1 Introduction

A three dimensional manifold M is called hyperbolic if its universal cover is three dimensional hyperbolic space $\mathbf{H}^3$, so M may be viewed as

$$M \cong \Gamma\backslash\mathbf{H}^3,$$

the orbit space of some lattice $\Gamma \cong \pi_1(M)$ in the group $Iso^+(\mathbf{H}^3)$ of orientation preserving isometries of hyperbolic space. For large classes of manifolds hyperbolicity is proven: In [Otal] the case of M fibering over the circle is settled, a case which conjecturally is fairly general. The proofs do not produce lattice representations of the funtamental group of the manifold in question. To do so for some classes of manifolds is the aim of the present paper. For further study of a hyperbolic manifold detailed information in this direction is indispensible. In addition although in general being non arithmetic hyperbolic lattices carry interesting arithmetic features, due to the fact that their trace fields are algebraic number fields.

The manifolds M treated here are assumed to fiber over the circle with fibre T_0 a once punctured torus. So

$$M = M(f) = (T_0 \times [0,1])/\sim$$

where f is some homeomorphism of T_0 onto itself which is the identity close to the puncture, and "$\sim$" means the identification

$$(p,0) = (f(p),1)$$

for all $p \in T_0$. The fundamental group $\pi_1(T_0)$ is free with rank 2; we normalize a pair $\{\xi,\eta\}$ of generators such that their commutator

$$\lambda = \eta^{-1}\xi^{-1}\eta\xi$$

is represented by a loop around the puncture. The homeomorphism f will induce an automorphism f_* of $\pi_1(T_0)$. It is easy to see that it is only f_* which matters for the construction of the threemanifold $M(f)$. So we understand f to be an automorphism of $\langle\xi,\eta\rangle$. It inherits, from its topological origin, that

$$f(\lambda) = \lambda.$$

The fundamental group $\pi_1(M(f))$ has the structure of a semidirect product:

$$\begin{array}{rcll}\pi_1(M(f)) & \cong & \langle \xi, \eta \rangle \rtimes \langle \mu \rangle & \\ & = & \langle \xi, \eta, \mu | & f(\xi) = \mu \xi \mu^{-1} \\ & & & f(\eta) = \mu \eta \mu \rangle\end{array}$$

and it is this group for which a lattice representation $\Gamma(f) < Iso^+(\mathbf{H}^3)$ has to be constructed and studied (in case $M(f)$ is hyperbolic). We proceed by in general first constructing the so called deformation variety $C(f)$ of $\pi_1(M(f))$ which is, whenever it exists, an algebraic curve, and then singling out that point on $C(f)$ which corresponds to the lattice representation. The last step requires, in general, computations in high degree algebraic number fields.

Our model for hyperbolic geometry in dimension three is Poincaré's upper half space model:

$$\mathbf{H}^3 = \{(z, r) | z \in \mathbf{C}, r > 0\}$$

with

$$Iso^+(\mathbf{H}^3) \cong PSL_2(\mathbf{C}).$$

When it comes down to constructing fundamental domains explicitly we use the action of $PSL_2(\mathbf{C})$ on $\mathbf{H}^3$ described by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z, r) = \left(\frac{(az + b)(\overline{cz + d}) + a\bar{c}r^2}{|cz + d|^2 + |c|^2 r^2}, \frac{r}{|cz + d|^2 + |c|^2 r^2} \right).$$

We shall deal with representations in $SL_2(\mathbf{C})$ instead of $PSL_2(\mathbf{C})$, exclusively, this being justified as the groups to be represented will not contain torsion.

Finally, for $f \in Aut\langle \xi, \eta \rangle$ we denote its Anosov monodromy by $A(f)$. It is defined to be the automorphism $\bar{f}$ on $\langle \xi, \eta \rangle$ abelianized, induced by f:

$$\bar{f} : (\bar{\xi}, \bar{\eta}) \longrightarrow (\bar{\xi}, \bar{\eta}) A(f)$$

where $\bar{\xi}, \bar{\eta}$ are ξ, respectively η, mod $\langle \xi, \eta \rangle'$. So $A(f) \in SL_2(\mathbf{Z})$ is a modular matrix.

2 Character varieties and deformation curves

We collect some standard material which is basic for the following.
A character, more precisely an ordinary, unimodular degree two character on

a group G is a complex valued function s on G which satisfies

$$\begin{array}{ll} (i) & s(abc)+s(cba)-s(a)s(bc) \\ & \quad -s(b)s(ac)-s(c)s(ab)+s(a)s(b)s(c)=0, \\ (ii) & s(a^2)=s^2(a)-2, \\ (iii) & s(e)=2 \end{array}$$

for all $a, b, c \in G$, and e the unit element in G. The set of characters on G is in one-one correspondance to the set of all conjugacy classes of semisimple representations

$$\rho : G \longrightarrow SL_2(\mathbf{C})$$

via

$$s = tr \rho.$$

The character set of a group carries, in a natural way, the structure of an affine algebraic set, see [Helling]. In case $G = \langle \xi, \eta \rangle$ is free on two generators it may be identified to $\mathbf{C}^3$ by associating to a character s its "coordinates"

$$s \longrightarrow \begin{pmatrix} s(\xi) \\ s(\eta) \\ s(\eta\xi) \end{pmatrix} \in \mathbf{C}^3.$$

The group $AutG$ of automorphisms of G canonically acts on the character set of G: For s a character and $f \in AutG$ the function $s \circ f$ again is a character on G. It can be shown, that this action is, in the affine algebraic model of the character space mentioned above, described by polynomial tranformations. We make this more explicit for $G = \langle \xi, \eta \rangle$ free:
Let $f_1, f_2 \in AutG$ be defined by

$$\begin{array}{lllll} f_1 & : & \begin{array}{c} \xi \\ \eta \end{array} & \longrightarrow & \begin{array}{c} \xi \\ \xi\eta \end{array}, \\ f_2 & : & \begin{array}{c} \xi \\ \eta \end{array} & \longrightarrow & \begin{array}{c} \eta\xi \\ \eta \end{array} \end{array}$$

with Anosov monodromies

$$A(f_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A(f_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Observe

$$f_1(\lambda) = f_2(\lambda) = \lambda.$$

We compute, for s a character on $\langle \xi, \eta \rangle$:

$$\begin{aligned} s \circ f_1(\xi) &= s(\xi) \\ s \circ f_1(\eta) &= s(\xi\eta) \\ s \circ f_1(\eta\xi) &= s(\xi^2\eta) = -s(\eta) + s(\xi)s(\eta\xi). \end{aligned}$$

So, f_1 transforms the model $\mathbf{C}^3$ for the character variety in the following way:

$$\mathbf{C}^3 \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ z \\ -y + xz \end{pmatrix} \in \mathbf{C}^3.$$

Similarly, for f_2 we have

$$\mathbf{C}^3 \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} z \\ y \\ -x + yz \end{pmatrix} \in \mathbf{C}^3.$$

Definition. Let $f \in Aut\langle \xi, \eta \rangle$. Then

$$C(f) := \left\{ p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{C}^3 \middle| f(p) = p \right\},$$

the fixed point set of f, regarded as a polynomial transformation on $\mathbf{C}^3$.

In view of the structure of the group $\pi_1(M(f))$ this affine algebraic set is intimately related to the character set of this group itself, and so we will work with it. It will, in general, be a curve in threespace. In order to find a faithful lattice representation of $\pi_1(M(f))$ we have to add the geometric condition that $\lambda = \eta^{-1}\xi^{-1}\eta\xi$ become paraolic:

$$s(\lambda) = 2 - (xyz - x^2 - y^2 - z^2 + 4) = -2,$$

so we have to intersect $C(f)$ with the surface

$$xyz = x^2 + y^2 + z^2$$

in $\mathbf{C}^3$.

(Remark: The trace of the parabolic matrix $L \in SL_2(\mathbf{C})$ representing λ cannot be 2 as the number 2 only trivially arises as trace of a commutator.)

We use

$$h(x,y,z) = xyz - x^2 - y^2 - z^2 + 4$$

as a function on $\mathbf{C}^3$, and normalize, on $\{(x,y,z) \in \mathbf{C}^3 | \, h(x,y,z) = 4, x \neq 0\}$ a pair of matrices $X, Y \in SL_2(\mathbf{C})$ such that

$$trX = x, \quad trY = y, \quad trYX = z$$

in the following way:

$$\begin{aligned} X &= \begin{pmatrix} x & -\frac{1}{x} \\ x & 0 \end{pmatrix}, \\ Y &= \begin{pmatrix} \frac{z}{x} & \frac{2xz-y}{x^2} \\ -y & \frac{xy+z}{x} \end{pmatrix}. \end{aligned}$$

We then have

$$YX = \begin{pmatrix} \frac{xz-y}{x} & \frac{z}{x^2} \\ z & \frac{y}{x} \end{pmatrix}$$

and

$$XY = \begin{pmatrix} \frac{-xz+y}{x} & \frac{2x^2z-2xy-z}{x^2} \\ -z & \frac{2xz-y}{x} \end{pmatrix}.$$

This normalization is made in order to achieve

$$L = Y^{-1}X^{-1}YX = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}.$$

3 Trace calculus

We collect some formulae which are often used when doing computations with characters; they mostly concern Čebyšev polynomials. Let, for $n \in \mathbf{Z}$,

$$p_n(x) = \frac{\sin n\vartheta}{\sin \vartheta}$$

written as a polynomial in terms of $x = 2\cos\vartheta$; similarly, let

$$q_n(x) = 2\cos n\vartheta,$$

again as a polynomial in $x = 2\cos\vartheta$. We use

$$\begin{array}{rcl} X^n & = & p_n(x)X - p_{n-1}(x)\mathbf{E}, \\ tr\,X^n & = & q_n(x) \end{array}$$

for $X \in SL_2,\ x = tr\,X,\ \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It is easy to see that, for $n \geq 1$,

$$\begin{array}{ll} p_n(x) & = \prod_{1\leq k\leq n-1}\left(x - 2\cos\frac{k\pi}{n}\right) \\ p_{n+1}(x) & = 1 + \prod_{1\leq l<\frac{n}{2}}\left(x - 2\cos\frac{2\pi l}{n}\right)\cdot\prod_{0\leq l<\frac{n+1}{2}}\left(x - 2\cos\frac{2l+1}{n+2}\pi\right). \end{array} \tag{1}$$

Both $p_n(x)$ and $q_n(x)$ satisfy the recursion

$$f_{n+2}(x) = xf_{n+1}(x) - f_n(x),$$

$n \in \mathbf{Z}$ arbitrary. We have

$$\begin{array}{ll} p_0(x) = 0, & q_0(x) = 2, \\ p_1(x) = 1, & q_1(x) = x, \end{array}$$

$$q_n(x) = p_{n+1}(x) - p_{n-1}(x),$$

$$\begin{array}{rcl} p_{-n}(x) & = & -p_n(x), \\ q_{-n}(x) & = & q_n(x). \end{array}$$

$p_n(x),\ q_n(x)$ are polynomials of degree $|n| - 1$ and $|n|$, respectively, except for $p_0(x)$.
The factorization of $p_n(x)$ and $p_{n+1}(x) - 1$ allows to detect, for $n \geq 1$,

$$d_n(x) = \prod_{i\leq l<\frac{n}{2}}\left(x - 2\cos\frac{2\pi l}{n}\right)$$

as greatest common divisor of $p_n(x)$ and $p_{n+1}(x) - 1$. So $d_n(x)$ has integer coefficients. For $n > 0f$ we set

$$d_{-n}(x) = -d_n(x).$$

From

$$\begin{aligned} p_{-n}(x) &= -p_n(x), \\ p_{-n+1}(x) - 1 &= -p_{n-1}(x) - 1 \\ &= -xp_n(x) + p_{n+1}(x) - 1 \end{aligned}$$

we see that $d_n(x)$ is the greatest common divisor of $p_n(x)$ and $p_{n+1}(x) - 1$ also in case n is negative.

We shall make use of the polynomials

$$\tilde{p}_n(x) = \frac{p_n(x)}{d_n(x)}$$

for $n \in \mathbf{Z}, n \neq 0$. We set

$$\tilde{p}_0(x) = 2.$$

$\tilde{p}_n(x)$ is, for n arbitrary, an integer coefficient polynomial; for $n \neq 0$ it has leading coefficient 1.
The degree of $d_n(x)$, for $n \neq 0$, is

$$deg\, d_n(x) = \begin{cases} \frac{n-1}{2}, & n \equiv 1(2) \\ \frac{n-2}{2}, & n \equiv 0(2) \end{cases},$$

so

$$deg\, \tilde{p}_n(x) = \begin{cases} \frac{n-1}{2}, & n \equiv 1(2) \\ \frac{n}{2}, & n \equiv 0(2) \end{cases}.$$

By definition we have

$$\tilde{p}_{-n}(x) = \tilde{p}_n(x)$$

for all n.
We produce a short list.

n	$p_n(x)$	$\tilde{p}_n(x)$
0	0	2
1	1	1
2	x	x
3	$x^2 - 1$	$x - 1$
4	$x^3 - 2x$	$x^2 - 2$
5	$x^4 - 3x^2 + 1$	$x^2 - x - 1$
6	$x^5 - 4x^3 + 3x$	$x^3 - 3x$

7	$x^6 - 5x^4 + 6x^2 - 1$	$x^3 - x^2 - 2x + 1$
8	$x^7 - 6x^5 + 10x^3 - 4x$	$x^4 - 4x^2 + 2$
9	$x^8 - 7x^6 + 15x^4 - 10x^2 + 1$	$x^4 - x^3 - 3x^2 + 2x + 1$
10	$x^9 - 8x^7 + 21x^5 - 20x^3 + 5x$	$x^5 - 5x^3 + 5x$
11	$x^{10} - 9x^8 + 28x^6 - 35x^4$ $+15x^2 - 1$	$x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1$
12	$x^{11} - 10x^9 + 36x^7 - 56x^5$ $+20x^3 - 6x$	$x^6 - 6x^4 + 9x^2 - 2$
13	$x^{12} - 11x^{10} + 45x^8 - 84x^6$ $+55x^4 - 21x^2 + 1$	$x^6 - x^5 - 5x^4 + 4x^3 + 6x^2$ $-3x - 1$
14	$x^{13} - 12x^{11} + 55x^9 - 120x^7$ $+111x^5 - 41x^3 + 7x$	$x^7 - 7x^5 + 14x^3 - 7x$
15	$x^{14} - 13x^{12} + 66x^{10} - 165x^8$ $+195x^6 - 96x^4 + 28x^2 - 1$	$x^7 - x^6 - 6x^5 + 5x^4 + 10x^3$ $-6x^2 - 4x + 1$
16	$x^{15} - 14x^{13} + 78x^{11} - 220x^9$ $+315x^7 - 207x^5 + 69x^3 - 8x$	$x^8 - 8x^6 + 20x^4 - 16x^2 + 2$
17	$x^{16} - 15x^{14} + 91x^{12} - 286x^{10}$ $+480x^8 - 402x^6 + 165x^4$ $-36x^2 + 1$	$x^8 - x^7 - 7x^6 + 6x^5 + 15x^4$ $-10x^3 - 10x^2 + 4x + 1$
18	$x^{17} - 16x^{15} + 105x^{13} - 364x^{11}$ $+700x^9 - 717x^7 + 372x^5$ $-105x^3 + 9x$	$x^9 - 9x^7 + 27x^5 - 30x^3 + 9x$
19	$x^{18} - 17x^{16} + 120x^{14} - 455x^{12}$ $+986x^{10} - 1197x^8 + 774x^6$ $-270x^4 + 45x^2 - 1$	$x^9 - x^8 - 8x^7 + 7x^6 + 21x^5$ $-15x^4 - 20x^3 + 10x^2 + 5x - 1$
20	$x^{19} - 18x^{17} + 136x^{15} - 560x^{13}$ $+1350x^{11} - 1897x^9 + 1491x^7$ $-642x^5 + 150x^3 - 10x$	$x^{10} - 10x^8 + 35x^6 - 50x^4$ $+25x^2 - 2$

Proposition. For $n \in \mathbf{Z}$ we have

$$\begin{aligned} \tilde{p}_{2n}(x) &= p_{n+1}(x) - p_{n-1}(x), \\ \tilde{p}_{2n+1}(x) &= p_{n+1}(x) - p_n(x). \end{aligned}$$

Proof. The statement is obvious for $\tilde{p}_0(x)$. We have to prove it for positive index, only. We do so by factorising the polynomials on both sides:

$$\begin{aligned} \tilde{p}_{2n}(x) &= \frac{\prod_{1\le k\le 2n-1}\left(x - 2\cos\frac{k\pi}{2n}\right)}{\prod_{\substack{1\le k\le 2n-1\\ k\equiv 0(2)}}\left(x - 2\cos\frac{k\pi}{2n}\right)} \\ &= \prod_{\substack{1\le k\le 2n-1\\ k\equiv 1(2)}}\left(x - 2\cos\frac{k\pi}{2n}\right). \end{aligned}$$

Similarly

$$\begin{aligned} \tilde{p}_{2n+1}(x) &= \frac{\prod_{1\le k\le 2n}\left(x - 2\cos\frac{k\pi}{2n+1}\right)}{\prod_{\substack{1\le k\le 2n\\ k\equiv 0(2)}}\left(x - 2\cos\frac{k\pi}{2n+1}\right)} \\ &= \prod_{\substack{1\le k\le 2n\\ k\equiv 1(2)}}\left(x - 2\cos\frac{k\pi}{2n+1}\right). \end{aligned}$$

We reinterpret x as $2\cos\vartheta$. Then

$$\begin{aligned} p_{n+1}(x) - p_{n-1}(x) &= \frac{\sin(n+1)\vartheta - \sin(n-1)\vartheta}{\sin\vartheta} \\ p_{n+1}(x) - p_n(x) &= \frac{\sin(n+1)\vartheta - \sin n\vartheta}{\sin\vartheta} \\ &. \end{aligned}$$

The zeros are, in terms of ϑ, located at $\vartheta_l = \frac{2l+1}{2n}\pi$ in the first case, and at $\vartheta_l = \frac{2l+1}{2n+1}\pi$ in the second case ($l \in \mathbf{Z}$). In terms of $x = 2\cos\vartheta$ these are exactly the zeros of $\tilde{p}_{2n}(x)$ and $\tilde{p}_{2n+1}(x)$, respectively. Comparing degree and leading coefficient then proves the proposition.

The recursion for p_n leads to a recursion for $\tilde{p}_n$:

Corollary. For $n \in \mathbf{Z}$ we have

$$\tilde{p}_{n+4}(x) = x\tilde{p}_{n+2}(x) - \tilde{p}_n(x).$$

The well known identity

$$p_n{}^2(x) - p_{n+1}(x)p_{n-1}(x) = 1$$

leads to a quadratic identity for $\tilde{p}_n(x)$:

Proposition.

$$\tilde{p}_n{}^2(x) - \tilde{p}_{n+2}(x)\tilde{p}_{n-2}(x) = \begin{cases} -(x-2), & n \equiv 1(2) \\ -\left(x^2-4\right), & n \equiv 0(2) \end{cases} .$$

Finally, note that from (1) we have

$$p_{n+1}(x) - 1 = d_n(x) \cdot \tilde{p}_{n+2}(x),$$

and, by the recursion formulae for p_n and $\tilde{p}_n$:

$$p_{n-1}(x) + 1 = d_n(x)\tilde{p}_{n-2}(x).$$

We shall, in what folows, mostly write x_n for $p_n(x)$, $\tilde{x}_n$ for $\tilde{p}_n(x)$, and we define

$$r_n(x) = \begin{cases} x-2, & n \equiv 1 \, mod \, 2 \\ x^2-4, & n \equiv 0 \, mod \, 2. \end{cases}$$

We now comment on some techniques which are useful for hyperbolic geometry modelled on upper half space $\mathbf{H} = \{(z,r)|z \in \mathbf{C}, r > 0\}$, especially for drawing fundamental domains. In this context it is advantageous to identify points, lines and planes with the isometric reflexions through these objects: A line reflexion is orientation preserving; it is, as a Möbius transformation, described by a trace 0 matrix L in $GL_2(\mathbf{C})$ whose set of fixed points is the corresponding line. Point and plane reflexions are orientation reversing. They can be described by

$$\mathbf{H} \ni (z,r) \quad \longrightarrow \quad M(\bar{z},r) \in \mathbf{H}^3 \tag{2}$$

where $M \in GL_2(\mathbf{C})$ is a matrix of the form

$$M = \begin{pmatrix} w & iu \\ iv & \bar{w} \end{pmatrix}$$

with $u, v \in \mathbf{R}$. Obviously the determinant of M is real. It is easy to see that, if $det\, M < 0$ then the transformation (2) has exactly one fixed point, namely

$\left(\frac{w}{iv}, \frac{1}{|v|}\sqrt{-\,det\,M}\right)$, and (2) describes the reflexion in this point. If $detM > 0$ the set of fixed points is the set of points $(z,r) \in \mathbf{H}^3$ for which

$$|-ivz + w|^2 + v^2r^2 = det\,M$$

under the condition that $v \neq 0$. If $v = 0$ this condition is replaced by

$$\bar{w}z - w\bar{z} = iu.$$

In any case the fixed point set is a hyperbolic plane, and (2) describes the reflexion in it. In case $v \neq 0$ one can read off Euclidean center and radius of this plane, regarded as a Euclidean hemisphere; the center is $\frac{w}{iv}$ and the radius $\frac{1}{|v|}\sqrt{det\,M}$.

For brevity we call $L \in GL_2(\mathbf{C})$ with $tr\,L = 0$ a line, a matrix

$$M = \begin{pmatrix} w & iu \\ iv & \bar{w} \end{pmatrix} \in GL_2(\mathbf{C})$$

with $u, v \in \mathbf{R}$ a point if $det\,M < 0$, and a plane if $det\,M > 0$. Two such matrices represent the same geometric object if and only if they are scalar multiples of each other.

The action of $PGL_2(\mathbf{C})$ on $\mathbf{H}^3$ translates into the following: To move a point or a plane by a matrix $S \in GL_2(\mathbf{C})$ means to replace the corresponding point and plane matrix P and K by $SP\bar{S}^{-1}$ and $SK\bar{S}^{-1}$, respectively, provided that S has been normalized to positive determinant. Line matrices, of course, transform traditionally from L to SLS^{-1}. There are many useful formulae involving manipulations on point, plane and line matrices. We only mention a few of them:

If K_1, K_2 are two plane matrices normalized to determinant 1 then $tr\,K_1\bar{K}_2 \in \mathbf{R}$, so $L = K_1, \bar{K}_2 - K_2\bar{K}_1$ is a line matrix provided it is nonsingular. The letter is the case if and only if the two planes are not parrallel; here "parallel" means to be identical or to be tangent to each other at the boundary, and singularity of L is equivalent to $tr\,K_1\bar{K}_2 = \pm 2$. If $-2 < tr\,K_1\bar{K}_2 < 2$ then K_1 and K_2 intersect, and L is their line of intersection, and $tr\,K_1\bar{K}_2 = 2\cos(K_1, K_2)$ where (K_1, K_2) denotes the angle of intersection of the two plains. If K_1, K_2, K_3 are planes with exactly one common point in $\mathbf{H}^3$, then

$$i(K_1\bar{K}_2K_3 - K_3\bar{K}_2K_1)$$

is a point matrix representing this point of intersection.

Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{C})$ be such that $c \neq 0$. Then the isometric sphere of S may be described by the equation

$$|cz + d|^2 + |c|^2 r^2 = |det\, S|.$$

A plain matrix description is given by the matrix

$$K = i \begin{pmatrix} -\bar{c}d & |det\, S| - d\bar{d} \\ c\bar{c} & c\bar{d} \end{pmatrix}$$

whose determinant is $|c|^2 |det\, S|$. We compute, for

$$S_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},\ S_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

in $GL_2(\mathbf{C})$ with $c_1 c_2 \neq 0$ with isometric sphere matrices K_1 and K_2:

$$\begin{aligned} tr\, K_1 \bar{K}_2 &= |c_1|^2 |det\, S_2| + |c_2|^2 |det\, S_1| - |c_1 d_2 - c_2 d_1|^2 \\ &= 2|c_1||c_2| \sqrt{|det\, S_1|\, |det\, S_2|} \cos(K_1, K_2) \end{aligned}$$

provided the isometric spheres intersect. Finally we present an explicit formula for the intersection point P of a triple of isometric spheres K_1, K_2, K_3 belonging to $S_1, S_2, S_3 \in SL_2(\mathbf{C})$ with second rows $(c_1, d_1), (c_2, d_2)$ and (c_3, d_3), respectively. We have

$$P = \begin{pmatrix} w & iu \\ iv & \bar{w} \end{pmatrix}$$

with

$$\begin{aligned} w &= det \begin{pmatrix} c_1\bar{c}_1 & \bar{c}_1 d_1 & d_1\bar{d}_1 - 1 \\ c_2\bar{c}_2 & \bar{c}_2 d_2 & d_2\bar{d}_2 - 1 \\ c_3\bar{c}_3 & \bar{c}_3 d_3 & d_3\bar{d}_3 - 1 \end{pmatrix}, \\ iu &= det \begin{pmatrix} c_1\bar{d}_1 & \bar{c}_1 d_1 & d_1\bar{d}_1 - 1 \\ c_2\bar{d}_2 & \bar{c}_2 d_2 & d_2\bar{d}_2 - 1 \\ c_3\bar{d}_3 & \bar{c}_3 d_3 & d_3\bar{d}_3 - 1 \end{pmatrix}, \\ iv &= det \begin{pmatrix} c_1\bar{c}_1 & c_1\bar{d}_1 & \bar{c}_1 d_1 \\ c_2\bar{c}_2 & c_2\bar{d}_2 & \bar{c}_2 d_2 \\ c_3\bar{c}_3 & c_3\bar{d}_3 & \bar{c}_3 d_3 \end{pmatrix}. \end{aligned}$$

We study the manifold $M(f)$ where

$$f = {f_2}^n \circ {f_1}^m$$

and m, n both positive. We have

$$f : \begin{array}{c} \xi \\ \eta \end{array} \longrightarrow \begin{array}{c} \eta^n \xi \\ (\eta^n \xi)^m \eta \end{array} ,$$

so the Anosov monodromy is

$$A(f) = \begin{pmatrix} 1 & m \\ n & 1 + mn \end{pmatrix} .$$

The equations for the curve $C(f)$ coming from $s \circ f = s$, then read

$$\begin{aligned} x &= y_n z - y_{n-1} x \\ y &= x_m (y_{n+1} z - y_n x) - x_{m-1} y \\ z &= x_{m+1} (y_{n+1} z - y_n x) - x_m y. \end{aligned}$$

So

$$z = \frac{y_{n-1} + 1}{y_n} x = \frac{\tilde{y}_{n-2}}{\tilde{y}_n} x$$

from the first equation, and

$$(x_{m-1} + 1) y y_n = (y_{n+1} - 1) x x_m \tag{3}$$

from the second. The third relation does not contain any further information. This polynomial equation in x and y has a series of zeros which are not interesting from the geometric point of view; these are the lines parallel to the coordinate axis defined by

$$d_m(x) = ggT(x_{m-1} + 1, x_m) = 0$$

or

$$d_n(y) = ggT(y_{n+1} - 1, y_n) = 0.$$

We cancel these zeros, the result being

$$\tilde{x}_{m-2} y \tilde{y}_n = \tilde{y}_{n+2} x \tilde{x}_m$$

or, finally

$$\tilde{x}_{m+2}\tilde{y}_{n+2} = \tilde{x}_{m-2}\tilde{y}_{n-2} \tag{4}$$

This is the component of $C(f)$ to be studied.
We, by the way, may change (3) by adding $xx_m yy_n$ on both sides. The result is an equivalent description of $C(f)$, namely

$$(x_{m+1} - 1)yy_n = (y_{n-1} + 1)xx_m \tag{5}$$

which reduces, after having cancelled uninteresting factors to

$$\tilde{x}_{m+2}y\tilde{y}_n = \tilde{y}_{n-2}x\tilde{x}_m.$$

So we have

$$z = \frac{\tilde{y}_{n-2}}{\tilde{y}_n}x = \frac{\tilde{x}_{m+2}}{\tilde{x}_m}y.$$

We now intersect the relevant part (4) of $C(f)$ with the surface

$$xyz = x^2 + y^2 + z^2.$$

The result is

$$(x\tilde{x}_m)^2 = r_m(x)y^2,$$

equivalently

$$(y\tilde{y}_n)^2 = r_n(y)x^2.$$

These equations allow to eliminate either x or y from (4). We end up with an algebraic equation for x (or y). We collect this material in a

Proposition. Let $m, n \geq 1$ be integers, $f = {f_2}^n \circ {f_1}^m$ and $\Gamma(f)$ the faithful lattice representation of

$$\pi_1(M(f)) \cong \left\langle \xi, \eta, \mu \,|\, f(\xi) = \mu\xi\mu^{-1}, f(\eta) = \mu\eta\mu^{-1} \right\rangle$$

in $SL_2(\mathbf{C})$. Let ξ, η be represented in $\Gamma(f)$ by the matrices X and Y respectively, and let

$$x_\infty = tr\, X, \quad y_\infty = tr\, Y, \quad z_\infty = tr\, YZ.$$

Then the deformation curve $C(f)$ of $\Gamma(f)$ is the space curve in $\mathbf{C}^3$ with (at most) finitely many points removed, defined by the polynomial equation

$$\tilde{p}_{m+2}(x)\tilde{p}_{n+2}(y) = \tilde{p}_{m-2}(x)\tilde{p}_{n-2}(y),$$

$$z = \frac{\tilde{p}_{n-2}(y)}{\tilde{p}_n(y)} x = \frac{\tilde{p}_{m+2}(x)}{\tilde{p}_m(x)} y.$$

The point $(x_\infty, y_\infty, z_\infty)$ then is one of the finitely many points of intersection of this curve with the surface

$$xyz = x^2 + y^2 + z^2.$$

This geometrical description is equivalent to either one of the two algebraic requirements

$$(x\tilde{p}_m(x))^2 = r_m(x) y^2$$

or

$$(y\tilde{p}_n(y))^2 = r_n(y) x^2.$$

(Of course, $C(f)$ really is a plane algebraic curve.)

In special cases it may be appropriate to rewrite the algebraic conditions on the coordinates (x, y, z) on the deformation curve $C(f)$ of $\Gamma(f)$. We have done so, e. g., in the case of $f = f_2 \circ {f_1}^m$.
We get, for $C(f)$:

$$y = x \frac{\tilde{p}_m(x)}{\tilde{p}_{m+2}(x)}, \quad z = x,$$

and to intersect this curve in $\mathbf{C}^3$ with $xyz = x^2 + y^2 + z^2$ leads to the algebraic equation

$$\tilde{p}^2_{m+2}(x) = r_m(x)$$

to be satisfied by x_∞.
This equation may be given the following form: Set

$$t_\infty = \tilde{p}_{m+2}(x_\infty).$$

Then simply

$$x_\infty = t^2_\infty + 2$$

if $m \equiv 1 \bmod 2$, and

$$x^2_\infty = t^2_\infty + 4$$

if $m \equiv 0 \bmod 2$. So in case m is odd we can immediately transform the equation for x_∞ into an equation for t_∞: t_∞ is a zero of

$$\tilde{p}_{m+2}(t^2 + 2) = t,$$

and something similar turns out to be possible in case of an even m. The resulting polynomials in terms of t are

(i) $m \equiv 1 \bmod 2$

$$t = \tilde{p}_{m+2}(x), \quad x = t^2 + 2$$

$$\sum_{0 \le j \le \frac{m+1}{2}} \begin{pmatrix} m+1-j \\ j \end{pmatrix} t^{m+1-2j} = t$$

(ii) $m \equiv 2 \bmod 4$

$$t = \tilde{p}_{m+2}(x), \quad x^2 = t^2 + 4$$

$$\sum_{0 \le j \le \frac{m+2}{4}} \left(2 \begin{pmatrix} \frac{m}{2} - j \\ j-1 \end{pmatrix} + \begin{pmatrix} \frac{m}{2} - j \\ j \end{pmatrix} \right) t^{\frac{m}{2}+1-2j} = t$$

(iii) $m \equiv 0 \bmod 4$

$$t = \tilde{p}_{m+2}(x), \quad x^2 = t^2 + 4$$

$$\sum_{0 \le j \le \frac{m}{2}} \left(2 \begin{pmatrix} m+1-j \\ j-1 \end{pmatrix} + \begin{pmatrix} m+1-j \\ j \end{pmatrix} \right) t^{m+2-2j} = t^2 + 4.$$

4 A two parameter series of lattices

We have succeeded in discovering a general principle which allows to construct a kind of canonical fundamental domain for $\Gamma(f)$. This dicovery was made by a careful analysis of the first two cases: $(m,n) = (1,1)$ and $(m,n) = (2,1)$. We move the cusp of $\Gamma(f)$ to ∞. Let $X, Y, L, M \in SL_2(\mathbf{C})$ represent ξ, η, λ, and $\mu \in \pi_1(M(f))$, respectively. L and M will be upper triangular commuting matrices with $tr\, L = -2$ and $tr\, M = \pm 2$. As μ enters into the presentation of $\pi_1(M(f))$ as a conjugator we may choose, without loss of generality, $tr\, M = 2$. The normalizations made until now allow a transformation by an arbitrary element from the stabilizer of ∞, so by an arbitrary (nonsingular) upper triangular matrix. So we may assume

$$L = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}.$$

Then

$$M = \begin{pmatrix} 1 & 2\tau \\ 0 & 1 \end{pmatrix}$$

will be fixed, τ being some complex (algebraic) number. This situation will not be destroyed by unipotent elements from the stabilizer of ∞, i.e. by Euclidean translations. So we may assume $X^{-1}(\infty) = 0$. Now $X, Y \in SL_2(\mathbf{C})$ are uniquely determined by

$$tr\, X = x_\infty,\ tr\, Y = y_\infty,\ trYX = z_\infty.$$

They solve the matrix equation

$$YX = LXY.$$

It is a matter of elementary computations that

$$X = \begin{pmatrix} x & -\frac{1}{x} \\ x & 0 \end{pmatrix},\ Y = \begin{pmatrix} -\frac{z}{x} & \frac{2xz-y}{x^2} \\ -y & \frac{xy+z}{x} \end{pmatrix}.$$

(We write x, y, z instead of $x_\infty, y_\infty, z_\infty$.)
The observation concerning the fundamental domain of $\Gamma(f)$ is this: Take the following finite collection of elements of the group

$$\pi_1(M(f)) = \left\langle \xi, \eta, \mu \,\middle|\, \mu\xi\mu^{-1} = \eta\xi, \mu\eta\mu^{-1} = (\eta^n\xi)^m\eta \right\rangle$$

which is faithfully represented by $\Gamma(f)$ as a lattice in $SL_2(\mathbf{C})$:

$$\left\{ \xi^{-j}\eta \,\middle|\, 0 \le j \le m-1 \right\} \cup \left\{ \eta^j\xi \,\middle|\, 0 \le j \le n-1 \right\}.$$

Compute the matrices $X^{-j}Y$, $0 \le j \le m-1$, and Y^jX, $0 \le j \le n-1$, representing this collection in $\Gamma(f)$. None of these $m \cdot n$ elements of $\Gamma(f)$ will fix the point at infinity. Compute, for the relevant indices j, the isometric spheres $K(X^{-j}Y)$ and $K(Y^jX)$ of these matrices,plus the isometric spheres $K\left((X^{-j}Y)^{-1}\right)$, $K\left((Y^jX)^{-1}\right)$ of there inverses, and form their translates by the stabilizer $\Gamma(f)_\infty = \{L^pM^q | p, q \in \mathbf{Z}\}$ of the point at infinity. Then the common exteriour of this doubly periodic system of hyperbolic planes is contained in some horoball at ∞, in other words, there is a horizontal (Euclidean) plane in $\mathbf{H}^3$ which bounds this region by below. The stabilizer of ∞ in $\Gamma(f)$ acts on this region as a group of horizontal Euclidean translations, and any fundamental domain of this action there will be a fundamental domain for $\Gamma(f)$.The proof of this requires nothing but an application of Poincaré's criterion. For this some more detailed information concerning the identifications of the various facettes on the boundary is necessary. We refer to the typical cases $(m,n)=(2,2),(3,2)$ and (4,2) worked out in figures 1,2, and 3. When looking down from the cusp at infinity one sees pieces of hyperbolic planes which play the rôle of isometric spheres constructed previously. As soon as two such spheres intersect one sees a Euclidean segment joining two triple points. These triple points are points where three isometric spheres intersect inside hyperbolic space. So down from ∞ one sees a collection of Euclidean polygons which doubly periodically covers the complex plane. They are all projections of planar hyperbolic polygons. The parallelogram shown in all pictures is a (noncanonical) fundamental domain for this periodicity; moving horizontally means a shift by (a power of) L, moving up means a shift by M. We have, in some cases, labelled polygons by the matrix in $\Gamma(f)$ whose isometric sphere is (partly) visible from above, the polygon being the projection into $\mathbf{C}$ of this face.

The combinatorial structure of this pattern is the same for all automorphisms f; it is shown in figure 4: We have on the boundary of a fundamental domain:

$$\begin{array}{llll}
2 & 2(m+2)-\text{gons} & \text{on} & K(X), K\left(X^{-1}\right), \\
2 & 2(n+2)-\text{gons} & \text{on} & K(Y), K\left(Y^{-1}\right), \\
2(n-1) & \text{quadrangles} & \text{on} & K\left(Y^{j}X\right), K\left(\left(Y^{j}X\right)^{-1}\right) \\
 & & & 1 \leq j \leq n-1, \\
2(m-1) & \text{quadrangles} & \text{on} & K\left(X^{-j}Y\right), K\left(\left(X^{-j}Y\right)^{-1}\right) \\
 & & & 1 \leq j \leq m-1.
\end{array}$$

The following pairs of isometric spheres intersect:

$$\begin{array}{lll}
K(X) & \text{and } K\left(\left(X^{-j}Y\right)^{-1}\right) & , 0 \leq j \leq m, \\
K(X^{-1}) & \text{and } K\left(\left(X^{-j}Y\right)^{-1}\right) & , 0 \leq j \leq m, \\
K(X^{-1}) & \text{and } K\left(X^{-j}Y\right) & , 0 \leq j \leq m, \\
K(LXL^{-1}) & \text{and } K\left(X^{-j}Y\right) & , 0 \leq j \leq m, \\
K(Y^{-1}) & \text{and } K\left(Y^{j}X\right) & , 0 \leq j \leq n, \\
K(Y^{-1}) & \text{and } K\left(\left(Y^{j}X\right)^{-1}\right) & , 0 \leq j \leq n, \\
K(Y) & \text{and } K\left(\left(Y^{j}X\right)^{-1}\right) & , 0 \leq j \leq n,
\end{array}$$

$$
\begin{array}{lll}
K(Y) & \text{and } K\left(LY^jXL^{-1}\right) & ,0 \le j \le n, \\
K(Y^jX) & \text{and } K\left(Y^{j+1}X\right) & ,0 \le j \le n-1, \\
K\left(\left(X^{-j-1}Y\right)^{-1}\right) & \text{and } K\left(\left(X^{-j}Y\right)^{-1}\right) & ,0 \le j \le m-1, \\
K\left(\left(Y^jX\right)^{-1}\right) & \text{and } K\left(\left(Y^{j+1}X\right)^{-1}\right) & ,0 \le j \le n-1, \\
K\left(X^{-j-1}Y\right) & \text{and } K\left(X^{-j}Y\right) & ,0 \le j \le m-1,
\end{array}
$$

So there are triple intersections among

$$
\begin{array}{ll}
K(X), K\left(\left(X^{-j}Y\right)^{-1}\right), K\left(\left(X^{-j-1}Y\right)^{-1}\right), & 0 \le j \le m-1, \\
K\left(Y^{-1}\right), K\left(Y^jX\right), K\left(Y^{j+1}X\right), & 0 \le j \le n-1 \\
K\left(Y^{-1}\right), K\left(\left(Y^jX\right)^{-1}\right), K\left(\left(Y^{j+1}X\right)^{-1}\right), & 0 \le j \le n-1 \\
K\left(X^{-1}\right), K\left(\left(X^{-j}Y\right)^{-1}\right), K\left(\left(X^{-j-1}Y\right)^{-1}\right), & 0 \le j \le m-1, \\
K\left(X^{-1}\right), K\left(X^{-j}Y\right), K\left(X^{-j-1}Y\right), & 0 \le j \le m-1, \\
K\left(LXL^{-1}\right), K\left(X^{-j}Y\right), K\left(X^{-j-1}Y\right), & 0 \le j \le m-1, \\
K(Y), K\left(\left(Y^jX\right)^{-1}\right), K\left(\left(Y^{j+1}X\right)^{-1}\right), & 0 \le j \le n-1, \\
K(Y), K\left(LY^jXL^{-1}\right), K\left(LY^{j+1}XL^{-1}\right), & 0 \le j \le n-1.
\end{array}
$$

The picture, as mentioned already, then repeats periodically along the Euclidean lattice $\{\, L^pM^q |\, p,q \in \mathbf{Z}\}$. We use a labelling of the triple intersection points in a fundamental domain which makes the geometry on the boundary combinatorially transparent; see the combinatorial diagram in figure 4 and the examples in figures 1,2,3. The labelling consists of two integer components (j,k) where $k \in \{1,2,3,4,5\}$ is the horizontal, $j \in \{0,1,2,...,m+n\}$ the vertical ("level") coordinate in the diagram. The first column in the following table contains the label of some triple point, the second the labels of the three isometric spheres which intersect in this point.
There is a slight difference between m even and m odd.
Remember

$$
\begin{aligned}
MXM^{-1} &= Y^nX \\
MYM^{-1} &= \left(Y^nX\right)^m Y.
\end{aligned}
$$

1. $m \equiv 0 \, mod \, 2$

$j1$	$X, \quad L^{-1}X^{j-\frac{m}{2}+1}YL, \quad L^{-1}X^{j-\frac{m}{2}}YL$ for $0 \leq j \leq \frac{m}{2}-1$ $Y^{j-\frac{m}{2}}X, \quad Y^{j-\frac{m}{2}+1}X, \quad L^{-1}YL$ for $\frac{m}{2} \leq j \leq \frac{m}{2}+n-1$ $MXM^{-1}, \quad (L^{-1}M)\,X^{j-\frac{3m}{2}-n+1}Y\,(L^{-1}M)^{-1},$ $(L^{-1}M)\,X^{j-\frac{3m}{2}-n}Y\,(L^{-1}M)^{-1}$ for $\frac{m}{2}+n \leq j \leq m+n$
$j2$	$\left(X^{j-\frac{m}{2}}Y\right)^{-1}, \quad X, \quad \left(X^{j-\frac{m}{2}-1}Y\right)^{-1}$ for $0 \leq j \leq \frac{m}{2}$ $Y^{-1}, \quad Y^{j-\frac{m}{2}}X, \quad Y^{j-\frac{m}{2}-1}X$ for $\frac{m}{2}+1 \leq j \leq \frac{m}{2}+n$ $M\left(X^{j-\frac{3m}{2}-n-1}Y\right)^{-1}M^{-1}, \quad M\left(X^{j-\frac{3m}{2}-n}Y\right)^{-1}M^{-1},$ MXM^{-1} for $\frac{m}{2}+n+1 \leq j \leq m+n$
$j3$	$X^{-1}, \quad \left(X^{j-\frac{m}{2}+1}Y\right)^{-1}, \quad \left(X^{j-\frac{m}{2}}Y\right)^{-1}$ for $0 \leq j \leq \frac{m}{2}-1$ $\left(Y^{j-\frac{m}{2}}X\right)^{-1}, \quad \left(Y^{j-\frac{m}{2}+1}X\right)^{-1}, \quad Y^{-1}$ for $\frac{m}{2} \leq j \leq \frac{m}{2}+n-1$ $MX^{-1}M^{-1}, \quad M\left(X^{j-\frac{3m}{2}-n+1}Y\right)^{-1}M^{-1},$ $M\left(X^{j-\frac{3m}{2}-n}Y\right)^{-1}M^{-1}$ for $\frac{m}{2}+n \leq j \leq m+n$

$j4$	$X^{j-\frac{m}{2}-1}Y, \quad X^{j-\frac{m}{2}}Y, \quad X^{-1}$ for $0 \leq j \leq \frac{m}{2}$ $Y, \quad \left(Y^{j-\frac{m}{2}}X\right)^{-1}, \quad \left(Y^{j-\frac{m}{2}-1}X\right)^{-1}$ for $\frac{m}{2}+1 \leq j \leq \frac{m}{2}+n$ $MX^{j-\frac{3m}{2}-n-1}YM^{-1}, \quad MX^{j-\frac{3m}{2}-n}YM^{-1},$ $MX^{-1}M^{-1}$ for $\frac{m}{2}+n+1 \leq j \leq m+n$
$j5$	$LXL^{-1}, \quad X^{j-\frac{m}{2}+1}Y, \quad X^{j-\frac{m}{2}}Y$ for $0 \leq j \leq \frac{m}{2}-1$ $LY^{j-\frac{m}{2}}XL^{-1}, \quad LY^{j-\frac{m}{2}+1}L^{-1}, \quad Y$ for $\frac{m}{2} \leq j \leq \frac{m}{2}+n-1$ $(LM)X(LM)^{-1}, \quad MX^{j-\frac{3m}{2}-n+1}YM^{-1},$ $MX^{j-\frac{3m}{2}-n}YM^{-1}$ for $\frac{m}{2}+n \leq j \leq m+n$

2. $m \equiv 1 \, mod \, 2$

$j1$	$X, \quad L^{-1}X^{j-\frac{m+1}{2}+1}YL, \quad L^{-1}X^{j-\frac{m+1}{2}}YL$ for $0 \leq j \leq \frac{m+1}{2}-1$ $Y^{j-\frac{m+1}{2}}X, \quad Y^{j-\frac{m+1}{2}+1}X, \quad L^{-1}YL$ for $\frac{m+1}{2} \leq j \leq \frac{m+1}{2}+n-1$ $MXM^{-1}, \quad (L^{-1}M)\, X^{j-\frac{3m+1}{2}-n+1}Y\, (L^{-1}M)^{-1},$ $(L^{-1}M)\, X^{j-\frac{3m+1}{2}-n}Y\, (L^{-1}M)^{-1}$ for $\frac{m+1}{2}+n \leq j \leq m+n$

$j2$	$\left(X^{j-\frac{m+1}{2}-1}Y\right)^{-1}$, $\left(X^{j-\frac{m+1}{2}}Y\right)^{-1}$ X, for $0 \le j \le \frac{m+1}{2}$ Y^{-1}, $Y^{j-\frac{m+1}{2}}X$, $Y^{j-\frac{m+1}{2}-1}X$ for $\frac{m+1}{2}+1 \le j \le \frac{m+1}{2}+n$ $M\left(X^{j-\frac{3m+1}{2}-n-1}Y\right)^{-1}M^{-1}$, $M\left(X^{j-\frac{3m+1}{2}-n}Y\right)^{-1}M^{-1}$, MXM^{-1} for $\frac{m+1}{2}+n+1 \le j \le m+n$
$j3$	X^{-1}, $\left(X^{j-\frac{m+1}{2}+1}Y\right)^{-1}$, $\left(X^{j-\frac{m+1}{2}}Y\right)^{-1}$ for $0 \le j \le \frac{m+1}{2}-1$ $\left(Y^{j-\frac{m+1}{2}}X\right)^{-1}$, $\left(Y^{j-\frac{m+1}{2}+1}X\right)^{-1}$, Y^{-1} for $\frac{m+1}{2} \le j \le \frac{m+1}{2}+n-1$ $MX^{-1}M^{-1}$, $M\left(X^{j-\frac{3m+1}{2}-n+1}Y\right)^{-1}M^{-1}$, $M\left(X^{j-\frac{3m+1}{2}-n}Y\right)^{-1}M^{-1}$ for $\frac{m+1}{2}+n \le j \le m+n$
$j4$	$X^{j-\frac{m+1}{2}-1}Y$, $X^{j-\frac{m+1}{2}}Y$, X^{-1} for $0 \le j \le \frac{m+1}{2}$ Y, $\left(Y^{j-\frac{m+1}{2}}X\right)^{-1}$, $\left(Y^{j-\frac{m+1}{2}-1}X\right)^{-1}$ for $\frac{m+1}{2}+1 \le j \le \frac{m+1}{2}+n$ $MX^{j-\frac{3m+1}{2}-n-1}YM^{-1}$, $MX^{j-\frac{3m+1}{2}-n}YM^{-1}$, $MX^{-1}M^{-1}$ for $\frac{m+1}{2}+n+1 \le j \le m+n$

$j5$	$LXL^{-1}, \qquad X^{j-\frac{m+1}{2}+1}Y, \qquad X^{j-\frac{m+1}{2}}Y$ for $0 \le j \le \frac{m+1}{2} - 1$ $LY^{j-\frac{m+1}{2}}XL^{-1}, \qquad LY^{j-\frac{m+1}{2}+1}L^{-1}, \qquad Y$ for $\frac{m+1}{2} \le j \le \frac{m+1}{2} + n - 1$ $(LM)X(LM)^{-1}, \qquad MX^{j-\frac{3m+1}{2}-n+1}YM^{-1},$ $MX^{j-\frac{3m+1}{2}-n}YM^{-1}$ for $\frac{m+1}{2} + n \le j \le m + n$