Superlinear Period Lengths in Some Subtraction Games

Ingo Althöfer
Fakultät für Mathematik
Universität Bielefeld
Postfach 100 131
33501 Bielefeld
Germany

Jörg Bültermann
Fakultät für Mathematik
Universität Bielefeld
Postfach 100 131
33501 Bielefeld
Germany

Abstract

Subtraction games are "simple" variants of the famous Nim game [Bou 01]. In this note we will show that in some subtraction games the sequences of Win/Loss states have superlinear period lengths. Our most prominent observation is:

> For all s with $1 \leq s \leq 26$ the (s,4s,12s+1,16s+1)-game has the cubic period length $56s^3 + 52s^2 + 9s + 1$.

Possibly, the (s,8s,30s+1,37s+1,38s+1)-games with $s \in \mathbb{N}$ have superpolynomial period lengths.

1 Introduction and Results

In the $(s_1, \ldots, s_k)$-subtraction game there is a heap of size $m > 0$ and natural numbers $s_1, \ldots, s_k$, the legal moves. Without loss of generality we always assume $0 < s_1 < s_2 < \ldots < s_k$ and $k \geq 2$.

Two players move alternately. A move consists in substracting some s_i from m. The player who makes the heap size nonpositive is the winner. Assuming optimal strategies for both players, we get a sequence $(v(m))_{m=1}^{\infty}$ of game theoretic values:

$v(m)$=1, if the starting position with heap size m is a win for the player to move.

$v(m)$=0, if the starting position with heap size m is a loss for the player to move.

The starting values are $v(1) = \ldots = v(s_k) = 1$, as the player wins directly by subtracting s_k . For all $m > s_k$ the values obey the recursive condition

$$v(m) = 1 - \min\{v(m - s_1), \ldots, v(m - s_k)\}. \tag{1}$$

A sequence $(v(m))$ is called periodic if there are some $M \geq 0$ and $P > 0$ such that $v(M + t) = v(M + P + t)$ for all $t \geq 0$. The smallest such values for M and P are called the starting point and the period, respectively.

The sequence $(v(m))_{m=s_{k+1}}^{\infty}$ in the $(s_1, \ldots, s_k)$-game has to become periodic with period $P(s_1, \ldots, s_k) \leq 2^{s_k}$. Let M be the starting point of the period. Then we call $PP(s_1, \ldots, s_k) = M - (s_k + 1)$ the preperiod of the $(s_1, \ldots, s_k)$-game. In the case of $PP = 0$ we say that the game has no preperiod.

We are interested in subtraction games with small numbers of legal moves and *long* periods and/or preperiods. In the rest of this section we list our theoretical and experimental results. Section 2 contains some remarks on the proofs. Section 3 demonstrates the structure of the $(s, 4s, 12s+1, 16s+1)$-games for $1 \leq$ s ≤ 26. In Section 4 we formulate conjectures and open questions.

Theorem 2:

For all $0 < s_1 < s_2$ the (s_1, s_2)-game has preperiod 0. If s_2 is an odd multiple of s_1, the period length is $2s_1$. In all other cases the period length is $s_1 + s_2$.

The next theorem is strongly related to [BCG 82, p.497] where a statement on the (quadratically long) periods of the Grundy sequences of $(a, b, a+b)$-games is made. (We have $a = s$ and $b = 2s + 1$.)

Theorem 3.1:

For all $\mathrm{s} \geq 1$ the $(\mathrm{s}, 2\mathrm{s}+1, 3\mathrm{s}+1)$-game has no preperiod and period $P = 4\mathrm{s}^2 + 3\mathrm{s}$.

Theorem 3.2:

For all $s \geq 2$ the $(\mathrm{s}, 2\mathrm{s}, 3\mathrm{s}+1)$-game has preperiod $PP = 3\mathrm{s}$ and period

$$P = \begin{cases} 3, & \text{if } \mathrm{s} = 2, \\ 4\mathrm{s}+1, & \text{if } \mathrm{s} > 2. \end{cases}$$

The $(2, 4, 7)$-game, which belongs to this family, is also mentioned in [BCG 82].

Theorem n*:

For all $s \geq 1$ and all $n \geq 3$ the $(\mathrm{s}, 2\mathrm{s}+1, +\dots, n\mathrm{s}+1)$-game has no preperiod and period

$$P = (n+1)\mathrm{s}^2 + 3\mathrm{s}.$$

Observation 4:

For all $1 \leq \mathrm{s} \leq 26$ the $(\mathrm{s}, 4\mathrm{s}, 12\mathrm{s}+1, 16\mathrm{s}+1)$-game has no preperiod and period

$$P = 56\mathrm{s}^3 + 52\mathrm{s}^2 + 9\mathrm{s} + 1.$$

Observation 5:

In the $(s, 8s, 30s+1, 37s+1, 38s+1)$-games with $s = 1, 2, 3$ there are no preperiods. The periods are

$$\begin{aligned} P &= 11{,}757 \quad \text{for} \quad s = 1, \\ P &= 3{,}539{,}830 \quad \text{for} \quad s = 2, \\ P &= 17{,}987{,}570{,}846 \quad \text{for} \quad s = 3. \end{aligned}$$

Many more results of subtraction games are reported in the encyclopedic book [BCG 82, p.83-86 and others]. In contrast to our approach, the authors do not allow moves which make the heap size negative. In their definition a player has lost if he does not have any more legal moves. This results in a sequence $(w(m))_{m=1}^{\infty}$ of game theoretic values which are related to our v-values by a simple shift:

$$w(m) = v(m + s_k + 1) \text{ for all } m \geq 0.$$

2 On the Proofs

We reached most of our results in the following way: With the help of a computer the periods of many subtraction games were determined. Then we selected interesting examples and tried to find infinite classes of subtraction tuples which covered these candidates. We tested whether "all" games in such a class had the desired periods. If so, we looked for structural properties in the periods and tried to prove them by induction. Of course the key parts were finding the right classes and the structures of their periods – after this the proofs by induction were merely a (boring) task of correct book-keeping.
Our key moment was when we found that games with linearly combined moves and only one free parameter s were good candidates for long periods.
Proofs of Theorem 2, Theorem 3.1, Theorem 3.2, and Theorem n* can be

found in [Bül 93]. In this note we only demonstrate the structure of the games mentioned in Observation 4.

3 On the $(s, 4s, 12s + 1, 16s + 1)$-Games

It is convenient to compress the $0,1$-string $v(s_k + 1), v(s_k + 2), \dots$ by *runlength* encoding [LH 87]: Each maximal substring with a constant symbol is represented by its length. As there are only two different symbols (0 and 1), no more information than the lengths are needed for unique representation. For example the string (from the $(2, 5, 7)$-game)

0011011111011001111110011......

becomes 2 2 1 5 1 2 2 7 2 2 ...

Look at the $(s, 4s, 12s + 1, 16s + 1)$-games. For $s = 1$ it can be checked by hand that the period is

$$56 \cdot 1^3 + 52 \cdot 1^2 + 9 \cdot 1 + 1 = 118.$$

For all $2 \leq s \leq 26$ the runlength encoding of the periodic Win/Loss sequence (starting with $v(16s + 2)$; so there is no preperiod) has the following structure. It is convenient to split it up in 13 blocks. (For $s = 1$ the structure is different.) Each block (and subblock) starts with a run of 0-positions and ends with a run of 1-positions.

Block 1: It has length 40s+1:

s s s (2s) s s s (2s) s s 1 s (s–1) (s+1) (s–1) (4s+1)

s (4s) 1 (2s–1) 1 s (s–1) s s s 1 s (s–1) s s (2s) s s.

Block 2: It consists of s–2 subblocks. Each subblock is of length 28s+1. So altogether this block has length (s–2)(28s+1).

For j=2,...,s–1 we have the subblocks

$$j\ s\ (s-j)\ (s+j)\ (s-j)\ (s+1)\ (j-1)\ (3s)\ (s-j+1)\ (4s)$$
$$j\ (2s-j)\ j\ s\ (s-j)\ s\ s\ s\ j\ s\ (s-j)\ s\ s\ (2s)\ s\ s.$$

Block 3: This block consists of s–1 subblocks, one for each j=0,...,s–2.

The j-th subblock itself is divided into the 4 subsubblocks 3a(j), 3b(j), 3c(j), 3d(j).

Sub^2-Block 3a(j): It has length 33s+2:

$$s\ (4s+1)\ (s-1)\ (3s-j)\ (j+1)\ (4s)\ s\ s\ s\ (2s)$$
$$s\ s\ (s-j)\ (2s+j)\ s\ (2s+1)\ (s-1)\ (s+1)\ s\ (4s).$$

Sub^2-Block 3b(j): It consists of j sub^3-blocks of length 28s+1 each. So altogether this sub^2-block has length j(28s+1).

For i=1,...,j we have the sub^3-blocks

$$s\ (3s-j+i-1)\ (j-i+1)\ s\ i\ (2s-i)\ i\ s\ (s-i)\ s\ s\ (2s+i)\ (s-i)\ s$$
$$(s-j+i)\ (2s+j-i)\ s\ (s+1)\ i\ s\ (s-i-1)\ (s+i+1)\ (s-i)\ s\ i\ (3s-i).$$

Sub^2-Block 3c(j): It has length 20s:

$$s\ (4s)\ (j+1)\ (2s-j-1)\ (j+1)\ s\ (s-j-1)$$
$$s\ s\ (2s+j+1)\ (s-j-1)\ s\ s\ (2s)\ s\ s.$$

Sub^2-Block 3d(j): It consists of s–j–2 sub^3-blocks each of which is of length 28s+1. So altogether this sub^2-block has length (s–j–2)(28s+1).

For i=j+2,...,s–1 we have the sub^3-blocks

i s (s–i) (s+i) (s–i) (s+1) (i–1) (3s–j–1) (s+j–i+2) (4s) i

(2s–i) i s (s–i) s s s (i–j–1) (s+j+1) (s–i) s s (2s) s s.

So the complete length of block 3 is

$$\sum_{\mathrm{j}=0}^{s-2}\left(\underbrace{33\mathrm{s}+2}_{3a(j)} + \underbrace{(28\mathrm{s}+1)\mathrm{j}}_{3b(j)} + \underbrace{20\mathrm{s}}_{3c(j)} + \underbrace{(28\mathrm{s}+1)(\mathrm{s}-\mathrm{j}-2)}_{3d(j)}\right)$$
$$= 28\mathrm{s}^3 - 30\mathrm{s}^2 + 2\mathrm{s}.$$

Block 4: It has length 33s+2:

s (4s+1) (s–1) (2s+1) s (4s) s s s (2s) s

s 1 (3s–1) s (2s+1) (s–1) (s+1) s (4s).

Block 5: It consists of s–2 subblocks of length 28s+1 each. So altogether this block has length (s–2)(28s+1).

For j=1,...,s–2 we have the subblocks

s (2s+j) (s–j) s j (2s–j) j s (s–j) s s (2s+j) (s–j) s (j+1)

(3s–j–1) s (s+1) j s (s–j–1) (s+j+1) (s–j) s j (3s–j).

Block 6: It has length 73s+3:

s (3s–1) 1 s (s–1) (s+1) (s–1) s 1 s s (3s–1) 1 s s (2s) s

(s+1) (s–1) (3s) 1 s (s–1) (2s+1) s (4s) s s s (2s) s (4s) s

(2s+1) (s–1) (s+1) s (4s) s (2s) s s 1 (3s) (s–1) s s (2s+1)

(s–1) s 1 (3s–1) s (s+1).

Block 7: It consists of s–2 subblocks each of which has length 28s+1. So altogether this block has length (s–2)(28s+1).

For j=1,...,s–2 we have the subblocks

$$j\ s\ (s-j-1)\ (s+j)\ (s-j+1)\ s\ j\ (3s-j)\ s\ (2s+j)\ (s-j)\ s\ (j+1)\ (2s-j-1)$$
$$j\ (s+1)\ (s-j-1)\ s\ s\ (2s+j+1)\ (s-j-1)\ s\ (j+1)\ (3s-j-1)\ s\ (s+1).$$

Block 8: It has length 32s:

$$(s-1)\ (3s-1)\ 2\ s\ (s-1)\ (2s+1)\ s\ (3s-1)\ 1\ s\ s\ s\ (s-1)$$
$$(2s+1)\ s\ (4s)\ s\ (2s+1)\ (s-1)\ (s+1)\ s\ (3s-1).$$

Block 9: This block consists of s–2 subblocks, one for each j=1,...,s–2.

The j-th subblock itself is divided into the 4 subsubblocks 9a(j), 9b(j), 9c(j), 9d(j).

Sub^2-Block 9a(j): It consists of j–1 sub^3-blocks. These sub^3-blocks have length 28s+2 each. So altogether the sub^2-block 9a(j) has length (j–1)(28s+2).

For i=0,...,j–2 we have the sub^3-blocks

$$(j-i)\ s\ s\ (2s)\ s\ s\ (i+1)\ (2s-j)\ (j-i-1)\ (s+i+1)\ (s-i-1)\ s$$
$$(s-j+i+1)\ (2s+j)\ (s-i-1)\ (4s)\ (s-j+i+2)\ (s+j-i-1)\ (i+1)\ s$$
$$(s-i-2)\ (s+1)\ s\ s\ (i+1)\ (2s-j).$$

Sub^2-Block 9b(j): It has length 21s+2:

$$1\ s\ s\ (2s)\ s\ s\ j\ (3s)\ (s-j)\ s$$
$$s\ (2s+j)\ (s-j)\ (4s)\ s\ (s+1).$$

Sub^2-Block 9c(j): It consists of s–j–1 sub^3-blocks, each of which has length 28s+1. So the sub^2-block 9c(j) has length (s–j–2)(28s+1).

For i=1,...,s–j–1 we have the sub^3-blocks

(j+i–1) s (s–j–i) A s (j+i–1) (3s–j–i+1) s (2s+i–1) (s–i+1) s (j+i)
B (s–j–i) s s (2s+j+i) (s–j–i) s i (3s–i) s (s+1).

In this representation A and B stand for the following strings.

$$\mathrm{A} = \begin{cases} (\mathrm{s}+1)\ \mathrm{s} & \text{if } \mathrm{i} = 1, 2, \\ (\mathrm{s}+\mathrm{i}-1)\ (\mathrm{s}-\mathrm{i}+2) & \text{otherwise.} \end{cases}$$

$$\mathrm{B} = \begin{cases} (3\mathrm{s}) & \text{if } \mathrm{i} = 1, \\ (2\mathrm{s}-\mathrm{j}-\mathrm{i})\ (\mathrm{i}-1)\ (\mathrm{s}+\mathrm{j}+1) & \text{otherwise.} \end{cases}$$

Sub^2-Block 9d(j): It has length 32s–j:

(s–1) (3s–j–1) (j+2) s (s–1) (2s+1) s (3s–j–1) (j+1) s s s (s–j–1)
(2s+j+1) s (4s) (s–j) (2s+j+1) (s–1) (s+1) s (3s–j–1).

So the complete length of block 9 is

$$\sum_{j=1}^{s-2} \left(\underbrace{(28\mathrm{s}+2)(\mathrm{j}-1)}_{9a(j)} + \underbrace{21\mathrm{s}+2}_{9b(j)} + \underbrace{(28\mathrm{s}+1)(\mathrm{s}-\mathrm{j}-1)}_{9c(j)} + \underbrace{32\mathrm{s}-\mathrm{j}}_{9d(j)} \right)$$
$$= 28\mathrm{s}^3 - 58\mathrm{s}^2 + 3\mathrm{s} + 2.$$

Block 10: It consists of s–2 subblocks of length 28s+2 each. So altogether this block has length (s–2)(28s+2).

For j=1,...,s–2 we have the subblocks

(s–j) s s (2s) s s j (s+1) (s–j–1) (s+j) (s–j) s (j+1) (3s–1)
(s–j) (4s) (j+2) (2s–j–1) j s (s–j–1) (s+1) s s j (s+1).

Block 11: It has length 60s+4:

1 s s (2s) s s (s–1) (3s) 1 s s (3s–1) 1 (4s) s (s+1)

(s–1) (2s+1) s s (s–1) (2s+1) s (2s) s s s (4s) s

(4s) 1 (3s) (s–1) (s+1) s (2s) s s s (2s) s s 1 s.

Block 12: This block consists of s–2 subblocks, each of which is of length 28s+2. So altogether this block is of length (s–2)(28s+2).

For j=1,...,s–2 we have the subblocks

(s–j) (s+j) (s–j) s j (3s) (s–j) (4s) (j+1) (2s–j)

j s (s–j–1) (s+1) s s j s (s–j) s s (2s) s s (j+1) s.

Block 13: It has length 39s+3:

1 (2s–1) 1 s (s–1) (3s) 1 (4s) s (s+1) (s–1) (2s+1)

s s (s–1) s 1 s s (2s) s s s (4s) s (7s+1).

Adding the lengths of all 13 blocks gives the desired length

$$56s^3 + 52s^2 + 9s + 1.$$

□

Finally, let us make a remark on Observation 5. We tested all games with 5 moves, where $s_1 = 1, s_2 < 10, s_3 < s_4 < s_5 < 50$. Among these candidates the $(1, 8, 31, 38, 39)$-game has the longest period, namely $11, 757$. This is more than 8 times longer than the period (=1,364) of the second best candidate.

4 Conjectures and Open Questions

Conjectures

(i) For all (s_1, s_2, s_3), $P(s_1, s_2, s_3)$ is bounded by a quadratic polynomial in s_3.

(ii) For all $s \in \mathbb{N}$ the periods of the $(s, 4s, 12s + 1, 16s + 1)$-games have the structure presented in Section 3.

(iii) The $(s, 8s, 30s+1, 37s+1, 38s+1)$-games have superpolynomial periods in s. For $s = 4$,5,6 computer runs have shown that the periods are longer than 50,000,000 if there are no preperiods.

Open Questions

(i) Are there families of 4-move games whose periods are not bounded by a polynomial of degree 3?

(ii) Are there subtraction games with superlinear preperiods?

(iii) How frequently do preperiods occur? There are $\binom{49}{3} = 18,424$ 3-move games with $s_3 < 50$. $5,785$ of them have a preperiod $\neq 0$. Is this ratio $\approx \frac{1}{3}$ typical?

(iv) More generally: What are the expected lengths of period and preperiod in the set of all k-move games with $s_k \leq n$?

(v) Do substrings of long periods give suitable pseudorandom numbers? How are the 0's and 1's distributed in "long" periods?

(vi) Consider one free parameter s and nonlinear (monotone) functions $f_1(s)$, $f_2(s)$, and $f_3(s)$, for instance polynomials like $f_1 = s$, $f_2 = s^2 + 1$, $f_3 = s^3 - 1$. What periods may such families have?

(vii) If we restrict ourselves to 3-move games: Are there some product inequalities concerning periods and preperiods, for instance

$$P(s_1, s_2, s_3)\ PP(s_1, s_2, s_3)\ \leq\ s_3^2$$

for all $s_1 < s_2 < s_3$?

(viii) In the case of some nonpositive moves s_i, for instance $s_1 < 0 < s_2$, our winning condition ("make the heap size nonpositive") will lead to a draw if the initial heap size is larger than s_2. (No player will diminish the heap to $m = s_2$ or a smaller value.) The losing condition of [BCG 82], however, ("no more legal moves") sometimes allows players to avoid draws. See for instance the $(-1, 4)$-game which leads to the Win/Loss sequence

m	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$v(m)$	0	0	1	0	1	1	0	1	1	0	1	1	0	...

with period $P = 3 \qquad (= s_1 + s_2)$.

Some games with negative s_i may have long periods.

(ix) The sequences of Grundy numbers in subtraction games may have longer periods than the Win/Loss - states. What are extremal examples?

(x) The recursive rule (1) in Section 1 may be applied to other starting strings in $\{0, 1\}^{s_k}$ than $(v(1), v(2), ..., v(s_k))$. How long may periods become in this more general situation, for instance in the case of (s_1, s_2, s_3)? Are the periods typically longer or shorter than in the corresponding subtraction game?

Imre Leader mentioned the possibility that some subtraction games might perhaps be interpreted as simulations of certain computing devices (for instance Turing machines). Such correspondences could help to prove the existence of subtraction tuples $(s_1, \ldots, s_k)$ for some fixed k with exponential period lengths.

(Confer also Conway's fraction games.) Especially in the context of this idea other starting strings than (1,...,1) might be of interest.

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