_{\alpha+\gamma} \langle \nabla u, \nabla u\rangle) \\
&= \int \langle \nabla u, \nabla u\rangle G'_\gamma g \, d\mu \; .
\end{aligned}$$

Hence by Theorem 2.5 we have

$$\mu_{\langle M^{[u]}\rangle} = \langle \nabla u, \nabla u\rangle \cdot d\mu \ .$$

Now, because of Lemma 2.4 and Lemma 1.18

$$\begin{aligned}
\int R'_\gamma g \, d\mu_{\langle M^{[u_i]}\rangle} &= \lim_{k\to\infty} \int R'_\gamma g \ 1_{D_k^{\hat{0}}} \, d\mu_{\langle M^{[u_i^{k+1}]}\rangle} \\
&= \lim_{k\to\infty} \int R'_\gamma g \ \langle \nabla u_i^{k+1}, \nabla u_i^{k+1}\rangle \ 1_{D_k^{\hat{0}}} \, d\mu \\
&= \lim_{k\to\infty} \int R'_\gamma g \ \langle \nabla u_i, \nabla u_i\rangle \ 1_{D_k^{\hat{0}}} \, d\mu \\
&= \int R'_\gamma g \ \langle \nabla u_i, \nabla u_i\rangle \, d\mu \ .
\end{aligned}$$

But also $\int R'_\gamma g \, d\mu_{\int_0^\cdot \langle \nabla u_i(X_s), \nabla u_i(X_s)\rangle \, ds} = \int R'_\gamma g \langle \nabla u_i, \nabla u_i\rangle \, d\mu$ and then we have $\langle M^{[u_i]}\rangle_t = \int_0^t \langle \nabla u_i(X_s), \nabla u_i(X_s)\rangle \, ds = t$. We obtain that $M^{[u_i]}$ is a Brownian motion by Levy's characterization. Similarly, we get $\langle M^{[u_i]}, M^{[u_j]}\rangle_t = \delta_{ij} \cdot t$ and therefore $M_t = (M_t^{[u_1]}, \ldots, M_t^{[u_n]})$ is the standard Brownian motion on $\mathbb{R}^d$. Now, let us take a look at $N^{[u_i]}$. Since

$$Lu_i^k = \frac{1}{2}\Delta u_i^k + \frac{1}{2}\langle B, \nabla u_i^k\rangle$$

we know that

$$N_{t\wedge\sigma_k}^{[u_i]} = \int_0^{t\wedge\sigma_k} \langle \frac{1}{2} B(X_s), \nabla u_i^{k+1}(X_s)\rangle \, ds + \int_0^{t\wedge\sigma_k} \frac{1}{2}\Delta u_i^{k+1}(X_s) \, ds = \int_0^{t\wedge\sigma_k} \frac{1}{2} B_i(X_s) \, ds$$

where $\sigma_k := \sigma_{D_k^c}$. Thus

$$N_t^{[u_i]} = \int_0^t \frac{1}{2} B_i(X_s) \, ds$$

because $B_i \in L^2_{\text{loc}}(\mathbb{R}^d, \mu)$ implies that $|B_i| \, \varphi^2 dx$ is a positive Radon measure and then by Remark 2.2(i) we can see the finiteness of the PCAF $\int_0^\cdot |B_i|(X_s) \, ds$. Hence summarizing, we have achieved the following

$$X_t - z = W_t + \int_0^t \frac{1}{2} B(X_s) \, ds, \ \ P_z\text{-a.s for } \mathcal{E}^0\text{-q.e z} \in \mathbb{R}^d, t \geq 0 \tag{44}$$

where $(W_t, \mathcal{F}_t, P_z)_{t\geq 0}$ is the standard Brownian motion on $\mathbb{R}^d$ starting at zero.
Instead of looking at the coassociated process (this will be done in the infinite dimensional case below) we want to see what the generalized Itô formula looks like for our process $\mathbf{M}$. Let $\Phi \in C^1(\mathbb{R}^d)$, $(u_i^k)_{k\in\mathbb{N}}$, $1 \leq i \leq k$, as above. By Theorem 3.6(iii) we have

$$\Phi(X_t) - \Phi(X_0) = M_t^{[\Phi]} + N_t^{[\Phi]}.$$

Then by Lemma 2.15(i), the chain rule, and since $M^{[u_i]}_{t\wedge\sigma_k} = M^{[u_i^k]}_{t\wedge\sigma_k} = M^{[u_i^{k+1}]}_{t\wedge\sigma_k}$ we have

$$\begin{aligned}
M^{[\Phi]}_{t\wedge\sigma_k} &= M^{[\Phi(u_1,\dots,u_d)]}_{t\wedge\sigma_k} \\
&= M^{[\Phi(u_1^{k+1},\dots,u_d^{k+1})]}_{t\wedge\sigma_k} \\
&= \sum_{i=1}^d \left(\Phi_{x_i}\left(u_1^{k+1},\dots,u_d^{k+1}\right) \bullet M^{[u_i^{k+1}]}\right)_{t\wedge\sigma_k} \\
&= \sum_{i=1}^d \lim_{n\to\infty}\sum_{j=0}^{n-1} \Phi_{x_i}\left(u_1^{k+1},\dots,u_d^{k+1}\right)(X_{t_j\wedge\sigma_k})\left(M^{u_i^{[k+1]}}_{t_{j+1}\wedge\sigma_k} - M^{[u_i^{k+1}]}_{t_j\wedge\sigma_k}\right) \\
&= \sum_{i=1}^d \lim_{n\to\infty}\sum_{j=0}^{n-1} \Phi_{x_i}(X_{t_j\wedge\sigma_k})\left(M^{[u_i]}_{t_{j+1}\wedge\sigma_k} - M^{[u_i]}_{t_j\wedge\sigma_k}\right) \\
&= \sum_{i=1}^d \int_0^{t\wedge\sigma_k} \Phi_{x_i}(X_s)\, dM^{[u_i]}_s
\end{aligned}$$

where the last term is a sum of usual stochastic integrals. Similarly, since $N^{[u_i^{k+1}]}_{t\wedge\sigma_k} = N^{[u_i^k]}_{t\wedge\sigma_k}$ we have

$$\begin{aligned}
N^{[\Phi]}_{t\wedge\sigma_k} &= N^{[\Phi(u_1^{k+1},\dots,u_d^{k+1})]}_{t\wedge\sigma_k} = \lim_n N^{[P_n^k(u_1^{k+1},\dots,u_d^{k+1})]}_{t\wedge\sigma_k} \\
&= \lim_{n\to\infty}\int_0^{t\wedge\sigma_k} Lp_n^k(X_s)\, ds \\
&= \int_0^{t\wedge\sigma_k} \langle \frac{1}{2}B(X_s), \nabla\Phi(X_s)\rangle\, ds + \lim_{n\to\infty}\int_0^{t\wedge\sigma_k} \frac{1}{2}\Delta p_n^k(X_s)\, ds
\end{aligned}$$

where $(p_n^k)_{n\in\mathbb{N}}$ are polynomials, such that $p_n^k \mapsto \Phi$, $\frac{\partial p_n^k}{\partial x_i} \mapsto \Phi_{x_i}$, uniformly on D_{k+1} as $n\to\infty$. Then the generalized Itô formula for our process is

$$\Phi(X_t) - \Phi(X_0) = \sum_{i=1}^d \int_0^t \Phi_{x_i}(X_s)\, dM^{[u_i]}_s + \int_0^t \langle\frac{1}{2}B, \nabla\Phi\rangle(X_s)\, ds + \lim_{n\to\infty}\int_0^t \frac{1}{2}\Delta p_n^k(X_s)\, ds,$$

for all $t \le \sigma_k$, $k\in\mathbb{N}$.

4.2 Weak solutions of SDE's in infinite dimensions

We will treat here the infinite dimensional analogon of the previous example.
Let E be a separable real Banach space and $(H, \langle\cdot,\cdot\rangle_H)$ a separable real Hilbert space such that $H \subset E$ densely and continuously. Identifying H with its topological dual H' we obtain that $E' \subset H \subset E$ densely and continuously. Define the linear space of finitely based smooth functions on E by

$$\mathcal{F}\mathcal{C}_b^\infty := \{f(l_1,\dots,l_m) \mid m\in\mathbb{N}, f\in\mathcal{C}_b^\infty(\mathbb{R}^m), l_1,\dots,l_m\in E'\}.$$

Here $\mathcal{C}_b^\infty(\mathbb{R}^m)$ denotes the set of all infinitely differentiable (real-valued) functions on $\mathbb{R}^m$ with all partial derivatives bounded. For $u \in \mathcal{F}\mathcal{C}_b^\infty$, $k \in E$ let

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + sk) \mid_{s=0},\ z \in E.$$

It follows that if $u = f(l_1, ..., l_m)$ and $k \in H$ we have that

$$\frac{\partial u}{\partial k}(z) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(l_1(z), ..., l_m(z))\langle l_i, k\rangle_H,\ z \in E.$$

Consequently, $k \mapsto \frac{\partial u}{\partial k}(z)$ is continuous on H and we can define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), k\rangle_H = \frac{\partial u}{\partial k}(z).$$

Let μ be finite positive measure on $(E, \mathcal{B}(E))$. Assume for simplicity $supp(\mu) \equiv E$. An element k in E is called *well-μ-admissible* if there exist $\beta_k^\mu \in L^2(E;\mu)$ such that for all u, v in $\mathcal{F}\mathcal{C}_b^\infty$

$$\int \frac{\partial u}{\partial k}\, d\mu = -\int u\beta_k^\mu\, d\mu.$$

Let us assume

(A.1) There exists a dense linear subspace K of E' consisting of well-μ-admissible elements.

Then it is well known that the densely defined positive definite symmetric bilinear form

$$\mathcal{E}^0(u, v) := \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_H\, d\mu \quad u, v \in \mathcal{F}\mathcal{C}_b^\infty$$

is closable on $L^2(E;\mu)$ and that the closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a symmetric quasi-regular Dirichlet form. Let $(L^0, D(L^0))$ be the associated generator. Let $\overline{\beta} \in L^2(E, H;\mu)$ (i.e. $\overline{\beta} : E \to E$ is $\mathcal{B}(E)/\mathcal{B}(E)$-measurable, $\overline{\beta}(E) \subset H$ and $\|\overline{\beta}\|_H \in L^2(E;\mu)$) be such that

$$\int \langle \overline{\beta}, \nabla u\rangle_H\, d\mu = 0 \ \text{ for all } u \in \mathcal{F}\mathcal{C}_b^\infty. \tag{45}$$

Since $\mathcal{F}\mathcal{C}_b^\infty$ is dense in $D(\mathcal{E}^0)$ (45) implies that $\int \langle \overline{\beta}, \nabla u\rangle_H\, d\mu = 0$ for all $u \in D(\mathcal{E}^0)$ and thus $\int \langle \overline{\beta}, \nabla u\rangle_H\, v\, d\mu = -\int \langle \overline{\beta}, \nabla v\rangle_H\, u\, d\mu$ for all u, $v \in D(\mathcal{E}^0)_b$. Let

$$Lu := L^0 u + \langle \overline{\beta}, \nabla u\rangle_H\ ,\quad u \in D(L^0)_b.$$

It then follows from [24, *Proposition* 4.1.] that $(L, D(L^0)_b)$ is closable on $L^1(E;\mu)$ and that the closure $(\overline{L}, D(\overline{L}))$ generates a Markovian C_0-semigroup of contractions. Furthermore $D(\overline{L})_b \subset D(\mathcal{E}^0)$ and for $u \in D(\overline{L})_b$, $v \in D(\mathcal{E}^0)_b$

$$\mathcal{E}^0(u, v) - \int \langle \overline{\beta}, \nabla u\rangle_H\, v\, d\mu = -\int \overline{L}u\, v\, d\mu. \tag{46}$$

Let $(L,D(L))$ with associated resolvent $(G_\alpha)_{\alpha>0}$ be the part of $(\overline{L},D(\overline{L}))$ on $L^2(E;\mu)$, $(L',D(L'))$ with associated resolvent $(G'_\alpha)_{\alpha>0}$ be the adjoint of $(L,D(L))$ in $L^2(E;\mu)$. According to Example 1.2(ii) $(L,D(L))$ is associated with the *generalized Dirichlet form*

$$\mathcal{E}(u,v) := \begin{cases} (-Lu,v) & \text{for } u\in D(L),\ v\in L^2(E;\mu) \\ (u,-L'v) & \text{for } u\in L^2(E;\mu),\ v\in D(L') \end{cases}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(E;\mu)$. There exists (cf. [24, *Th.*4.6., *Prop.*4.7.]) a μ-tight special standard process $\mathbf{M}=(\Omega,\mathcal{F}_\infty,(X_t)_{t\geq 0},(P_z)_{z\in E_\Delta})$ with life time ζ that is associated with $(L,D(L))$ in the sense that $R_\alpha f(\cdot) := E_\cdot[\int_0^\infty e^{-\alpha t} f(X_t)dt]$ is an $\mathcal{E}^0$-q.c. m-version of $G_\alpha f$ for all $f\in\mathcal{B}_b\cap L^2(E;\mu)$, $\alpha>0$. Furthermore $P_z[\zeta=+\infty]=1$, $P_z[t\mapsto X_t$ is continuous on $[0,\infty)]=1$ for $\mathcal{E}^0$-q.e $z\in E$.
Note that by [24, *Lemma* 4.5] L-nests and $\mathcal{E}^0$-nests coincide. Therefore $\mathcal{E}$-exceptional and $\mathcal{E}^0$-exceptional sets coincide.

Remark 4.3 *Let $L_{sym}(H)$ denote the linear space of all symmetric and bounded operators on H. Let $A: E\to L_{sym}(H)$ be measurable such that for some positive constant C*

$$C\, Id_H \leq A(z) \leq C^{-1} Id_H \ \ for\ all\ z\in E.$$

The inequalities here are to be understood in the quadratic form sense. The case where

$$\mathcal{E}^0(u,v) := \frac{1}{2}\int \langle A\nabla u,\nabla v\rangle_H\, d\mu \quad u,v\in\mathcal{F}\mathcal{C}_b^\infty$$

and $A\neq Id_H$ will not be treated here. We only remark that in this case the martingale part of the decomposition below is not necessarily a Brownian motion and that a representation for the martingale part can be carried out as in [21].

Since $-\overline{\beta}$ satisfies the same assumptions as $\overline{\beta}$ the closure $(\overline{L}',D(\overline{L}'))$ of $L'u := L^0u - \langle\overline{\beta},\nabla u\rangle_H$, $u\in D(L^0)_b$ on $L^1(E;\mu)$ generates a Markovian C_0-semigroup of contractions too, $D(\overline{L}')_b\subset D(\mathcal{E}^0)$ and for $u\in D(\overline{L}')_b$, $v\in D(\mathcal{E}^0)_b$ $\mathcal{E}^0(u,v)+\int\langle\overline{\beta},\nabla u\rangle_H v\,d\mu = -\int \overline{L}'u\,v\,d\mu$. It is easy to see that the part of $(\overline{L}',D(\overline{L}'))$ on $L^2(E;\mu)$ is $(L',D(L'))$. Let $(R'_\alpha)_{\alpha>0}$ denote the resolvent of the associated coprocess. Since $(G'_\alpha)_{\alpha>0}$ is sub-Markovian and strongly continuous on $\mathcal{V}=L^2(E;\mu)$, Theorem 2.11 applies for $u\in D(L)$ with $N_t^{[u]}=\int_0^t Lu(X_s)ds$. Let $v\in D(L)_b$, $g\in L^2(E;\mu)\cap\mathcal{B}_b^+$, $\gamma>0$, then by (23)

$$\begin{aligned}\int R'_\gamma g\, d\mu_{<v>} &= 2\mathcal{E}^0(v,vR'_\gamma g)-\mathcal{E}^0(v^2,R'_\gamma g)\\ &= \int R'_\gamma g\,\langle\nabla v,\nabla v\rangle_H\, d\mu.\end{aligned}$$

Now let $u_n\in D(L)_b$ such that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $D(L)$. Since by (46) $u_n \underset{n\to\infty}{\longrightarrow} u$ in $D(\mathcal{E}^0)$ we have $\int R'_\gamma g\,d\mu_{<u>}=\int R'_\gamma g\,\langle\nabla u,\nabla u\rangle_H\,d\mu$. In particular $\int R'_\gamma g\,d\mu_{<u>}=\int R'_\gamma g\,d\mu_A$ where $A=\int_0^\cdot\langle\nabla u(X_s),\nabla u(X_s)\rangle_H\,ds$. Therefore by Remark 2.14 and Theorem 2.5 it follows that $\langle M^{[u]}\rangle_t = \int_0^t \langle \nabla u(X_s), \nabla u(X_s)\rangle_H \, ds$. Note that $\langle M^{[u]}\rangle$ is finite since $\langle \nabla u, \nabla u\rangle_H \in L^1(E;\mu)$. Assume

(A.2) $\quad u_k(\cdot) :=_{E'} \langle k, \cdot\rangle_E \in L^2(E;\mu)$ for all $k \in K$.

Here $_{E'}\langle \cdot,\cdot\rangle_E$ denotes the dualization between E and E'. Then clearly $u_k \in D(L)$,

$$Lu_k \;=\; \frac{1}{2}\beta_k^{\mu} + \langle \overline{\beta}, k\rangle_H, \quad k \in K, \tag{47}$$

and

$$\langle M^{[u_k]}, M^{[u_{k'}]}\rangle_t \;=\; t\langle k, k'\rangle_H, \quad k, k' \in K. \tag{48}$$

Choosing an ONB $K_0 \subset K$ of H which separates the points of E by Theorem 2.11 applied to u_k, $k \in K_0$ we get a countable system of 1-dimensional SDE's with independent 1-dimensional Brownian motions according to (48) and drifts given according to (47). If we assume

(A.3) For one (and hence all) $t > 0$ there exists a probability measure μ_t on $(E, \mathcal{B}(E))$, such that

$$\int e^{i_{E'}\langle k,z\rangle_E}\mu_t(dz) = e^{-\frac{1}{2}t\|k\|_H^2} \text{ for all } k \in E'$$

similar to [1, Theorem 6.6] it is then possible to lift the countable system of 1-dimensional equations to a single equation on E, namely we have

Theorem 4.4 *There exist maps $W, N^0 : \Omega \longrightarrow C([0,\infty), E)$ with the following properties:*
(i) $\omega \mapsto W_t(\omega) := W(\omega)(t)$ and $\omega \mapsto N_t^0(\omega) := N^0(\omega)(t)$ are both $\mathcal{F}_t/\mathcal{B}(E)$ measurable for $t \geq 0$.
(ii) There exists an $\mathcal{E}^0$-exceptional set $S \subset E$ such that under each P_z, $z \in E\backslash S$, $W = (W_t)_{t\geq 0}$ is an E-valued $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion starting at $0 \in E$ with covariance $\langle \cdot,\cdot\rangle_H$ (i.e. under each P_z, $z \in E\backslash S$, for all $0 \leq s < t$ $\;W_t - W_s$ is independent of $\mathcal{F}_s$ and $\langle k, W_t - W_s\rangle_E$ is mean zero Gaussian with variance $(t-s)\|k\|_H^2$).
(iii) For each $k \in K$, $t \geq 0$ and $\mathcal{E}^0$-q.e. $z \in E$ we have P_z-a.s.

$$_{E'}\langle k, W_t\rangle_E = M_t^{[u_k]} \text{ and } _{E'}\langle k, N_t^0\rangle_E = \frac{1}{2}\int_0^t \beta_k^{\mu}(X_s)ds.$$

(iv) For $\mathcal{E}^0$-q.e. $z \in E$ we have P_z-a.s.

$$X_t = z + W_t + N_t^0 + \int_0^t \overline{\beta}(X_s)ds \tag{49}$$

where the last integral is in the sense of Bochner (cf. the following Remark 4.5) and where for $k \in K$ $\langle k, \int_0^t \overline{\beta}(X_s)ds\rangle_H = \int_0^t \langle \overline{\beta}(X_s), k\rangle_H \, ds$.

Remark 4.5 *(i) The assumption that the Gaussian measures satisfying (A.3) exist is of course, necessary. It just means that there exists (cf. [10, p.74]) a Brownian semigroup on E with covariance $\langle\cdot,\cdot\rangle_H$, i.e., there is a Brownian motion on E over H. Hence (A.3) is the best condition one can hope for.*
(ii) In the above general situation there is no guarantee that $k \mapsto \beta_k^\mu(z)$, $k \in K$, is represented by an element in E for μ-a.e. $z \in E$. But if we assume

(A.4) *There exists a $\mathcal{B}(E)/\mathcal{B}(E)$-measurable map $\beta_H^\mu : E \to E$ such that*
(a) ${}_{E'}\langle k, \beta_H^\mu\rangle_E = \beta_k^\mu$ *μ-a.s. for each $k \in K$,*
(b) $\|\beta_H^\mu\|_E \in L^1(E;\mu)$

then we may define the process N^0 in Theorem 4.4 as a Bochner integral. In fact, it is easy to see that $\|\beta_H^\mu\|_E \in L^1(E;\mu)\cap\mathcal{B}$ implies the finiteness of the AF $\int_0^\cdot \|\beta_H^\mu\|_E(X_s)\,ds$. *Hence, by [28, Theorem 1, p.133, Corollary 2, p.134], $N_t^0 := \frac{1}{2}\int_0^t \beta_H^\mu(X_s)\,ds$, $t \geq 0$ (where the integral is in the sense of Bochner P_z-a.s for $\mathcal{E}^0$-q.e. $z \in E$) has the desired properties.*
(iii) It is easy to see that (A.4) is equivalent to the following assumption:

(A.4') *There exists a $\mathcal{B}(E)/\mathcal{B}(E)$-measurable map $B : E \to E$ such that*
(a) $\frac{1}{2}{}_{E'}\langle k, B\rangle_E = Lu_k$ *μ-a.s. for each $k \in K$,*
(b) $\|B\|_E \in L^1(E;\mu)$.

Analogous to (*ii*) *we may then replace $N_t^0 + \int_0^t \overline{\beta}(X_s)ds$ in* (49) *by the Bochner integral $\frac{1}{2}\int_0^t B(X_s)ds$.*

Applications

In this subsection we assume that E is a separable real Hilbert space with inner product $\|\cdot\|_E := \langle\cdot,\cdot\rangle_E^{\frac{1}{2}}$ and that $H \subset E$ densely by a Hilbert-Schmidt map. Then there exists a nonnegative definite injective self-adjoint Hilbert-Schmidt operator T on E such that $H = T(E)$ and $\|\cdot\|_H = \|T^{-1}\cdot\|_E$. Analogous to [10, *Theorem* 4.4 *Step* 3.] we see that $\|\cdot\|_E$ is measurable over H, hence *(A.3)* holds. Let $B : E \to E$ be a Borel measurable vector field satisfying the following conditions:

(B.1) $\lim_{\|z\|_E\to\infty}\langle B(z), z\rangle_E = -\infty$,
(B.2) ${}_{E'}\langle l, B\rangle_E : E \to \mathbb{R}$ is weakly continuous for all $l \in E'$.
(B.3) There exist C_1, C_2, $d \in (0,\infty)$, such that $\|B(z)\|_E \leq C_1 + C_2\|z\|_E^d$ for all $z \in E$.

Then by [2, *Theorem* 5.2.] there exists a probability measure μ on $(E, \mathcal{B}(E))$ such that ${}_{E'}\langle l, B\rangle_E \in L^2(E;\mu)$ for all $l \in E'$ and such that

$$\int \frac{1}{2}\Delta_H u + \frac{1}{2}{}_{E'}\langle\nabla u, B\rangle_E\, d\mu \quad = \quad 0 \ \text{ for all } \ u \in \mathcal{F}C_b^\infty \tag{50}$$

where Δ_H is the Gross-Laplacian, i.e., $\Delta_H u = \sum_{i,j=1}^m \frac{\partial f}{\partial x_i \partial x_j}(l_1(z),...,l_m(z))\langle l_i, l_j\rangle_H$ if $u = f(l_1,...,l_m) \in \mathcal{F}C_b^\infty$. Assume that $B(z) = -z + v(z)$, $v : E \to H$. Because of $(B.1)$,

$(B.3)$ it follows by $[2, Lemma\,5.1.]$ that $v \in L^2(E,H;\mu)$. In particular (using Young's and the logarithmic Sobolev inequality) we have $\|z\|_E \in L^p(E,\mu)$ for all $p \geq 1$. Let γ be a Gaussian measure on E with covariance $\langle\cdot,\cdot\rangle_H$. By $[2, Theorem\,3.5.]$ $d\mu = \varphi^2\,d\gamma$ where φ is in the Sobolev space $H^{1,2}(E;\gamma)$. Furthermore the logarithmic derivative β_H^μ of μ associated with H exists and admits the representation $\beta_H^\mu(z) = -z + \frac{2\nabla\varphi}{\varphi}(z)$. Note that possibly $supp(\mu) \not\equiv E$. Nevertheless, since every $k \in E'$ is well-μ-admissible (thus in particular *(A.1)* holds)

$$\mathcal{E}^0(u,v) := \frac{1}{2}\int \langle\nabla u, \nabla v\rangle_H\, d\mu, \quad u,v \in \mathcal{FC}_b^\infty,$$

is well-defined and closable on $L^2(E;\mu)$ and the closure is a symmetric quasi-regular Dirichlet form. Let $(L^0, D(L^0))$ be the associated generator. It is easy to see that $\mathcal{FC}_b^\infty \subset D(L^0)$ and

$$L^0 u \;=\; \frac{1}{2}\Delta_H u + \frac{1}{2}{}_{E'}\langle\nabla u, \beta_H^\mu\rangle_E, u \in \mathcal{FC}_b^\infty.$$

Set $\overline{\beta} := \frac{1}{2}(B - \beta_H^\mu)$. Clearly $\overline{\beta} \in L^2(E,H;\mu)$ and by (50) since $\int L^0 u\, d\mu = 0$, $u \in \mathcal{FC}_b^\infty$

$$\int \langle\overline{\beta}, \nabla u\rangle_H\, d\mu \;=\; 0 \text{ for all } \; u \in \mathcal{FC}_b^\infty. \tag{51}$$

As in section 4.2 we then construct a conservative diffusion $\mathbf{M} = (\Omega, \mathcal{F}_\infty, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ associated to the part on $L^2(E;\mu)$ (which we denote by $(L, D(L))$) of the closure on $L^1(E,\mu)$ of $L^0 u + \langle\overline{\beta}, \nabla u\rangle_H$, $u \in D(L^0)_b$. Note that $Lu = \frac{1}{2}\Delta_H u + \frac{1}{2}{}_{E'}\langle\nabla u, B\rangle_E$, $u \in \mathcal{FC}_b^\infty$. Surely *(A.2)* is satisfied and clearly $Lu_k = \frac{1}{2}{}_{E'}\langle k, B\rangle_E$ hence *(A.4')* holds. By Theorem 4.4 and Remark 4.5(ii) we then have P_z-a.s. for $\mathcal{E}^0$-q.e. $z \in E$ (thus in particular P_μ-a.s.)

$$X_t \;=\; z + W_t - \frac{1}{2}\int_0^t X_s ds + \frac{1}{2}\int_0^t v(X_s) ds \tag{52}$$

where $(W_t)_{t\geq 0}$ is an E-valued $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion starting at $0 \in E$ with covariance $\langle\cdot,\cdot\rangle_H$ and where $\frac{1}{2}\int_0^t X_s ds$, $\frac{1}{2}\int_0^t v(X_s) ds$ are in the sense of Bochner P_z-a.s. for $\mathcal{E}^0$-q.e. $z \in E$. Note that $\int \langle v - 2\frac{\nabla\varphi}{\varphi}, \nabla u\rangle_H\, d\mu = 0$ for all u in $\mathcal{FC}_b^\infty$.
Let $(L', D(L'))$ denote the adjoint operator of $(L, D(L))$ on $L^2(E;\mu)$. Clearly, since $L' u_k = {}_{E'}\langle k, \frac{1}{2} id_E + \frac{2\nabla\varphi}{\varphi} - \frac{1}{2} v\rangle_E$ and $\|id_E\|_E$, $\|\frac{2\nabla\varphi}{\varphi}\|_E$, $\|v\|_E \in L^2(E;\mu)$ the coprocess $\widehat{\mathbf{M}} = (\widehat{\Omega}, \widehat{\mathcal{F}}_\infty, (\widehat{X}_t)_{t\geq 0}, (\widehat{P}_z)_{z\in E_\Delta})$ associated to $(L', D(L'))$ weakly solves

$$\widehat{X}_t \;=\; z + \widehat{W}_t - \frac{1}{2}\int_0^t \widehat{X}_s ds + \int_0^t 2\frac{\nabla\varphi}{\varphi}(\widehat{X}_s) ds - \frac{1}{2}\int_0^t v(\widehat{X}_s) ds$$

for $\mathcal{E}^0$-q.e. $z \in E$ where $(\widehat{W}_t)_{t\geq 0}$ is an E-valued $(\widehat{F}_t)_{t\geq 0}$-Brownian motion starting at $0 \in E$ with covariance $\langle\cdot,\cdot\rangle_H$.

An Itô-type formula

Let $(\overline{G}_\alpha)_{\alpha>0}$ be the resolvent associated to $(\overline{L}, D(\overline{L}))$. Since $\overline{G}_{\alpha|L^2} = G_\alpha$, $\alpha > 0$, it follows that $D(L)_b \subset D(\overline{L})_b$ is dense w.r.t. the L^1-graph norm. Note that since $1 \in D(L)$, the 1-reduced function e_f exists for all $f \in L^\infty(E;\mu)$. Let $u_n := n\overline{G}_n u$, $u \in D(\overline{L})_b$. By [24, *Lemma* 4.4.(*iii*)] we have that $e_{u-u_n} + e_{u_n-u} \underset{n\to\infty}{\longrightarrow} 0$ in $D(\mathcal{E}^0)$, hence in $L^2(E;\mu)$. Furthermore $\overline{e}(A^{[u-u_n]}) = (-\overline{L}(u-u_n), u-u_n) \underset{n\to\infty}{\longrightarrow} 0$. Now (cf. Remark 2.9) by Theorem 2.11 (ii) the decomposition (30) extends to $A^{[u]}$, $u \in D(\overline{L})_b$. Similarly to the finite dimensional case $D(\overline{L})_b$ is an algebra. Hence **Alg** is satisfied with $\mathcal{Y} = D(\overline{L})_b$ and we can apply Proposition 3.4(i) to extend decomposition (30) to $A^{[\Phi(f)]}$, $\Phi \in C^1(\mathbb{R}^n)$, $\Phi(0) = 0$, $f = (f_1, ..., f_n)$, $f_1, ..., f_n \in D(\overline{L})_b$.
Now let $\Phi \in C^1(\mathbb{R}^n)$, $\Phi(0) \neq 0$. Set $\Psi(x) := \Phi(x) - \Phi(0)$. Since $1 \in D(L)$ we have $\Phi(f) \in \mathcal{H}^{dec}$. Obviously $A^{[\Phi(f)]} = A^{[\Psi(f)]}$, hence $M^{[\Phi(f)]} = M^{[\Psi(f)]}$. Then using Theorem 2.18, Theorem 2.11(i) for $A^{[\Psi(f)]}$ and the uniqueness of the decomposition (30) we obtain $M^{[\Phi(f)]} = \sum_{i=1}^n \frac{\partial\Psi}{\partial x_i}(\widetilde{f}_1, \ldots, \widetilde{f}_n) \bullet M^{[f_i]} = \sum_{i=1}^n \frac{\partial\Phi}{\partial x_i}(\widetilde{f}_1, \ldots, \widetilde{f}_n) \bullet M^{[f_i]}$ and hence the following Itô-type formula

$$\Phi(\widetilde{f}_1, \ldots, \widetilde{f}_n)(Y_t) - \Phi(\widetilde{f}_1, \ldots, \widetilde{f}_n)(Y_0) = \sum_{i=1}^n \frac{\partial\Phi}{\partial x_i}(\widetilde{f}_1, \ldots, \widetilde{f}_n) \bullet M_t^{[f_i]} + N_t^{[\Phi(f_1,\ldots,f_n)]}.$$

In particular $N_t^{[\Phi(f_1,\ldots,f_n)]} = N_t^{[\Psi(f_1,\ldots,f_n)]} = \lim_{n\to\infty} N_t^{[p_n(f_1,\ldots,f_n)]}$ where $(p_n)_{n\in\mathbb{N}}$ are polynomials as specified in the proof of Proposition 3.4(i) and the martingale part is a version of the usual stochastic integral (cf. Lemma 2.15(i)).

5 Appendix

We shall give here below the exact definition of an m-tight special standard process.

Let E be a Hausdorff topological space and assume that its Borel σ-algebra is generated by the set $C(E)$ of all continuous functions on $(E, \mathcal{B}(E))$ such that $\mathcal{H} := L^2(E;m)$ is a separable real Hilbert space. Adjoin Δ as an isolated point to E and let $E_\Delta := E \cup \{\Delta\}$. Denote by $\mathcal{P}(E_\Delta)$ the set of all probability measures on $(E_\Delta, \mathcal{B}(E_\Delta))$. As usual we extend every function f defined on a subset $A \subset E$ to $A \cup \{\Delta\}$ by setting $f(\Delta) := 0$. Denote by $\mathcal{B}(E_\Delta)^*$ the σ-algebra of universally measurable sets.

Definition 5.1 $\mathbf{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ *is called a (time-homogeneous) Markov process with state space E, life time* ζ*, and corresponding filtration* $(\mathcal{M}_t)_{t\geq 0}$ *if*

(M.1) $X_t : \Omega \to E_\Delta$ *is* $\mathcal{M}_t/\mathcal{B}(E_\Delta)$*-measurable for all* $t \geq 0$*, and* $X_t(\omega) = \Delta \Leftrightarrow t \geq \zeta(\omega)$ *for all* $\omega \in \Omega$.

(M.2) For all $t \geq 0$ *there exists a map* $\theta_t : \Omega \to \Omega$ *such that* $X_s \circ \theta_t = X_{s+t}$ *for all* $s \geq 0$.

(M.3) $(P_x)_{x\in E_\Delta}$ *is a family of probability measures on* $(\Omega, \mathcal{M})$*, such that* $x \mapsto P_x[B]$ *is* $\mathcal{B}(E_\Delta)^*$*–measurable for all* $B \in \mathcal{M}$ *and* $\mathcal{B}(E_\Delta)$*–measurable for all* $B \in \sigma(X_t | t \geq 0)$ *and* $P_\Delta[X_0 = \Delta] = 1$.

(M.4) For all $A \in \mathcal{B}(E_\Delta), s, t \geq 0$*, and* $x \in E_\Delta$

$$P_x[X_{t+s} \in A | \mathcal{M}_t] = P_{X_t}[X_s \in A] \;\; P_x - a.s.$$

If $\mathbf{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ is a Markov process and μ a positive measure on $(E_\Delta, \mathcal{B}(E_\Delta))$ let $P_\mu := \int P_x \mu(dx)$. For a sub-$\sigma$-algebra $\mathcal{A} \subset \mathcal{M}$ let $\mathcal{A}^{P_\mu}$ be its P_μ-completion in $\mathcal{M}$.

Definition 5.2 *A Markov process* $\mathbf{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ *with state space E, life time* ζ*, and corresponding filtration* $(\mathcal{M}_t)_{t\geq 0}$ *is called a right process if*

(M.5) $P_x[X_0 = x] = 1$ *for all* $x \in E_\Delta$.

(M.6) $t \mapsto X_t(\omega)$ *is right continuous on* $[0, \infty)$ *for all* $\omega \in \Omega$.

(M.7) $(\mathcal{M}_t)_{t\geq 0}$ *is right continuous and for any* $(\mathcal{M}_t)_{t\geq 0}$*–stopping time* τ *and* $\mu \in \mathcal{P}(E_\Delta)$

$$P_\mu[X_{\tau+s} \in A | \mathcal{M}_\tau] = P_{X_\tau}[X_s \in A] \;\; P_\mu - a.s.$$

for all $A \in \mathcal{B}(E_\Delta)$ *and* $s \geq 0$.

Given a right process $\mathbf{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ with state space E the family $(p_t)_{t\geq 0}$ (resp. $(R_\alpha)_{\alpha>0}$) of kernels on $(E, \mathcal{B}(E))$ defined by $p_t f(x) := E_x[f(X_t)]$ (resp. $R_\alpha f(x) := \int_0^\infty e^{-\alpha t} p_t f(x) dt$), $x \in E$, $f \in \mathcal{B}_b$, is called the *transition semigroup* (resp. *resolvent*) of $\mathbf{M}$. Note that $R_\alpha f$ is well–defined and $\mathcal{B}(E)$–measurable for all $f \in \mathcal{B}_b$ because of our assumption $\mathcal{B}(E) = \sigma(C(E))$.

For a subset $A \in \mathcal{B}(E)$ let $\sigma_A := \inf\{t > 0 \mid X_t \in A\}$ be the *first hitting time* (w.r.t. $\mathbf{M}$).

Remark 5.3 *(i) Given a right process* $\mathbf{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ *with state space* E *and life time* ζ*, the filtration* $(\mathcal{F}_t)_{t\geq 0}$ *defined by* $\mathcal{F}_t := \cap_{\mu\in\mathcal{P}(E_\Delta)}(\mathcal{F}_t^0)^{P_\mu}$*, where* $\mathcal{F}_t^0 := \sigma(X_s|s \in [0,t])$*, is called the natural filtration. A right process w.r.t. some filtration* $(\mathcal{M}_t)_{t\geq 0}$ *is always a right process w.r.t. the natural filtration.*
(ii) If E *is a Radon topological space in the sense of [22] (i.e. homeomorphic to a universally measurable subset of a compact metric space, e.g. a locally compact separable metric space) then any right process* $\mathbf{M}$ *with state space* E *w.r.t. some filtration* $(\mathcal{M}_t)_{t\geq 0}$ *(satisfying* $\theta_{t+s} = \theta_t \circ \theta_s$ *for all* $s, t \geq 0$*) is a right process w.r.t. some filtration* $(\mathcal{M}_t)_{t\geq 0}$ *in the sense of [22]. Conversely, any right process* $\mathbf{M}$ *w.r.t.* $(\mathcal{M}_t)_{t\geq 0}$ *in the sense of [22] is a right process w.r.t.* $(\mathcal{M}_t)_{t\geq 0}$ *in the sense of Definition 1.2 if the corresponding transition semigroup* $(p_t)_{t\geq 0}$ *satisfies* $p_t(\mathcal{B}_b(E_\Delta)) \subset \mathcal{B}_b(E_\Delta)$*,* $t \geq 0$*.*

Definition 5.4 *A right process* $\mathbf{M}$ *with state space* E *and resolvent* $(R_\alpha)_{\alpha>0}$ *is called associated with* $\mathcal{E}$ *if* $R_\alpha f$ *is an* m*-version of* $G_\alpha f$ *for all* $\alpha > 0$ *and* $f \in \mathcal{B}_b \cap \mathcal{H}$*.* $\mathbf{M}$ *is called properly associated in the resolvent sense with* $\mathcal{E}$ *if in addition* $R_\alpha f$ *is* $\mathcal{E}$*-q.c. for* $\alpha > 0$ *and* $f \in \mathcal{B}_b \cap \mathcal{H}$*.*

If $\mathbf{M}$ is a right process with state space E, and $(p_t)_{t\geq 0}$ denotes the transition semigroup of $\mathbf{M}$, it is easy to see that $\mathbf{M}$ is associated with $\mathcal{E}$ if and only if $p_t f$ is an m-version of $T_t f$ for all $t \geq 0$ and $f \in \mathcal{B}_b \cap \mathcal{H}$.

Definition 5.5 *Let* $\mathbf{M} = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta}$ *be a right process with state space* E *and life time* ζ*. Let* μ *be a* σ*-finite positive measure on* $(E_\Delta, \mathcal{B}(E_\Delta))$*.*

(i) $\mathbf{M}$ *is called* μ*-tight, if there exists an increasing sequence* $(K_n)_{n\geq 1}$ *of compact metrizable sets in* E *such that*
$$P_\mu\left[\lim_{n\to\infty} \sigma_{E\setminus K_n} < \zeta\right] = 0 \ .$$

(ii) $\mathbf{M}$ *is called* μ*-special standard, if*

(M.8) $X_{t-} := \lim_{\substack{s\uparrow t\\ s<t}} X_s$ *exists in* E *for all* $t \in (0,\zeta) P_\mu$*-a.s.*

(M.9) $\lim_{n\to\infty} X_{\tau_n} = X_\tau P_\mu$*–a.s. on* $\{\tau < \zeta\}$ *and* X_τ *is* $\bigvee_{n\geq 1} \mathcal{F}_{\tau_n}^{P_\mu}$*-measurable for every increasing sequence* $(\tau_n)_{n\geq 1}$ *of* $(\mathcal{F}_t^{P_\mu})_{t\geq 0}$*-stopping times with limit* τ*.*

(iii) $\mathbf{M}$ *is called special standard, if* $\mathbf{M}$ *is* μ*-special standard for all* $\mu \in P(E_\Delta)$*.*

(iv) $\mathbf{M}$ *is called a Hunt process, if (M.8) and (M.9) hold with* ζ *replaced by* ∞ *and* E *by* E_Δ *for all* $\mu \in \mathcal{P}(E_\Delta)$*.*

References

[1] Albeverio, S., Röckner, M.:Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms, Probab. Th. Rel. Fields **89** (1991), 347-386.

[2] Bogachev, V.I., Röckner, M.: Regularity of invariant measures on finite and infinite dimensional spaces and applications, J. Funct. Anal. **133** (1995), 168-223.

[3] Bouleau, N., Hirsch, F.: Dirichlet forms and Analysis on Wiener space, Walter de Gruyter, Berlin, 1991.

[4] Courant R., Hilbert D.: Methoden der mathematischen Physik I. Heidelberger Taschenbücher. Berlin-Heidelberg-New-York: Springer 1968.

[5] Z. Dong, Z.M. Ma: An integral representation theorem for quasi-regular Dirichlet spaces. Chinese Sci. Bull. **38** (1993), 1355-1358.

[6] Föllmer, H.: Random fields and diffusion processes, Lecture Notes in Math. **1362** (1988), 101-203.

[7] Fukushima, M.: On a stochastic calculus related to Dirichlet forms and distorted Brownian motions, Physics Reports 77, No.3 (1981), 255-262.

[8] Fukushima, M.,Oshima, Y., Takeda, M.: Dirichlet forms and Symmetric Markov processes. Berlin-New York: Walter de Gruyter 1994.

[9] Kim, J.H.: Stochastic calculus related to nonsymmetric Dirichlet forms. Osaka J. Math. 24 (1987), no. 2, 331–371.

[10] Kuo, H. H.: Gaussian measures in Banach spaces, Lecture Notes in Math. **463**, Berlin-Heidelberg-New-York: Springer 1975.

[11] LeJan, Y.: Mesures associées à une forme de Dirichlet. Applications. Bull. Soc. math. France **106** (1978), 61-112.

[12] Lions, J.L., Magenes, E.: Non-homogeneous boundary value problems and applications I, Springer, Berlin, 1972.

[13] Meyer, P. A.: Intégrales stochastiques I-III. In: Séminaire de Probabilités I. Lecture Notes in Math. **39**, 72-141, Berlin-Heidelberg-New-York: Springer 1967.

[14] Ma, Z.M., Overbeck, L.,Röckner, M.: Markov processes associated with semi-Dirichlet forms. Osaka Journal of Mathematics **32** (1995), 97-119.

[15] Ma, Z.M., Röckner, M.: Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Berlin: Springer 1992.

[16] Oshima, Y.: Lectures on Dirichlet forms. Preprint Erlangen-Nürnberg 1988.

[17] Y. Oshima: On a construction of Markov processes associated with time dependent Dirichlet spaces , Forum Mathematicum **4** (1992), 395-415.

[18] Oshima, Y.: Some properties of Markov processes associated with time dependent Dirichlet forms. Osaka J. Math. **29** (1992), 103-127.

[19] Reed, M., Simon, B.: Methods of modern mathematical physics II. Fourier Analysis, Academic Press, New York-San Francisco-London, 1975.

[20] Revuz, D.: Mesures associées aux fonctionelles additives de Markov I. Trans. Amer. Math. Soc. **148** (1970), 501-531.

[21] Röckner, M., Zhang, T.-S.: Decomposition of Dirichlet processes on Hilbert space. Stochastic analysis (Durham, 1990), 321–332, London Math. Soc. Lecture Note Ser., 167, Cambridge Univ. Press, Cambridge, 1991.

[22] Sharpe, M.T.: General theory of Markov processes. New York: Academic Press 1988.

[23] Stannat, W.: The theory of generalized Dirichlet forms and its applications in analysis and stochastics, Preprint 97-101, SFB 343, Universität Bielefeld, and Memoirs of the AMS, Vol. 142, No. 678, 1999.

[24] Stannat, W.: (Nonsymmetric) Dirichlet operators on L^1: Existence, Uniqueness and associated Markov processes, Preprint 97-090, SFB 343, Universität Bielefeld. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), Vol. 28, No. 1, 99-140, 1999.

[25] Trutnau, G.: Stochastic calculus of generalized Dirichlet forms, Preprint 98-001, SFB 343, Universität Bielefeld.

[26] Trutnau, G.: Fukushima's decomposition and an Itô-type formula for generalized Dirichlet forms, Preprint Oberwolfach, 1998.

[27] Trutnau, G.: Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions Preprint 98-087, SFB 343, Universität Bielefeld, and Osaka J. Math. 37 (2000), 315-343.

[28] Yosida, K.: Functional analysis. Berlin Heidelberg New York. Springer 1974.

[29] Widder, D.V.: The Laplace transform. Princeton: Princeton University Press 1946.