Questions about tame algebras

Collected by Peter Dräxler

Introduction

This is a collection of questions about tame algebras which emerged from (formal and informal) discussions with colleagues during the last 2 years after the workshop on *The Notion of Tameness* in Bielefeld, May of 1993. The presentation is an extended version of the one used by the author for a problem session at the Oberwolfach conference about *Darstellungstheorie endlich-dimensionaler Algebren* in July of 1995. In particular some sections are added which due to lack of time could not be presented in the session.

Since this paper is only meant as a hopefully helpful list of problems for interested people working close to the area, known results and definitions are only provided if necessary and no completeness is intended. In the references we include a list of survey papers on tameness or various related aspects which a reader interested in more details and literature should use.

Although the author is indebted and very grateful to many colleagues for communicating the listed problems to him, no problem will be related to a specific person. One reason is that the author does not always remember who posed which problem for the first time. The other is that the author may have misunderstood questions and does not want to embarrass others for his errors in the formulations arising in this way.

The questions were collected concentrating on representations of algebras. In particular, we looked for problems related to the definition of tameness and for parallels to the well-understood finite type case. Following this concept, we will only touch or even omit some important topics like selfinjective algebras (see [Sk3]), the infinite radical (see [Sk2]), modules over orders (see [Di]), representations of bocses and subspace categories of vector space categories (see [CB1], [Si]). Moreover, by restricting to an algebraically closed base field all problems related to a definition of tameness for arbitrary fields or even artin algebras are not touched at all.

1 The definition of tameness

Let us fix the algebraically closed field k and a finite-dimensional basic k-algebra A which we may suppose to be given as $A = k[\Delta]/\langle\rho\rangle$ were Δ is a finite quiver and ρ is a (finite) set of admissible relations.

For a given dimension vector $d = (d_x) \in \mathbb{N}_0^{\Delta_0}$ we denote by $(\Delta, R)-\mathrm{rep}_d$ the set $\prod_{\alpha: x\to y\in\Delta_1} R^{d_y\times d_x}$ where $R^{d_y\times d_x}$ is the set of matrices with d_y rows and d_x columns over another (possibly infinite-dimensional) k-algebra R. The subset of $(\Delta, R)-\mathrm{rep}_d$ given by all tuples of matrices M such that $M(r) = 0$ for each $r \in \rho$ will be called $(A, R)-\mathrm{rep}_d$. Obviously any element of $(A, R)-\mathrm{rep}$ can be considered as a A-R-bimodule which is free of rank $\sum d_x$ as right R-module.

Note that $(A, k)-\mathrm{rep}_d$ is an algebraic subset of the linear variety $(\Delta, k)-\mathrm{rep}_d$ on which the algebraic group $Gl(d) = \prod_{x\in\Delta_0} Gl(d_x)$ acts by conjugation. On the other hand A-k-bimodules are just left A-modules and consequently $(A, k)-\mathrm{rep}_d$ can be identified with the set $A-\mathrm{mod}_d$ of left A-modules of dimension vector d. Via this identification the isomorphism classes correspond to the $Gl(d)$-orbits.

The algebra A is said to be tame provided that for any dimension vector $d \in \mathbb{N}_0^{\Delta_0}$ there exists a finite sequence of non-zero polynomials $f_1^{(d)}, \ldots, f_\nu^{(d)}$ together with a finite sequence $B_1^{(d)}, \ldots, B_\nu^{(d)}$ of bimodules $B_i^{(d)} \in (A, k[X, (f_i^{(d)})^{-1}])-\mathrm{rep}_d$ with the following three properties:

(i) $B_i^{(d)} \otimes - : k[X, (f_i^{(d)})^{-1}]-\mathrm{mod} \to A-\mathrm{mod}$ preserves indecomposability and isoclasses.

(ii) For any dimension vector $d' \in \mathbb{N}_0^{\Delta_0}$ there are only finitely many isoclasses of indecomposable modules in $A-\mathrm{mod}_{d'}$ which have empty intersection with the image of each of the functors $B_i^{(d)} \otimes -$.

(iii) For $(d, i) \neq (d', i')$ the images of the functors $B_i^{(d)} \otimes -$ and $B_{i'}^{(d')} \otimes -$ do not contain non-zero isomorphic modules.

Note that the indecomposable modules in the image of the functor $B_i^{(d)} \otimes -$ are just the modules $B_i^{(d)} \otimes k[X]/(X-\lambda)^n \cong B_i^{(d)}(J(n,\lambda))$ where $\lambda \in k$ is not a zero of $f_i^{(d)}$ and $J(n, \lambda)$ is the Jordan block matrix of size $n \in \mathbb{N}$.

On the other hand A is said to be wild if there exists $B \in (A, k\langle X, Y\rangle)-\mathrm{rep}$ such that the functor $B\otimes - : k\langle X, Y\rangle-\mathrm{mod} \to A-\mathrm{mod}$ preserves indecomposability and isoclasses. A is called strictly wild if this functor is actually full.

The fundamental theorem on tameness proved first by Drozd [Dd] (see also [CB1]) says that any finite-dimensional algebra A over an algebraically closed field k is either tame or wild but not both. Note that the original definition of tameness is weaker. To get the above definition of tameness one also has to invoke [CB2].

The proofs of the tame and wild dichotomy in [Dd] and [CB1] rest on a reduction to bocses, a very general class of matrix problems. [G-V] presents an alternative access based on a reduction to vector space categories, a much smaller class of one-sided matrix problems. This also allows to characterize tameness and wildness by involving only linear bimodules. To be more precise:

- In the definition of tameness one may require that the entries of the matrices appearing in the $B_i^{(d)}$ are not arbitrary elements of the ring $k[X,(f_i^{(d)})^{-1}]$ but linear polynomials.
- Similarly in the definition of wildness one may require that the entries of the matrices appearing in B are not arbitrary elements of $k\langle X,Y\rangle$ but again linear polynomials.

Questions

Q1.1 It is shown in [CB1] that for a tame algebra and a fixed dimension vector d up to finitely many exceptions almost all isoclasses of indecomposables of dimension vector d consist of modules which are stable under the Auslander-Reiten translation (and hence lie in homogeneous tubes).

Can this theorem also be proved using the access from [G-V]? Is the converse also true, i.e. does an algebra where almost all indecomposables of dimension d lie in homogeneous tubes have to be tame?

Q1.2 Which polynomials $f_i^{(d)}$ are really needed if we consider a linear bimodule $B_i^{(d)} \in (A, k[X,(f_i^{(d))^{-1}}])-\mathrm{rep}_d$? Can the $B_i^{(d)}$ be arranged such that the $B_i^{(d)}(J(n,\lambda))$ are all indecomposable for all $\lambda \in k$? Can one even always insert an indecomposable module at ∞ (like e.g. for the Kronecker algebra)?

Q1.3 For the study of tameness in [CB2] generic modules are introduced which are just indecomposable infinite-dimensional modules which are of finite length over their endomorphism rings. Given a generic module M over a tame algebra it is shown that the radical factor algebra of the endomorphism algebra is the function field $k(X)$ and that the radical splits. Consequently M may be considered as a A-$k(X)$-bimodule. It is explained in [CB2] how in this fashion the bimodules $B_i^{(d)}$ can be constructed as realizations of the generic modules.

What does the existence of linear bimodules following from [G-V] say about generic modules?

Q1.4 The generic modules belong to the Ziegler spectrum of an algebra which as set consists of the isomorphism classes of pure-injective modules.

What is the Ziegler spectrum of a tame algebra?

Q1.5 For a strictly wild algebra any other finite-dimensional algebra appears as endomorphism algebra of some module.

What are the possible endomorphism algebras of modules over tame algebras? Can they be used to characterize tameness?

2 The growth function

The number d of bimodules $B_i^{(d)}$ appearing in the above definition of tameness obviously defines a function $\nu : \mathbb{N}_0^{\Delta_0} \to \mathbb{N}_0$. To pass from dimension vectors to dimensions we consider the function $\bar{\nu} : \mathbb{N} \to \mathbb{N}_0$ defined by $\bar{\nu}(D') = \sum \nu(d)$ where we sum over all dimension vectors d such that $\sum d_x = D'$. The growth function for the tame algebra A which reflects the number of bimodules one really needs to parametrize almost all indecomposables of dimension D is then given as $\mu : \mathbb{N} \to \mathbb{N}_0$, $D \mapsto \sum_{D'|D} \bar{\nu}(D')$.

Consequently A is said to be of constant resp. linear resp. polynomial growth if there exist constants $C, n \in \mathbb{N}$ such that the growth function satisfies $\mu(D) \leq C$ resp. $\mu(D) \leq CD$ resp. $\mu(D) \leq CD^n$ for all $D \in \mathbb{N}$. The algebras of constant growth are called domestic.

Questions

Q2.1 Using the methods from [G-V] Brüstle has shown that for a tame algebra A always $\mu(D) \leq (\dim A)e^{4(D4)^{2D}}$ holds (see [Be]).

What is the optimal bound? Which kind of growth behavior does occur? Does polynomial growth imply linear growth?

Q2.2 It is known that there exists a function $F : \mathbb{N} \to \mathbb{N}$ such that a given algebra is of finite type if there are only finitely many isoclasses of indecomposable A-modules of dimension at most $F(\dim A)$.

Is there an analogous function $T : \mathbb{N} \to \mathbb{N}$ for tame type (i.e. A is tame provided there does not occur any 2-parametric family of indecomposables in $A-\mathrm{mod}_d$ for all d satisfying $D = \sum d_x \leq T(\dim A)$)?

Q2.3 The algebras closest to the representation finite algebras should be the domestic ones.

Can the domestic algebras be classified? How do their module categories look like?

Q2.4 The module category of any finite-dimensional algebra can be embedded into the module category of a wild algebra via an exact functor which preserves indecomposability and isoclasses (see [Br]).

Which module categories can be embedded in an exact way into the module category of a tame, say non-domestic algebra?

3 Tubular type and the Auslander-Reiten quiver

As stated before for a tame algebra almost all indecomposables of a fixed dimension vector lie in homogeneous tubes. In particular this happens for almost all modules $B_i^{(d)}(J(n,\lambda))$ where $B_i^{(d)} \in (A, k[X, (f_i^{(d)})^{-1}])-\mathrm{rep}_d$ is a bimodule as in the definition of tameness.

In lot of examples of tame algebras like the tame concealed, tubular and coil algebras one has more informations about this behavior namely that besides the preprojective and preinjective ones all other indecomposables lie in families of tubes. The information about the non-homogeneous tubes is stored in the so-called extension or tubular type. This is by definition a finite graph which is a star and the lengths of the arms are just the ranks of the homogeneous tubes. It turns out the occurring graphs are always Dynkin or Euclidian.

Questions

Q3.1 How can one define tubular type in general? What is its relation to generic modules?

Q3.2 What can one say about the indecomposables which do not occur in any of the images of the functors $B_i^{(d)} \otimes -$? What is known about the indecomposables X such that $X \ncong \tau X$? How many of them do exist for fixed dimension vector d?

Q3.3 What kind of stable components are possible for a tame algebra besides tubes? Is $\mathbb{Z}\mathbb{A}_\infty$ possible? Of which kind there may be infinitely many components for a fixed algebra? Are there more possibilities than tubes (i.e. $\mathbb{Z}\mathbb{A}_\infty/(\tau^{q})$), $\mathbb{Z}\mathbb{A}_\infty^\infty$, $\mathbb{Z}\mathbb{D}_\infty$?

Q3.4 It is known that the middle term of an Auslander-Reiten sequence over a representation finite algebra can decompose in at most four indecomposable direct summands (see [BB]).

How many summands may occur for a tame algebra?

Q3.5 Does tameness imply $\limsup_{n\to\infty}(\dim_k \tau^n X)^{\frac{1}{n}} \leq 1$ for all indecomposables X?

4 Coverings and deformations

For finite representation type there is an algorithm (see [BGRS]) to check if a given algebra A has this property: One only has to consider the case that A has a finite ideal lattice and consequently the standard form $\bar{A}$ is defined which is a degeneration of A in the variety of algebras of a fixed dimension. $\bar{A}$ is the k-linear hull of a ray-category P which can be assumed zigzag free. For such a P there is an interval finite (in particular directed) universal Galois covering $\tilde{P}$. The point is that in order to prove that A is of finite type it suffices to prove that $\tilde{P}$ is locally representation finite. To investigate $\tilde{P}$ there were developed powerful tools like critical algebras, quadratic forms and vector space categories.

Questions

Q4.1 There are several results allowing to push tameness down from certain Galois coverings (see [DS], [DO]).

Is there a similar algorithm to decide tameness like for finite type by passing to a suitable 'standard' degeneration and then to an 'appropriate' covering where the above mentioned tools can be applied effectively? What is the right notion of 'standard' and of 'appropriate'?

Q4.2 In [Ga] it is shown that to be of finite type is an open condition. Concerning tameness for the time being it is only known that deformations of tame algebras are tame (see [Ge]).

Is tameness also an open condition?

Q4.3 Deformations were used to show the tameness of the quaternion algebras and other algebras in the lists of candidates for tame algebras in [Er].

Are all algebras in this lists actually tame?

Q4.4 The tameness proof via deformations does not give the information how the module category of the deformed algebra looks like.

What does the degeneration of an algebra do to the module category? Is every degeneration of a tame non-polynomial growth algebra again not of polynomial growth? What is the module category of a quaternion algebra?

5 Strongly simply connected algebras

An algebra with a directed quiver is called strongly simply connected provided that every convex subalgebra is simply connected. By [Sk4] the following properties are equivalent for such an algebra A:

a) A is tame of polynomial growth.

b) A is tame of linear growth.

c) A does not contain a pg-critical (see [NS]) or hypercritical algebra (see [Un]) as a convex full subcategory.

d) A is a multicoil algebra.

For schurian strongly simply connected algebras A which do not contain a pg-critical convex subcategory the above description yields that also the following properties are equivalent:

i) A is tame.

ii) The Tits form q_A is weakly non-negative (see [Pe1]).

iii) A does not contain a hypercritical algebra as a convex subcategory.

iv) The posets of thin start modules for A are tame (see [DN]).

(Note that the equivalence of i),ii),iii) is also true without the the ‘schurian’ assumption.)

Questions

Q5.1 What is the role of the strongly simply connected algebras? Are they the ’appropriate coverings’ of ’standard’ forms for tame algebras (like the simply connected, zigzag free algebras are for finite type)?

Q5.2 Are the properties i)-iv) equivalent for arbitrary (schurian) strongly simply connected algebras?

Q5.3 By [PS] a strongly simply connected sincere tame algebra which is not of polynomial growth must not contain a tubular or representation infinite tilted algebra of type $\mathbb{E}_p$ as a convex subcategory. On the other hand these algebras seem to be rather typical examples of polynomial growth tame algebras.

Does such a disjointness of polynomial and non-polynomial growth hold in more generality?

References

[BB] R. Bautista, S. Brenner, *Replication number for non-Dynkin sectional subgraphs in finite Auslander-Reiten quivers and some properties of Weyl roots,* Proc. London Math. Soc. 47 (1983), 429-462.

[Br] S. Brenner, *Decomposition properties of some small diagrams of modules,* Symposia Math. Inst. Naz. Alta Mat. 13 (1974), 127-141.

[Be] T. Brüstle, *On the growth function of a tame algebra,* to appear in C. R. Acad. Sci. Paris.

[CB1] W.W. Crawley-Boevey, *On tame algebras and bocses,* Proc. London Math. Soc. 56 (1988), 451-483.

[CB2] W.W. Crawley-Boevey, *Tame algebras and generic modules,* Proc. London Math. Soc. 63 (1991), 241-265.

[DS] P. Dowbor, A. Skowroński, *Galois coverings of representation infinite algebras,* Comment. Math. Helv. 62 (1987), 311-337.

[DN] P. Dräxler, R. Nörenberg, *Thin start modules and tame representation type,* CMS Conference Proceedings 14 (1993), 149-163.

[Dd] Ju.A. Drozd, *Tame and wild matrix problems,* Lecture Notes in Math. 832 (1980), 242-258.

[Di] E. Dieterich, *Tame orders,* in :Topics in Algebra, Banach Center Publications 26, Part I, (1990), 233-261.

[DO] Ju.A. Drozd, S.A. Ovsienko, *Coverings of tame algebras and boxes,* talk at the Oberwolfach conference 'Darstellungstheorie endlichdimensionaler Algebren', July 23-29, 1995.

[Er] K. Erdmann, *Blocks of tame representation type and related topic,* Lecture Notes in Math. 1428 (1980).

[Ga] P. Gabriel, *Finite representation type is open,* Lecture Notes in Math. 488 (1975), 132-155.

[GP] P. Gabriel, J.A. de la Peña, *On algebras, wild and tame*, in: Duration and change, fifty years at Oberwolfach, Springer (1994), 177-210.

[BGRS] R. Bautista, P. Gabriel, A.V. Roiter, L. Salmerón, *Representation-finite algebras and multiplicative bases,* Invent. Math. 81 (1985), 217-285.

[GR] P. Gabriel, A.V. Roiter *Representations of finite-dimensional algebras,* Encyclopaedia of Math. Sci. 73, Algebra VIII, Berlin, Heidelberg, New York (1992).

[G-V] P. Gabriel, L.A. Nazarova, A.V. Roiter, V.V. Sergejchuk, D. Vossieck, *Tame and wild subspace problems,* Ukr. Math. J. 45 (1993), 313-352.

[Ge] Ch. Geiß , *On degenerations of tame and wild algebras,* Arch. Math. 64 (1995), 11-16.

[NS] R. Nörenberg, A. Skowroński, *Tame minimal non-polynomial growth strongly simply connected algebras,* Preprint.

[Pe1] J.A. de la Peña, *Algebras with hypercritical Tits form,* in: Topics in Algebra, Banach Center Publ. 26, part I (1990), 353-369.

[Pe2] J.A. de la Peña, *The Tits form of a tame algebra,* Preprint.

[PS] J.A. de la Peña, A. Skowroński, *Forbidden subalgebras of non-polynomial growth tame simply connected algebras,* To appear Bull. Canadian Math. Soc.

[Ri1] C.M. Ringel, *Tame algebras,* in Representation Theory I, Lecture Notes in Math. 831 (1980), 134-287.

[Ri2] C.M. Ringel, *Tame algebras and integral quadratic forms,* Lecture Notes in Math. 1099 (1984).

[Si] D. Simson, Linear representations of partially ordered sets and vector space categories, Algebra, Logic and Applications Series 4, Amsterdam, 1992.

[Sk1] A. Skowroński, *Algebras of polynomial growth,* in :Topics in Algebra, Banach Center Publications 26, Part I, (1990), 535-568.

[Sk2] A. Skowroński, *Cycles in module categories*, in: Finite Dimensional Algebras and Related Topics, ed.: V. Dlab, L.L Scott, Dordrecht/Boston/London (1994), 309-345.

[Sk3] A. Skowroński, *Module categories over tame algebras*, Preprint.

[Sk4] A. Skowroński, *Tame algebras with simply connected Galois coverings*, Preprint.

[Un] L. Unger, *The concealed algebras of the minimal wild hereditary algebras,* Bayreuther Math. Schr. 31 (1990), 145-154.

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