Selfinjective and iterated tilted algebras of type E_6 , E_7 , E_8

Bernd Roggon

Abstract. This paper is devoted to the construction of a system of representatives of the selfinjective / the iterated tilted algebras of type E_6, E_7, E_8. The two related Pascal programs are integrated into the program package $CREP$ on (combinatorial) representation theory. A few of the output lists are contained in the Appendix.

Introduction

One of the present highlights of the representation theory of finite- dimensional algebras (over an algebraically closed field) is, surely, the theory of the representation- finite (r.f.) selfinjective algebras. According to their stable AR- quivers, these algebras are distributed into classes indexed by the Dynkin graphs, A_n, D_n, and E_6, E_7, E_8 (cf. [Ri1], [Ri2]). In this way, we speak of the *type* of such an algebra. There are general descriptions of the selfinjective algebras of the 'regular' types A_n, D_n, due to Bretscher/ Riedtmann (cf. [BLR] $6, 7$). One aim of this paper is to construct a system of representatives of the selfinjective algebras of the 'exceptional' types E_6, E_7, E_8.

An important subclass of the r.f. selfinjective algebras is given by the r.f. trivial extensions. By grouping the standard r.f. selfinjective algebras, say of Dynkin type Δ, according to the isoclass of their universal covering, these algebras can the parametrized by the (isoclasses of the) trivial extensions of type Δ.

Linking the two items in the title of this paper, we note that, due to Assem/ Happel/ Roldan, the algebras whose trivial extension is of Dynkin type Δ coincide with the iterated tilted algebras of type Δ (see [AHR]). We refer to general descriptions of the iterated tilted algebras of type A_n resp. D_n, given by Assem/ Happel resp. Assem/ Skowronski (see [AH], [AS]). The second aim of this paper is to determine a system of representatives of the iterated tilted algebras of type E_6, E_7, E_8.

In this way, this paper completes the description of

- *all representation-finite selfinjective algebras*
- *all representation-finite trivial extensions*
- *all the algebras whose trivial extension is representation-finite*

0.1 Now let us fix some notation which will be valid for the remainder of this paper. Denote by k a fixed algebraically closed field. All the algebras will be finite- dimensional k- algebras (associative, with 1), and we will furthermore assume that they are basic and connected. Moreover, if an algebra is assumed to be selfinjective, we demand that it is $\neq k$. By a module of an algebra A (resp. a locally bounded category Λ) we will mean a finite-dimensional A- right- module (resp. a finitely generated covariant functor $\Lambda \longrightarrow mod{-}k$). Maps (as well as paths in quivers) will be composed from the left to the right, that is, the composition of two maps $f : X \to Y$, $g : Y \to Z$ will be written as $fg : X \to Y \to Z$ (however, when using an argument $x \in X$, we prefer the more conventional notation fx, $gf\,x$). The *Cartan matrix* C_A of an algbra A is defined as $C_A := (dim_k e_i A e_j)_{i,j} \in \mathcal{M}_n(\mathbf{Z})$, where $1_A = e_1 + \ldots + e_n$ is a fixed decomposition of 1_A into pairwise orthogonal primitive idempotents. Finally, when speaking of a '*system of representatives*' (of certain algebras) we always mean a system of representatives of the isomorphism classes of these algebras.

0.2 We will adopt the language of [R] 2.1, 2.4 resp. 4.1, 4.2 (concerning basic concepts resp. tilting theory) and [BG] 2.1, 2.2 resp. 1 , 3 (on locally bounded categories resp. covering theory).

0.3 The present paper is organized as follows. (We remark that, apart from section 1 , this paper is essentially a rather complete summary of [Ro] 4.1, 6.1, 6.2, 6.3 .)
In *section 1* we will present a few essential facts about simply connected (repetitive) categories and, moreover, an outline of the classifications of the (standard) r.f. selfinjective algebras, due to Bretscher/ Riedtmann resp. Hughes/ Waschbuesch. Combining these two theories, which will be done in *section 2*, leads to a description of the selfinjective algebras of type E_n, $n = 6, 7, 8$, which needs as input data 'just' a system of representatives of the repetitive categories of type E_n and, in case $n = 6$, for each of the categories of this system a mark whether its associated description by quiver and relations allows an involution. In this way, the construction of a system of representatives of the selfinjective algebras of type E_n is reduced to the consideration of certain simply connected algebras which in turn are uniquely determined by their Cartan matrix, and, moreover, that it is possible (and easy) to construct from these various Cartan matrices the associated repetitive categories of type E_n. However, these two fundamental observations were

the starting point of trying to carry out the classification of these repetitive categories (given in [HW]) onto the level of these Cartan matrices (in this way making the construction of the needed input data computer accessable), which is done in *section 3* . The according model allows to construct from certain 'garbage' data a system of representatives of the algebras whose trivial extension is (r.f.) of type E_n (that is, the iterated tilted algebras of type E_n), and the algebras of this system are grouped by the isoclass of their trivial extension resp. their repetitive category. (An application to this grouping will be given in section 5 .) In *section 4* we describe the resulting two algorithms which will achieve the two aims of this paper.
The corresponding two Pascal programs are integrated into the program package $CREP$ on representation theory.
In *section 5* we will present a few aspects from the output. Furthermore, the *Appendix* contains the output list of the repetitive categories of type E_6,E_7,E_8 and of the iterated tilted algebras of type E_6. The other list concerning type E_7 can be found in the Appendix of [Ro], the list of the iterated tilted algebras of type E_8 is contained on the discs attached to [Ro].

1. Preliminary results

This section sketches the classification of the r.f. selfinjective algebras due to Bretscher/ Riedtmann resp. Hughes/ Waschbuesch.

1.1 First, we collect a few important properties of *simply connected k-*categories (i.e. locally representation- finite k- categories having a simply connected AR- quiver , cf. [BG]).

Proposition. *([BG] 3.1, 6.1) Let Λ be a simply connected category. Then the quiver of Λ contains no oriented cycles, and Λ is standard.*

Theorem. *([BLR] 5.1) Assume Λ is a simply connected category with quiver Q. Then there exists an ideal $I \subseteq (kQ^+)^2$ s.t. $\Lambda \tilde{\to} k(Q,I)$ and for (each) two objects x,y in Λ either $I(x,y) = kQ(x,y)$ or $I(x,y)$ is spanned by the differences of paths $(x \rightsquigarrow y)$.*

We emphasize that the above construction just depends on the *Cartan invariants* $(dim_k\Lambda(x,y))_{x,y}$ of Λ (x,y ranging over all objects of Λ).
Indeed, since $I = \bigoplus_{x,y} I(x,y)$ and since each k- space $I(x,y)$ is already determined by $kQ(x,y)$ and $c_{x,y} := dim_k\Lambda(x,y) \in \{0,1\}$, I is uniquely determined by the quiver Q and the Cartan invariants of Λ. Moreover, Q is uniquely determined by the Cartan invariants of Λ , because :

For two objects $x,\ y$ of Λ there is an arrow $x \to y$ in Q, iff $x \neq y,\ c_{x,y} \neq 0$ and $c_{x,z} \cdot c_{z,y} = 0$ for all objects $z \neq x, y$.

(Observe that this provides a powerful **recipe of constructing a simply connected category directly from its Cartan invariants** .)

In fact, assume there is an arrow $x \to y$ in Q (in particular, $c_{x,y} = 1$). Thus we have $\Lambda(x, z^\star)\,\Lambda(z^\star, y) = 0$ for all objects $z^\star \neq x, y$ of Λ. Furthermore x is non-equal to y, since Q contains no oriented cycles. Assume there is a $z \neq x, y$ satisfying $c_{x,z} = c_{z,y} = 1$. Consequently there are paths $(v : x \rightsquigarrow z)$, $(w : z \rightsquigarrow y)$ in Q which are both not lying in I. But since $\Lambda(x, z)\,\Lambda(z, y) = 0$ its composition $(vw : x \rightsquigarrow z \rightsquigarrow y)$ is contained in $I(x, y)$, and thus, by the Theorem, $I(x, y) = kQ(x, y)$ or equivalently $c_{x,y} = 0$, a contradiction.

1.2 Now, lets encounter the summary of the results of [BLR]. Let us start with the description of the AR- quivers of r.f. selfinjective algebras (the notion of a *configuration* can be picked up from [BLR]) :

Theorem. *The AR-quivers of the selfinjective algebras of Dynkin type Δ coincide with the translation quivers of the form $\mathbf{Z}\Delta_C/\Pi$, where C is a configuration of $\mathbf{Z}\Delta$ and Π an infinte cyclic subgroup of $Aut(\mathbf{Z}\Delta_C)$. Two such translation quivers $\mathbf{Z}\Delta_C/\Pi$, $\mathbf{Z}\Delta^\star_{C^\star}/\Pi^\star$ are isomorphic, iff there is an isomorphism $\xi : \mathbf{Z}\Delta \tilde{\to} \mathbf{Z}\ \Delta^\star$ satisfying $\xi C = C^\star$ and $\xi^{-1}\Pi\xi = \Pi^\star$*

Call a configuration C of $\mathbf{Z}\Delta$ *symmetric* provided there is an automorphism ϕ of $\mathbf{Z}\Delta$ which leaves C invariant and fixes at least one vertex of C. Note that, in case $\Delta \neq D_4$, ϕ will be an involution (i.e. $\phi^2 = 1$, $\phi \neq 1$).

Recall that $\tau^{m_\Delta}C = C$, where $m_\Delta = n, 2n-3$ resp. the primes $11, 17, 29$ case of $\Delta = A_n, D_n$ resp. E_6, E_7, E_8 ([BLR] 1.2), denote by $d(C)$ be the smallest positive integer d satisfying $\tau^d C = C$, observe that $d(C)$ divides m_Δ. Recall also that the resulting residue quiver $\mathbf{Z}\Delta_C/\tau^{m_\Delta\mathbf{Z}}$ contains exactly $\#\Delta_0$ projective vertices ([BLR] 2.3).

Starting with the description of $Aut\mathbf{Z}\Delta$ given in [Ri1] 4.3 we obtain from these two properties that for an arbitrary configuration C of $\mathbf{Z}\Delta$, $\Delta \neq D_4$ (for $\Delta = D_4$ see [BLR] 7.6) we have $Aut\,\mathbf{Z}\Delta_C\ = \phi^{\mathbf{Z}} \times \tau^{d(C)\mathbf{Z}}$ (for a unique involution ϕ) in case C is symmetric, resp. $\tau^{d(C)\mathbf{Z}}$ else. Let us stress the cases $\Delta = E_n$: here $Aut(\mathbf{Z}E_n)_C = \phi^{\mathbf{Z}} \times \tau^{m_{E_n}\mathbf{Z}}$ in case $n = 6$ and C symmetric (for a unique involution ϕ), or $= \tau^{m_{E_n}\mathbf{Z}}$ otherwise.

Since a standard algebra is uniquely determined by its AR- quiver ([BG] 5.1), the above Theorem gives a classification of the standard r.f. selfinjective algebras. Thus it remains to describe these algebras by quiver and relations,

having as input data the various AR- quivers.
Let A be a standard selfinjective algebra with the AR- quiver $\mathbf{Z}\Delta_C/\Pi$. Associate to C the k-category $\mathbf{\Lambda_C}$ s.t. the opposite category Λ_C^{op} is the full subcategory of the mesh category $k(\mathbf{Z}\Delta_C)$ whose objects are given by the projective vertices of $\mathbf{Z}\Delta_C$. By [BLR] 1.2 we have a canonical isomorphism $ind\Lambda_C \tilde{\rightarrow} k(\mathbf{Z}\Delta_C)$ of locally bounded categories. The AR- quiver of Λ_C is isomorphic to $\mathbf{Z}\Delta_C$ ([BG] 2.6a), and the latter one is simply connected. Consequently, the k- category Λ_C is a **simply connected** one. Thus we may choose a description $\Lambda_C \tilde{\rightarrow} k(Q,I)$ as in the Theorem of 1.1 . The group $\Pi \subseteq Aut\mathbf{Z}\Delta_C$ acts canonically on (Q,I), and by [BLR] 5.3 : $\quad A \tilde{\longrightarrow} k((Q,I)/\Pi)$
Finally, note that, in case $\Delta = A_n, D_n$, the access of Bretscher/ Riedtmann directly leads to an algorithm for computing a system of representatives of the selfinjective algebras of these types (for details, cf. [Ro] 2.2) .

1.3 This subsection is devoted to introduce repetitive categories, all the subsequent results are due to Hughes/ Waschbuesch and taken from [HW].
Let B be an algebra. Denote by $\mathcal{D} := Hom(-,k)$ the standard duality on B-mod. Consider the doubly infinite matrix algebra

$$\hat{B} := \begin{bmatrix} \ddots & & & & & \\ \ddots & \ddots & & & 0 & \\ & \mathcal{D}B & B & & & \\ & & \mathcal{D}B & B & & \\ & & & \mathcal{D}B & B & \\ & 0 & & & \ddots & \ddots \\ & & & & & \ddots \end{bmatrix}$$

in which matrices are assumed to have only finitely many entries $\neq 0$, and the entries outside the two indicated diagonals are $= 0$. The multiplication in $\hat{B}$ is induced by the multiplication in B, the natural bimodule structure of $\mathcal{D}B$ and the zero map $\mathcal{D}B \otimes \mathcal{D}B \to 0$. Choosing a complete set of primitive idempotents for B turns $\hat{B}$ (uniquely) into a locally bounded k- category which we call the *repetitive category* to B.
Clearly, $\hat{B}$ is selfinjective (i.e. the projective $\hat{B}$- modules coincide with the injective ones), thus $\hat{B}$ allows a unique automorphism $\nu = \nu_{\hat{B}}$ (called the *Nakayama automorphism*) satisfying $top\hat{B}(x,-) \cong soc\hat{B}(\nu^{-1}x,-)$ for each

object x in $\hat{B}$. In the above interpretation of $\hat{B}$ this automorphism is just the canonical diagonal shift into the lower right corner by one position. Observe that the resulting residue category $\hat{B}/\nu^{\mathbf{Z}} \cong T(B)$ is isomorphic to the *trivial extension* $T(B)$ of B, which is a symmetric algebra whose k- dimension is twice the one of B.

Clearly, $\nu^{\mathbf{Z}}$ acts freely on the objects of $\hat{B}$, and since $\nu^{\mathbf{Z}} \cong \mathbf{Z}$ is torsion-free, $\nu^{\mathbf{Z}}$ acts freely even on $modB$. Considering the AR- quiver $\Gamma_{\hat{B}}$ of $\hat{B}$ the resulting residue quiver $\Gamma_{\hat{B}}/\nu^{\mathbf{Z}}$ constitutes a system of complete connected components of the AR- quiver of $T(B)$.

Now we turn entirely to the case that $T(B)$ is **representation- finite**.

Proposition 1 . *Let B be an algebra s.t. $T(B)$ is r.f. of type Δ. Then the AR- quiver of $\hat{B}$ is connected, and its stable part is the universal covering of the stable AR- quiver of $T(B)$, thus $\cong \mathbf{Z}\Delta$.*

Recall that the isoclass of $\mathbf{Z}\Delta$ is uniquely determined by the Dynkin graph Δ ([Ri1] 1.3) , in this way we say that $\hat{B}$ is *of type* Δ.

The main properties of repetitive categories resp. trivial extensions of Dynkin type are given in the two Theorems below.

Theorem 1 . *Let A, B be algebras, and assume that the trivial extension $T(A)$ of A is r.f.. Then the following assertions are equivalent: (i) $T(A) \cong T(B)$, (ii) $\hat{A} \cong \hat{B}$, (iii) There exists an admissable sequence $(i_1, \ldots, i_t)$ of negative ν- reflections for A s.t. $B \cong S^-_{i_t} \ldots S^-_{i_1} A$.*

(Note that the last two items are already equivalent under the (much weaker) assumption that the quiver of A does not contain oriented cycles.)

Theorem 2 . *Let $\vec{\Delta}$ be an arbitrary orientation of the Dynkin graph Δ. Then the trivial extensions of type Δ coincide with the trivial extensions of tilted algebras of $k\vec{\Delta}$.*

Using the above two Theorems and parts of [AHR] 2.2 it is not difficult to derive from the simply connectedness of the tilted algebras of $k\vec{\Delta}$

Proposition 2 . *Let B be an algebra with r.f. trivial extension. Then B is simply connected.*

Now let us turn our attention to the associated repetitive category $\hat{B}$. Note that $\hat{B}$ is the *universal covering* of $T(B)$ in the sense of [W1]. Then [W1] ensures that $\hat{B}$ is locally representation- finite, and that, by Proposition 1 resp. the description of the AR- quivers of r.f. selfinjective algebras, there is a configuration C s.t. the (simply connected) translation quiver $\mathbf{Z}\Delta_C$ is isomorphic to the AR- quiver of $\hat{B}$. Altogether we have

Proposition 3 . *Let B be an algebra with r.f. trivial extension. Then the repetitive category $\hat{B}$ is simply connected.*

Let $\hat{B} \xrightarrow{\sim} k(Q_{\hat{B}}, I_{\hat{B}})$ be the *usual description* of $\hat{B}$ resulting from the Theorem of the first subsection. Recall from 1.1 the recipe for computing the quiver $Q_{\hat{B}}$ from the Cartan invariants of $\hat{B}$, and, thus, from the Cartan matrix of B. Note that the ideal $I_{\hat{B}}$ has an *alternative description*: $I_{\hat{B}}$ is generated by the relations of the following kind : • full commutativity , • all paths $(v : x \rightsquigarrow y) \in \hat{I}$, unless there is an extension $vw : x \rightsquigarrow y \rightsquigarrow \nu x$.
In fact, assume there is an extension $(vw : x \rightsquigarrow y \rightsquigarrow \nu x)$, then we have $v \notin \hat{I}$, since otherwise $vw \in \hat{I}$ and thus $\hat{B}(x, \nu x) = 0$, a contradiction. Conversely, assume $v \notin \hat{I}$, then $0 \neq \hat{B}(x, y) \cong_k \hat{B}(y, \nu x)$, thus there is a path $(\tilde{v} : y \rightsquigarrow \nu x) \notin \hat{I}$, and in this way $(v\tilde{v} : x \rightsquigarrow y \rightsquigarrow \nu x)$ cannot lie in $\hat{I}$, since otherwise $\hat{B}(x, \nu x) = 0$.

Let us add the following characterization of the algebras whose trivial extension is r.f. .

Theorem 3 . *([HW], [AHR], [G]) For an algebra B the following properties are equivalent : (i) $\hat{B}$ is locally r.f. , (ii) $T(B)$ is r.f. , (iii) B is iterated tilted of Dynkin type. Moreover, in case B satisfies one of these properties, then the several Dynkin types arising in this equivalence coincide.*

1.4 Now we are going to outline the classification of the r.f. selfinjective algebras, due to Hughes / Waschbuesch.

We perform the following subdivision : in case *char* $k = 2$ denote the class of algebras being isomorphic to an algebra $B(T, x, 1)$ (where T is a Brauer tree with a distinguished vertex x), in the sense of [W2], by $\Re$, and call by $\Re^\star$ the class consisting of the other r.f. selfinjective algebras; case of *char* $k \neq 2$, we denote by $\Re^\star$ the class of all r.f. selfinjective algebras.
The algebras of $\Re$ are classified in [W2], they have stable AR- quivers of the form $\mathbf{Z}D_{3m}/\tau^{(2m-1)\mathbf{Z}}$, moreover, these algebras are the **only** r.f. selfinjective algebras having no non- trivial covering resp. whose universal covering (in the sense of [W1]) does not exist (cf. [W1]). As a conclusion of [W1], [HW] (for details see [HW] 4.1, 4.2) we expose the

Theorem. *Let $\vec{\Delta}$ be an arbitrary orientation of the Dynkin graph Δ.*
Then the algebras in $\Re^\star$ which are of Dynkin type Δ coincide, up to isomorphism, with the algebras of the form $k((\hat{Q}, \hat{I})/\Pi)$, where $B = k(Q, I)$ runs

over all algebras with (r.f.) trivial extension of type Δ (resp. over all tilted algebras of $k\vec{\Delta}$), and for each such B the bound quiver $(\hat{Q}, \hat{I})$ denotes the usual description of $\hat{B}$ by quiver and relations, and Π runs over all infinte cyclic subgroups of $(\hat{Q}, \hat{I})$.
Moreover, two such algebras $k((\hat{Q}, \hat{I})/\Pi)$, $k((\widehat{Q^\star}, \widehat{I^\star})/\Pi^\star)$ are isomorphic, iff there is an isomorphism $\xi : (\hat{Q}, \hat{I}) \tilde{\to} (\widehat{Q^\star}, \widehat{I^\star})$ s.t. $\Pi = \xi \Pi^\star \xi^{-1}$.

2. A first approach

Combining the classifications of the selfinjective algebras by Bretscher/ Riedtmann resp. Hughes/Waschbuesch will give a first approach to the explicit computation of the selfinjective algebras of type E_6, E_7, E_8.

2.1 First observe that the universal covering structures in both classifications are essentially the same:

Proposition. *Let Δ be a Dynkin graph. Then the k-categories of the form Λ_C (C some configuration of $\mathbf{Z}\Delta$, see 1.2) coincide with the repetitive categories of type Δ, up to isomorphism.*

Proof. Denote by $\mathcal{C}_\Delta$ the set of isoclasses of configurations of $\mathbf{Z}\Delta$, and by $\mathcal{R}_\Delta$ the set of isoclasses of repetitive categories of type Δ. It is not difficult to derive from the results of the preceeding section that there are maps $\mathbf{\Psi}$ resp. $\mathbf{\Psi}^-$ s.t. the first associates to a repetitive category $\hat{B}$ a fixed configuration C s.t. $\mathbf{Z}\Delta_C$ is isomorphic to the AR- quiver of $\hat{B}$ (cf. 1.3), and that, by [W1], the second one which maps a configuration of $\mathbf{Z}\Delta$ to the associated simply connected category Λ_C is in fact a map into the class of repetitive categories of type Δ. Clearly the induced map $\Psi^- : \mathcal{C}_\Delta \longrightarrow \mathcal{R}_\Delta$ is injective, which is also true for the induced map $\Psi : \mathcal{R}_\Delta \longrightarrow \mathcal{C}_\Delta$, since a simply connected category is uniquely determined by its Cartan invariants (1.1), and these, by standardness, result from the associated mesh category, say $k(\mathbf{Z}\Delta_C)$, and thus (only) from the isoclass of C. Altogether we have $\#\mathcal{C}_\Delta \leq \#\mathcal{R}_\Delta \leq \#\mathcal{C}_\Delta < \infty$, by 1.2. Therefore the maps induced by Ψ, Ψ^- are mutually converse bijections. Consequently, each repetitive category $\hat{B}$ of type Δ satisfies: $\hat{B} \xrightarrow{\sim} \Lambda_{\Psi\hat{B}}$ $\square$

2.2 Now let us consider the above situation $\hat{B} \xrightarrow{\sim} k(\hat{Q}, \hat{I}) \xleftarrow{\sim} \Lambda_C$, where $(\hat{Q}, \hat{I})$ denotes the usual description of $\hat{B}$ provided by the Theorem of 1.1. Observe the canonical **identifications** (projective vertices in $\mathbf{Z}\Delta_C$) $\rightleftharpoons$ (vertices $x \in \hat{Q}_0$) $\rightleftharpoons$ ($(\hat{Q}, \hat{I})(x, -) \in proj(\hat{Q}, \hat{I})$, $x \in \hat{Q}_0$). As in 1.2 , each automorphism of $\mathbf{Z}\Delta_C$ acts naturally on $(\hat{Q}, \hat{I})$. This yields a canonical group homomorphism $\mathbf{\Phi}$: $Aut\mathbf{Z}\Delta_C \longrightarrow Aut(\hat{Q}, \hat{I})$ which is moreover injective, since an automorphism of $\mathbf{Z}\Delta_C$ fixing all the projective vertices can only be the identity (1.2).

Next we will show that Φ is also surjective, thus a group isomorphism.

In fact, each $\sigma \in Aut(\hat{Q}, \hat{I})$ acts on $ind(\hat{Q}, \hat{I}) \approx ind\Lambda_C \approx k(\mathbf{Z}\Delta_C)$ (and thus on $\mathbf{Z}\Delta_C$) as follows : For $\mathcal{F} : (\hat{Q}, \hat{I}) \to mod - k$ define $\sigma\mathcal{F} : x \mapsto \mathcal{F}\sigma^{-1}x$, (f : x$\to$ $y) \mapsto (\mathcal{F}\sigma^{-1}f : \mathcal{F}\sigma^{-1}x \to \mathcal{F}\sigma^{-1}y)$. Denote the resulting group homomorphism $Aut(\hat{Q}, \hat{I}) \longrightarrow Aut\mathbf{Z}\Delta_C$ by $\mathbf{\Theta}$. Now take some $\sigma \in Aut(\hat{Q}, \hat{I})$. Then the indecomposable projective $(\hat{Q}, \hat{I})$- modules satisfy $(\Theta\sigma)(\hat{Q}, \hat{I})(x, -) = (\hat{Q}, \hat{I})(x, \sigma^{-1}-) \cong (\hat{Q}, \hat{I})(\sigma x, -)$. Therefore σ, $\Phi\Theta\,\sigma \in Aut(\hat{Q}, \hat{I})$ act identically on $\hat{Q}_0$. But since the ideal $\hat{I}$ contains full commutativity, σ and $\Phi\Theta\,\sigma$ also act identically on the various morphism spaces, and thus $\sigma = \Phi\Theta\,\sigma = \sigma\,\Theta\Phi$. We conclude $\Theta\Phi = 1$, and altogether we see that $\Phi = \Theta^{-1}$ is an **isomorphism of groups**.

2.3 Considering the two classifications of the selfinjective algebras sketched in the preceeding section we now obtain the following **characterization of the standard algebras** among the r.f. selfinjective algebras:

Proposition. *Assume that A is a representation- finite selfinjective algebra. Then the following assertions are equivalent: (i) A is standard. (ii) A posesses a non-trivial covering. (iii) The universal covering of A exists. (iv) char $k \neq 2$ or (char $k = 2$ and $A \notin \Re$)*

In particular, all the selfinjective algebras of type E_6, E_7, E_8 are standard.

2.4 For further illustration of the above group isomorphism Φ we will show : *If considering $\Lambda_C \cong \hat{B}$ as a repetitive category then $\tau^{m_\Delta} = \nu$ coincides with the the Nakayama automorphism.*

In fact, observe that the two automorphisms τ^{m_Δ} , τ^{-m_Δ} are the only automorphisms $\psi \in Aut\mathbf{Z}\Delta_C$ s.t. the resulting residue category $\Lambda_C/\psi^{\mathbf{Z}}$ consists of $\#\Delta_0$ objects and all the diagonal entries of its Cartan matrix are nonequal to 1 (cf. 1.2). But $T(B) \cong \hat{B}/\nu^{\mathbf{Z}}$ posesses these two properties, and therefore $\nu = \Phi\,\tau^{\varepsilon\, m_\Delta}$ for some $\varepsilon = \pm 1$. Considering the Cartan invariants of Λ_C resp. $\hat{B}$ shows that τ^{m_Δ} corresponds to the Nakayama automorphism ν and that, in fact, $\nu = \Phi\,\tau^{m_\Delta}$.

In particular, the trivial extensions of type Δ coincide with the selfinjective algebras of type Δ having a stable AR- quiver $\cong \mathbf{Z}\Delta/\tau^{m_\Delta \mathbf{Z}}$, consequently this class of algebras is closed under passing to the opposite algebra.

Furthermore we conclude that for a given repetitive category $k(\hat{Q},\hat{I})$ of Dynkin type Δ (described as usual) the Nakayama automorphism ν is the *unique* automorphism ζ of $(\hat{Q},\hat{I})$ s.t. the associated residue category contains exactly $\#\Delta_0$ objects, its Cartan matrix has only diagonal entries $\neq 1$, and there is an object x of $\hat{B}$ satisfying $\hat{B}(x,\zeta x) \neq 0$.

2.5 Now it is clear how to carry over the results of Bretscher/ Riedtmann concerning $Aut(\mathbf{Z}\Delta_{\Psi k(\hat{Q},\hat{I})})$ to describe the infinite cyclic subgroups of $Aut(\hat{Q},\hat{I})$ directly. Completing the train of thought of the last two sections, we are arrive at the main result of this section (which, in fact, is a first version of the *Main Theorem* of this paper, for the types A_n, D_n cf. [Ro] 4.1).

Theorem. *Let $n = 6,7,8$. If $(\hat{Q},\hat{I})$ is the usual description of a repetitive category of type E_n, then the infinite cyclic subgroups of $Aut(\hat{Q},\hat{I})$ are given by $\nu^{s\mathbf{Z}}$, $s \in \mathbf{N}$, and, additionally, if in case $n = 6$ the associated configuration $\Psi k(\hat{Q},\hat{I})$ is symmetric, the groups $(\phi_{(\hat{Q},\hat{I})}\nu^s)^{\mathbf{Z}}$, $s \in \mathbf{N}$, where $\phi_{(\hat{Q},\hat{I})}$ denotes the unique involution in $Aut(\hat{Q},\hat{I})$.*
Let $\mathcal{Q}_{E_n}$ be a system of such bound quivers constituting a system of representatives of (the isoclasses of) the repetitive categories of type E_n.
Then the algebras associated to the following bound quivers form a **system of representatives of the isomorphism classes of the selfinjective algebras of type** *E_n :*

$$(\hat{Q},\hat{I})/\nu^{s\mathbf{Z}}, \ \ s \in \mathbf{N}, \ \ (\hat{Q},\hat{I}) \in \mathcal{Q}_{E_n}$$

and moreover in case $n = 6$:

$$(\hat{Q},\hat{I})/(\phi_{(\hat{Q},\hat{I})}\nu^s)^{\mathbf{Z}}, \ \ s \in \mathbf{N}, \ \ (\hat{Q},\hat{I}) \in \mathcal{Q}_{E_6} \ \text{ and } \ \Psi k(\hat{Q},\hat{I}) \ \text{ symmetric}$$

3. S - matrices and associated algebras

In this section we are going to construct a mathematical model making the construction of a system of representatives of the selfinjective algebras resp. the iterated tilted algebras of type E_6, E_7, E_8 perfectly accessable to the use of computers.

3.1 Let $n \in \mathbf{N}$. $C = (c_{ij})_{i,j} \in \mathcal{M}_n(\mathbf{Z})$ is called an $\mathcal{S} - matrix$, iff the following is satisfied: *(i)* All the entries of C are equal to 0 or 1. *(ii)* C is invertible and upper triangular. *(iii)* There is no permutation matrix P s.t.

PCP^T is decomposable (i.e. $PCP^T = \mathrm{C}_1 \oplus C_2$ in a non-trivial way).
We associate to the $\mathcal{S}$-Matrix C the following bound quiver $(\mathbf{Q_C}, \mathbf{I_C})$. Let $(Q_C)_0 := \{1, \ldots, n\}$, $(Q_C)_1$ is defined as follows: there is an arrow $i \to j$ iff $i \neq j$, $c_{ij} \neq 0$, and $c_{iz}c_{zj} = 0$ for all $z \neq i, j$. The ideal I_C in kQ_C is generated by the relations of the form: • full commutativity, • for each two vertices i, j s.t. $c_{ij} = 0$ all the paths $(i \rightsquigarrow j)$ are contained in I_C.
Let $\mathcal{A}(\mathbf{C}) := \mathbf{k}(\mathbf{Q_C}, \mathbf{I_C})$. The algebras arising in this form (for arbitrary $n \in \mathbf{N}$) will be called *$\mathcal{S}$-algebras.*

Observe the following obvious, but basic properties of $\mathcal{S}$- algebras :

Lemma 1 . *Let $C \in \mathcal{M}_n(\mathbf{Z})$ be an $\mathcal{S}$- matrix. Then I_C is contained in $(kQ_C^+)^2$, the algebra $\mathcal{A}(C)$ is connected and basic, its quiver Q_C does not contain oriented cycles, and its Cartan matrix is equal to C.*

Examples of $\mathcal{S}$-algebras are the hereditary tree algebras (especially all three representation types appear among the $\mathcal{S}$-algebras) and the simply connected algebras (1.1) .

3.2 Reflecting the uniqueness property of the Cartan matrix of a given algebra we define an equivalence relation "$\sim$" on the set of upper triangular square matrices with integer entries

$$M \sim M^\star :\Leftrightarrow \text{ there is some permutation matrix } P \text{ s.t. } P\, M\, P^T = M^\star.$$

Obviously the property of being an $\mathcal{S}$- matrix is compatible with this equivalence relation, and we conclude from Lemma 1

Proposition. *The map Ω : $C \mapsto A(C)$ being defined on the set of $\mathcal{S}$- matrices induces a bijection between the $\sim$- equivalence classes of $\mathcal{S}$-matrices and the isoclasses of $\mathcal{S}$-algebras.*

Corollary. *Let C be in $\mathcal{M}_n(\mathbf{Z})$, and assume that there is a simply connected algebra A s.t. C_A satisfies $C_A \sim C$. Then $\mathcal{A}(C) \cong A$, in particular $\mathcal{A}(C)$ is simply connected.*

3.3 In the following we will restrict our attention to certain subclasses of $\mathcal{S}$- matrices resp. $\mathcal{S}$- algebras. Accordingly, for the remainder of this section let Δ be some Dynkin graph, and let $n := \#\Delta_0$ its number of vertices.

An $\mathcal{S}$-matrix C is called *$\mathcal{S}_\Delta$-matrix*, if there is an algebra A whose trivial extension is (r.f.) of type Δ s.t. $C_A \sim C$. Therefore $C \in \mathcal{M}_n(\mathbf{Z})$, and thus the set of $\mathcal{S}_\Delta$-matrices is **finite** . An $\mathcal{S}$- algebra is called *$\mathcal{S}_\Delta$-algebra* , provided its Cartan matrix is an $\mathcal{S}_\Delta$-matrix. Since by 1.3 an algebra with r.f. trivial extension is simply connected, the above Corollary gives

Lemma 2 . *The $\mathcal{S}_\Delta$-algebras coincide with the algebras whose trivial extension is of type Δ .*

Thus the class of $\mathcal{S}_\Delta$-algebras is closed under passing to the opposite algebra, since $T(A^{op}) = T(A)^{op}$ for each algebra A and, as shown in 2.4 , the class of trivial extensions of type Δ has this closure property. In the same manner as above we get

Lemma 3 . *The assignment $\mathbf{\Omega_\Delta} : C \mapsto A(C)$ being defined on the set of $\mathcal{S}_\Delta$-matrices induces a bijection between the $\sim$- equivalence classes of $\mathcal{S}_\Delta$-matrices and the isoclasses of $\mathcal{S}_\Delta$-algebras.*

3.4 We will now introduce certain matrix manipulations being analogous to the Hughes/ Waschbuesch - reflections. In this way we will simply call them (matrix) *reflections*. These reflections will be the main tool in the forth-coming classification of the repetitive categories of type Δ in terms of $\mathcal{S}_\Delta$-matrices.

Assume $C = (c_{ij})_{i,j}$ to be an $\mathcal{S}_\Delta$-matrix. A row index $\kappa = 1, \ldots, n$ is called *sink in C* iff $c_{\kappa j} = 0$ for all $j = 1, \ldots, n$, $j \neq \kappa$. We observe immediately: *The sinks in C coincide with (the indices of) the sinks in the quiver of the algebra $\mathcal{A}(C)$.*

For a sink κ in C consider $x^{(\kappa)} := (c_{1,\kappa}, \ldots, c_{\kappa-1,\kappa}, c_{\kappa+1,\kappa}, \ldots, c_{n,\kappa}) \in \mathbf{Z}^{n-1}$, denote by $C^{(\kappa)}$ the matrix in $\mathcal{M}_{n-1}(\mathbf{Z})$ which arises from C by deleting row and column no. κ, and define

$$S_\kappa^- C := \begin{pmatrix} 1 & x^{(\kappa)} \\ 0 & C^{(\kappa)} \end{pmatrix} \in \mathcal{M}_n(\mathbf{Z})$$

3.5 Now consider some fixed $\mathcal{S}_\Delta$-algebra A. Assume that the vertices in Q_A are numbered by $1, \ldots, n$ and that the Cartan matrix C_A of A is given in this way. Let κ be some sink in Q_A , thus $B := S_\kappa^- A$ is defined. (Recall that the class of $\mathcal{S}_\Delta$-algebras is closed under all admissable reflections S_i^-.) It is easy to determine the Cartan matrix of B and in this way to verify the first fundamental property of the matrix reflections:

$$C_{S_\kappa^- A} \; = \; S_\kappa^- C_A$$

Especially, the class of $\mathcal{S}_\Delta$-matrices is closed under all admissable reflections S_j^- , and by virtue of Lemma 2 and the Corollary to Lemma 1 we arrive at the the second fundamental property of the matrix reflections :

$$(\diamond) \quad \mathcal{A}(S^-_\kappa C^\star) \;\cong\; S^-_\kappa \mathcal{A}(C^\star)$$

3.6 A sequence $(\kappa_1, \dots, \kappa_t)$, $t \in \mathbf{N_0}$, of integers in $\{1, \dots, n\}$ is called an *admissable sequence* of the $\mathcal{S}_\Delta$- matrix C iff , in case $t > 0$, κ_1 is a sink in C and for all $i = 2, \dots, t$ the index κ_i is a sink in $S^-_{\kappa_{i-1}} \dots S^-_{\kappa_1} C$, which, of course, means exactly that $S^-_{\kappa_t} \dots S^-_{\kappa_1} C$ is actually defined. An important property of such sequences consists in the fact that each matrix which is $\sim$-equivalent to C is already given by a suitable admissable sequence of C :

Lemma 4 . *Let C be an $\mathcal{S}_\Delta$-matrix. Assume that the matrix $C^\star$ satisfies $C \sim C^\star$. Then there is an admissable sequence $(\kappa_1, \dots, \kappa_t)$ of the matrix $C^\star$ s.t. $S^-_{\kappa_t} \dots S^-_{\kappa_1} C^\star = C$.*

Proof. Denote by S_n the group of permutations of the letters $1, \dots, n$. Consider the map $g : S_n \longrightarrow \mathbf{N_0}$, which associates each permutation $\sigma \in S_n$ its maximal "defect" , i.e. $g\sigma := max\{0, i \mid i = 1, \dots, n;\, \sigma i \neq i\}$. Assume C is an $\mathcal{S}_\Delta$-matrix, and let $(c_{ij})_{ij}$ be the entries of C. Given some $\pi \in S_n$, we call the integer sequence $(\pi 1 , \dots, \pi n)$ an *admissable sequence* of the row indices of C iff the matrix $(c_{\pi i, \pi j})_{ij}$ is upper triangular. Now define a map $h : \{C' \mid C' \sim C\} \longrightarrow \mathbf{N_0}$ as follows : Let C' be a matrix satisfying $C' \sim C$, denote by $X_{C'} \subseteq S_n$ the (non- empty) set of permutations $\pi \in S_n$ s.t. $(c_{\pi i, \pi j})_{ij} = C'$, and set $h\, C' := min\{g\sigma \mid \sigma \in X_{C'}\}$. The proof of the Lemma will go by induction on the values of the mapping h.
Now consider a matrix $C^\star$ which is in the same $\sim$- equivalence class as C. In case $h\, C^\star = 0$ we have $C^\star = C$, and the claim follows. Thus we may assume $h\, C^\star \neq 0$ and therefore $h\, C^\star \geq 2$. Choose $\vartheta \in X_{C^\star}$ s.t. $g\vartheta$ is minimal, and let $m := g\vartheta = h\, C^\star \geq 2$. Consider

$$\begin{array}{llll} (1, \dots, \text{m-1, m} , & \text{m+1} , \dots, \text{n}) & \text{original sequence} \\ (\;\underbrace{\vartheta 1, \dots, \vartheta m}_{\text{the integers } 1,\dots,m} \;, & \text{m+1} , \dots, \text{n}) & \text{admiss. sequence assoc. to } \vartheta \end{array}$$

Since ϑ permutes the indices $1, \dots, m$, we conclude from the upper triangular form of C that $c_{m, \vartheta j} = 0$ for all $j \in \{1, \dots, m\} \setminus \{\vartheta^{-1} m\}$. Consequently the index $\tilde{m}$ in $C' := (\prod_{j=m+1}^n S^-_n) C^\star$ which corresponds to the index m in $C^\star$ is a sink in C' , and $C^\diamondsuit := \left(\prod_{j=1}^{m-1} S^-_n\right) S^-_{\tilde{m}} \left(\prod_{j=m+1}^n S^-_n\right) \; C^\star$ is in fact defined. Iterated application of $(\diamond)$ shows that $\mathcal{A}(C^\star)$ and $\mathcal{A}(C^\diamondsuit)$ must be isomorphic,

because $\mathcal{A}(C^\diamondsuit)$ arises from $\mathcal{A}(C^\star)$ by shifting successively "all of its vertices". Consequently $C^\diamondsuit \sim C^\star$ and therefore $C^\diamondsuit \sim C$.
The above construction provides some $\eta \in S_n$ s.t. $C^\diamondsuit = (c_{\eta i, \eta j})_{ij}$ and

$$\begin{array}{rll} (1, \ldots, \text{m-1}, & \text{m}, \text{m+1}, \ldots, \text{n}) & \text{original sequence} \\ (\eta 1, \ldots, \eta(m-1), & \text{m}, \text{m+1}, \ldots, \text{n}) & \text{admiss. sequence assoc. to } \eta \end{array}$$

thus $g\eta \leq m-1 = g\vartheta - 1 = h\,C^\star - 1$ and therefore $h\,C^\diamondsuit < h\,C^\star$. Via induction on the value of h there is an admissable sequence $(\alpha_1, \ldots, \alpha_t)$ of the matrix $C^\diamondsuit$ s.t. $C = S^-_{\alpha_t} \ldots S^-_{\alpha_1} C^\diamondsuit$, and collecting the equalities we get $C = S^-_{\alpha_t} \ldots S^-_{\alpha_1} \left(\prod_{j=1}^{m-1} S^-_n\right) S^-_{\tilde{m}} \left(\prod_{j=m+1}^{n} S^-_n\right) C^\star$. □

3.7 Now we define a relation " $\simeq$ " on the set of $\mathcal{S}_\Delta$-matrices by the rule

C$^\star \simeq C$:$\Leftrightarrow$ there is an admissable sequence $(\kappa_1, \ldots, \kappa_t)$ of the matrix C s.t. C$^\star = S^-_{\kappa_t} \ldots S^-_{\kappa_1} C$

Lemma 5 . *The relation " $\simeq$ " is an* **equivalence relation**, *and it is finer than the equivalence relation " $\sim$ ".*

Proof. The second part of the claim (i.e. $C^\star \sim C$, then $C^\star \simeq C$) is exactly Lemma 4, thus it remains to show the symmetry of " $\simeq$ ".
Consider two $\mathcal{S}_\Delta$-matrices C, $C^\star$ satisfying $C^\star \simeq C$. Therefore there exists an admissable sequence $(\kappa_1, \ldots, \kappa_t)$ of the matrix C s.t. $C^\star = S^-_{\kappa_t} \ldots S^-_{\kappa_1} C$, thus $(\diamond)$ yields $\mathcal{A}(C^\star) \cong S^-_{\kappa_t} \ldots S^-_{\kappa_1} \mathcal{A}(C)$, and by means of Theorem 1 of 1.3 we arrive at $\widehat{\mathcal{A}(C^\star)} \cong \widehat{\mathcal{A}(C)}$, applying 1.3, Theorem 1 once more we have $\mathcal{A}(C) \cong S^-_{\alpha_r} \ldots S^-_{\alpha_1} \mathcal{A}(C^\star)$ for some admissable sequence $(\alpha_1, \ldots, \alpha_r)$ of negative ν-reflections of the quiver of $\mathcal{A}(C^\star)$. Now $(\diamond)$ yields $\mathcal{A}(S^-_{\alpha_r} \ldots S^-_{\alpha_1} C^\star) \cong S^-_{\alpha_r} \ldots S^-_{\alpha_1} \mathcal{A}(C^\star) \cong \mathcal{A}(C)$. By Lemma 3 we get $C \sim S^-_{\alpha_r} \ldots S^-_{\alpha_1} C^\star$, and by applying Lemma 4 we conclude $C \simeq C^\star$. □

3.8 Now we are in position to apply the results of the preceeding two sections to the actual setting.

First define an equivalence relation " $\bowtie$ " on the class of all k-algebras by setting $A \bowtie B$ iff $T(A) \cong T(B)$. Due to Theorem 1 of 1.3 the $\bowtie$- equivalence class of an $\mathcal{S}_\Delta$-algebra consists exactly of the algebras which result from A by an admissable sequence of ν- reflections. The following Theorem shows that something analogous holds for the Cartan matrix of A.

Main Theorem (Part 1). *Let Δ be a Dynkin graph, and let $\vec{\Delta}$ an arbitrary orientation of Δ. Then the following holds:*
The mapping $\Omega_\Delta \ : \ C \mapsto A(C)$ induces a bijection between the $\simeq$- equivalence classes of $\mathcal{S}_\Delta$- matrices and the $\bowtie$- equivalence classes of algebras with trivial extension of type Δ.
Furthermore for each of these $\simeq$- equivalence classes $\mathcal{K}$ we have:
(i) There is a tilted algebra $B_\mathcal{K}$ of $k\vec{\Delta}$ whose Cartan matrix lies in $\mathcal{K}$, consequently the algebras in $\Omega_\Delta\, \mathcal{K}$ are (up to isomorphism) exactly the algebras whose trivial extension is isomorphic to $T(B_\mathcal{K})$.
(ii) The restricted map $\Omega_\Delta \mid_\mathcal{K}$ induces a bijection between the $\sim$- equivalence classes of matrices in $\mathcal{K}$ and the isoclasses of algebras whose trivial extension is isomorphic to $T(B_\mathcal{K})$.

Proof. Assume $C, C^\star$ to be $\mathcal{S}_\Delta$- matrices. In case $C \simeq C^\star$ we conclude by $(\diamond)$ that $\mathcal{A}(C) \cong S^-_{\alpha_r} \dots S^-_{\alpha_1} \mathcal{A}(C^\star)$ for some admissable sequence $(\alpha_1, \dots, \alpha_r)$ of negative ν-reflections of the quiver of $\mathcal{A}(C^\star)$, and using 1.3, Theorem 1 we arrive at $T(\mathcal{A}(C)) \cong T(\mathcal{A}(C^\star))$, that is $\mathcal{A}(C) \bowtie \mathcal{A}(C^\star)$. Conversely, if $\mathcal{A}(C) \bowtie \mathcal{A}(C^\star)$, then Theorem 1 of 1.3 provides $\mathcal{A}(C) \cong S^-_{\beta_t} \dots S^-_{\beta_1} \mathcal{A}(C^\star)$ for some admissable sequence $(\beta_1, \dots, \beta_t)$. Iterated application of $(\diamond)$ implies $S^-_{\beta_t} \dots S^-_{\beta_1} \mathcal{A}(C^\star) \cong \mathcal{A}(S^-_{\beta_t} \dots S^-_{\beta_1} C^\star)$ and therefore $\mathcal{A}(C) \cong \mathcal{A}(S^-_{\beta_t} \dots S^-_{\beta_1} C^\star)$, thus by Lemma 3 we have $C \sim S^-_{\beta_t} \dots S^-_{\beta_1} C^\star$ and using Lemma 4 we end with $C \simeq C^\star$. The rest of the claim follows easily by the two Theorems of 1.3 and by Lemma 3. $\square$

3.9 Recall from 1.3, 1.1 the way how to construct for a given algebra B (with r.f. trivial extension) the associated repetitive category $\hat{B}$ by quiver and relations, just by the knowledge of the Cartan matrix of B. In this manner we can (easily) give for each $\mathcal{S}_\Delta$-Matrix $C \ \ (= C_{\mathcal{A}(C)})$ a description $(\hat{\mathbf{Q}}_\mathbf{C}, \hat{\mathbf{I}}_\mathbf{C})$ of the repetitive category $\widehat{\mathcal{A}(C)}$ by quiver and relations, and for sake of brevity we set $\mathcal{R}(\mathbf{C}) := k(\hat{Q}_C, \hat{I}_C) \ \ (\cong \widehat{\mathcal{A}(C)})$.

3.10 Now choose two systems of representatives $\mathcal{S}_{\mathbf{\Delta},\sim}$ resp. $\mathcal{S}_{\mathbf{\Delta},\simeq}$ of the $\sim$- resp. the $\simeq$- equivalence classes of $\mathcal{S}_\Delta$-matrices.

By the Main Theorem of [AHR] we conclude from Lemma 2, Lemma 3

Corollary. *Let Δ be a Dynkin graph. Then the (pairwise non-isomorphic) algebras $\mathcal{A}(C)\,, C \in \mathcal{S}_{\Delta,\sim}$, form a system of representatives of (the isoclasses of) iterated tilted algebras of type Δ.*

Using Theorem 1 of 1.3 we derive from the above Theorem that the (pairwise non-isomorphic) repetitive categories $\mathcal{R}(C)$, $C \in \mathcal{S}_{\Delta,\simeq}$, constitute a system of representatives of the repetitive categories of type Δ.

3.11 In the following we restrict our attention to the three exceptional cases $\mathbf{\Delta = E_n}$, $\mathbf{n = 6, 7, 8}$ (for the general case cf. [Ro] 6.1).

In order to construct from the above system of repetitive categories a system of representatives of all the selfinjective algebras of type E_n (of course, by use of the Theorem of the last section) , we need an effective criterion which in case $n = 6$ decides for a given $\mathcal{S}_{E_6}$-matrix C whether the associated configuration $\Psi\mathcal{R}(C)$ (see 2.1) is symmetric or not.

Lemma 6 . *Let C be an $\mathcal{S}_{E_6}$-matrix. Then the configuration $\Psi\mathcal{R}(C)$ of $\mathbf{Z}E_6$ is symmetric, iff* ($\sharp$) *: There is a matrix $C^\star = (c^\star_{ij})_{ij} \simeq C$ s.t. $(c^\star_{\pi i,\pi j})_{ij} = C$ for some $\pi \in \{(2,3),(2,3)(4,5),(2,3)(5,6)\} \subseteq S_6$.*
In case $\Psi\mathcal{R}(C)$ is symmetric, there exits a unique *involution in $Aut(\hat{Q}_C, \hat{I}_C)$, denoted by $\phi_{\mathbf{C}}$.*

Proof. The last assertion already appears in the Theorem of the last section, and by considering the actual construction $C \rightsquigarrow (\hat{Q}_C, \hat{I}_C)$ the implication from point 2 to point 1 becomes obvious.
Thus let C be an S_{E_6}-matrix s.t. $\Psi\mathcal{R}(C)$ is symmetric, and let ϕ_C be the unique involution in $Aut(\hat{Q}_C, \hat{I}_C)$. By passing to a suitable ν- slice R of $\hat{Q}_C$ ([HW] 1.6) we can realize the following: ϕ_C stabilizes each subset $\{(i,z) \mid i = 1,\ldots,6\}$, $z \in \mathbf{Z}$, of $(\hat{Q}_C)_0$ (where the vertices of $\hat{Q}_C$ are numbered according to this slice R, in the sense of [HW]) and induces an involution σ_z on it s.t. there is an involution $\sigma \in S_6$ satisfying $\sigma_z(i,z) = (\sigma i, z)$ for all i, z, and we can further assume that $\sigma 1 = 1$ and $\sigma 2 = 3$ and that the Cartan matrix of the algebra determined by R ([HW] 2.4) is upper triangular. Consequently, σ either fixes all the vertices $4, 5, 6$ (and we are done), or leaves exactly one of the vertices $4, 5, 6$ invariant. Then a suitable renumbering of two of the these three vertices which moreover keeps the upper triangularity of the Cartan matrix of B, where B denotes the algebra determined by R, achieves that σ moves the vertex 5. By construction C_B satisfies criterion ($\sharp$), and since $\hat{B} \cong \mathcal{R}(C)$, thus $T(B) \cong T(\mathcal{A}(C))$ (by Theorem 1 of 1.3), the $\mathcal{S}_{E_6}$-matrix C_B is $\simeq$- equivalent to C , by the above Theorem. $\square$

3.12 Using the results established in this section we derive from the Theorem of the last section

Main Theorem (Part2). *Let $n = 6, 7, 8$. Then the algebras associated to the subsequent bound quivers form a* **system of representatives of (the isomorphism classes of) the selfinjective algebras of type** E_n *:*

$$(\hat{Q}_C, \hat{I}_C)/\nu^{s\mathbf{Z}}, \;\; s \in \mathbf{N}, \;\; C \in \mathcal{S}_{E_n,\simeq}$$

and moreover in case of $n = 6$:

$$(\hat{Q}_C, \hat{I}_C)/(\phi_C \nu^s)^{\mathbf{Z}}, \;\; s \in \mathbf{N}, \;\; C \in \mathcal{S}_{E_6,\simeq} \text{ and } C \text{ satisfies criterion } (\sharp)$$

4. The computational realization

4.1 The considerations of the preceeding section lead directly to an algorithm ($\mathcal{A}$) which provides a system of representatives of the repetitive categories of type E_n, $n = 6, 7, 8$. It is easy to deduce from the computer output a system of representatives of all the selfinjective algebras of type E_n, by applying the Main Theorem, part 2, cf. the Appendix. Moreover, the previous section yields an algorithm ($\mathcal{B}$) which, by further exploitation of certain 'garbage' data arising in the former computation, provides a system of representatives of the iterated tilted algebras of type E_n, and the algebras in this system are grouped by the isoclass of their trivial extension resp. their repetitive category.
The two resulting Pascal programs are part of the program package *CREP*. For detailed information on them we refer to [Ro] 6.2.

4.2 Let us first describe the **algorithm** ($\mathcal{A}$). The first part of this algorithm consists in the computation of the basic data. For a certain orientation $\vec{E_n}$ the program computes the (preprojective) AR- quiver of $k\vec{E_n}$, and afterwards it computes the n-tupels of $\underline{dim}$-vectors which correspond to tilting modules of $k\vec{E_n}$ having a projective part $\neq 0$ (cf. [HV]), denote these tilting modules by $T_1, \ldots, T_{r_n}$. Clearly, up to $\sim$- equivalence all the Cartan matrices of tilted algebras of $k\vec{E_n}$ appear in the sequence of Cartan matrices resulting from the sequence $T_1, \ldots, T_{r_n}$.
In the second (and main) part of ($\mathcal{A}$) these Cartan matrices are classified according to $\simeq$:
(i) Compute the Cartan matrix C_1 of the tilted algebra $(End\, T_1)^{op}$ ([R] 4.2) and determine the $\simeq$- equivalence class $\mathcal{L}_1$ of C_1 (of course, recursively and in the obvious way), set $t := 1$, $i_1 := 1$.

(ii) Let $s = 1, \ldots, r_n$ s.t. the Cartan matrices $C_1, \ldots, C_s$ associated to $T_1, \ldots, T_s$ are already computed. Further let $(i_1, \ldots, i_t)$ be the indices s.t. the $\simeq$- equivalence classes $\mathcal{L}_{i_\alpha}$ of C_{i_α}, $\alpha = 1, \ldots, t$ have been determined where either $t = s = 1$ or $t \geq 2$ and $1 = i_1 < \ldots < i_t \leq s$.
(ii.a) Compute the Cartan matrix C_{s+1} of the tilted algebra $(End\, T_{s+1})^{op}$
(ii.b) Check whether $C_{s+1} \in \mathcal{L}_{i_\alpha}$ for some $\alpha = 1, \ldots, t$.
(ii.c) If not, determine the $\simeq$- equivalence class of C_{s+1} , set $t := t + 1$, then $i_t := s + 1$, and denote this $\simeq$- equivalence class by $\mathcal{L}_{i_t}$.

Observe that in the course of this classification **all** the $\simeq$- equivalence classes of $\mathcal{S}_{E_n}$- matrices, and especially all the $\mathcal{S}_{E_n}$- matrices, are determined. These will be the garbage data necessary for algorithm $(\mathcal{B})$.
Finally, pick from each of these equivalence classes an arbitrary matrix C and compute the description $(\hat{Q}_C, \hat{I}_C)$ of the resulting repetitive category (cf. 3.9) . If in case of $n = 6$ the actual equivalence class contains a matrix satisfying the invariance property in criterion $(\sharp)$, then choose such a matrix for the computation of $\hat{Q}_C$ (for the effect of this choice see Appendix).

4.3 Now we are going to describe **algorithm** $(\mathcal{B})$.
For $i = 1, \ldots, t_n$ (where t_n denotes the number of the above classes $\mathcal{L}_{i_\alpha}$) :
(i) Set $r := 1$ and $\mathcal{L}_i^0 := \mathcal{L}_i$.
(ii) Let $r \geq 1$. If $\mathcal{L}_i^{r-1} = \emptyset$ then pass to $i + 1$, else :
Choose $C_{i,r} =: (c_{\alpha,\beta}^{(i,r)})_{\alpha,\beta} \in \mathcal{L}_i^{r-1}$. Determine all $\sigma \in S_n$ s.t. $C_{i,r}^\sigma := (c_{\sigma\alpha,\sigma\beta}^{(i,r)})_{\alpha,\beta}$ is upper triangular, and delete all these $C_{i,r}^\sigma$ from $\mathcal{L}_i^{r-1}$, denote the resulting set of matrices by $\mathcal{L}_i^r$. Pass to $r + 1$.

This provides a sequence $C_{1,1}, \ldots, C_{1,d_1}, C_{2,1}, \ldots, C_{2,d_2}, \ \ldots\ , C_{t_n,1}, \ldots, C_{t_n,d_{t_n}}$ of $\mathcal{S}_{E_n}$-matrices (for some positive integers $d_1, \ldots, d_{t_n}$ depending on n) which, by Lemma 4 of the previous section, form a set of representatives of the $\sim$-equivalence classes of **all** the $\mathcal{S}_{E_n}$-matrices. By the Corollary to the Main Theorem, part1 the (pairwise non-isomorphic) $\mathcal{S}_{E_n}$-algebras
$\mathcal{A}(C_{1,1}), \ldots, \mathcal{A}(C_{1,d_1}), \mathcal{A}(C_{2,1}), \ldots, \mathcal{A}(C_{2,d_2}),\ \ldots\ , \mathcal{A}(C_{t_n,1}), \ldots, \mathcal{A}(C_{t_n,d_{t_n}})$
(of course, given by the easily computable data $(Q_{C_{i,\beta}}, I_{C_{i,\beta}})$) form a system of representatives of the iterated tilted algebras of type E_n.

4.4 Note that the algebras of this system are naturally grouped by the isoclass of their repetitive category resp. their trivial extension, namely, by the first index of the several matrices $C_{i,\beta}$.

5. A few results

In this final section we present a few results taken from the output of the two Pascal programs realizing the algorithms $(\mathcal{A})$, $(\mathcal{B})$ from the last section. Some of the complete output lists can be found in the Appendix.

5.1 Let us begin with the various numbers of isomorphism classes :

# isomorphism classes	E_6	E_7	E_8	altogether
repetitive categories	22	143	598	763
iterated tilted algebras	298	3341	21948	25587

Collecting the iterated tilted algebras of type E_n to *frames* (cf. [D]), we arrived at 86, 793, 4747 frames for $n = 6, 7, 8$. Illustrating the effort of the algorithms we remark that the number of $\mathcal{S}_{E_n}$- matrices is equal to 2598, 81004, 1579144 in case $n = 6, 7, 8$.

5.2 Closing this paper, we consider sincere directed algebras and in this context give a first application of the special grouping in the computer list of iterated tilted algebras of type E_n (cf. 4.4).

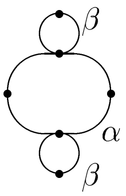
Recall that the sincere directed algebras coincide with the sincere r.f. tilted algebras, they are even simply connected ([R] 6.1). The sincere directed algebras are well-known, their classification due to Bongartz splits into the regular case, i.e. 24 infinite families of algebras containing all the sincere directed algebras having at least 14 vertices ([R] 6.3), and the exceptional case, that is 16344 (isoclasses of) algebras determined by Draexler ([D]).

5.3 We used a $CREP$ program computing the AR-quiver of a simply connected algebra to mark the sincere algebras in the computer list of iterated tilted algebras of type E_n. Ensuring known results we got 61, 425, 2026 sincere directed algebras of type E_6, E_7, E_8, with 20, 45, 48 regulars among them. Furthermore the program output reveals:

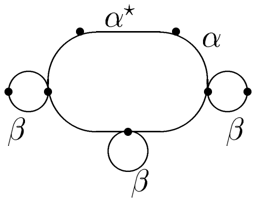
Remark. Let $n = 6, 7, 8$. Assume that B is a trivial extension of E_n. Then there is a *sincere* directed algebra A (of type E_n) s.t. $T(A) \cong B$.

Using the descriptions of the sincere directed algebras A_n, D_n (see e.g. [R] 6.3) and the descriptions of the trivial extensions of these types by Bretscher/Riedtmann ([BLR] 6.,7.) it easy to find a counterexample to the above assertion for type A_6 and D_9 (and in this way to refute this assertion for the types A_n, $n \geq 6$ resp. D_m, $m \geq 9$) and, moreover, to verify this assertion for type $A_1, \ldots, A_5$ resp. for type D_4.

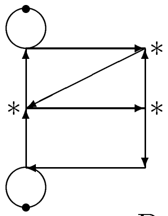
Using the three mentioned computer programs we checked the assertion for the $8, 24, 72$ trivial extensions of type D_5, D_6, D_7, and we found three counterexamples among the 231 trivial extensions of type D_8.
We present the several counterexamples below using the notation of [BLR]. (In the following Brauer quivers we left out the loops.)



type A_6



type D_9
two- cornered,
replacement at $\alpha^\star$



type D_8
three- cornered,
around stared triangle

Altogether we are now able to complete the above Remark to the

Theorem. *Let Δ be a Dynkin graph. Then each trivial extension of type Δ is isomorphic to the trivial extension of a sincere (directed) algebra, iff Δ is one of the graphs $E_6, E_7, E_8,\ A_1, \ldots, A_5,\ D_4, \ldots, D_7$*

Finally, we wish to remark that the types $D_4, D_5,\ E_6,\ E_7,\ E_8$ are the **only** Dynkin types having the property that each trivial extension of this type results from two non- isomorphic sincere (directed) algebras. While this observation is easily explained for type D_4, D_5 (namely, by self- duality), it still waits for an explanation for the types E_6, E_7, E_8.

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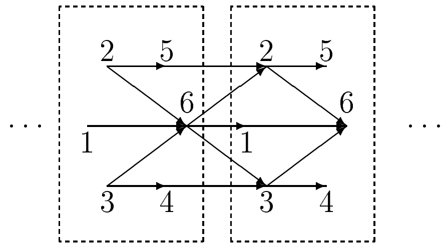
Appendix

The Appendix lists a system of representatives (of the isoclasses) of the repetitive categories (resp. the trivial extensions) of type E_6, E_7, E_8 and of the iterated tilted algebras of type E_6. (The list of iterated tilted algebras of type E_7,E_8 is part of [Ro].) Now we describe how to read these lists.

First example : a repetitive category of type E_6 , the expression

" 1.) ** 16 25 26 34 36 43 52 61 62 63 , k- dim. = 22 "

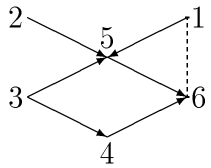
encodes the below repetitive category of type E_6 in the sense of 1.3 (e.g. 16 denotes $1 \to 6$), its associated bound quiver, as indicated by ** , allows an involution (whose structure is same as described the proof of 3., Lemma 6 , with $\sigma = (2,3)(4,5)$) , the k- dimension of the resulting trivial extension is equal to 22.
In the lists concerning type E_7,E_8 we printed just *one* algebra out of each pair A, A^{op}, the selfdual algebras are marked by an "s".



Second example : an iterated tilted algebra of type E_6 , the expression

" 068.) ** [30] 15 25 34 35 46 56 rel 16 , index : 7 , k- dim. = 14 "

encodes the below iterated tilted algebra of type E_6 which is, as indicated by ** , even sincere directed, its number of indecomposable modules is equal to 30, its k- dimension is equal to 14, and its repetitive category (resp. trivial extension) is, up to isomorphism, the algebra no. 7 in the related list of repetitive categories (resp. trivial extensions) of type E_6.
(We printed just *one* algebra from each pair A, A^{op}, the selfdual algebras are marked by an "s".)



, $(1 \rightsquigarrow 6) = 0$

The repetitive algebras / trivial extensions of type E_6 :

** = the assoc. configuration is symmetric

k- dimension = 22

01.) ** 16 25 26 34 36 43 52 61 62 63

k- dimension = 24

02.) 12 14 15 23 31 46 41 51 64

k- dimension = 26

03.) ** 14 26 35 45 46 51 53 61 62

04.) ** 14 24 34 45 46 41 53 62

05.) 12 13 15 24 26 34 41 51 62

06.) 12 15 23 24 36 31 41 51 63

k- dimension = 28

07.) 12 15 23 24 36 31 41 51 62

08.) 12 14 23 25 26 31 45 51 62

09.) 12 14 16 23 25 31 45 51 61

k- dimension = 30

10.) ** 14 24 34 45 46 51 53 61 62

11.) 12 14 16 23 35 45 51 61

12.) 12 23 25 34 41 56 51 65

13.) 12 14 23 25 36 31 45 51 62

14.) ** 12 13 24 34 45 46 41 53 62

15.) 12 16 23 25 34 41 51 61

k- dimension = 32

16.) 12 23 34 35 46 41 51 64

17.) 12 15 23 24 36 41 56 61

k- dimension = 34

18.) 12 14 23 25 31 45 56 61

19.) 12 15 23 34 36 41 56 61

20.) ** 12 13 25 34 46 56 61

k- dimension = 36

21.) 12 23 25 34 36 41 56 61

k- dimension = 38

22.) 12 23 34 36 45 51 61

The iterated tilted alg. of type E_6 :

s = selfdual

** = sincere directed

first datum = number of indec. modules

index: 1 , k- dimension = 11

001.) ** [36] 16 25 34 35 36

002.) [15] 16 23 25 34 36 rel 24 26

003.) [22] 16 23 25 26 34 rel 24

004.) [17] 16 23 35 36 45 rel 25 26

005.) [14] 16 23 34 36 45 rel 24 25 26 35

006.) [14] 16 23 34 45 46 rel 24 25 26 35 36

007.) [12] 12 23 34 35 46 rel 13 16 24 25 26 36

008.) [12] 12 23 24 35 46 rel 13 14 15 16 25 26

009.) [13] 12 16 23 25 34 rel 13 14 15 24

010.) [16] 12 14 16 23 45 rel 13 15

index: 2 , k- dimension = 12

011.) [23] 16 23 34 35 36 rel 24 26

012.) ** [36] 16 23 24 26 35

013.) [17] 16 23 34 36 45 rel 24 25 26

014.) [16] 16 24 34 45 46 rel 25 26 36

015.) [15] 16 23 34 45 46 rel 25 26 35 36

016.) [14] 14 23 34 45 46 rel 15 24 25 26 35 36

017.) [14] 12 23 34 35 46 rel 13 16 24 25 26

018.) [14] 12 23 24 35 46 rel 13 14 15 16 25

019.) [17] 12 23 25 26 34 rel 13 14 15 24

020.) [15] 12 16 23 24 35 rel 13 14 15

021.) [22] 12 14 15 23 46 rel 13

022.) [13] 12 23 34 35 46 rel 14 15 24 25 26 36

index: 3 , k- dimension = 13

023.) s [17] 16 23 25 34 46 rel 26

024.) ** [36] 16 23 35 36 45

025.) [22] 16 23 34 36 45 rel 25 35

026.) s [19] 14 23 24 35 36 46 rel 16 25

027.) [20] 12 14 23 26 45 46 rel 13 15

028.) [15] 12 16 23 34 45 rel 14 15 25 35

029.) [16] 12 23 24 35 46 rel 15 16 25 26

030.) s [13] 12 23 34 45 56 rel 13 25 26 36 46

index: 4 , k- dimension = 13

031.) [19] 14 24 34 45 46 rel 15 16 25 36

032.) [16] 14 23 34 45 46 rel 15 25 26 35 36

033.) [17] 12 23 24 35 46 rel 13 14 15 16

034.) [23] 12 23 24 26 35 rel 13 14 15

035.) [15] 12 23 34 35 46 rel 14 15 16 24 25 26

036.) ** [36] 12 13 14 26 35

index: 5 , k- dimension = 13

037.) ** [35] 16 23 34 35 36 rel 24

038.) ** [36] 16 24 34 35 46

039.) [23] 16 24 34 45 46 rel 25 26

040.) [17] 16 23 34 45 46 rel 25 26 35

041.) [18] 14 23 25 34 46 rel 16 26

042.) [26] 14 23 24 25 36 46 rel 16

043.) [17] 13 23 34 35 46 rel 14 15 16 26 36

044.) [22] 13 23 25 34 46 rel 16 26 36
045.) [17] 12 23 24 35 36 46 rel 13 14 15 16 25
046.) [14] 12 23 34 45 46 rel 13 15 16 25 26 35
047.) [21] 12 23 24 26 45 rel 13 15 25
048.) [22] 12 14 15 23 26 46 rel 13
049.) [14] 12 23 34 35 46 rel 14 15 16 25 26 36
050.) [16] 12 16 23 34 35 rel 14 15 24
051.) [17] 16 23 35 45 56 rel 26 46
052.) [22] 16 25 34 45 56 rel 35 36
053.) [17] 15 23 34 35 46 56 rel 16 24 25 26
054.) [15] 15 23 34 45 56 rel 16 24 25 26 36
055.) [15] 14 23 34 45 56 rel 15 16 25 26 36 46
056.) [16] 14 23 34 45 56 rel 16 24 25 26 36 46
index: 6 , the opposite algebras to index 5
index: 7 , k- dimension = 14
057.) ** [34] 16 23 24 35 36 46
058.) [24] 14 23 34 35 46 rel 25 26
059.) [21] 14 23 34 45 46 rel 15 24 25 26
060.) [23] 14 23 24 36 45 46 rel 15 25
061.) ** [35] 14 23 24 45 46 rel 15
062.) [18] 12 23 34 35 46 rel 13 14 15 16 26
063.) [16] 12 23 25 34 46 rel 14 15 16 26
064.) [24] 12 15 23 26 34 rel 14
065.) [19] 12 13 24 34 35 46 rel 15 16 26
066.) [21] 12 13 24 34 36 45 rel 15 25 35
067.) [21] 16 23 34 35 46 56 rel 24 26
068.) ** [30] 15 25 34 35 46 56 rel 16
069.) [19] 15 24 34 45 56 rel 25 26 36
070.) [18] 14 23 24 35 45 56 rel 15 16 26 36
071.) [15] 13 23 34 45 56 rel 14 15 16 25 26 36
072.) [16] 12 23 24 35 45 56 rel 13 26 36 46
index: 8 , the opposite algebras to index 7
index: 9 , k- dimension = 14
073.) [26] 14 24 34 45 46 rel 15 16 25
074.) [19] 14 23 34 45 46 rel 15 25 26 35
075.) [21] 12 23 24 35 36 46 rel 13 14 15 16
076.) [21] 12 23 24 25 36 46 rel 13 15 16
077.) [27] 12 23 24 26 45 rel 13 15
078.) [16] 12 23 34 35 46 rel 14 15 16 25 26
079.) [17] 12 13 24 34 45 46 rel 15 16 25 26 35
080.) ** [34] 12 13 14 26 35 36
index: 10 , k- dimension = 15
081.) ** [34] 14 24 34 45 46 rel 15 26
082.) [23] 13 23 34 36 45 rel 14 15 25
083.) [22] 12 23 24 35 36 46 rel 13 15 16
084.) [21] 12 23 24 35 46 rel 15 16
085.) ** [35] 12 13 25 26 34 36
086.) [25] 15 23 24 35 45 56 rel 26 46
087.) ** [30] 14 15 23 25 36 46 56
088.) [18] 13 23 34 35 46 56 rel 14 16 25 26
089.) [17] 12 13 24 34 45 56 rel 15 16 25 26 36
index: 11 , k- dimension = 15
090.) [20] 14 23 34 45 46 rel 15 16 25
091.) [23] 13 23 34 35 46 rel 14 15 16
092.) [19] 13 23 34 45 46 rel 15 16 25 35
093.) ** [35] 12 23 24 25 46 rel 13
094.) [17] 12 23 34 35 46 rel 14 15 16 25
095.) [23] 12 23 34 35 36 rel 14 15 24
096.) [16] 12 23 34 45 46 rel 15 16 25 26 35
097.) ** [28] 12 13 15 26 34 46
098.) ** [36] 16 25 34 45 56
099.) [24] 15 24 34 45 56 rel 16 26
100.) [18] 15 23 34 45 56 rel 16 26 36
101.) [17] 14 23 34 45 56 rel 15 16 25 26
102.) [22] 14 23 34 45 56 rel 16 26 36 46
103.) [19] 12 23 24 35 46 56 rel 13 14 15 16
index: 12 , k- dimension = 15
104.) ** [36] 16 23 34 36 45
105.) [24] 16 23 34 45 46 rel 26
106.) [18] 12 23 34 36 45 rel 14 15 25 35
107.) [17] 12 16 23 34 45 rel 14 15
108.) [22] 12 13 25 26 34 46 rel 15
109.) [22] 12 23 24 35 46 rel 15 25
110.) [15] 12 23 34 45 56 rel 13 14 15 16 25 26
index: 13 , k- dimension = 15
111.) s ** [31] 14 23 34 35 46 rel 26
112.) s ** [35] 14 23 24 35 36 46
113.) ** [30] 14 23 24 35 45 46 rel 15
114.) [19] 12 23 26 34 45 rel 14 15 25
115.) [24] 12 14 23 26 35 46 rel 15
116.) [22] 12 13 24 26 34 45 rel 15 25
117.) s [17] 12 23 24 35 45 56 rel 13 26 46
118.) s [20] 12 14 23 25 36 45 56 rel 13 16 46
119.) s [15] 12 23 34 45 56 rel 14 25 26 36
index: 14 , the opposite algebras to index 10
index: 15 , the opposite algebras to index 11
index: 16 , k- dimension = 16
120.) ** [36] 16 23 34 45 46

121.) [26] 12 13 24 34 35 46 rel 15
122.) [22] 12 23 24 35 36 46 rel 15 25
123.) [22] 12 23 34 36 45 rel 15 25 35
124.) [19] 12 16 23 34 45 rel 15
125.) [16] 12 23 34 45 56 rel 13 14 15 16 26
index: 17 , k- dimension = 16
126.) [27] 14 23 34 45 46 rel 15 25
127.) [19] 12 23 34 45 46 rel 14 15 16 25
128.) ** [31] 12 23 25 34 36 rel 14
129.) [20] 12 23 34 35 46 rel 15 16 26
130.) ** [32] 12 13 26 34 35 46
131.) [20] 12 23 24 36 45 56 rel 13 15 16
index: 18 , k- dimension = 17
132.) ** [35] 14 23 34 45 46 rel 15
133.) [26] 12 23 24 35 36 46 rel 15
134.) [24] 12 23 34 36 45 rel 15 25
135.) ** [34] 12 13 24 34 35 46
136.) [25] 15 23 34 45 56 rel 26
137.) ** [32] 14 23 24 36 45 56
138.) [25] 13 23 34 35 46 56 rel 14 16
139.) [21] 12 23 24 35 45 56 rel 13 15 16
140.) [17] 12 23 34 45 56 rel 15 16 26 36
141.) [21] 12 14 23 35 45 56 rel 16 46
index: 19 , the opposite algebras to index 18
index: 20 , k- dimension = 17
142.) s [23] 13 23 34 45 46 rel 15 26
143.) [24] 12 23 34 36 45 rel 14 15
144.) ** [36] 12 23 24 35 46
145.) s [17] 12 23 34 45 56 rel 15 16 25 26
146.) s ** [25] 12 13 25 34 46 56
index: 21 , k- dimension = 18
147.) s ** [31] 13 23 34 45 46 rel 15
148.) [28] 12 23 34 36 45 rel 15
149.) ** [34] 12 23 24 35 36 46
150.) [22] 12 23 34 35 46 56 rel 14 16
151.) s [18] 12 23 34 45 56 rel 15 16 26
152.) s ** [29] 12 13 25 34 35 46 56
index: 22 , k- dimension = 19
153.) [25] 12 23 34 45 46 rel 15
154.) ** [36] 12 23 34 35 46
155.) [19] 12 23 34 45 56 rel 15 16
156.) ** [28] 12 23 24 35 46 56

The repetitive algebras /
trivial extensions of type E_7 :

s = selfdual

k- dimension = 26
01.) s 12 13 14 25 21 36 31 41 57 52 63 75
k- dimension = 28
02.) s 12 14 15 23 31 46 41 51 67 64 76
03.) 12 14 15 23 26 31 47 41 51 62 74
04.) s 12 14 23 31 45 46 41 57 54 64 75
k- dimension = 30
05.) 12 14 16 23 37 31 45 51 61 73
06.) 12 13 15 24 26 34 41 57 51 62 75
07.) 12 15 23 24 36 31 41 51 67 63 76
08.) s 12 14 15 23 31 46 41 51 67 74
09.) s 12 23 24 35 31 46 41 57 53 64 75
10.) s 12 15 23 24 26 37 31 41 51 62 73
k- dimension = 32
11.) 12 15 23 24 36 31 41 51 67 62 76
12.) 12 15 23 24 36 31 41 51 67 73
13.) 12 15 23 24 36 31 47 41 51 62 74
14.) 12 14 23 25 26 31 45 51 67 62 76
15.) 12 14 15 23 31 46 47 56 61 74
16.) s 12 15 16 23 34 41 57 51 61 75
17.) s 12 14 23 36 31 45 51 67 62 76
18.) 12 14 16 23 25 37 31 45 51 61 73
19.) s 12 14 16 23 25 31 45 51 67 61 76
20.) 12 14 16 23 25 27 31 45 51 61 72
k- dimension = 34
21.) 12 13 15 24 26 34 37 41 56 61 73
22.) 12 13 15 24 26 34 47 41 56 61 74
23.) s 12 16 23 25 27 34 41 51 61 72
24.) s 12 15 17 23 34 41 56 61 71
25.) 12 23 25 34 46 41 57 51 64 75
26.) s 12 14 16 23 25 31 45 51 67 71
27.) s 12 14 16 23 25 31 45 57 51 61 72
28.) 12 14 23 25 36 37 31 45 51 62 73
29.) 12 16 23 25 34 41 51 67 61 76
30.) 12 14 15 23 31 46 56 67 61 74
31.) 12 14 16 23 27 35 45 51 61 72
32.) 12 14 23 25 36 31 45 51 67 62 76
33.) 12 16 23 25 34 37 41 51 61 73
34.) s 12 23 25 34 41 56 51 67 65 76
k- dimension = 36

35.) 12 14 23 25 37 31 45 46 51 61 72
36.) s 12 14 16 23 25 31 45 47 51 67 71
37.) s 12 16 23 24 35 37 45 51 61 73
38.) 12 16 23 25 34 41 51 67 71
39.) 12 14 23 25 31 45 46 57 51 61 72
40.) 12 16 23 25 34 41 57 51 61 72
41.) 12 14 23 35 45 46 47 51 61 74
42.) s 12 23 34 35 46 41 51 67 64 76
43.) 12 15 17 23 26 34 41 56 61 71
44.) 12 15 23 24 36 41 56 57 61 75
45.) s 12 15 23 24 36 37 41 56 61 73
46.) 12 13 16 24 27 34 45 51 61 72
47.) s 12 23 25 34 41 56 51 67 75
48.) 12 15 23 24 27 36 41 56 61 72
49.) s 12 14 23 25 36 31 45 51 67 62 73
k- dimension = 38
50.) 12 23 25 34 37 41 56 61 73
51.) 12 15 17 23 34 36 41 56 61 71
52.) 12 15 23 34 36 41 56 57 61 75
53.) 12 15 23 26 34 41 56 67 61 75
54.) 12 15 23 34 36 37 41 56 61 73
55.) s 12 15 23 34 47 41 56 61 72
56.) 12 15 23 34 36 47 41 56 61 74
57.) 12 15 23 24 36 47 41 56 61 72
58.) 12 15 23 26 34 41 56 57 61 71
59.) 12 14 16 23 27 35 45 51 67 71
60.) 12 16 23 24 35 45 57 51 61 73
k- dimension = 40
61.) 12 15 23 34 36 41 56 57 61 71
62.) 12 23 25 34 36 37 41 56 61 73
63.) 12 15 23 26 34 47 41 56 61 72
64.) s 12 23 26 34 45 51 67 61 76
65.) 12 15 17 23 34 46 56 61 71
66.) 12 17 23 25 34 36 41 56 61 71
67.) s 12 15 23 24 36 41 56 67 61 72
68.) s 12 15 23 34 36 47 41 56 61 73
69.) 12 23 25 34 36 41 56 57 61 75
70.) 12 16 23 25 34 41 57 67 71
k- dimension = 42
71.) 12 14 17 23 35 45 56 61 71
72.) 12 16 23 24 35 37 45 51 67 71
73.) s 12 16 23 25 34 47 51 67 71
74.) s 12 17 23 25 34 46 56 61 71
75.) 12 15 23 26 34 41 56 67 71
76.) 12 23 34 36 45 57 51 61 75
77.) s 12 23 25 34 36 41 56 67 61 73
k- dimension = 44
78.) s 12 15 23 26 34 37 41 56 67 71
79.) 12 16 23 34 35 47 51 67 71
80.) s 12 23 34 45 46 57 51 61 75
81.) 12 16 23 34 37 45 51 67 71
82.) s 12 23 26 34 45 51 67 71
k- dimension = 46
83.) 12 23 25 34 46 56 57 61 71
84.) 12 16 23 34 45 47 51 67 71
k- dimension = 48
85.) 12 23 25 34 36 41 56 67 71
k- dimension = 50
86.) s 12 23 34 37 45 56 61 71
87.) s 12 23 34 36 45 47 51 67 71
k- dimension = 52
88.) s 12 23 34 45 47 56 61 71

The repetitive algebras / trivial extensions of type E_8 :

s = selfdual

k- dimension = 30
001.) s 12 13 14 25 21 36 31 41 57 52 63 78 75 87
k- dimension = 32
002.) s 12 14 23 31 45 41 56 57 54 68 65 75 86
003.) s 12 14 15 23 31 46 41 51 67 64 78 76 87
004.) 12 14 15 23 26 31 47 41 51 68 62 74 86
005.) 12 14 23 25 31 46 47 41 52 68 64 74 86
006.) s 12 14 15 23 26 37 31 48 41 51 62 73 84
k- dimension = 34
007.) 12 14 16 23 37 31 45 51 61 78 73 87
008.) 12 13 15 24 26 34 41 57 51 62 78 75 87
009.) 12 14 15 23 31 46 41 51 67 78 74 87
010.) 12 15 23 24 36 31 41 51 67 63 78 76 87
011.) s 12 14 15 23 31 46 41 51 67 64 78 86
012.) 12 14 15 23 26 31 48 41 51 67 72 84
013.) s 12 23 24 35 31 46 41 57 53 64 78 75 87
014.) 12 15 23 24 26 37 31 41 58 51 62 73 85
015.) s 12 14 16 23 27 38 31 45 51 61 72 83
k- dimension = 36

016.) 12 14 16 23 27 31 45 51 61 78 82
017.) 12 13 15 24 26 34 41 57 51 62 78 85
018.) 12 15 23 24 36 31 41 57 51 62 78 75 87
019.) 12 15 23 24 36 31 41 51 67 63 78 86
020.) 12 15 23 24 36 31 41 51 67 68 73 86
021.) 12 14 23 36 31 45 47 51 68 62 74 86
022.) 12 15 23 24 36 31 41 51 67 62 78 76 87
023.) 12 15 23 24 36 31 41 51 67 78 73 87
024.) 12 15 16 23 27 34 41 58 51 61 72 85
025.) 12 15 23 24 36 31 47 41 51 68 62 74 86
026.) s 12 23 24 35 31 46 41 57 53 64 78 85
027.) s 12 15 23 34 41 56 57 51 68 65 75 86
028.) 12 14 23 25 26 37 31 45 51 62 78 73 87
029.) 12 15 23 24 37 31 41 56 58 61 73 85
030.) 12 14 16 23 25 37 31 45 51 61 78 73 87
031.) s 12 14 16 23 25 31 45 51 67 61 78 76 87
032.) 12 14 16 23 25 27 31 45 51 61 78 72 87
033.) s 12 15 16 23 34 37 41 58 51 61 73 85
034.) 12 14 15 23 27 31 46 48 56 61 72 84
035.) 12 15 23 24 27 38 31 41 56 61 72 83
036.) 12 14 16 23 25 27 38 31 45 51 61 72 83
037.) 12 14 16 23 25 31 45 57 51 68 61 75 86
038.) s 12 14 16 23 25 37 31 45 48 51 61 73 84

k- dimension = 38

039.) 12 14 16 23 25 31 45 51 67 78 71 87
040.) 12 14 16 23 25 38 31 45 57 51 61 72 83
041.) 12 13 15 24 26 34 37 41 56 68 61 73 86
042.) 12 16 23 25 27 34 41 51 61 78 72 87
043.) 12 14 23 25 37 31 45 46 51 61 78 73 87
044.) 12 15 17 23 34 48 41 56 61 71 84
045.) 12 14 23 25 36 31 45 47 51 62 78 74 87
046.) s 12 15 17 23 34 38 41 56 61 71 83
047.) 12 23 25 34 46 41 57 51 68 64 75 86
048.) 12 14 23 25 36 37 31 45 51 62 78 73 87
049.) 12 15 23 24 36 31 47 41 51 62 78 84
050.) 12 14 16 23 27 35 45 51 61 78 72 87
051.) 12 16 23 25 34 41 51 67 61 78 76 87
052.) 12 14 23 25 26 31 45 51 67 62 78 86
053.) 12 14 15 23 37 31 46 56 68 61 73 84
054.) 12 14 15 23 27 31 46 56 68 61 72 84
055.) 12 16 23 25 34 37 48 41 51 61 73 84
056.) 12 14 23 25 26 31 45 47 51 62 78 84
057.) 12 15 23 24 28 37 31 41 56 61 72 82
058.) 12 14 16 23 25 31 45 58 51 67 71 85
059.) 12 16 23 25 34 37 41 51 61 78 73 87
060.) s 12 23 25 34 36 47 41 58 51 63 74 85
061.) 12 15 23 24 36 31 41 51 67 62 78 73 87
062.) s 12 23 25 34 41 56 51 67 65 78 76 87
063.) 12 14 16 23 25 37 31 45 51 61 78 83
064.) s 12 14 16 23 25 31 45 51 67 61 78 86
065.) s 12 15 23 34 41 56 58 51 67 75 85
066.) 12 13 15 24 26 34 47 41 56 61 78 74 87
067.) s 12 14 16 23 25 31 45 57 51 61 78 72 87
068.) s 12 15 23 24 36 31 41 51 67 68 62 73 86

k- dimension = 40

069.) 12 14 16 23 35 45 51 67 68 71 86
070.) 12 15 16 23 34 41 57 58 67 71 85
071.) 12 16 23 25 28 34 41 51 67 71 82
072.) 12 16 23 25 34 37 41 51 61 78 83
073.) 12 15 23 24 37 31 41 56 61 78 72 83
074.) 12 15 23 26 27 34 41 56 58 61 72 85
075.) 12 13 15 24 26 34 47 41 56 61 78 72 87
076.) 12 23 25 34 41 56 51 67 68 75 86
077.) s 12 15 23 34 41 56 67 61 78 75 87
078.) 12 15 23 24 36 41 56 57 61 78 75 87
079.) s 12 15 23 24 36 37 41 56 61 78 73 87
080.) 12 14 23 25 36 31 45 47 51 62 78 84
081.) 12 16 23 34 35 47 41 51 68 61 74 86
082.) 12 15 23 26 34 37 41 56 68 61 73 86
083.) 12 15 23 34 37 41 56 61 78 72 87
084.) 12 15 17 23 26 28 34 41 56 61 71 82
085.) s 12 23 25 34 41 56 51 67 65 78 86
086.) 12 14 17 23 35 45 46 51 68 61 71 86
087.) 12 13 15 24 26 34 47 41 56 68 61 72 86
088.) 12 16 23 25 34 37 48 41 51 61 72 84
089.) 12 14 23 25 28 31 45 46 57 51 61 72 82
090.) 12 15 23 24 27 36 41 56 58 61 72 85
091.) 12 13 15 24 26 34 47 41 56 61 78 73 87
092.) 12 14 16 23 25 31 45 47 51 67 68 71 86
093.) 12 14 23 25 38 31 45 46 57 51 61 72 83
094.) 12 13 15 24 26 34 37 41 56 61 78 83
095.) 12 15 23 24 27 38 31 41 56 61 78 82
096.) 12 15 23 34 41 56 67 68 61 75 86
097.) 12 15 23 26 34 47 41 56 68 61 74 86
098.) s 12 23 34 35 46 41 51 67 64 78 76 87
099.) 12 15 23 34 47 48 41 56 61 73 84
100.) 12 15 23 26 34 41 56 57 68 61 75 86
101.) 12 15 23 26 27 34 38 41 56 61 72 83

102.) s 12 16 23 24 35 37 45 51 68 61 73 86
103.) 12 16 23 25 34 41 51 67 68 71 86
104.) 12 15 23 26 27 34 48 41 56 61 72 84
105.) 12 15 17 23 26 34 48 41 56 61 71 84
106.) 12 15 17 23 26 34 38 41 56 61 71 83
107.) 12 14 23 25 37 31 45 46 51 68 61 72 86
108.) 12 16 23 25 34 37 41 51 68 61 72 86
109.) 12 16 23 24 27 35 38 45 51 61 72 83
110.) s 12 15 23 24 36 47 41 56 58 61 74 85
111.) 12 14 23 25 37 31 45 46 48 51 61 72 84
112.) 12 14 16 23 25 28 31 45 47 51 67 71 82
113.) s 12 16 23 34 35 37 48 41 51 61 73 84
114.) 12 14 16 23 35 45 51 67 61 78 86
115.) 12 14 23 25 36 31 45 48 51 67 62 73 84
k- dimension $= 42$
116.) 12 14 16 23 35 38 45 47 51 67 71 83
117.) 12 15 16 23 34 41 57 67 78 71 85
118.) 12 15 23 26 34 48 41 56 57 61 71 84
119.) 12 23 25 34 47 41 56 61 78 74 87
120.) 12 15 17 23 34 36 38 41 56 61 71 83
121.) 12 15 23 34 36 41 56 57 61 78 75 87
122.) 12 14 23 35 45 46 51 67 68 61 74 86
123.) 12 15 23 26 34 41 56 57 58 61 71 85
124.) 12 15 23 34 36 47 41 56 61 78 74 87
125.) 12 15 23 24 36 38 47 41 56 61 72 83
126.) 12 16 23 24 27 35 45 58 51 61 72 83
127.) 12 14 16 23 27 35 45 51 67 68 71 86
128.) 12 15 23 26 34 37 41 56 68 61 73 85
129.) s 12 15 17 23 34 41 56 58 61 78 81
130.) 12 15 23 34 36 37 41 56 61 78 73 87
131.) 12 15 23 24 36 47 41 56 61 78 84
132.) 12 16 23 25 34 38 41 51 67 71 82
133.) 12 14 17 23 35 45 46 58 51 61 71 84
134.) s 12 15 23 24 36 37 41 56 61 78 83
135.) 12 16 23 34 35 48 41 51 67 71 84
136.) 12 15 17 23 34 36 41 56 61 78 71 87
137.) 12 14 23 25 37 31 45 48 56 61 73 84
138.) s 12 23 25 34 37 48 41 56 61 73 84
139.) 12 15 17 23 34 36 48 41 56 61 71 84
140.) 12 15 23 24 36 38 41 56 57 61 71 83
141.) 12 15 23 26 34 47 41 56 68 61 74 85
142.) 12 14 23 25 37 31 45 46 58 51 61 72 84
143.) 12 14 16 23 25 31 45 47 58 51 67 71 82
144.) 12 15 17 23 28 34 36 41 56 61 71 82
145.) 12 15 23 34 47 41 56 68 61 72 86
146.) 12 15 23 26 34 37 41 56 58 61 72 85
147.) 12 15 23 24 28 36 41 56 57 61 71 82
148.) s 12 23 34 35 46 41 51 67 64 78 86
149.) 12 14 23 25 37 31 45 46 51 61 78 72 83
150.) s 12 14 23 25 27 38 31 45 46 51 61 78 82
151.) 12 16 23 25 34 37 41 51 61 78 72 83
152.) s 12 15 23 26 34 41 56 67 68 61 72 86
153.) s 12 14 23 25 36 31 45 58 51 67 62 73 84
154.) s 12 16 17 23 34 45 51 68 61 71 86
155.) 12 16 23 24 35 37 45 51 68 61 72 86
156.) 12 15 23 24 36 41 56 57 58 61 71 85
157.) s 12 16 23 24 35 38 45 51 67 71 83
k- dimension $= 44$
158.) 12 15 23 26 34 41 56 57 61 78 71 85
159.) 12 15 23 26 34 41 56 67 61 78 72 86
160.) 12 15 17 23 28 34 46 56 61 71 82
161.) 12 16 23 34 35 47 41 51 61 78 73 84
162.) 12 16 23 24 35 37 45 58 51 61 78 83
163.) 12 14 15 23 31 46 56 67 68 71 84
164.) 12 15 17 23 26 34 41 56 58 61 78 81
165.) 12 15 23 34 36 38 41 56 57 61 71 83
166.) 12 14 23 25 37 31 45 56 61 78 72 87
167.) 12 15 23 26 34 41 56 57 68 61 71 82
168.) 12 16 23 25 34 41 51 67 78 81
169.) 12 14 17 23 25 31 45 56 61 78 81
170.) 12 15 23 24 37 48 41 56 67 71 84
171.) 12 16 23 24 35 45 58 51 67 71 83
172.) s 12 15 18 23 27 34 41 56 67 71 81
173.) 12 23 25 34 47 41 56 61 78 84
174.) 12 15 17 23 34 38 46 56 61 71 83
175.) 12 15 23 34 36 47 41 56 61 78 84
176.) 12 15 23 24 36 41 56 57 68 61 71 85
177.) 12 15 23 34 36 41 56 57 61 78 85
178.) 12 17 23 25 34 36 41 56 61 78 71 87
179.) 12 17 23 26 34 38 45 51 61 71 83
180.) 12 23 25 34 36 41 56 67 61 78 76 87
181.) 12 15 23 24 36 41 56 58 67 61 72 85
182.) 12 15 23 24 37 41 56 58 67 71 85
183.) 12 15 23 28 34 36 41 56 57 61 71 82
184.) 12 15 23 27 34 41 56 67 78 71 87
185.) s 12 15 23 26 34 37 41 56 61 78 72 83
186.) s 12 16 23 24 35 37 45 58 51 61 72 84
187.) 12 17 23 25 34 36 38 41 56 61 71 83

188.) 12 15 17 23 34 36 41 56 68 61 71 83
189.) 12 23 25 34 36 37 48 41 56 61 73 84
190.) 12 15 23 24 36 48 41 56 57 61 71 82
191.) 12 15 23 24 28 37 41 56 67 71 82
192.) 12 14 17 23 25 28 36 45 56 61 71 82
193.) 12 15 23 24 36 47 41 56 61 78 72 84
194.) 12 14 23 35 45 46 47 51 68 61 78 84
195.) 12 15 23 34 36 47 41 56 61 78 73 87
196.) s 12 23 25 34 36 47 41 56 58 61 74 85
197.) 12 15 23 24 37 41 56 67 68 71 86
198.) s 12 17 23 25 28 34 36 41 56 61 71 82
199.) s 12 15 23 24 36 41 56 67 61 78 72 87
200.) 12 14 16 23 35 45 47 58 51 67 71 83
201.) 12 13 16 24 27 34 45 51 67 68 71 86
202.) 12 23 25 34 36 47 41 56 61 78 74 87
203.) s 12 16 18 23 34 45 51 67 71 81
204.) 12 15 23 26 34 47 41 56 68 61 72 86
205.) s 12 15 23 34 41 56 67 61 78 85
206.) s 12 23 26 34 45 47 51 68 61 74 86
207.) 12 23 26 34 45 57 51 68 61 75 86
208.) 12 14 23 35 45 46 51 67 61 78 74 86
209.) s 12 14 16 23 27 35 45 48 51 67 71 81

k- dimension = 46

210.) 12 15 23 34 36 41 56 57 61 78 71 85
211.) 12 23 25 34 36 48 41 56 67 61 73 84
212.) 12 15 23 34 36 41 56 57 68 61 71 85
213.) s 12 17 23 34 36 38 45 51 61 71 83
214.) s 12 17 23 25 34 46 56 61 78 71 87
215.) 12 16 23 25 28 34 47 51 67 71 82
216.) 12 16 23 27 34 45 51 67 78 71 87
217.) 12 16 23 24 35 37 45 51 67 68 71 86
218.) s 12 15 23 34 48 41 56 67 71 82
219.) 12 23 25 34 36 41 56 57 61 78 85
220.) 12 16 23 24 35 37 45 48 51 67 71 84
221.) 12 15 23 34 36 47 41 56 61 78 73 84
222.) 12 16 23 24 35 37 45 58 51 67 71 85
223.) 12 14 15 23 31 46 56 67 78 71 84
224.) 12 17 23 26 34 45 51 68 61 71 82
225.) 12 15 23 26 34 41 56 58 67 71 85
226.) 12 15 23 34 37 41 56 67 68 71 86
227.) 12 15 23 34 36 48 41 56 57 61 71 83
228.) 12 15 17 23 34 36 41 56 58 61 78 81
229.) s 12 23 34 36 45 47 58 51 61 74 85
230.) 12 15 23 27 34 38 41 56 67 71 82
231.) 12 17 23 28 34 36 45 51 61 71 82
232.) 12 14 17 23 25 36 45 56 61 78 81
233.) 12 14 17 23 35 45 56 61 78 71 87
234.) 12 15 23 24 36 41 56 57 61 78 81
235.) 12 16 18 23 27 34 45 51 67 71 81
236.) 12 16 23 25 34 38 47 51 67 71 83
237.) 12 15 23 26 34 48 41 56 67 71 84
238.) 12 23 34 36 45 57 51 61 78 75 87
239.) s 12 23 25 34 36 41 56 67 61 78 73 87
240.) 12 17 23 25 34 36 41 56 58 61 71 82
241.) s 12 16 23 25 34 38 41 57 67 71 82

k- dimension = 48

242.) 12 15 23 26 34 37 41 56 58 67 71 85
243.) 12 15 23 26 34 37 41 56 67 68 71 86
244.) 12 16 23 27 34 45 51 67 68 71 81
245.) s 12 16 23 25 34 41 58 67 78 81
246.) s 12 23 26 34 45 48 51 67 71 84
247.) 12 14 23 35 38 45 47 56 61 71 83
248.) 12 15 23 27 34 41 56 67 78 71 85
249.) 12 15 23 34 46 56 57 61 78 71 85
250.) 12 15 23 26 34 41 56 67 68 71 82
251.) 12 14 23 35 45 47 56 61 78 71 87
252.) 12 17 23 34 36 45 48 51 61 71 83
253.) 12 13 17 24 28 34 45 56 61 71 82
254.) 12 15 23 34 36 41 56 57 68 61 71 82
255.) s 12 17 23 25 34 46 56 61 78 81
256.) 12 16 23 34 35 47 51 67 68 71 86
257.) 12 17 23 24 35 38 45 56 61 71 83
258.) 12 15 23 34 37 41 56 58 67 71 81
259.) 12 16 23 24 35 37 45 51 67 68 71 81
260.) 12 15 23 26 34 41 56 67 68 71 85
261.) 12 16 23 24 35 37 45 58 51 67 71 83
262.) 12 16 23 34 45 58 51 67 71 83
263.) 12 16 23 27 34 38 45 51 67 71 82
264.) 12 16 23 34 35 47 48 51 67 71 84
265.) 12 14 23 25 36 45 47 56 61 78 71 84
266.) 12 14 17 23 35 45 56 61 78 81
267.) 12 17 23 25 34 46 56 68 61 71 84
268.) 12 14 16 23 35 45 47 51 67 78 81
269.) 12 14 23 28 35 45 47 56 61 71 82
270.) 12 16 18 23 34 37 45 51 67 71 81
271.) 12 23 26 34 45 58 51 67 71 85
272.) 12 16 23 25 34 47 51 67 68 71 81
273.) s 12 15 18 23 26 34 37 41 56 67 71 81

k- dimension = 50
274.) 12 23 25 34 46 56 57 68 61 71 86
275.) s 12 15 23 26 34 41 56 67 78 71 82
276.) 12 23 26 34 37 45 58 51 67 71 85
277.) 12 16 23 24 35 38 45 51 67 78 81
278.) 12 16 23 34 35 47 58 51 67 71 83
279.) 12 16 23 34 35 47 51 67 68 71 81
280.) 12 23 26 34 37 45 48 51 67 71 84
281.) 12 23 25 34 46 56 57 61 78 71 87
282.) 12 15 23 24 36 48 41 56 67 71 82
283.) 12 16 23 34 45 47 48 51 67 71 84
284.) 12 14 18 23 25 31 45 56 67 71 81
285.) 12 17 23 26 34 45 51 68 78 81
286.) s 12 15 23 26 34 48 41 56 67 71 82
287.) 12 15 23 26 34 37 41 56 67 68 71 82
288.) s 12 16 23 34 37 45 51 67 78 71 83
289.) 12 17 23 34 35 46 56 68 61 71 84
290.) 12 17 23 24 35 45 56 58 61 71 83
k- dimension = 52
291.) 12 23 25 34 36 41 56 67 68 71 86
292.) 12 18 23 26 34 45 47 51 67 71 81
293.) 12 16 23 34 38 45 51 67 78 81
294.) s 12 23 27 34 45 56 61 78 71 87
295.) 12 23 26 34 37 45 51 67 78 71 83
296.) 12 15 23 24 36 41 56 57 68 78 81
297.) 12 23 25 34 36 41 56 67 78 71 87
298.) 12 16 23 27 34 45 51 67 78 81
299.) 12 18 23 27 34 45 56 61 71 81
300.) 12 23 26 34 45 47 58 51 67 71 85
301.) 12 17 23 25 34 46 56 58 61 78 81
302.) s 12 16 23 34 45 47 58 51 67 71 84
303.) 12 16 23 34 35 48 51 67 78 81
304.) 12 18 23 25 34 36 41 56 67 71 81
305.) 12 16 23 34 45 47 51 67 68 71 81
306.) 12 16 23 27 34 45 58 51 67 71 82
k- dimension = 54
307.) 12 15 23 27 34 38 41 56 67 78 81
308.) s 12 18 23 25 34 36 47 56 67 71 81
309.) 12 15 23 34 36 47 56 58 67 71 81
310.) s 12 23 34 37 45 56 58 61 71 85
311.) 12 23 34 37 45 48 56 61 71 84
312.) 12 23 25 34 36 41 56 67 68 71 83
313.) 12 23 34 36 45 47 51 67 78 71 87
314.) 12 14 18 23 25 36 45 56 67 71 81
315.) s 12 17 23 26 34 45 58 61 78 81
k- dimension = 56
316.) 12 16 23 27 34 45 48 51 67 78 81
317.) 12 18 23 34 45 47 56 61 71 81
318.) 12 15 23 34 46 56 58 67 71 81
319.) 12 14 23 25 36 45 48 56 67 71 81
320.) s 12 23 26 34 38 45 51 67 78 81
321.) 12 17 23 34 36 45 58 61 78 81
322.) 12 18 23 34 36 45 57 67 71 81
323.) s 12 23 27 34 45 56 61 78 81
k- dimension = 58
324.) 12 23 26 34 45 57 67 68 71 81
325.) 12 15 23 24 36 41 56 67 78 81
326.) s 12 23 34 45 56 57 68 61 71 86
k- dimension = 60
327.) 12 23 27 34 45 48 56 61 78 81
328.) 12 14 23 25 31 45 56 67 78 81
329.) s 12 23 26 34 37 45 48 51 67 78 81
k- dimension = 62
330.) 12 23 25 34 36 41 56 67 78 81
k- dimension = 64
331.) 12 23 34 37 45 56 58 61 78 81
332.) s 12 23 34 38 45 56 67 71 81
k- dimension = 66
333.) s 12 23 34 45 47 56 58 61 78 81
k- dimension = 68
334.) s 12 23 34 45 56 58 67 71 81