Stable homotopy of algebraic theories

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0 Introduction

The original motivation for this paper came from the attempt to understand how a rational result about the stable homotopy theory of commutative rings might generalize to arbitrary characteristic. We showed in [Sch, 3.2] that for a commutative $\mathbb{Q}$-algebra B, the stable homotopy theory of augmented commutative simplicial B-algebras is equivalent to the homotopy theory of simplicial B-modules. This theorem does not hold, as it stands, for general B. However, the stable homotopy category of commutative simplicial B-algebras is still a category of modules if one allows brave new rings rather than simplicial rings, i.e., if one works with a suitable notion of homotopy ring.

It turned out that the right notion to consider here is that of a 'Functor with Smash Product' (FSP), introduced in [Bö]. There is currently no Quillen model category structure available for modules over an FSP, hence we chose to work with a variant of this notion, which we call DFSPs. DFSPs are based on a symmetric monoidal smash product for Γ-spaces with good homotopical properties ([Se],[BF],[Ly]). They are 'FSPs defined on finite sets' and they represent all homotopy types of connective FSPs. We give model category structures for modules over a fixed DFSP, for the whole category of DFSPs and more generally for algebras over a commutative DFSP. If $H\mathbb{Z}$ denotes the (commutative DFSP model of the) Eilenberg-MacLane ring spectrum of the integers, we show that the homotopy theory of $H\mathbb{Z}$-algebras is equivalent to that of simplicial rings.

The generalization of the rational result mentioned at the beginning then reads:

Theorem: Let B be a commutative ring, possibly simplicial. Then the stable homotopy theory of augmented commutative simplicial B-algebras is equivalent to the homotopy theory of modules over a certain DFSP DB. The graded ring of homotopy groups of DB is isomorphic to the ring of stable homotopy operations in the category of commutative augmented simplicial B-algebras. If B is a $\mathbb{Q}$-algebra, this DB is weakly equivalent to (the DFSP model of) B.

We give more information about the homotopy type of DB in section 3. The theorem is appropriately dealt with in a more general framework. We are led to consider pointed simplicial algebraic theories, or just simplicial theories for short. A simplicial theory T has a category of algebras; definitions will be given later, but thinking of a T-algebra as a simplicial set with certain algebraic operations and equational relations is a good guide line. Examples include simplicial sets, simplicial sets with an action of a simplicial group, (commutative) simplicial groups, modules over a simplicial ring, augmented (commutative) algebras over a (commutative) simplicial ring, and many more. The category of T-algebras is naturally a closed simplicial model category. There are further structural properties that the homotopy theory of T-algebras shares with the ordinary homotopy theory of spaces. These include for example Blakers-Massey, Hurewicz and Whitehead theorems. The category of spectra of T-algebras can be formed and also forms a closed simplicial model category. Its homotopy theory will be referred to as the stable homotopy theory of T. The above theorem then becomes a special case of

Theorem: To a simplicial theory T, there is functorially associated a DFSP T^s. The graded ring of homotopy groups of T^s is isomorphic to the ring of stable homotopy operations in the category of T-algebras. The stable homotopy theory of T-algebras is equivalent to the homotopy theory of modules over T^s.

We can classify linear functors (see section 5.1) in this framework and prove a chain rule which expresses the derivative of a composite functor in terms of the derivatives of its constituents.

Theorem (Classification of linear functors): Let T_1 and T_2 be simplicial theories and $L\colon T_1\text{-alg} \longrightarrow T_2\text{-alg}$ a connective linear functor. Then there is a T_2^s-T_1^s-bimodule $L^!$ and a natural chain of weak equivalences of T_2-algebras

$$L(X) \quad \simeq \quad \Omega^\infty(L^! \wedge_{T_1^s} \Sigma^\infty X^{\mathrm{c}}) \ .$$

The precise definitions of all expressions in the classification will be given in due time. The notation is meant to be suggestive: Ω^∞ means infinite loop object, $\wedge_{T_1^s}$ is the smash product over T_1^s and Σ^∞ denotes the suspension spectrum of a T_1-algebra. Given the classification of linear functors, one might guess what the chain rule will look like. If $F\colon T_1\text{-alg} \longrightarrow T_2\text{-alg}$ is a pointed homotopy functor that is stably excisive (in the sense of [Gw1, 1.8]), denote by $\partial_* F$ the T_2^s-T_1^s-bimodule that classifies its differential at the zero object. Then we have

Theorem (Chain rule): Let T_1, T_2, T_3 be simplicial theories and $F\colon T_1\text{-alg} \longrightarrow T_2\text{-alg}$, $G\colon T_2\text{-alg} \longrightarrow T_3\text{-alg}$ stably excisive connective reduced homotopy functors. Then there is a natural chain of weak equivalences of T_3^s-T_1^s-bimodules

$$\partial_*(G \circ F) \quad \simeq \quad \partial_* G \wedge^L_{T_2^s} \partial_* F$$

This chain rule specializes to a chain rule of J. Klein and J. Rognes [KR] for functors on the category of spaces. One takes the theories to be simplicial sets with actions of the simplicial loop groups of the respective spaces involved.

We assume that the reader is familiar with the language of homotopical algebra (cf. [Q1], [DS]) and with the basic ideas concerning monoidal and symmetric monoidal categories (cf. [MacL, VII], [Bor, 6]) and triples (also called monads, cf. [MacL, VI.1], [Bor, 4]).

This paper is a slightly modified version of the author's thesis. It could not have been written without many discussions that I had with various people. I would like to thank Greg Arone, John Klein and Manos Lydakis for many discussions on related and unrelated topics during my time as a graduate student at Bielefeld. I am particularly indebted to Manos Lydakis for bringing to my attention the smash product of Γ-spaces (cf. [Ly]); this device was crucial in turning ideas into theorems. It is a pleasure to acknowledge various helpful discussions I had with Bjorn Dundas, Tom Goodwillie, Teimuraz Pirashvili, Charles Rezk and Brooke Shipley. Lastly, I would like to thank my adviser, Friedhelm Waldhausen, for his support of all my mathematical efforts.

1 DFSPs and their modules

1.1 Review of Γ-spaces

Our model for the category of connective spectra will be the category of Γ-spaces. These were introduced by G. Segal [Se] who showed that they give rise to a homotopy category equivalent to the usual homotopy category of connective spectra. A. K. Bousfield and E. M. Friedlander [BF] considered a bigger category of Γ-spaces in which the ones introduced by Segal appeared as the *special* Γ-spaces. Their category admits a closed simplicial model category structure with a notion of stable weak equivalence giving rise again to the homotopy theory of connective spectra. Then M. Lydakis [Ly] showed that Γ-spaces admit internal function objects and a symmetric monoidal smash product with good homotopical properties.

The category Γ is the skeletal category of the category of finite pointed sets. There is one object n of Γ for every non-negative integer n and the morphisms from n to m are the maps of sets $f: \{0, 1, \ldots, n\} \longrightarrow \{0, 1, \ldots, m\}$ with $f(0) = 0$. This convention agrees with that of [Ly] but differs from the one in [Se] in that we use the symbol Γ to denote the opposite of Segal's category Γ. Denote by $\mathcal{S}$ the category of pointed simplicial sets. Then a Γ-space is a functor $\Gamma \longrightarrow \mathcal{S}$ taking 0 to a discrete one point set. A morphism of Γ-spaces is a natural transformation of functors. Denote the category of Γ-spaces by $\mathcal{GS}$. A choice of equivalence of Γ with the category of all finite pointed sets gives a symmetric monoidal smash product functor $\wedge: \Gamma \times \Gamma \longrightarrow \Gamma$. We give names to certain morphisms in Γ that will occur repeatedly. First there are the projections $p_i : n \longrightarrow 1$ defined by $p_i(i) = 1$ and $p_i(j) = 0$ for $j \neq i$. Furthermore, we need the fold map $\nabla: 2 \longrightarrow 1$ defined by $\nabla(1) = \nabla(2) = 1$. The representable Γ-spaces $\Gamma^n = \mathrm{Hom}_\Gamma(n, -)$ play a role analogous to that of the standard simplices in the category of simlicial sets. We use the notation S^0 as another name for Γ^1, since the spectrum associated to the Γ-space Γ^1 (see below) is the sphere spectrum.

A Γ-space X can be prolonged, by direct limit, to a functor from the category of pointed sets to pointed simplicial sets. By degreewise evaluation and formation of the diagonal of the resulting bisimplicial sets, it can furthermore be promoted to a functor from the category of pointed simplicial sets to itself [BF, §4]. This extension process has another description as the following coend [MacL, IX.6]. If X is a Γ-space and K a pointed simplicial set, the value of the extended functor on K is given by

$$\int^{n \in \Gamma} K^n \times X(n) \ .$$

We will not distinguish notationally between the original Γ-space and the extended functor. The resulting functor preserves weak equivalences of simplicial sets [BF, Prop. 4.9] and it is automatically simplicial, i.e., it comes with coherent natural maps $K \wedge X(L) \longrightarrow X(K \wedge L)$.

A Γ-space then gives rise to a spectrum of simplicial sets [BF, Def. 2.1] by evaluating it on spheres. We let $S^1 = \Delta^1/\partial\Delta^1$ be the quotient of the standard simplicial 1-simplex by its boundary, and we set $S^n = S^1 \wedge \ldots \wedge S^1$ (n factors). A spectrum X then consists of a sequence of pointed simplicial sets $X_n, n = 0, 1, \ldots$ together with maps $S^1 \wedge X_n \longrightarrow X_{n+1}$. A map of spectra $X \longrightarrow Y$ consists of maps $X_n \longrightarrow Y_n$ strictly commuting with the suspension maps. Given a Γ-space X, extend it to a simplicial functor from all pointed simplicial sets and define its associated spectrum $X\underline{S}$ by $(X\underline{S})_n = X(S^n)$ with structure maps given by the simplicial structure of X.

Bousfield and Friedlander introduce two model category structures for both Γ-spaces and spectra called the *strict* and the *stable* model categories. To give the definitions we recall the notions of left and right lifting properties: let $i: A \longrightarrow B$ and $p: X \longrightarrow Y$ be morphisms in some category. i has the **left lifting property** (LLP) with respect to p and p has the **right lifting property** (RLP) with respect to i, if given any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with unbroken arrows, there exists the dotted morphism $B \longrightarrow X$ such that the resulting diagram is commutative.

A map of spectra $X \longrightarrow Y$ is a strict fibration (resp. strict weak equivalence) if all the maps $X_n \longrightarrow Y_n$ are fibrations (resp. weak equivalences) of simplicial sets. It is a strict cofibration if $X_0 \longrightarrow Y_0$ and

$$X_n \cup_{\Sigma X_{n-1}} \Sigma Y_{n-1} \;\longrightarrow\; Y_n$$

(for $n \geq 1$) are cofibrations of simplicial sets. The homotopy groups of a spectrum Y are defined as

$$\pi_n Y \;=\; \operatorname{colim}_i \pi_{n+i}|Y_i| \;.$$

A map of spectra is a stable weak equivalence if it induces isomorphisms on homotopy groups. The stable cofibrations coincide with the strict cofibrations and the stable fibrations are the maps with the RLP with respect to stable acyclic cofibrations. There is a more explicit characterizations of the stable fibrations in [BF, 2.2].

To define the corresponding notions for Γ-spaces, we need the skeleton and coskeleton functors. Let $\Gamma_{\leq n}$ denote the full subcategory of Γ generated by the objects $0, 1, \ldots, n$. Every Γ-space can be restricted to a functor with domain $\Gamma_{\leq n}$ and restricting has both a left and a right adjoint given by Kan extension. So for a Γ-space X, the Γ-space $\mathrm{sk}_n X$ (resp. $\mathrm{csk}_n X$) is defined by restricting X to $\Gamma_{\leq n}$ and then taking the left (resp. right) Kan extension along the inclusion $\Gamma_{\leq n} \longrightarrow \Gamma$. A map $X \longrightarrow Y$ of Γ-spaces is a strict cofibration if it is injective and if for every $n \in \mathrm{Ob}\,\Gamma$, the action of the symmetric group Σ_n on $Y(n)$ is free off the image of $(X \cup_{\mathrm{sk}_{n-1} X} \mathrm{sk}_{n-1} Y)(n)$. It is a strict fibration if for all n, the map

$$X(n) \longrightarrow (\mathrm{csk}_{n-1} X \times_{\mathrm{csk}_{n-1} Y} Y)(n)$$

is a fibration of simplicial sets. It is a strict weak equivalence if all the maps $X(n) \longrightarrow Y(n)$ are weak equivalences of simplicial sets. The homotopy groups of a Γ-spaces are defined to be the homotopy groups of the associated spectrum. The spectrum associated to a Γ-space is always connective in the sense that homotopy groups in negative dimensions are trivial. A map of Γ-spaces is a stable weak equivalence if it induces isomorphisms on homotopy groups. The stable cofibrations coincide with the strict cofibrations and the stable fibrations are the maps having the RLP with respect to all stable acyclic cofibrations. Again there is a more explicit characterization of the stable fibrations in [BF, 5.7].

For our purposes, the stable model category structures are more important than the strict ones. Hence we adopt the convention that whenever we use model category notions for spectra or Γ-space without qualifying, we are refering to the **stable** notions.

Theorem 1.1.1 [BF, Thm. 2.2, 2.3, 3.5, 5.2, 5.8] *The strict and the stable notions of cofibrations, fibrations and weak equivalences each make the category of Γ-spaces and the category of spectra into closed simplicial model categories. The functor $X \mapsto X\underline{S}$ from Γ-spaces to spectra has a right adjoint. The total derived functors of these two functors are inverse equivalences of the stable homotopy theories of Γ-spaces and connective spectra.*

M. Lydakis [Ly] introduces smash products $X \wedge Y$ and internal function objects $\mathrm{Hom}(X, Y)$ for Γ-spaces. Two Γ-spaces X and Y have an external smash product $X \tilde{\wedge} Y$ which is the bi-Γ-space defined as

$$(X \tilde{\wedge} Y)(k, l) \quad = \quad X(k) \wedge Y(l)\ .$$

The internal smash product is then obtained by left Kan-extension along the smash product functor $\wedge : \Gamma \times \Gamma \longrightarrow \Gamma$, i.e.,

$$(X \wedge Y)(n) \ = \ \mathrm{colim}_{k \wedge l \to n}\, X(k) \wedge Y(l)\ .$$

The smash product is thus characterized by the universal property that Γ-space maps $X \wedge Y \longrightarrow Z$ are in bijective correspondence with maps

$$X(k) \wedge Y(l) \quad \longrightarrow \quad Z(k \wedge l)$$

which are natural in both k and l. The internal hom object is defined by

$$\mathrm{Hom}(X, Y)(n) \ = \ \mathrm{hom}_{\mathcal{GS}}(X, Y_{n\wedge -})\ .$$

On the right hand side, $\mathrm{hom}_{\mathcal{GS}}$ stands for the simplicial hom set of Γ-spaces and $Y_{n\wedge -}$ is the Γ-space obtained from Y by precomposing with $n \wedge - : \Gamma \longrightarrow \Gamma$. In particular, the internal Hom-Γ-space evaluated at $1 \in \mathrm{Ob}\,\Gamma$ gives the simplicial hom set.

Theorem 1.1.2 [Ly, 2.10, 3.23, 4.18, 5.1] *The category of Γ-spaces forms a symmetric monoidal closed category with respect to $\wedge$ and Hom and with unit S^0. If $A \rightarrowtail B$ and $K \rightarrowtail L$ are cofibrations of Γ-spaces then so is the map*

$$A \wedge L \cup_{A \wedge K} B \wedge K \rightarrowtail B \wedge L\ .$$

If in addition one of the former maps is a stable weak equivalence, so is the latter map. Smashing with a cofibrant Γ-space preserves stable weak equivalences. Smashing with an arbitrary Γ-space preserves injective maps.

A Γ-space X is called *special* if the map $X(k \vee l) \longrightarrow X(k) \times X(l)$ induced by the projections $k \vee l \longrightarrow k$ resp. l is a weak equivalence for all $k, l \in \mathrm{Ob}\,\Gamma$. In this case, the weak map

$$X(1) \times X(1) \xleftarrow[\sim]{(p_1, p_2)} X(2) \xrightarrow{\ \nabla\ } X(1)$$

induces an abelian monoid structure on $\pi_0(X(1))$. X is called *very special* if this monoid is actually a group, i.e., if it has inverses. Every stably fibrant Γ-space is very special. The converse holds for strictly fibrant Γ-spaces [BF, 5.7].

Abelian groups give rise to very special Γ-spaces. More generally, a simplicial abelian monoid A defines a Γ-space $\widetilde{A}$ as follows. $\widetilde{A}(k) = A^k$; if $f : k \longrightarrow n$ is a map in Γ, then

$$f_*(a_1, \ldots, a_k) \;=\; \Big(\sum_{f(j)=1} a_j \,,\; \ldots \,,\; \sum_{f(j)=n} a_j \Big) \;.$$

The Γ-space $\widetilde{A}$ is special and it is very special if A is group-like, i.e., if $\pi_0 A$ is a group. The functor $\widetilde{\ }$ in fact embeds simplicial abelian monoids as a full subcategory of Γ-spaces.

There is an explicit presentation of the 0-th stable homotopy group of a Γ-space. This group depends only on the path-components of the 2-skeleton of the Γ-space:

Lemma 1.1.3 *There is an exact sequence*

$$\mathbb{Z}[\pi_0(X(2))] \xrightarrow{p_1+p_2-\nabla} \mathbb{Z}[\pi_0(X(1))] \longrightarrow \pi_0 X \longrightarrow 0$$

natural in the Γ*-space* X*.*

Proof: We analyse the set of Γ-space maps $X \longrightarrow \widetilde{M}$ for M a discrete abelian group. $\widetilde{M}(k)$ is the k-th cartesian power of $\widetilde{M}(1) = M$ with respect to the maps induced by the projections $p_i : n \longrightarrow 1$ in Γ. Thus any map of Γ-spaces $X \longrightarrow \widetilde{M}$ is determined by its value at $1 \in \mathrm{Ob}\,\Gamma$. Since M is discrete, any map $X(1) \longrightarrow M$ factors through $\pi_0(X(1))$. Since the map $M^2 = \widetilde{M}(2) \longrightarrow \widetilde{M}(1) = M$ induced by the fold map $\nabla : 2 \longrightarrow 1$ is addition, a necessary condition for a map $f : \pi_0(X(1)) \longrightarrow M$ to extend to a map of Γ-spaces is that $f \circ p_1 + f \circ p_2 = f \circ \nabla$ holds. We omit the verification that this equality already implies that f does in fact extend to a (necessarily unique) map of Γ-spaces $X \longrightarrow \widetilde{M}$. Granted this fact, we have shown that the functor $M \mapsto \widetilde{M}$ from abelian groups to Γ-spaces has as left adjoint the functor

$$X \;\mapsto\; \mathbb{Z}[\pi_0(X(1))] \,/\, \{p_1 + p_2 = \nabla\}.$$

By [BF, Lemma 5.5], the 0-th stable homotopy group functor is also a left adjoint. Since any two left adjoints are naturally isomorphic, this proves the lemma. $\square$

We will have several occasions to use the following

Lemma 1.1.4 *Let* $A \longrightarrow B$ *be an injective map of* Γ*-spaces. Then there is a natural long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_i A \longrightarrow \pi_i B \longrightarrow \pi_i B/A \longrightarrow \pi_{i-1} A \longrightarrow \cdots \pi_0 B/A \longrightarrow 0$$

where B/A *denotes the cofibre (quotient) of* B *by* A*. Let* $X \longrightarrow Y$ *be a stable fibration of* Γ*-spaces which induces an epimorphism on* π_0*. Then there is a natural long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_i F \longrightarrow \pi_i X \longrightarrow \pi_i Y \longrightarrow \pi_{i-1} F \longrightarrow \cdots \pi_0 Y \longrightarrow 0$$

where F *denotes the fibre of* $X \longrightarrow Y$*.*

Proof: For every n, the sequence

$$A(S^n) \longrightarrow B(S^n) \longrightarrow (A/B)(S^n)$$

is a cofibre sequence of simplicial sets in which all objects are $(n-1)$-connected. Hence for these simplicial sets there is a long exact sequence of homotopy groups in a range of roughly $2n$. As filtered direct limits of abelian groups are exact, the desired long exact sequence is obtained by letting n go to infinity.

For the second part recall the functor T of [BF, Thm. 5.2]. T associates to a Γ-space X the very special Γ-space TX and a natural stable weak equivalence $X \xrightarrow{\sim} TX$. Let Φ denote the homotopy fibre of the map $TX \longrightarrow TY$ in the strict model category structure of Γ-spaces. So Φ is the categorical fibre of the latter map turned into a strict fibration. By [BF, 5.7], the induced map $F \longrightarrow \Phi$ is a strict weak equivalence. Since F is very special, so is Φ. TX and TY are also very special, so the stable homotopy groups of X, Y and F are canonically identified with the ordinary homotopy groups of $TX(1), TY(1)$ and $\Phi(1)$. Since the latter simplicial sets form a homotopy fibration sequence, we get the advertised long exact sequence of homotopy groups. □

The comparison map. Given two Γ-spaces X and Y, there is a natural map $X \wedge Y \longrightarrow X \circ Y$ from the smash product to the composition product. Despite its simple definition, that map is of fundamental importance to this paper.

Since Γ-spaces extend to functors defined on the category of pointed simplicial sets, they can be composed. Explicitly, for Γ-spaces X and Y, we set $(X \circ Y)(n) = X(Y(n))$. This composition $\circ$ is a monoidal (though not symmetric monoidal) product on the category of Γ-spaces. The unit is the same as for the smash product, it is the Γ-space S^0 (alias Γ^1) which as a functor is the inclusion of Γ into all pointed simplicial sets.

The comparison map

$$X \wedge Y \longrightarrow X \circ Y$$

is obtained as follows. Γ-spaces are naturally simplicial functors after extension to the category of all pointed simplicial sets [BF, §3]. This means that there are natural coherent maps $X(K) \wedge L \longrightarrow X(K \wedge L)$. This simplicial structure gives maps

$$X(n) \wedge Y(m) \longrightarrow X(n \wedge Y(m)) \longrightarrow X(Y(n \wedge m))$$

natural in $n, m \in \mathrm{Ob}\,\Gamma$. From this the comparison map is obtained by the universal property of the smash product of Γ-spaces. The comparison map is associative and unital, S^0 being the unit for both $\wedge$ and $\circ$. In technical terms: the identity functor on the category of Γ-spaces becomes a lax monoidal functor from $(\mathcal{GS}, \wedge)$ to $(\mathcal{GS}, \circ)$. The composition product $X \circ Y$ preserves weak equivalences in both variables, the smash product $X \wedge Y$ preserves weak equivalences when one of the constituents is cofibrant. Furthermore, the comparison map is a weak equivalence whenever it has a chance:

Lemma 1.1.5 *Let X, Y be Γ-spaces one of which is cofibrant. Then the comparison map $X \wedge Y \longrightarrow X \circ Y$ is a stable weak equivalence.*

Proof: We treat the case where Y is cofibrant, the other case being similar. By [Ly, 4.20], the map $X(S^n) \wedge Y(S^n) \longrightarrow (X \wedge Y)(S^{2n})$ is roughly $3n$-connected and we claim the same is true when the target $X \wedge Y$ replaced by $X \circ Y$. Taking $A = \Gamma^1$ and $B = Y$ in [Ly, 4.20] shows that the map $S^n \wedge Y(S^n) \longrightarrow Y(S^{2n})$ is roughly $3n$-connected. Since functors arising from Γ-spaces preserve connectivity, $X(S^n \wedge Y(S^n)) \longrightarrow X(Y(S^{2n}))$ is also approximately $3n$-connected. By [Ly, Remark 4.23], the map $X(S^n) \wedge K \longrightarrow X(S^n \wedge K)$ is roughly $3n$-connected for every $(n-1)$-connected simplicial set K. Taking $K = Y(S^n)$ gives that $X(S^n) \wedge Y(S^n) \longrightarrow X(Y(S^{2n}))$ is roughly $3n$-connected. $\square$

1.2 Cofibrant generation

The model category structures for DFSPs and modules over a DFSP that we will introduce in section 1.3 rely on the fact that the stable model category structure for Γ-spaces is cofibrantly generated (see Def. A.2). The stable model category of spectra of simplicial sets even admits the small object argument in the sense of [Sch, Def. 1.3.1], which is a special case of 'cofibrantly generated'. However, for Γ-spaces we need the more general notion. Both strict model category structures are also cofibrantly generated, but we will not use this fact.

To show that the stable model category of Γ-spaces is cofibrantly generated, we adapt the cardinality argument of A. K. Bousfield [B2, §11]. Call a Γ-space X **countable** if the disjoint union of all simplices in all the simplicial sets $X(n), n \in \mathrm{Ob}\,\Gamma$ is countable.

Lemma 1.2.1 *Let* $D \overset{\sim}{\rightarrowtail} B$ *be an inclusion of* Γ*-spaces which is also a stable acyclic cofibration. If* $D \neq B$*, then there exists a countable sub-*Γ*-space* $M \subset B$ *not contained in* D *such that* $D \cap M \overset{\sim}{\rightarrowtail} M$ *is a stable acyclic cofibration.*

Proof: We follow the proof of [B2, Lemma 11.2]. The desired M is obtained as the ascending union of countable sub-Γ-spaces $M_i \subset B, i \in \mathbb{N}$. Choose M_0 to be any countable sub-Γ-space which is not contained in D. Assume that M_i has been constructed and is countable. B/D is weakly contractible and it is the filtered union of its sub-Γ-spaces $N/(N \cap D)$ where N runs over all countable sub-Γ-spaces of B. Homotopy groups commute with filtered colimits over inclusions of Γ-spaces, so for any element $x \in \pi_*(M_i/(M_i \cap D))$ there exists a countable $N_x \subset B$ such that the element x maps to 0 in $N_x/(N_x \cap D)$. Since $M_i/(M_i \cap D)$ is countable, so are its homotopy groups, so we can define M_{i+1} to be the union of M_i with all the $N_x, x \in \pi_*(M_i/(M_i \cap D))$. M_{i+1} is again countable and by construction the map $M_i/(M_i \cap D) \longrightarrow M_{i+1}/(M_{i+1} \cap D)$ induces the trivial map on homotopy groups. Hence if we define M as the union of the M_i, then M is countable and $M/(M \cap D)$ is weakly contractible, i.e., $(D \cap M) \longrightarrow M$ is a weak equivalence. $(D \cap M) \longrightarrow M$ is a cofibration since it is injective and its cofibre $M/(D \cap M)$ is cofibrant (since it is a sub-Γ-space of the cofibrant Γ-space B/D). $\square$

Lemma 1.2.2 *Let* $X \longrightarrow Y$ *be a map of* Γ*-spaces which has the RLP with respect to all stable acyclic cofibrations between countable* Γ*-spaces. Then the map is a stable fibration.*

Proof: We follow the proof of [B2, Lemma 11.3]. We have to show that $X \longrightarrow Y$ has the RLP with respect to stable acyclic cofibrations between arbitrarily large Γ-spaces. So let $A \overset{\sim}{\rightarrowtail} B$ be an arbitrary stable acyclic cofibration, assume that it is in fact the inclusion of a sub-Γ-space, and consider a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "\sim"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

for which we have to produce a lifting. Consider the set of all $(D, f_D{:}D \longrightarrow X)$ with the properties that

- D is a sub-Γ-space of B containing A.
- the inclusion $A \longrightarrow D$ is an acyclic cofibration.
- f_D is a partial lifting, i.e., it makes the following diagram commute

$$\begin{tikzcd} A \arrow[rr] \arrow[d] & & X \arrow[d] \\ D \arrow[r] \arrow[urr, dashed, "f_D"] & B \arrow[r] & Y \end{tikzcd}$$

The set of these (D, f_D) is non-empty and is preordered by declaring: $(D, f_D) \leq (D', f_{D'}) \iff D \subseteq D'$ and $f_{D'}$ extends f_D. Every chain in this preorder has an upper bound, namely the union of all the sub-Γ-spaces with the unique partial lifting extending all the given ones. By Zorn's lemma, there is thus a maximal element (D_0, f_{D_0}) and we claim that $D_0 = B$.

If D_0 was a proper subobject of B, then by lemma 1.2.1 applied to the inclusion of D_0 into B, we could find a countable M inside B, not contained in D_0 such that $M \cap D_0 \overset{\sim}{\rightarrowtail} M$ is a stable acyclic cofibration. By assumption, $X \longrightarrow Y$ has the RLP for that latter acyclic cofibration. Since $D_0 \longrightarrow M \cup D_0$ is obtained from it by cobase change, $X \longrightarrow Y$ also has the RLP with respect to that map. This contradicts the maximality of (D_0, f_{D_0}). $\square$

Lemma 1.2.3 *Let I be a small category (i.e., the objects of I form a set). Then every object in the functor category* Funct(I,(sets)) *is small relative to the whole functor category (cf. App. A).*

Proof: Fix a functor $X : I \longrightarrow$(sets) and choose a regular cardinal κ which is bigger than the cardinality of the disjoint union of the sets $X(\mathrm{domain}(f))$ where f runs through the disjoint union of the morphism sets of I. We claim that X is κ-small relative to the whole functor category. So let $\lambda \geq \kappa$ be a regular cardinal and consider a λ-sequence $V : \lambda \longrightarrow \mathrm{Funct}(I,(\mathrm{sets}))$. We have to show that the map

$$\mathrm{colim}_{\beta<\lambda} \mathrm{Hom}\,(X, V(\beta)) \quad \longrightarrow \quad \mathrm{Hom}\,(X, \mathrm{colim}_{\beta<\lambda} V(\beta))$$

is bijective.

To show surjectivity we fix a map from X into the colimit. The image of every element σ of one of the sets $X(i)$, $i \in \mathrm{Ob}I$ is contained in the image of $V(\beta_\sigma)$ for some $\beta_\sigma < \lambda$. Setting $\beta_0 = \bigcup \beta_\sigma$, we get $\beta_0 < \kappa \leq \lambda$ by the choice of κ, so the image of X is contained in the image of $V(\beta_0)$. Hence we can choose maps $X(i) \longrightarrow V(\beta_0)(i)$ lifting the maps $X(i) \longrightarrow \mathrm{colim}_{\beta<\lambda} V(\beta)(i)$. These maps do not yet assemble into a natural transformation from X to $V(\beta_0)$, i.e., the diagrams

$$\begin{tikzcd} X(i) \arrow[r] \arrow[d, "X(f)"'] & V(\beta_0)(i) \arrow[d, "V(\beta_0)(f)"] \\ X(j) \arrow[r] & V(\beta_0)(j) \end{tikzcd}$$

do not necessarily commute for morphisms $f : i \longrightarrow j$ in I. However, since the corresponding diagrams commute with $V(\beta_0)$ replaced by the colimit, for every pair $(f : i \longrightarrow j, \sigma \in X(i))$ there exists $\beta_{(f,\sigma)} \geq \beta_0$ such that the element σ is sent to the same element in $V(\beta_{(f,\sigma)})$ both ways around the square. Setting $\beta_1 = \bigcup \beta_{(f,\sigma)}$, we still have $\beta_1 < \kappa \leq \lambda$ by the choice of κ, so all relevant squares commute for the composite maps $X(i) \longrightarrow V(\beta_1)(i)$ and we have produced a natural transformation $X \longrightarrow V(\beta_1)$ lifting the one from X to the colimit. This proves surjectivity; the proof of injectivity is similar, but easier, and we omit it. $\square$

Theorem 1.2.4 *The stable model category structure for Γ-spaces is cofibrantly generated.*

Proof: We saw in lemma 1.2.2 that a map of Γ-spaces is a stable fibration if it has the right lifting property with respect to all stable acyclic cofibrations between countable Γ-spaces. The isomorphism classes of maps of the latter kind form a set, so we choose the generating acyclic cofibrations to be some set of representatives of the isomorphism classes of stable acyclic cofibrations between countable Γ-spaces.

The generating cofibrations can be described more explicitly. They come in two flavors. Consider the cofibrations of Γ-spaces

$$(A) \qquad (\mathrm{sk}_{n-1}\Gamma^n \wedge \Delta^i_+) \cup_{(\mathrm{sk}_{n-1}\Gamma^n \wedge \Lambda^{i,k}_+)} (\Gamma^n \wedge \Lambda^{i,k}_+) \xrightarrow{\simeq} \Gamma^n \wedge \Delta^i_+$$

$$(B) \qquad \Gamma^n \wedge \partial\Delta^i_+ \rightarrowtail \Gamma^n \wedge \Delta^i_+$$

for varying n, i, k ($\Lambda^{i,k}$ is the k-th horn and $\partial\Delta^i$ the boundary of the standard simplicial i-simplex Δ^i, the subscript "$_+$" denotes an additional disjoint basepoint). We claim that a map is a stable acyclic fibration if it has the RLP with respect to all maps of type (A) and (B). These maps thus qualify as a set of generating cofibrations.

The skeleton functor and the coskeleton functor are adjoint and there is a natural Yoneda isomorphism of simplicial sets $\hom_{\mathcal{GS}}(\Gamma^n, X) \cong X(n)$. Hence a map $X \longrightarrow Y$ of Γ-spaces has the RLP with respect to all maps of type (A) if and only if the maps $X(n) \longrightarrow (\mathrm{csk}_{n-1}X \times_{\mathrm{csk}_{n-1}Y} Y)(n)$ have the RLP with respect to all inclusions of horns into simplices, i.e., if and only if they are fibrations of simplicial sets. This is the definition of $X \longrightarrow Y$ being a strict fibration. If in addition the map $X \longrightarrow Y$ has the RLP with respect to all maps of type (B), it follows again by adjointness that all $X(n) \longrightarrow Y(n)$ are acyclic fibrations of simplicial sets, so the map is also a strict weak equivalence of Γ-spaces. Since the cofibrations coincide in the strict and the stable model category structures, so do the acyclic fibrations, so $X \longrightarrow Y$ is a stable acyclic fibration.

A Γ-space is a functor $X : \Gamma \times \Delta^{\mathrm{op}} \longrightarrow (\mathrm{sets})$ with the property that $X(0,n) \cong *$ for all $n \in \mathrm{Ob}\Delta^{\mathrm{op}}$. So the argument of lemma 1.2.3 shows that every Γ-space is small relative to the whole category of Γ-spaces. $\square$

Lemma 1.2.5 *The stable model category structure for spectra of simplicial sets is cofibrantly generated.*

Lemma 1.2.6 *A map $X \longrightarrow Y$ of spectra is a stable fibration if it is a strict fibration and the squares*

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & \Omega^j X_{n+j} \arrow[d] \\ Y_n \arrow[r] & \Omega^j Y_{n+j} \end{tikzcd}$$

are homotopy cartesian for all $n, j \geq 0$ (here $\Omega = \hom_{\mathcal{S}}(S^1, -)$).

Proof: By the criterion of [BF, A.7] it suffices to show that the squares

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & (QX)_n \arrow[d] \\ Y_n \arrow[r] & (QY)_n \end{tikzcd}$$

are homotopy cartesian where $(QX)_n = \mathrm{colim}_j\, \Omega^j \,\mathrm{Sing}\, |X_{n+j}|$. Since $X \longrightarrow Y$ is a strict fibration, the squares of simplicial sets

$$\begin{tikzcd} \Omega^j X_{n+j} \arrow[r] \arrow[d] & \Omega^j \,\mathrm{Sing}\, |X_{n+j}| \arrow[d] \\ \Omega^j Y_{n+j} \arrow[r] & \Omega^j \,\mathrm{Sing}\, |Y_{n+j}| \end{tikzcd}$$

are homotopy cartesian, hence so are all composed the squares

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & \Omega^j \,\mathrm{Sing}\, |X_{n+j}| \arrow[d] \\ Y_n \arrow[r] & \Omega^j \,\mathrm{Sing}\, |Y_{n+j}| \end{tikzcd}$$

Since $X \longrightarrow Y$ is a strict fibration, the right vertical maps in the latter squares are fibrations, so we get homotopy cartesian squares in the colimit. $\square$

Proof of lemma 1.2.5. The functor that takes a spectrum to its n-th term has a left adjoint which we will denote F_n. $F_n K$ is a shift desuspension of the suspension spectrum; explicitly

$$(F_n K)_i \;=\; \begin{cases} * & \text{if} \quad i < n \\ S^{i-n} \wedge K & \text{if} \quad i \geq n \end{cases}$$

There are three types of generating (acyclic) cofibrations:

$$(A) \qquad F_n\, \partial\Delta^i_+ \rightarrowtail F_n\, \Delta^i_+$$

$$(B) \qquad F_n\, \Lambda^{i,k}_+ \xrightarrow{\simeq} F_n\, \Delta^i_+$$

$$(C) \qquad (F_{n+j}S^j)\wedge\Delta^i_+ \cup_{(F_{n+j}S^j)\wedge\partial\Delta^i_+} C\wedge\partial\Delta^i_+ \xrightarrow{\simeq} C\wedge\Delta^i_+$$

where

$$C \;=\; (F_{n+j}S^j \wedge\Delta^1_+) \cup_{F_{n+j}S^j\times 1} F_n S^0$$

is the reduced mapping cylinder of $F_{n+j}S^j \longrightarrow F_n S^0$. The map $F_{n+j}S^j \longrightarrow F_n S^0$ is an isomorphism in spectrum degrees above $n+j$, hence a stable weak equivalence, but it is not a cofibration. Since both source and target of this map are cofibrant, the inclusion of the source into the mapping cylinder is a cofibration – this is why the mapping cylinder C is used. The map $C \longrightarrow F_n S^0$ is a simplicial homotopy equivalence.

The first observation is that a map of spectra $X \longrightarrow Y$ has the RLP for maps of type (A) if and only if all maps $X_n \longrightarrow Y_n$ are acyclic fibrations of simplicial sets. So RLP for maps of type (A) is equivalent to being a strict acyclic fibration. Since cofibrations coincide in the strict and stable model category structures, so do the acyclic fibrations. Hence the maps of type (A) qualify as generating cofibrations.

Similarly, RLP for all maps of type (B) is equivalent to being a strict fibration. We claim that if in addition the RLP for maps of type (C) holds, then the squares of lemma 1.2.6 are homotopy cartesian. That lemma thus shows that the maps of type (B) and (C) qualify as generating acyclic cofibrations.

Since $X \longrightarrow Y$ is a strict, i.e., degreewise, fibration, we have to show that the map $X_n \longrightarrow Y_n \times_{\Omega^j Y_{n+j}} \Omega^j X_{n+j}$ is a weak equivalence. Note that $\hom(F_k S^l, X) \cong \Omega^l X_k$ where the left hand side denotes simplicial hom sets of spectra. Since $C \xrightarrow{\sim} F_n S^0$ is a simplicial homotopy equivalence, so is $X_n = \hom(F_n S^0, X) \xrightarrow{\sim} \hom(C, X)$. We obtain a commutative square

$$\begin{tikzcd}
\hom(F_n S^0, X) \arrow[r] \arrow[d, "\sim"'] & \hom(F_n S^0, Y) \times_{\hom(F_{n+j}S^j, Y)} \hom(F_{n+j}S^j, X) \arrow[d, "\sim"] \\
\hom(C, X) \arrow[r] & \hom(C, Y) \times_{\hom(F_{n+j}S^j, Y)} \hom(F_{n+j}S^j, X)
\end{tikzcd}$$

in which the upper horizontal map is isomorphic to $X_n \longrightarrow Y_n \times_{\Omega^j Y_{n+j}} \Omega^j X_{n+j}$. The left vertical map is a simplicial homotopy equivalence. The right vertical map is obtained from the weak equivalence $\hom(F_n S^0, Y) \xrightarrow{\sim} \hom(C, Y)$ by base change along the fibration

$$\hom(C, Y) \times_{\hom(F_{n+j} S^j, Y)} \hom(F_{n+j} S^j, X) \longrightarrow\!\!\!\!\rightarrow \hom(C, Y) \quad .$$

The right map is thus a weak equivalence itself. The fact that $X \longrightarrow Y$ has the RLP for all maps of type (C) is equivalent to the fact that the lower horizontal map is an acyclic fibration of simplicial sets, hence the upper horizontal map is a weak equivalence. $\square$

1.3 Model category structures for modules and DFSPs

Our version of brave new rings (i.e., homotopy rings) will be the monoids in the symmetric monoidal category of Γ-spaces with respect to smash product. M. Lydakis notices [Ly, 4.5] that monoids with respect to the smash product of Γ-spaces are special cases of 'Functors with Smash Product' (FSPs, cf. [Bö, 1.1], [PW, 2.2]). One can describe these monoids as 'FSPs defined on finite sets'. We chose to call these objects 'Degreewise Functors with Smash Product' (DFSPs) because the behavior of the associated FSPs on simplicial sets is determined by degreewise application. Suggestions for better terminology are appreciated. Every connective FSP is related through a chain of weak equivalences to a DFSP. DFSPs have the advantage, however, that they form a closed model category and that modules over a fixed DFSP form a closed model category. The analogous properties for FSPs are not known at present.

Many results of this section are formal in the sense that they follow from the existence and a few homotopical properties of the symmetric monoidal smash product. An axiomatic approach to modules and algebras in symmetric monoidal model categories will be explored in [SS].

Definition 1.3.1 *A* **Degreewise Functor with Smash Product (DFSP)** *is a monoid in the symmetric monoidal category of Γ-spaces with respect to the smash product. Explicitly, a DFSP is a Γ-space R equipped with maps*

$$\mu : R \wedge R \longrightarrow R \quad \textit{and} \quad \eta : S^0 \longrightarrow R$$

making the following diagrams commute

(Associativity)
$$\begin{tikzcd} R \wedge R \wedge R \arrow[r, "\mu \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge \mu"'] & R \wedge R \arrow[d, "\mu"] \\ R \wedge R \arrow[r, "\mu"'] & R \end{tikzcd}$$

(Unit)
$$\begin{tikzcd} R \cong S^0 \wedge R \arrow[r, "\eta \wedge \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & R \wedge R \arrow[d, "\mu"] & R \wedge S^0 \cong R \arrow[l, "\mathrm{id} \wedge \eta"'] \arrow[dl, "\mathrm{id}"] \\ & R & \end{tikzcd}$$

μ and η will be referred to as the multiplication and the unit map respectively. R is called commutative if the diagram

$$\begin{tikzcd} R \wedge R \arrow[rr, "\text{switch}"] \arrow[dr, "\mu"'] & & R \wedge R \arrow[dl, "\mu"] \\ & R & \end{tikzcd}$$

commutes. A map of DFSPs is a map of Γ-spaces commuting with the multiplication and unit maps.

First examples of DFSP are monoid rings over the sphere spectrum and Eilenberg-MacLane ring spectra of classical rings. Both these kinds of DFSPs are in fact the restrictions to finite sets of the standard FSP models (cf. [BHM, 3.2], [PW, 2.3]). If M is a simplicial monoid, define a Γ-space $S^0[M]$ by

$$S^0[M]\,(n) \;=\; n \wedge M_+ \;.$$

There is a unit map $S^0 \longrightarrow S^0[M]$ induced by the unit of M and a multiplication map $S^0[M] \wedge S^0[M] \longrightarrow S^0[M]$ induced by the multiplication of M. If R is a simplicial ring, the very special Γ-space $\tilde{R}$ associated to the underlying simplicial abelian group of R also has a unit map and a multiplication. We denote the DFSP thus obtained by HR and refer to it as the Eilenberg-MacLane DFSP for R. We remark that the functor H is fully faithful and embeds simplicial rings as a reflexive subcategory of DFSPs. The reflector is obtained by considering a DFSP as a simplicial object of Γ-spaces and applying π_0 in every simplicial degree. More examples of DFSPs arise from simplicial algebraic theories as described in section 2.3. We indicate in Appendix C that in fact every DFSP is weakly equivalent to one arising from such a theory.

When the underlying Γ-space of a DFSP is extended to a functor from the category of pointed simplicial sets, it inherits the structure of an FSP. Furthermore, up to weak equivalence every connective FSP arises this way. Here is a brief sketch of the argument: a given FSP F can first be stabilized to an FSP QF defined via

$$QF\,(K) = \mathrm{hocolim}_I \; \mathrm{Sing}\,\Omega^n |F(S^n \wedge K)|$$

where I is the category of finite sets and injective maps (details of this construction can be found in [Bö], [PW, 2] or [BHM, 3]). There is a map $F \longrightarrow QF$ of FSPs which is a weak equivalence. The spectrum underlying QF is an Ω-spectrum in the sense that the maps $|QF(K)| \longrightarrow \Omega|QF(\Sigma K)|$ adjoint to $\Sigma QF(K) \longrightarrow QF(\Sigma K)$ are weak equivalences (not necessarily homeomorphisms). QF can thus be restricted to finite pointed sets without losing information about the stable homotopy type. More precisely, let DF denote the DFSP obtained by restricting the domain of the functor QF to the category Γ of finite pointed sets. Then there is a weak equivalence of FSPs from the FSP obtained by extending DF back again to all pointed simplicial sets to QF.

Definition 1.3.2 *A left module over a DFSP R is a Γ-space M together with a map of Γ-spaces $\alpha : R \wedge M \longrightarrow M$ which makes the following diagrams commute*

(Associativity)
$$\begin{array}{ccc} R \wedge R \wedge M & \xrightarrow{\mu \wedge \mathrm{id}} & R \wedge M \\ {\scriptstyle \mathrm{id} \wedge \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ R \wedge M & \xrightarrow[\alpha]{} & M \end{array}$$

(Unit)
$$\begin{array}{ccc} M \cong S^0 \wedge M & \xrightarrow{\eta \wedge \mathrm{id}} & R \wedge M \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \alpha} \\ & & M \end{array}$$

A morphism of left R-modules is a map of Γ-spaces $f : M \longrightarrow N$ such that the diagram

$$\begin{array}{ccc} R \wedge M & \xrightarrow{\alpha} & M \\ {\scriptstyle \mathrm{id} \wedge f}\downarrow & & \downarrow{\scriptstyle f} \\ R \wedge N & \xrightarrow[\alpha]{} & N \end{array}$$

*commutes. We denote the category of left R-modules by R-*mod*.*

One similarly defines right modules and bimodules. The category of S^0-modules is isomorphic to the category of Γ-spaces. For a DFSP R one defines the opposite DFSP R^{op} by twisting the multiplication map with the switch map $R \wedge R \longrightarrow R \wedge R$. Then the category of right R-modules is isomorphic to the category of left R^{op}-modules. For DFSPs R and S, the Γ-space $R \wedge S$ is naturally a DFSP and an R-S-bimodule is the same thing as an $(R \wedge S^{\mathrm{op}})$-module.

Another formal consequence of having a closed symmetric monoidal smash product is that the category of R-modules inherits a smash product and function objects. More precisely, let M be a right R-module, N a left R-module. Define the smash product $M \wedge_R N$ as the coequalizer, in the category of Γ-spaces, of the two maps $M \wedge R \wedge N \rightrightarrows M \wedge N$ given by the action of R on M and N respectively. If M happens to be an S-R-bimodule and N an R-T-bimodule, then $M \wedge_R N$ is naturally an S-T-bimodule. In particular, if R is a commutative DFSP, the notions of left and right module coincide and agree with the notion of a symmetric bimodule. In this case $\wedge_R$ is an internal symmetric monoidal smash product for R-modules.

The analogous phenomenon happens for internal function objects. Given two left R-modules M and N, there are two maps of Γ-spaces $\mathrm{Hom}\,(M, N) \rightrightarrows \mathrm{Hom}\,(R \wedge M, N)$. The first is induced by the action map α for M, the other is adjoint to the map

$$R \wedge (M \wedge \mathrm{Hom}(M, N)) \xrightarrow{\mathrm{id} \wedge \mathrm{eval}} R \wedge N \xrightarrow{\alpha} N\ .$$

Here the evaluation map $M \wedge \mathrm{Hom}(M,N) \longrightarrow N$ is adjoint to the identity of $\mathrm{Hom}(M,N)$. Now define the function Γ-space $\mathrm{Hom}_R(M,N)$ of R-maps between M and N to be the equalizer of these two maps

$$\mathrm{Hom}(M,N) \rightrightarrows \mathrm{Hom}(R \wedge M, N) \ .$$

As was the case with the smash product, the usual properties known from algebra carry over: if M is an R-S-bimodule and N an R-T-bimodule, then $\mathrm{Hom}_R(M,N)$ is naturally an S-T-bimodule. In particular, over a commutative DFSP we get an internal function object on the module category.

Finally, the adjunction between smash and function Γ-spaces extends as follows: consider a left S-module N, an R-S-bimodule M and a left R-module W. Then there is an isomorphism of Γ-spaces, natural in all three variables

$$\mathrm{Hom}_R(M \wedge_S N, W) \ \cong \ \mathrm{Hom}_S(N, \mathrm{Hom}_R(M,W)) \ .$$

In particular, over a commutative DFSP the internal smash and function objects are again adjoint.

The functor π_0 which assigns to a Γ-space its zeroth homotopy group (i.e., the zeroth stable homotopy group of the associated spectrum), has as right adjoint the functor which takes an abelian group A to the very special Γ-space $\widetilde{A}$ representing the Eilenberg-MacLane spectrum [BF, 5.5]. These functors are actually symmetric monoidal functors with respect to the smash product of Γ-spaces and the tensor product of abelian groups. This means that there are natural maps

$$\pi_0 X \otimes \pi_0 Y \ \longrightarrow \ \pi_0(X \wedge Y)$$
$$\widetilde{A} \wedge \widetilde{B} \ \longrightarrow \ \widetilde{A \otimes B}$$

which are associative, commutative and unital (with respect to the isomorphism $\pi_0 S^0 \cong \mathbb{Z}$).

Consequently, for a DFSP R, $\pi_0 R$ is an ordinary ring and the adjunction map

$$R \ \longrightarrow \ \widetilde{\pi_0 R} = H(\pi_0 R)$$

is a map of DFSPs. Furthermore, for an R-module M, $\pi_0 M$ is a $\pi_0 R$-module. In fact, we have

Lemma 1.3.3 *The functors π_0 and $\tilde{}$ pass to adjoint functors*

$$\text{(DFSPs)} \ \rightleftarrows \ \text{(discrete rings)}$$
$$R\text{-mod} \ \rightleftarrows \ \text{(discrete } \pi_0 R\text{-modules)}$$

We now establish the model category structure for modules over a DFSP. It is created by the forgetful functor R-mod$\longrightarrow \mathcal{GS}$. By this we mean that a map of R-modules is called a weak equivalence (resp. fibration) if it is a stable weak equivalence (resp. stable fibration) as a map of Γ-spaces. A map of R-modules will be called a cofibration if it has the LLP with respect to all acyclic fibrations in R-mod. We will see later (Thm. 2.3.4 applied to the theory of R-modules) that for a simplicial ring R, the homotopy theory of HR-modules is equivalent to that of simplicial R-modules.

Theorem 1.3.4 *For a DFSP R, the category of left R-modules is a cofibrantly generated closed simplicial model category. The forgetful functor to Γ-spaces preserves limits, colimits, products with and mapping objects from simplicial sets. Every cofibration of R-modules is injective. Every R-module is small relative to the whole category of R-modules. If M is a cofibrant left R-module then the functor $-\wedge_R M$ maps weak equivalences of right R-modules to weak equivalences of Γ-spaces.*

Proof: We use that the category of R-modules can be viewed as the category of algebras over the triple $R\wedge -$ on the category of Γ-spaces. This allows us to apply lemma A.3. Γ-spaces are a cofibrantly generated model category in which all objects are small relative to the whole category (lemma 1.2.4). Viewed as an endofunctor on the category of Γ-spaces, $R\wedge -$ has a right adjoint $\mathrm{Hom}(R, -)$, thus it preserves all colimits. Note that the functor $\mathrm{Hom}(R, -)$ naturally takes values in R-modules and that it provides a right adjoint to the forgetful functor $R\text{-mod}\longrightarrow \mathcal{GS}$. This forgetful functor thus has both a left and a right adjoint, so it preserves both limits and colimits.

We now verify condition (1) of lemma A.3. If $A \overset{\sim}{\rightarrowtail} B$ is an acyclic cofibration of Γ-spaces, then B/A is cofibrant and weakly contractible. Thus $R\wedge(B/A)$ is weakly contractible [Ly, 4.18]. By [Ly, 3.23], $R\wedge A \longrightarrow R\wedge B$ is still injective. Since its cofibre is contractible, this map is thus a weak equivalence by the long exact sequence 1.1.4. A pushout square of R-modules

$$\begin{tikzcd} R\wedge A \arrow[r, "\sim"] \arrow[d] & R\wedge B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is also a pushout when considered as a square of Γ-spaces. Hence the square is pointwise a pushout of simplicial sets. So the lower horizontal map is injective since the upper map is. By the five lemma applied to the long exact sequences on homotopy groups (lemma 1.1.4), the lower map is thus a weak equivalence. Furthermore, any transfinite composite of maps of Γ-spaces which are injective weak equivalences is again a weak equivalence since homotopy groups of Γ-spaces commute with such filtered direct limits. So if J_R is a set of R-module maps of the form $R\wedge A \longrightarrow R\wedge B$ for $A \overset{\sim}{\rightarrowtail} B$ an acyclic cofibration of Γ-spaces, then any map in $J_R\text{-cof}_{\text{reg}}$ is a weak equivalence. This proves hypothesis (1) of lemma A.3 which then shows that the category of R-modules is a cofibrantly generated closed simplicial model category.

Since the functor $R\wedge -$ commutes with $-\otimes K$ for K a simplicial set, products of R-modules with simplicial sets are also formed on underlying Γ-spaces. Every generating cofibration is injective and injectivity is closed under pushouts, transfinite composition and retracts. Since every cofibration of R-modules can be obtained using these operations, cofibrations of R-modules are injective.

It remains to show that smashing over R with a cofibrant module M preserves weak equivalences. To establish this property, we show that the class of left R-modules M for which $-\wedge_R M$ maps weak equivalences of right R-modules to weak equivalences of Γ-spaces is closed under retracts, cobase extensions along maps of the type $R\wedge A \longrightarrow R\wedge B$ for $A \rightarrowtail B$ a cofibration Γ-spaces and transfinite composites. By the small object argument, every cofibrant left R-module can be obtained from the trivial module by performing these kinds of operations, so every cofibrant R-module belongs to this class. Closure under retracts holds because weak equivalences are closed under retracts. Next consider a cofibration of Γ-spaces $A \rightarrowtail B$ and a pushout square of R-modules

$$\begin{array}{ccc} R \wedge A & \rightarrowtail & R \wedge B \\ \downarrow & & \downarrow \\ M & \rightarrowtail & M' \end{array}$$

and suppose that smashing with M over R preserves weak equivalences. We claim that then M' has the same property. To see this, smash the square with a weak equivalence to obtain a map between pushout squares of Γ-spaces in which all horizontal maps are injective. This provides long exact sequences of homotopy groups (lemma 1.1.4) and we conclude by the five lemma. Finally, homotopy groups commute with filtered colimits over injective maps and smashing over R commutes with colimits and maps cofibrations to injective maps. Hence the class of maps in question is closed under transfinite composition.

□

If $R \longrightarrow S$ is a map of DFSPs, we get an adjoint functor pair analogous to restriction and extension of scalars: any S-module becomes an R-module if we let R act through the map. This functor has a left adjoint mapping an R-module M to the S-module $S \wedge_R M$. Since smashing with cofibrant modules preserves weak equivalence, an application of Quillen's adjoint functor theorem ([Q1, II 4 Thm. 3], [DS, Thm. 9.7]) gives

Corollary 1.3.5 *Given a weak equivalence of DFSPs $R \xrightarrow{\sim} S$, the total derived functors of restriction and extension of scalars provide inverse equivalences of homotopy theories*

$$\mathrm{Ho}\,(R\text{-mod}) \;\cong\; \mathrm{Ho}\,(S\text{-mod}) \;.$$

We define the derived smash product $M \wedge_R^L N$ of a left R-module M and a right R-module N in the usual way: choose a cofibrant right R-module N^{c} and a weak equivalence $N^{\mathrm{c}} \xrightarrow{\sim} N$ and set $M \wedge_R^L N = M \wedge_R N^{\mathrm{c}}$. It follows from the homotopy invariance statement in Theorem 1.3.4 that the derived smash product is well defined up to weak equivalence.

There is a standard Tor spectral sequence calculating the homotopy of $M \wedge_R^L N$. The idea is the same as in [Q1, II.6 b] or [EKMM, IV 5], namely geometrically realizing a free resolution of $\pi_* M$ by fibration sequences of R-modules. The technical details have to be adapted to the Γ-space setup.

Lemma 1.3.6 *Let R be a DFSP, M a right R-module and N a left R-module. There is a natural convergent first quadrant spectral sequence*

$$\mathrm{E}^2_{p,q} \;=\; \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)_q \quad\Longrightarrow\quad \pi_{p+q}\,(M \wedge_R^L N)$$

with differential of standard homological form.

Proof: By functorial replacement we can assume that M is fibrant and N is cofibrant. Denote by $F(M)$ the wedge

$$F(M) = \bigvee_{\mathrm{Hom}_R(\Sigma^n R, M)} \Sigma^n R$$

of copies of suspensions of R considered as a right module over itself; the copies are indexed by all R-module maps $\Sigma^n R \longrightarrow M$ for varying n. There is a natural evaluation map $F(M) \longrightarrow M$. Since M is fibrant, every element in $\pi_n M$ is represented by a map of simplicial sets $S^n \longrightarrow M(1)$, hence by a map of R-modules $\Sigma^n R \longrightarrow M$. We conclude that the map $F(M) \longrightarrow M$ is surjective on homotopy groups. Furthermore, $\pi_* F(M)$ is free over $\pi_* R$.

Set $P_0 = M$ and define P_{i+1} inductively as the homotopy fibre, in R-mod, of the map $F(P_i) \longrightarrow P_i$ as follows. P_{i+1} is the categorical fibre of the fibration which is part of the functorial factorization

$$F(P_i) \overset{\sim}{\rightarrowtail} E(P_i) \twoheadrightarrow P_i$$

of $F(P_i) \longrightarrow P_i$ given by the small object argument. Then P_{i+1} is fibrant, hence $F(P_{i+1}) \longrightarrow P_{i+1}$ again induces epimorphisms on homotopy groups. Thus taking homotopy groups of the sequence of R-modules

$$\cdots \longrightarrow E(P_{i+1}) \longrightarrow E(P_i) \longrightarrow \cdots \longrightarrow E(P_0) \longrightarrow M$$

gives a functorial free resolution of $\pi_* M$ over $\pi_* R$.

$E(P_i) \longrightarrow P_i$ is surjective on π_0. So by lemma 1.1.4 the fibration sequence

$$P_{i+1} \longrightarrow E(P_i) \twoheadrightarrow P_i$$

is a homotopy cofibre sequence, i.e., the map from the homotopy cofibre of $P_{i+1} \longrightarrow E(P_i)$ to P_i is a weak equivalence. The sequence thus remains a homotopy cofibre sequence upon smashing with N over R. The long exact sequences of homotopy groups of the homotopy cofibre sequences

$$P_{i+1} \wedge_R N \longrightarrow E(P_i) \wedge_R N \longrightarrow P_i \wedge_R N$$

assemble into an exact couple of standard homological form with $\mathrm{E}^1_{p,q} \cong \pi_{p+q}(E(P_p) \wedge_R N)$. Since $E(P_p) \wedge_R N$ is weakly equivalent to

$$F(P_p) \wedge_R N \cong \bigvee_{\mathrm{Hom}_R(\Sigma^n R, P_p)} \Sigma^n N ,$$

we have $\mathrm{E}^1_{p,q} \cong (\pi_* E(P_p) \otimes_{\pi_* R} \pi_* N)_{p+q}$. Under this isomorphism, the differential d^1 is isomorphic to the map obtained by tensoring the above free resolution of $\pi_* M$ with $\pi_* N$. Thus the E^2-term of the spectral sequence calculates the advertised Tor groups. There are no problems with convergence since all homotopy groups vanish in negative dimensions.

□

Our next task is to show that DFSPs form a model category with fibrations and weak equivalences defined on the underlying Γ-spaces. We will in fact prove the more general result that algebras over a commutative DFSP form a model category. If that commutative DFSP is the sphere spectrum S^0, this gives the case of the category of DFSPs.

Definition 1.3.7 *Let R be a commutative DFSP. An R-algebra is a monoid in the symmetric monoidal category of R-modules with respect to smash product over R. Explicitly, an R-algebra is an R-module A together with R-module maps*

$$R \longrightarrow A \qquad \text{and} \qquad A \wedge_R A \longrightarrow A$$

making the analogous associativity and unit diagrams as for a DFSP commute. A map of R-algebras is a map of R-modules which commutes with the unit and multiplication map.

It is a formal property of symmetric monoidal categories that specifying an R-algebra structure on a Γ-space A is the same as giving A a DFSP structure together with a map of DFSPs $f\colon R \longrightarrow A$ which is central in the sense that the following diagram commutes

$$\begin{tikzcd}
R \wedge A \arrow[r, "\text{switch}"] \arrow[d, "f \wedge \mathrm{id}"'] & A \wedge R \arrow[r, "\mathrm{id} \wedge f"] & A \wedge A \arrow[d, "\mu"] \\
A \wedge A \arrow[rr, "\mu"'] & & A
\end{tikzcd}$$

Call a map of R-algebras a fibration or weak equivalence if it is a fibration or weak equivalence on underlying R-modules (equivalently: a stable fibration or stable weak equivalence on underlying Γ-spaces). Call a map of R-algebras a cofibration if it has the LLP with respect to all acyclic fibrations.

Theorem 1.3.8 *The category of R-algebras is a cofibrantly generated closed simplicial model category. Every cofibrant R-algebra is also cofibrant as an R-module. In particular, the category of DFSPs is a cofibrantly generated closed simplicial model category.*

Proof: We apply Theorem B.1 which gives a condition to ensure that the monoids in a symmetric monoidal model category again form a model category. This criterion is applied to the symmetric monoidal category of R-modules with respect to smash product over R. This is a cofibrantly generated closed model category in which every object is small relative to the whole category (Theorem 1.3.4). It remains to verify the smashout axiom (2) and the monoid axiom (3).

Ad (2). Consider two cofibrations of R-modules $A \rightarrowtail B$ and $K \rightarrowtail L$. We have to show that the smashout

$$A \wedge_R L \cup_{A \wedge_R K} B \wedge_R K \longrightarrow B \wedge_R L$$

is a cofibration and that it is an acyclic cofibration if one of the former maps is. Suppose first that both maps are induced from cofibrations of Γ-spaces, i.e., that they are isomorphic to maps $R \wedge (\bar{A} \rightarrowtail \bar{B})$ and $R \wedge (\bar{K} \rightarrowtail \bar{L})$ where $\bar{A} \rightarrowtail \bar{B}$ and

$\bar{K} \rightarrowtail \bar{L}$ are cofibrations of Γ-spaces. Then the smashout of R-modules in question is induced from the map of Γ-spaces

$$\bar{A}\wedge\bar{L} \cup_{\bar{A}\wedge\bar{K}} \bar{B}\wedge\bar{K} \rightarrowtail \bar{B}\wedge\bar{L} \quad .$$

Since the latter is a cofibration of Γ-spaces by [Ly, 5.1], the smashout of R-modules is a cofibration of R-modules. In particular, the smashout property holds for all generating cofibrations in the category of R-modules.

For a cofibration $i\colon A \rightarrowtail B$ of R-modules denote by $G(i)$ the class of maps of R-modules for which the smashout property with respect to i holds. By adjointness of $\wedge_R$ with Hom_R, a map is in $G(i)$ if and only if for all acyclic fibrations of R-modules $X \xrightarrow{\sim} Y$ it has the left lifting property with respect to the map

$$\mathrm{Hom}_R(B,X) \longrightarrow \mathrm{Hom}_R(B,Y) \times_{\mathrm{Hom}_R(A,Y)} \mathrm{Hom}_R(A,X) \ .$$

$G(i)$ is thus closed under cobase change, transfinite composition and retracts. If $i\colon A \rightarrowtail B$ is a generating cofibration, $G(i)$ contains all generating cofibrations by the previous paragraph. Because of the closure properties it thus contains all cofibrations of R-modules. Reversing the roles of i and an arbitrary cofibration $j\colon K \rightarrowtail L$, we thus know that $G(j)$ contains all generating cofibrations. Again by the closure properties, $G(j)$ contains all cofibrations, which proves the smashout axiom for two cofibrations of R-modules. A similar proof shows that the smashout is an acyclic cofibration when one of the constituents is.

Ad (3): We have to consider an acyclic cofibration of R-modules $A \xrightarrow{\sim} B$, R-modules W, X, P and a pushout square

$$\begin{array}{ccc} A\wedge_R W & \longrightarrow & B\wedge_R W \\ \downarrow & & \downarrow \\ X & \longrightarrow & P \end{array}$$

of R-modules. Note that the class of R-module maps $A \longrightarrow B$ for which the map $A\wedge_R W \longrightarrow B\wedge_R W$ is an injective weak equivalence is closed under cobase change, transfinite composition and retracts. It furthermore contains the generating acyclic cofibrations. This class thus contains all acyclic cofibrations of R-modules, so the upper map in the square is an injective weak equivalence. Since pushouts of R-modules are formed on underlying Γ-spaces, the map $X \longrightarrow P$ is thus also an injective weak equivalence as is any transfinite composition of such maps. This proves the monoid axiom (3) needed in Theorem B.1. □

A concluding remark about commutativity of multiplication for DFSPs is in order. One can consider the full subcategory of the *commutative* DFSPs. This category is complete and cocomplete and still has a notion of weak equivalences (on underlying Γ-spaces). However, the following variation of a well-known argument shows that with weak equivalences and fibrations defined on underlying Γ-spaces, the category of commutative DFSPs is definitely *not* a model category. If commutative DFSPs became a model category this way, the sphere spectrum DFSP S^0 had a weakly equivalent fibrant model inside this category. Evaluating this fibrant representative on $1 \in \mathrm{Ob}\,\Gamma$ would give a commutative simplicial monoid weakly equivalent to QS^0. But it is known that the multiplicative E_∞-structure on QS^0 is not equivalent to such a commutative simplicial monoid.

2 Homotopy of algebraic theories

2.1 Algebraic theories

F. W. Lawvere has introduced the notion of an algebraic theory [Lw]. It formalizes the concept of an algebraic object as a set together with n-ary operations for various $n \in \mathbb{N}$ and equational relations. A detailed exposition of algebraic theories can be found in [Bor, Sec. 3], which will also serve as our main reference. To do homotopy theory, we use algebraic theories which are enriched over the category of simplicial sets; compare [Bck, (6)], [BV, Chpt. II] or [SV1,2] for the version enriched over topological spaces. For many purposes, topological and simplicial theories can be used interchangeably: the geometric realization and singular complex functors commute with finite products, so applying these to the simplicial hom set or the hom spaces respectively gets one from simplicial to topological theories and vice versa. We will discuss two examples of algebraic theories in some detail at the end of this subsection. A selection of further examples appears in section 3.

Definition 2.1.1 *A* **simplicial theory** *is a simplicial category T with the following properties:*

- *T has a discrete set of objects equal to the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.*
- *The object n is the n-fold product of the object 1: there are morphisms $p_i^n \in \mathrm{Hom}_T(n,1)_0$, $i = 1, \ldots, n$, such that the induced map*

$$\mathrm{Hom}_T(k,n) \longrightarrow \mathrm{Hom}_T(k,1)^n$$

  is an isomorphism of simplicial sets. In particular, 0 is a terminal object.
- *T is a pointed category: $\mathrm{Hom}_T(0,1)$ is a (simplicial) point.*

A morphism of simplicial theories is a product preserving simplicial functor which is the identity on objects.

The term "finitary pointed simplicial algebraic theory" would be a more precise, but too clumsy, name for what we call a simplicial theory. If all the simplicial hom sets are discrete and if one omits the condition that $\mathrm{Hom}_T(0,1)$ be a point, one recovers the original definition of an algebraic theory. One should think of the simplicial set $\mathrm{Hom}_T(n,1)$ as the space of n-ary operations in the theory T. A simplicial theory will be called **discrete** if all the simplicial hom sets are discrete. Since morphisms of theories are always the identity on objects, a simplicial theory is the same as a simplicial object of pointed discrete theories.

Definition 2.1.2 *If T is a simplicial theory, a T-algebra X is a product preserving simplicial functor $X : T \longrightarrow \mathcal{S}$ to the category of pointed simplicial sets. A morphism of T-algebras is a natural transformation of functors. $X(1)$ is called the underlying simplicial set of the T-algebra X.*

What we call a T-algebra is sometimes called a model of T in the category of pointed simplicial sets. Note that even if T happens to be a discrete theory, T-algebras are still allowed to have a simplicial direction, unless otherwise stated. For a T-algebra X, each morphism $\phi \in \mathrm{Hom}_T(n,1)_0$ gives rise to a map $X(1)^n \cong X(n) \xrightarrow{X(\phi)} X(1)$. This justifies thinking of a T-algebra as the underlying simplicial set together with n-ary operations parametrized by the simplicial set $\mathrm{Hom}_T(n,1)$.

Free T-algebras. The forgetful functor T-alg$\longrightarrow \mathcal{S}$, $X \mapsto X(1)$ has a left adjoint, the free T-algebra functor

$$F^T : \mathcal{S} \longrightarrow T\text{-alg} .$$

For a pointed simplicial set Y, the underlying simplicial set of $F^T(Y)$ is given by

$$F^T(Y)(1) \;=\; \coprod_{n\geq 0} \mathrm{Hom}_T(n,1) \times Y^n / \sim$$

(One can take the wedge instead of the coproduct since everything in sight is pointed). The indicated relations come from the set operations which are present in any theory. If n runs through the objects of the category Γ of finite pointed sets, then $\mathrm{Hom}_T(n,1)$ is covariant in n whereas Y^n is contravariant in n, so $F^T(Y)(1)$ is the coend of

$$\Gamma \times \Gamma^{\mathrm{op}} \longrightarrow \mathcal{S}, \qquad (n,m) \mapsto \mathrm{Hom}_T(n,1) \times Y^m$$

The composite of the free T-algebra functor with the forgetful functor has the structure of a triple on the category of pointed simplicial sets. We will usually denote this triple by the same letter T. A triple also has a notion of algebras over it; however the category of algebras over the triple derived from T is equivalent to the category of T-algebras, so our abuse of notation creates no ambiguity as to what a T-algebra is. Note that the triple, considered as a functor from the category of pointed simplicial sets to itself, is degreewise evaluable, i.e., it comes from a Γ-space.

For $n \in \mathbb{N}$, the representable functor $\mathrm{Hom}_T(n,-)$ is a T-algebra isomorphic to the free T-algebra generated by a finite set with n non-basepoints. In fact, this gives an equivalence of simplicial categories of T^{op} with the full subcategory of the finitely generated free T-algebras inside T-alg (cf. [Bor, Prop. 3.8.5]).

Algebraic functors. A morphism of simplicial theories $\phi : R \longrightarrow T$ induces an adjoint functor pair analogous to restriction and extension of scalars: if X is a T-algebra, i.e., a product preserving simplicial functor $X : T \longrightarrow \mathcal{S}$, then the composite $X\phi : R \longrightarrow \mathcal{S}$ is again product preserving and simplicial, so it defines an R-algebra denoted by $\phi^* X$. Note that the underlying simplicial set of $\phi^* X$ is the same as for X. Functors of the form ϕ^* are called algebraic functors. ϕ^* has a left adjoint ϕ_* ([Bor, Thm. 3.7.7] states this for discrete theories, for simplicial theories it follows by degreewise application of the discrete case). Looking at the induced algebraic functors is in fact a convenient way of describing morphisms of theories: an arbitrary simplicial functor $W : R\text{-alg} \longrightarrow T\text{-alg}$ is algebraic (i.e., of the form $W = \phi^*$ for a unique theory morphism ϕ) if and only if the diagram of functors

$$\begin{tikzcd} T\text{-alg} \arrow[rr, "W"] \arrow[dr, "\text{forget}"'] & & R\text{-alg} \arrow[dl, "\text{forget}"] \\ & \mathcal{S} & \end{tikzcd}$$

commutes [Bor, Prop. 3.9.2].

Two examples. To illustrate the above definitions, we discuss how two familiar examples of algebraic theories fit into the abstract framework. We start with the theory of groups which we denote by $\mathbb{G}$. This is a discrete theory with $\mathrm{Hom}_{\mathbb{G}}(k, n)$ equal to the set of n-tuples of elements of the free group on k generators $\gamma_1, \ldots, \gamma_k$. The identity morphism of $\mathrm{Hom}_{\mathbb{G}}(k, k)$ is the tuple $(\gamma_1, \ldots, \gamma_k)$ whose i-th component is the word consisiting only of the i-th generator. Composition

$$\mathrm{Hom}_{\mathbb{G}}(k, n) \times \mathrm{Hom}_{\mathbb{G}}(m, k) \longrightarrow \mathrm{Hom}_{\mathbb{G}}(m, n)$$

is given by substitution. This means if $(w_1, \ldots, w_n)$ and $(v_1, \ldots, v_k)$ are tuples of words in k resp. m generators, their composite is the tuple

$$(w_1(v_1, \ldots, v_k), \ldots, w_n(v_1, \ldots, v_k)) \ .$$

Here $w_i(v_1, \ldots, v_k)$ means that in the word w_i each generator γ_j is substituted by the entire word v_j.

We claim that the category of $\mathbb{G}$-algebras is equivalent to the category of simplicial groups. So let $X : \mathbb{G} \longrightarrow \mathcal{S}$ be a $\mathbb{G}$-algebra, i.e., a product preserving functor. Then the underlying simplicial set $X(1)$ has a group structure as follows. The word $\gamma_1\gamma_2$ is an element of $\mathrm{Hom}_{\mathbb{G}}(2, 1)$, so it gives rise to a multiplication map

$$\mu = X(\gamma_1\gamma_2) : X(1)^2 \cong X(2) \longrightarrow X(1) \ .$$

The word γ_1^{-1} is an element of $\mathrm{Hom}_{\mathbb{G}}(1, 1)$, so it gives rise to an inverse map

$$\iota = X(\gamma_1^{-1}) : X(1) \longrightarrow X(1) \ .$$

The associativity and inverse conditions are codified in the category $\mathbb{G}$. We explain this for associativity. We consider the two elements $(\gamma_1\gamma_2, \gamma_3)$ and $(\gamma_1, \gamma_2\gamma_3)$ of $\mathrm{Hom}_{\mathbb{G}}(3, 2)$. Since multiplication in the free group on 3 generators is associative, the equality

$$(\gamma_1\gamma_2) \circ (\gamma_1\gamma_2, \gamma_3) \ = \ \gamma_1\gamma_2\gamma_3 \ = \ (\gamma_1\gamma_2) \circ (\gamma_1, \gamma_2\gamma_3)$$

holds in $\mathrm{Hom}_{\mathbb{G}}(3, 1)$. The product preserving functor X takes this relation to the associtivity condition $\mu \circ (\mu \times \mathrm{id}) = \mu \circ (\mathrm{id} \times \mu)$. Hence the underlying simplicial set of a $\mathbb{G}$-algebra is naturally a group.

For the converse let H be a simplicial group. Define a functor $\bar{H} : \mathbb{G} \longrightarrow \mathcal{S}$ on objects by $\bar{H}(n) = H^n$. The behaviour on morphisms is again given by substitution: for $w = (w_1, \ldots, w_n)$ from $\mathrm{Hom}_{\mathbb{G}}(k, n)$, define

$$\bar{H}(w)(h_1, \ldots, h_k) \ = \ (w_1(h_1, \ldots, h_k), \ldots, w_n(h_1, \ldots, h_k)) \ .$$

Here $w_i(h_1, \ldots, h_k)$ means that the elements $h_j \in H$ are substituted for the generators γ_j into the word w_i, and then multiplication is carried out in the group H. We omit the verification that this $\bar{H}$ is a functor and that we in fact described an equivalence of categories

$$\mathbb{G}\text{-alg} \;\cong\; (\text{simplicial groups}) \ .$$

For the second example we fix a ring R, possibly simplicial, and we define the theory R^{mod} of left R-modules. In this case $\mathrm{Hom}_{R^{mod}}(k, n)$ is the (simplicial) set of $(n \times k)$-matrices of elements from R. The identity morphism is the identity matrix and composition is given by matrix multiplication. We claim that an R^{mod}-algebra is the same thing as a (simplicial) left R-module. So given an R^{mod}-algebra $X : R^{mod} \longrightarrow \mathcal{S}$, then underlying simplicial set $X(1)$ inherits a left R-module structure as follows. The column matrix

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathrm{Hom}_{R^{mod}}(2, 1)$$

gives rise to an addition map

$$\alpha = X(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) : X(1)^2 \cong X(2) \longrightarrow X(1) \ .$$

For every element $r \in R$, the $(1{\times}1)$-matrix (r) in $\mathrm{Hom}_{R^{mod}}(1, 1)$ gives rise to a left scalar multiplication map

$$\lambda_r = X(r) : X(1) \longrightarrow X(1) \ .$$

Again, all the laws for the action of a ring on an abelian group are encoded in the category R^{mod}. We exemplify this with the law of distributivity. The functor X takes the matrix equality

$$(r) \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} r \\ r \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

in $\mathrm{Hom}_{R^{mod}}(2, 1)$ to the law of distributivity $\lambda_r \circ \alpha = \alpha \circ (\lambda_r \times \lambda_r)$.

Conversely, let M be a (simplicial) left R-module. A functor $\bar{M} : R^{mod} \longrightarrow \mathcal{S}$ is defined on objects by $\bar{M}(n) = M^n$. $\bar{M}$ takes a morphism in R^{mod} to left multiplication with the representing matrix. This defines an equivalence of categories

$$R^{mod}\text{-alg} \;\cong\; R\text{-mod} \ .$$

The construction $R \mapsto R^{mod}$ is functorial in the (simplicial) ring R. Given a ring homomorphism $\phi : R \longrightarrow S$, a product preserving functor $\phi^{mod} : R^{mod} \longrightarrow S^{mod}$ is defined by sending a matrix $(a_{ij}) \in \mathrm{Hom}_{R^{mod}}(k, n)$ to the matrix $(\phi(a_{ij})) \in \mathrm{Hom}_{S^{mod}}(k, n)$. The resulting functor

$$-^{mod} \;:\; (\text{rings}) \;\longrightarrow\; (\text{theories})$$

is fully faithful. Because of this fact we will follow the standard practice of identifying rings with the theories of their left modules and we omit the superscript ^{mod}. Also, under the above identifications of R^{mod}-algebras with R-modules, the functor

$$\phi_*^{mod} : R^{mod} \longrightarrow S^{mod}$$

corresponds to

$$S \otimes_R - : R\text{-mod} \longrightarrow S\text{-mod} .$$

These two examples illustrate the general pattern: an arbitrary algebraic theory can be recovered from its category of algebras as the opposite of the full subcategory of finitely generated free objects. There is also a criterion for when a category with an adjoint functor pair to the category of sets is (equivalent to) the category of algebras over some algebraic theory, see [Bor, Thm. 3.9.1]. The two above examples are somewhat special because in each case the operations are generated by unary and binary operations, and all relations involved at most three generators. This need not be true for general theories. In fact, what makes Lawvere's notion of algebraic theories so elegant is that generating operations and relations are not mentioned at all. Instead, the category underlying the theory encodes *all* possible operations and their relations at the same time.

2.2 Unstable homotopy

We now want to show that algebras over a simplicial theory T admit a reasonable homotopy theory. Apart from having a model category structure, the homotopy theory of T-algebras shares a number of structural features with the homotopy theory of spaces. E.g. we will establish Blakers-Massey, Hurewicz and Whitehead theorems for T-algebras.

We are not claiming any particular originality as far as the existence of a model category structure is concerned. For discrete theories, this is due to Quillen [Q1, II.4 Thm. 4] and for topological theories it is proven in [SV2, Thm. B]. Call a map of T-algebras a weak equivalence or fibration if it is a weak equivalence or fibration on underlying simplicial sets respectively. Call a map of T-algebras a cofibration if it has the LLP with respect to all acyclic fibrations.

Theorem 2.2.1 *Let T be a simplicial theory. Then the category of T-algebras is a cofibrantly generated (Def. A.2) closed simplicial model category. Every cofibration of T-algebras is injective on underlying simplicial sets.*

Proof: We noted above that the category of T-algebras is (equivalent to) the category of algebras over the free T-algebra triple. The fact that the latter functor commutes with filtered colimits can either be seen from its explicit description (after definition 2.1.2) or by recalling that filtered colimits of sets commute with finite products [MacL, IX 2, Thm. 1]. The category of pointed simplicial sets is a cofibrantly generated closed simplicial model category, so the above theorem is an application of lemma A.3. To verify criterion (2) of that lemma, we have to produce a functor $Q : T\text{-alg} \longrightarrow T\text{-alg}$ and a natural weak equivalence $X \xrightarrow{\sim} QX$ such that QX is fibrant for all X. We use the adjoint functor pair between simplicial sets and topological spaces given by geometric realization and singular complex and set $QX = \mathrm{Sing}\,|X|$. The key point is that both functors commute with finite products (when giving spaces and products the compactly generated topology). This property implies that if X is underlying a T-algebra, then the singular complex of the realization of X has a natural T-algebra structure such that the map $X \longrightarrow \mathrm{Sing}\,|X|$ is a map of T-algebras. But the singular complex of any space is fibrant as a simplicial set and the map $X \xrightarrow{\sim} QX$ is a weak equivalence.

It remains to verify that cofibrations are injective on underlying simplicial sets. The cofibrations in the model category of T-algebras are generated (by pushout, transfinite composition and retract) by the T-algebra maps freely generated by the inclusions of the boundaries into the simplicial simplices. So it suffices to check that the lower map in a pushout square of T-algebras

$$\begin{tikzcd} F^T(\partial\Delta^n_+) \arrow[r, tail] \arrow[d] & F^T(\Delta^n_+) \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

is injective on underlying simplicial sets. This can be checked in every simplicial degree k separately, where we have a pushout square of T_k-algebras (we view T as a simplicial object of discrete theories). But in every simplicial degree there exist a retraction to the upper horizontal map (not compatible with the simplicial structure), hence also to the lower map, which is thus injective. □

Since T-algebras have underlying simplicial sets, we can introduce the usual notions of connectivity of maps and objects. By the homotopy groups of a T-algebra we always mean the homotopy groups of the geometric realization of the underlying simplicial set. A T algebra will be called n-connected if all homotopy groups below and including dimension n are trivial. A map of T-algebras is called n-connected if it induces isomorphisms on homotopy groups below dimension n and an epimorphism in dimension n (for all choices of basepoint in the underlying simplicial set of the source). A square of T-algebras

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

will be called n-homotopy cartesian if for some factorization

$$B \xrightarrow{\sim} Z \twoheadrightarrow D$$

of the right vertical map into a weak equivalence and a fibration the induced map $A \longrightarrow C \times_D Z$ is n-connected. Since the model category of simplicial sets is proper [BF, Def. 1.2], the pullback part of properness also holds in the model category of T-algebras. Hence the weak equivalence type of the pullback $C \times_D Z$ and thus the notion of n-homotopy cartesian, is independent of the factorization chosen and can in fact be more symmetrically described by choosing factorizations of both the right and the lower map (cf. [BF, A.2]). A pullback of the form $C \times_D Z$ for a factorization as above will be referred to as a homotopy pullback. The square is homotopy cartesian if it is n-homotopy cartesian for all n, i.e., if the map to the homotopy pullback is a weak equivalence.

The model category of T-algebras does not necessarily satisfy the pushout part of the properness definition. For squares involving only cofibrant objects, the pushout part of properness holds in any model category. This is a special case of the following gluing lemma. The restriction to *simplicial* model categories is not necessary, but it simplifies the proof.

Lemma 2.2.2 *Consider a commutative diagram of cofibrant objects in a closed simplicial model category*

$$\begin{tikzcd} B \arrow[d, "\sim"'] & A \arrow[l] \arrow[r, tail] \arrow[d, "\sim"'] & C \arrow[d, "\sim"] \\ B' & A' \arrow[l] \arrow[r, tail] & C' \end{tikzcd}$$

Assume that the vertical maps are weak equivalences and that two of the horizontal maps are cofibrations as indicated. Then the induced map on pushouts

$$B \cup_A C \xrightarrow{\sim} B' \cup_{A'} C'$$

is a weak equivalence.

Proof: A map between cofibrant objects in a simplicial model category is a weak equivalence if and only if it induces weak equivalences on simplicial hom sets $\hom(-, X)$ into all fibrant objects X [BF, Lemma 4.5]. But for any such fibrant X, the square

$$\begin{tikzcd} \hom(C, X) \arrow[d, two heads] & \hom(B \cup_A C, X) \arrow[l] \arrow[d] \\ \hom(A, X) & \hom(B, X) \arrow[l] \end{tikzcd}$$

is a pullback square of simplicial sets, with the left vertical map being a fibration (since $A \rightarrowtail C$ is a cofibration). There is furthermore a map from the analogous square with A, B, C replaced by A', B', C'. Since the vertical maps in the original diagram are weak equivalences between cofibrant objects, we have weak equivalence on all corners of the (homotopy) pullback squares except possibly the initial one. But then the map on the initial corners is also a weak equivalence. □

Back in the simplicial model category of T-algebras, a square of cofibrant objects as before will be called n-homotopy cocartesian ($n = 0, 1, \ldots$ or $n = \infty$) if for some (hence any) factorization

$$A \rightarrowtail W \xrightarrow{\sim} C$$

of the left vertical map into a cofibration and a weak equivalence the induced map $W \cup_A B \longrightarrow D$ is n-connected. An arbitrary square of T-algebras will be called n-homotopy cocartesian if the functorial replacement by a square of cofibrant objects is.

Geometric realization of simplicial T-algebras. Given a simplicial object of T-algebras X_*, its geometric realization $|X_*|$ is defined in the standard fashion as the coend of the functor $\Delta^{\mathrm{op}} \times \Delta \longrightarrow T\text{-alg}$ which takes (n, m) to $X_n \otimes \Delta^m$, i.e.,

$$|X_*| = \int^{n \in \Delta} X_n \otimes \Delta^n .$$

Hence $|X_*|$ is a T-algebra. The geometric realization can also be described inductively over skeleta. Define the n-skeleton of the realization $|X_*|_{(n)}$ as the coend of the above functor truncated to $\Delta^{\mathrm{op}}_{(n)} \times \Delta_{(n)}$, where $\Delta_{(n)}$ is the full subcategory of Δ of objects of cardinality $\leq n$. Then $|X_*|_{(0)} = X_0$ and the n-skeleton can be obtained from the $(n-1)$-skeleton by attaching $X_n \otimes \Delta^n$ along the degenerate part, i.e., there is a pushout square of T-algebras

$$\begin{tikzcd} L_n(X_*) \otimes \Delta^n \cup_{L_n(X_*) \otimes \partial\Delta^n} X_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & X_n \otimes \Delta^n \arrow[d] \\ |X_*|_{(n-1)} \arrow[r] & |X_*|_{(n)} \end{tikzcd}$$

Here $L_n(X_*)$ is the n-th latching object defined as the colimit

$$L_n(X_*) \ = \ \mathrm{colim}_{[n]\to[k],\, k<n} \ X_k \ ;$$

the indexing category for the colimit has as objects all maps $[n] \longrightarrow [k]$ in Δ with $k < n$.

As for bisimplicial sets, realization of simplicial T-algebras is diagonalization. A simplicial theory T can be viewed as a simplicial object of discrete theories. If X is a T-algebra, then the n-simplices X_n are a discrete algebra over the discrete theory T_n. Furthermore, the simplicial structure maps in the simplicial set X are twisted morphisms of algebras. This means that if $\alpha : [n] \longrightarrow [m]$ is a map in the simplicial category Δ, then $X(\alpha) : X_m \longrightarrow X_n$ is a morphisms of T_m-algebras where X_n is given a T_m-algebras structure via the morphisms of theories $T(\alpha) : T_m \longrightarrow T_n$. Algebras over a simplicial theory have underlying simplicial sets. Simplicial algebras over a simplicial theory thus have underlying bisimplicial sets. By what we just said the diagonal of the underlying bisimplicial set of a simplicial T-algebra is naturally a T-algebras and we have

Lemma 2.2.3 *Let X_* be a simplicial T-algebra. Then the realization $|X_*|$ is naturally isomorphic to the diagonal T-algebra* $\mathrm{diag}\,(X_*)$.

Proof: There is a natural map $|X_*| \longrightarrow \mathrm{diag}\,(X_*)$. Suppose first that X_* is freely generated by a bisimplicial set Y_{**}, i.e., $X_n = F^T(Y_{*,n})$. The free functor commutes with realization and with diagonalization, i.e.,

$$|X_*| \ \cong \ F^T(|Y_{**}|) \qquad \text{and} \qquad \mathrm{diag}\,(X_*) \ \cong \ F^T(\mathrm{diag}\, Y_{**}) \ .$$

The map is thus an isomorphism since realization is diagonalization for bisimplicial sets ([BF, Prop. B.1]; realization is denoted 'Tot' there).

An arbitrary T-algebra is a coequalizer of free T-algebras: for a T-algebras Y,

$$F^T(F^T(Y)) \ \rightrightarrows \ F^T(Y) \longrightarrow Y$$

is a coequalizer diagram of T-algebras. Applying this observation degreewise to the simplicial T-algebra X_* and noting that both realization and diagonal commute with colimits gives the desired result in general. □

The standard resolution. It will be convenient to have at our disposal the standard cotriple resolution for T-algebras, which we now recall. In our context, the standard resolution for a T-algebra X is a simplicial T-algebra which is degreewise free and whose realization provides a cofibrant replacement of X. With the standard resolution, homotopical properties of free objects can often be extended to all cofibrant objects.

Denote by Φ^T the composite of the forgetful functor $T\text{-alg} \longrightarrow \mathcal{S}$ with the free T-algebra functor $F^T : \mathcal{S} \longrightarrow T\text{-alg}$. This Φ^T is a cotriple on the category of T-algebras, which means that there are natural transformations

$$\eta : \Phi^T \longrightarrow \mathrm{Id}_{T\text{-alg}} \quad \text{and} \quad \mu : \Phi^T \longrightarrow \Phi^T \circ \Phi^T$$

making a co-unit and a co-associativity diagram commute. Consider a T-algebra X and define a simplicial object $\Phi(X)_*$ of T-algebras as follows: in simplicial degree n, we set it to be $(\Phi^T)^{n+1}(X)$. The i-th face map is set to be

$$d_i = \Phi^i(\eta_{\Phi^{n-i}(X)}) : (\Phi^T)^{n+1}(X) \longrightarrow (\Phi^T)^n(X)$$

for $0 \leq i \leq n$. The degeneracy maps are defined to be

$$s_i = \Phi^i(\mu_{\Phi^{n-1-i}(X)}) : (\Phi^T)^n(X) \longrightarrow (\Phi^T)^{n+1}(X)$$

for $0 \leq i \leq n-1$. We omit the verification of the simplicial identities. By construction, $\Phi(X)_n$ is a free T-algebra generated by the underlying simplicial set of $(\Phi^T)^n(X)$ and all structure maps except the first face maps are free maps with respect to these generators.

The map $\eta_X : \Phi(X)_0 = \Phi^T(X) \longrightarrow X$ provides an augmentation, i.e., a map to the constant simplicial T-algebra X. Since the realization of a constant simplicial object is isomorphic to the object in degree 0, we get a map of T-algebras $|\Phi(X)_*| \longrightarrow X$. The fact that we have obtained a resolution of X appears in various contexts in the literature; we state it in the form

Lemma 2.2.4 *The map $|\Phi(X)_*| \xrightarrow{\sim} X$ is a weak equivalence of T-algebras. If $X \longrightarrow Y$ is a map of T-algebras which is injective on underlying simplicial sets, then $|\Phi(X)_*| \longrightarrow |\Phi(Y)_*|$ is a cofibration of T-algebras. In particular, $|\Phi(X)_*|$ is cofibrant as a T-algebra.*

Proof: The question whether a map of T-algebras is a weak equivalence can be checked on underlying simplicial sets. We know that the underlying simplicial sets of the realization of a simplicial T-algebra is isomorphic to the diagonal of the bisimplicial set underlying the simplicial T-algebra. So for the first claim in the lemma, we can consider $\Phi(X)_*$ as a bisimplicial set and we have to check that $\Phi(X)_* \longrightarrow X$ induces a weak equivalence on diagonal simplicial sets. But after disregarding the T-algebra structure, the 'inclusion of generators' provide a section to the map in question as well as a simplicial homotopy showing that $\Phi(X)_*$ contains the constant simplicial object X as a deformation retract (see [May, Prop. 9.8] for more details).

It remains to establish that $|\Phi(-)_*|$ transforms injective maps into cofibrations. By induction on the skeleta of the realization and using the simplicial model category structure it suffices to show that the map

$$L_n(\Phi(Y)_*) \cup_{L_n(\Phi(X)_*)} \Phi(X)_n \longrightarrow \Phi(Y)_n$$

is a cofibration of T-algebras. The latching objects L_n and the maps appearing above only depend on the degeneracies of the simplicial structure.

$\Phi(X)_*$ is degreewise freely generated by a simplicial set and most of the simplicial structure maps are free. This has been formalized in the notion of an almost free object. Denote by $\widetilde{\Delta}$ the subcategory of the simplicial category Δ of those order preserving maps which send 0 to 0. An almost simplicial object in a category is a contravariant functor from $\widetilde{\Delta}$. Hence an almost simplicial object is almost a simplicial object, except that it lacks the 0-th face maps. Now $\Phi(X)_*$ is almost free in the sense that the restriction to an almost simplicial T-algebra is the composite

$$\widetilde{\Delta}^{\mathrm{op}} \xrightarrow{G^X} \mathcal{S} \xrightarrow{F^T} T\text{-alg}$$

for an almost simplicial object of pointed simplicial sets G^X. In simplicial degree n, G^X is the underlying simplicial set of $(\Phi^T)^n(X)$ and the structure maps are analogous to those of $\Phi(X)_*$, just with one fewer application of Φ^T, which is why the face map d_0 is not defined for G^X. The free T-algebra functor maps cofibrations of simplicial sets to cofibrations of T-algebras which are injective by Theorem 2.2.1. Using this observation repeatedly shows that the maps $(\Phi^T)^n(X) \longrightarrow (\Phi^T)^n(Y)$ are injective on underlying simplicial sets.

Define the latching object of G^X as

$$L_n(G^X) \;=\; \operatorname{colim}_{\substack{[n]\to[k] \\ \text{surj.},\, k<n}} G^X_k \ .$$

Since the latching object of $\Phi(X)_*$ only depends on the degeneracies of the simplicial structure, we have $F^T(L_n(G^X)) \cong L_n(\Phi(X)_*)$. Since the free T-algebra functor takes cofibrations of simplicial sets to cofibrations of T-algebras, we are reduced to showing that

$$L_n(G^Y) \cup_{L_n(G^X)} G^X_n \;\longrightarrow\; G^Y_n$$

is a cofibration of simplicial sets, i.e., that it is injective. This question can be handled degreewise, so it is an instance of the following elementary property of almost simplicial sets. We omit the proof; it makes use of the fact that although d_0 has been taken away, every surjective morphism still has a section in $\widetilde{\Delta}$.

Lemma: If $V_* \longrightarrow W_*$ is a map of almost simplicial sets which is degreewise injective, then

$$L_n(W_*) \cup_{L_n(V_*)} V_n \;\longrightarrow\; W_n$$

is also injective. □

Here are a few applications of the standard resolution. The first is the following homotopy invariance property. In the context of topological theories, results of a similar kind can be found in [Bck, Thm. (8)] and [BV, Thm. 4.58].

Lemma 2.2.5 *Let $\phi : T \longrightarrow R$ be a morphism of simplicial theories which is a weak equivalence on all simplicial hom sets. Then the total derived functors of ϕ_* and ϕ^* are inverse equivalences of homotopy theories*

$$\mathrm{Ho}\,(T\text{-alg}) \quad\cong\quad \mathrm{Ho}\,(R\text{-alg})\ .$$

Proof: This is another application of Quillen's adjoint functor theorem ([Q1, II 4 Thm. 3], [DS, Thm. 9.7]). ϕ^* preserves underlying simplicial sets, hence weak equivalences and fibrations, so ϕ_* preserves cofibrations, acyclic cofibrations and weak equivalences between cofibrant objects. Since ϕ^* detects and preserves all weak equivalences, it remains to check that for every cofibrant T-algebra A, the unit of the adjunction $A \longrightarrow \phi^*\phi_*A$ is a weak equivalence.

ϕ_* maps free T-algebras to free T'-algebras on the same generating simplicial sets. So if A is freely generated by a finite set, the above unit map is a weak equivalence by assumption. If A is freely generated by an arbitrary set, it is the filtered colimit, over cofibrations, of finitely generated free T-algebras. Similarly, ϕ_* is then a filtered colimit of finitely generated free T'-algebras. ϕ^* preserves filtered colimits, and since both are taken over maps which are cofibrations of underlying simplicial sets, homotopy groups commute with these filtered colimits. Thus the map in question is a weak equivalence when A is free on an arbitrary set. If A is freely generated by a simplicial set, which is not necessarily discrete, the result holds by the realization lemma (degreewise weak equivalences of bisimplicial sets induce weak equivalences on the diagonal simplicial sets).

To get from free T-algebras to arbitrary T-algebras we use the standard resolution. By the above lemma 2.2.4, the map $|\Phi(A)_*| \xrightarrow{\sim} A$ is a weak equivalence of T-algebras. Since both source and target are cofibrant, the map will still be a weak equivalence after applying ϕ_*. We arrive at a commutative square of T-algebras

$$\begin{tikzcd}
|\Phi(A)_*| \arrow[r, "\sim"] \arrow[d] & A \arrow[d] \\
\phi^*\phi_*|\Phi(A)_*| \arrow[r, "\sim"] & \phi^*\phi_*A
\end{tikzcd}$$

ϕ_* commutes with the realization because it preserves enriched colimits, ϕ^* commutes with the realization because the latter is passage to the diagonal simplicial set, hence $\phi^*\phi_*$ can be put inside the realization in the lower left corner of the square. Since $\Phi(A)_*$ is free in every simplicial degree, the left vertical map is the realization of a degreewise weak equivalence. So the left map, hence also $A \longrightarrow \phi^*\phi_*A$, is a weak equivalence. $\square$

Theorem 2.2.6 *(Blakers-Massey theorem) Consider a pushout square of cofibrant T-algebras*

$$\begin{tikzcd}
A \arrow[r, tail, "n"] \arrow[d, "m"'] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd} .$$

Assume further that all objects are connected, that the upper horizontal map is an n-connected cofibration and that the left vertical map is m-connected. Then the square is $(n+m-1)$-homotopy cartesian.

The main ingredients for the proof of the Blakers-Massey theorem for T-algebras are the standard resolution and the fact that functors obtained from Γ-spaces by degreewise extension are always stably excisive ([Gw1, 1.8]; Def. 5.2.1). Say that a homotopy functor F from the category of pointed simplicial sets to itself satisfies the property $E(1)$ if the following holds: let

$$\begin{tikzcd} Y \arrow[r, "n"] \arrow[d, "m"'] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y_{12} \end{tikzcd}$$

be a homotopy cocartesian square of pointed connected simplicial sets. Assume further that the upper horizontal map is n-connected and that the left vertical map is m-connected. Then the square of simplicial sets

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & F(Y_1) \arrow[d] \\ F(Y_2) \arrow[r] & F(Y_{12}) \end{tikzcd}$$

is $(n+m-1)$-homotopy cartesian ($E(1)$ is a special case of stable excision [Gw1, 1.8], Def. 5.2.1).

We recall that diagonalizing bisimplicial sets preserves k-cartesianness:

Lemma 2.2.7 *Consider a square*

$$\begin{tikzcd} V_{*,*} \arrow[r] \arrow[d] & X_{*,*} \arrow[d] \\ W_{*,*} \arrow[r] & Y_{*,*} \end{tikzcd}$$

of bisimplicial sets. Assume that for every $m \geq 0$, $X_{m,*}$ and $Y_{m,*}$ are connected and that the square of simplicial sets consisting of $V_{m,*}, W_{m,*}, X_{m,*}$ and $Y_{m,*}$ is k-homotopy cartesian ($k \geq 0$). Then the square consisting of the diagonal simplicial sets is also k-homotopy cartesian.

Proof: One introduces a bisimplicial set $P_{*,*}$ which for fixed m is the homotopy pullback of

$$W_{m,*} \longrightarrow Y_{m,*} \longleftarrow X_{m,*} .$$

Then the map $V_{m,*} \longrightarrow P_{m,*}$ is k-connected for every m, so $\mathrm{diag}(V_{*,*}) \longrightarrow \mathrm{diag}(P_{*,*})$ is also k-connected. Furthermore, by [BF, Thm. B.4] the square with $V_{*,*}$ replaced by $P_{*,*}$ is homotopy cartesian after taking diagonals. $\square$

Lemma 2.2.8 (T. Goodwillie) *The functor associated (by degreewise application) to any Γ-space satisfies $E(1)$.*

Proof: Let A be a Γ-space. Choose a strict weak equivalence $A^c \xrightarrow{\sim} A$ from a cofibrant Γ-space to A. Since $A^c(K) \xrightarrow{\sim} A(K)$ is then a weak equivalence for all simplicial sets K, we can assume that A is a cofibrant Γ-space. Viewing A as a simplicial object of Γ-sets, lemma 2.2.7 reduces the question to the case where A is a cofibrant Γ-set. By direct limit we can then assume that A is finite dimensional, i.e., the map $\mathrm{sk}_n A \longrightarrow A$ is an isomorphism for some n. The proof is completed by induction on n, the case $n = 0$ being trivial.

If A is n-dimensional, it can be obtained from its $(n-1)$-skeleton as a pushout

$$\begin{tikzcd} S \wedge \mathrm{sk}_{n-1}\Gamma^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}A \arrow[d] \\ S \wedge \Gamma^n \arrow[r] & A \end{tikzcd}$$

for a suitable pointed set S by [Ly, 3.21]. By induction, $E(1)$ holds for the two upper Γ-spaces. Since $(S \wedge \Gamma^n)(K) \cong S \wedge K^n$, it also holds for $S \wedge \Gamma^n$. To finish the proof we thus need the following lemma. □

Lemma 2.2.9 *Let*

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & F_1 \arrow[d] \\ F_2 \arrow[r] & F_{12} \end{tikzcd}$$

be a commutative square of homotopy functors from the category of pointed simplicial sets to itself. Assume that all functors preserve connectivity and that the square is homotopy cocartesian when evaluated at any simplicial set. If F_0, F_1 and F_2 satisfy $E(1)$, then so does F_{12}.

Proof: Start with a homotopy cocartesian square

$$\begin{tikzcd} Y \arrow[r, "n"] \arrow[d, "m"'] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y_{12} \end{tikzcd}$$

of connected pointed simplicial sets where the upper map is n-connected and the left map is m-connected. Applying F_i $(i = 0, 1, 2)$ gives a square which is $(n+m-1)$-homotopy cartesian and in which the two terminal maps are respectively n and m-connected. Let P_i $(i = 0, 1, 2)$ denote the homotopy pullback of the diagram

$$\begin{tikzcd} & F_i(Y_1) \arrow[d] \\ F_i(Y_2) \arrow[r] & F_i(Y_{12}) \end{tikzcd},$$

so that the map $F_i(Y_0) \longrightarrow P_i$ is $(m+n-1)$-connected. The map from the homotopy pushout of

$$F_i(Y_2) \longleftarrow F_i(Y_0) \longrightarrow F_i(Y_1)$$

to the homotopy pushout of

$$F_i(Y_2) \longleftarrow P_i \longrightarrow F_i(Y_1)$$

is thus $(n+m)$-connected (this connectivity increase by 1 in crucial). By T. Goodwillie's dual version of the Blakers-Massey theorem for spaces [Gw2, Thm. 2.4], the homotopy pullback square

$$\begin{tikzcd} P_i \arrow[r] \arrow[d] & F_i(Y_1) \arrow[d] \\ F_i(Y_2) \arrow[r] & F_i(Y_{12}) \end{tikzcd}$$

is $(n+m+1)$-homotopy cocartesian, so that the square

$$\begin{tikzcd} F_i(Y_0) \arrow[r] \arrow[d] & F_i(Y_1) \arrow[d] \\ F_i(Y_2) \arrow[r] & F_i(Y_{12}) \end{tikzcd}$$

is still $(n+m)$-homotopy cocartesian.

Applying the square of functor made of the F_i's and F_{12} to the square of spaces gives a 4-dimensional cube to which we apply Goodwillie's proposition [Gw2, 1.12] with $k=n+m$. The total cube is homotopy cocartesian since the original square of homotopy functors was assumed to be pointwise homotopy cocartesian. By the last paragraph, the square obtained by applying F_i to the Y's is $(n+m)$-homotopy cocartesian for $i=0,1$ and 2. Goodwillie's proposition [Gw2, 1.12] tells us that the square obtained by applying F_{12} to the Y's is then also $(n+m)$-homotopy cocartesian.

Since F_{12} preserves connectivity, the two initial maps of the square

$$\begin{tikzcd} F_{12}(Y_0) \arrow[r] \arrow[d] & F_{12}(Y_1) \arrow[d] \\ F_{12}(Y_2) \arrow[r] & F_{12}(Y_{12}) \end{tikzcd}$$

are m and n-connected respectively. Since this square is also $(n+m)$-homotopy cocartesian, the Blakers-Massey theorem for simplicial sets implies that it is $(n+m-1)$-homotopy cartesian, which finishes the proof. $\square$

Proof of the Blakers-Massey theorem for T-algebras. The square in question admits a map from the square

$$\begin{tikzcd} |\Phi(A)_*| \arrow[r, tail] \arrow[d] & |\Phi(B)_*| \arrow[d] \\ |\Phi(C)_*| \arrow[r] & P \end{tikzcd}$$

where P is defined to be the pushout. The transformation between the squares is a weak equivalence at all corners except possibly the terminal one. As a cofibration of T-algebras, the map $A \rightarrowtail B$ is injective on underlying simplicial sets. Thus the upper horizontal map in this second square is a cofibration of T-algebras (lemma 2.2.4), so the map on terminal vertices is also a weak equivalence by the gluing lemma 2.2.2. Hence it is enough to show that this second square is $(n + m - 1)$-homotopy cartesian.

Since realization of simplicial objects commutes with pushout, P is isomorphic to the realization of a simplicial T-algebra P_* which in simplicial degree k is the pushout of the diagram

$$\begin{tikzcd} (\Phi^T)^{k+1}(A) \arrow[r, tail] \arrow[d] & (\Phi^T)^{k+1}(B) \arrow[d] \\ (\Phi^T)^{k+1}(C) \arrow[r] & P_k \end{tikzcd} .$$

In this diagram, all objects are freely generated by certain simplicial sets, all maps are free maps, the upper horizontal map being free on a cofibration of simplicial sets. Furthermore, the free T-algebra functor preserves connectivity of objects and maps, hence so does Φ^T and any iteration of it. Since the composite of the free T-algebra functor with the forgetful functor comes from a Γ-space, lemma 2.2.8 shows that this square is $(n+m-1)$-homotopy cartesian. By lemma 2.2.7 we get the asserted amount of cartesianness after realizing the simplicial objects.

□

2.3 Stable homotopy

Given a simplicial theory T, we will set up a model category of spectra of T-algebras, thus defining the stable homotopy theory of T-algebras. The main result of this section is the proof that this stable homotopy theory is equivalent to the homotopy theory of modules over a certain DFSP T^s.

Before defining spectra, we need the definition of the suspension of a T-algebra X. Since T-algebras form a simplicial model category, the product $X \otimes S^1$ with the simplicial circle is defined. The suspension of X is then obtained as the following pushout, in the category of T-algebras

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \otimes S^1 \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

(the upper map is induced by the inclusion of the unique vertex into S^1). ΣX is a kind of bar construction with respect to the coproduct of T-algebras. By this we mean that it is the geometric realization of the simplicial T-algebra which in simplicial degree k consists of the coproduct of k copies of X:

$$\Sigma X \;=\; |\, k \mapsto \underbrace{X \amalg \ldots \amalg X}_{k} \,|$$

The suspension functor has an adjoint Ω which is defined dually, using X^{S^1} instead of $X \otimes S^1$. The loop functor commutes with the forgetful functor, i.e., the underlying simplicial set of ΩX is the simplicial set of pointed maps of S^1 into the underlying simplicial set of X. Σ and Ω are a Quillen pair, i.e., Σ preserves cofibrations, acyclic cofibrations and weak equivalences between cofibrant objects. Ω preserves fibrations, acyclic fibrations and weak equivalences between fibrant objects. The total derived functors of Σ and Ω are the suspension and loop functors on the homotopy category of T-algebras [Q1, I.2]. The simplicial structure thus provides liftings of these functors from the homotopy category to functors on the actual model category. One has to remember that ΣX can have the 'wrong' homotopy type if X is not cofibrant just as ΩY can have the 'wrong' homotopy type if Y is not fibrant.

For our purposes, the naive definition of a spectrum suffices. The following spectra category is used in [Sch, 2.1] and it is an elaboration on the construction of [BF, §2]. These spectra are sometimes called prespectra by other authors.

Definition 2.3.1 *A* **spectrum** X *of T-algebras is a collection of T-algebras X_n $(n = 0, 1, 2, \ldots)$ and maps $\Sigma X_n \longrightarrow X_{n+1}$. A morphism of spectra $f : X \longrightarrow Y$ is a collection of maps $f_n \colon X_n \longrightarrow Y_n$ such that all the diagrams*

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\Sigma f_n"] \arrow[d] & \Sigma Y_n \arrow[d] \\ X_{n+1} \arrow[r, "f_{n+1}"'] & Y_{n+1} \end{tikzcd}$$

commute. We denote the category of spectra by $(T\text{-alg})^\infty$.

If T is the theory of sets, a T-algebra is just a simplicial set and the above definition reduces to the notion of a spectrum as in [BF, §2]. In general, the forgetful functor from T-algebras to pointed simplicial sets induces a forgetful functor from $(T\text{-alg})^\infty$ to that category of spectra of pointed simplicial sets. Call a map of spectra of T-algebras a weak equivalence (resp. fibration) if it is a weak equivalence (resp. fibration) as a map of spectra of simplicial sets. Call a map a cofibration if it has the LLP with respect to all acyclic fibrations of spectra of T-algebras. In order to show that with these definitions the category of spectra is again a closed simplicial model category, we could appeal to [Sch, Prop. 2.1.5] where a sufficient condition is given. However, to apply this theorem, we would have to make the assumption that the model category of T-algebras is proper [BF, Def. 1.2]. To avoid this unnecessary restriction, we give a different proof.

Theorem 2.3.2 *If T is a simplicial theory, then $(T\text{-alg})^\infty$ becomes a cofibrantly generated (Def. A.2) closed simplicial model category.*

Proof: We note that the category $(T\text{-alg})^\infty$ can be viewed as a category of algebras over a suitable triple on the category of spectra of simplicial sets. By our usual abuse of notation, we let $T : \mathcal{S} \longrightarrow \mathcal{S}$ also denote the composite of the free T-algebra functor with the forgetful functor. We have used before that this T is in fact a triple on $\mathcal{S}$ and that algebras over this triple are (equivalent to) T-algebras (hence our abuse of notation does not create any ambiguity in what T-algebras are). This functor T is also a simplicial functor, hence it can be applied degreewise to spectra of simplicial sets. In accordance with the notation of [Sch, 2.1], this functor on spectra is denoted T^∞. T^∞ inherits the structure of a triple, this time on the category of spectra of simplicial sets. Then an algebra over the triple T^∞ is the same as a spectrum of simplicial sets which is in every spectrum degree an algebra over the triple T and whose structure maps are maps of T-algebras. In other words, algebras over T^∞ are spectra of T-algebras, we get an equivalence of categories

$$T^\infty\text{-alg} \quad \cong \quad (T\text{-alg})^\infty \ .$$

The model category of spectra of simplicial sets is cofibrantly generated (lemma 1.2.5) so this change of point of view allows us to apply lemma A.3. We now verify condition (2) of that lemma by exhibiting a functor Q that provides fibrant replacements.

For the category of spectra of simplicial sets, Bousfield and Friedlander use of the functor Q given by

$$(QX)_n \;=\; \mathrm{colim}_i\ \Omega^i \,\mathrm{Sing}\,|X_{n+i}| \ .$$

For every spectrum X, QX is an Ω-spectrum and degreewise a fibrant simplicial set. It is thus fibrant in the stable model category of spectra by [BF, A.7]. This functor Q is furthermore simplicial and it commutes with finite products. So we note that spectra of T-algebras can be viewed from yet another point of view, namely as product-preserving simplicial functors from the category T to the category of spectra of simplicial sets. Since Q is simplicial and product preserving, it passes to a functor on the category of spectra of T-algebras. $\square$

Lemma 2.3.3 *Let* $A \rightarrowtail B$ *be a cofibration of cofibrant spectra of T-algebras. Then there is a natural long exact sequence of homotopy groups (infinite in both directions)*

$$\cdots \longrightarrow \pi_n\, A \longrightarrow \pi_n\, B \longrightarrow \pi_n\, B/A \longrightarrow \pi_{n-1}\, A \longrightarrow \cdots$$

Proof: Every spectrum is the colimit of a sequence of spectra which are eventually suspension spectra. Homotopy groups commute with such colimits and direct limit is an exact functor. This reduces to the case of suspension spectra. By the Blakers-Massey theorem for T-algebras 2.2.6, a cofibration sequences of n-connected T-algebras has exact sequences of homotopy groups in a range of roughly $2n$. The result follows since again direct limit is an exact functor for abelian groups. $\square$

The goal now is to show that the stable homotopy theory of T-algebras is equivalent to the homotopy theory of modules over a certain DFSP. The theorem we are heading for has a well known algebraic analog which we first recall. Suppose T is an ordinary (i.e., discrete and not necessarily pointed) algebraic theory. One can consider discrete T-algebras (i.e., models of T in the category of sets as opposed to simplicial sets). The notion of an abelian group object in the category of T-algebras is defined and it coincides with that of a model of T in the category of abelian groups. By general arguments, the forgetful functor from abelian group objects to T-alg has a left adjoint and the category $(T\text{-alg})^{\mathrm{ab}}$ of abelian group objects is in fact an abelian category. Moreover, the category $(T\text{-alg})^{\mathrm{ab}}$ is equivalent to the category of modules over a certain ring T^{ab}, namely the endomorphism ring of the free abelian group object on one generator:

$$(T\text{-alg})^{\mathrm{ab}} \quad \cong \quad T^{\mathrm{ab}}\text{-mod}$$

We are proving the homotopy theoretic analog of this fact. The idea is to replace abelian group objects by its homotopy analogue, namely infinite loop objects or spectra. This forces one to introduce a topological or simplicial direction where there wasn't one already. Spectra of T-algebras form a model category $(T\text{-alg})^{\infty}$. If we also allow brave new rings (which for us are DFSPs) instead of ordinary rings, we get

Theorem 2.3.4 *To a simplicial theory, there is functorially associated a DFSP T^s. $\pi_* T^s$ is isomorphic, as a graded ring, to the ring of stable homotopy operations of T-algebras. There is an adjoint functor pair*

$$(T\text{-alg})^{\infty} \quad \rightleftarrows \quad T^s\text{-mod}$$

whose total derived functors are inverse equivalences between the homotopy theory of connective spectra of T-algebras and the homotopy theory of T^s-modules:

$$\mathrm{Ho}\,(T\text{-alg})^{\infty}_{\mathrm{conn}} \quad \cong \quad \mathrm{Ho}\,(T^s\text{-mod})$$

The fact that we only get connective spectra of T-algebras stems from the fact that Γ-spaces only represent connective spectra. Working with other notions of brave new rings which allow non-connective modules, one should in fact obtain an equivalence with the full stable homotopy theory of T-algebras. The proof of the theorem with its various parts will occupy the rest of this section.

The DFSP T^s. The construction of T^s from the simplicial theory T is a special case of the observation made by T. Gunnarson [Gu, 6.3] that a triple on the category of spaces gives rise to an FSP. This and related constructions of FSP models for ordinary rings and group rings over the sphere spectrum ([BHM, ex. 3.2], [PW, ex. 2.3]) implicitly use a comparison map like the one we introduced at the end of section 1.1. Making the use of the comparison map explicit allows us to exploit its formal and homotopical properties.

The Γ-space underlying the DFSP T^s is defined as the composite of the free T-algebra functor, restricted to the category Γ, with the forgetful functor from T-algebras to pointed simplicial sets. Hence as a simplicial set, $T^s(n)$ is isomorphic to $\hom_T(n, 1)$. The composite of the free T-algebra functor with the forgetful functor is a triple on the category of simplicial sets. Hence its restriction T^s to Γ comes with maps

$$T^s \circ T^s \longrightarrow T^s \qquad \text{and} \qquad S^0 \longrightarrow T^s$$

which are associative and unital. This just says that T^s becomes a monoid in the category of Γ-spaces with respect to the composition product $\circ$. Composition with the comparison map

$$T^s \wedge T^s \longrightarrow T^s \circ T^s \longrightarrow T^s$$

gives T^s a multiplication with respect to the smash product. Since the comparison map is associative and unital, T^s becomes a DFSP. The multiplication of T^s can be reinterpreted in terms of composition in the category underlying the theory T. We omit the proof of the following lemma.

Lemma 2.3.5 *Under the isomorphism of pointed simplicial sets*

$$T^s(n) \cong \hom_T(n,1)$$

the multiplication map $T^s(n) \wedge T^s(m) \longrightarrow T^s(n \wedge m)$ corresponds to the composition product

$$\begin{array}{ccccc} \hom_T(n,1) & \wedge & \hom_T(m,1) & \longrightarrow & \hom_T(nm,1) \\ f & \wedge & g & \longmapsto & g \circ (\underbrace{f \times \ldots \times f}_{m}) \end{array}$$

using the isomorphism $n^m \cong nm$ in T.

Remark: The functor that sends a simplicial theory T to the Γ-space T^s with the composition product structure is an equivalence of categories

$$(\text{simplicial theories}) \cong (\text{monoids in } (\mathcal{GS}, \circ)) \ .$$

This result can be found in [Bor, Prop. 4.6.2] for discrete theories, and in [BV, Prop. 2.30] for topological theories. The inverse functor is given as follows: a monoid M in $(\mathcal{GS}, \circ)$ becomes a triple on the category of pointed simplicial sets when M is extended from Γ to pointed simplicial sets by colimit and degreewise application. So the category of M-algebras is defined. The simplicial theory associated to M is the opposite of the full subcategory of M-algebras generated by the M-algebras free on the sets n for $n \in \mathbb{N}$.

Comparison of the stable categories. We now proceed to compare the two categories $(T\text{-alg})^\infty$ and T^s-mod. This will be done through an intermediate category $(T\text{-alg})^\Gamma$, the category of pointed functors $\Gamma \longrightarrow T$-alg. We obtain three model categories with Quillen adjoint functor pairs

$$T^s\text{-mod} \underset{\Psi^*}{\overset{\Psi_*}{\rightleftarrows}} (T\text{-alg})^\Gamma \underset{\Phi(S,-)}{\overset{(-)\underline{S}}{\rightleftarrows}} (T\text{-alg})^\infty$$

The left pair passes to an equivalence of homotopy theories, the right pair passes to an equivalence of the homotopy theory of $(T\text{-alg})^\Gamma$ with that of connective spectra of T-algebras.

Step 1. We first establish the model category structure for $(T\text{-alg})^\Gamma$. Since T^s is a monoid for the composition product of Γ-spaces, $T^s \circ -$ is a triple on the category of Γ-spaces. By a reversal of priorities, a $(T^s \circ -)$-algebra is the same as a Γ-object in T-alg, i.e., we have an equivalence of categories

$$(T^s \circ -)\text{-alg} \quad \cong \quad (T\text{-alg})^\Gamma \ .$$

We already used in 2.3.2 that similarly spectra of T-algebras are algebras over a triple in the Bousfield-Friedlander category of spectra of simplicial sets: degreewise application of the free T-algebra functor to a spectrum of simplicial sets gives an endofunctor T^∞ on the Bousfield-Friedlander category of spectra. T^∞ inherits the structure of a triple on the category of spectra of simplicial sets and algebra over the triple T^∞ is nothing but a spectrum in the category of T-algebras, i.e., there is another equivalence of categories

$$T^\infty\text{-alg} \quad \cong \quad (T\text{-alg})^\infty \ .$$

Bousfield and Friedlander [BF, §5] construct a pair of adjoint functors

$$\mathcal{GS} \quad \underset{\Phi(\underline{S},-)}{\overset{(-)\underline{S}}{\rightleftarrows}} \quad (\text{spectra in } \mathcal{S}) \ .$$

We note that these functors pass to an adjoint functor pair between the categories of $(T^s \circ -)$-algebras and T^∞-algebras. The left adjoint functor matches the triples on the two categories: for a Γ-space X, the two spectra

$$(T^s \circ X)\underline{S} \qquad \text{and} \qquad T^\infty((X)\underline{S})$$

are naturally isomorphic – the n-th term of both is the underlying simplicial set of the free T-algebra on $X(S^n)$. The fact that we get an induced adjoint functor pair between spectra and Γ-objects of T-algebras is now a consequence of the following formal lemma whose proof we omit.

Lemma 2.3.6 *Consider an adjoint pair of functors*

$$\mathcal{C} \quad \underset{R}{\overset{L}{\rightleftarrows}} \quad \mathcal{D}$$

L being the left adjoint. Suppose further that S is a triple on the category $\mathcal{C}$ and T is a triple on $\mathcal{D}$ and that there is a natural isomorphism $LS \cong TL$ compatible with the triple structure. Then the functors L and R pass to adjoint functors between the category of S-algebras and the category of T-algebras.

So if we equip $(T\text{-alg})^\Gamma$ with the stable notions of weak equivalences and (co-)fibrations on the underlying Γ-spaces, it becomes a closed simplicial model category by lemma A.3 if we can verify condition (2) of that lemma. This conditions asks for a functorial fibrant replacement of objects, which we obtain by setting $\bar{Q}X = \Phi(\underline{S}, Q(X\underline{S}))$. Here Q is the fibrant replacement functor used in proving that spectra of T-algebras form a model category (Thm. 2.3.2). $\bar{Q}X$ is in fact fibrant since $\Phi(\underline{S}, -)$ preserves fibrations; also the map $X \longrightarrow \bar{Q}X$ is a stable weak equivalence.

Since $\Phi(\underline{S}, -)$ preserves weak equivalences and fibrations, it still forms a Quillen adjoint functor pair together with $(-)\underline{S}$. If X a connective fibrant spectrum of T-algebras, then the adjunction map $\Phi(\underline{S}, X)\underline{S} \xrightarrow{\sim} X$ is a weak equivalence. So if A is a cofibrant object in $(T\text{-alg})^\Gamma$, then a map $A\underline{S} \longrightarrow X$ is a weak equivalence if and only if the adjoint map $A \longrightarrow \Phi(\underline{S}, X)$ is. Thus the functors in fact pass to an equivalence of the homotopy theory of $(T\text{-alg})^\Gamma$ with that of connective spectra of T-algebras (by [DS, Thm. 9.7]).

Step 2. The associative and unital comparison map

$$X \wedge Y \longrightarrow X \circ Y$$

from the end of section 1.1 gives a morphism

$$\Psi : T^s \wedge - \longrightarrow T^s \circ -$$

of triples on the category of Γ-spaces. An algebra over the triple $T^s \wedge -$ is nothing but a T^s-module, an algebra over the triple $(T^s \circ -)$ is a Γ-object of T-algebras. Pulling back along the triple morphism Ψ gives a functor $\Psi^* : (T^s \circ -)\text{-alg} \longrightarrow T^s\text{-mod}$. This functor has a left adjoint Ψ_* by [Li, Cor. 1]. We want to apply the adjoint functor criterion [DS, Thm. 9.7] to Ψ^* and Ψ_*. The right adjoint functor preserves fibrations and weak equivalences since these are defined everywhere on underlying Γ-spaces. Since the right adjoint in fact detects and preserves all weak equivalences, it suffices to show

Lemma 2.3.7 *For a cofibrant T^s-module A, the unit map $A \longrightarrow \Psi^*\Psi_*A$ is a stable weak equivalence.*

Proof: Assume first that the cofibrant T^s-module A is induced, i.e., it is of the form $A = T^s \wedge Y$ for some cofibrant Γ-space Y. Pushforward along a map of triples maps free objects to free objects, so in this case the map in question is the comparison map

$$T^s \wedge Y = A \quad \longrightarrow \quad \Psi^*\Psi_*A = T^s \circ Y \ .$$

This is a weak equivalence by lemma 1.1.5. Next we argue that if lemma 2.3.7 holds for cofibrant T^s-modules A, K, L and if $K \rightarrowtail L$ is a cofibration of T^s-modules, then the lemma also holds for the pushout of the diagram

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d] & L \\ A & \end{tikzcd}$$

Cofibrations of T^s-modules are injective and cofibres of T^s-modules are calculated on underlying Γ-spaces (Theorem 1.3.4). So any cofibre sequence of T^s-modules gives rise to a long exact sequence of homotopy groups by lemma 1.1.4. Ψ_* is a left adjoint, so it preserves pushout and cofibres. Hence the five lemma gives the desired conclusion once we know that the sequence

$$\Psi_*K \longrightarrow \Psi_*L \longrightarrow \Psi_*(L/K)$$

also gives rise to a long exact sequence of homotopy groups. By definition, the homotopy groups of Ψ_*K are the homotopy groups of the spectrum $(\Psi_*K)\underline{S}$. Evaluation at $\underline{S}$ is also a left adjoint, hence it preserves cofibre sequences, but any cofibre sequence of cofibrant spectra of T-algebras gives rise to a long exact sequence of homotopy groups by lemma 2.3.3.

The previous two paragraphs together show that the conclusion of lemma 2.3.7 holds for all T^s-modules which can be obtained from the trivial module by finitely many cobase changes along cofibrations between modules that are induced from Γ-spaces. Then it also holds for modules which are filtered direct limits of such modules. But an arbitrary cofibrant T^s-module is a retract of one of this sort by the small object argument A.1. $\square$

Stable homotopy operations. Interpreting $\pi_* T^s$ in terms of stable homotopy operations is a standard representability argument. We have added this interpretation to give a more complete picture, but it will not be used in the rest of this paper.

A homotopy operation of T-algebras is a natural transformation $\pi_n \longrightarrow \pi_m$ of functors T-alg$\longrightarrow$(sets) for some n, m. Homotopy operations can be composed if the source of one is the target of the other, and in fact they form a category with objects the natural numbers. The functor π_n is corepresented by $F^T(S^n)$ in the homotopy category of T-algebras, i.e., $\pi_n X \cong [F^T(S^n), X]$ (the right hand side denotes maps in the homotopy category). Consequently, the category of homotopy operations is isomorphic the opposite of the full subcategory of $\mathrm{Ho}\,(T\text{-alg})$ generated by the $F^T(S^n)$. In particular, homotopy operations $\pi_n \longrightarrow \pi_m$ are in bijective correspondence with elements of $[F^T(S^m), F^T(S^n)]$. Homotopy operations can be suspended, i.e., if $\tau : \pi_n \longrightarrow \pi_m$ is one, define

$$\Sigma\tau : \pi_{n+1} \longrightarrow \pi_{m+1}$$

on a T-algebra X as the composite

$$\pi_{n+1}X \;\cong\; \pi_n \Omega X \quad \xrightarrow{\tau\Omega X} \quad \pi_m \Omega X \;\cong\; \pi_{m+1} X$$

(in fact, one uses a fibrant replacement before applying Ω). The suspension of operations corresponds to the suspension

$$\Sigma : [F^T(S^m), F^T(S^n)] \;\longrightarrow\; [F^T(S^{m+1}), F^T(S^{n+1})]$$

in the homotopy category of T-algebras. Homotopy operations need not be additive (or homomorphisms in the case of π_1) in general, but the suspension of a homotopy operation is always additive (resp. a homomorphism).

A stable homotopy operation of degree n is, by definition, a sequence $\tau = (\tau_i)_{i \geq i_0}$ of homotopy operations $\tau_i : \pi_i \longrightarrow \pi_{i+n}$ with the property that $\tau_i = \Sigma\tau_{i+1}$. Two such sequences define the same stable operation if the components eventually agree, i.e., if almost all components are equal. Stable homotopy operations can always be composed, the degrees add under composition, and they form a graded ring.

We now identify stable homotopy operations with $\pi_* T^s$ additively and then check that the multiplications agree. We have

$$\begin{aligned} \pi_{n+i}|F^T(S^i)| &\cong \pi_0\, \Omega^{n+i}|F^T(S^i)| \\ &\cong \pi_0\, \hom_{T\text{-alg}}(F^T(S^{n+i}), F^T(S^i)^{\mathrm{f}}) \\ &\cong [F^T(S^{n+i}), F^T(S^i)] \end{aligned}$$

where $\hom_{T\text{-alg}}$ is short for the simplicial set of T-algebra maps and the superscript $^{\mathrm{f}}$ denotes a fibrant replacement. Consequently

$$\pi_n T^s \quad = \quad \mathrm{colim}_i\ \pi_{n+i}|F^T(S^i)| \quad \cong \quad \mathrm{colim}_i\ [F^T(S^{n+i}), F^T(S^i)]\ ,$$

colimits being taken over suspension. A stable homotopy operation of degree n is precisely an element of right hand side of the last equation.

It remains to check that the multiplication of $\pi_* T^s$ coincides with composition of operations. By lemma 2.3.5 and the fact that as a simplicial category, T is equivalent to the dual of the category of finitely generated free T-algebras, we know that the product of T^s corresponds to

$$\begin{array}{ccccc} \hom_{T\text{-alg}}(F^T(1), F^T(n)) & \wedge & \hom_{T\text{-alg}}(F^T(1), F^T(m)) & \longrightarrow & \hom_{T\text{-alg}}(F^T(1), F^T(n \wedge m)) \\ f & \wedge & g & \longmapsto & (\underbrace{f \amalg \ldots \amalg f}_{m}) \circ g \ = \ (m \wedge f) \circ g \end{array}$$

using the isomorphism $\underbrace{F^T(n) \amalg \ldots \amalg F^T(n)}_{m} \cong F^T(n \wedge m)$.

This reinterpretation of the product of T^s in terms of composition is inherited by the extended functor, i.e., for K, L pointed simplicial sets there is still an isomorphism

$$T^s(K) \ \cong \ \hom_{T\text{-alg}}(F^T(S^0), F^T(K))$$

under which the multiplication map $T^s(K) \wedge T^s(L) \longrightarrow T^s(K \wedge L)$ corresponds to the composition product

$$\begin{array}{ccccc} \hom_{T\text{-alg}}(F^T(1), F^T(K)) & \wedge & \hom_{T\text{-alg}}(F^T(1), F^T(L)) & \longrightarrow & \hom_{T\text{-alg}}(F^T(1), F^T(K \wedge L)) \\ f & \wedge & g & \longmapsto & (L \wedge_T f) \circ g \end{array}$$

using the isomorphism $F^T(K) \wedge_T L \ \cong \ F^T(K \wedge L)$. Here $\wedge_T$ refers to the simplicial enrichment of the category of T-algebras. Taking K and L to be simplicial spheres and passing to homotopy classes gives the required multiplicative isomorphism

$$\pi_* T^s \ \cong \mathrm{colim}_i\ [F^T(S^{*+i}), F^T(S^i)]\ .$$

2.4 Homotopy versus homology

We have seen that a simplicial theory T gives rise to two "rings" and a multiplicative map between them. There is the simplicial ring T^{ab} whose modules are the abelian group objects in the category of T-algebras. And there is the DFSP T^s whose modules are spectra of T-algebras. The forgetful functor from abelian group objects to T-algebras induces a DFSP map $T^s \longrightarrow HT^{\mathrm{ab}}$. This map encodes the relationship between stable homotopy and homology in the homotopy theory of T-algebras.

Theorem 2.4.1 *The map $T^s \longrightarrow HT^{\mathrm{ab}}$ induces an isomorphism on π_0. In particular, if T is a discrete theory, then $T^{\mathrm{ab}} \cong \pi_0 T^s$.*

Proof: We consider the case of a discrete theory first. The functors that send a discrete theory T to the rings T^{ab} and $\pi_0 T^s$ respectively are both left inverse, hence also left adjoint, to the fully faithful embedding of discrete rings into discrete theories. Since any two left adjoints are naturally isomorphic, we have $T^{\mathrm{ab}} \cong \pi_0 T^s$.

The case of an arbitrary, i.e., not necessarily discrete, simplicial theory T can be reduced to the discrete case as follows. Define πT as the discrete category with objects the natural numbers and hom sets the components of the simplicial hom sets of the original T. Since π_0 preserves products, πT is again a theory and the projection $T \longrightarrow \pi T$ is a morphisms of theories. By lemma 1.1.3, T^s and $(\pi T)^s$ have isomorphic 0-th stable homotopy groups. By the first part of this theorem, $\pi_0 (\pi T)^s \cong (\pi T)^{\mathrm{ab}}$, so it remains to show that the map $T^{\mathrm{ab}} \longrightarrow (\pi T)^{\mathrm{ab}}$ also induces an isomorphism on π_0. Viewing the simplicial theory T as a simplicial object of discrete theories, the passage $T \mapsto T^{\mathrm{ab}}$ is just application of the 0-th stable homotopy group functor in every simplicial degree by the first part. We thus conclude with the following lemma showing that the latter does not change the global 0-th stable homotopy group. $\square$

Lemma 2.4.2 *Consider a Γ-space X and view it as a simplicial object of Γ-sets. Denote by X_k the Γ-set in simplicial degree k. Then $\pi_0 X$ is naturally isomorphic to the 0-th homotopy group of the simplicial abelian group $(k \mapsto \pi_0 X_k)$.*

Proof: The sequence of free abelian groups

$$\mathbb{Z}[X_1(1)] \xrightarrow{d_0 - d_1} \mathbb{Z}[X_0(1)] \longrightarrow \mathbb{Z}[\pi_0 (X(1))] \longrightarrow 0$$

is exact. By lemma 1.1.3, dividing out the relations coming from $X_1(2), X_0(2)$ and $\pi_0 (X(2))$ gives the groups $\pi_0 X_1, \pi_0 X_0$ and $\pi_0 X$ respectively. Since $X_0(2) \longrightarrow \pi_0 (X(2))$ is surjective, the subgroup of $\mathbb{Z}[X_0(1)]$ generated by the relations surjects onto the corresponding subgroup in $\mathbb{Z}[\pi_0 (X(1))]$, so the sequence

$$\pi_0 X_1 \xrightarrow{d_0 - d_1} \pi_0 X_0 \longrightarrow \pi_0 X \longrightarrow 0$$

is again exact which proves the lemma. $\square$

Theorem 2.4.1 implies that T-algebras have a Hurewicz and a Whitehead Theorem. We recall Quillen's general approach to homology as the left derived functor of abelianization [Q1, II.5] in the case of T-algebras. By definition, the abelianization functor

$$-_{\mathrm{ab}} : T\text{-alg} \longrightarrow (T\text{-alg})^{\mathrm{ab}}$$

is the left adjoint to the forgetful functor. If X is a T-algebra, choose a cofibrant replacement $X^{\mathrm{c}} \xrightarrow{\sim} X$ and define the homology of X to be the homotopy of the abelianization of the replacement:

$$\mathrm{H}_* X = \pi_*(X^{\mathrm{c}}_{\mathrm{ab}}) .$$

There is a natural weak map

$$X \xleftarrow{\sim} X^{\mathrm{c}} \longrightarrow X^{\mathrm{c}}_{\mathrm{ab}}$$

which we refer to as the Hurewicz map. In the case of the theory of sets this homology specializes to singular homology of simplicial sets. For commutative rings it specializes to André-Quillen homology [Q2, Sec. 4].

Corollary 2.4.3 *(Hurewicz theorem) Let T be a discrete theory, X an $(n-1)$-connected T-algebra ($n \geq 2$). Then the homology of X vanishes below dimension n and the Hurewicz map induces an isomorphism $\pi_n X \cong \mathrm{H}_n X$.*

Proof: In a first step we reduce to the case where X is $(n-1)$-reduced, i.e., where it consists only of the basepoint in simplicial dimensions $< n$. Recall the subcomplex $\mathrm{Sing}_{(n)}(Y)$ of the singular complex of a pointed space Y. The k-simplices of $\mathrm{Sing}_{(n)}(Y)$ consist of those singular simplices $\Delta^k \longrightarrow Y$ which map the entire $(n-1)$-skeleton of Δ^k to the basepoint of Y. In particular, $\mathrm{Sing}_{(n)}(Y)$ is always $(n-1)$-reduced. It is well known that for $(n-1)$-connected Y, the inclusion $\mathrm{Sing}_{(n)}(Y) \longrightarrow \mathrm{Sing}\,(Y)$ is a weak equivalence of simplicial sets. The functor $X \mapsto \mathrm{Sing}_{(n)}(|X|)$ on the category of pointed simplicial sets preserves finite products, so it passes to a functor on the category of T-algebras. Note that $\mathrm{Sing}_{(n)}$ is not a simplicial functor (unless n=0), but since we currently consider only discrete theories, this presents no problem. To summarize, any $(n-1)$-connected T-algebra X is connected by a chain of two weak equivalences to an $(n-1)$-reduced T-algebra:

$$X \xrightarrow{\sim} \mathrm{Sing}\,(|X|) \xleftarrow{\sim} \mathrm{Sing}_{(n)}(|X|)\ .$$

If X is $(n-1)$-reduced, so is its standard resolution (2.2.4), so we can assume that in addition X is cofibrant. But then the free abelian group object generated by X is also $(n-1)$-reduced, so the homology vanishes below dimension n.

Applying the Blakers-Massey Theorem for T-algebras (2.2.6) to the pushout square

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] & CX \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

(C denotes the cone) gives a Freudenthal Theorem for T-algebras. Since $n \geq 2$, the n-th homotopy group of X is thus isomorphic to the n-th stable homotopy group: $\pi_n X \cong \pi_n(\Sigma^\infty X)$. Since homology is a linear functor (section 5.1), there is a weak equivalence

$$X_{\mathrm{ab}} \simeq \Omega^\infty(HT^{\mathrm{ab}} \wedge^L_{T^s} \Sigma^\infty X)$$

by theorem 5.1.2. Since $T^{\mathrm{ab}} \cong \pi_0 T^s$, the Tor spectral sequence converging to the homotopy of the derived smash product (lemma 1.3.6) gives the desired answer for the first non-trivial homotopy group of X_{ab}. □

Corollary 2.4.4 *(Whitehead theorem) Let T be a discrete theory, $X \longrightarrow Y$ a map of simply connected T-algebras which induces an isomorphism on homology. Then the map is a weak equivalence.*

Proof: We can assume that X and Y are 1-reduced and cofibrant and that the map is a cofibration. Then the cofibre Y/X is 1-connected and has vanishing homology, hence is weakly contractible by the Hurewicz theorem 2.4.3. Applying the Blakers-Massey theorem 2.2.6 to the square of T-algebras

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, "\text{2-conn.}"'] & Y \arrow[d] \\ * \arrow[r, "\sim"'] & Y/X \end{tikzcd}$$

and induction gives that the map $X \longrightarrow Y$ is n-connected for all n $\square$

3 Examples

Sets. In the theory of pointed sets, the algebras are the pointed simplicial sets and the stable category is (a model for) the usual stable homotopy category. The associated DFSP is the sphere spectrum, so Theorem 2.3.4 reduces back to [BF, Thm. 5.8] saying that the homotopy theory of Γ-spaces is equivalent to that of connective spectra.

Simplicial sets with G-action. Let G be a simplicial monoid and consider the theory of pointed simplicial sets with pointed G-action. The stable category is the category of spectra with G-action (not to be confused with the more sophisticated notion of G-spectra when G is a compact Lie-group). The associated DFSP is the standard model for the monoid ring of G over the sphere spectrum $S^0[G]$. The map from stable homotopy to homology is represented by the map of monoid rings $S^0[G] \longrightarrow \mathbb{Z}[G]$.

If X is a reduced simplicial set, let GX be the Kan loop group of X. Pullback along a universal principal GX-bundle induces an equivalence between the homotopy theory of retractive spaces over X and the homotopy theory of pointed GX-spaces. So Theorem 2.3.4 can be interpreted as saying that the stable homotopy theory of spaces augmented over X is equivalent to the homotopy theory of $S^0[GX]$ modules, or spectra with an action of GX. This is exploited by J. Klein and J. Rognes to prove a chain rule for the Calculus of Functors [KR] (cf. the remarks at the end of section 5.2).

Monoids and groups. The theories of sets, monoids and groups have equivalent stable homotopy theories. This follows from the fact that the free monoid and the free group generated by a connected simplicial set are weakly equivalent. In fact, both are weakly equivalent to the loop space on the suspension of the simplicial set. Since the map from a simplicial set to the loop space of its suspension is roughly twice as highly connected as the space itself, the DFSPs associated to the theories of monoid and groups are both weakly equivalent to the sphere spectrum S^0.

In the setting of topological theories, [Bck, (7), (9)] or [SV1, Def. 1.8] show how one defines theories whose algebras are things like homotopy associative H-spaces, Stasheff's

A_∞-spaces or E_∞ monoid or ring spaces. For the A_∞-monoid theory, the discretization map $A_\infty \longrightarrow$(Associative monoids) is a weak equivalence of topological theories.

Nilpotent groups. We recall the lower central series

$$\cdots \subseteq \Gamma_r G \subseteq \Gamma_{r-1} G \subseteq \cdots \subseteq \Gamma_1 G = G$$

of a group G. By definition $\Gamma_r G = [\Gamma_{r-1}G, G]$, the subgroup generated by commutators (this is automatically a normal subgroup of G). A group is called nilpotent of class r if $\Gamma_r G$ is trivial. We denote by Nil^r the theory of class r nilpotent groups. We obtain a diagram of theories

$$(\text{groups}) \longrightarrow \cdots \longrightarrow \mathrm{Nil}^r \longrightarrow \mathrm{Nil}^{r-1} \longrightarrow \cdots \longrightarrow \mathrm{Nil}^2 = (\text{abelian groups})$$

which all have equivalent abelian group objects. It follows from a theorem of E. Curtis [Cu, Thm. 1.4] that the map $S^0 \longrightarrow (\mathrm{Nil}^r)^s$ is $(\log_2 r - 1)$-connected. So the associated sequence of DFSPs interpolates between S^0 and $H\mathbb{Z}$:

$$S^0 \longrightarrow \cdots \longrightarrow (\mathrm{Nil}^r)^s \longrightarrow (\mathrm{Nil}^{r-1})^s \longrightarrow \cdots \longrightarrow (\mathrm{Nil}^2)^s = H\mathbb{Z} .$$

p-local spaces via p-local groups. Fix a prime number p. By considering p-local nilpotent groups we can obtain a model for the homotopy theory of p-local spaces. In particular, we will obtain a DFSP model for the p-local sphere spectrum, together with a 'multiplicative filtration'. This provides a different view at the mod p-lower central series spectral sequence of [BCKQRS].

A nilpotent group G is called **p-local** if for all primes $q \neq p$ the set map $x \mapsto x^q$ is a bijection of G onto itself. On the category of nilpotent groups there exists a p-localization functor $G \mapsto G_{(p)}$ which is left adjoint to the inclusion of nilpotent p-local groups [Wf, Sec. 8]. p-localization is exact and commutes with the terms in the lower central series, i.e.,

$$(G/\Gamma_r G)_{(p)} \cong G_{(p)}/\Gamma_r(G_{(p)}) .$$

p-local groups of fixed nilpotence class r form a theory which we denote by $\mathrm{Nil}^r_{(p)}$. The map of DFSPs

$$(\mathrm{Nil}^r)^s \longrightarrow (\mathrm{Nil}^r_{(p)})^s$$

is the p-localization map on underlying spectra.

The category of simplicial theories has inverse limits and these are calculated pointwise [Bor, Prop. 3.11.1]. Denote by $\mathrm{Nil}^\wedge_{(p)}$ the inverse limit theory of the $\mathrm{Nil}^r_{(p)}$. Algebras over this theory are closely related, but not equivalent to, p-local nilpotent groups of arbitrary nilpotence class.

Bousfield and Kan show [BK, Ch. IV Prop. 4.1] that if X is a reduced simplicial set, GX its Kan loop group, then the inverse limit of the simplicial groups $(GX/\Gamma_r(GX))_{(p)}$ is weakly equivalent to the loop group of the $\mathbb{Z}_{(p)}$-completion of X. Hence if X is nilpotent, then this inverse limit models the p-localization of ΩX. In this sense, the algebraic theory $\mathrm{Nil}^\wedge_{(p)}$ models p-local spaces. In particular, the free $\mathrm{Nil}^\wedge_{(p)}$-algebra generated by a simplicial set X is a model for the p-localization of $\Omega|\Sigma X|$. Hence the DFSP $(\mathrm{Nil}^{\wedge}_{(p)})^s$ is a (non-commutative) DFSP model for the p-local sphere spectrum.

By Curtis' theorem [Cu, Thm. 1.4], the tower of DFSPs consisting of the $\mathrm{Nil}^r_{(p)}$ becomes eventually constant on homotopy groups of a fixed dimension. We claim that furthermore most of the maps are stable equivalences of DFSP and that the only changes occur at powers of the prime (compare [BCKQRS, 2.3]). This suggests forgetting most of the stages and reindexing the rest (compare [BCKQRS, 2.1]). Define F_i as the pointwise fibre of the map of DFSPs associated to $\mathrm{Nil}^{\wedge}_{(p)} \longrightarrow \mathrm{Nil}^{p^i+1}_{(p)}$.

$$F_i \;\longrightarrow\; (\mathrm{Nil}^{\wedge}_{(p)})^s \;\longrightarrow\; (\mathrm{Nil}^{p^i+1}_{(p)})^s$$

is a fibration sequence of simplicial sets when evaluated at any simplicial set since the right map is a surjective homomorphism of simplicial groups. In particular it induces a long exact sequence of stable homotopy groups. Furthermore, since F_i is the fibre of a multiplicative map between DFSPs, it inherits a multiplication (but no unit), and it behaves like an ideal of $(\mathrm{Nil}^{\wedge}_{(p)})^s$. We claim that the F_i's even form a multiplicative filtration of $(\mathrm{Nil}^{\wedge}_{(p)})^s$:

Claim: The image of $F_i \wedge F_j$ in $(\mathrm{Nil}^{\wedge}_{(p)})^s$ under the DFSP multiplication is contained in F_{i+j}.

The proof of the claim follows from the reinterpretation of DFSP multiplication in terms of composition of maps of free algebras (lemma 2.3.5) and the following lemma from group theory

Lemma: Let $f : F \longrightarrow G$ and $g : G \longrightarrow H$ be group homomorphisms and assume that the image of f is contained in $\Gamma_r G$, and similarly $\mathrm{Im}(g) \subseteq \Gamma_s H$. Then the image of the composite gf is contained in $\Gamma_{rs} H$.

Let us summarize what we obtained: looking at nilpotent p-local groups gives an algebraic theory $\mathrm{Nil}^{\wedge}_{(p)}$ that models p-local spaces. The DFSP associated to this theory is a model for the p-local sphere spectrum and it admits a convergent multiplicative filtration. This filtration in turn gives rise to a multiplicative spectral sequence. If one works with the p-lower central series, one gets a mod-p version of the p-local situation we just described. In that case, the spectral sequence obtained from the filtration is the mod-p lower central series spectral sequence of [BCKQRS]. These authors show that it is isomorphic, from the E^2-term on, to the Adams spectral sequence.

Infinite loop spaces. In our simplicial set-up, the Barratt-Eccles model [BE 1,2] gives an algebraic theory modeling infinite loop spaces. M. G. Barratt and P. J. Eccles define a functor Γ^+ from the category of pointed simplicial sets to itself [BE 1, Def. 3.1]. To avoid notational confusion with the category Γ of finite pointed sets, we use the notation γ^+ for the functor of Barratt and Eccles. The properties of γ^+ which matter from our point of view are:

- γ^+ is degreewise defined and commutes with filtered colimits, i.e., it comes from a Γ-space.
- γ^+ has the structure of a triple [BE 1, Prop. 3.6].

- for connected pointed simplicial sets X, $\gamma^+ X$ is a model for $\Omega^\infty \Sigma^\infty |X|$

The last property is proved for Kan complexes in [BE 1, Thm. 4.10, 5.4]. Since the functor γ^+ stems from a Γ-space , it preserves weak equivalences of simplicial sets [BF, 4.9] so that property holds for arbitrary X. The first two properties are equivalent to the fact that γ^+ is the free algebra functor associated to a simplicial algebraic theory. The theory, which we still denote γ^+, is obtained as usual as the dual of the category of finitely generated free γ^+-algebras (γ^+-algebras are called 'simplicial set with Γ^+-structure' in [BE 1]).

Every γ^+-algebra X is naturally a simplicial monoid. Barratt and Eccles show furthermore [BE 2, Thm. A] that if $\pi_0 X$ is a group, then the γ^+-structure provides natural infinite deloopings of X. In this sense, the algebraic theory γ^+ models infinite loop spaces. Note that the DFSP $(\gamma^+)^s$ arising from the theory γ^+ is yet another model for the sphere spectrum. This one has the property that its underlying Γ-space is special.

Modules. Let B be a ring, possibly simplicial, and consider the theory of simplicial left B-modules. The DFSP obtained from this theory is the standard model for the Eilenberg-MacLane spectrum associated to B; we denoted it HB in section 1.3. The homotopy theory of B-modules remains unchanged under stabilization (cf. [Sch, 2.2.2]). Theorem 2.3.4 thus says that the homotopy theory of simplicial B-modules is equivalent to the homotopy theory of HB-modules (as it should be).

We want to emphasize that the equivalence of the homotopy theories of simplicial B-modules with HB-modules comes from adjoint functor pairs on the level of model categories. This will allows us, for commutative B, to compare monoids with respect to the symmetric monoidal structures giving by $\wedge_{HB}$ and $\otimes_B$ respectively. This will be carried out in the next section. It leads to a comparison of the homotopy theory of simplicial B-algebras with that of HB-algebras.

Rings and algebras. Fix a commutative ring B, possibly simplicial. We will now discuss different types of associative B-algebras in our context. The examples are summarized in the following diagram of theories and maps:

$$(\text{Associative}) \longrightarrow (E_\infty) \longrightarrow (\text{Commutative}) \longrightarrow (\text{Nil}_B^\wedge)$$

Our approach takes as input *pointed* algebraic theories (i.e., there is a unique constant, or equivalently: the category of algebras has a zero object), but the above theories are not of this kind. Hence we have to make a slight modification and trade the units for augmentations: we note that the category of *augmented* associative B-algebras is equivalent to the category of associative B-algebras without unit; the augmentation ideal functor provides an equivalence. Associative B-algebras without unit are the algebras over a pointed simplicial algebraic theory, so this is the category we really describe. Hence the 'underlying simplicial set' of an augmented associative B-algebra is really the underlying simplicial set of the augmentation ideal. Similarly for E_∞ (which we will define below) and commutative B-algebras.

In fact, all these theories of B-algebras except for the last one arise from operads, so we briefly recall that notion (cf. [May, Def. 1.1]). Operads can be formed in any symmetric monoidal category, but we restrict our attention to the case of simplicial B-modules with respect to the tensor product. Then an operad A consists of a sequence A_n ($n \in \mathbb{N}$) of simplicial B-modules, together with the following structure:

- A linear Σ_n-action on A_n.
- A module map $B \longrightarrow A_1$
- B-module maps
$$A_n \otimes_B A_{i_1} \otimes_B \cdots \otimes_B A_{i_n} \quad\longrightarrow\quad A_{i_1+\cdots+i_n}$$

this data is required to satisfy certain equivariance, associativity and unit conditions that can be found in [May, Def. 1.1]. The property of the algebraic theory of being pointed corresponds to the module A_0 being trivial. Such an operad A gives rise to a functor on the category of simplicial B-modules

$$M \;\mapsto\; \bigoplus_{n=0}^{\infty} A_n \otimes_{B\,[\Sigma_n]} \underbrace{M \otimes_B \cdots \otimes_B M}_{n}$$

which is in fact a triple on B-mod. An algebra over the operad A is, by definition, an algebra over this triple.

All the above theories have a map from and to the theory of simplicial B-modules. In terms of algebraic functors, one of the maps comes from the forgetful functor (algebras)$\longrightarrow$(modules) and the other one from the functor (modules)$\longrightarrow$(algebras) which endows a module with the trivial multiplication. In particular, the DFSPs arising from any of the above theories of algebras always contain HB as a retract.

Associative algebras. Augmented associative B-algebras (alias associative B-algebras without unit) are the algebras over the associative operad given by $\mathrm{Ass}_n = B\,[\Sigma_n]$, the regular representation of the symmetric group. We claim that the map from the theory of B-modules to the theory of augmented associative B-algebras induces a weak equivalence on associated DFSP

$$HB \quad\xrightarrow{\sim}\quad (\text{Ass. } B\text{-alg})^s$$

The connective stable homotopy theory of augmented associative B-algebras is thus equivalent to the homotopy theory of simplicial B-modules.

To prove the claim we note that the free associative non-unital B-algebra generated by a pointed simplicial set K decomposes as the direct sum

$$\bigoplus_{n=1}^{\infty} \tilde{B}\,[\underbrace{K\wedge \cdots \wedge K}_{n}]$$

where $\tilde{B}\,[-]$ denotes the reduced free B-module. If K is taken to be a k-dimensional sphere, all homogenous components of degree ≥ 2 are at least $(2k-1)$-connected, so the connectivity of the map from the free B-module on S^k to the free non-unital associative B-algebra on S^k is roughly $2k$.

The fact that stable augmented associative B-algebras are equivalent to simplicial B-modules could have been proven without ever introducing DFSPs by the methods of [Sch, Sec. 3].

Commutative algebras. Augmented commutative B-algebras (alias commutative B-algebras without unit) are the algebras over the commutative operad given by $\mathrm{Com}_n = B$, with trivial action of the symmetric group. Commutative simplicial algebras have been the object of much study ([Q2], [Dw], [Go1,2], [Sch]). For example, the homology theory arising as the derived functor of abelianization in this case is known as André-Quillen-homology for commutative rings.

We denote by DB the DFSP arising from the theory of augmented commutative B-algebras. If B is a $\mathbb{Q}$-algebra, the map $HB \longrightarrow DB$ induced from the forgetful functor is a weak equivalence (cf. [Sch, Thm. 3.2.3]). In general, the Eilenberg-MacLane spectrum splits off DB, but the category of commutative augmented B-algebras can have higher stable homotopy operations, in which case DB is not equivalent to HB. We expect that then DB is not equivalent to any other (DFSP coming from a) simplicial ring.

We claim that as a Γ-space, DB is weakly equivalent to $HB \wedge^L H\mathbb{Z}$. In particular, the homotopy groups of DB are additively isomorphic to the integral spectrum homology of the Eilenberg-MacLane spectrum HB. To prove our claim we note that the free commutative B-algebra without unit on a pointed set X is additively isomorphic to the reduced free B-module on the free abelian monoid (infinite symmetric product) of X. Hence if we let CM denote the theory of abelian monoids, then as a Γ-space, DB is isomorphic to the composite of HB with CM^s. The forgetful functor from abelian groups to abelian monoids induces a stable weak equivalence $\mathrm{CM}^s \xrightarrow{\sim} H\mathbb{Z}$, so by lemma 1.1.5, DB is weakly equivalent to the derived smash product of HB and $H\mathbb{Z}$. $HB \wedge^L H\mathbb{Z}$ can be constructed as an E_∞-ring spectrum, but the weak equivalence to DB can not be multiplicative in any sense since the ring of homotopy groups of $HB \wedge H\mathbb{Z}$ is graded commutative, that of DB is generally not.

$\pi_* DB$ is the 'stable algebra of the functor algebra' of symmetric powers in the sense of [B1, §12]. For prime fields $\mathbb{F}_p$, the calculations of A. K. Bousfield determine the ring of homotopy groups $\pi_* D\mathbb{F}_p$. For $p = 2$ [B1, Thm. 12.3] it is the associative unital $\mathbb{F}_2$-algebra with generators α_i for $i \geq 2$ subject to the relations

$$\sum_{i+j=n} \binom{n}{i} \alpha_{2m+i}\,\alpha_{1+m+j} \;=\; 0$$

and

$$\sum_{i+j=n} \binom{n}{i} \alpha_{1+2m+i}\,\alpha_{1+m+j} \;=\; 0$$

for $m, n \geq 0$. [B1, Thm. 12.6] gives a similar but more complicated description for odd primes.

Completed commutative algebras and formal group laws. By suitably completing the theory of commutative B-algebras we now describe a way of constructing weak DFSP maps $H\mathbb{Z} \longrightarrow DB$ from commutative formal group laws. Denote by Nil^r_B the theory of augmented commutative B-algebras for which the r-th power of the augmentation ideal is trivial (this theory also comes from an operad, namely the commutative operad, truncated from the r-th term on). The forgetful functors give rise to theory maps

$$\cdots \longrightarrow \mathrm{Nil}^r_B \longrightarrow \mathrm{Nil}^{r-1}_B \longrightarrow \cdots \longrightarrow \mathrm{Nil}^2_B \cong B\text{-mod} .$$

Let $\mathrm{Nil}^\wedge_B$ denote the inverse limit theory. The category of $\mathrm{Nil}^\wedge_B$-algebras contains, but is bigger than, the category of complete augmented commutative B-algebras. For example, the free $\mathrm{Nil}^\wedge_B$-algebra generated by an infinite set is the filtered union of the formal power series rings generated by the finite subsets. Although the finite formal power series rings are complete, their union is not.

The theory of commutative augmented B-algebras maps to all the Nil^r_B's, so it also maps to the inverse limit theory. One can show that the induced map on DFSPs $DB \longrightarrow (\mathrm{Nil}^\wedge_B)^s$ is a stable weak equivalence. Now consider a one-dimensional commutative formal group law F over B. For every $\mathrm{Nil}^\wedge_B$-algebra, formal addition in the augmentation ideal is defined and gives an algebraic functor

$$\mathrm{Nil}^\wedge_B\text{-alg} \quad \xrightarrow{F^*} \quad (\text{simpl. abelian groups}) .$$

This algebraic functor corresponds to a morphism of theories in the opposite direction, hence gives a DFSP map $F_* : H\mathbb{Z} \longrightarrow (\mathrm{Nil}^\wedge_B)^s$. In view of the relation to commutative B-algebras, this is a weak map of DFSPs to DB,

$$H\mathbb{Z} \longrightarrow (\mathrm{Nil}^\wedge_B)^s \xleftarrow{\sim} DB .$$

If the commutative ring B is an algebra over the rational numbers, the situation is not very interesting: on the one hand, DB is then weakly equivalent to the Eilenberg-MacLane ring-spectrum HB, on the other hand all commutative formal group laws are then isomorphic to the additive formal group law. If B is a field of characteristic p, an important invariant of the formal group law F is its height. One can then show that formal group laws of different height give different (in the homotopy category of DFSPs, that is) maps $H\mathbb{Z} \longrightarrow DB$.

E_∞-algebras. Let $E\Sigma_n$ denote the standard model of a free weakly contractible simplicial Σ_n-set obtained as the nerve of the translation category of the symmetric group Σ_n. The collection of the $E\Sigma_n$'s has the structure of an operad on the category of pointed simplicial sets with respect to cartesian product (this is the operad that gives rise to the Barratt-Eccles model). By taking free B-modules we get an operad E on simplicial B-modules

$$E_n = \begin{cases} B\,[E\Sigma_n] & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

The operad map $E \longrightarrow \mathrm{Com}$ is degreewise a weak equivalence of $B\,[\Sigma_n]$-modules, and E_n is cofibrant as a $B\,[\Sigma_n]$-module, Com_n is generally not. An algebra over the operad E thus deserves to be called an E_∞-algebra (over the commutative ring B).

The free (non-unital) E_∞-algebra on a simplicial B-module M decomposes into homogenous pieces

$$E_\infty(M) = \bigoplus_{n=1}^{\infty} B\,[E\Sigma_n] \otimes_{B\,[\Sigma_n]} \underbrace{M \otimes_B \cdots \otimes_B M}_{n} .$$

If M is n connected and cofibrant as a B-module, all homogenous pieces from degree 2 on are $(2n+1)$-connected. Hence the map of DFSPs from the Eilenberg-MacLane spectrum HB to the DFSP associated to the theory of augmented E_∞-algebras is a stable weak equivalence for general B (as in the associative case, but in contrast to the strict commutative case). So theorem 2.3.4 implies that the (connective) stable homotopy theory of augmented E_∞-algebras is equivalent to the homotopy theory of simplicial B-modules. Again, as for the associative case, this result could have been proved by the methods of [Sch, Sec. 3] without ever mentioning DFSPs.

4 $H\mathbb{Z}$-algebras are simplicial rings

The current setup allows us to make precise the idea that simplicial rings are the same as connective brave new rings which are algebras over the Eilenberg-MacLane ring spectrum of integers. The $H\mathbb{Z}$-algebras (monoids in $(H\mathbb{Z}\text{-mod}, \wedge_{H\mathbb{Z}})$) form a model category by Theorem 1.3.8. We mention that the arguments of this section go through unchanged when the integers are replaced by any other commutative simplicial ring B and 'simplicial rings' are replaced by 'simplicial B-algebras'.

Theorem 4.1 *The model categories of simplicial rings and $H\mathbb{Z}$-algebras are related by two pairs of adjoint functors whose total derived functors are equivalences of homotopy theories*

$$\mathrm{Ho}\ (H\mathbb{Z}\text{-alg}) \;\cong\; \mathrm{Ho}\ (\text{simpl. rings})$$

This result was motivated by a construction of B. Dundas. Dundas shows in [Du, 3.4] that certain FSPs which should be (equivalent to FSPs arising from) simplicial rings actually are so. Here Dundas' explicit construction is replaced by the use of derived functors, which are available since all our categories are in fact model categories.

Most of the work has already been done. In fact, when applied to the theory of abelian groups, Theorem 2.3.4 says that the category $\mathbb{Z}$-mod of simplicial abelian groups has the same homotopy theory as the category of $H\mathbb{Z}$-modules. The equivalence is induced by Quillen adjoint functor pairs that compare both categories to the intermediate category $(\mathbb{Z}\text{-mod})^\Gamma$ of Γ-objects of simplicial abelian groups. Now all we have to do is to note that the category $(\mathbb{Z}\text{-mod})^\Gamma$ also has a symmetric monoidal product, that the associative monoids in there form a model category, and that the functors used for comparing the 'additive' structures behave well with respect to the products, i.e., that they are monoidal functors.

Step 1: Model category structure for DFTPs. We define a symmetric monoidal product $\otimes$ in the category $(\mathbb{Z}\text{-mod})^\Gamma$ and show that the associative monoids again form a model category. This tensor product is analogous to the smash product of Γ-spaces. Given two Γ-objects in $\mathbb{Z}$-mod M and N, we first form their external tensor product $M\tilde{\otimes}N$ using the tensor product of simplicial abelian groups:

$$M\tilde{\otimes}N : \Gamma\times\Gamma \longrightarrow \mathbb{Z}\text{-mod}, \quad (k,l)\mapsto M(k)\otimes_{\mathbb{Z}} N(l)$$

To obtain the internal tensor product $M \otimes N$ we take the left Kan extension along the smash product functor $\wedge : \Gamma \times \Gamma \longrightarrow \Gamma$. This internal tensor product is associative and commutative up to coherent isomorphism. It has a unit given by the Γ-abelian group $\tilde{\mathbb{Z}}$ which maps an object of Γ to the reduced free abelian group it generates.

A monoid in $((\mathbb{Z}\text{-mod})^\Gamma, \otimes)$ consists of a Γ-simplicial abelian group R together with multiplication and unit maps

$$R \otimes R \longrightarrow R, \qquad \tilde{\mathbb{Z}} \longrightarrow R$$

in the category $(\mathbb{Z}\text{-mod})^\Gamma$. These maps are required to satisfy the associativity and unit conditions analogous to those in the definition of DFSPs (Def. 1.3.1). The multiplication map amounts to having maps

$$R(n) \otimes_{\mathbb{Z}} R(m) \longrightarrow R(n \wedge m)$$

of simplicial abelian groups natural in $m, n \in \mathrm{Ob}\,\Gamma$. In analogy to DFSPs we call these monoids 'Degreewise Functors with Tensor Product' (DFTPs).

As usual, a map of DFTPs will be called a weak equivalence (resp. fibration) if the underlying map in the category $(\mathbb{Z}\text{-mod})^\Gamma$ is a weak equivalence (resp. fibration). Recall that weak equivalences and fibrations in $(\mathbb{Z}\text{-mod})^\Gamma$ were in turn defined on underlying Γ-spaces (sec. 2.3, step 1 in the comparison of the stable categories). A map of DFTPs is a cofibration if it has the LLP with respect to all acyclic fibrations. The model category structure of $(\mathbb{Z}\text{-mod})^\Gamma$ is cofibrantly generated and every object is small relative to the whole category. To establish the model category structure for DFTPs, we appeal again to theorem B.1. It remains to verify the smashout and the monoid axiom. The arguments for this are very similar to those used in [Ly, Sec. 3-5] and Thm. 1.3.8 to show that the smash product of Γ-spaces has the analogous properties.

The tensor product has an adjoint internal Hom, so it is enough to verify the smashout axiom for the generating (acyclic) cofibrations. These are all obtained from (acyclic) cofibrations of Γ-spaces by application of the reduced free abelian group functor. The latter functor transforms the smash product of Γ-spaces to the tensor product of Γ-abelian groups. Hence the smashout axiom follows from the smashout axiom for Γ-spaces.

The proof of the monoid axiom relies on the following two properties of the tensor product in $(\mathbb{Z}\text{-mod})^\Gamma$:

Lemma 4.2 *Let* $A \rightarrowtail B$ *be a cofibration and* W *an arbitrary object in* $(\mathbb{Z}\text{-mod})^\Gamma$*. Then the map* $A \otimes W \longrightarrow B \otimes W$ *is injective. If* $A \overset{\sim}{\rightarrowtail} B$ *is an acyclic cofibration, then* $A \otimes W \longrightarrow B \otimes W$ *is a weak equivalence.*

Given this lemma, the monoid axiom is proved as follows. For every acyclic cofibration $A \overset{\sim}{\rightarrowtail} B$ in $(\mathbb{Z}\text{-mod})^\Gamma$, the map $A \otimes W \longrightarrow B \otimes W$ is an injective weak equivalence. But the class of injective weak equivalences is closed under retracts, pushouts and transfinite composition. Because of the similarity of the arguments with those of [Ly], we omit the proof of lemma 4.2. For the convenience of the reader, we provide one important ingredient where the analogy between Γ-spaces and Γ-abelian groups is less obvious:

Γ-objects of abelian groups have skeleta defined by restriction to $\Gamma_{\leq n}$ and left Kan extension along the inclusion $\Gamma_{\leq n} \longrightarrow \Gamma$. These skeleta have to be distiguished from the skeleta of the underlying Γ-spaces. For W in $(\mathbb{Z}\text{-mod})^\Gamma$, let W_n^d denote the value of the $(n-1)$-skeleton at $n \in \mathrm{Ob}\,\Gamma$. ($W_n^d$ is the 'Γ-degenerate' part of $W(n)$). The proof of lemma 4.2 repeatedly uses induction over skeleta. For this one needs

Lemma 4.3 *For every object W in $(\mathbb{Z}\text{-mod})^\Gamma$ and every $n = 1, 2, \ldots$, the canonical map $W_n^d \longrightarrow W(n)$ is injective.*

Proof: We can assume that W is a Γ-object of discrete abelian groups. The abelian group W_n^d can be described as a certain quotient of the direct sum of the groups $W(i)$ indexed over all maps $f\colon i \longrightarrow n$ in Γ with $i < n$. Let us denote by $[f, x]$ the element $x \in W(i)$ in the summand indexed by f. Then the subgroup that is being divided out to obtain W_n^d is generated by the elements $[f\alpha, y] - [f, \alpha_*(y)]$ for all $\alpha\colon j \longrightarrow i$ with $j < n$, $y \in W(j)$. The map to $W(n)$ sends the element represented by $[f, x]$ to $f_*(x)$.

For $j = 1, \ldots, n$ let $d^j\colon (n-1) \longrightarrow n$ denote the map which is given by

$$d^j(k) \;=\; \begin{cases} k & \text{if } k < j \\ k+1 & \text{if } k \geq j \end{cases}$$

Any map $f\colon i \longrightarrow n$ in Γ with $i < n$ can be factored $f = d^j f'$ for some j and $f'\colon i \longrightarrow (n-1)$. Every element in W_n^d can thus be represented by a sum

$$\sum_{j=1}^{n} [d^j, x_j] \qquad (*)$$

Consider a sum $(*)$ which represents an element in the kernel of the map to $W(n)$, i.e., for which $\sum_{j=1}^n d^j_*(x_j) = 0$ holds in $W(n)$. We want to show that this sum actually represents 0 in W_n^d. We can assume that one of the x_j is non-zero and we let k be the smallest index for which $x_k \neq 0$.

We introduce two more maps $s^k : n \longrightarrow (n-1)$ and $c^k : (n-1) \longrightarrow (n-1)$ in Γ. s^k is determined by the properties that $s^k(k) = 0$ and $s^k d^k = \mathrm{id}_{n-1}$. c^k sends k to 0 and is the identity otherwise. For $j > k$, the relation $d^k s^k d^j = d^j c^k$ holds. We apply the map s^k_* to the relation $\sum_{j=k}^n d^j_*(x_j) = 0$ in $W(n)$ and obtain the equality

$$x_k \;=\; -\sum_{j>k}^{n} s^k_* d^j_*(x_j)\ .$$

But this implies that in the group W_n^d the relation

$$\begin{aligned} [d^k, x_k] &= -\sum_{j>k}^{n} [d^k, s^k_* d^j_*(x_j)] \\ &= -\sum_{j>k}^{n} [d^k s^k d^j, x_j] \\ &= -\sum_{j>k}^{n} [d^j, c^k_*(x_j)] \end{aligned}$$

holds. This last equality means that the original representative $(*)$ for the element in the kernel of $W_n^d \longrightarrow W(n)$ can be replaced by a different representative whose components below and including k are trivial. Working our way up inductively we obtain that the whole element is trivial in W_n^d. $\square$

Step 2: Simplicial rings versus DFTPs. The functor

$$-(1) : (\mathbb{Z}\text{-mod})^\Gamma \longrightarrow \mathbb{Z}\text{-mod}$$

which evaluates a Γ-object at $1 \in \mathrm{Ob}\,\Gamma$ is right adjoint to the functor $\tilde{\ }$ which maps a simplicial abelian group A to the Γ-simplicial abelian group $\widetilde{A}$ given by

$$(\widetilde{A})\,(k) \;=\; \underbrace{A \oplus \cdots \oplus A}_{k}\;;$$

(the underlying Γ-space of $\widetilde{A}$ is the very special Γ-space associated to A that was introduced in section 1.1).

The functors $-(1)$ and $\tilde{\ }$ form a Quillen adjoint pair: evaluation at 1 preserves fibrations and acyclic fibration, so its left adjoint preserves cofibrations and acyclic cofibrations. Furthermore, if A is any simplicial abelian group and X a *fibrant* object in $(\mathbb{Z}\text{-mod})^\Gamma$, then a map $\widetilde{A} \longrightarrow X$ is a weak equivalence if and only if its adjoint $A \longrightarrow X(1)$ is. Hence the total derived functors of $\tilde{\ }$ and $-(1)$ provide equivalences of homotopy theories ([D-S, Thm. 9.3]).

To finish this step, we note that the left adjoint $\tilde{\ }$ is in fact a strong symmetric monoidal functor with respect to the two kinds of tensor product. This means that there are natural isomorphisms

$$\tilde{\mathbb{Z}} \cong \widetilde{(\mathbb{Z})} \qquad \text{and} \qquad \tilde{M} \otimes \tilde{N} \;\cong\; \widetilde{(M \otimes_{\mathbb{Z}} N)}$$

compatible with the unit, associativity and commutativity isomorphisms. Since both tensor products commute with colimits in each of their variables, the monoids (simplicial rings resp. DFTPs) are the algebras over the free monoid triple

$$M \;\mapsto\; \bigoplus_{n=0}^{\infty} M^{\otimes n}\;.$$

Since it is a strong monoidal functor, $\tilde{\ }$ takes free monoids to free monoids, thus lemma 2.3.6 shows that $\tilde{\ }$ and $-(1)$ pass to an adjoint functor pair between the respective categories of monoids. These functors on the monoid categories are still Quillen pairs, so their derived functors provide equivalences of homotopy theories

$$\mathrm{Ho}\ (\mathrm{DFTPs}) \;\;\cong\;\; \mathrm{Ho}\ (\mathrm{simpl.\ rings})\;.$$

Step 3: DFTPs versus $H\mathbb{Z}$-algebras. Recall from the proof of Theorem 2.3.4, in the special case of the theory of abelian groups, that the category of Γ-objects in $\mathbb{Z}$-mod can also be viewed as the category of algebras over the triple $\tilde{\mathbb{Z}} \circ -$ on Γ-spaces:

$$(\tilde{\mathbb{Z}} \circ -)\text{-alg} \;\;\cong\;\; (\mathbb{Z}\text{-mod})^\Gamma\;.$$

$H\mathbb{Z}$ has the same underlying Γ-space as $\tilde{\mathbb{Z}}$, but it is viewed as a DFSP (= monoid with respect to $\wedge$) rather than as a triple (=monoid with respect to $\circ$).

In the same proof we noted that the comparison map from the smash to the composition product of Γ-spaces gives rise to a natural transformation

$$\Psi : H\mathbb{Z} \wedge - \longrightarrow \tilde{\mathbb{Z}} \circ -$$

which constitutes a map of triples on the category of Γ-spaces. So the pushforward Ψ_* gives a functor from $H\mathbb{Z}$-modules to $(\tilde{\mathbb{Z}} \circ -)$-algebras, i.e., Γ-objects in $\mathbb{Z}$-mod. Both categories have a symmetric monoidal structure given by $\wedge_{H\mathbb{Z}}$ and $\otimes$ respectively. Monoids in the first category are $H\mathbb{Z}$-algebras, monoids in the second category are DFTPs. Again, the pushforward functor Ψ_* is compatible with the symmetric monoidal structure in the best possible way: the next lemma will show that it is a strong symmetric monoidal functor. As in step 2 this implies that Ψ_* and Ψ^* pass to adjoint functors between the categories of $H\mathbb{Z}$-algebras and DFTPs. Every cofibrant $H\mathbb{Z}$-algebra is also cofibrant as a $H\mathbb{Z}$-module (Theorem 1.3.8). So by lemma 2.3.7, the unit of the adjunction is a weak equivalence for cofibrant $H\mathbb{Z}$-algebras. So the two functors provide the desired equivalences of homotopy theories

$$\mathrm{Ho}\,(H\mathbb{Z}\text{-alg}) \underset{R\Psi^*}{\overset{L\Psi_*}{\rightleftarrows}} \mathrm{Ho}\ (\mathrm{DFTPs})$$

Lemma 4.4 *Ψ_* is a strong symmetric monoidal functor, i.e., there is a unital, associative and commutative natural isomorphism*

$$\Psi_* M \otimes \Psi_* N \quad \cong \quad \Psi_* \left(M \wedge_{H\mathbb{Z}} N\right) \qquad\qquad (M, N \in H\mathbb{Z}\text{-mod})\ .$$

Proof: We note first that every $H\mathbb{Z}$-module M is the coequalizer, in the category of left $H\mathbb{Z}$-modules, of the two maps

$$\mu \wedge \mathrm{id}, \mathrm{id} \wedge \alpha : H\mathbb{Z} \wedge H\mathbb{Z} \wedge M \rightrightarrows H\mathbb{Z} \wedge M\ .$$

Since both sides of the isomorphism we want to establish commute with colimits, it is enough to consider $H\mathbb{Z}$-modules which are induced, i.e., which are of the form $H\mathbb{Z} \wedge X$ for some Γ-space X.

So assume $M = H\mathbb{Z} \wedge X$ and $N = H\mathbb{Z} \wedge Y$ for Γ-spaces X and Y. Then we have to establish a natural, unital, associative and commutative isomorphism of

$$\Psi_*(H\mathbb{Z} \wedge X) \otimes \Psi_*(H\mathbb{Z} \wedge Y) \quad \cong \quad (\tilde{\mathbb{Z}} \circ X) \otimes (\tilde{\mathbb{Z}} \circ Y)$$

with

$$\Psi_*((H\mathbb{Z} \wedge X) \wedge_{H\mathbb{Z}} (H\mathbb{Z} \wedge X)) \quad \cong \quad \Psi_*(H\mathbb{Z} \wedge X \wedge Y) \quad \cong \quad \tilde{\mathbb{Z}} \circ (X \wedge Y)\ .$$

This in turn comes from the isomorphism of simplicial abelian groups

$$\tilde{\mathbb{Z}}(K) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}(L) \quad \cong \quad \tilde{\mathbb{Z}}(K \wedge L)$$

for K, L pointed simplicial sets. $\square$

5 Linear functors

5.1 Classification

In this section we will classify connective linear functors from the category of S-algebras to the category of T-algebras for simplicial theories S and T. The result is that every such linear functor is given by smashing with a T^s-S^s-bimodule. This generalizes the classical fact that every linear functor from spaces to spaces is given by smashing with some spectrum. Before going into the details of the classification theorem, it might be illustrative to look at the analogous statement in algebra, which is well known.

Proposition 5.1.1 *Let A, B be discrete rings and $F : A\text{-mod} \longrightarrow B\text{-mod}$ an additive functor between the categories of left modules. Letting A act on $F(A)$ from the right through F, $F(A)$ becomes a B-A-bimodule such that there is a natural transformation*

$$F(A) \otimes_A - \longrightarrow F$$

of functors $A\text{-mod} \longrightarrow B\text{-mod}$. Furthermore, this map is an isomorphism for all finitely generated projective A-modules.

In homotopical algebra, the analogue of additive functors are the linear functors. Consider two simplicial theories S and T. A functor

$$L : S\text{-alg} \longrightarrow T\text{-alg} .$$

is called **linear** if

- L is a homotopy functor, i.e., it takes weak equivalences to weak equivalences.
- L is reduced, i.e., $L(*)$ is weakly contractible.
- L is excisive, i.e., it takes homotopy cocartesian squares to homotopy cartesian squares.
- L satisfies the limit axiom: let I be a filtered poset and $\phi : I \longrightarrow S\text{-alg}$ a functor that takes all morphisms to cofibrations. Then the map

  $$\operatorname{hocolim}_I L \circ \phi \longrightarrow L(\operatorname{colim}_I \phi)$$

  is a weak equivalence.

Given an S-algebra X, we need a model for its suspension spectrum as an S^s-module. Define a Γ-object of S-algebras $\Sigma^\infty X$ by

$$(\Sigma^\infty X)(n) \quad = \quad X \wedge n \quad = \quad \underbrace{X \amalg \cdots \amalg X}_{n} ,$$

the coproduct being taken in the category of S-algebras. Note that when $\Sigma^\infty X$ is extended (by direct limit and degreewise application) to a functor from simplicial sets to S-algebras, then we get $(\Sigma^\infty X)(K) = X \otimes K$. Hence the spectrum associated to $\Sigma^\infty X$ is isomorphic to the suspension spectrum of X, which justifies using the same name for these two different objects. Recall from Theorem 2.3.4 that Γ-objects in S-alg can be pulled back to S^s-modules along the triple map

$$\Psi : S^s \wedge - \quad \longrightarrow \quad S^s \circ - \ .$$

Explicitly, this means that the underlying Γ-space of the Γ-S-algebra $\Sigma^\infty X$ is endowed with a left S^s-action via the comparison map

$$S^s \wedge \Sigma^\infty X \quad \longrightarrow \quad S^s \circ \Sigma^\infty X \quad \longrightarrow \quad \Sigma^\infty X \ .$$

We also need the infinite loop object functor

$$\Omega^\infty : T^s\text{-mod} \ \longrightarrow \ T\text{-alg}$$

which is defined as the following composite: take a T^s-module M, replace it by a cofibrant approximation M^{c}, push it forward to a Γ-object of T-algebras using the functor Ψ_* of step 2 in comparison of stable homotopy theories (Theorem 2.3.4), then approximate by a fibrant Γ-object of T-algebras and evaluate the latter at $1 \in \mathrm{Ob}\,\Gamma$. In formulas

$$\Omega^\infty M \quad = \quad (\Psi_*(M^{\mathrm{c}})^{\mathrm{f}})\,(1) \ .$$

By the small object argument (Lemma A.1) replacements by (co-)fibrant objects can be made functorially. This complicated definition is necessary because left (resp. right) adjoint functors have the tendency to produce 'wrong' homotopy types unless objects are made cofibrant (resp. fibrant) first. The important property of Ω^∞ is that it preserves weak equivalences and homotopy groups.

Since Γ-spaces only represent connective spectra we can only expect to classify *connective* linear functors L, i.e., those L which preserve connectivity of maps and objects. To a connective linear functor L we will functorially associate a T^s-S^s-bimodule $L^!$. The classification of linear functors then reads

Theorem 5.1.2 *Let $L : S$-alg$\longrightarrow T$-alg be a connective linear functor. Then there is a chain of weak equivalences of T-algebras, natural in X and L,*

$$\Omega^\infty\,(L^! \wedge_{S^s} \Sigma^\infty X^{\mathrm{c}}) \ \simeq \ L\,(X) \ .$$

Under the assumptions of this theorem, the Tor-spectral sequence for the derived smash product (lemma 1.3.6) translates to a universal coefficient spectral sequence that calculates the homotopy of $L\,(X)$ from the stable homotopy of X and the coefficients of the linear functor L

$$\mathrm{E}^2_{p,q} \ = \ \mathrm{Tor}_p^{\pi_* S^s}(\pi^s_* X, \pi_* L^!)_q \quad \Longrightarrow \quad \pi_{p+q}\, L\,(X) \ .$$

Remark. If S and T are the theories of left modules over discrete rings A and B respectively, there are now two kinds of classification theorems. The algebraic version says that additive functors A-mod$\longrightarrow B$-mod are classified by algebraic B-A-bimodules, i.e., by $(A^{\mathrm{op}} \otimes_{\mathbb{Z}} B)$-modules. Every additive functor gives rise, by degreewise application to simplicial modules, to a linear functor between the categories of simplicial modules. The homotopy theoretic classification says that the (bigger class of) linear functors A-mod$\longrightarrow B$-mod are classified by brave new B-A-bimodules, i.e., by $(A^{\mathrm{op}} \wedge_{S^0} B)$-modules.

The **proof** of Theorem 5.1.2 follows a familiar pattern: we show that the left hand side of the desired equivalence is also a connective linear functor; we construct a (weak) map between the two expressions; we verify that the (weak) map is an equivalence when evaluated at the free S-algebra on a single generator. Linearity then implies that the map is a weak equivalence for arbitrary S-algebras. Unfortunately, a certain amount of technicalities are involved in getting this idea to work.

Lemma 5.1.3 *For any T^s-S^s-bimodule M, the functor*

$$\Omega^\infty \left(M \wedge_{S^s} \Sigma^\infty X^{\mathrm{c}}\right)$$

is a connective linear homotopy functor in X.

Proof: The functor in question is a reduced homotopy functor. The functor $\Sigma^\infty : S\text{-alg} \longrightarrow S^s\text{-mod}$ preserves cofibrations and weak equivalences between cofibrant objects. Smashing with M over S^s preserves pushouts and weak equivalences between cofibrant S^s-modules and it takes cofibrations to injective maps (Theorem 1.3.4). Hence the composite

$$X \quad \mapsto \quad M \wedge_{S^s} \Sigma^\infty X^{\mathrm{c}}$$

maps homotopy cocartesian squares of S-algebras to squares of T^s-modules whose underlying squares of Γ-spaces are pointwise homotopy cocartesian. This is enough to conclude long exact sequences of homotopy groups by lemma 1.1.4, so the five-lemma shows that such squares are homotopy cartesian. Ω^∞ preserves homotopy groups, hence it preserves homotopy cartesian squares. □

In the next step we replace L by a weakly equivalent functor with additional properties. This construction will be needed again in the next section and it works more generally for reduced homotopy functors. We state it as the following

Lemma 5.1.4 *Every reduced homotopy functor $F : S\text{-alg} \longrightarrow T\text{-alg}$ admits a natural chain of weak equivalences to a functor which is pointed, simplicial and takes values in fibrant objects.*

Proof: The first step makes F into a pointed functor. Recall that the standard resolution (lemma 2.2.4) gives an endofunctor on the category of T-algebras which is weakly equivalent to the identity and which turns injective maps into cofibrations. For any S-algebra X, the map $F(*) \longrightarrow F(X)$ is injective since it has a retraction. Thus $|\Phi(F(*))_*| \longrightarrow |\Phi(F(X))_*|$ is a cofibration between cofibrant T-algebras whose source is weakly contractible. We get a chain of two natural weak equivalences

$$F(X) \quad \overset{\sim}{\longleftarrow} \quad |\Phi(F(X))_*| \quad \overset{\sim}{\longrightarrow} \quad \text{cofibre } \left(\ |\Phi(F(*))_*| \rightarrowtail |\Phi(F(X))_*| \ \right)$$

such that the rightmost functor is pointed.

We can now assume that F is pointed and we also make it simplicial by a device from [W, sec. 3.1]. Define $\bar{F}(X)$ to be the realization of the simplicial T-algebra which in simplicial degree n is equal to $F(X^{\Delta^n})$. The maps $\Delta^n \longrightarrow *$ of simplicial sets induce a map from the constant simplicial object with value $F(X)$ to $\bar{F}(X)$. Since F is a homotopy functor, this map is the realization of a degreewise weak equivalence, hence it is a weak equivalence. The functor $\bar{F}$ is simplicial and it is pointed if F is.

The composite of the geometric realization and singular complex functors lifts to an endofunctor on the category of T-algebras (cf. proof of Theorem 2.2.1) which is weakly equivalent to the identity and takes values in fibrant objects. Composing a given homotopy functor F with geometric realization and singular complex thus gives an equivalent functor with values in fibrant T-algebras. If F is pointed and simplicial, so is the composite.

□

We can thus assume that the given linear functor L is pointed and simplicial (the fibrancy condition will not be needed until the next section). We now describe how an arbitrary pointed functor $G\colon S\text{-alg}\longrightarrow T\text{-alg}$ gives rise to a T^s-S^s-bimodule $G^!$ (the notation is taken from [PW, 2.6] where this construction is used in the case of the theory of modules over a ring). As a Γ-space, $G^!$ takes the object n to $G\,(F^S(n))$, the underlying simplicial set of G applied to the free S-algebra on n generators. Since G is defined for S-algebras and takes values in T-algebras, there are maps of Γ-spaces

$$T^s(G^!(S^s(n))) = T^s(G\,(F^S(S^s(n)))) \quad\longrightarrow\quad G\,(F^S(n)) = G^!(n)\ .$$

So the Γ-space $G^!$ becomes a bimodule via the composite with the comparison map

$$T^s \wedge G^! \wedge S^s \quad\longrightarrow\quad T^s \circ G^! \circ S^s \quad\longrightarrow\quad G^!\ .$$

Now a linear functor L can be recovered, up to weak equivalence, from the Γ-space $L^!$ and the bimodule structure. We first construct a map of T-algebras

$$(L^! \wedge_{S^s} \Sigma^\infty X)\,(K) \;\longrightarrow\; L\,(X \wedge K)\ ,$$

functorial in the S-algebra X and the simplicial set K, as follows: the map

$$L^!\,(X \wedge n) \;=\; L\,(F^S\,(X \wedge n)) \quad\longrightarrow\quad L\,(X \wedge n)$$

comes from the fact that $X \wedge n$ is a S-algebra. We get a map of Γ-spaces

$$L^! \wedge \Sigma^\infty X \quad\longrightarrow\quad L^! \circ \Sigma^\infty X \quad\longrightarrow\quad L\,(X\wedge -)$$

which equalizes the two maps $L^! \wedge S^s \wedge \Sigma^\infty X \implies L^! \wedge \Sigma^\infty X$ coming from the actions of S^s from the left and the right of the two respective smash factors. Hence, by the definition of the smash product over S^s, the last map factors over a map of Γ-spaces

$$L^! \wedge_{S^s} \Sigma^\infty X \;\longrightarrow\; L\,(X\wedge -)$$

which is in fact a map of T^s-modules. Since L is a pointed simplicial functor there is a map, natural in simplicial sets K,

$$L\,(X\wedge -)\,(K) \;\longrightarrow\; L\,(X \wedge K)\ .$$

Here the left hand side is a Γ-space extended to simplicial sets degreewise which is in general different from the right hand side (we did not assume that L is degreewise evaluable). Evaluating on spheres we get a map of spectra of T-algebras

$$(L^! \wedge_{S^s} \Sigma^\infty X)\,\underline{S} \quad\longrightarrow\quad L^\infty(\Sigma^\infty X)$$

of which we can take infinite loop objects to arrive at a natural map

$$\Omega^\infty \left(L^! \wedge_{S^s} \Sigma^\infty X\right) \quad \longrightarrow \quad \Omega^\infty L^\infty(\Sigma^\infty X) \; .$$

The right hand side admits a map from $L(X)$ since L is pointed and simplicial. For X cofibrant that map is a weak equivalence since L is linear. Functorially replacing X by a weakly equivalent cofibrant X^c gives a weak map

$$\Omega^\infty \left(L^! \wedge_{S^s} \Sigma^\infty X^c\right) \quad \longrightarrow \quad \Omega^\infty L^\infty(\Sigma^\infty X^c) \quad \xleftarrow{\sim} \quad L(X^c) \quad \xrightarrow{\sim} \quad L(X)$$

between the two connective linear functors whose equivalence we want to establish.

In the special case where $X = F^S(S^0)$ is the free S-algebra on a single generator, we note that the map

$$(L^!)\,\underline{S} \quad \longrightarrow \quad L^\infty(\Sigma^\infty F^S(S^0))$$

is a weak equivalence of spectra. In fact, since L is linear, the Γ-space $L^!$ is very special, so the left hand side is a connective Ω-spectrum by [BF, Thm. 4.2]. But the right hand side also is, again since L is linear. Since the given map is a weak equivalence on the zeroth term (in fact an isomorphism), it is a weak equivalence of spectra.

So all that is left to show is

Lemma 5.1.5 *Let $L \longrightarrow L'$ be a natural transformation between connective linear functors from S-algebras to T-algebras. Suppose the map is a weak equivalence for the free S-algebra on one generator. Then it is a weak equivalence for all S-algebras.*

Proof: The argument is well known in the case of simplicial sets, so we only give a brief outline. By assumption, L and L' agree on the free S-algebra generated by the boundary of the simplicial 1-simplex. The boundary of the simplicial n-simplex can be written as the union of two contractible simplicial subsets D^+ and D^- with intersection the boundary of an $(n-1)$-simplex. We apply the excisiveness condition for L to the pushout square of free S-algebras generated by these simplicial sets to get a homotopy cartesian square

$$\begin{tikzcd}
L(F^S(\partial\Delta^{n-1})) \arrow[r] \arrow[d] & L(F^s(D^+)) \arrow[d] \\
L(F^S(D^-)) \arrow[r] & L(F^S(\partial\Delta^n))
\end{tikzcd}$$

Since L is reduced, the upper left corner is thus weakly equivalent to the loop object of the lower right corner. A similar reasoning applies to L'. By induction, L and L' agree on $F^S(\partial\Delta^{n-1})$, so since both are connective, they agree on $F^S(\partial\Delta^n)$.

Consider an arbitrary cofibrant S-algebra X and apply the linear functor L to the homotopy cocartesian square

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[d] & CX \arrow[d] \\
* \arrow[r] & \Sigma X
\end{tikzcd}$$

(C denotes the cone). By excisiveness, $L(X)$ is weakly equivalent to the loop object of $L(\Sigma X)$. So it is enough to show that L and L' agree on connected S-algebras. Suppose $L(X) \longrightarrow L'(X)$ is a weak equivalence for some cofibrant connected S-algebra X and assume X' is obtained from X by gluing on a free S-algebra generated by the simplicial n-simplex along its boundary ($n \geq 1$). If we apply L and L' to the gluing diagram we obtain a map of homotopy cartesian squares between *connected* objects which is a weak equivalence on three of the four vertices. Then $L(X') \longrightarrow L'(X')$ is also a weak equivalence. Every cofibrant S-algebra is a retract of an S-algebra which is obtained by a (possibly transfinite) sequence of such gluings of free algebras generated by simplices. By the limit axiom, L and L' thus agree on all cofibrant objects, by homotopy invariance they agree on all S-algebras.

□

5.2 Chain rule

In the last section we showed how linear functors between categories of algebras over theories are classified by bimodules over the associated DFSPs . In this section we will prove that composition of linear functors corresponds to smashing the classifying bimodules. This gives a chain rule that expresses the derivative of the composite of two homotopy functors in terms of the derivatives of the constituent functors.

We recall the notions of stable excision and derivative for a homotopy functor. These were introduced by T. Goodwillie in the framework of Calculus of functors [Gw1,2]. The extension of the definitions from topological spaces to algebras over a simplicial theory is straightforward:

Definition 5.2.1 *(cf.* [Gw1, 1.8, 1.13]) *Let S, T be simplicial theories and $F, H : S\text{-alg} \longrightarrow T\text{-alg}$ homotopy functors. F is called stably excisive if there are constants c, κ for which F satisfies the following property called $E(c,\kappa)$: every homotopy cocartesian square of connected S-algebras*

$$\begin{array}{ccc} Y & \xrightarrow{k_1\text{-conn.}} & Y_1 \\ {\scriptstyle k_2\text{-conn.}}\downarrow & & \downarrow \\ Y_2 & \longrightarrow & Y_{12} \end{array}$$

in which the upper map is k_1-connected and the left map is k_2-connected (with $k_1, k_2 \geq \kappa$) is also $(k_1 + k_2 - c)$-homotopy cartesian.

F and H are said to agree to first order via a natural transformation $\tau : F \longrightarrow H$ if there are constants c, κ for which the following property holds: for every k-connected S-algebra Y with $k \geq \kappa$ the map $\tau_Y : F(Y) \longrightarrow H(Y)$ is $(2k - c)$-connected.

Our definition of stable excision is not the exact analogue of [Gw1, 1.8] since we only require the excision property for homotopy cocartesian squares of *connected* S-algebras. However, the arguments of [Gw1] go through with this slightly more general notion.

So for example if F satisfies $E(-n, -1)$ for all n, then F is excisive. The Blakers-Massey theorem for T-algebras (2.2.6) says that the identity functor of the category of T-algebras satisfies $E(1,1)$. By slight abuse of language we will also say that two functors agree to first order when they are related by a finite chain of natural transformation (usually suppressed from the notation) each of which satisfies the above first order property for some constants c, κ.

Goodwillie considers functors which are defined on the category of objects over a fixed object X. We restrict our attention to the case where $X = *$, the zero object in the category of S-algebras and we will only consider derivatives and differentials of functors at the zero object (except for the concluding remarks at the end of this section). We thus consider only reduced homotopy functors F. By lemma 5.1.4, we can make further simplifying assumptions on F. Replacing F by a weakly equivalent functor, we can assume it is pointed, simplicial and that it takes values in fibrant objects. With these assumptions, the construction of differential and derivative of F [Gw1, 1.11, 1.16] at the zero object can be simplified as follows:

Since F is pointed and simplicial, there is a natural map $F(Y) \longrightarrow \Omega F(\Sigma Y)$ adjoint to the map $\Sigma F(Y) \longrightarrow F(\Sigma Y)$. Define $DF(Y)$, the differential of F evaluated at Y, as the colimit of the sequence

$$F(Y^c) \quad \longrightarrow \quad \Omega F(\Sigma Y^c) \quad \longrightarrow \quad \cdots \quad \longrightarrow \quad \Omega^n F(\Sigma^n Y^c) \quad \longrightarrow \quad \cdots .$$

where Y^c is a cofibrant replacement of Y (there is no need for a homotopy colimit here since we consider a sequence of maps between fibrant simplicial sets. For such sequences the categorical colimit has the 'correct' homotopy type, e.g. it commutes with homotopy groups). There is thus a weak map

$$F(Y) \quad \xleftarrow{\sim} \quad F(Y^c) \quad \longrightarrow \quad DF(Y)$$

and as in [Gw1, 1.15], one proves:

Proposition 5.2.2 *Let $F : S\text{-alg} \longrightarrow T\text{-alg}$ be a stably excisive pointed homotopy functor as above. Then DF is excisive and F agrees with DF to first order via the (weak) map $F \longrightarrow DF$. If F satisfies the limit axiom, so does DF and hence DF is then linear. If two such functors agree to first order, then their derivatives are weakly equivalent.*

We can now define the derivative of F, denoted $\partial_* F$, as the T^s-S^s-bimodule that classifies the linear functor DF by theorem 5.1.2. So $\partial_* F = (DF)^!$ in the notation of the previous section. Recall that a functor is called connective if it preserves connectivity of maps.

Theorem 5.2.3 (Chain rule) *Let T_1, T_2, T_3 be simplicial theories and $F : T_1\text{-alg} \longrightarrow T_2\text{-alg}$, $H : T_2\text{-alg} \longrightarrow T_3\text{-alg}$ be stably excisive connective pointed homotopy functors. Then $H \circ F$ is stably excisive. If H satisfies the limit axiom, then there is a natural chain of weak equivalences of T_3^s-T_1^s-bimodules*

$$\partial_*(H \circ F) \quad \simeq \quad \partial_* H \wedge^L_{T_2^s} \partial_* F$$

Before starting the proof of the chain rule, we need the

Lemma 5.2.4 *Let S be a simplicial theory. Then the identity functor of the category of S^s-modules agrees to first order with the functor $M \mapsto \Sigma^\infty \Omega^\infty M$.*

Proof: Consider an n-connected S^s-module M. Then the S-algebra $\Omega^\infty M$ is n-connected and by the Freudenthal Theorem for S-algebras (cf. lemma 2.2.6), the map $\Omega^\infty M \longrightarrow \Omega^k \Sigma^k \Omega^\infty M$ is roughly $2n$-connected for all $k \geq 0$ (since in fact objects have to be replaced by cofibrant equivalent ones before applying Σ and by fibrant ones before applying Ω, the latter is only a weak map, but that does not affect the argument). Hence the (weak) map of S^s-modules $\Sigma^\infty \Omega^\infty M \longrightarrow M$ is also roughly $2n$-connected. $\square$

Proof of the chain rule. Suppose that F and H satisfy $E(c_F, \kappa_F)$ and $E(c_H, \kappa_H)$ respectively. We claim that $H \circ F$ then satisfies $E(c, \kappa)$ with constants $c = \max(c_F, c_H, 1)$ and $\kappa = \max(\kappa_F, \kappa_H)$. To see this, consider a homotopy cocartesian square of connected T_1-algebras

$$\begin{tikzcd} Y \arrow[r, "k_1\text{-conn.}"] \arrow[d, "k_2\text{-conn.}"'] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y_{12} \end{tikzcd}$$

where the maps have connectivity as indicated, $k_1, k_2 \geq \kappa$. Because F is stably excisive, the square

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & F(Y_1) \arrow[d] \\ F(Y_2) \arrow[r] & F(Y_{12}) \end{tikzcd}$$

is $(k_1 + k_2 - c_F)$-homotopy cartesian. Denote by P the homotopy pushout, in the category of T_2-algebras, of the above square with $F(Y_{12})$ deleted. By the Blakers-Massey Theorem 2.2.6 for T_2-algebras, the square with $F(Y_{12})$ replaced by P is $(k_1 + k_2 - 1)$-homotopy cartesian. Hence the map $P \longrightarrow F(Y_{12})$ is $(k_1 + k_2 - \max(c_F, -1))$-connected. Applying H gives a commutative diagram

$$\begin{tikzcd} H(F(Y)) \arrow[r] \arrow[d] & H(F(Y_1)) \arrow[d] \arrow[ddr] & \\ H(F(Y_2)) \arrow[r] \arrow[drr] & H(P) \arrow[dr] & \\ & & H(F(Y_{12})) \end{tikzcd}$$

Since H is stably excisive, the inner square is $(k_1 + k_2 - c_H)$-homotopy cartesian. The map $H(P) \longrightarrow H(F(Y_{12}))$ is again $(k_1 + k_2 - \max(c_F, 1))$-connected since H preserves connectivity. Hence the outer square is $(k_1 + k_2 - c)$-homotopy cartesian and thus $H \circ F$ is stably excisive.

Replacing F and H by their differentials DF and DH only changes the derivatives by a weak equivalence. Since H preserves connectivity, the derivative of the composite is also unchanged, up to weak equivalence. Hence we can assume that F is excisive and H is linear.

By the classification of linear functors, the composite $H \circ F$ is then related by a chain of maps to the functor

$$K \quad \mapsto \quad \Omega^\infty \left(\partial_* H \wedge_{T_2^s} \Sigma^\infty \Omega^\infty \left(\partial_* F \wedge_{T_1^s} \Sigma^\infty K\right)\right)$$

such that for finite cell-S-algebras K, the chain consists of weak equivalences. To simplify notation we are suppressing the fact that before applying Σ^∞ objects are to be replaced by weakly equivalent cofibrant ones.

By the above lemma 5.2.4, this functor agrees to first order with the functor

$$K \quad \mapsto \quad \Omega^\infty \left(\left(\partial_* H \wedge^L_{T_2^s} \partial_* F\right) \wedge^L_{T_1^s} \Sigma^\infty K\right) .$$

This last functor is linear, and since it agrees to first order with the functor $H \circ F$, is is weakly equivalent to the differential of $H \circ F$. The classification of linear functors thus provides the desired chain of weak equivalences of T_3-T_1-bimodules

$$\partial_*(H \circ F) \quad \simeq \quad \partial_* H \wedge^L_{T_2^s} \partial_* F .$$

□

J. Klein and J. Rognes [KR] have obtained a chain rule for the derivatives of functors from the category of pointed spaces to itself. We sketch how their chain rule can be viewed as a special case of ours. For a reduced simplicial set X, denote by $R(X)$ the category of simplicial sets which are retractive over X. For a functor $F\colon \mathcal{S}_{\mathrm{red}} \longrightarrow \mathcal{S}_{\mathrm{red}}$ on the category of reduced simplicial sets, denote by F_X the functor defined on the category of retractive spaces

$$F_X : R(X) \longrightarrow R(F(X))$$

that sends Y to $F(Y)$ with induced retraction and section. By construction, F_X preserves zero objects. Now the homotopy theory of the category $R(X)$ is equivalent to the homotopy theory of pointed simplicial sets with action of the Kan loop group GX of X (cf. [W, 2.1.4], [KR, 1.3]). This way F_X gives rise to a functor

$$F_X^\omega : GX\text{-simpl. sets} \quad \longrightarrow \quad G\left(F(X)\right)\text{-simpl. sets}$$

If F is stably excisive, connective and preserves weak equivalences, so does F_X^ω. This latter functor is pointed, so it has a derivative $\partial_* F_X^\omega$ in our sense which is a $S^0[G(F(X))]$-$S^0[GX]$-bimodule. We define $\partial_X F$ to be this bimodule and call it the derivative of F at X.

If $H\colon \mathcal{S} \longrightarrow \mathcal{S}$ is another stably excisive connective homotopy functor, we can apply the same construction to H and the composite $H \circ F$. Since $(H \circ F)_X^\omega \;\simeq\; H_{F(X)}^\omega \circ F_X^\omega$, our chain rule gives a chain of weak equivalences

$$\partial_X(H \circ F) \quad \simeq \quad \partial_{F(X)} H \wedge^L_{G(F(X))} \partial_X F$$

of $S^0[G(H(F(X))]$-$S^0[GX]$-bimodules.

To compare this to the chain rule of [KR], one has to make the following observations: a module over a group ring $S^0[G]$ is just a Γ-space with pointwise G-action. Furthermore, derived smash product of a left and a right module over $S^0[G]$ corresponds, up to weak equivalence, to passage to homotopy orbits of the smash product of the underlying Γ-spaces with diagonal G-action. Finally, the space $X \vee_x S^n$ used to define the derivative in [KR] corresponds to $S^n \wedge (GX)_+$ under the equivalence of homotopy theories between $R(X)$ and simplicial sets with GX-action.

A Cofibrantly generated model categories

We will need to transfer model category structures to categories of algebras over triples. In [Q, p. II 3.4], Quillen formulates his *small object argument*, which is now the standard device for such purposes. After Quillen, several authors have axiomatized and generalized the small object argument (see e.g. [Bl, Def. 4.4], [Cr, Def. 3.2] or [Sch, Def. 1.3.1]). In our context we will need a transfinite version of the small object argument. An axiomatization suitable for our purposes are the 'cofibrantly generated model categories' of [DHK], which we now recall. If a model category is cofibrantly generated, its model category structure is determined by a set of cofibrations and a set of acyclic cofibrations. A transfinite version of Quillen's small object argument [Q1, p. II 3.4] allows functorial factorization of maps as cofibrations followed by acyclic fibrations and as acyclic cofibrations followed by fibrations. Most of the model categories in the literature are cofibrantly generated, e.g. compactly generated topological spaces or simplicial sets are, as are all model categories that appear in this paper. The author is grateful to Bill Dwyer, Phil Hirschhorn and Daniel Kan for giving him access to early drafts of their book [DHK].

Ordinals and cardinals. An ordinal γ is an ordered isomorphism class of well ordered sets. It can be identified with the well ordered set of all preceding ordinals. For an ordinal γ, the same symbol will denote the associated poset category. The latter has an initial object $\emptyset$, the empty ordinal. An ordinal κ is a cardinal if its cardinality is bigger than that of any preceding ordinal. A cardinal κ is called regular if the following holds: for every set of sets $\{X_j\}_{j \in J}$ indexed by a set J of cardinality less that κ and such that the cardinality of each X_j is less than that of κ, then the cardinality of the union $\bigcup_J X_j$ is also less than that of κ. The successor cardinal (the smallest cardinal of bigger cardinality) of every cardinal is regular.

Transfinite composition. Let $\mathcal{C}$ be a cocomplete category and γ a well ordered set which we identify with its poset category. A functor $V : \gamma \longrightarrow \mathcal{C}$ is called a γ-sequence if for every limit ordinal $\beta < \gamma$ the natural map $\mathrm{colim} V|_\beta \longrightarrow V(\beta)$ is an isomorphism. The map $V(\emptyset) \longrightarrow \mathrm{colim}_\gamma V$ is called the transfinite composition of the maps of V. A subcategory $\mathcal{C}_1 \subset \mathcal{C}$ is said to be closed under transfinite composition if for every ordinal γ the following holds: given a γ-sequence $V : \gamma \longrightarrow \mathcal{C}$ such that $V(\alpha) \longrightarrow V(\alpha+1)$ is in $\mathcal{C}_1$ for every $\alpha < \gamma$, then the induced map $V(\emptyset) \longrightarrow \mathrm{colim}_\gamma V$ is also in $\mathcal{C}_1$. Examples of such subcategories are the cofibrations or the acyclic cofibrations in a closed model category.

Relatively small objects. Consider a cocomplete category $\mathcal{C}$ and a subcategory $\mathcal{C}_1 \subset \mathcal{C}$ closed under transfinite composition. If κ is a regular cardinal, an object $C \in \mathcal{C}$ is called

κ-small relative to $\mathcal{C}_1$ if for every regular cardinal $\lambda \geq \kappa$ and every functor $V : \lambda \longrightarrow \mathcal{C}_1$ which is a λ-sequence in $\mathcal{C}$, the map

$$\mathrm{colim}_\lambda \mathrm{Hom}_{\mathcal{C}}(C, V) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, \mathrm{colim}_\lambda V)$$

is an isomorphism. An object $C \in \mathcal{C}$ is called small relative to $\mathcal{C}_1$ if there exists a regular cardinal κ such that C is κ-small relative $\mathcal{C}_1$.

I-injectives, I-cofibrations and regular I-cofibrations. Given a cocomplete category $\mathcal{C}$ and a set I of maps, we denote

- by I-inj the subcategory of $\mathcal{C}$ consisting of maps which have the right lifting property with respect to the maps in I. Maps in I-inj are referred to as I-injectives.
- by I-cof the subcategory of $\mathcal{C}$ consisting of maps which have the left lifting property with respect to the maps in I-inj. Maps in I-cof are referred to as I-cofibrations.
- by $I\text{-cof}_{\mathrm{reg}} \subset I\text{-cof}$ the subcategory of the (possibly transfinite) compositions of maps that can be obtained via cobase change from maps in I. Maps in $I\text{-cof}_{\mathrm{reg}}$ are referred to as regular I-cofibrations.

Quillen's small object argument [Q1, p. II 3.4] has the following transfinite analog:

Lemma A.1 [DHK, II Prop. 7.5] *Let $\mathcal{C}$ be a cocomplete category and I a set of maps in $\mathcal{C}$ whose domains are small relative $I\text{-cof}_{\mathrm{reg}}$. Then*

- *there is a functorial factorization of maps $f \in \mathrm{Mor}\,(\mathcal{C})$ as $f = qi$ with $q \in I\text{-inj}$ and $i \in I\text{-cof}_{\mathrm{reg}}$ and thus*
- *every I-cofibration is a retract of a regular I-cofibration.*

Definition A.2 [DHK, II Def. 7.7] *A closed model category $\mathcal{C}$ is called cofibrantly generated if it is complete and cocomplete and there exist a set of cofibrations I and a set of acyclic cofibrations J such that*

- *the fibrations are precisely the J-injectives;*
- *the acyclic fibrations are precisely the I-injectives;*
- *the domain of each map in I (resp. in J) is small relative to $I\text{-cof}_{\mathrm{reg}}$ (resp. $J\text{-cof}_{\mathrm{reg}}$).*

For a specific choice of I and J as in the definition of a cofibrantly generated model category, the maps in I (resp. J) will be referred to as generating cofibrations (resp. generating acyclic cofibrations). In cofibrantly generated model categories, the factorization of maps as acyclic cofibrations followed by fibrations and as cofibrations followed by acyclic fibrations is thus possible in a functorial way.

Special cases of cofibrantly generated model categories are model categories that admit the small object argument in the sense of [Sch, Def. 1.3.1]. These include simplicial sets, algebras over a simplicial algebraic theory, and spectra of algebras over a simplicial algebraic theory. That definition in fact requires that one can choose generating cofibrations and generating acyclic cofibrations with the special property that sources and targets are $\aleph_0$-small relative to the category of all cofibrations (where $\aleph_0$ is the first infinite cardinal). In this case the transfinite small object argument can be replaced by a countable small object argument. In this paper we work with the more general notion since the stable model category of Γ-spaces is cofibrantly generated (Theorem 1.2.4) but the generators can not be chosen to be $\aleph_0$-small.

The application of cofibrant generation which is most important for our purposes is the following criterion for lifting a model category structure to a category of algebras over a triple. Other lifting theorems for model category structures can be found in [Bl, Thm. 4.14], [CG, Thm. 2.5], [Cr, Thm. 3.3], [DHK, II 8.2], [EKMM, VII Thm. 4.7, 4.9].

Let $\mathcal{C}$ be a cofibrantly generated model category and T a triple on $\mathcal{C}$. Call a map of T-algebras a weak equivalence (resp. fibration) if the underlying map in $\mathcal{C}$ is a weak equivalence (resp. fibration). Call a map of T-algebras a cofibration if it has the LLP with respect to all acyclic fibrations. The forgetful functor $T\text{-alg} \longrightarrow \mathcal{C}$ has a left adjoint, the free functor F^T.

Lemma A.3 *Assume that the underlying functor of T commutes with filtered direct limits and that every object in $\mathcal{C}$ is small relative to the whole category $\mathcal{C}$. Let I (resp. J) be a set of generating cofibrations (resp. acyclic cofibrations) and let I_T (resp. J_T) be the image under the free T-algebra functor F^T. Then the category of T-algebras becomes a cofibrantly generated closed model category with a generating set of cofibrations I_T and a generating set of acyclic cofibrations J_T if one of the following conditions holds:*

1. *Every regular J_T-cofibration is a weak equivalence.*
2. *$\mathcal{C}$ is a simplicial model category, T is a simplicial triple and there is a functor $Q\colon T\text{-alg} \longrightarrow T\text{-alg}$ and a natural weak equivalence $X \xrightarrow{\sim} QX$ such that QX is fibrant for all X.*

Proof: We use the numbering of the model category axioms as given in [BF, 1.1]. All those kinds of limits that exist in $\mathcal{C}$ also exist in T-alg, and limits are created in the underlying category $\mathcal{C}$ [Bor, Prop. 4.3.1]. Colimits are more subtle, but since the underlying functor of T commutes with filtered colimits, they exist by [Bor, Prop. 4.3.6]. Model category axioms CM2 (saturation) and CM3 (closure properties under retracts) are clear. One half of CM4 (lifting properties) holds by definition of cofibrations of T-algebras.

The proof of the remaining axioms uses the transfinite small object argument. We start with the factorization axiom CM5. Since the underlying functor of T commutes with filtered colimits, so does the forgetful functor $T\text{-alg} \longrightarrow \mathcal{C}$. Since every object in $\mathcal{C}$ is small relative to the whole of $\mathcal{C}$, all free T-algebras will be small relative to all of T-alg by adjointness. Also, every map in I_T is a cofibration of T-algebras by adjointness. Hence all I_T-cofibrations are cofibrations of T-algebras. Since I is a generating set of cofibrations for $\mathcal{C}$, the I_T-injectives are precisely the maps of T-algebras which are acyclic fibrations on underlying objects (adjointness again), i.e., the acyclic fibrations of T-algebras. Hence the small object argument applied to the set I_T gives a (functorial) factorization of maps in T-alg as a cofibration followed by an acyclic fibration.

The other half of the factorization axiom CM5 needs the additional assumptions we made. Applying the small object argument to the set of maps J_T gives a functorial factorization of maps in T-alg as regular J_T-cofibrations followed by J_T-injectives. Since J is a generating set for the acyclic cofibrations in $\mathcal{C}$, the J_T-injectives are precisely the fibrations among the T-algebra maps, one more time by adjointness. In case (1) we assumed that every regular J_T-cofibration is a weak equivalence on underlying objects in $\mathcal{C}$, so we are done. In case (2) we can adapt the argument of [Q1, p. 4.9] as follows.

Assuming (2) we first establish that the category of T-algebras inherits a simplicial structure. Since T was assumed to be a simplicial functor, there are natural maps

$$T(X^K) \longrightarrow T(X)^K$$

in the underlying category $\mathcal{C}$, where X is an object of $\mathcal{C}$ and K is a simplicial set. Now suppose that X is a T-algebra via the map $\alpha : T(X) \longrightarrow X$. Then X^K becomes a T-algebra via

$$T(X^K) \longrightarrow T(X)^K \xrightarrow{\alpha^K} X^K .$$

In defining products with simplicial sets for T-algebras, we will use the notation $X \otimes_T K$ to distinguish it from the product of X with K in the underlying category $\mathcal{C}$. Since T is simplicial, there are natural maps

$$T(X) \otimes K \longrightarrow T(X \otimes K)$$

in $\mathcal{C}$. These give maps of T-algebras

$$T(T(X) \otimes K) \longrightarrow T(T(X \otimes K)) \longrightarrow T(X \otimes K)$$

the second one coming from the triple structure of the functor T. There is a different map of T-algebras with the same source and target, namely

$$T(\alpha \otimes K) : T(T(X) \otimes K) \longrightarrow T(X \otimes K) .$$

We define $X \otimes_T K$ to be the coequalizer, in the category of T-algebras, of these two maps. We can then finally define the Hom simplicial sets of T-algebra maps between two T-algebras X and Y by

$$\mathrm{Hom}_T(X, Y)_n = \mathrm{Hom}_{T\text{-alg}}(X \otimes_T \Delta^n, Y) .$$

We omit the verification of the adjointness properties which shows that we in fact get a simplicial category this way.

We continue with the proof that every J_T-cofibration $i : X \longrightarrow Y$ is a weak equivalence, assuming hypothesis (2). Since QX is fibrant, the weak equivalence $X \xrightarrow{\sim} QX$ can be factored through Y by a lifting

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d, "i"'] & QX \arrow[d, two heads] \\ Y \arrow[r] \arrow[ur, dashed, "r"] & * \end{tikzcd}$$

Then $r \circ i$ is a weak equivalence. We will show that $(Qi) \circ r$ also is a weak equivalence. Since maps in $\mathcal{C}$ are weak equivalences if and only if they become isomorphisms in the homotopy category of $\mathcal{C}$, this will prove that i is a weak equivalence.

By the above, the path space QY^{Δ^1} has a canonical T-algebra structure, and the "inclusion of constant paths" $QY \longrightarrow QY^{\Delta^1}$ is a map of T-algebras. $(Qi)\circ r$ and the weak equivalence $Y \xrightarrow{\sim} QY$ together give a map $Y \longrightarrow QY \times QY$. The inclusion $\partial\Delta^1 \longrightarrow \Delta^1$ induces a map $QY^{\Delta^1} \longrightarrow QY \times QY$ which is a fibration since QY is fibrant. Hence a lifting exists in the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & QY^{\Delta^1} \arrow[d, two heads] \\ Y \arrow[r] \arrow[ur, dashed] & QY \times QY \end{tikzcd}$$

This shows that $(Qi) \circ r$ and the weak equivalence $Y \xrightarrow{\sim} QY$ descend to the same map in the homotopy category of $\mathcal{C}$, hence $(Qi) \circ r$ also is a weak equivalence. This finishes the proof of the factorization axiom CM5.

It remains to prove the other half of CM4, i.e., that any acyclic cofibration $A \overset{\sim}{\rightarrowtail} B$ has the LLP with respect to fibrations. In other words: the acyclic cofibrations are contained in the J_T-cofibrations. The small object argument provides a factorization

$$A \overset{\sim}{\rightarrowtail} W \twoheadrightarrow B$$

with $A \overset{\sim}{\rightarrowtail} W$ a J_T-cofibration and $W \twoheadrightarrow B$ a fibration. In addition, $W \longrightarrow B$ is a weak equivalence since $A \overset{\sim}{\rightarrowtail} B$ is. Since $A \overset{\sim}{\rightarrowtail} B$ is a cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & W \arrow[d, two heads, "\sim"] \\ B \arrow[r, "="'] \arrow[ur, dashed] & B \end{tikzcd}$$

exists. Thus $A \overset{\sim}{\rightarrowtail} B$ is a retract of a J_T-cofibration, so it is one itself.

The axiom SM7a that relates the model category structure to the simplicial structure is now easy. We just recall that weak equivalences, fibrations, pullbacks and mapping objects from simplicial sets in T-alg are all actually formed in the underlying category $\mathcal{C}$. So SM7a for T-alg follows from SM7a for $\mathcal{C}$ $\square$

B Monoids in symmetric monoidal model categories

The results in this section are joint work with B. Shipley. We plan to write an extended version of this appendix as a seperate paper [SS]. We provide a condition under which associative monoids in a symmetric monoidal, cofibrantly generated model category again form a model category. This applies e.g. to the well known model category structures for simplicial monoids or simplicial rings, established by Quillen in a different fashion. The new examples we apply this theorem to are DFSPs, i.e., the monoids in the category of Γ-spaces with respect to the smash product, and 'FSPs on spheres', which are the monoids in the category of symmetric spectra, investigated in [HSS].

So in this section we consider a cofibrantly generated closed model category $\mathcal{C}$ which is also a symmetric monoidal category ([MacL, VII.7], [Bor, 6.1]) under a product denoted $\wedge$ with unit $\mathbb{I}$. The model structure and the monoidal structure should be related by the following axioms:

1. Every object in $\mathcal{C}$ is small relative to the whole category $\mathcal{C}$.
2. (Smashout axiom) The smash product commutes with colimits in both variables. Given cofibrations $A \rightarrowtail B$ and $K \rightarrowtail L$ in $\mathcal{C}$, then the map
$$A{\wedge}L \cup_{A\wedge K} B{\wedge}K \rightarrowtail B{\wedge}L$$
is a cofibration. If in addition one of the former maps is a weak equivalence, so is the latter map.
3. (monoid axiom) Given an acyclic cofibration $A \overset{\sim}{\rightarrowtail} B$ in $\mathcal{C}$, objects W, X, P and a pushout square
$$\begin{tikzcd} A{\wedge}W \arrow[r] \arrow[d] & B{\wedge}W \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$
in $\mathcal{C}$, then the lower map $X \xrightarrow{\sim} P$ is a weak equivalence. Furthermore, every transfinite composite of maps obtained this way is again a weak equivalence in $\mathcal{C}$.

A monoid in $(\mathcal{C}, \wedge)$ is an object M together with maps
$$M \wedge M \longrightarrow M \quad \text{and} \quad \mathbb{I} \longrightarrow M$$
making the usual associativity and unit diagrams commute (see [MacL, VII.3]). These maps will be referred to as the multiplication and unit map of M. Morphisms of monoids are maps in $\mathcal{C}$ commuting with the structure maps (the terminology 'monoid in $(\mathcal{C}, \wedge)$' is a bit sloppy as it suppresses the rest of the symmetric monoidal structure from the notation). For example, a monoid in (simpl. sets, $\times$) is a simplicial monoid in the usual sense, a monoid in (simpl. abelian groups, $\otimes$) is a simplicial ring.

Call a map of monoids a weak equivalence or fibration if it is a weak equivalence or fibration in the underlying model category $\mathcal{C}$. Call a map of monoids a cofibration if it has the left lifting property with respect to all acyclic fibrations. We will prove

Theorem B.1 *Under the above assumptions, the category of monoids is a cofibrantly generated closed model category. If the unit of the smash product is cofibrant in the underlying category $\mathcal{C}$, then all cofibrant monoids are also cofibrant in the underlying category.*

The proof proceeds by using the fact that monoids are the algebras over the free monoid triple on $\mathcal{C}$, and then appealing to the general procedure of transporting a model category structure given by lemma A.3. We will gain the necessary homotopical control over certain pushout required by condition (1) of that lemma by using a kind of word length filtration.

For an object K of $\mathcal{C}$, define $T(K)$ to be

$$T(K) \;=\; \mathbb{I} \amalg K \amalg (K \wedge K) \amalg \;\ldots\; \amalg K^{\wedge n} \amalg \;\ldots$$

(T is reminiscent of 'tensor algebra'). Using that $\wedge$ distributes over the coproduct, $T(K)$ can in fact be given a monoid structure by concatenation. Then the functor T is left adjoint to the forget functor functor from monoids to $\mathcal{C}$. But T is also a triple on the category $\mathcal{C}$ and the T-algebras are precisely the monoids. Furthermore, the underlying functor of T commutes with filtered colimits.

Now we describe the filtration of certain kinds of pushouts in the monoid category, which is the crucial ingredient for making lemma A.3 applicable. Consider a map $K \longrightarrow L$ in $\mathcal{C}$, a monoid M and a monoid map $T(K) \longrightarrow M$. We want to describe the pushout in the monoid category of the diagram

$$\begin{tikzcd} T(K) \arrow[r] \arrow[d] & T(L) \\ X & \end{tikzcd}$$

The pushout P will be described as the colimit, in the underlying category $\mathcal{C}$, of a sequence

$$X = P_0 \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_n \longrightarrow \cdots \;.$$

If one thinks of P as consisting of formal products of elements from X and from L, with relations coming from the elements of K and the multiplication in X, then P_n consists of those products where the total number of factors from L is less than or equal to n. For ordinary monoids, this is in fact a valid description, and we will now translate this idea into the element-free form which applies to general symmetric monoidal categories.

As indicated above we set $P_0 = X$ and describe P_n inductively as a pushout in $\mathcal{C}$. We first describe an n-dimensional cube in $\mathcal{C}$. By definition, such a cube is a functor

$$W : \mathcal{P}(\{1, 2, \ldots, n\}) \;\longrightarrow\; \mathcal{C}$$

from the poset category of subsets of $\{1, 2, \ldots, n\}$ to $\mathcal{C}$. If $S \subseteq \{1, 2, \ldots\}$ is a subset, the vertex of the cube at S is defined to be

$$W(S) \;= X \wedge C_1 \wedge X \wedge C_2 \wedge \;\ldots\; \wedge C_n \wedge X$$

with

$$C_i \;= \begin{cases} K & \text{if } i \notin S \\ L & \text{if } i \in S \end{cases}$$

So a total of $n+1$ smash factors of X alternate with n smash factors of either K or L. The initial vertex corresponding to the empty subset has all C_i's equal to K and the terminal vertex corresponding to whole set has all C_i's equal to L. For example for $n = 2$, the cube is a square and looks like

$$\begin{tikzcd} X \wedge K \wedge X \wedge K \wedge X \arrow[r] \arrow[d] & X \wedge K \wedge X \wedge L \wedge X \arrow[d] \\ X \wedge L \wedge X \wedge K \wedge X \arrow[r] & X \wedge L \wedge X \wedge L \wedge X \end{tikzcd}$$

All maps in the cube W are induced from the map $K \longrightarrow L$ and the identity on the X factors.

Denote by Q_n the colimit of the punctured cube, i.e., of the cube with the terminal vertex removed, and define P_n via the pushout in $\mathcal{C}$

$$\begin{tikzcd} Q_n \arrow[r] \arrow[d] & (X \wedge L)^{\wedge n} \wedge X \arrow[d] \\ P_{n-1} \arrow[r] & P_n \end{tikzcd}$$

This is not a complete definition until we say what the left vertical map is. We define that map from Q_n to P_{n-1} by saying how it maps a vertex $W(S)$ to P_{n-1}, for S a proper subset of $\{1, 2, \dots, n\}$. All the smash factors of $W(S)$ which are equal to K are first mapped into X and then adjacent smash factors of X are multiplied. This gives a map

$$W(S) \longrightarrow X \wedge L \wedge X \wedge \ldots \wedge L \wedge X \ ,$$

where the right hand side has $|S|+1$ smash factors X and $|S|$ smash factors L. So the right hand side maps further to $P_{|S|}$, hence to P_{n-1} since S is a proper subset. We have to check that these maps on the vertices of the punctured cube W are compatible so that they assemble to a map from the colimit Q_n. So let S be again a proper subset of $\{1, 2, \dots, n\}$ and take $i \notin S$. We have to verify commutativity of the diagram

$$\begin{tikzcd} W(S) \arrow[r] \arrow[d] & (X \wedge L)^{\wedge |S|} \wedge X \arrow[r] & P_{|S|} \arrow[d] \\ W(S \cup \{i\}) \arrow[r] & (X \wedge L)^{\wedge (|S|+1)} \wedge X \arrow[r] & P_{|S|+1} \end{tikzcd}$$

By definition, $W(S)$ and $W(S \cup \{i\})$ differ in exactly one smash factor in the $2i$-th position which is equal to K for the former and equal to L for the latter. The upper left map factors as

$$W(S) \longrightarrow (X \wedge L)^{\wedge a} \wedge X \wedge K \wedge (X \wedge L)^{\wedge b} \wedge X \longrightarrow (X \wedge L)^{\wedge |S|} \wedge X$$

where a (resp. b) is the number of elements in S which are smaller (resp. bigger) than i; in particular $a + b = |S|$. The right map in this factorization pushes K into X and multiplies the three adjacent smash factors of X. Hence the diagram in question is the composite of two commutative squares

$$\begin{tikzcd}
W(S) \arrow[r] \arrow[d] & (X\wedge L)^{\wedge a} \wedge X \wedge K \wedge (X\wedge L)^{\wedge b} \wedge X \arrow[r] \arrow[d] & P_{|S|} \arrow[d] \\
W(S\cup\{i\}) \arrow[r] & (X\wedge L)^{\wedge(|S|+1)} \wedge X \arrow[r] & P_{|S|+1}
\end{tikzcd}$$

The right square commutes by the definition of $P_{|S|+1}$.

We have now completed the inductive definition of the P_n and we can set $P = \operatorname{colim} P_n$, the colimit being taken in $\mathcal{C}$. P comes equipped with $\mathcal{C}$-morphisms $X = P_0 \longrightarrow P$ and

$$L \cong \mathbb{I} \wedge L \wedge \mathbb{I} \longrightarrow X \wedge L \wedge X \longrightarrow P_1 \longrightarrow P$$

which make the diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & L \arrow[d] \\
X \arrow[r] & P
\end{tikzcd}$$

commute. There are several things to check:

(i) P is naturally a monoid so that
(ii) the map $X \longrightarrow P$ is multiplicative and
(iii) P has the universal property of the pushout.

Define the unit of P as the composite of $X \longrightarrow P$ with the unit of X. The multiplication of P is defined from compatible maps $P_n \wedge P_m \longrightarrow P_{n+m}$ by passage to the colimit. The latter maps are defined by induction on $n+m$ as follows. Note that $P_n \wedge P_m$ is the pushout in $\mathcal{C}$

$$\begin{tikzcd}
Q_n \wedge ((X\wedge L)^m \wedge X) \cup_{(Q_n \wedge Q_m)} ((X\wedge L)^n \wedge X) \wedge Q_m \arrow[r] \arrow[d] & ((X\wedge L)^n \wedge X) \wedge ((X\wedge L)^m \wedge X) \arrow[d] \\
(P_{n-1} \wedge P_m) \cup_{(P_{n-1} \wedge P_{m-1})} (P_n \wedge P_{m-1}) \arrow[r] & P_n \wedge P_m
\end{tikzcd}$$

The lower left corner already has a map to P_{n+m} by induction, the upper left corner is mapped there by multiplying the two adjacent X factors followed by the map $(X\wedge L)^{n+m} \wedge X \longrightarrow P_{n+m}$. We omit the tedious verification that this in fact gives a well defined multiplication map and that associativity and unit diagrams commute. This then shows that P is a monoid. The unit and multiplication were arranged so that $X \longrightarrow P$ is multiplicative.

For (iii) suppose we are given another monoid M, a multiplicative map $X \longrightarrow M$, and a

$\mathcal{C}$-map $L \longrightarrow M$ such that the outer square in

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & L \arrow[d] \arrow[ddr] & \\
X \arrow[r] \arrow[drr] & P \arrow[dr, dashed] & \\
 & & M
\end{tikzcd}$$

commutes. We have to show that there is a unique multiplicative map $P \longrightarrow M$ making the entire square commute. These conditions in fact force the behavior of the composite map $W(S) \longrightarrow P_n \longrightarrow P \longrightarrow M$. Since P is obtained by various colimit constructions from these basic building blocks, uniqueness follows. We again omit the tedious verification that the maps $W(S) \longrightarrow M$ are compatible and assemble to a multiplicative map $P \longrightarrow M$.

Now that we have established that P is the pushout of the original diagram of monoids, we continue with the homotopical analysis of the constructed filtration, i.e., we will verify condition (1) of lemma A.3. Assume now that $K \overset{\sim}{\rightarrowtail} L$ is an acyclic cofibration in $\mathcal{C}$. The cube W used in the inductive definition of P_n has $n+1$ smash factors of X at every vertex which map by the identity everywhere. Using the symmetry isomorphism for $\wedge$, these can all be shuffled to one side and we get that the map $Q_n \longrightarrow (X \wedge L)^{\wedge n} \wedge X$ is isomorphic to

$$\bar{Q}_n \wedge X^{\wedge(n+1)} \longrightarrow L^{\wedge n} \wedge X^{\wedge(n+1)} .$$

Here $\bar{Q}_n$ is the colimit of a punctured cube analogous to W, but with all the X smash factors in the vertices deleted. By iterated application of the smashout axiom (2), the map $\bar{Q}_n \overset{\sim}{\rightarrowtail} L^{\wedge n}$ is an acyclic cofibration, so by the monoid axiom (3), the map $P_{n-1} \overset{\sim}{\longrightarrow} P_n$ is a weak equivalence. The map $X = P_0 \overset{\sim}{\longrightarrow} P$ is an instance of a transfinite composite (indexed by the first infinite ordinal) of the kind of maps considered in axiom (3), so by assumption it is also a weak equivalence.

The candidates J_T for the generating acyclic cofibrations in the monoid category are the monoid maps freely generated by the generating acyclic cofibrations in the underlying category $\mathcal{C}$. With the use of the filtration we just established that any pushout, in the category of monoids, of a coproduct of maps in J_T is a countable composite of maps of the kind considered in axiom (3). Since a transfinite composite of transfinite composites is again a transfinite composite, this shows that regular J_T-cofibrations are weak equivalences. Thus condition (1) of lemma A.3 holds and this lemma shows that monoids in $(\mathcal{C}, \wedge)$ are in fact a cofibrantly generated closed model category with the notions of fibrations and weak equivalences defined on the underlying category.

It remains to prove the statement about cofibrant monoids. We note that if in the above pushout diagram $K \rightarrowtail L$ is a cofibration and the monoid X is cofibrant in the underlying category, then $\bar{Q}_n \wedge X^{\wedge(n+1)} \longrightarrow L^{\wedge n} \wedge X^{\wedge(n+1)}$ is a cofibration in the underlying category (by several applications of the smashout axiom (2)). Thus also the maps $P_{n-1} \longrightarrow P_n$ and finally $X = P_0 \longrightarrow P$ are cofibrations in the underlying category.

Since the forgetful functor commutes with filtered colimits, transfinite composites of such pushouts in the monoid category are still cofibrations in the underlying category $\mathcal{C}$.

Saying that a monoid X is cofibrant means, by definition, that the map $\mathbb{I} \longrightarrow X$ from the initial monoid is a cofibration of monoids. By the small object argument, it is thus a retract of a regular I_T-cofibration with source $\mathbb{I}$. Since $\mathbb{I}$ is cofibrant in the underlying category, such a map is a cofibration in the underlying category $\mathcal{C}$ by the last paragraph. Thus X is a retract of a monoid which is cofibrant in $\mathcal{C}$, so X itself is cofibrant in $\mathcal{C}$.

C All connective brave new rings arise from theories

In this section we close the circle of the relationship between universal algebra and brave new algebra. We indicate how an arbitrary connective brave new ring R gives rise to a simplicial theory whose stable DFSP is weakly equivalent to R. We emphasize that we are only concerned with associative brave new rings and that our approach does not adress questions involving commutativity of multiplication. This section should be regarded as an epilogue as there are no rigorous proofs, few details and the ideas of this section are not used in the rest of the paper.

We use the machinery of [EKMM], and the construction relies on certain formal properties of the model categories of R-modules (R an S-algebra in the sense of [EKMM, Def. II 3.1]) established in [EKMM, Cor. VII 4.8]. S-algebras are the monoids in another symmetric monoidal model category that represents the stable homotopy category. The objects of this symmetric monoidal category are called S-modules and the category is denoted $\mathcal{M}_S$ [EKMM, II 1.1]. S-algebras thus represent another approach to brave new rings and we sketch how an S-algebra R gives rise to a simplicial theory T_R.

The (left, say) modules over an S-algebra R form a topological model category $\mathcal{M}_R$ [EKMM, Thm. VII 4.8]. This model category has the special property that *every object is fibrant* (the model category of modules over a DFSP does not have this property). The construction of the simplicial theory T_R from the S-algebra R is then as follows: choose a cofibrant approximation R^c to R in $\mathcal{M}_R$ (R is not cofibrant when considered as a module over itself). For example, the module $\mathbb{F}_R S^0$ of [EKMM, Def. III 1.2] provides a choice which is furthermore natural in R. First, $\bar{T}_R$ is defined as the opposite of the full topological subcategory of the category of R-modules generated by the finite coproducts of the module R^c. Explicitly, if $n \times R^c$ denotes the n-fold coproduct of copies of R^c, we set

$$\mathrm{Hom}_{\bar{T}_R}(m, n) \;=\; \mathcal{M}_R(n \times R^c, m \times R^c)\;,$$

where the right hand side is the topological space of R-module maps. $\bar{T}_R$ is a topological theory and we obtain the simplicial theory T_R by applying the singular complex functor to the morphism spaces.

The object $n \times R^c$ is a fibrant and cofibrant representative of the free R-module of rank n, so the space $\bar{T}_R(m, n)$ has the 'correct' homotopy type. For example, the homotopy groups of $\bar{T}_R(m, n)$ are naturally isomorphic to the $(n \times m)$-matrices over the connective part of $\pi_* R$. Also, the Γ-space underlying the simplicial theory T_R is very special and for connective R, the DFSP T_R^s represents R.

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