Hammocks for string algebras

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Introduction

0.1 In 1986 S. Brenner introduced the concept of hammocks in order to decide when a finite translation quiver is the Auslander-Reiten quiver of a representation-finite algebra. These hammocks are partially ordered sets (posets) which describe the factorizations of the canonical homomorphism $f_S : P \longrightarrow I$. Here P (respectively I) is the projective cover (respectively injective envelope) of the simple module S, and the image of f_S is S. After that C.M. Ringel, D. Vossieck and Y. Lin gave an axiomatic treatment of these posets in the representation-finite case (see for example [RV] and [YL]).

In this paper we consider the class of string algebras and deal with the corresponding problem. These algebras are usually representation-infinite and are regarded as an important class of tame algebras. We introduce a generalized notion of hammocks for string algebras and prove various combinatorial properties of these posets. In case one is dealing with a representation-finite string algebra, our definition and the definition given in [RV] coincide.

As a main application we compute the index of nilpotency (up to a small error term) of the radical of A-mod where A is a string algebra. We also construct examples which show that every possible index occurs. Only very few examples of this type were known before.

0.2 Throughout, let k be a fixed algebraically closed field. An algebra A is a finite-dimensional associative k-algebra with a unit element. An A-module is a finite-dimensional unitary left module. The category of A-modules is denoted by A-mod. Call A tame if for each $d \geq 0$ there exist finitely many one-parameter families of indecomposable A-modules which contain almost all d-dimensional indecomposable A-modules. Thus we get a map

$$\mu_A : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$$

where $\mu_A(d)$ is the minimal number of one-parameter families which are needed to parametrize almost all d-dimensional indecomposables. We call A domestic if there exists a constant $c = c(A)$ such that $\mu_A(d) \leq c$ for all d. The growth behaviour of this function is regarded as one important subject in representation theory. We show that there is a strong relation between the degree of nilpotency of the radical of A-mod and the growth of μ_A in case A

is a string algebra.

0.3 Let A be an algebra and $\mathcal{I}$ an ideal in A-mod. For $n \geq 1$ define

$$\mathcal{I}^n = \{\sum_{i=1}^{l} f_n^{(i)} \cdots f_1^{(i)} \mid f_j^{(i)} \in \mathcal{I}, l \geq 1\}.$$

Furthermore, let $\mathcal{I}^0 = A$-mod. Let rad_A be the radical of A-mod and α an ordinal number. If α is finite then rad_A^α is defined as above. Next, let α be a limit number. Then

$$\mathrm{rad}_A^\alpha = \bigcap_{\beta<\alpha} \mathrm{rad}_A^\beta.$$

Finally, let α be an arbitrary infinite ordinal. Thus $\alpha = \beta + n$ for some limit number β and some finite ordinal n. In this case, define

$$\mathrm{rad}_A^\alpha = (\mathrm{rad}_A^\beta)^{n+1}.$$

Let

$$\mathrm{rad}_A^\infty = \bigcap_\alpha \mathrm{rad}_A^\alpha$$

where we intersect over all ordinal powers of rad_A. Call rad_A^∞ the transfinite radical of A-mod. Let $\mathrm{nil}(A) = \infty$ if $\mathrm{rad}_A^\alpha \neq 0$ for all ordinals α. Define $\mathrm{nil}(A) = \min\{\alpha \mid \mathrm{rad}_A^\alpha = 0\}$, else. Let f be an A-module homomorphism. Define $\mathrm{nil}(f) = \infty$ if $f \in \mathrm{rad}_A^\alpha$ for all α, and let $\mathrm{nil}(f) = \min\{\alpha \mid f \notin \mathrm{rad}_A^\alpha\}$, else.

Denote the first limit number by ω, the second by $\omega 2$ and so on. The smallest limit number which is not of the form ωn for some $n \in \mathbb{N}$ is denoted by ω^2.

0.4 As main results of this paper we obtain the following theorems:

Theorem. *Let A be a string algebra. Then the following are equivalent:*

1. *A is domestic;*
2. $\mathrm{nil}(A) < \infty$*;*
3. $\mathrm{nil}(A) < \omega^2$.

The next theorem shows that each possible degree of nilpotency occurs:

Theorem. *Let* $(1,1) \neq (n,d) \in \mathbb{N} \times \mathbb{N}$. *Then there exists a string algebra* A *such that* $\mathrm{nil}(A) = \omega n + d$.

Examples of this type were rather rare before. In the literature one can only find examples with $\mathrm{nil}(A) \leq \omega + 4$. In private communications with C. Geiß, J.A. de la Peña and A. Skowroński we learned that examples with $\mathrm{nil}(A) \leq \omega + 8$ had been known.

In section 4.2 we associate to a given domestic string algebra A and a simple A-module S a finite oriented graph $\Gamma_{\mathcal{B}}(S)$ without oriented cycles which contains at least two points $1_{(S,1)}$ and $1_{(S,-1)}$. This graph is in practice easy to compute. Let n_S be the maximal length of a path in $\Gamma_{\mathcal{B}}(S)$ starting at $1_{(S,1)}$ or $1_{(S,-1)}$ and let d_S be the number of arrows b such that there exists a path of length n_S starting at $1_{(S,1)}$ or $1_{(S,-1)}$ with first arrow b.

Theorem. *Let* A *be a domestic string algebra and* S *a simple* A*-module. If* $n_S \geq 1$ *then* $\omega n_S + d_S \leq \mathrm{nil}(f_S) \leq \omega n_S + d_S + l$ *where* l *is a finite ordinal.*

For $n_S = 0$ we get $\mathrm{nil}(f_S) = d < \omega$ and d can be computed easily. Since $\mathrm{nil}(A) = \max\{\mathrm{nil}(f_S) \mid S \text{ simple }\}$, we have an efficient method to compute the index of nilpotency of rad_A for every string algebra A up to a finite error term l. The number l is usually very small and often equal to 0.

0.5 Additionally, we obtain results about the factorization behaviour of important homomorphisms between indecomposable modules over a domestic string algebra. In particular, the factorizations of the canonical map f_S are investigated.

0.6 In the appendix we outline the relation of the above results with the Krull-Gabriel dimension of a string algebra. Combining the second theorem with a recent result of H. Krause one gets the following: For each $n \in \mathbb{N}$ there exists a domestic string algebra A such that the Krull-Gabriel dimension of A is at least n.

0.7 Furthermore, one can use the hammock approach to get results about the tilting theory of string algebras. In [Sch] we prove that the class of gentle algebras is closed under tilting and cotilting. In [PS] and [AS] it is shown that an algebra $A = kQ/\langle\rho\rangle$, where Q has no oriented cycles, is gentle if and only if its repetitive category is special biserial. On the other hand there exist examples of string algebras where a sequence of tilts and cotilts leads to a wild algebra.

0.8 General conventions

By $\mathbb{N}$ we denote the natural numbers without 0. The integers are denoted by $\mathbb{Z}$ and the rational numbers by $\mathbb{Q}$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Q}_0^+ = \{q \in \mathbb{Q} \mid q \geq 0\}$. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are maps then the composition is denoted by $gf : X \longrightarrow Z$. As a general reference for all unexplained terminology we use C.M. Ringel's book [R3] and also [AR1] and [AR2].

Acknowledgments.

I am indebted to Claus Michael Ringel who convinced me that representation theory is as interesting as playing Go. I thank him for explanations and advice and also for his suggestion to contemplate about hammocks which led to this thesis. I thank Peter Dräxler and Henning Krause for helpful remarks and discussions. Furthermore, I am indebted to the *Instituto de Mathemáticas, UNAM, Mexico* for one month of friendly hospitality where parts of this thesis were written.

1 String algebras (Known results)

In this chapter we repeat the most important definitions and results concerning the representation theory of string algebras. As a main reference we use [BR].

1.1 Definition of a string algebra

Let $Q = (Q_0, Q_1)$ be a quiver with vertices Q_0 and arrows Q_1. Thus Q is a finite directed graph. For $\alpha \in Q_1$ let $s(\alpha)$ be its starting point and $e(\alpha)$ its end point. A *path* of length $n \geq 1$ in Q is of the form $\alpha_1\alpha_2 \cdots \alpha_n$ where the α_i's are arrows with $s(\alpha_i) = e(\alpha_{i+1})$ for $1 \leq i \leq n-1$. To each $S \in Q_0$ we associate a path e_S of length 0. A *zero-relation* in Q is given by a path of lengths ≥ 2. Let ρ be a set of zero-relations in Q. The pair (Q, ρ) is said to be a *pair of type* $\tilde{A}$ if the following hold:

1. There are only finitely many paths in Q which do not have an element in ρ as a subpath;

2. Any vertex of Q is starting point of at most two arrows and also end point of at most two arrows;

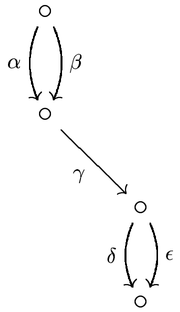
3. Given an arrow β, there is at most one arrow γ with $s(\gamma) = e(\beta)$ and $\gamma\beta \notin \rho$ and at most one arrow α with $s(\beta) = e(\alpha)$ and $\beta\alpha \notin \rho$.

Throughout, let (Q, ρ) be a pair of type $\tilde{A}$, and let $A = kQ/\langle\rho\rangle$ be the corresponding k-algebra. This type of algebra is called *string algebra.*

Remark. Maybe one should call the class of string algebras just algebras of type $\tilde{A}$ since the representation theory of string algebras is based on the representation theory of an euclidian quiver of type $\tilde{A}$. For the same reason, the socalled clannish algebras should be referred to as algebras of type $\tilde{D}$. Nevertheless, we will use more common term string algebra.

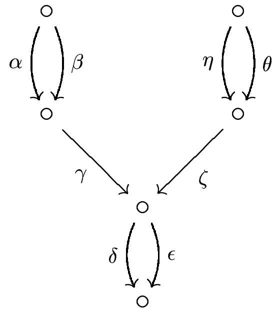
1.2 Examples

(1)



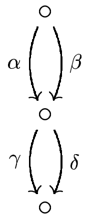
$\rho = \{\gamma\beta, \delta\gamma\}$

(2)



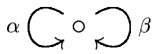
$\rho = \{\gamma\beta, \delta\gamma, \zeta\eta, \epsilon\zeta\}$

(3)



$\rho = \{\gamma\alpha, \delta\beta\}$

(4) $GP_{n,m}, \quad n, m \geq 2$



$\rho = \{\alpha\beta, \beta\alpha, \alpha^n, \beta^m\}$

1.3 Strings

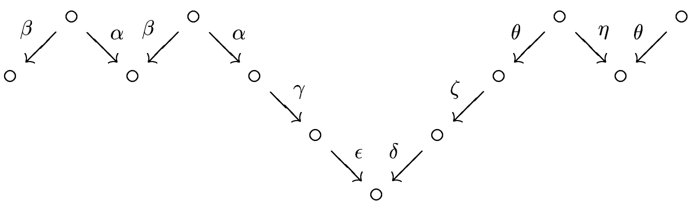
We need the notion of a formal inverse of an arrow. Given an arrow α in Q denote by α^- a formal inverse where $s(\alpha^-) = e(\alpha)$ and $e(\alpha^-) = s(\alpha)$. Also let $(\alpha^-)^- = \alpha$. The set of formal inverses of arrows is denoted by Q_1^-. Sometimes we call the elements in $Q_1 \cup Q_1^-$ *letters.* A *string* of length $n \geq 1$ is of the form $c_1 c_2 \cdots c_n$ with the following properties:

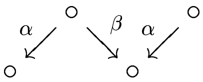
1. Each c_i is contained in $Q_1 \cup Q_1^-$ for $1 \leq i \leq n$;
2. We have $s(c_i) = e(c_{i+1})$ for $1 \leq i \leq n-1$;
3. We have $c_i \neq c_{i+1}^-$ for $1 \leq i \leq n-1$;
4. Neither $c_i c_{i+1} \cdots c_{i+t}$ nor $c_{i+t}^- \cdots c_{i+1}^- c_i^-$ belong to ρ for $1 \leq i < i+t \leq n$.

Thus a string is a special kind of word with letters in $Q_1 \cup Q_1^-$. Define $s(c_1 \cdots c_n) = s(c_n)$ and $e(c_1 \cdots c_n) = e(c_1)$. Additionally, we define for every vertex $S \in Q_0$ two strings $1_{(S,1)}$ and $1_{(S,-1)}$ of length 0 with $s(1_{(S,t)}) = e(1_{(S,t)}) = S$ and $1^-_{(S,t)} = 1_{(S,-t)}$ for $t \in \{-1, 1\}$. The length of a string C is denoted by $|C|$. The set of all strings is denoted by $\mathcal{S}$. A string C is called *direct* if $C = 1_{(S,t)}$ for some $S \in Q_0$ or $C = c_1 \cdots c_n$ with $c_i \in Q_1$ for $1 \leq i \leq n$. It is called *inverse* if $C = 1_{(S,t)}$ for some $S \in Q_0$ or $C = c_1 \cdots c_n$ with $c_i \in Q_1^-$ for $1 \leq i \leq n$.

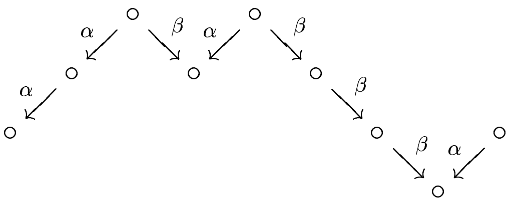
1.4 Examples

Let A be the algebra (2) from section 1.2. Then $\beta\alpha^-\beta\alpha^-\gamma^-\epsilon^-\delta\zeta\theta\eta^-\theta$ and also $\alpha\beta^-\alpha$ are strings. One can visualize these strings in the following way:





Next, let $A = GP_{3,4}$ (see section 1.2). Then for example $\alpha\alpha\beta^-\alpha\beta^-\beta^-\beta^-\alpha$ is a string. The corresponding picture looks as follows:



Note that we draw a direct arrow always from upper right to lower left and an inverse arrow from upper left to lower right.

1.5 The maps σ and ε

Similarly as in [BR] we can define two functions $\sigma, \varepsilon : Q_1 \longrightarrow \{-1, 1\}$ such that the following hold:

1. If $\beta_1 \neq \beta_2$ are arrows with $s(\beta_1) = s(\beta_2)$ then $\sigma(\beta_1) = -\sigma(\beta_2)$;
2. If $\gamma_1 \neq \gamma_2$ are arrows with $e(\gamma_1) = e(\gamma_2)$ then $\varepsilon(\gamma_1) = -\varepsilon(\gamma_2)$;
3. If β and γ are arrows with $s(\beta) = e(\gamma)$ and $\beta\gamma \notin \rho$ then $\sigma(\beta) = -\varepsilon(\gamma)$;
4. Assume there is only one arrow β with $s(\beta) = S$ and only one arrow α with $e(\alpha) = S$ for some $S \in Q_0$. If $\beta\alpha \in \rho$ then $\varepsilon(\alpha) = \sigma(\beta) = 1$.

We extend these functions as follows: If β is an arrow then define $\sigma(\beta^-) = \varepsilon(\beta)$ and $\varepsilon(\beta^-) = \sigma(\beta)$. If $C = c_1 \cdots c_n$ is a string of length ≥ 1 let $\sigma(C) = \sigma(c_n)$ and $\varepsilon(C) = \varepsilon(c_1)$. Finally, for $S \in Q_0$ define $\sigma(1_{(S,t)}) = -t$ and $\varepsilon(1_{(S,t)}) = t$.

We need these functions for technical reasons. From now on we assume that a string algebra is given together with such maps σ and ε.

1.6 Concatenation of strings

Let $C = c_1 \cdots c_n$ and $D = d_1 \cdots d_m$ be two strings of length ≥ 1. If $c_1 \cdots c_n d_1 \cdots d_m$ is a string again then we call it the *concatenation* of C and D and write $C \cdot D$ or CD. For a string $C = c_1 \cdots c_n$ define $C \cdot 1_{(s(C),t)} = C$ if $\sigma(C) = -t$ and $1_{(e(C),t)} \cdot C = C$ if $\varepsilon(C) = t$. Otherwise the concatenation of a string of length 0 and a string of length ≥ 1 is not defined. The concatenation $1_{(S,t)} \cdot 1_{(S',t')}$ is defined if and only if $S = S'$ and $t = t'$. In this case $1_{(S,t)} \cdot 1_{(S,t)} = 1_{(S,t)}$. By C^m we denote the m-fold concatenation $C \cdot C \cdots C$ if it is defined. If the concatenation CD of two arbitrary strings C and D is defined then we have $s(C) = e(D)$ and $\sigma(C) = -\varepsilon(D)$.

1.7 Bands

Next, we define an important subset of $\mathcal{S}$, namely the set of bands. A string $C = c_1 \cdots c_n$ of length ≥ 2 is called *band* if the following hold:

1. All powers C^m are defined;
2. The string C itself is not the power of a string of smaller length;
3. We have $c_1 \in Q_1$ and $c_n \in Q_1^-$.

Note that usually one defines bands without the third assumption (compare [BR]).

Call a string algebra *domestic* if there are only finitely many bands in $\mathcal{S}$.

1.8 Examples

Let A be the algebra (2) from section 1.2. Then $\{\alpha\beta^-, \beta\alpha^-, \eta\theta^-, \theta\eta^-, \delta\epsilon^-, \epsilon\delta^-\}$ is the set of all bands of A. Next, let $A = GP_{2,3}$. Then, for example, $(\alpha\beta^-)^n\beta^-$ is a band of A for all $n \geq 1$.

1.9 String modules

Let $C = c_1 \cdots c_n$ be a string of length ≥ 1 and $S \in Q_0$. Define $C(S) = \{i \mid s(c_i) = S \text{ for } 1 \leq i \leq n\} \cup \{0 \mid e(c_1) = S\}$. Now we construct an $(n+1)$-dimensional A-module $M(C)$ corresponding to the string C. Fix a basis $\{z_0, \cdots, z_n\}$ of $M(C)$. Let e_S be a path of length 0 in Q. Then define

$$e_S \times z_j = \begin{cases} z_j & : \quad j \in C(S); \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Given $\alpha \in Q_1$ let

$$\alpha \times z_j = \begin{cases} z_{j-1} & : \quad \alpha = c_j \text{ and } 1 \leq j \leq n; \\ z_{j+1} & : \quad \alpha^- = c_{j+1} \text{ and } 0 \leq j \leq n-1; \\ 0 & : \quad \text{otherwise.} \end{cases}$$

This defines an A-module structure on $M(C)$. In case $C = 1_{(S,t)}$ for some $S \in Q_0$, let $C(S) = \{0\}$ and $M(C)$ the simple module with basis $\{z_0\}$ corresponding to S. The basis $\{z_0, \cdots, z_n\}$ is called the *canonical basis* of $M(C)$. Modules of the form $M(C)$ are called *string modules*.

1.10 Examples

Again, let A be the algebra (2) from section 1.2 and $C = \beta\alpha^{-}\beta\alpha^{-}\gamma^{-}\epsilon^{-}\delta\zeta\theta\eta^{-}\theta$. Then the string module $M(C)$ looks as follows:

$$\begin{tikzcd}
k^2 \arrow[d, shift right, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}"'] \arrow[d, shift left, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}"] & & k^2 \arrow[d, shift right, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}"'] \arrow[d, shift left, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"] \\
k^3 \arrow[dr, "\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}"'] & & k^2 \arrow[dl, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}"] \\
& k^2 \arrow[d, shift right, "\begin{pmatrix} 0 & 1 \end{pmatrix}"'] \arrow[d, shift left, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & \\
& k &
\end{tikzcd}$$

Remember that we defined a canonical basis $\{z_0, \cdots, z_{11}\}$ of $M(C)$. Another possibility to write down the string module $M(C)$ is the following:

$$\begin{tikzcd}
& z_1 \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & z_3 \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & & & & & z_9 \arrow[dl, "\theta"'] \arrow[dr, "\eta"] & & z_{11} \arrow[dl, "\theta"'] \\
z_0 & & z_2 & & z_4 \arrow[dr, "\gamma"] & & & & z_8 \arrow[dl, "\zeta"'] & & z_{10} & \\
& & & & & z_5 \arrow[dr, "\epsilon"] & & z_7 \arrow[dl, "\delta"'] & & & & \\
& & & & & & z_6 & & & & &
\end{tikzcd}$$

The arrows indicate how the arrows of the algebra operate on the given basis.

1.11 Band modules

Next, we associate to a given band $B = b_1 \cdots b_m$ a family $\{M(B, \lambda, n) \mid \lambda \in k^*, n \geq 1\}$ of A-modules. Let $\{z_1^{(i)}, \cdots, z_m^{(i)} \mid 1 \leq i \leq n\}$ be a basis of $M(B, \lambda, n)$. Let e_S be a path of length 0 in Q. Define

$$e_S \times z_j^{(l)} = \begin{cases} z_j^{(l)} & : \quad j \in B(S); \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Let $\alpha \in Q_1$. Define

$$\alpha \times z_1^{(i)} = \begin{cases} \lambda z_m^{(1)} & : \quad \alpha = b_1 \text{ and } i = 1; \\ \lambda z_m^{(i)} + z_m^{(i-1)} & : \quad \alpha = b_1 \text{ and } 2 \leq i \leq n; \\ z_2^{(i)} & : \quad \alpha^- = b_2; \\ 0 & : \quad \text{otherwise.} \end{cases}$$

For $2 \leq j \leq n-1$ we define

$$\alpha \times z_j^{(i)} = \begin{cases} z_{j-1}^{(i)} & : \quad \alpha = b_j; \\ z_{j+1}^{(i)} & : \quad \alpha^- = b_{j+1}; \\ 0 & : \quad \text{otherwise.} \end{cases}$$

The basis $\{z_1^{(i)}, \cdots, z_m^{(i)} \mid 1 \leq i \leq n\}$ is called the *canonical basis* of $M(B, \lambda, n)$. Modules of this type are called *band modules.*

1.12 Examples

Let $A = GP_{2,3}$. Take the band $B = \alpha\beta^-\alpha\beta^-\beta^-$. We want to describe the band module $M(B, \lambda, 3)$. The vertices of the following graph correspond to the elements of the canonical basis of $M(B, \lambda, 3)$. The arrows describe the operation of the arrows of the algebra on them. If there occur scalar multiples which are unequal to one then they are written on the corresponding arrow:

$$\begin{tikzcd}
z_5^{(3)} & & & z_1^{(3)} \arrow[lll, "\lambda"', "\alpha"] \arrow[rrr, "\beta"'] \arrow[dll, "\alpha"] & & & z_2^{(3)} \\
& z_5^{(2)} & & z_1^{(2)} \arrow[ll, "\lambda"', "\alpha"] \arrow[rr, "\beta"'] \arrow[dl, "\alpha"] & & z_2^{(2)} & \\
& & z_5^{(1)} & z_1^{(1)} \arrow[l, "\lambda"', "\alpha"] \arrow[r, "\beta"'] & z_2^{(1)} & & \\
& & z_4^{(1)} \arrow[u, "\beta"'] & & z_3^{(1)} \arrow[ll, "\beta"] \arrow[u, "\alpha"'] & & \\
& z_4^{(2)} \arrow[uuu, "\beta"] & & & & z_3^{(2)} \arrow[llll, "\beta"] \arrow[uuu, "\alpha"'] & \\
z_4^{(3)} \arrow[uuuuu, "\beta"] & & & & & & z_3^{(3)} \arrow[llllll, "\beta"] \arrow[uuuuu, "\alpha"']
\end{tikzcd}$$

For example, the canonical basis vector $z_1^{(2)}$ is mapped to $z_5^{(1)} + \lambda z_5^{(2)}$ under α.

1.13 Equivalence relations on strings and bands

Let C and D be strings. We write $C \sim D$ if $C = D$ or $C = D^-$. This defines an equivalence relation on $\mathcal{S}$. For bands B_1 and $B_2 = b_1 \cdots b_n$ we write $B_1 \sim_r B_2$ if $B_1 = B_2$ or $B_1 = b_i \cdots b_n b_1 \cdots b_{i-1}$ for some $2 \leq i \leq n$. This defines an equivalence relation on the set of bands. We define another equivalence relation on the set of bands as follows: Let $B_1 \sim B_2$ if $B_1 \sim_r B_2$ or $B_1 \sim_r B_2^-$.

1.14 Classification of the indecomposable modules

Lemma. *Given strings C and D and bands B_1 and B_2. Then the following hold:*

1. *The modules $M(C)$ and $M(D)$ are isomorphic if and only if $C \sim D$;*
2. *The modules $M(B_1, \lambda, n)$ and $M(B_2, \mu, m)$ are isomorphic if and only if $B_1 \sim B_2$, $\lambda = \mu$ and $n = m$;*

3. *No string module is isomorphic to a band module;*

4. *String modules and band modules are indecomposable.*

Theorem. *The string and the band modules provide a complete list of indecomposable A-modules.*

1.15 The Auslander-Reiten quiver

We need some more notation in order to deal with irreducible maps between string modules and with Auslander-Reiten sequences:

We say that a string C *starts on a peak* if there is no arrow α with $C\alpha \in \mathcal{S}$, and that C *starts in a deep* if there is no arrow β with $C\beta^- \in \mathcal{S}$. Dually, C *ends on a peak* if there is no arrow α with $\alpha^- C \in \mathcal{S}$, and C *ends in a deep* if there is no arrow β with $\beta C \in \mathcal{S}$. For every arrow α let $N_\alpha = U_\alpha \alpha V_\alpha$ such that U_α and V_α are inverse and N_α starts in a deep and ends on a peak. Denote the corresponding string modules by $U(\alpha)$, $V(\alpha)$ and $N(\alpha)$, respectively. Let C be a string with $C\alpha \in \mathcal{S}$ for some $\alpha \in Q_1$. Define $C_h = C\alpha V_\alpha$. We say C_h is obtained from C by adding a hook from the right and C is obtained from C_h by deleting a hook from the right. Dually, let C be a string with $\alpha^- C \in \mathcal{S}$ for some $\alpha^- \in Q_1^-$. Define ${}_hC = V_\alpha^- \alpha^- C$. Next, we define what it means to add or delete socalled cohooks from the right or left. Let C be a string with $C\alpha^- \in \mathcal{S}$ for some $\alpha^- \in Q_1^-$. Let $C_c = C\alpha^- U_\alpha^-$. We obtained C_c from C by adding a cohook from the right and C from C_c by deleting a cohook from the right. Finally, let $\alpha C \in \mathcal{S}$ for some $\alpha \in Q_1$. Let ${}_cC = U_\alpha \alpha C$.

Obviously, there are canonical embeddings $\iota : M(C) \longrightarrow M(C_h)$ and $\iota : M(C) \longrightarrow M({}_hC)$ whenever C_h and ${}_hC$ are defined. Dually, we have canonical projections $\pi : M(C_c) \longrightarrow M(C)$ and $\pi : M({}_cC) \longrightarrow M(C)$ provided C_c and ${}_cC$ are defined. For details see [BR] and also [CB2] and [K1].

The following short exact sequences are called *canonical exact sequences*:
Let C be a string such that $M(C)$ is not isomorphic to any $U(\alpha)$.

Assume C neither starts nor ends on a peak.

$$0 \longrightarrow M(C) \xrightarrow{\left[\begin{smallmatrix}\iota\\ \iota\end{smallmatrix}\right]} M(C_h) \oplus M({}_hC) \xrightarrow{[\iota, -\iota]} M({}_hC_h) \longrightarrow 0$$

Assume C does not start on a peak but ends on a peak.

$$0 \longrightarrow M({}_cC) \xrightarrow{\left[\begin{smallmatrix}\pi\\ \iota\end{smallmatrix}\right]} M({}_cC_h) \oplus M(C) \xrightarrow{[\iota,-\pi]} M(C_h) \longrightarrow 0$$

Assume C starts on a peak but does not end on a peak.

$$0 \longrightarrow M(C_c) \xrightarrow{\left[\begin{smallmatrix}\iota\\ \pi\end{smallmatrix}\right]} M(C) \oplus M({}_hC_c) \xrightarrow{[\pi,-\iota]} M({}_hC) \longrightarrow 0$$

Finally, assume C starts and ends on a peak.

$$0 \longrightarrow M({}_cC_c) \xrightarrow{\left[\begin{smallmatrix}\pi\\ \pi\end{smallmatrix}\right]} M({}_cC) \oplus M(C_c) \xrightarrow{[\pi,-\pi]} M(C) \longrightarrow 0$$

Additionally, for every $\alpha \in Q_1$ we have

$$0 \longrightarrow U(\alpha) \xrightarrow{\iota} N(\alpha) \xrightarrow{\pi} V(\alpha) \longrightarrow 0$$

where ι is the obvious canonical inclusion and π the obvious canonical projection.

Theorem. *The canonical exact sequences are the Auslander-Reiten sequences containing string modules.*

Theorem. *For a given band B and $\lambda \in k^*$ the band modules $\{M(B,\lambda,n) \mid n \geq 1\}$ form a homogeneous tube.*

We call an Auslander-Reiten component of A *non-regular* if it contains projective or injective modules or a canonical exact sequence with just one indecomposable middle term. The other components are called *regular*. Note that a middle term of an Auslander-Reiten sequence of A has at most two indecomposable direct summands. There are exactly $|Q_1|$ canonical exact sequences with indecomposable middle term. As a consequence we have only finitely many non-regular components.

Theorem. *The regular components of the Auslander-Reiten quiver of A are of the form $\mathbb{Z}A_\infty^\infty$ or $\mathbb{Z}A_\infty/\langle\tau\rangle$.*

Some remarks on the history of these theorems: In 1968 I.M. Gelfand and V.A. Ponomarev classified the indecomposable modules over the algebras

$GP_{n,m}$ which are defined in section 1.2. They introduced the notion of string and band modules and developed a method to prove that one gets all modules in this way. Then there came a series of papers like [R1], [DS], [SW], [WW] and [BR] where these results were generalized. All these papers make use of the results and methods of I.M. Gelfand and V.A. Ponomarev.

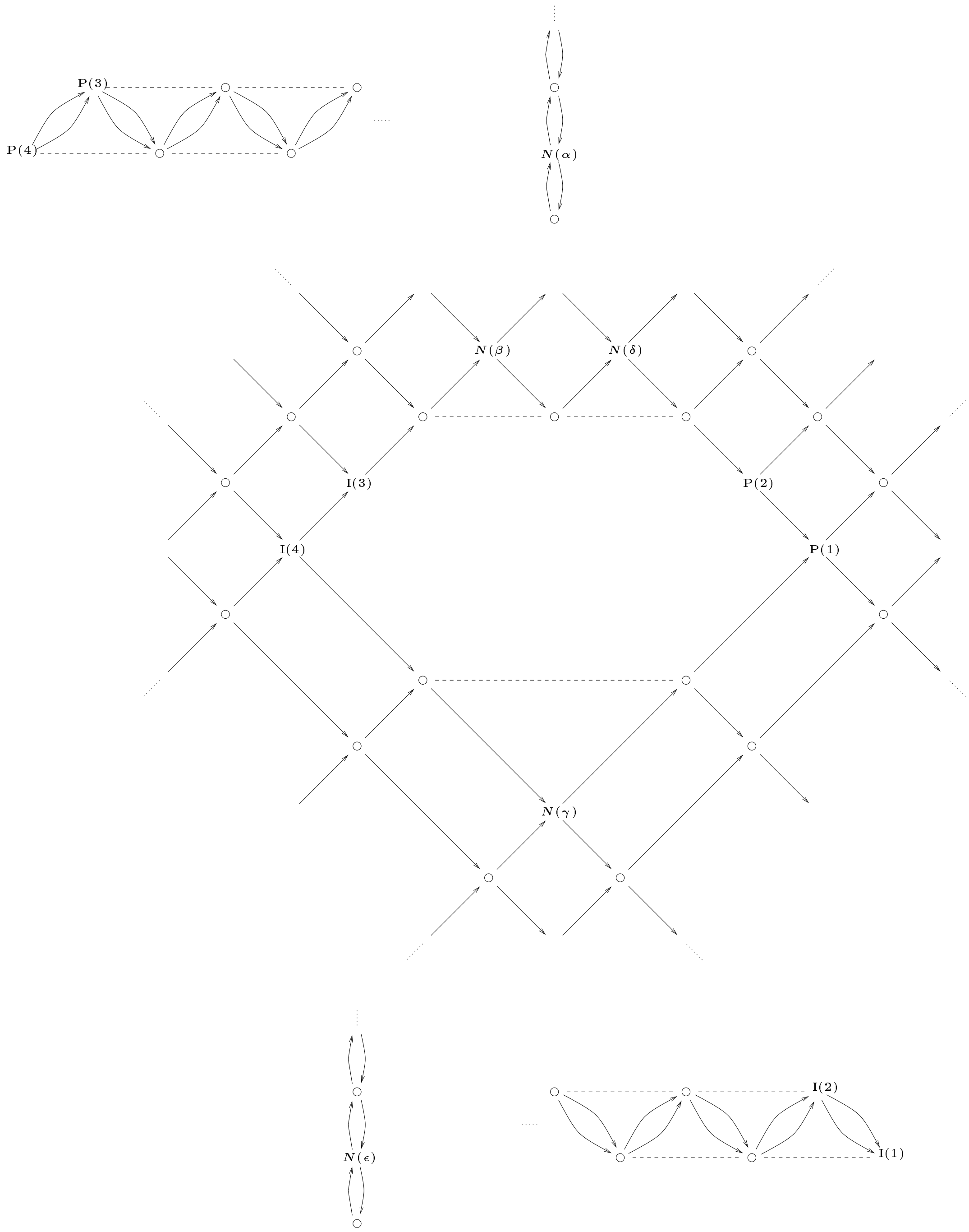
1.16 Example

Let A be the following algebra:

$$\begin{tikzcd}
1 \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & \\
2 \arrow[dr, "\gamma"'] & \\
& 3 \arrow[d, shift right, "\delta"'] \arrow[d, shift left, "\epsilon"] \\
& 4
\end{tikzcd}$$

$\rho = \{\gamma\beta, \delta\gamma\}$

Then we have two k^*-families of homogeneous tubes consisting of band modules. The Auslander-Reiten components containing string modules are displayed on the next page. Usually, one draws a dotted line from an indecomposable module to its Auslander-Reiten translate. Mostly, we omit these dotted lines. We draw them only in case we are dealing with an Auslander-Reiten sequence with indecomposable middle term or if multiple arrows between two points occur. By $P(i)$ (respectively $I(i)$) we denote the projective cover (respectively injective envelope) of the simple A-module i.



P(3)
P(4)
$N(\alpha)$
$N(\beta)$
$N(\delta)$
I(3)
P(2)
I(4)
P(1)
$N(\gamma)$
$N(\epsilon)$
I(2)
I(1)

2 Hammocks for string algebras

In this chapter let $A = kQ/\langle\rho\rangle$ be a string algebra, and let S be a fixed simple A-module with projective cover P and injective envelope I. Note that there exists an A-module homomorphism $f_S : P \longrightarrow I$ with image S. This homomorphism is unique up to multiplication with non-zero scalars. We do not distinguish between f_S and its non-zero scalar multiples and call *f_S the canonical A-module homomorphism corresponding to S.* With S we also denote the corresponding vertex of Q.
The aim of this section is to introduce posets $h(S)$, $H(S)$ and $\mathcal{H}$(S) and to prove some basic properties of them. These posets turn out to be useful for the description of factorizations of f_S. Later on, we express $\mathrm{nil}(f_S)$ in terms of these posets.

2.1 Notations

A *poset* is a set together with a partial order. A subset of a poset together with the induced partial order is called *subposet*. A *linear poset* is a poset such that each two elements of the poset are comparable. Next, we define some procedures which allow the construction of new posets from given ones.

For a poset H we denote by $^*H$ the poset with the same underlying set but with opposite partial order.

Let H_1 and H_2 be posets. Then let $H_1 \cdot H_2$ be the disjoint union of H_1 and H_2. The partial order on the subsets H_i is the same as before and we have $h_1 < h_2$ for all $h_1 \in H_1$ and $h_2 \in H_2$.

Let T_1 be a linear poset which has a maximal element t_1, and let T_2 a linear poset which has a minimal element t_2. The linear poset $T_1 \circ T_2$ is obtained from the linear poset $T_1 \cdot T_2$ by identifying t_1 and t_2.

Let H_1 and H_2 be posets and $h \in H_1$. Let H be the disjoint union of $H_1 \setminus \{h\}$ and H_2. The partial order on $H_1 \setminus \{h\}$ and on H_2 is the same as before. Let $h_1 \in H_1 \setminus \{h\}$ and $h_2 \in H_2$. Then let $h_1 < h_2$ if and only if $h_1 < h$, and let $h_2 < h_1$ if and only if $h < h_1$. We say that *the poset H is obtained from H_1 by replacing h by H_2.*

Let H_1 and H_2 be posets. Then the cartesian product $H_1 \times H_2$ is also a poset via $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. We call this the *product order* on $H_1 \times H_2$.

2.2 Strings of infinite length

An $\mathbb{N}$-*string* is an infinite word of the form $c_1 \cdots c_n \cdots$ with letters $c_i \in Q_1 \cup Q_1^-$ for $i \geq 1$ such that $c_1 \cdots c_m$ is a string for all $m \geq 1$. An ${}^*\mathbb{N}$-*string* is an infinite word of the form $\cdots c_{-n} \cdots c_{-1}$ such that $c_{-m} \cdots c_{-1}$ is a string for all $m \geq 1$. The inverse of an $\mathbb{N}$-string or an ${}^*\mathbb{N}$-string is defined in the obvious way. Note that the inverse of some $\mathbb{N}$-string is an ${}^*\mathbb{N}$-string and vice versa. Let C be a string of length ≥ 1. Assume that the n-fold concatenation $C \cdots C$ is defined for all $n \geq 1$. Define $C^\infty = C \cdot C \cdot C \cdots$ and ${}^\infty C = \cdots C \cdot C \cdot C$. Thus C^∞ is an $\mathbb{N}$-string and ${}^\infty C$ an ${}^*\mathbb{N}$-string. The set of $\mathbb{N}$-strings is denoted by $\mathcal{S}_{\mathbb{N}}$ and the set of ${}^*\mathbb{N}$-strings by $\mathcal{S}_{{}^*\mathbb{N}}$.

2.3 Partial orders on $\mathcal{S} \cup \mathcal{S}_{\mathbb{N}}$ and $\mathcal{S} \cup \mathcal{S}_{{}^*\mathbb{N}}$

Let $C \in \mathcal{S} \cup \mathcal{S}_{\mathbb{N}}$. Define $C_l = 1_{(S,t)}$ if $C = 1_{(S,t)}$ and $C_l = c_1$ if $C = c_1 \cdots$. We define a partial order $\leq_1$ on $\mathcal{S} \cup \mathcal{S}_{\mathbb{N}}$ as follows: Let $C, D \in \mathcal{S} \cup \mathcal{S}_{\mathbb{N}}$. The elements C and D are comparable if and only if there exists an $M \in \mathcal{S}$ such that $C = M \cdot C'$ and $D = M \cdot D'$. Now assume that C and D are comparable. If $C \neq D$ we can assume that $C = M \cdot C'$, $D = M \cdot D'$ and M is of maximal length with this property. Then define $C <_1 D$ if C'_l is inverse or of length 0 and D'_l is direct or of length 0. It is easy to check that this defines a partial order on $\mathcal{S} \cup \mathcal{S}_{\mathbb{N}}$.

Next, we define a partial order $\leq_{-1}$ on $\mathcal{S} \cup \mathcal{S}_{{}^*\mathbb{N}}$ as follows: Let $C, D \in \mathcal{S} \cup \mathcal{S}_{{}^*\mathbb{N}}$. Define $C \leq_{-1} D$ if and only if $C^- \leq_1 D^-$.

Note that partial orders of this kind were already introduced in [GP] and [R1].

2.4 Definition of $h(S)$

Let $h(S)_{-1} = \{C \in \mathcal{S} \mid s(C) = S \text{ and } \sigma(C) = -1\}$ and $h(S)_1 = \{C \in \mathcal{S} \mid e(C) = S \text{ and } \varepsilon(C) = 1\}$.

From now on we regard $h(S)_{-1}$ as a poset with partial order $\leq_{-1}$ and $h(S)_1$

as a poset with partial order $\leq_1$.

Let

$$h(S) = \{(C, C') \in h(S)_{-1} \times h(S)_1 \mid CC' \in \mathcal{S}\}.$$

This definition is essential and we use it quite often.

The set $h(S)$ is contained in the cartesian product $h(S)_{-1} \times h(S)_1$. Take the product order on $h(S)_{-1} \times h(S)_1$ and regard $h(S)$ as a subposet.

2.5 Description of $h(S)_i$

Lemma. *The posets $h(S)_i$ are linear posets.*

Proof. Each element of $h(S)_{-1}$ is of the form $C1_{(S,1)}$ for some $C \in \mathcal{S}$. The elements of $h(S)_1$ are of the form $1_{(S,1)}C$. The lemma follows then immediately from the definition of the partial orders $\leq_i$.

Let a_{-1} be the direct string of maximal length and z_{-1} the inverse string of maximal length in $h(S)_{-1}$. Let a_1 be the inverse string of maximal length and z_1 the direct string of maximal length in $h(S)_1$.

Lemma. *The element a_i is the minimal and z_i the maximal element of $h(S)_i$.*

Proof. Obvious.

Lemma. *Each element of $h(S)_i$ (except a_i) has a direct predecessor and each element (except z_i) has a direct successor.*

Proof. Let $i = 1$. Let $C \in h(S)_1$ and assume $C \neq z_1$. First, assume that C does not start on a peak. Thus there exists an $\alpha \in Q_1$ such that $C\alpha$ is a string. We claim that $C_h = C\alpha V_\alpha$ is the direct successor of C in $h(S)_1$. Assume there exists a $D \in h(S)_1$ such that $C <_1 D <_1 C_h$. This implies $D = CD'$ for some string D'. Since $C <_1 D$, we get $D' = \alpha D''$. From $D <_1 C_h$ we deduce $D' = \alpha V_\alpha \beta^- D'''$ for some $\beta^- \in Q_1^-$ and some string D'''. This is a contradiction because V_α starts in a deep.
Next, assume that C starts on a peak. Since $C \neq z_1$, we know that $C = C'\alpha^- U_\alpha^-$ for some $\alpha^- \in Q_1^-$ and some string C'. We claim that C' is the

direct successor of C in $h(S)_1$. This is proved similarly as above. Now let $C \in h(S)_1$ and $C \neq a_1$. If C starts in a deep we get $C = C'\alpha V_\alpha$ for some $\alpha \in Q_1$ and some string C'. Then C' is the direct predecessor of C in $h(S)_1$. Next, assume C does not start in a deep. Thus there exists some $\alpha^- \in Q_1^-$ such that $C\alpha^-$ is a string. Then $C\alpha^- U_{\alpha^-}$ is the direct predecessor of C in $h(S)_1$.
The case $i = -1$ is similar. One just has to add or delete hooks or cohooks from the left. This finishes the proof.

Corollary. *Let C be the direct predecessor of some element D in $h(S)_i$. Then there exists an irreducible map from $M(C)$ to $M(D)$.*

Proof. This follows from the description of irreducible maps in section 1.15 and the proof of the previous lemma.

2.6 Corner elements

In this section we introduce the socalled corner elements. The reason for this name will become clear in the next section.

Let $\{(D_1^{(-1)}, D_1^{(1)}), \cdots, (D_{n_1}^{(-1)}, D_{n_1}^{(1)})\}$ be the set of elements of $h(S)$ satisfying the following properties:

1. The string $D_i^{(-1)} \cdot D_i^{(1)}$ is direct and starts on a peak and ends in a deep for $1 \leq i \leq n_1$;
2. We have $(D_i^{(-1)}, D_i^{(1)}) < (D_{i+1}^{(-1)}, D_{i+1}^{(1)})$ for $1 \leq i \leq n_1 - 1$.

Remark. It is easy to see that there are only finitely many elements $(C, C') \in h(S)$ which satisfy the first property. It is also obvious that these elements are linearly ordered in $h(S)$.

Dually, define $\{(I_1^{(-1)}, I_1^{(1)}), \cdots, (I_{n_2}^{(-1)}, I_{n_2}^{(1)})\}$ such that the following hold:

1. The string $I_i^{(-1)} \cdot I_i^{(1)}$ is inverse and starts in a deep and ends on a peak for $1 \leq i \leq n_2$;
2. We have $(I_i^{(-1)}, I_i^{(1)}) < (I_{i+1}^{(-1)}, I_{i+1}^{(1)})$ for $1 \leq i \leq n_2 - 1$.

Remark. We have $a_1 = I_1^{(1)}$, $z_1 = D_{n_1}^{(1)}$, $a_{-1} = D_1^{(-1)}$ and $z_{-1} = I_{n_2}^{(-1)}$.

Let

$$
\begin{aligned}
\mathcal{C}(S) \quad = \quad & \{(D_1^{(-1)}, D_1^{(1)}), \cdots, (D_{n_1}^{(-1)}, D_{n_1}^{(1)})\} \\
& \cup\{(D_2^{(-1)}, D_1^{(1)}), \cdots, (D_{n_1}^{(-1)}, D_{n_1-1}^{(1)})\} \\
& \cup\{(I_1^{(-1)}, I_1^{(1)}), \cdots, (I_{n_2}^{(-1)}, I_{n_2}^{(1)})\} \\
& \cup\{(I_1^{(-1)}, I_2^{(1)}), \cdots, (I_{n_2-1}^{(-1)}, I_{n_2}^{(1)})\}.
\end{aligned}
$$

We call $\mathcal{C}(S)$ the set of *corner elements* of $h(S)$.

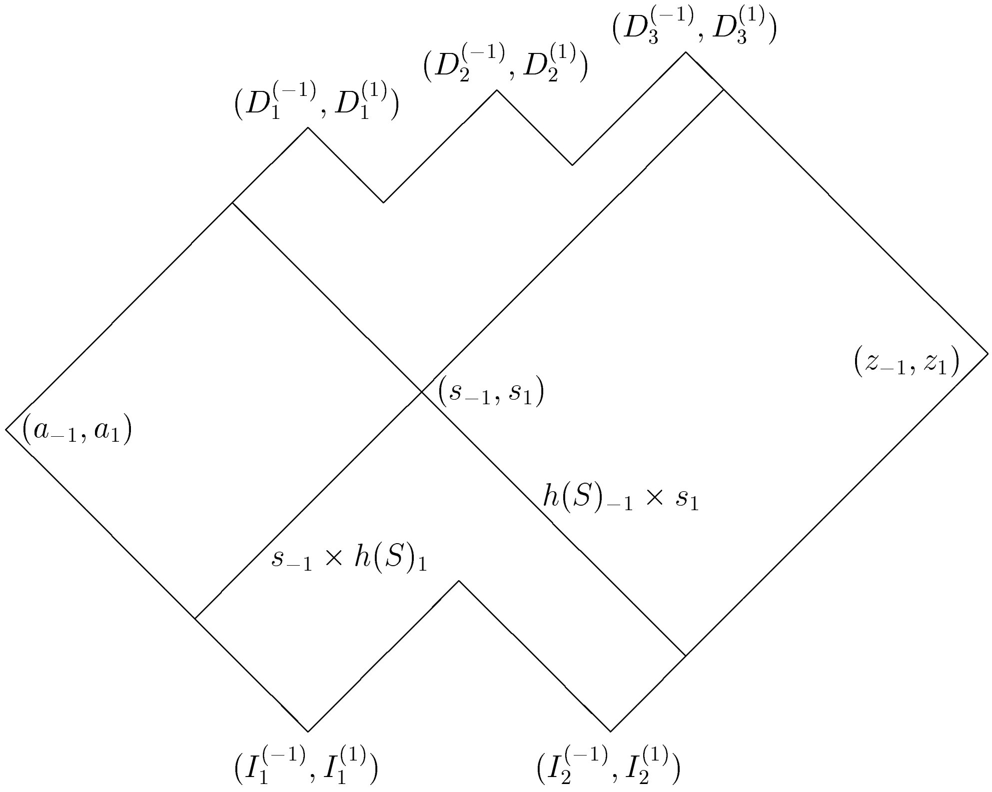
2.7 Description of $h(S)$

Proposition. *Let $(C, C') \in h(S)_{-1} \times h(S)_1$. Then $(C, C') \notin h(S)$ if and only if ($C <_{-1} D_{i+1}^{(-1)}$ and $C' >_1 D_i^{(1)}$ for some $1 \leq i \leq n_1 - 1$) or ($C >_{-1} I_i^{(-1)}$ and $C' <_1 I_{i+1}^{(1)}$ for some $1 \leq i \leq n_2 - 1$).*

Proof. Assume $(C, C') \in h(S)_{-1} \times h(S)_1$ but $(C, C') \notin h(S)$. Thus CC' is not a string. Let D and D' be direct strings of maximal length such that $C = MD$ and $C' = D'M'$ for some strings M and M'. Dually, let I and I' be inverse strings of maximal length such that $C = NI$ and $C' = I'N'$ for some N and N'.
Case 1: The length of D and D' is greater or equal to 0.
We know that DD' is not a string. Let D'_1 be a direct string of maximal length such that $D'_1 D'$ is a string. Next, let D'_2 be a direct string of maximal length such that $D'_1 D' D'_2$ is a string. Then $(D'_1, D'D'_2) = (D_{i+1}^{(-1)}, D_{i+1}^{(1)})$ for some corner element $(D_{i+1}^{(-1)}, D_{i+1}^{(1)})$ of $h(S)$. Since $D <_{-1} D'_1$, we know that there exists a corner element $(D_i^{(-1)}, D_i^{(1)})$ such that $D_i^{(-1)} D_i^{(1)}$ is direct and $(D_i^{(-1)}, D_i^{(1)}) < (D_{i+1}^{(-1)}, D_{i+1}^{(1)})$. From $D <_{-1} D'_1 = D_{i+1}^{(-1)}$ we get $C <_{-1} D_{i+1}^{(-1)}$. Next, we claim that $D_i^{(1)} <_1 D'$. Assume $D_i^{(1)} \geq D'$. Thus $D_i^{(1)} = D'D''$ for some string D''. Since $D_i^{(-1)} <_{-1} D_{i+1}^{(-1)} = D'_1$, we get $D_i^{(-1)} = E\alpha D'_1$ for some $\alpha \in Q_1$ and some string E. Thus $\alpha D'_1 D'$ is a string. But this is a contradiction to the maximality of the length of D'_1. This implies $D_i^{(1)} <_1 D'$ and $D_i^{(1)} <_1 C'$.
Case 2: The length of I and I' is greater than 0. In this case, one shows similarly that $C >_{-1} I_i^{(-1)}$ and $C' <_1 I_{i+1}^{(1)}$ for some $1 \leq i \leq n_2 - 1$.

Now we prove that the converse holds as well. First, assume $(C, C') \in h(S)_{-1} \times h(S)_1$ and $C <_{-1} D_{i+1}^{(-1)}$ and $C' >_1 D_i^{(1)}$ for some $1 \leq i \leq n_1 - 1$. Again let D and D' be direct strings of maximal length such that $C = MD$ and $C' = D'M'$. This implies $D <_{-1} D_{i+1}^{(-1)}$ and $D' >_1 D_i^{(1)}$. Thus $D = E\alpha D_{i+1}^{(-1)}$ and $D' = D_i^{(1)}\alpha' E'$ for some $\alpha, \alpha' \in Q_1$ and some $E, E' \in \mathcal{S}$. Assume CC' is a string. Then also DD' is a string. In particular, the concatenation $\alpha D_{i+1}^{(-1)} D_i^{(1)} \alpha'$ is defined. Thus $D_i^{(-1)} \leq_{-1} \alpha D_{i+1}^{(-1)} <_{-1} D_{i+1}^{(-1)}$ and $D_i^{(1)} <_1 D_i^{(1)}\alpha' \leq_1 D_{i+1}^{(1)}$. If $D_i^{(-1)} = \alpha D_{i+1}^{-1}$ we get a contradiction because $D_i^{(-1)} D_i^{(1)} \alpha'$ is not a string. Similar $D_i^{(1)}\alpha' = D_{i+1}^{(1)}$ contradicts the fact that $\alpha D_{i+1}^{(-1)} D_{i+1}^{(1)}$ is not a string. Let F' be a direct string of maximal length such that $\alpha D_{i+1}^{(-1)} F'$ is a string. We know that $D_i^{(1)} <_1 D_i^{(1)}\alpha' \leq_1 F <_1 D_{i+1}^{(1)}$. Next, let F be a direct string of maximal length such that $F\alpha D_{i+1}^{(-1)} F'$ is a string. This implies that $x = (F\alpha D_{i+1}^{(-1)}, F')$ is a corner element. Since $D_i^{(-1)} \leq F_{-1}\alpha D_{i+1}^{(-1)} \leq_{-1} \alpha D_{i+1}^{(-1)} <_{-1} D_{i+1}^{(-1)}$, we constructed a corner element x such that $(D_i^{(-1)}, D_i^{(1)}) < x < (D_{i+1}^{(-1)}, D_{i+1}^{(1)})$. This is a contradiction. If $C >_{-1} I_i^{(-1)}$ and $C' <_{-1} I_{i+1}^{(1)}$ for some $1 \leq i \leq n_2 - 1$ then one proves in the same way that $(C, C') \notin h(S)$. This finishes the proof.

Let $(s_{-1}, s_1) = (1_{(S,1)}, 1_{(S,1)})$. To be more concrete let $n_1 = 3$ and $n_2 = 2$. Then $h(S)$ has the following form:



We call an element $(C, C') \in h(S)$ *non-projective* if the following hold:

1. We have $C \neq a_{-1}$ and $C' \neq a_1$;
2. Let D be the direct predecessor of C in $h(S)_{-1}$ and D' the direct predecessor of C' in $h(S)_1$. Then $(D, D') \in h(S)$.

In this case, define $\tau_{h(S)}((C, C')) = (D, D')$.

An element $(C, C') \in h(S)$ is *non-injective* if the following hold:

1. We have $C \neq z_{-1}$ and $C' \neq z_1$;
2. Let D be the direct successor of C in $h(S)_{-1}$ and D' the direct successor of C' in $h(S)_1$. Then $(D, D') \in h(S)$.

In this case, define $\tau^-_{h(S)}((C,C')) = (D,D')$.

It is easy to check that this defines a translation quiver structure on $h(S)$.

2.8 Definition of $H(S)$

Up to now we only looked at strings and constructed the poset $h(S)$. Next, we want to enlarge this poset by using also $\mathbb{N}$-strings and ${}^*\mathbb{N}$-strings.

Let $\mathcal{B}$ be a set of representatives of $\sim_r$-equivalence classes of bands. Define $\mathcal{B}(S) = \{({}^\infty(BB')B, B'(BB')^\infty) \mid BB' \in \mathcal{B}, (B,B') \in h(S) \text{ and } |B| \geq 1\}$. Thus $\mathcal{B}(S) \subseteq \mathcal{S}_{^*\mathbb{N}} \times \mathcal{S}_{\mathbb{N}}$. Let

$$H(S) = h(S) \cup \mathcal{B}(S).$$

Note that $H(S) \subseteq (\mathcal{S} \cup \mathcal{S}_{^*\mathbb{N}}) \times (\mathcal{S} \cup \mathcal{S}_{\mathbb{N}})$. Thus $H(S)$ is also a poset with partial order induced from the product order. Define $H(S)_{-1} = h(S)_{-1} \cup \{x \mid (x,y) \in \mathcal{B}(S)\}$ and $H(S)_1 = h(S)_1 \cup \{y \mid (x,y) \in \mathcal{B}(S)\}$. Of course, $H(S)_i$ is a linear poset with respect to $\leq_i$ for $i \in \{-1,1\}$.

Call the elements in $H(S) \setminus h(S)$ (respectively in $H(S)_i \setminus h(S)_i$) *exceptional elements* of $H(S)$ (respectively of $H(S)_i$).

2.9 Definition of $\mathcal{H}(S)$

For $B \in \mathcal{B}(S)$ and $\lambda \in k^*$ let $B(\lambda) = {}^*\mathbb{N} \times \mathbb{N}$ with the product order. The poset $\mathcal{H}(S)$ is obtained from $H(S)$ by replacing each $B \in \mathcal{B}(S)$ by the disjoint union

$$\bigcup_{\lambda \in k^*} B(\lambda)$$

where $x \in B(\lambda_1)$ and $y \in B(\lambda_2)$ are incomparable if $\lambda_1 \neq \lambda_2$. Also $\mathcal{H}(S)$ is a translation quiver: The restriction of $\tau_{\mathcal{H}(S)}$ to the subset $h(S)$ is just $\tau_{h(S)}$. Let $(m,n) \in B(\lambda)$. Then (m,n) is called *non-projective* if $n \neq 1$ and *non-injective* if $m \neq 1$. Assume that (m,n) is non-projective. Then define $\tau_{\mathcal{H}(S)}((m,n)) = (m+1, n-1)$. In case (m,n) is non-injective let $\tau^-_{\mathcal{H}(S)}((m,n)) = (m-1, n+1)$.

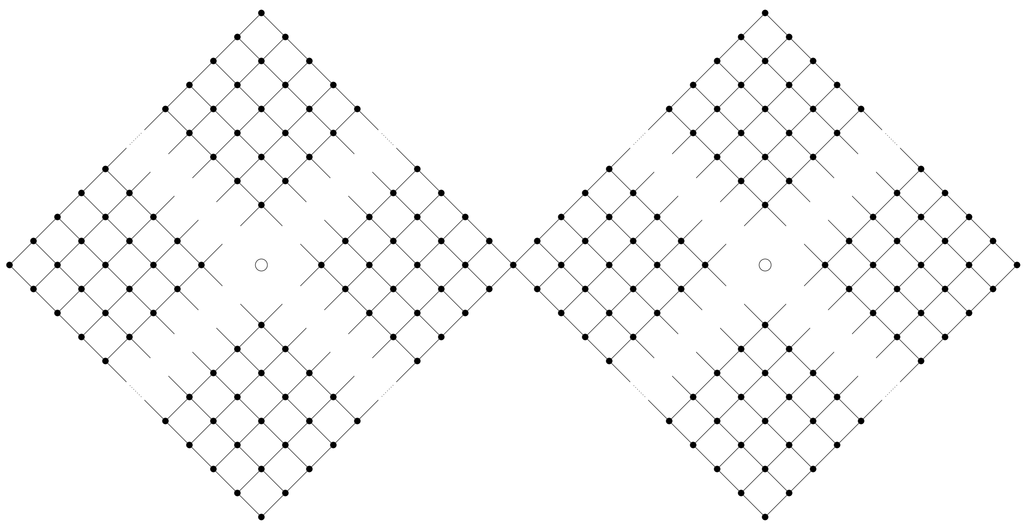
Altogether, we associated to a given string algebra A and a simple A-module S three posets $h(S)$, $H(S)$ and $\mathcal{H}(S)$. All three posets are called *hammocks*.

2.10 Examples

1.) Let A be the algebra $GP_{2,2}$ and S the simple A-module. Denote the exceptional elements of $H(S)$ by (x', x'') and (y', y''). Then, regarded as linear posets, we have

$$\begin{aligned} h(S)_{-1} &= \mathbb{N} \cdot \mathbb{Z} \cdot {}^*\mathbb{N} \\ h(S)_1 &= \mathbb{N} \cdot \mathbb{Z} \cdot {}^*\mathbb{N} \\ H(S)_{-1} &= \mathbb{N} \cdot \{x'\} \cdot \mathbb{Z} \cdot \{y'\} \cdot {}^*\mathbb{N} \\ H(S)_1 &= \mathbb{N} \cdot \{x''\} \cdot \mathbb{Z} \cdot \{y''\} \cdot {}^*\mathbb{N}. \end{aligned}$$

Assume $\varepsilon(\alpha) = \sigma(\beta) = 1$ and $\varepsilon(\beta) = \sigma(\alpha) = -1$. As corner elements we have $(D_1^{(-1)}, D_1^{(1)}) = (\alpha, 1_{(S,1)})$, $(D_2^{(-1)}, D_1^{(1)}) = (1_{(S,1)}, 1_{(S,1)})$ and $(D_2^{(-1)}, D_2^{(1)}) = (1_{(S,1)}, \alpha)$ on the upper side and $(I_1^{(-1)}, I_1^{(1)}) = (1_{(S,1)}, \beta^-)$, $(I_1^{(-1)}, I_2^{(1)}) = (1_{(S,1)}, 1_{(S,1)})$ and $(I_2^{(-1)}, I_2^{(1)}) = (\beta^-, 1_{(S,1)})$ on the lower side. The two exceptional elements of $H(S)$ are denoted by non-filled points. The hammock $H(S)$ has the following shape:



2.) Let A be the algebra from section 1.16 and $S = 2$. Assume $\varepsilon(\beta) = \sigma(\gamma) = 1$ and $\varepsilon(\alpha) = -1$. Denote the exceptional element of $H(S)$ by (x', x''). Then, regarded as linear posets, we get

$$\begin{aligned} h(S)_{-1} &= \mathbb{N} \cdot {}^*\mathbb{N} \\ h(S)_1 &= \mathbb{N} \cdot \bigcup_{i \in \mathbb{N}} \mathbb{Z}_i \cdot {}^*\mathbb{N} \\ H(S)_{-1} &= \mathbb{N} \cdot \{x'\} \cdot {}^*\mathbb{N} \\ H(S)_1 &= \mathbb{N} \cdot \bigcup_{i \in \mathbb{N}} \mathbb{Z}_i \cdot \{x''\} \cdot {}^*\mathbb{N} \end{aligned}$$

where $\mathbb{Z}_i = \mathbb{Z}$ for all i and $z_i < z_j$ for all $z_i \in \mathbb{Z}_i$ and $z_j \in \mathbb{Z}_j$ with $i < j$. The corner elements of $h(S)$ are $(D_1^{(-1)}, D_1^{(1)}) = (1_{(S,1)}, \beta)$ and $(I_1^{(-1)}, I_1^{(1)}) = (\alpha^-, \gamma^- \epsilon^-)$. This implies $h(S) = h(S)_{-1} \times h(S)_1$.

Remark. Remember that we associated to each string algebra A the two maps σ and ε. Obviously, the sets $h(S)_{-1}$ and $h(S)_1$ depend on the choice of these maps. But $h(S)$ regarded as a poset is up to isomorphism of posets independent of σ and ε.

2.11 Some basic properties of hammocks

The global structure of $h(S)$ depends only on the local shape of (Q, ρ) around the vertex S. In this section we describe the shape of $h(S)$ in case $kQ/\langle\rho\rangle$ is a gentle algebra.

Let (Q, ρ) be a pair of type $\tilde{A}$ and $S \in Q_0$. We say that a zero-relation $\alpha_1 \cdots \alpha_n \in \rho$ *passes* S if $e(\alpha_i) = S$ for some $2 \leq i \leq n$. By p_S we denote the number of zero-relations in ρ which pass S. Furthermore, let $n_1^S = |\{\alpha \in Q_1 \mid e(\alpha) = S\}|$ and $n_2^S = |\{\alpha \in Q_1 \mid s(\alpha) = S\}|$.

Lemma. *Assume we are in one of the following situations:*

1. $n_1^S + n_2^S = 1$*;*
2. $n_1^S + n_2^S = 2$ *and* $p_S = 0$*;*
3. $n_1^S + n_2^S = 3$ *and* $p_S = 1$*;*
4. $n_1^S + n_2^S = 4$ *and* $p_S = 2$*.*

Then $h(S) = h(S)_{-1} \times h(S)_1$.

Proof. It is easy to check that there are exactly two corner elements in each case. But this implies $h(S) = h(S)_{-1} \times h(S)_1$. This finishes the proof.

Lemma. *Let* $n_1^S + n_2^S = 2$ *and* $p_S = 1$. *Assume further that the zero-relation which passes* S *is of length 2. Then* $h(S) = h(S)_{-1} \times h(S)_1 = 1_{(S,1)} \times h(S)_1$.

Proof. Obvious.

An algebra A is called *gentle* if the following hold:

1. We have $A = kQ/\langle\rho\rangle$ where (Q, ρ) is a pair of type $\tilde{A}$;

2. All zero-relations in ρ are of length 2;

3. Let $\beta \in Q_1$. Then there is at most one α such that $\beta\alpha \in \rho$ and at most one γ such that $\gamma\beta \in \rho$.

This definition goes back to I. Assem and A. Skowroński (see [AS]). But note that we allow oriented cycles in Q.

Proposition. *Let* A *be a gentle algebra and* S *a simple* A*-module. Then* $h(S) = h(S)_{-1} \times h(S)_1$.

Proof. The proposition follows directly from the previous two lemmas and the definition of a gentle algebra.

Corollary. *Let* A *be a gentle algebra and* S *a simple* A*-module. Assume* $(C, C') < (D, D')$ *in* $h(S)$ *such that* $C <_{-1} D$ *and* $C' <_1 D'$. *Then* (D, D') *is not projective and* $(C, C') \leq \tau_{h(S)}((D, D'))$.

Proof. This follows from the above proposition and the definition of $\tau_{h(S)}$.

3 Combinatorics I: Hammock posets

In the last chapter we associated to a given string algebra A and a simple A-module S posets $h(S)$, $H(S)$ and $\mathcal{H}(S)$. One of our aims is to understand the behaviour of the transfinite radical of A-mod. The poset $H(S)$ will be the main tool to deal with this problem. In this chapter we develop some theory of posets and introduce a special class of posets, namely the hammock posets. The main theorem of this chapter deals with the structure of these hammock posets.

3.1 Ordinal numbers

Recall that a poset is called *well-ordered* if every non-empty subset of it has a smallest element. Thus a well-ordered set is a linear poset. An *ordinal number* is a well-ordered set α such that $s(\xi) = \xi$ for all $\xi \in \alpha$. Here $s(\xi)$ is by definition the initial segment $\{x \in \alpha \mid x < \xi\}$. Sometimes we call an ordinal number just *ordinal*. For every set x we define $x^+ = x \cdot \{x\}$. For example, $\mathbb{N}_0$ can be defined as follows:

$$\begin{array}{rcl} 0 & = & \emptyset \\ 1 & = & 0^+ \\ 2 & = & 1^+ \\ & \cdots & \end{array}$$

By ω we denote the smallest set that contains 0 and that contains x^+ whenever it contains x. Thus every natural number and also ω are examples of ordinal numbers. For convenience we mention the names of the first few ordinal numbers:

$$\begin{array}{l} 0,\ 1,\ 2,\ \cdots \\ \omega,\ \omega+1,\ \omega+2,\ \cdots \\ \omega 2,\ \omega 2+1,\ \omega 2+2,\ \cdots \\ \omega 3,\ \omega 3+1,\ \omega 3+2,\ \cdots \\ \cdots \\ \omega n,\ \omega n+1,\ \omega n+2,\ \cdots \\ \cdots \\ \omega^2,\ \omega^2+1,\ \cdots \\ \omega^2+\omega,\ \omega^2+\omega+1,\ \cdots \end{array}$$

$\cdots$

Using the notation from section 2.1 we get:

$$\begin{aligned} \omega &= \mathbb{N}_0 \\ \omega + 1 &= \mathbb{N}_0 \cdot \{x\} \\ \omega + 2 &= \mathbb{N}_0 \cdot \{x_1\} \cdot \{x_2\} \\ \omega 4 + 2 &= \mathbb{N}_0 \cdot \mathbb{N}_0 \cdot \mathbb{N}_0 \cdot \mathbb{N}_0 \cdot \{x_1\} \cdot \{x_2\} \end{aligned}$$

All ordinals $\geq \omega$ are called *infinite*, the others are called *finite*. We call an infinite ordinal β a *limit number* if β has no direct predecessor. For example, ω and $\omega^2+\omega$ are limit numbers but $\omega+1$ is not a limit number. The smallest limit number is ω. Note that we can write every infinite ordinal α uniquely as $\beta + n = ((\beta^+)^+ \cdots)^+$ where $n \in \omega$, β is a limit number and we applied $^+$ exactly n times. We will not give an introduction to ordinal arithmetic (for details see [Ro]). Just note that $\omega 0 = 0$, $\omega n + 0 = \omega n$ and $\omega 1 = \omega$.
We introduce an additional symbol ∞ and as a convention $\alpha < \infty$ for every ordinal α.

3.2 The density of a linear poset (Definition)

Let H be a poset. For $h_1, h_2 \in H$ we define a subposet $H_{h_1,h_2} = \{h \in H \mid h_1 \leq h \leq h_2 \text{ or } h_2 \leq h \leq h_1\}$.
Now let T be a linear poset and $t_1, t_2 \in T$. We write $t_1 \sim t_2$ if T_{t_1,t_2} is a finite set. This defines an equivalence relation on T. The equivalence class of some $t \in T$ is denoted by $[t]$. For a linear poset T we define linear posets $T^{(i)}$, $i \geq 0$, associated to T as follows: Let $T^{(0)} = T$. Assume that $T^{(n)}$ is already defined. Then let $T^{(n+1)} = \{[t] \mid t \in T^{(n)}\}$ with the obvious induced order (that is $[t_1] < [t_2]$ if $[t_1] \neq [t_2]$ and $t_1 < t_2$). Assume that $T^{(i)}$ is finite for some i. Define $d(T) = \omega n + d$ where $n = \min\{i \mid T^{(i)} \text{ is finite}\}$ and $d = |T^{(n)}| - 1$. We write $d(T) = \infty$ if $T^{(i)}$ is infinite for all $i \geq 0$. Call $d(T)$ the *density* of the linear poset T.
For $t \in T^{(n)}$ and $i \geq 1$ define $[t]^i = [\cdots [t] \cdots]$ where the bracket is applied i times. Thus $[t]^i$ is an element in $T^{(n+i)}$. On the other hand for $t \in T^{(n)}$ and $1 \leq i \leq n$ define $[t]^{-i} = \{t' \in T^{(n-i)} \mid [t']^i = t\}$. For $n = 0$ let $[t]^{-0} = [t]^0 = t$.

3.3 Examples

- $d(\mathbb{N}) = d(\mathbb{Z}) = \omega$
- $d(\mathbb{N} \cdot \{x\}) = d(\mathbb{Z} \cdot \{x\}) = \omega + 1$
- $d(\{x\} \cdot \mathbb{Z} \cdot \{y\}) = \omega + 2$
- $d(\{x\} \cdot \mathbb{N} \cdot \{y\}) = \omega + 1$
- $d(\mathbb{Q}) \geq \omega^2$

3.4 Posets with an α-structure (Definition)

Let H be a poset and $d \geq 1$. A *d-structure* of H is given by subposets $H_{y_0,y_1}, \cdots, H_{y_{d-1},y_d}$ such that $y_{i-1} < y_i$ for $1 \leq i \leq d$. Next, let $n \geq 1$. The poset H has an *$(\omega n + d)$-structure* if there exists a d-structure $H_{y_0,y_1}, \cdots, H_{y_{d-1},y_d}$ such that every H_{y_{i-1},y_i} has an $(\omega(n-1) + d')$-structure for every $d' \geq 1$.

3.5 Examples

- The set $\mathbb{N}$ has a d-structure for every $d \geq 1$. But it does not have an $(\omega + 1)$-structure.
- The set $\mathbb{N} \cdot \{x\}$ has also d-structure for every $d \geq 1$. It has an $(\omega + 1)$-structure, but no $(\omega + 2)$-structure.
- The set $\mathbb{Q}$ has an $(\omega n + d)$-structure for every $n \geq 0$, $d \geq 1$.

Remark. It is essential to understand the connection between α-structures of a poset and the existence of linear subposets which have a corresponding density.

3.6 The density of a linear poset (Elementary results)

Lemma. *Let T be a linear poset with a minimal element x and a maximal element y. Assume $x \neq y$. Let $d(T) = \omega n + d$. Then $d \geq 1$.*

Proof. Let $n \geq 1$. Since T has a minimal and a maximal element, we know that each $T^{(i)}$ has a minimal and a maximal element. Especially $T^{(n-1)}$ has this property. We have $d(T^{(n-1)}) = \omega + d$. The only linear posets with density ω are $\mathbb{N}$, ${}^*\mathbb{N}$ and $\mathbb{Z}$. Thus $d \geq 1$.

Lemma. *Let T be a linear poset with $d(T) = \omega n + d$ where $d \geq 1$. Then $T = T' \circ T_1 \circ \cdots \circ T_d \circ T''$ where $d(T_i) = \omega n + 1$ and T_i has a minimal and a maximal element.*

Proof. We know that $T^{(n)}$ contains exactly $d+1$ elements, say $\{t_0, \cdots, t_d\}$ where $t_{i-1} < t_i$ for $1 \leq i \leq d$. Choose some $t'_i \in [t_i]^{-n}$ for each i. Thus $t'_i \in T$. We have $d(T_{t'_{i-1}, t'_i}) = \omega n + 1$ and $T = \{t \in T \mid t \leq t'_0\} \circ T_{t'_0, t'_1} \circ \cdots \circ T_{t'_{d-1}, t'_d} \circ \{t \in T \mid t \geq t'_d\}$.

Corollary. *Let T be a linear poset which has a minimal and a maximal element with $d(T) = \omega n + d$. Then $T = T_1 \circ \cdots \circ T_d$ where $d(T_i) = \omega n + 1$ and T_i has a minimal and maximal element for each i.*

Proof. This follows directly from the previous lemma.

Lemma. *Let $T_1, \cdots, T_d$ be linear posets which have a minimal and a maximal element with $d(T_i) = \omega n + d_i$. Then*

$$d(T_1 \circ \cdots \circ T_d) = \omega n + \sum_{i=1}^{d} d_i.$$

Proof. For $d = 1$ there is nothing to show. Let $d = 2$ and assume that T_1 and T_2 contain both at least two elements (otherwise the statement is trivial). Let $x = \min T_1$, $y_1 = \max T_1$, $y_2 = \min T_2$ and $z = \max T_2$. The elements y_1 and y_2 are identified in $T_1 \circ T_2$. Denote the corresponding element by y. Note that $(T_1 \circ T_2)^{(n)} = T_1^{(n)} \circ T_2^{(n)}$, and $[y]^n$ is obtained from identifying $[y_1]^n$ and $[y_2]^n$. Thus the elements $[x]^n$, $[y]^n$ and $[z]^n$ are pairwise different in $(T_1 \circ T_2)^{(n)}$. This implies $d(T_1 \circ T_2) = \omega n + d_1 + d_2$. The generalization to $d > 2$ is trivial.

Lemma. *Let T_1 and T_2 be linear posets which have a minimal and a maximal element. Let $d(T_i) = \omega n_i + d_i$ such that $n_1 > n_2$. Then $d(T_1 \circ T_2) = d(T_2 \circ T_1) = \omega n_1 + d_1$.*

Proof. Obvious.

Corollary. *Let $T_1, \cdots, T_d$ be linear posets which have a minimal and a maximal element, and let σ be a permutation of $\{1, \cdots, d\}$. Then $d(T_1 \circ \cdots \circ T_d) = d(T_{\sigma(1)} \circ \cdots \circ T_{\sigma(d)})$.*

Proof. This follows directly from the previous two lemmas.

Lemma. *Let T be a linear poset and $\omega n + d \leq d(T)$. Then there exists a subposet of density $\omega n + d$.*

Proof. For $n = 0$ the statement is trivial. Assume $n \geq 1$ and $d = 0$. Since $n \geq 1$, we know that $T^{(n-1)}$ contains infinitely many elements. Every infinite linear poset contains $\mathbb{N}$ or ${}^*\mathbb{N}$ as a subposet. Assume that $T^{(n-1)}$ contains $\mathbb{N} = \{t_1 < t_2 < \cdots\}$ as a subposet. Choose some $t'_i \in [t_i]^{-(n-1)}$ for each i. Thus $d(T_{t'_{i-1},t'_i}) \geq \omega(n-1)+1$. By induction assumption there is a subposet T'_i of $T_{t'_{i-1},t'_i}$ with $d(T'_i) = \omega(n-1)+1$. We can assume that the minimal element of T'_i is t'_{i-1} and the maximal element is t'_i. We get $d(T'_1 \circ \cdots T'_l) = \omega(n-1) + l$ for all $l \geq 1$. It is easy to see that the subposet $\bigcup\limits_{i \geq 1} T'_i$ has density ωn.
If $T^{(n-1)}$ contains ${}^*\mathbb{N}$ as a subposet the arguments are nearly the same.
Now let $d = 1$. We can find a subposet of the form $\mathbb{N} \cdot \{t\}$ or $\{t\} \cdot^* \mathbb{N}$ in $T^{(n-1)}$ and construct similarly as for $d = 0$ a subposet of density $\omega n + 1$ in T.
Finally, let $d \geq 2$. We know that $T^{(n)}$ contains at least $d+1$ elements. Choose such elements say $\{t_0 < \cdots < t_d\}$. Let $t'_i \in [t_i]^{-n}$. Thus $d(T_{t'_{i-1},t'_i}) \geq \omega n + 1$. Again $T_{t'_{i-1},t'_i}$ contains a subposet of density $\omega n + 1$ with minimal element t'_{i-1} and maximal element t'_i. The concatenation gives a subposet of density $\omega n + d$. This finishes the proof.

Corollary. *Let T be a linear poset. Assume that for every $d \geq 1$ there exists a subposet T_d of T with $d(T_d) = \omega n + d$. Then there exists a subposet of density $\omega(n+1)$ in T.*

Proof. It is obvious that $d(T) \geq \omega n + d$ for all $d \geq 1$. This implies $d(T) \geq \omega(n+1)$. By the previous lemma there exists a subposet of density $\omega(n+1)$.

3.7 Some relations between density and α-structures

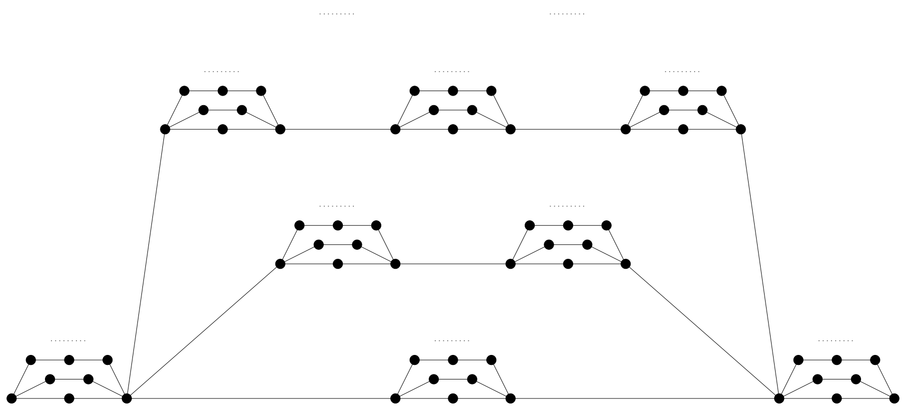
Lemma. *Let H be a poset such that there exists a linear subposet T with $d(T) = \omega n + d$ where $d \geq 1$. Then H has an $(\omega n + d)$-structure.*

Proof. We know that $T = T' \circ T_1 \circ \cdots \circ T_d \circ T''$ with $d(T_i) = \omega n + 1$, and each T_i has a minimal and a maximal element. The case $n = 0$ is trivial. Thus assume $n \geq 1$. Let $d = 1$. Then for every $d' \geq 1$ there is a subposet $T_{d'}$ in T_1 with $d(T_{d'}) = \omega(n-1) + d'$. Thus by induction we get an $(\omega(n-1) + d')$-structure of T_1 for every $d' \geq 1$. Since T_1 has a minimal and a maximal element we get a $(\omega n + 1)$-structure of T_1. This induces an $(\omega n + 1)$-structure for the poset H. The generalization to $d \geq 2$ is obvious. This finishes the proof.

The converse of this lemma is in general not true as the following example shows:

Let $[1, n] = \{1, \cdots, n\} \subset \mathbb{N}$. Let $H_1' = \bigcup_{n \geq 3} [1, n]$ be a disjoint union of posets. Thus elements $i_1 \in [1, n_1]$ and $i_2 \in [1, n_2]$ are incomparable if $n_1 \neq n_2$. The poset H_1 is obtained by identifying all the minimal elements of H_1' and by identifying all the maximal elements.

Now we replace every element in H_1 by a copy of the poset H_1 itself and get the following poset which we call H_2:



Note that the elements become larger from left to right. We go on inductively and obtain the poset H_n by replacing every element of H_1 by a copy of H_{n-1}.

Let T be a linear subposet of some H_n. Then T is finite and consequently $d(T) < \omega$. But H_n has an $(\omega n + 1)$-structure.

Lemma. *Let T be a linear poset which has an $(\omega n + d)$-structure. Then there exists a subposet of density $\omega n + d$.*

Proof. The case $n = 0$ is trivial. Thus assume $n \geq 1$. Let $T_{t_0,t_1}, \cdots T_{t_{d-1},t_d}$ be an $(\omega n + d)$-structure of T.
For $d = 1$ we get by induction assumption subposets $T_{d'}$ of T_{t_0,t_1} with $d(T_{d'}) = \omega(n-1) + d'$ for every $d' \geq 1$. This implies the existence of a subposet of density $\omega n + 1$ in T.
Next, let $d \geq 2$. For all i we get a subposet of density $\omega n + 1$ inside T_{t_{i-1},t_i}. which has a minimal and a maximal element. By concatenating them we get a subposet of T of density $\omega n + d$.

Remark. The aim of the next sections is to introduce a special class of posets, the socalled hammock posets. We show that for hammock posets there is a strong connection between α-structures and the existence of linear subposets with a corresponding density.

3.8 Hammock posets (Definition)

For $i \in \{-1, 1\}$ let H_i be a linear poset. Fix some element $s_i \in H_i$ and a set $E_i = \{e_1^{(i)}, \cdots, e_l^{(i)}\} \subseteq H_i$ such that the following hold:

1. There is a minimal element a_i and a maximal element z_i in H_i;
2. We have $a_i, s_i, z_i \notin E_i$;
3. The elements in E_i have no direct predecessor or successor;
4. Every element in $H_i \setminus \{a_i, e_1^{(i)}, \cdots, e_l^{(i)}\}$ has a direct predecessor and every element in $H_i \setminus \{z_i, e_1^{(i)}, \cdots, e_l^{(i)}\}$ has a direct successor.

The set E_i is also allowed to be empty. The elements in E_i are called *exceptional elements* of H_i.

Next, fix $\{D_1^{(i)}, \cdots, D_{n_1}^{(i)}\} \subseteq H_i \setminus E_i$ such that the following hold:

1. $a_{-1} \leq D_1^{(-1)} < \cdots < D_{n_1}^{(-1)} \leq s_{-1}$;
2. $s_1 \leq D_1^{(1)} < \cdots < D_{n_1}^{(1)} \leq z_1$;
3. $e_j^{(-1)} > D_{i+1}^{(-1)}$ or $e_j^{(1)} < D_i^{(1)}$ for all i and j.

Similarly, fix some subset $\{I_1^{(i)}, \cdots, I_{n_2}^{(i)}\} \subseteq H_i \setminus E_i$ such that the following hold:

1. $s_{-1} \leq I_1^{(-1)} < \cdots < I_{n_2}^{(-1)} \leq z_{-1}$;
2. $a_1 \leq I_1^{(1)} < \cdots < I_{n_2}^{(1)} \leq s_1$;
3. $e_j^{(-1)} < I_i^{(-1)}$ or $e_j^{(1)} > I_{i+1}^{(1)}$ for all i and j.

Define

$$\begin{aligned} H &= H_{-1} \times H_1 \setminus \{\{(e_i^{(-1)}, h) \mid h \neq e_i^{(1)}\} \\ &\quad \cup \{(h, e_i^{(1)} \mid h \neq e_i^{(-1)}\} \\ &\quad \cup \{(h_1, h_2) \mid h_1 < D_{i+1}^{(-1)} \text{ and } h_2 > D_i^{(1)} \text{ for some } 1 \leq i \leq n_1 - 1\} \\ &\quad \cup \{(h_1, h_2) \mid h_1 > I_i^{(-1)} \text{ and } h_2 < I_{i+1}^{(1)} \text{ for some } 1 \leq i \leq n_2 - 1\}\}. \end{aligned}$$

We call the poset H a *hammock poset* and $E = \{(e_1^{(-1)}, e_1^{(1)}) \cdots, (e_l^{(-1)}, e_l^{(1)})\}$ the set of *exceptional elements* of H. The reason for introducing this class of posets is the following:

Proposition. *Let A be a domestic string algebra and S a simple A-module. Then $H(S)$ is a hammock poset.*

Proof. First, use the description of $h(S)$ as given in section 2.7. Then the domesticity of A implies that $\mathcal{B}(S)$ is a finite set. Let $E = \mathcal{B}(S)$.

For the rest of the chapter let H be a hammock poset. We also need the associated sets H_{-1}, H_1, E_{-1}, E_1 and E as defined above.

3.9 Linear subposets of hammock posets

For a linear subposet T of H we define two projection maps

$$\pi_i : T \longrightarrow H_i$$

with $\pi_i(x_{-1}, x_1) = x_i$. If T has a minimal and a maximal element then also $\pi_i(T)$ has this property.

Remark. Let $T = T_1 \circ \cdots \circ T_d$ be a linear subposet of H such that T_i has a minimal and a maximal element for all i. Then obviously $\pi_i(T_1 \circ \cdots \circ T_d) = \pi_i(T_1) \circ \cdots \circ \pi_i(T_d)$.

Lemma. *Let T be a linear subposet of H with $d(T) = \omega n + d$. Assume that T has a minimal and a maximal element. Then $d(\pi_{-1}(T) \circ \pi_1(T)) \geq d(T)$.*

Proof. The case $n = 0$ is obvious. Thus assume $n \geq 1$. From section 3.6 we know that $d \geq 1$. First, let $d = 1$. Thus for every $d' \geq 1$ there exists a subposet $T_{d'}$ of T which has a minimal and a maximal element with $d(T_{d'}) = \omega(n-1) + d'$. By the induction assumption we get $d(\pi_{-1}(T) \circ \pi_1(T)) \geq d(\pi_{-1}(T_{d'}) \circ \pi_1(T_{d'})) \geq d(T_{d'}) = \omega(n-1) + d'$. This implies $d(\pi_{-1}(T) \circ \pi_1(T)) \geq \omega n$. Since $\pi_{-1}(T) \circ \pi_1(T)$ has a minimal and a maximal element we get $d(\pi_{-1}(T) \circ \pi_1(T)) \geq \omega n + 1 = d(T)$.
Next, let $d \geq 2$. Then $T = T_1 \circ \cdots \circ T_d$ where T_i has a minimal and maximal element and $d(T_i) = \omega n + 1$ for each i. By induction we know that $d(\pi_{-1}(T_i) \circ \pi_1(T_i)) \geq d(T_i)$. This implies

$$\begin{aligned}
d(\pi_{-1}(T) \circ \pi_1(T)) &= d(\pi_{-1}(T_1 \circ \cdots \circ T_d) \circ \pi_1(T_1 \circ \cdots \circ T_d)) \\
&= d(\pi_{-1}(T_1) \circ \cdots \circ \pi_{-1}(T_d) \circ \pi_1(T_1) \circ \cdots \circ \pi_1(T_d)) \\
&= d(\pi_{-1}(T_1) \circ \pi_1(T_1) \circ \cdots \circ \pi_{-1}(T_d) \circ \pi_1(T_d)) \\
&\geq d(T_1 \circ \cdots \circ T_d) \\
&= d(T).
\end{aligned}$$

This finishes the proof.

Corollary. *Let H be a hammock poset. Then $d(H_{-1} \circ H_1) \geq d(T)$ for every linear subposet T of H.*

Proof. This is obvious since $\pi_{-1}(T) \circ \pi_1(T)$ is a subset of $H_{-1} \circ H_1$.

Corollary. *Let H be a hammock poset. Then $d((H_{-1} \setminus E_{-1}) \circ (H_1 \setminus E_1)) \geq d(T)$ for every linear subposet T of $H \setminus E$.*

Proof. Let T be a linear subposet of $H \setminus E$. Then $\pi_i(T) \subseteq H_i \setminus E_i$.

Lemma. *Let H be a hammock poset. Then there exists a linear subposet T of $H \setminus E$ such that $d(T) = d((H_{-1} \setminus E_{-1}) \circ (H_1 \setminus E_1))$.*

Proof. We have $d((H_{-1} \setminus E_{-1}) \circ (H_1 \setminus E_1)) = d((H_{-1} \setminus E_{-1})_{a_{-1},s_{-1}} \circ (H_{-1} \setminus E_{-1})_{s_{-1},z_{-1}} \circ (H_1 \setminus E_1)) = d((H_{-1} \setminus E_{-1})_{a_{-1},s_{-1}} \circ (H_1 \setminus E_1) \circ (H_{-1} \setminus E_{-1})_{s_{-1},z_{-1}})$. But $(H_{-1} \setminus E_{-1})_{a_{-1},s_{-1}} \circ (H_1 \setminus E_1) \circ (H_{-1} \setminus E_{-1})_{s_{-1},z_{-1}}$ is a linear poset which is isomorphic to $T = H_{(a_{-1},a_1),(s_{-1},a_1)} \circ H_{(s_{-1},a_1),(s_{-1},z_1)} \circ H_{(s_{-1},z_1),(z_{-1},z_1)}$. Note that T does not contain exceptional elements. This finishes the proof.

The previous lemma enables us to define $d_T(H \setminus E) = \max\{d(T) \mid T \text{ is a linear subposet of } H \setminus E\}$. Thus $d_T(H \setminus E) = d((H_{-1} \setminus E_{-1}) \circ (H_1 \setminus E_1))$. Since E is a finite set, we can define

$$d_T(H) = \max\{d(T) \mid T \text{ is a linear subposet of } H\}.$$

Lemma. *Let H be a hammock poset. Then $0 \leq d_T(H) - d_T(H \setminus E) \leq |E|$.*

Proof. Inserting n new elements into a linear poset T increases the density of T by at most n.

Lemma. *Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements in H with $x < y$. Let T be a linear subposet of H_i which does not contain exceptional elements. Assume $\pi_i(x) \leq t \leq \pi_i(y)$ for all $t \in T$. Then there exists a subposet T' of $H_{x,y}$ which is isomorphic to T.*

Proof. Without loss of generality assume $i = 1$. Let l be minimal such that $x_1 \leq_{-1} I_l^{(-1)}$. Choose some $h \in H_{-1}$ such that $x_1 \leq_{-1} h \leq_{-1} y_1$, $h \leq_{-1} I_l^{(-1)}$ and $h \notin E_{-1}$. Next, let l' be maximal such that $D_{l'}^{(-1)} \leq_{-1} y_1$. Choose some $h' \in H_{-1}$ such that $h \leq_{-1} h' \leq_{-1} y_1$, $D_{l'}^{(-1)} \leq_{-1} h'$ and $h' \notin E_{-1}$. It is easy to see that we can choose h and h' in such a way. Define $T' = \{(h, t) \mid t \in T \text{ and } t \leq s_1\} \cup \{(h', t) \mid t \in T \text{ and } t > s_1\}$. Thus $T' \subseteq (H_{-1} \setminus E_{-1}) \times (H_1 \setminus E_1)$. It is obvious that T' is a linear subposet of $(H_{-1} \setminus E_{-1}) \times (H_1 \setminus E_1)$ which is isomorphic to T. We have to show that T' is also contained in $H \setminus E$. First, look at $(h,t) \in T'$. We have $t \leq_1 s_1 \leq_1 D_i^{(1)}$ for all i. Furthermore, $h \leq_{-1} I_i^{(-1)}$ for $l \leq i \leq n_2$. It remains to show that $t \geq_{-1} I_i^{(-1)}$ for $1 \leq i \leq l$. We have $x_1 >_{-1} I_{l-1}^{(-1)}$. Since $(x_1, x_2) \in H$, we get $x_2 \geq_1 I_l^{(1)} \geq_1 \cdots \geq_1 I_1^{(1)}$. Since $t \geq_1 x_2$, we get $(h,t) \in H \setminus E$. Here we used the definition of hammock poset. Similarly, one shows that (h', t) is also contained in $H \setminus E$. Furthermore, it is obvious that $x \leq t' \leq y$ for all $t' \in T'$. Thus T' is a linear subposet of $H_{x,y}$. This finishes the proof.

Lemma. *Let $x, y \in H$ and $n \geq 0$, and for every $d \geq 1$ let T_d be a linear subposet of $H_{x,y}$ with $d(T_d) = \omega n + d$. Then there exists a linear subposet T of $H_{x,y}$ with $d(T) = \omega(n+1) + 1$.*

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We know that $d(\pi_{-1}(T_d) \circ \pi_1(T_d)) \geq \omega n + d$. This implies the existence of a subposet $T_{d,1} \circ \cdots \circ T_{d,d}$ of $\pi_{-1}(T_d) \circ \pi_1(T_d)$ with $d(T_{d,j}) = \omega n + 1$ for $1 \leq j \leq d$. We have $x_i \leq \pi_i(t) \leq y_i$ for all $t \in T_d$. Thus we can regard $\pi_{-1}(T_d) \circ \pi_1(T_d)$ as a subposet of $H_{-1 x_1, y_1} \circ H_{1 x_2, y_2}$. Consequently, for either $i = -1$ or $i = 1$ there exists for every $l \geq 1$ a subposet T'_l of $H_{i x_i, y_i}$ with $d(T'_l) = \omega n + l$.
Again we can write T'_l in the form $T'_{l,1} \circ \cdots \circ T'_{l,l}$ with $d(T_{l,j}) = \omega n + 1$. But at most $2|E|$ of these $T_{l,j}$ contain exceptional elements.
This implies the existence of a linear subposet T''_l of $H_{i x_i, y_i}$, which does not contain exceptional elements, with $d(T''_l) = \omega n + l$ for every $l \geq 1$. Thus there exists a subposet of density $\omega(n+1)$ in $H_{i x_i, y_i}$ which does not contain exceptional elements.
By the previous lemma we get a linear subposet of $H_{x,y}$ with density $\omega(n+1)$ which does not contain exceptional elements. This implies the existence of a linear subposet of $H_{x,y}$ with density $\omega(n+1)+1$.
Note that we can not guarantee the existence of a linear subposet of $H_{x,y}$ of density $\omega(n+1)+1$ which does not contain exceptional elements. The reason is that x or y itself can be exceptional. This finishes the proof.

3.10 From α-structures to linear subposets with density α

Theorem. *Let $x, y \in H$. Assume that $H_{x,y}$ has an $(\omega n + d)$-structure. Then there exists a linear subposet T of $H_{x,y}$ with $d(T) = \omega n + d$.*

Proof. The case $n = 0$ is trivial. Assume $n \geq 1$ and $d = 1$. Thus $H_{x,y}$ has an

$(\omega n+1)$-structure. This implies that $H_{x,y}$ has an $(\omega(n-1)+d')$-structure for every $d' \geq 1$. By induction assumption there exists for every $d' \geq 1$ a linear subposet $T_{d'}$ of $H_{x,y}$ with $d(T_{d'}) = \omega(n-1)+d'$. As a consequence of the previous lemma we get a linear subposet T of $H_{x,y}$ with $d(T) = \omega n + 1$. Next, let $d \geq 2$. Thus there exists a d-structure $H_{x_0,x_1}, \cdots, H_{x_{d-1},x_d}$ of $H_{x,y}$ such that every H_{x_{i-1},x_i} has an $(\omega n+1)$-structure. Thus there exists a linear subposet of density $\omega n + 1$ with minimal element x_{i-1} and maximal element x_i for $1 \leq i \leq d$. The concatenation of these orders gives a linear subposet T of $H_{x,y}$ with $d(T) = \omega n + d$.

The most important special case of the theorem is the following:

Corollary. *Assume that H has an $(\omega n + d)$-structure. Then there exists a linear subposet of density $\omega n + d$ in H.*

Proof. We know that $H = H_{(a_{-1},a_1),(z_{-1},z_1)}$.

Let H be a hammock poset. Remember that the set of exceptional elements of H is denoted by E. Let H' be a poset which does not have an $(\omega+1)$-structure. Let $\mathcal{H}$ be the poset obtained from H by replacing each $x \in E$ by the poset H'. We call $\mathcal{H}$ an *extended hammock poset* of H. The elements in the copies of H' are called *exceptional elements* of $\mathcal{H}$.

Remark. Given a domestic string algebra A and a simple A-module S. Then the poset $\mathcal{H}(S)$ as defined in section 2.9 is an extended hammock poset of $H(S)$.

Lemma. *Let H be a hammock poset and $\mathcal{H}$ an extended hammock poset of H. Let $x < y$ in $\mathcal{H}$ and let $\pi : \mathcal{H} \longrightarrow H$ be the canonical surjective poset map. Assume that $\mathcal{H}_{x,y}$ has an $(\omega n+d)$-structure for some $n \geq 1$. Then also $H_{\pi(x),\pi(y)}$ has an $(\omega n + d)$-structure.*

Proof. Assume that $\mathcal{H}_{x,y}$ has an $(\omega+1)$-structure. Thus $\mathcal{H}_{x,y}$ contains infinitely many elements. First, assume that x and y are exceptional. Then $\pi(x)$ and $\pi(y)$ are different exceptional elements in H. Thus $H_{\pi(x),\pi(y)}$ is infinite. Next, let either x or y be exceptional. Then also one of $\pi(x)$ and $\pi(y)$ is exceptional and the other not. Thus $H_{\pi(x),\pi(y)}$ is infinite again. Finally, assume that neither x nor y are exceptional in $\mathcal{H}$. Assume $H_{\pi(x),\pi(y)}$ is finite. Then $\mathcal{H}_{x,y}$ contains only finitely many non-exceptional elements. But this implies that $\mathcal{H}_{x,y}$ contains no exceptional element. Thus $\mathcal{H}_{x,y}$ is finite, a contradiction. Thus we showed that in all cases $H_{\pi(x),\pi(y)}$ has an $(\omega+1)$-structure. Now let $n \geq 2$ and $d=1$. Thus $\mathcal{H}_{x,y}$ has an $(\omega(n-1)+d')$-structure for every $d' \geq 1$. By induction assumption we get an $(\omega(n-1)+d')$-structure of $H_{\pi(x),\pi(y)}$ for every $d' \geq 1$. This implies that $H_{\pi(x),\pi(y)}$ has an $(\omega n+1)$-structure. The case $n \geq 1$ and $d \geq 2$ is obvious. This finishes the proof.

Corollary. *Let H be a hammock poset and $\mathcal{H}$ an extended hammock poset of H. Assume that $\mathcal{H}$ has an $(\omega n+d)$-structure. Then also H has an $(\omega n+d)$-structure.*

Proof. The case $n=0$ is obvious. For $n \geq 1$ the statement follows immediately from the previous lemma since $\mathcal{H}=\mathcal{H}_{(a_{-1},a_1),(z_{-1},z_1)}$.

4 Combinatorics II: Bridge-quivers

In this chapter we introduce the socalled bridge-quiver. It will be the main combinatorial tool to compute $d_T(h(S))$. For the whole chapter $A = kQ/\langle\rho\rangle$ denotes a string algebra.

4.1 Preliminary results

Let $\mathcal{B}(\alpha)$ be the set of bands $b_1 \cdots b_n$ with $b_1 = \alpha$. Let $b_1 \cdots b_n \in \mathcal{B}(\alpha)$. Then the letter b_n is of the form β^- where α and β are two different arrows ending at the same vertex. Define $\mathcal{B}_1(\alpha)$ as the subset of all elements in $\mathcal{B}(\alpha)$ which do not have $\beta^-\alpha$ as a substring.

Proposition. *The following statements are equivalent:*

1. *A is domestic;*
2. *The set $\mathcal{B}(\alpha)$ contains at most one element for each $\alpha \in Q_1$;*
3. *We have $\mathcal{B}_1(\alpha) = \mathcal{B}(\alpha)$ for each $\alpha \in Q_1$.*

Proof. See [R5].

Let $C = c_1 \cdots c_n$ be a string. A *sink* (respectively *peak*) of C is a substring of the form $c_i c_{i+1}$ where $c_i \in Q_1^-$ and $c_{i+1} \in Q_1$ (respectively $c_i \in Q_1$ and $c_{i+1} \in Q_1^-$) and $1 \leq i \leq n-1$.
Let $B = b_1 \cdots b_l$ be a band and $S \in Q_0$. Let m (respectively n) be the number of sinks (respectively peaks) $b_i b_{i+1}$ of B such that $e(b_{i+1}) = S$. Define $\operatorname{soc}(B)_S = m$ if $e(b_1) \neq S$, and let $\operatorname{soc}(B)_S = m+1$ if $e(b_1) = S$. Define $\operatorname{soc}(B) = \{S \in Q_0 \mid \operatorname{soc}(B)_S \neq 0\}$. Similarly, let $\operatorname{top}(B)_S = n$ and $\operatorname{top}(B) = \{S \in Q_0 \mid \operatorname{top}(B)_S \neq 0\}$.

Corollary. *Let A be domestic. Let $B_1 \not\sim B_2$ be two non-equivalent bands. Then* $\operatorname{soc}(B_1) \cap \operatorname{soc}(B_2) = \emptyset$ *and* $\operatorname{top}(B_1) \cap \operatorname{top}(B_2) = \emptyset$.

Proof. See [R5].

Corollary. *Let A be domestic. Then the number of $\sim$-equivalence classes of bands is bounded by $|Q_0|$.*

Proof. See [R5].

Corollary. *Assume $\mathcal{B}_1(\alpha) = \{B = b_1 \cdots b_n\}$ for some $\alpha \in Q_1$. Then B has no substring of the form $b_1^- b_n^-$.*

Proof. This follows easily from the last proposition and its corollaries.

Corollary. *Let A be domestic and B a band. Then $\mathrm{soc}(B)_S \leq 1$ and $\mathrm{top}(B)_S \leq 1$ for all $S \in Q_0$.*

Proof. This follows from the definition of $\mathcal{B}_1(\alpha)$ and from the previous corollary.

4.2 Definition of the bridge-quiver

Let A be domestic and let $\mathcal{B}$ be a set of representatives of $\sim_r$-equivalence classes of bands.
Let $v, w \in \mathcal{B}$. A string b is called *bridge* from v to w if the following hold:

1. We have $|b| > 0$;
2. The concatenation vbw is defined;
3. We have $b \neq b_1 u b_2$ for all $b_i \in \mathcal{S}$ and $u \in \mathcal{B}$;
4. If $b = b_1 b_2$ for some strings b_i of length ≥ 1 then $b_1 u b_2$ is not defined for all $u \in \mathcal{B}$.

The *bridge-quiver* $\Gamma_{\mathcal{B}}$ of A is defined as follows: The vertices of $\Gamma_{\mathcal{B}}$ are just the elements of $\mathcal{B}$. Let $v, w \in \mathcal{B}$. For each bridge b from v to w we draw an arrow $v \xrightarrow{b} w$.

Note that the bridge-quiver is defined with respect to a chosen set of representatives of $\sim_r$-equivalence classes of bands. Of course, the definition of $\Gamma_{\mathcal{B}}$ depends on the choice of $\mathcal{B}$. From now on we assume that a domestic string algebra is given together with a bridge-quiver $\Gamma_{\mathcal{B}}$.

This definition goes back to C.M. Ringel (listen [R4]). He defined bridge-quivers also for non-domestic string algebras and used them to distinguish domestic and non-domestic ones.

4.3 Examples

1.) Let A be the algebra (2) from section 1.2. Then we have

$$\mathcal{B} = \{\beta\alpha^{-}, \alpha\beta^{-}, \theta\eta^{-}, \eta\theta^{-}, \epsilon\delta^{-}, \delta\epsilon^{-}\}.$$

Thus the bridge-quiver $\Gamma_{\mathcal{B}}$ looks as follows:

$$\begin{tikzcd}
\beta\alpha^{-} \arrow[rr, "\gamma^{-}\epsilon^{-}"] & & \delta\epsilon^{-} \arrow[rr, "\delta\zeta"] & & \theta\eta^{-} \\
\eta\theta^{-} \arrow[rr, "\zeta^{-}\delta^{-}"] & & \epsilon\delta^{-} \arrow[rr, "\epsilon\gamma"] & & \alpha\beta^{-}
\end{tikzcd}$$

2.) For $A = GP_{2,2}$ we have $\mathcal{B} = \{\alpha\beta^{-}, \beta\alpha^{-}\}$. Then $\Gamma_{\mathcal{B}}$ consists of two single points:

$$\alpha\beta^{-} \qquad\qquad \beta\alpha^{-}$$

3.) Let A be the following string algebra:

$$\begin{tikzcd}
\circ \arrow[d, bend right, "\alpha_1"'] \arrow[d, bend left, "\alpha_2"] & \circ \arrow[l, "\alpha_3"'] \arrow[d, "\alpha_5"] & \circ \arrow[l, "\alpha_4"'] \arrow[d, "\alpha_6"] & \\
\circ & \circ & \circ \arrow[l, "\alpha_7"'] \arrow[r, "\alpha_8"] & \circ \arrow[d, bend right, "\alpha_9"'] \arrow[d, bend left, "\alpha_{10}"] \\
 & & & \circ
\end{tikzcd}$$

$$\rho = \{\alpha_2\alpha_3, \alpha_5\alpha_4, \alpha_7\alpha_6, \alpha_9\alpha_8\}$$

Then $\mathcal{B} = \{\alpha_1\alpha_2^{-}, \alpha_2\alpha_1^{-}, \alpha_9\alpha_{10}^{-}, \alpha_{10}\alpha_9^{-}\}$ and $\Gamma_{\mathcal{B}}$ has the following shape:

$$\begin{tikzcd} \alpha_1\alpha_2^- \arrow[r, bend left=10, "\alpha_1\alpha_3\alpha_4\alpha_6^-\alpha_8^-\alpha_{10}^-"] \arrow[r, bend right=10, "\alpha_1\alpha_3\alpha_5^-\alpha_7\alpha_8^-\alpha_{10}^-"'] & \alpha_9\alpha_{10}^- \end{tikzcd}$$

$$\begin{tikzcd} \alpha_{10}\alpha_9^- \arrow[r, bend left=10, "\alpha_{10}\alpha_8\alpha_6\alpha_4^-\alpha_3^-\alpha_1^-"] \arrow[r, bend right=10, "\alpha_{10}\alpha_8\alpha_7^-\alpha_5\alpha_3^-\alpha_1^-"'] & \alpha_2\alpha_1^- \end{tikzcd}$$

4.4 The shape of a bridge-quiver

Remark. It follows from the second corollary of the proposition in section 4.1 that the bridge-quiver of a domestic string algebra $A = kQ/\langle\rho\rangle$ has at most $2 \cdot |Q_0|$ vertices.

Lemma. *Let A be domestic. Let $v, w \in \mathcal{B}$. Then there are only finitely many bridges from v to w.*

Proof. Assume there are infinitely many bridges from v to w. Since there are for each m only finitely many strings of length m, there are bridges of arbitrary length. Since the length of direct and inverse strings is bounded, we know that there are bridges with arbitrary many sinks. Let n be the number of arrows of Q and b a bridge with $4n+1$ sinks. Since there exist at most $2n$ different sinks, we know that at least three of the sinks of b coincide. But this implies that b is of the form b_1BBb_2 for some band B. Here we use the fact that A is domestic. The string BB contains a substring which is $\sim_r$-equivalent to some $B' \in \mathcal{B}$. This is a contradiction to the third condition in the definition of a bridge.

Lemma. *The bridge-quiver of a domestic string algebra has no oriented cycles.*

Proof. Assume that $v_0 \xrightarrow{b_1} v_1 \xrightarrow{b_2} \cdots \xrightarrow{b_m} v_0$ is a cycle in $\Gamma_{\mathcal{B}}$. Assume further that it is of minimal length. Thus $v_i \neq v_0$ for all $i \geq 1$. Let $v = v_0b_1v_1v_1b_2\cdots b_mv_0$ in case $m > 1$ and $v = v_0b_1v_0$ else. Obviously, all powers v^n are defined. Assume $v = w^l$ for some $l \geq 2$ and some string w. Let w be of minimal length with this property. This implies that w is a band.

Since w starts with the same letter as v_0, the domesticity of A implies $w = v_0$. In case $m = 1$ we get $b_1 = v_0^{l'}$ for some $l' \geq 1$. This is a contradiction because b_1 is a bridge. Next, let $m > 1$. Then $v = v_0^l$ implies that $\mathrm{soc}(v_0)$ and $\mathrm{soc}(v_1)$ are not disjoint. In case $v_0 \not\sim v_1$ this is a contradiction to the first corollary in section 4.1. In case $v_0 = v_1^-$ we get a contradiction to the third corollary. Thus we proved that v is a band. But v and v_0 are both in $\mathcal{B}(\alpha)$ for some α and obviously $v \neq v_0$. This is a contradiction again. This finishes the proof.

Altogether, we proved that $\Gamma_{\mathcal{B}}$ is a finite oriented graph without oriented cycles. Again the previous two lemmas are similar to results of C.M. Ringel. Since they are not published up to now, we decided to give a proof here.

4.5 Generating strings with the bridge-quiver

One of the main properties of bridge-quivers is that one can use them to generate certain sets of strings.

Let v and w be vertices in $\Gamma_{\mathcal{B}}$. We call a string b a *half-bridge* starting at v (respectively ending at w) if the following hold:

1. The concatenation vb (respectively bw) is defined;
2. We have $b \neq b_1 u b_2$ for all $b_i \in \mathcal{S}$ and $u \in \mathcal{B}$;
3. If $b = b_1 b_2$ for some strings b_i of length ≥ 1 then $b_1 u b_2$ is not defined for all $u \in \mathcal{B}$.

Thus every bridge is also a half-bridge. One shows similarly as before that there are only finitely many half-bridges starting at v or ending at w.

Let $v \in \mathcal{B}$. Let $v \xrightarrow{b_1} v_1 \xrightarrow{b_2} \cdots \xrightarrow{b_m} v_m$ be a path of length $m \geq 1$ in $\Gamma_{\mathcal{B}}$ which starts in v. We say that *a string C is generated by this path* if $C = v^{n_0} b_1 v_1^{n_1} b_2 \cdots b_m v_m^{n_m} b$ where b is a half-bridge starting at v_m and $n_i \geq 0$.

Remark. If $n_i \geq 1$ for each i then $v_0^{n_0} b_1 v_1^{n_1} b_2 \cdots b_m v_m^{n_m} b$ is a string.

A path of length 0 starting at v is just the vertex v itself. The strings generated by such a path are of the form $v^n b$ where b is a half-bridge starting at

v and $n \geq 0$.

By $\mathcal{S}(v)$ we denote the set of all strings which are generated by a path starting at v.

Lemma. *Let $\Gamma_{\mathcal{B}}$ be the bridge-quiver of a domestic string algebra and $v \in \mathcal{B}$. Then $\mathcal{S}(v)$ is a linear poset with respect to $\leq_1$.*

Proof. Let b be a half-bridge starting at v. Let $v = v_1 \cdots v_m$. Then $\varepsilon(v) = \varepsilon(v_1) = -\varepsilon(v_m^-) = -\sigma(v_m) = -\sigma(v) = \varepsilon(b)$. Thus $1_{(e(v),\varepsilon(v))}b$ and $1_{(e(v),\varepsilon(v))}v$ are strings. Since every element in $\mathcal{S}(v)$ is of the form bC or vC for some string C, we get that the elements in $\mathcal{S}(v)$ are linearly ordered with respect to $\leq_1$.

4.6 Computing the density of $\mathcal{S}(v)$

The following easy lemma is often quite useful:

Lemma. *Let $C <_1 D$ be strings, and let M be a string of maximal length such that $C = MC'$ and $D = MD'$ for some strings C' and D'. Then the following hold:*

1. *If $|C'| > 0$ then $CN <_1 D$ for all strings N such that CN is a string;*
2. *If $|D'| > 0$ then $C <_1 DL$ for all strings L such that DL is a string;*
3. *If $|C'| > 0$ and $|D'| > 0$ then $CN <_1 DL$ for all strings N and L such that CN and DL are strings.*

Proof. This follows directly from the definition of $\leq_1$.

Let $v = v_1 \cdots v_n \in \mathcal{B}$. To v we associate some important subsets of $\mathcal{S}(v)$: Define

$\mathcal{M}(v) = \{b_1 \cdots b_m b \in \mathcal{S} \mid v \xrightarrow{b_1} \cdots \xrightarrow{b_m} w$ is a path in $\Gamma_{\mathcal{B}}$ and b is a half-bridge starting at $w, m \geq 1\} \cup \{b \mid b$ is a half-bridge starting at $v\}$

$\mathcal{M}'(v) = \{b_1 \cdots b_m w \in \mathcal{S} \mid v \xrightarrow{b_1} \cdots \xrightarrow{b_m} w$ is a path in $\Gamma_{\mathcal{B}}, m \geq 1\}$

$\mathcal{N}(v) = \{v^l C \in \mathcal{S} \mid C \in \mathcal{M}(v), l \geq 1\} \cup \mathcal{M}(v)$

$\mathcal{N}'(v) = \{v^l D \in \mathcal{S} \mid D \in \mathcal{M}'(v), l \geq 1\} \cup \mathcal{M}'(v)$.

Note that the sets $\mathcal{M}(v)$ and $\mathcal{M}'(v)$ are both finite. Let $\mathcal{M}(v) \cup \mathcal{M}'(v) = \{x_1 < \cdots < x_q\}$. Then there exists some $i \geq 1$ such that $x_1 < \cdots < x_i < v < x_{i+1} < \cdots < x_q$. This is a consequence of the fact that $1_{(e(v),\varepsilon(v))}$ and $v_1 \cdots v_{n-1}$ are half-bridges starting at v and $1_{(e(v),\varepsilon(v))} <_1 v <_1 v_1 \cdots v_{n-1}$.

Lemma. *Let A be domestic with bridge-quiver $\Gamma_{\mathcal{B}}$. Let $C \in \mathcal{M}(v)$ for some $v \in \mathcal{B}$, and let $w \in \mathcal{B}$. Then w is not a substring of C.*

Proof. In case C is a half-bridge starting at v the statement is trivial. Thus let $C = b_1 \cdots b_m b$ where $v \xrightarrow{b_1} \cdots \xrightarrow{b_m} v_m$ is a path in $\Gamma_{\mathcal{B}}$ and b is a half-bridge starting at v_m. Assume that $b_1 \cdots b_m b$ contains w as a substring. Without loss of generality we can assume $b_1 \cdots b_m b = b_1' v b''$ where $b_1 = b_1' b_1''$ and $b = b' b''$ such that $|b_1''| \geq 1$ and $|b'| \geq 1$. This implies $w = b_1'' b_2 \cdots b_m b'$. Since $b_1 \cdots b_m$ is a path in $\Gamma_{\mathcal{B}}$, we know that $b_1 v_1 b_2 \cdots b_m v_m v_m b$ is a string. Thus also $u = b_1'' v_1 b_2 v_2 \cdots b_m v_m v_m b'$ is a string. Obviously, all powers u^n are defined. Assume $u = y^l$ for some string y and let y be of minimal length with this property. This implies that y is a band. Since y and w are both in $\mathcal{B}(\alpha)$ for some α, we get $y = w$ because A is domestic. Since $\operatorname{soc}(y) \cap \operatorname{soc}(v_m) \neq \emptyset$, we get $y = v_m$ or $y \sim_r v_m^-$. Here we use the first corollary of the proposition in section 4.1. In the first case we get $b' = v_m^l$ which is a contradiction. In case $y \sim_r v_m^-$ we get a contradiction to the third corollary in section 4.1. Thus $b_1 \cdots b_m b$ can not contain w as a substring. This finishes the proof.

Lemma. *Let A be domestic with bridge-quiver $\Gamma_{\mathcal{B}}$. Let $C \in \mathcal{M}'(v)$ for some $v \in \mathcal{B}$. Then v is not a substring of C.*

Proof. Let $C = b_1 \cdots b_m w$ where $v \xrightarrow{b_1} \cdots \xrightarrow{b_m} w$ is a path of length ≥ 1 in $\Gamma_{\mathcal{B}}$. Assume that v is a substring of C. Since $\Gamma_{\mathcal{B}}$ does not contain an oriented cycle, we know that $v \neq w$. Let D_v be an inverse string of maximal length such that $v = v' D_v$ for some v'. Thus $v = v'' \alpha D_v$ for some string v'' and some $\alpha \in Q_1$. Furthermore, the previous lemma implies that $b_1 \cdots b_m$ does not contain v as a substring. This implies $w = w' \alpha D_v w''$ for some w' and w''. The domesticity of A and $v \neq w$ imply $|w''| \geq 1$. Thus the intersection of $\mathrm{top}(v)$ and $\mathrm{top}(w)$ is not empty which implies $v \sim_r w^-$. Thus w contains $D_v^- \alpha^-$ and αD_v as a substring. This is a contradiction to the fourth corollary in section 4.1. Thus v can not be a substring of C.

Lemma. *Let A be domestic with bridge-quiver $\Gamma_{\mathcal{B}}$. Let $C <_1 D$ be elements in $\mathcal{M}'(u)$ for some $u \in \mathcal{B}$. Let M be a string of maximal length such that $C = MC'$ and $D = MD'$ for some strings C' and D'. Then $|C'| > 0$ and $|D'| > 0$.*

Proof. Let $C = b_1 \cdots b_m v$ and $D = c_1 \cdots c_n w$ where $u \xrightarrow{b_1} \cdots \xrightarrow{b_m} v$ and $u \xrightarrow{c_1} \cdots \xrightarrow{c_n} w$ are paths in $\Gamma_{\mathcal{B}}$.
Assume $|C'| = 0$. Since $b_1 \cdots b_m v \neq c_1 \cdots c_n w$, we have $|D'| > 0$. Let D_v be an inverse string of maximal length such that $v = v' D_v$ for some string v'. Since v is a band, we know $v = v'' \alpha D_v$ for some $\alpha \in Q_1$ and some string v''. From the second lemma of this section we know that v can not be a substring of $c_1 \cdots c_n$. This implies that w is of the form $w = w_1 \alpha D_v \beta w_2$ where $\beta \in Q_1$. Thus $\mathrm{top}(v)$ and $\mathrm{top}(w)$ are not disjoint. We get $v = w$ or $v \sim_r w^-$. In both cases we get $\mathrm{top}(w)_{s(\alpha)} \geq 2$ which is a contradiction to the fourth corollary in section 4.1. Thus we proved that $|C'| > 0$.

Next, assume $|D'| = 0$. Similarly as before, let D_w be an inverse string of maximal length such that $w = w' D_w$ for some string w'. Thus $w = w'' \gamma D_w$ for some $\gamma \in Q_1$. Since w can not be a substring of $b_1 \cdots b_m$, we get $v = v_1 \gamma D_w \delta^- v_2$ for some $\delta^- \in Q_1^-$. Thus $\mathrm{top}(v)$ and $\mathrm{top}(w)$ are not disjoint. This implies $v = w$ or $v \sim_r w^-$ which is again a contradiction. Thus we proved $|D'| > 0$. Altogether, we showed that $|C'| > 0$ and $|D'| > 0$.

Proposition. *Let A be domestic with bridge-quiver $\Gamma_{\mathcal{B}}$. Let $v \in \mathcal{B}$ and let $\mathcal{M}(v)$, $\mathcal{M}'(v)$, $\mathcal{N}(v)$ and $\mathcal{N}'(v)$ be defined as above. Let $\mathcal{M}(v) \cup \mathcal{M}'(v) = \{x_1, \cdots, x_q\}$ such that $x_1 < \cdots < x_i < v < x_{i+1} < \cdots x_q$. Then the linear subposet $\mathcal{N}(v) \cup \mathcal{N}'(v)$ of $\mathcal{S}(v)$ looks as follows:*

$$x_1, \cdots, x_i, vx_1, \cdots, vx_i, vvx_1, \cdots, vvx_i, vvvx_1, \cdots$$
$$\cdots vvvx_q, vvx_{i+1}, \cdots, vvx_q, vx_{i+1}, \cdots, vx_q, x_{i+1}, \cdots, x_q.$$

Thus, regarded as a linear poset, it is isomorphic to $\mathbb{N} \cdot {}^*\mathbb{N}$.

Proof. In the first step we show $x_i <_1 vx_1$. This implies obviously $v^n x_i <_1$

$v^{n+1}x_1$ for all $n \geq 1$. Since $x_i <_1 v$ we can find a string M of maximal length such that $x_i = Mx_i'$ and $v = Mv'$ for some strings x_i' and v'. There are two cases to consider:

Case 1: Assume $x_i \in \mathcal{M}(v)$.
Let $|v'| = 0$. This implies that v is a substring of x_i which is a contradiction to the second lemma of this section. Then apply the first lemma.

Case 2: Assume $x_i \in \mathcal{M}'(v)$.
In case $|v'| = 0$ we get a contradiction to the third lemma of this section. Then again apply the first lemma.

In the second step we prove $vx_q <_1 x_{i+1}$ which implies $v^{n+1}x_q <_1 v^n x_{i+1}$ for all $n \geq 1$. Let M be of maximal length such that $v = Mv'$ and $x_{i+1} = Mx_{i+1}'$ for some v' and x_{i+1}'. Again we have to consider two cases:

Case 1: Assume $x_{i+1} \in \mathcal{M}(v)$.
In case $|v'| = 0$ we get that v is a substring of x_{i+1} which is a contradiction again.

Case 2: Assume $x_{i+1} \in \mathcal{M}'(v)$.
If $|v'| = 0$ we know that v is a substring of x_{i+1}. This is also a contradiction.

This finishes the proof.

Corollary. *Let A be domestic with bridge-quiver $\Gamma_{\mathcal{B}}$. Let $C \in \mathcal{N}'(v)$ for some $v \in \mathcal{B}$ and $D \in \mathcal{N}(v) \cup \mathcal{N}'(v)$ with $C <_1 D$. Let M be of maximal length such that $C = MC'$ and $D = MD'$ for some strings C' and D'. Then $|C'| > 0$. If $D \in \mathcal{N}'(v)$ then additionally $|D'| > 0$.*

Proof. This follows immediately from the proof of the previous proposition.

Theorem. *Let A be domestic with bridge-quiver $\Gamma_{\mathcal{B}}$. For $v \in \mathcal{B}$ let n_v be the maximum of the length of all paths in $\Gamma_{\mathcal{B}}$ which start at v. Then $d(\mathcal{S}(v)) = \omega(n_v + 1) + 1$.*

Proof. We prove the theorem by induction on n_v. For $n_v = 0$ we calculated $\mathcal{S}(v)$ already in the previous proposition. Namely $\mathcal{S}(v) = \mathcal{N}(v) = \mathbb{N} \cdot {}^*\mathbb{N}$.

This implies $d(\mathcal{S}(v)) = \omega(0+1)+1 = \omega+1$.

Next, assume $n_v \geq 1$.
Let $\mathcal{M}'(v) = \{y_1 = b_{1,1}\cdots b_{1,m_1}w_1, \cdots, y_l = b_{l,1}\cdots b_{l,m_l}w_l\}$ where $v \xrightarrow{b_{i,1}} \cdots \xrightarrow{b_{i,m_i}} w_i$ is a path in $\Gamma_{\mathcal{B}}$ for $1 \leq i \leq l$. Then the previous corollary implies that $\mathcal{S}(v)$ is obtained from $\mathcal{N}(v) \cup \mathcal{N}'(v)$ by replacing each element of the form $v^n y_i$ where $n \geq 0$ by the linear poset $v^n y_i \mathcal{S}(w_i) = \{v^n y_i C \mid C \in \mathcal{S}(w_i)\}$. Note that the concatenation $v^n y_i C$ is always defined for $C \in \mathcal{S}(w_i)$. It is easy to see that $v^n y_i \mathcal{S}(w_i)$ has a minimal and a maximal element. By the induction assumption we know that $d(\mathcal{S}(w_i)) \leq \omega n_v + 1$ for all i. We know that there is a path $v \xrightarrow{b_1} v_1 \xrightarrow{b_2} \cdots \xrightarrow{b_{n_v}} v_{n_v}$ of maximal length in $\Gamma_{\mathcal{B}}$. Without loss of generality assume $v_1 = w_1$. Thus by the induction assumption $d(\mathcal{S}(w_1)) = \omega n_v + 1$.

Let I be an infinite subset of the linear poset $\mathbb{N} \cdot {}^*\mathbb{N}$. Let $\{T_1, \cdots, T_m\}$ be linear posets with $d(T_i) \leq \omega n + 1$ and $d(T_1) = \omega n + 1$ for some n and all i. For each $i \in I$ replace the element i in $\mathbb{N} \cdot {}^*\mathbb{N}$ by one of the orders T_j. We assume additionally that T_1 is inserted infinitely many times. Denote the new order by T. Then obviously $d(T) = \omega(n+1)+1$. Since we are exactly in this situation we get $d(\mathcal{S}(v)) = \omega(n_v+1)+1$. This finishes the proof.

4.7 The extended bridge-quiver

Next, we extend the bridge-quiver as follows: Let $S \in Q_0$. We add two vertices $1_{(S,1)}$ and $1_{(S,-1)}$ to the bridge-quiver $\Gamma_{\mathcal{B}}$. Let $v \in \mathcal{B}$ and $b \in \mathcal{S}$. Then we draw an additional arrow $1_{(S,i)} \xrightarrow{b} v$ if the following hold:

1. The concatenation $1_{(S,i)}b$ is defined;
2. The string b is a half-bridge ending at v;
3. The vertex v is a source in the bridge-quiver.

We denote the resulting quiver by $\Gamma_{\mathcal{B}}(S)$.

4.8 Examples

1.) Again, let A be the algebra (2) from section 1.2. Let $S = e(\alpha)$. Assume $\varepsilon(\alpha) = -1$ and $\varepsilon(\beta) = \sigma(\gamma) = 1$. Then the extended bridge-quiver $\Gamma_{\mathcal{B}}(S)$ is of the following form:

$$\begin{tikzcd}
1_{(S,1)} \arrow[r, "1_{(S,1)}"] & \beta\alpha^- \arrow[r, "\gamma^-\epsilon^-"] & \delta\epsilon^- \arrow[r, "\delta\zeta"] & \theta\eta^- \\
\eta\theta^- \arrow[r, "\zeta^-\delta^-"] & \epsilon\delta^- \arrow[r, "\epsilon\gamma"] & \alpha\beta^- & 1_{(S,-1)} \arrow[l, "1_{(S,-1)}"']
\end{tikzcd}$$

2.) Next, let $A = GP_{2,2}$ with $\varepsilon(\alpha) = -1$ and $\sigma(\alpha) = 1$. Then $\Gamma_{\mathcal{B}}(S)$ looks as follows:

$$\begin{tikzcd}
 & \alpha\beta^- & \\
1_{(S,1)} \arrow[ur, "1_{(S,1)}"] \arrow[dr, "\alpha^-"'] & & 1_{(S,-1)} \arrow[ul, "\beta^-"'] \arrow[dl, "1_{(S,-1)}"] \\
 & \beta\alpha^- &
\end{tikzcd}$$

4.9 Generating strings with the extended bridge-quiver

For $i \in \{-1, 1\}$ let $1_{(S,i)} \xrightarrow{b_1} v_1 \xrightarrow{b_2} v_2 \xrightarrow{b_3} \cdots \xrightarrow{b_m} v_m$ be a path of length $m \geq 1$ in $\Gamma_{\mathcal{B}}(S)$ starting at $1_{(S,i)}$. Similar as for $\Gamma_{\mathcal{B}}$, we say that *a string C is generated by this path* if $C = b_1 v_1^{n_1} b_2 v_2^{n_2} b_3 \cdots b_m v_m^{n_m} b$ where b is a half-bridge starting at v_m and $n_i \geq 0$. A path of length 0 starting at $1_{(S,i)}$ is the vertex $1_{(S,i)}$ itself. The strings generated by $1_{(S,i)}$ are strings C such that $1_{(S,i)}C$ is a string and the second and third condition in the definition of half-bridge are satisfied by C. Call these strings *half-bridges starting at* $1_{(S,i)}$. The strings generated by paths starting at $1_{(S,i)}$ are denoted by $\mathcal{S}(1_{(S,i)})$.

Lemma. *We have $\mathcal{S}(1_{(S,1)}) = h(S)_1$ and $\mathcal{S}(1_{(S,-1)}) = \{C^- \mid C \in h(S)_{-1}\}$.*

Proof. Obvious

4.10 Computing $d_T(h(S))$

Let n_S be the maximal length of a path in $\Gamma_{\mathcal{B}}(S)$ which starts at $1_{(S,1)}$ or $1_{(S,-1)}$. Let d_S be the number of arrows b in $\Gamma_{\mathcal{B}}(S)$ which start at $1_{(S,1)}$ or $1_{(S,-1)}$ such that there exists a path of length n_S with first arrow b.

Theorem. *Let A be domestic and S a simple A-module. Assume $n_S \geq 1$. Then $d_T(h(S)) = \omega n_S + d_S$.*

Proof. In section 3.9 we proved that $d_T(h(S)) = d(h(S)_{-1} \circ h(S)_1)$. Note that $d(h(S)_{-1}) = d(\mathcal{S}(1_{(S,-1)}))$ and $d(h(S)_1) = d(\mathcal{S}(1_{(S,1)}))$. For $i \in \{-1,1\}$ the computation of $d(\mathcal{S}(1_{(S,i)}))$ is done in a similar way as the computation of $d(\mathcal{S}(v))$ for $v \in \Gamma_{\mathcal{B}}$. Define

$\mathcal{M}(1_{(S,i)}) = \{b_1 \cdots b_m b \in \mathcal{S} \mid 1_{(S,i)} \xrightarrow{b_1} \cdots \xrightarrow{b_m} w$ is a path in $\Gamma_{\mathcal{B}}(S)$ and b is a half-bridge starting at $w, m \geq 1\} \cup \{b \mid b$ is a half-bridge starting at $1_{(S,i)}\}$

$$\mathcal{M}'(1_{(S,i)}) = \{b_1 \cdots b_m w \in \mathcal{S} \mid 1_{(S,i)} \xrightarrow{b_1} \cdots \xrightarrow{b_m} w \text{ is a path in } \Gamma_{\mathcal{B}}, m \geq 1\}.$$

Then $\mathcal{S}(1_{(S,i)})$ is obtained from $\mathcal{M}(1_{(S,i)}) \cup \mathcal{M}'(1_{(S,i)})$ by replacing each element $b_1 \cdots b_m w \in \mathcal{M}'(1_{(S,i)})$ by the linear poset $\{b_1 \cdots b_m w C \mid C \in \mathcal{S}(w)\}$. By $n_S(i)$ we denote the maximal length of a path starting at $1_{(S,i)}$. Let $\{1_{(S,i)} \xrightarrow{b_1} w_1, \cdots, 1_{(S,i)} \xrightarrow{b_1} w_{d_S(i)}\}$ be the set of arrows b_i starting at $1_{(S,i)}$ such that there exists a path of length $n_S(i)$ with first arrow b_i. Then the last theorem states that $d(\mathcal{S}(w_i)) = \omega n_S(i) + 1$. This implies $d(\mathcal{S}(1_{(S,i)}) = \omega n_S(i) + d_S(i)$. From $d(\mathcal{S}(1_{(S,-1)}) \circ \mathcal{S}(1_{(S,1)})) = d_T(h(S))$ we get $d_T(h(S)) = \omega n_S + d_S$. This finishes the proof.

Remark. In case $n_S = 0$ we get $d_T(h(S)) = d < \omega$ where d can be computed easily since $h(S)$ is finite.

Remark. Computing the quiver $\Gamma_{\mathcal{B}}(S)$ is quite easy. Thus the theorem gives an efficient method to compute $d_T(h(S))$. We also get the value $d_T(H(S))$ up to the finite error term $|\mathcal{B}(S)|$.

This finishes the combinatorial part of the thesis. The next chapter will show how the results, which we obtained so far, can be interpreted in terms of representation theory.

5 Representation theoretic interpretation

The aim of this chapter is to give a representation theoretic interpretation of the combinatorics which we did so far. We recall the definition of arbitrary ordinal powers of rad_A and give a recipe to compute the degree of nilpotency of rad_A in case A is a string algebra.

5.1 Definition of the transfinite radical

Let A be an algebra. An *ideal* $\mathcal{I}$ in A-mod is a class of morphisms which is an abelian group with respect to sums of morphisms, and for f in $\mathcal{I}$ we have ff' and $f''f$ are in $\mathcal{I}$ for all morphisms f' and f'' such that the compositions are defined. For $n \geq 1$ define

$$\mathcal{I}^n = \{\sum_{i=1}^{l} f_n^{(i)} \cdots f_1^{(i)} \mid f_j^{(i)} \in \mathcal{I}, l \geq 1\}.$$

Furthermore, let $\mathcal{I}^0 = A$-mod.

Let X and Y be indecomposable A-modules. By $\mathrm{rad}_A(X,Y)$ we denote the set of non-isomorphisms from X to Y. If $X = \bigoplus_{i=1}^{n} X_i$ and $Y = \bigoplus_{j=1}^{m} Y_j$, where X_i and Y_j are indecomposable A-modules, then any A-module homomorphism $f : X \longrightarrow Y$ can be written in the form $f = (f_{ij})_{i,j}$ with $f_{ij} \in \mathrm{Hom}_A(X_i, Y_j)$. Then, by definition, f belongs to $\mathrm{rad}_A(X,Y)$ if $f_{ij} \in \mathrm{rad}_A(X_i, Y_j)$ for all i and j. By rad_A we denote the ideal of all homomorphisms which lie in $\mathrm{rad}_A(X,Y)$ for some X, Y in A-mod. We call rad_A the *radical* of A-mod.
Let β be a limit number. Define

$$\mathrm{rad}_A^{\beta} = \bigcap_{\alpha<\beta} \mathrm{rad}_A^{\alpha}.$$

Next, let α be an arbitrary infinite ordinal. Thus $\alpha = \beta + n$ for some limit number β and some finite ordinal n. In this case, define

$$\mathrm{rad}_A^{\alpha} = (\mathrm{rad}_A^{\beta})^{n+1}.$$

Finally, let

$$\mathrm{rad}_A^{\infty} = \bigcap_{\alpha} \mathrm{rad}_A^{\alpha}$$

where we intersect over all ordinal powers of rad_A. We call rad_A^∞ the *transfinite radical* of A-mod.

Remark. What is usually called the *infinite radical* of A-mod is in our notation just rad_A^ω and not rad_A^∞. In particular, we have the following:

$$\begin{aligned}
\mathrm{rad}_A^{\omega 2} &= \bigcap_{d\in\mathbb{N}} (\mathrm{rad}_A^\omega)^d \\
\mathrm{rad}_A^{\omega 3} &= \bigcap_{d\in\mathbb{N}} (\mathrm{rad}_A^{\omega 2})^d \\
&\cdots \\
\mathrm{rad}_A^{\omega^2} &= \bigcap_{n,d\in\mathbb{N}} (\mathrm{rad}_A^{\omega n})^d
\end{aligned}$$

We will mainly investigate radical powers rad_A^α where $\alpha < \omega^2$.

We call rad_A *transfinite nilpotent* if $\mathrm{rad}_A^\infty = 0$. Let $\mathrm{nil}(A) = \infty$ if $\mathrm{rad}_A^\alpha \neq 0$ for all ordinals α, and let $\mathrm{nil}(A) = \min\{\alpha \mid \mathrm{rad}_A^\alpha = 0\}$, else. Let f be an A-module homomorphism. Define $\mathrm{nil}(f) = \infty$ if $f \in \mathrm{rad}_A^\alpha$ for all α, and let $\mathrm{nil}(f) = \min\{\alpha \mid f \notin \mathrm{rad}_A^\alpha\}$, else. Thus $\mathrm{nil}(f) = \infty$ if and only if $f \in \mathrm{rad}_A^\infty$.

Remark. Note that $\mathrm{nil}(f)$ is never a limit number.

References for the definition of the transfinite radical are given in the appendix.

5.2 Basic properties of the transfinite radical

The following question is due to A. Skowroński (see [S]): Assume

$$\mathrm{rad}_A^{\omega 2} = \bigcap_{m\geq 1} (\mathrm{rad}_A^\omega)^m = 0.$$

Does there exist an $n \geq 1$ such that

$$(\mathrm{rad}_A^\omega)^n = 0\ ?$$

A reformulation of this question is: Does there exist some algebra A such that $\mathrm{nil}(A) = \omega 2$?
The following proposition answers this question:

Proposition. *Let A be an algebra such that* $\mathrm{rad}_A^\infty = 0$. *Then there exists an ordinal α such that* $\mathrm{rad}_A^\alpha \neq 0$ *and* $\mathrm{rad}_A^{\alpha+1} = 0$.

Proof. For each simple A-module S let $f_S : P \longrightarrow I$ be the canonical map from the projective cover of S to the injective envelope of S with image S. Since $\mathrm{rad}_A^\infty = 0$, we know that $\mathrm{nil}(f_S) < \infty$. We know also that $\mathrm{nil}(f_S)$ is not a limit number. Define $\mathrm{nil}'(A) = \max\{\mathrm{nil}(f_S) \mid S \in Q_0\}$. Let $\beta > \mathrm{nil}'(A)$ such that $\mathrm{rad}_A^\beta \neq 0$. Let $f : X \longrightarrow Y$ be a non-zero homomorphism in rad_A^β. Assume that S is a composition factor of the image of f. Since P is projective and I injective, we can find homomorphisms $f' : P \longrightarrow X$ and $f'' : Y \longrightarrow I$ such that $f_S = f''ff'$. But this implies $\mathrm{nil}(f_S) \geq \mathrm{nil}(f) \geq \beta$ which is a contradiction.

Corollary. *Let A be an algebra. Then the following hold:*

1. *We have* $\mathrm{rad}_A^\infty = 0$ *if and only if* $\mathrm{nil}(A) < \infty$*;*
2. *If* $\mathrm{nil}(A) < \infty$ *then* $\mathrm{nil}(A)$ *is not a limit number;*
3. *We have* $\mathrm{nil}(A) = \max\{\mathrm{nil}(f_S) \mid S \in Q_0\}$.

Proof. This follows directly from the proof of the above proposition.

In particular, we get $\mathrm{nil}(A) \neq \omega 2$ for every algebra A.

Call an algebra A *representation-finite* if A-mod contains only finitely many isomorphism classes of indecomposable modules.

Theorem (F.U. Coelho, E.N. Marcos, H.A. Merklen, A. Skowroński). *An algebra A is representation-finite if and only if* $\mathrm{rad}_A^{\omega+1} = 0$.

Proof. See [CMMS].

Corollary. *For every algebra A we have* $\mathrm{nil}(A) \neq \omega + 1$.

Proof. Assume $\mathrm{nil}(A) = \omega + 1$. Then $\mathrm{rad}_A^\omega \neq 0$ and $\mathrm{rad}_A^{\omega+1} = 0$. But obviously an algebra A with $\mathrm{rad}_A^\omega \neq 0$ is not representation-finite. This is a contradiction to the previous theorem.

The following proposition and its corollary seem to be well known:

Proposition. *Let A be an algebra. Then there exists some ordinal α such that* $\mathrm{rad}_A^\alpha = \mathrm{rad}_A^\infty$.

Proof. For set theoretic reasons we have to choose representatives of isoclasses of A-modules. Then it is easy to see that the A-module homomorphisms form a set. By the well-ordering theorem every set can be well-ordered. Furthermore, it is well known that for every well-ordered set there exists an ordinal number with the same cardinality. See [Ro] for details. In case rad_A and the ordinal α have the same cardinality, we get $\mathrm{rad}_A^\alpha = \mathrm{rad}_A^\infty$. This finishes the proof.

Corollary. *Let A be an algebra. Then* $\mathrm{rad}_A^\infty \neq 0$ *if and only if there exists a non-zero ideal $\mathcal{I}$ in* rad_A *such that* $\mathcal{I}^2 = \mathcal{I}$.

Proof. If $\mathrm{rad}_A^\infty = 0$ then there exists some α such that $\mathrm{rad}_A^\alpha = 0$. Let $\mathcal{I}$ be a non-zero ideal in rad_A. Then also $\mathcal{I}^\alpha = 0$ which implies that $\mathcal{I}$ can not be idempotent. Next, assume $\mathrm{rad}_A^\infty \neq 0$. Then by the proposition there exists an α such that $\mathrm{rad}_A^\alpha = \mathrm{rad}_A^\infty$. We have $\mathrm{rad}_A^\infty = \mathrm{rad}_A^{\alpha+1} = (\mathrm{rad}_A^\alpha)^2 = (\mathrm{rad}_A^\infty)^2$. Thus rad_A^∞ is an idempotent ideal. This finishes the proof.

Corollary. *Let A be an algebra. Then* $(\mathrm{rad}_A^\infty)^2 = \mathrm{rad}_A^\infty$*, and every other idempotent ideal in* rad_A *is contained in* rad_A^∞.

Proof. This follows directly from the previous corollary and its proof.

5.3 Factorization systems

Let T be a linear poset with minimal element a and maximal element z. To every $t \in T$ we associate an A-module M_t, and to every $t < t'$ in T we associate a non-zero A-module homomorphism $f_{t,t'} : M_t \longrightarrow M_{t'}$ which is in rad_A. Assume that for $t_1 < t_2 < t_3$ in T we have

$$f_{t_2,t_3} f_{t_1,t_2} = f_{t_1,t_3}.$$

Then we call (T, f) together with the associated modules and homomorphisms a *factorization system* of $f = f_{a,z} : M_a \longrightarrow M_z$.

Remark. In [Pr] M. Prest introduced socalled factorizable systems. In contrast to our definition, he does not assume that T has a minimal and a maximal element. In fact, he takes always $T = \mathbb{Q}$ and assumes not that the homomorphisms occurring in the factorizable system are in the radical of A-mod.

Proposition. *Let A be an algebra, and let (T, f) be a factorization system of some A-module homomorphism f such that $d(T) = \omega n + d$. Then $f \in \mathrm{rad}_A^{\omega n+d-1}$ if $n \geq 1$, and $f \in \mathrm{rad}_A^d$, else.*

Proof. For $n = 0$ the statement is trivial. Thus let $n \geq 1$. First, assume $d = 1$.
Thus $T^{(n-1)}$ is of one of the following forms:

1. $\mathbb{N} \cdot {}^*\mathbb{N} = \{x_0 < x_1 < x_2 < \cdots\} \cdot {}^*\mathbb{N}$
2. $\mathbb{N} \cdot \{y_0 < \cdots < y_m\} = \{x_0 < x_1 < x_2 < \cdots\} \cdot \{y_0 < \cdots < y_m\}$
3. $\{y_0 < \cdots < y_m\} \cdot {}^*\mathbb{N} = \{y_0 < \cdots < y_m\} \cdot \{\cdots < x_2 < x_1 < x_0\}$

Assume we are in case 1 or 2. Let $a = t_0 = \min T$ and $z = \max T$. Choose some $t_i \in [x_i]^{-(n-1)}$ for every $i \geq 1$. Then $T = T_{a,t_1} \circ T_{t_1,t_2} \cdots T_{t_{l-1},t_l} \circ T_{t_l,z}$ for every $l \geq 1$. This implies $f = f_{t_l,z} f_{t_{l-1},t_l} \cdots f_{t_0,t_1}$. Furthermore, we have $d(T_{t_i,t_{i+1}}) = \omega(n-1) + 1$ for $0 \leq i \leq l-1$. Thus by induction assumption we get $f_{t_i,t_{i+1}} \in \mathrm{rad}_A^{\omega(n-1)}$ for $0 \leq i \leq l-1$. This implies $f \in \mathrm{rad}_A^{\omega(n-1)+l-1}$. But l can be chosen arbitrary large. Thus $f \in \mathrm{rad}_A^{\omega n}$.
The third case is handled in a similar way. The generalization to $d \geq 2$ is also trivial. This finishes the proof.

Theorem (M. Prest). *Let A be an algebra. Assume there is a factorization system (T, f) of some A-module homomorphism f with $T = \mathbb{Q}_0^+ \cup \{\infty\}$. Then $f \in \mathrm{rad}_A^\infty$.*

Proof. See [Pr].

5.4 A first interpretation of the posets $h(S), H(S)$ and $\mathcal{H}(S)$

In [CB2] and [K1] W.W. Crawley-Boevey and H. Krause construct basis elements for $\mathrm{Hom}_A(X, Y)$ where X and Y are indecomposable modules over some string algebra A. The aim of this section is to show some relations between these results and the posets $h(S)$, $H(S)$ and $\mathcal{H}(S)$ which we defined in the second chapter.

For the rest of this section let $A = kQ/\langle\rho\rangle$ be a string algebra with hammocks $h(S)$, $H(S)$ and $\mathcal{H}(S)$ for each simple A-module S.

We start by associating an indecomposable A-module to each element in the hammock $\mathcal{H}(S)$. Define

$$\theta_S : \mathcal{H}(S) \longrightarrow A\text{-mod}$$

as follows: Let $x = (X', X'') \in h(S) \subseteq \mathcal{H}(S)$. Then let

$$\theta_S(x) = M(X'X'').$$

Next, let $x \in \mathcal{H}(S) \setminus h(S)$. Thus $x = (m_1, m_2) \in B(\lambda) = {}^*\mathbb{N} \times \mathbb{N}$ for some $B = ({}^\infty(B', B'')B', B''(B', B'')^\infty) \in \mathcal{B}(S)$ and $\lambda \in k^*$. Define

$$\theta_S(x) = M(B'B'', \lambda, m_1 + m_2 - 1).$$

Proposition. *Let x be a non-projective element in the translation quiver $\mathcal{H}(S)$. Then $\theta_S(\tau_{\mathcal{H}(S)}(x)) = \tau_A(\theta_S(x))$.*

Proof. This follows immediately from the construction of the irreducible maps of a string algebra, which we described in the first chapter, and the corollary in section 2.5.

Roughly speaking, the hammock $\mathcal{H}(S)$ is constructed by sewing Auslander-Reiten components or parts of Auslander-Reiten components. But note that one component or parts of it may occur several times in the hammock.

Let $x \in \mathcal{H}(S)$. First, assume $x = (X', X'') \in h(S) \subseteq \mathcal{H}(S)$. Thus $\theta_S(x) = M(X'X'')$ is a string module. Let $\{z_0, \cdots, z_{|X'X''|}\}$ be the canonical basis of $M(X'X'')$. Then define $z_x = z_{|X'|}$. Next, let $x \in \mathcal{H}(S) \setminus h(S)$. Thus $x = (m_1, m_2) \in B(\lambda)$ for some $B = ({}^\infty(B', B'')B', B''(B', B'')^\infty) \in \mathcal{B}(S)$ and $\lambda \in k^*$. Let $\{z_1^{(i)}, \cdots, z_n^{(i)} \mid 1 \leq i \leq m_1 + m_2 - 1\}$ be the canonical basis of $M(B'B'', \lambda, m_1 + m_2 - 1)$. Define $z_x = z_{|B'|}^{(m_1)}$.

Altogether, we associated to each $x \in \mathcal{H}(S)$ a module $\theta_S(x)$ and a canonical basis vector z_x of $\theta_S(x)$.

Let $x \leq y$ be in $\mathcal{H}(S)$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We have $x_1 \leq_{-1} y_1$ and $x_2 \leq_1 y_2$. Thus let M and N be strings of maximal length such that $x_1 = x_1'M$, $y_1 = y_1'M$, $x_2 = Nx_2'$ and $y_2 = Ny_2'$ for some strings x_1', x_2', y_1' and y_2'. Then let U be the string $M \cdot N$. Let σ (respectively τ) be the quiver homomorphism which embeds the string U in the obvious way into the (possibly infinite) string associated to x (respectively y). Note that we associate to an element $x \in \mathcal{H}(S) \setminus h(S)$ an infinite string whereas H. Krause interprets it as a cyclic string. But this causes no difficulties. It is obvious that we get an admissible triple (U, σ, τ) in the sense of [K1] in this way. The vertex a_0 of U, which appears in the proof of the main theorem in [K1], is just the vertex between the substrings M and N. Then, according to [CB2] and [K1], one gets an A-module homomorphism

$$f_{x,y}^S : \theta_S(x) \longrightarrow \theta_S(y)$$

which is induced from the admissible tripel (U, σ, τ) such that $f_{x,y}^S(z_x) = z_y$.

Let X and Y be indecomposable A-modules. Define $B(X, Y) = \{f_{x,y}^S \mid x \leq y$ in $\mathcal{H}(S)$ and $\theta_S(x) = X, \theta_S(y) = Y, S \in Q_0\}$. We call $B(X, Y)$ the set of canonical maps from X to Y. In particular, we have $f_S = f_{(a_{-1},a_1),(z_{-1},z_1)}^S \in B(P, I)$ where P is the projective cover and I the injective envelope of S.

Remark. In [CB2] W.W. Crawley-Boevey constructs a basis for $\mathrm{Hom}_A(X, Y)$ where X and Y are tree modules over a monomial algebra. This result is generalized in [K1] by H. Krause. As a main reference we use [K1]. Note that string modules are special tree modules.

Theorem (W.W. Crawley-Boevey, H.Krause). *Let X, Y and $B(X, Y)$ be defined as above. Then the following hold:*

1. *The homomorphisms in $B(X, Y)$ generate $\mathrm{Hom}_A(X, Y)$ regarded as a k-vectorspace;*

2. *Let $x \leq y \leq z$ in $\mathcal{H}(S)$. Then $f_{y,z}^S f_{x,y}^S = f_{x,z}^S$;*

3. *Let $f_{x_i,y_i}^{S_i}$ be a non-invertible canonical map for $1 \leq i \leq n$ such that $\theta_{S_i}(y_i) = \theta_{S_{i+1}}(x_{i+1})$ for $1 \leq i \leq n-1$. Assume that the image of the*

composition

$$f^{S_n}_{x_n,y_n}\cdots f^{S_2}_{x_2,y_2}f^{S_1}_{x_1,y_1}$$

contains a composition factor of the form S. Then there exists a chain $z_1<\cdots<z_n$ in $\mathcal{H}(S)$ such that $\theta_S(z_i)=\theta_{S_i}(x_i)$ for $1\le i\le n-1$ and $\theta_S(z_n)=\theta_{S_n}(y_n)$.

Proof. Interpret any canonical map as an admissible triple (U,σ,τ) in the way we described before. Then use the main result in [K1] and its proof.

Remark. Let A be a string algebra, S a simple A-modules and $x\le y$ in $\mathcal{H}(S)$. Then every linear subposet of $\mathcal{H}_{x,y}(S)$ gives rise to a factorization system of the canonical map $f^S_{x,y}$. In particular, every linear subposet of $\mathcal{H}(S)$ induces a factorization system of f_S.

5.5 The transfinite radical of a non-domestic string algebra

Let $A=kQ/\langle\rho\rangle$ be a non-domestic string algebra. In section 4.1 we saw that there exists some $\alpha\in Q_1$ such that $\mathcal{B}_1(\alpha)$ contains more than one element. Let $B=b_1\cdots b_n$ and $C=c_1\cdots c_m$ be two different elements in $\mathcal{B}_1(\alpha)$. Assume $B<_1 C$.

Remark. Let M be of maximal length such that $B=MB'$ and $C=MC'$ for some strings B' and C'. Then we have $|B'|>0$.

Namely assume $|B'|=0$. Then $C=b_1\cdots b_nc_{n+1}\cdots c_m$ with $c_{n+1}=b_1=\alpha$. But this is not allowed according to the definition of $\mathcal{B}_1(\alpha)$.

Let $\mathcal{W}_{B,C}$ be the set of finite words in the letters B and C. Define $\mathcal{W}=BC\mathcal{W}_{B,C}C=\{BCWC\mid W\in\mathcal{W}_{B,C}\}$. The set $\mathcal{W}$ is linearly ordered with respect to $\le_1$.

Lemma. *Let $x<_1 z$ be elements in $\mathcal{W}$. Then there exists some $y\in\mathcal{W}$ such that $x<_1 y<_1 z$.*

Proof. Let $x=x_1\cdots x_n$ and $z=z_1\cdots z_m$ where $x_i,z_j\in\{B,C\}$ for all i and j. Let l be maximal such that $x_i=z_i$ for all $1\le i\le l$.
Assume first $l<n$ and $l<m$. This implies $x_{l+1}=B$ and $z_{l+1}=C$. Define $y=x_1\cdots x_nC$. Then obviously $x<_1 y$. Because of the previous remark we

also get $y <_1 z$.
Next, let $l = n$ and $l < m$. Thus $z = x_1 \cdots x_n z_{n+1} \cdots z_m$. Define $y = x_1 \cdots x_n z_{n+1} \cdots z_{m-1} BC$. Then obviously $x <_1 y$. Since $z_m = C$, we also get $y <_1 z$.
Finally, assume $l < n$ and $l = m$. This implies $x = z_1 \cdots z_m x_{m+1} \cdots x_n$. Thus $x = z_1 \cdots z_m \alpha x'$ for some string x'. This is a contradiction to $x <_1 z$.
The case $l = n$ and $l = m$ is of course also not possible, since this would imply $x = z$. This finishes the proof.

We call a linear poset T *dense* if for $x < z$ in T there exists always some $y \in T$ such that $x < y < z$. There is the following well known fact about linear posets:

Lemma. *Any infinite dense linear poset contains* $\mathbb{Q}$ *as a subposet.*

Proof. See [Ro].

Corollary. *The linear poset* $\mathcal{W}$ *contains* $\mathbb{Q}$ *as a subposet.*

Proof. Combine the last two lemmas.

Proposition. *Let A be a non-domestic string algebra. Then there exists some non-zero A-module homomorphism $f : X \longrightarrow X$ where X is indecomposable and $f \in \mathrm{rad}_A^\infty$. In particular, we have* $\mathrm{rad}_A^\infty \neq 0$.

Proof. Define $\mathcal{W}$ as before. Let $S = e(\alpha)$. Assume $\varepsilon(\alpha) = 1$. Then we can embed $\mathcal{W}$ into the hammock $h(S)$ as follows: $\theta : \mathcal{W} \longrightarrow h(S), w \mapsto (1_{(S,1)}, w)$. Denote the image of θ by T. Note that $(1_{(S,1)}, BBC) < t < (BB, C)$ for all $t \in T$. Here $<$ denotes the order of $h(S)$. This gives rise to a factorization system of the canonical map $f = f_{(1_{(S,1)},BBC),(BB,C)} : M(BBC) \longrightarrow M(BBC)$ with underlying linear order $\mathbb{Q}_0^+ \cup \{\infty\}$. The theorem of M. Prest implies that f lies in the transfinite radical of A-mod. This finishes the proof.

Remark. C.M. Ringel proved this proposition by constructing some idempotent ideal in the radical of A-mod (listen [R4]). However, up to our knowledge there does not exist a written version of this. For the special case $A = GP_{n,m}$ this proposition was proved by M. Prest in [Pr]. It seems to be new that one gets always a non-zero endomorphism of some indecomposable module which lies in rad_A^∞.

5.6 The representation-type of an algebra

Let A be an algebra. Let $k[T]$ be the polynomial ring in one variable, and let $\{S(\lambda) \mid \lambda \in k\}$ be the set of simple $k[T]$-modules. Then A is called *tame* if for each $d \geq 0$ there exist finitely many A-$k[T]$-bimodules $M_1, \cdots, M_{d'}$, which are free and of finite rank as $k[T]$-modules, such that all but finitely many indecomposable A-module of dimension d are of the form $M_i \otimes_{k[T]} S(\lambda)$ for some $1 \leq i \leq d'$ and some $\lambda \in k$. This defines a map

$$\mu_A : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$$

where $\mu_A(d)$ is the minimal number of bimodules which are needed to get almost all indecomposable A-modules of dimension d in the described way.

Call A *domestic* if $\mu_A(d) \leq c$ for all d and some constant $c = c(A)$. Note that the definition of domesticity of a string algebra as presented in section 1.7 coincides in this case.

Call A to be of *polynomial growth* if there exists some $c = c(A)$ such that $\mu_A(d) \leq d^c$ for all d.

Let $k\langle X, Y\rangle$ be the polynomial ring in two non-commuting variables. We call A *wild* if there exists an A-$k\langle X, Y\rangle$-bimodule M, which is free and of finite rank as a $k\langle X, Y\rangle$-module, such that the functor

$$F = M \otimes_{k\langle X,Y\rangle} - : k\langle X, Y\rangle\text{-mod} \longrightarrow A\text{-mod}$$

preserves isoclasses and indecomposables. In particular, this implies that F is faithful and exact.

It is well known that there exists a full, faithful and exact functor

$$G : B\text{-mod} \longrightarrow k\langle X, Y\rangle\text{-mod}$$

for every k-algebra B. Thus we get an exact functor $FG : B\text{-mod} \longrightarrow A\text{-mod}$ which preserves isoclasses and indecomposables. In particular, FG is faithful.

Theorem (W.W. Crawley-Boevey, Yu.A. Drozd). *Let A be an algebra. Then A is either tame or wild.*

Proof. See [CB1] and [Dr].

The next proposition seems to be well known:

Proposition. *Let A be an algebra. If A is wild then* $\mathrm{rad}_A^\infty \neq 0$.

Proof. Suppose A is wild. Let B be a non-domestic string algebra and $(T = \mathbb{Q}_0^+ \cup \{\infty\}, f)$ a factorization system of some B-module homomorphism $f : X \longrightarrow X$ where X is indecomposable. Note that such a homomorphism was constructed in section 5.5. Then construct a functor FG : B-mod $\longrightarrow$ A-mod as described above. This functor maps the factorization system (T, f) again onto a factorization system $(T, FG(f))$ of the A-module homomorphism $FG(f) : FG(X) \longrightarrow FG(X)$. Then the theorem of M. Prest from section 5.3 implies $\mathrm{rad}_A^\infty \neq 0$. This finishes the proof.

5.7 Main results

In this section we finally apply the results of the previous chapters in order to compute $\mathrm{nil}(A)$ where A is a string algebra.

For the rest of this section let A be a string algebra with hammocks $h(S)$, $H(S)$ and $\mathcal{H}(S)$ for each simple A-module S.

Proposition. *For $1 \leq i \leq d$ let $f_i : X_{i-1} \longrightarrow X_i$ be an A-module homomorphism such that the following hold:*

1. *The module X_i is indecomposable for each i;*
2. *We have $f_i \in \mathrm{rad}_A$ for each i;*
3. *The image of $f_d \cdots f_1$ has a composition factor of the form S.*

Then there exists a chain $x_0 < \cdots < x_d$ in $\mathcal{H}(S)$ such that $\theta_S(x_i) = X_i$.

Proof. Since the set $B(X_{i-1}, X_i)$ generates $\mathrm{Hom}_A(X_{i-1}, X_i)$ as a k-vectorspace, we can write f_i in the form

$$f_i = \sum_{j=1}^{l_i} \lambda_j f_{ji}$$

where $f_{ij} \in B(X_{i-1}, X_i)$ for all i and j. But then for every i there exists some $1 \leq j_i \leq l_i$ such that

$$f_{j_d d} \cdots f_{j_1 1} \neq 0,$$

and the image of $f_{j_d d} \cdots f_{j_1 1}$ contains a composition factor of the form S. Then apply the theorem in section 5.4.

Theorem. *Let $n \geq 1$. For $0 \leq i \leq d$ let $f_i : X_{i-1} \longrightarrow X_i$ be an A-module homomorphism such that the following hold:*

1. *The module X_i is indecomposable for each i;*
2. *We have $f_i \in \mathrm{rad}_A^{\omega n}$;*
3. *The image of $f_d \cdots f_1$ has a composition factor of the form S.*

Then there exists a chain $x_0 < \cdots < x_d$ in $\mathcal{H}(S)$ such that $\theta_S(x_i) = X_i$ and $\mathcal{H}_{x_{i-1},x_i}$ has an $(\omega n + 1)$-structure for all i.

Proof. Let $d = 1$. By induction assumption for every $d' \geq 1$ there exist $x_0 < x_1$ in $\mathcal{H}(S)$ such that $\mathcal{H}_{x_0,x_1}$ has an $(\omega(n-1) + d')$-structure and $\theta_S(x_0) = X_0$ and $\theta_S(x_1) = X_1$. Note that there are only finitely many pairs (x_0, x_1) in $\mathcal{H}(S) \times \mathcal{H}(S)$ such that $\theta(x_0) = X_0$ and $\theta_S(x_1) = X_1$. It follows that there exists such a pair such that H_{x_0,x_1} has an $(\omega(n-1) + d')$-structure for every $d' \geq 1$. This implies that H_{x_0,x_1} has also an $(\omega n + 1)$-structure. Note that the case $n = 1$ follows from the previous proposition.
Next, let $d \geq 2$. Since $f_i \in \mathrm{rad}_A^{\omega n}$, we can find for every $d' \geq 1$ homomorphisms $f_{i,j} : X_{i,j-1} \longrightarrow X_{i,j}$ where $0 \leq i \leq d$ and $1 \leq j \leq d'$ such that the following hold:

1. The module $X_{i,j}$ is indecomposable for all i and j;
2. We have $f_{i,j} \in \mathrm{rad}_A^{\omega(n-1)}$ for all i and j;
3. We have $X_{i+1,0} = X_{i,d'}$ and $X_{i,0} = X_i$ for all i;

4. The image of $f_{d,d'} \cdots f_{d,1} f_{d-1,d'} \cdots f_{d-1,1} \cdots f_{1,d'} \cdots f_{1,1}$ contains a composition factor of the form S.

Since $f_{j,d'} \cdots f_{j,1}$ is in $\mathrm{rad}_A^{\omega(n-1)+d'-1}$ (respectively in $\mathrm{rad}_A^{d'}$ in case $n = 1$) for all j, we know that there exists a chain $x_0 < \cdots < x_d$ in $\mathcal{H}(S)$ such that $\theta_S(x_i) = X_i$ and $\mathcal{H}_{x_{i-1},x_i}$ has an $(\omega(n-1) + d')$-structure.
Since there are only finitely many tupels $(x_0, \cdots x_d)$ with this property, we get that for one of them we have an $(\omega(n-1) + d')$-structure of $\mathcal{H}_{x_{i-1},x_i}$ for every $d' \geq 1$. But this means that $\mathcal{H}_{x_{i-1},x_i}$ has an $(\omega n + 1)$-structure. This finishes the proof.

Corollary. *Let $f : X \longrightarrow Y$ be an A-module homomorphism such that the following hold:*

1. *The modules X and Y are indecomposable;*
2. *We have $f \in \mathrm{rad}_A^{\omega n+d-1}$ where $n \geq 1$;*
3. *The image of f has a composition factor of the form S.*

Then there exist $x < y$ in $\mathcal{H}(S)$ such that $\theta_S(x) = X$ and $\theta_S(y) = Y$ and $\mathcal{H}_{x,y}$ has an $(\omega n + d)$-structure.

Proof. This follows easily from the above theorem.

Corollary. *Let $f_S \in \mathrm{rad}_A^{\omega n+d-1}$ where $n \geq 1$. Then $\mathcal{H}(S)$ has an $(\omega n + d)$-structure.*

Proof. This is just a special case of the corollary before.

Theorem. *Let A be domestic and S a simple A-module. Assume $n \geq 1$. Then $f_S \in \mathrm{rad}_A^{\omega n+d-1}$ if and only if there exists a linear subposet T of $H(S)$ with $d(T) = \omega n + d$.*

Proof. Assume $f_S \in \mathrm{rad}_A^{\omega n+d-1}$. According to the previous corollary we get an $(\omega n + d)$-structure for $\mathcal{H}(S)$. Since A is domestic, we know that $H(S)$ is a hammock poset. Then the last corollary in section 3.10 ensures that $H(S)$ has an $(\omega n + d)$-structure, too. The first corollary in section 3.10 implies the existence of a linear subposet T of $H(S)$ with $d(T) = \omega n + d$.

Next, assume that there exists a linear subposet T of density $\omega n + d$ in $H(S)$. This subposet induces a factorization system (T, f_S). Then the statement follows directly from the propostion in section 5.3.

Remark. The theorem is also true if one replaces f_S by an arbitrary canonical map $f^S_{x,y}$ and $H(S)$ by $H(S)_{\pi(x),\pi(y)}$ where $\pi : \mathcal{H}(S) \longrightarrow H(S)$ denotes the canonical surjective poset map.

Lemma. *Let A be domestic and S a simple A-module. Then the following hold:*

1. *We have $d_T(h(S)) = d(h(S)_{-1} \circ h(S)_1)$;*
2. *We have $0 \leq d_T(H(S)) - d_T(h(S)) \leq |\mathcal{B}(S)|$.*

Proof. The statement follows from the fact that $H(S)$ is a hammock poset.

Theorem. *Let A be domestic and S a simple A-module. Then* $\mathrm{nil}(f_S) = d_T(H(S))$.

Proof. This follows easily from the theorem before.

Theorem. *Let A be domestic and S a simple A-module. Let n_S be the maximal length of a path in $\Gamma_{\mathcal{B}}(S)$ starting at $1_{(S,1)}$ or $1_{(S,-1)}$. Let d_S be the number of arrows b in $\Gamma_{\mathcal{B}}(S)$ starting at $1_{(S,1)}$ or $1_{(S,-1)}$ such that there is a path of length n_S with first arrow b. Then*

$$\omega n_S + d_S \leq \mathrm{nil}(f_S) \leq \omega n_S + d_S + |\mathcal{B}(S)|.$$

Proof. Combine the previous theorem with the theorem in section 4.10.

Theorem. *Let A be a string algebra. Then the following are equivalent:*

1. *A is domestic;*
2. $\mathrm{nil}(A) < \infty$*;*
3. $\mathrm{nil}(A) < \omega^2$.

Proof. Use the proposition in section 5.5 and the last two theorems.

Thus we have an efficient way to compute $\mathrm{nil}(A)$ for any given string algebra A up to the small error term $\max\{|\mathcal{B}(S)| \mid S \text{ simple}\}$. In case $d(H(S)_{-1} \circ H(S)_1) = d(h(S)_{-1} \circ h(S)_1)$ we get the exact value for $\mathrm{nil}(A)$.

6 Examples

Let A be a domestic string algebra. From the previous chapter we know the following:

1. $\mathrm{nil}(A) < \omega^2$;

2. $\mathrm{nil}(A)$ is not a limit number;

3. $\mathrm{nil}(A) \neq \omega + 1$.

In this chapter we construct for each ordinal number α, which satisfies the above three conditions, a domestic string algebra A with $\mathrm{nil}(A) = \alpha$.

6.1 Definition of $A_{\omega n+d}$

Let $(n, d) \in \mathbb{N} \times \mathbb{N}$ with $d \geq 2$. Then let $A_{\omega n+d}$ be the following algebra:

$$\begin{tikzcd}
S \arrow[d, "\beta_d"] & \circ \arrow[l, "\alpha_{d-1}"'] \arrow[d, "\beta_{d-1}"] & \cdots \arrow[l, "\alpha_{d-2}"'] & \circ \arrow[l, "\alpha_2"'] \arrow[d, "\beta_2"] & \circ \arrow[l, "\alpha_1"'] \arrow[d, "\beta_1"] \\
\circ & \circ \arrow[l, "\gamma_{d-1}"] & \cdots \arrow[l, "\gamma_{d-2}"] & \circ \arrow[l, "\gamma_2"] & \circ \arrow[l, "\gamma_1"] \arrow[r, "\delta_1"'] & \circ \arrow[r, bend left, "\delta'_2"] \arrow[r, bend right, "\delta_2"'] & \circ \arrow[r, "\delta_3"'] & \circ \arrow[r, bend left, "\delta'_4"] \arrow[r, bend right, "\delta_4"'] & \circ \arrow[r, "\delta_5"'] & \cdots \arrow[r, "\delta_{2n-1}"'] & \circ \arrow[r, bend left, "\delta'_{2n}"] \arrow[r, bend right, "\delta_{2n}"'] & \circ
\end{tikzcd}$$

$$\rho = \{\beta_{i+1}\alpha_i, \gamma_i\beta_i, \delta_{j+1}\delta_j \mid 1 \leq i \leq d-1 \text{ and } 1 \leq j \leq 2n-1\}$$

For $n \geq 2$ define $A_{\omega n+1}$ as follows:

$$\begin{tikzcd}
S \arrow[r, "\delta_1"'] & \circ \arrow[r, bend left, "\delta'_2"] \arrow[r, bend right, "\delta_2"'] & \circ \arrow[r, "\delta_3"'] & \circ \arrow[r, bend left, "\delta'_4"] \arrow[r, bend right, "\delta_4"'] & \circ \arrow[r, "\delta_5"'] & \cdots \arrow[r, "\delta_{2n-1}"'] & \circ \arrow[r, bend left, "\delta'_{2n}"] \arrow[r, bend right, "\delta_{2n}"'] & \circ
\end{tikzcd}$$

$$\rho = \{\delta_{j+1}\delta_j \mid 1 \leq j \leq 2n-1\}$$

The vertex S in the upper left will play an important role in the computation of $\mathrm{nil}(A_{\omega n+d})$.

6.2 Computing $\mathrm{nil}(A_{\omega n+d})$

The bridge-quiver $\Gamma_{\mathcal{B}}$ of $A_{\omega n+d}$ looks as follows:

$$\begin{tikzcd}
\delta_2\delta_2'^{-} \arrow[r, "\delta_3^{-}\delta_4'^{-}"] & \delta_4\delta_4'^{-} \arrow[r, "\delta_5^{-}\delta_6'^{-}"] & \cdots \arrow[r, "\delta_{2n-1}^{-}\delta_{2n}'^{-}"] & \delta_{2n}\delta_{2n}'^{-}
\end{tikzcd}$$

$$\begin{tikzcd}
\delta_{2n}'\delta_{2n}^{-} \arrow[r, "\delta_{2n}'\delta_{2n-1}"] & \delta_{2n-2}'\delta_{2n-2}^{-} \arrow[r, "\delta_{2n-2}'\delta_{2n-3}"] & \cdots \arrow[r, "\delta_4'\delta_3"] & \delta_2'\delta_2^{-}
\end{tikzcd}$$

Assume that $\varepsilon(\alpha_{d-1}) = \sigma(\beta_d) = 1$ in case $d \geq 2$. Then the extended bridge-quiver $\Gamma_{\mathcal{B}}(S)$ has the following shape:

$$\begin{tikzcd}
1_{(S,1)} \arrow[r, bend left, "b_1"] \arrow[r, phantom, "\cdots"] \arrow[r, bend right, "b_d"'] & \delta_2\delta_2'^{-} \arrow[r, "\delta_3^{-}\delta_4'^{-}"] & \cdots \arrow[r, "\delta_{2n-1}^{-}\delta_{2n}'^{-}"] & \delta_{2n}\delta_{2n}'^{-} \\
1_{(S,-1)} & \delta_{2n}'\delta_{2n}^{-} \arrow[r, "\delta_{2n}'\delta_{2n-1}"] & \cdots \arrow[r, "\delta_4'\delta_3"] & \delta_2'\delta_2^{-}
\end{tikzcd}$$

If $d \geq 2$ then the arrows starting at $1_{(S,1)}$ look as follows:

$$b_1 = \beta_d^{-}\gamma_{d-1}\cdots\gamma_1\delta_1^{-}\delta_2'^{-}$$

and

$$b_i = \alpha_{d-1}\cdots\alpha_{d-i+1}\beta_{d-i+1}^{-}\gamma_{d-i}\cdots\gamma_1\delta_1^{-}\delta_2'^{-}$$

where $2 \leq i \leq d$.

For $d = 1$ we have $b_1 = \delta_1^{-}\delta_2'^{-}$.

Thus, we get $n_S = n$ and $d_S = d$. Remember that the numbers n_S and d_S are independent of the choice of the maps σ and ε.

Lemma. *Let V be a vertex of $A_{\omega n+d}$ such that $V \notin \{S, e(\delta_1), e(\delta_2)\}$. Then $d_T(H(V)) < d_T(H(S))$.*

Proof. Computing the extended bridge-quiver $\Gamma_{\mathcal{B}}(V)$ yields $n_V < n_S$.

Lemma. *Let $V \in \{e(\delta_1), e(\delta_2)\}$. Then $d_T(H(V)) \leq d_T(H(S))$.*

Proof. First, let $n \geq 2$. It is easy to see that $n_V = n_S$ and $d_V = 1$. Furthermore, the set $\mathcal{B}(V)$ contains exactly one element. This implies $\omega n_S + 1 \leq d_T(H(V)) \leq \omega n_S + 2$. Since $d_T(H(S)) = \omega n_S + d_S$, the lemma is obvious in case $d_S \geq 3$. Let $\mathcal{B}(V) = \{x = (x', x'')\}$. Then after appropriate choice of σ and ε one easily calculates that $H(V)_{-1} = \mathbb{N} \cdot \{x'\} \cdot {}^*\mathbb{N}$ and $H(V)_1 = H_1 \cdot \{x''\} \cdot {}^*\mathbb{N}$ for some linear poset H_1. We know that $d_T(H(V)_{-1} \circ H(V)_1) \geq \omega 2 + 1$. This implies $d_T(H(V)_{-1} \circ H(V)_1) = d_T(\mathbb{N} \cdot \{x'\} \cdot {}^*\mathbb{N} \circ H_1 \cdot \{x''\} \cdot {}^*\mathbb{N}) = d_T(\mathbb{N} \cdot {}^*\mathbb{N} \circ H_1 \cdot {}^*\mathbb{N}) = d_T(h(V)_{-1} \circ h(V)_1) = d_T(h(V)) = \omega n_S + 1$.
Next, assume $n = 1$. This implies $d_V = 2$. Thus $\omega n_S + 2 \leq d_T(H(V)) \leq \omega n_S + 3$. In case $d_S \geq 3$ we are done. Let $d_S = 2$. Since $n = 1$, we get $H(V)_{-1} = \mathbb{N} \cdot \{x'\} \cdot {}^*\mathbb{N}$ and $H(V)_{-1} = \mathbb{N} \cdot \{x''\} \cdot {}^*\mathbb{N}$. It is obvious that a linear subposet of $H(V)$ can have a density of at most $\omega + 2$. Thus we get $d_T(H(V)) = \omega n_S + 2$. This finishes the proof.

Theorem. *Let $(1,1) \neq (n,d) \in \mathbb{N} \times \mathbb{N}$. Then $\mathrm{nil}(A_{\omega n + d}) = \omega n + d$.*

Proof. We showed before that $d_T(H(S)) \geq d_T(H(V))$ for all vertices V of $A_{\omega n+d}$. Since $\mathrm{nil}(f_S) = d_T(H(S)) = d_T(h(S)) = \omega n + d$, we get $\mathrm{nil}(A_{\omega n+d}) = \omega n + d$.

6.3 The Auslander-Reiten quiver of $A_{\omega n+d}$

By Λ_n we denote the following algebra:

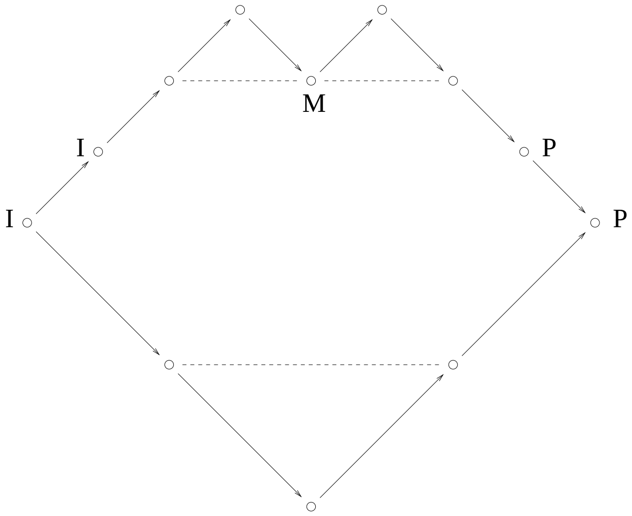
$$\begin{tikzcd} \circ \arrow[r, bend left, "\delta_2'"] \arrow[r, bend right, "\delta_2"'] & \circ \arrow[r, "\delta_3"'] & \circ \arrow[r, bend left, "\delta_4'"] \arrow[r, bend right, "\delta_4"'] & \circ \arrow[r, "\delta_5"'] & \cdots \arrow[r, "\delta_{2n-1}"'] & \circ \arrow[r, bend left, "\delta_{2n}'"] \arrow[r, bend right, "\delta_{2n}"'] & \circ \end{tikzcd}$$

$\rho = \{\delta_{i+1}\delta_i \mid 2 \leq i \leq 2n-1\}$

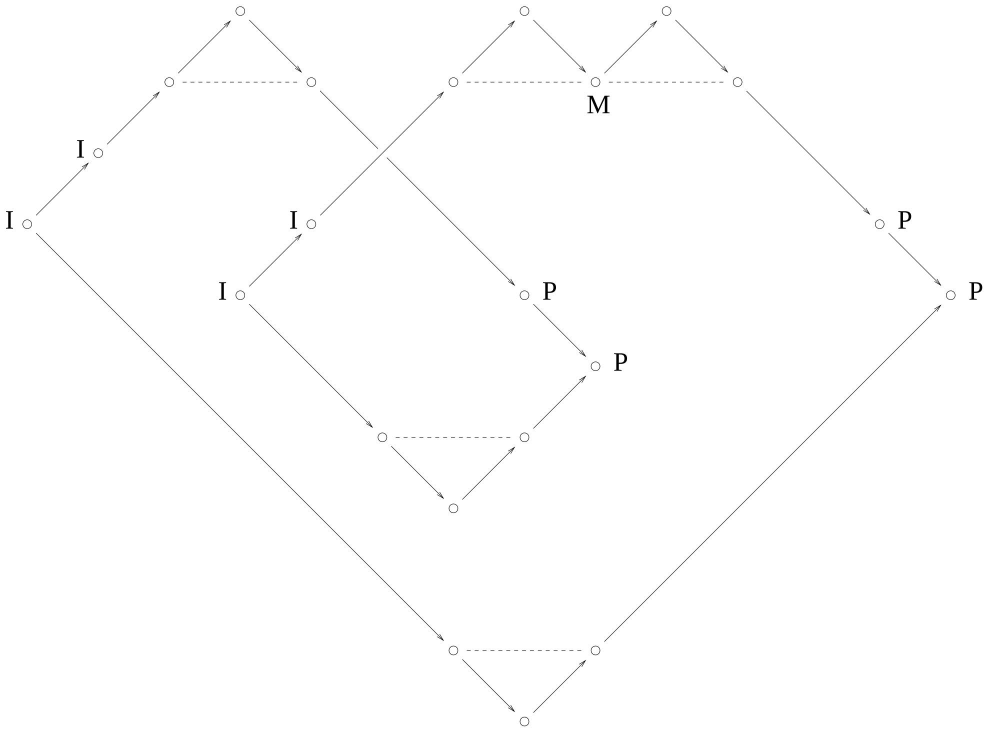
The Auslander-Reiten quiver of Λ_n consists of a preprojective component and a preinjective component which are the same as the preprojective respectively the preinjective component of the Kronecker algebra with two arrows. Then we have n families of homogeneous tubes, and for $n \geq 3$ there is an infinite number of $\mathbb{Z}A_\infty^\infty$ components. For $n \geq 2$ we have additionally one non-regular component which contains all the remaining projective and injective modules. We call this component *the exceptional component of* Λ_n. Note that this component is already determined by its boundary. This means we only have to calculate the Auslander-Reiten sequences with only one indecomposable middle term and the projective and injective modules together with the irreducible maps between them. By M we denote the string module $M(\delta_2'\delta_3\cdots\delta_{2n-1}\delta_{2n}')$. For $n=1$ the module M is on the mouth of a homogeneous tube. In this case, call this tube *the exceptional component of* Λ_1. For all n the module M is contained in the exceptional component of Λ_n.

Examples

The boundary of the exceptional component of Λ_2 looks as follows:



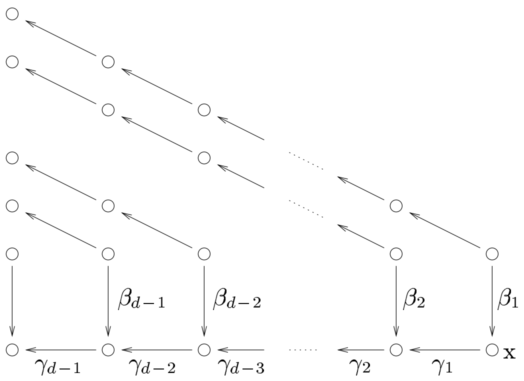
The boundary of the exceptional component of Λ_3 is this:



The boundary for larger n behaves correspondingly.

Every module indexed by P (respectively I) is projective (respectively injective).

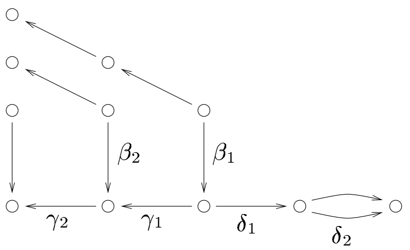
By B_d, $d \geq 2$, we denote the following branch in x (compare [R3]):



$\rho = \{\gamma_i\beta_i \mid 1 \leq i \leq d-1\}$

The branch B_1 is by definition the vertex x itself.

For $(1,1) \neq (n,d) \in \mathbb{N} \times \mathbb{N}$ define $A'_{\omega n+d} = \Lambda_n[M, B_d]$. Here we use the same notation as in [R3]. Thus, the algebra $A'_{\omega n+d}$ is by definition the branch-extension of the algebra Λ_n with the module M and the branch B_d. Call the component of $A'_{\omega n+d}$, which we obtain from the exceptional component of Λ_n, *the exceptional component of* $A'_{\omega n+d}$. For the theory of branch-extensions we refer to [R3]. For example, the algebra $A'_{\omega+3}$ is the following:



$\rho = \{\gamma_2\beta_2, \gamma_1\beta_1, \delta_2\delta_1\}$

Remark. Let $d \geq 2$. It is easy to see that the algebra $A'_{\omega n+d}$ is obtained from $A_{\omega n+d}$ by removing socalled splitting zero-relations. This implies that the Auslander-Reiten quiver of $A_{\omega n+d}$ is obtained from the Auslander-Reiten quiver of $A'_{\omega n+d}$ by sewing certain Auslander-Reiten components. In our case we just have to sew the preinjective component of $A'_{\omega n+d}$ with the exceptional component of $A'_{\omega n+d}$. Call the resulting component *the exceptional component of* $A_{\omega n+d}$. The elementary technique of removing splitting zero-relations is described for example in [R2].

Remark. The Auslander-Reiten components of $A_{\omega n+d}$ are the following:

1. The exceptional component;
2. A preprojective component of the same shape as the preprojective component of the Kronecker quiver with two arrows;
3. If $d = 1$ then there is one preinjective component;
4. We have n homogeneous tubes consisting of string modules;
5. We have n families of homogeneous tubes containing only band modules;
6. For $n \geq 3$ there is an infinite number of $\mathbb{Z}A_\infty^\infty$ components.

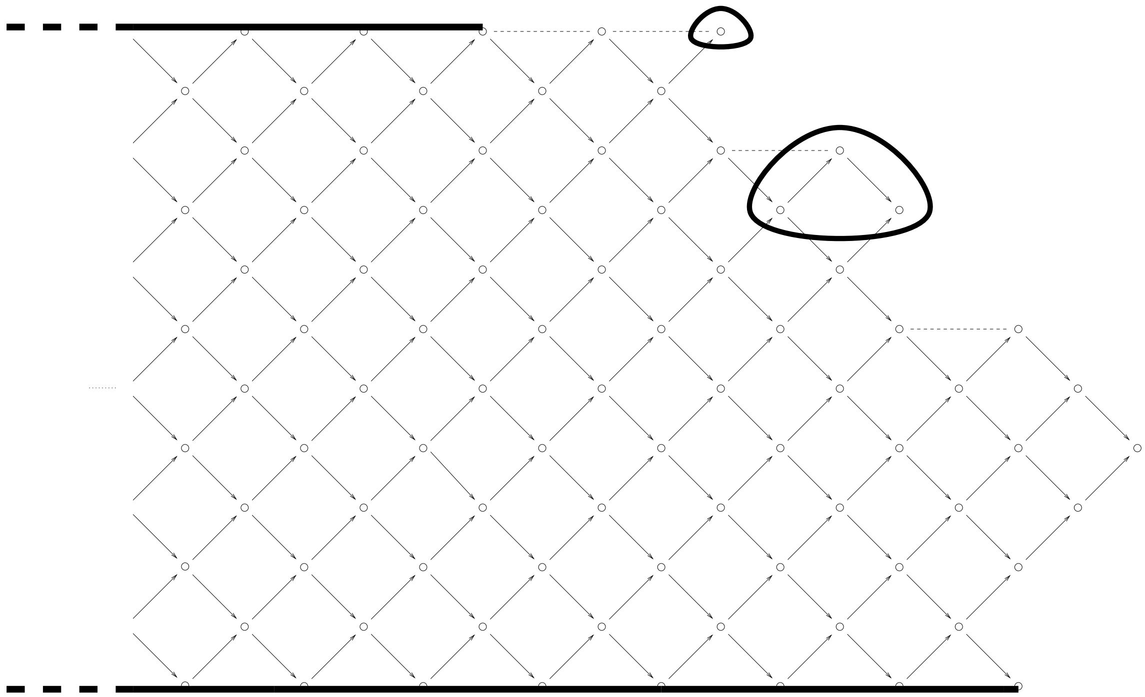
Proposition. *Let $A_{\omega n+d}$ and S be defined as before. Then the following hold:*

1. *If $n = 1$ and $d \geq 2$ then all indecomposable modules with composition factor S lie in the exceptional component of $A_{\omega n+d}$;*
2. *If $n = 1$ then $\mathcal{H}(S) = H(S) = h(S) = \mathbb{N} \cdot \underbrace{\mathbb{Z} \cdots \mathbb{Z}}_{d-1} \cdot {}^*\mathbb{N}$;*
3. *The simple module S occurs with multiplicity at most one in a composition series of a given indecomposable module.*

Proof. Compute the Auslander-Reiten quiver.

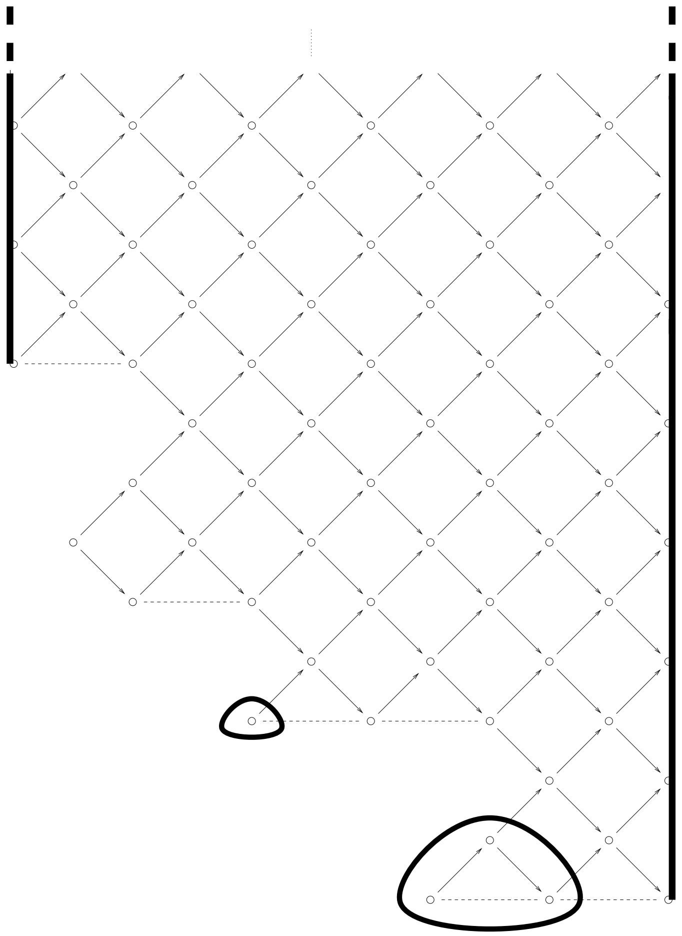
Next, we give some examples of exceptional components:

Let $n \geq 1$. Then the algebra $A'_{\omega n+3}$ has the following preinjective component:



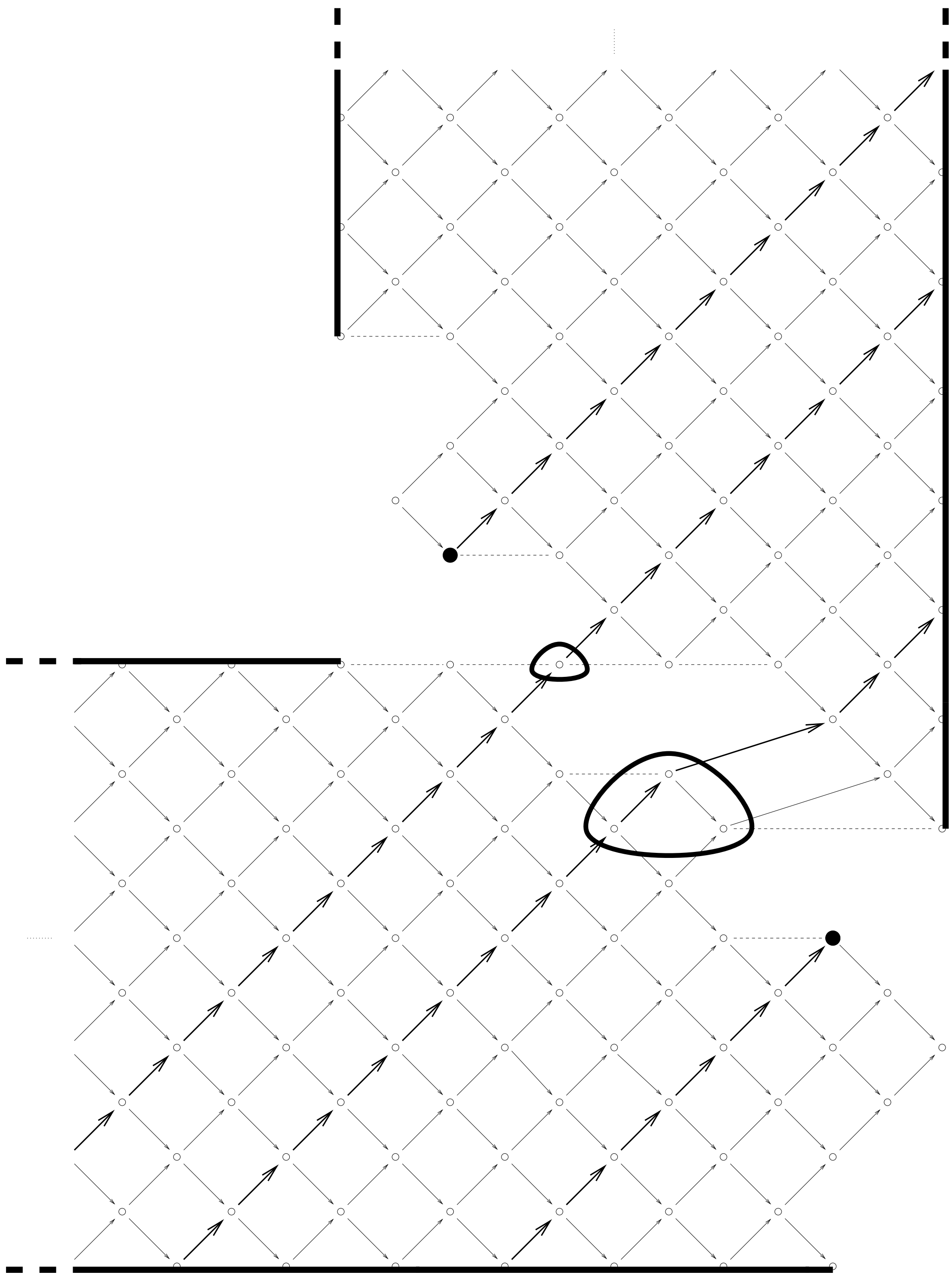
The two thick horizontal lines have to be identified.

The exceptional component of $A'_{\omega+3}$ is the following:



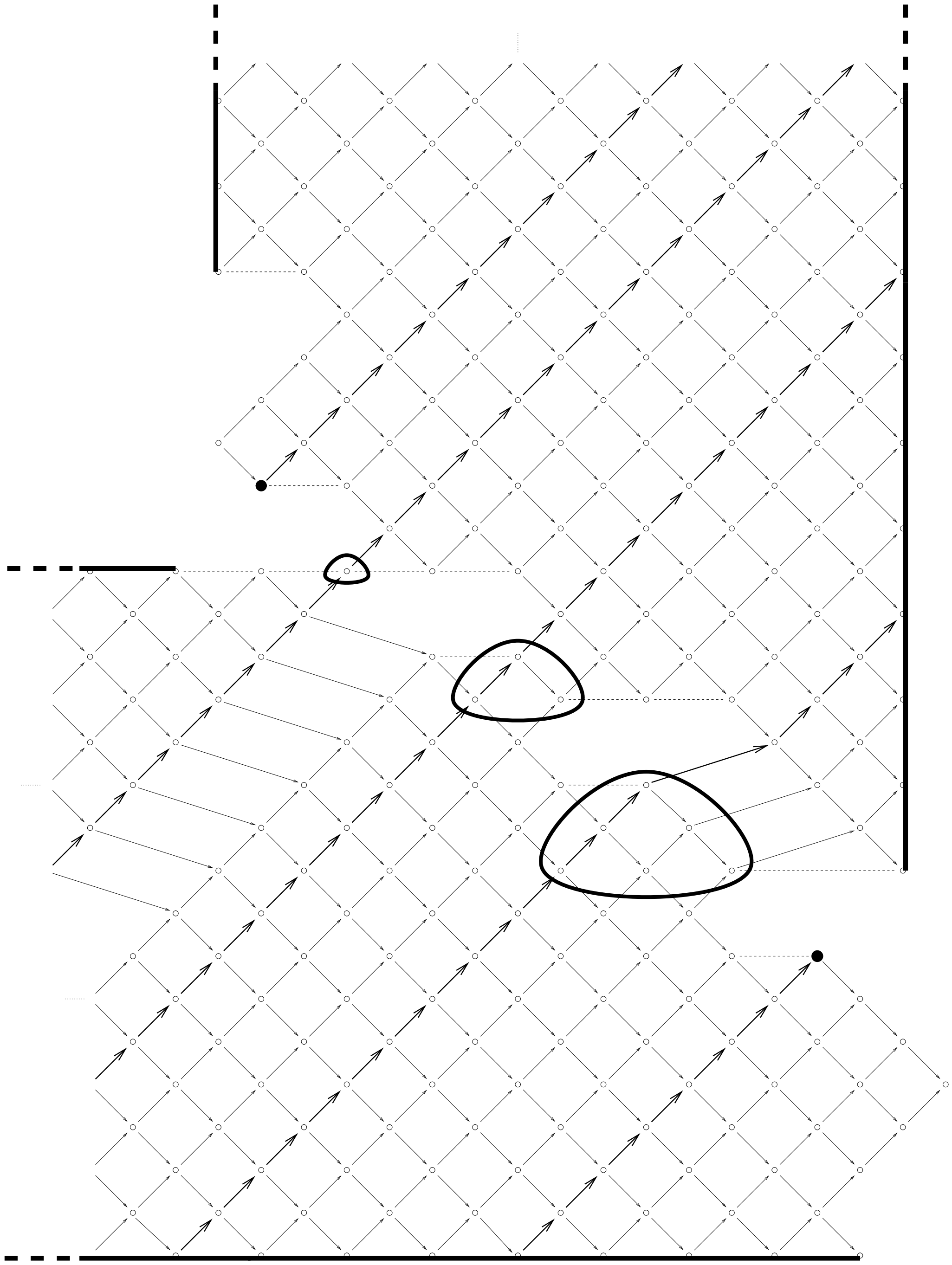
Again identify the two thick lines.

Then the sewing process identifies the corresponding encircled triangles and we get the exceptional component of $A_{\omega+3}$:



Identify the two vertical thick lines and the two horizontal thick lines.

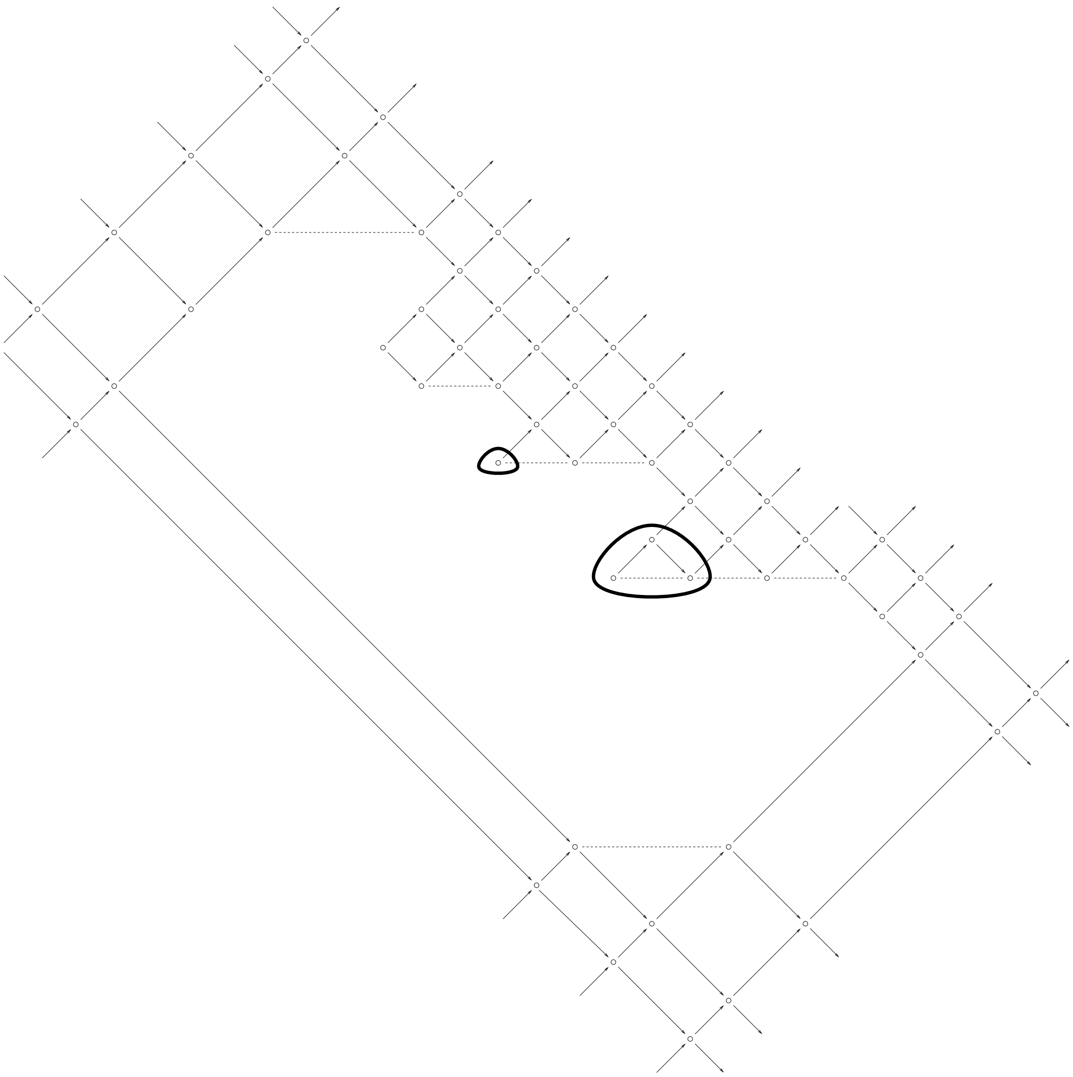
Similarly, we get the following exceptional component of $A_{\omega+4}$:



Identify the two vertical thick lines and the two horizontal thick lines.

Note that in our pictures of the exceptional component of $A_{\omega+3}$ and $A_{\omega+4}$ we have drawn thick arrows between modules which have a composition factor S. The two large filled vertices correspond to the projective cover and the injective envelope of S. The exceptional component of $A_{\omega+d}$ for larger d has almost the same form. One just has to sew $d-1$ triangles. Thus for $A_{\omega+d}$ we have a complete description of the possible factorizations of the canonical map f_S. Remember that we have $\mathrm{nil}(A_{\omega+d}) = \mathrm{nil}(f_S)$.

The next picture shows the shape of the exceptional component of $A'_{\omega 2+3}$:



The corresponding component of $A_{\omega 2+3}$ is again obtained by identifying the encircled parts of this component with the two encircled parts of the preinjective component. The structure of the Auslander-Reiten-quiver of $A_{\omega 2+d}$ for larger d is similar.

Remark: A recent result of C. Geiß allows to find a good parametrization for $\mathbb{Z}A_\infty^\infty$ components of a string algebra (see [G]). He shows that each

$\mathbb{Z}A_{\infty}^{\infty}$ component of a string algebra contains a unique module with minimal dimension. He gives also a recipe to determine these minimal modules.

7 Appendix

7.1 Nilpotency of rad_A and domesticity of A

The aim of this section is to give some survey of older results which are related to the results of this thesis.

Let A be an algebra. Remember that the usual *infinite radical* of A-mod is in our notation just rad_A^{ω} whereas the intersection over all ordinal powers of rad_A is denoted by rad_A^{∞}. The infinite radical is called *nilpotent* if $\mathrm{rad}_A^{\omega+n} = 0$ for some $n \in \mathbb{N}$. This is equivalent to the condition $\mathrm{rad}_A^{\omega 2} = 0$. In the context of representation theory of algebras this is the most studied nilpotency phenomenon. Some years ago there arose the feeling that $\mathrm{rad}_A^{\omega 2} = 0$ should be related to the domesticity of A. There is for example the following result:

Theorem. *Let A be an algebra. Assume that A satisfies one of the following assumptions:*

1. *A is a tilted algebra;*
2. *A is standard selfinjective;*
3. *A is special biserial selfinjective;*
4. *A is cycle-finite;*
5. *A is strongly simply connected.*

Then the following two conditions are equivalent:

1. *A is domestic;*
2. $\mathrm{rad}_A^{\omega 2} = 0$.

The proof of this theorem is due to several people. For a survey and complete references see [S].

On the other hand, it was known that there exist domestic algebras A with $\mathrm{rad}_A^{\omega 2} \neq 0$.

Let A be a domestic algebra which satisfies the assumption of the theorem. Then $\mathrm{nil}(A)$ is not known in general. There is however the conjecture that $\mathrm{nil}(A) \leq \omega + 8$. From this point of view the domestic algebras occurring in the theorem are not too difficult to understand.

The results, which we obtained for string algebras, indicate that one should not look at the phenomenon $\mathrm{rad}_A^{\omega 2} = 0$ but at $\mathrm{rad}_A^{\infty} = 0$ in order to characterize domestic algebras. In fact, there is the following general conjecture which seems to be due to M. Prest and C.M. Ringel:

Conjecture (M. Prest, C.M. Ringel). *Let A be an algebra. Then the following are equivalent:*

1. *A is domestic;*
2. $\mathrm{rad}_A^{\infty} = 0$.

In case A is a domestic algebra with $\mu_A(d) \leq c$ for all $d \in \mathbb{N}_0$ one might ask whether $\mathrm{nil}(A) = \omega c' + d$ where the number c' is related to c.

Conjecture. *Let A be an algebra. Then the following are equivalent:*

1. $\mathrm{nil}(A) < \infty$*;*
2. $\mathrm{nil}(A) < \omega^2$.

7.2 Factorization behaviour of homomorphisms

Question. *Let A be an algebra and f an A-module homomorphism with $f \in \mathrm{rad}_A^{\omega n + d}$ where $n \geq 1$. Does there exist a factorization system (T, f) such that $d(T) = \omega n + d + 1$?*

In case A is a domestic string algebra and S a simple A-module, we answered this question for the canonical map f_S. With the same techniques one can extend this result to any $f \in B(X, Y)$ where X and Y are indecomposable A-modules.

7.3 Remarks about the definition of the transfinite radical

Let A be an algebra and $\mathcal{I}$ an ideal in A-mod. For $n \geq 1$ define

$$\mathcal{I}^n = \{\sum_{i=1}^{l} f_n^{(i)} \cdots f_1^{(i)} \mid f_j^{(i)} \in \mathcal{I}, l \geq 1\}.$$

Already in this case, there exist different notions of nilpotency: Call $\mathcal{I}$ *nilpotent* if $\mathcal{I}^n = 0$ for some $n \in \mathbb{N}$. Call it *left T-nilpotent* (respectively *right T-nilpotent*) if for each sequence $(f_i \in \mathcal{I})_{i \in \mathbb{N}}$ there exists an $n \in \mathbb{N}$ such that $f_n \cdots f_1 = 0$ (respectively $f_1 \cdots f_n = 0$) whenever the compositions are defined. These definitions are well known.

Also several constructions of arbitrary ordinal powers of $\mathcal{I}$ seem to be quite old. Let α be a limit number. Then define

$$\mathcal{I}^\alpha = \bigcap_{\beta < \alpha} \mathcal{I}^\beta.$$

Next, let β be an arbitrary infinite ordinal. Thus $\beta = \alpha + n$ for some limit number α and some finite ordinal n. Then there are three possibilities to define $\mathcal{I}^\beta$:

1. $\mathcal{I}^\beta = \mathcal{I}^n \mathcal{I}^\alpha$;
2. $\mathcal{I}^\beta = \mathcal{I}^\alpha \mathcal{I}^n$;
3. $\mathcal{I}^\beta = (\mathcal{I}^\alpha)^{n+1}$.

Choose one of these three definitions and then define

$$\mathcal{I}^\infty = \bigcap_\alpha \mathcal{I}^\alpha$$

where we intersect over all ordinal powers of $\mathcal{I}$. Call $\mathcal{I}^\infty$ the *transfinite power* of the ideal $\mathcal{I}$. Note that we always used the third definition. It seems that M.C.R. Butler and M. Prest introduced it about ten years ago in this context. The analogue construction for ideals of rings is however older (see for example [KL]).

The next proposition shows that the first and the second definition do not make sense in case one is looking at ideals in module categories of algebras.

Proposition. *Let A be an algebra. Then* $\mathrm{rad}_A^\omega = \mathrm{rad}_A \mathrm{rad}_A^\omega = \mathrm{rad}_A^\omega \mathrm{rad}_A$.

Proof. Let $f : X \longrightarrow Z$ be a homomorphism in rad_A. First, we show that $f = hg$ where $h \in \mathrm{rad}_A$ and $g \in \mathrm{rad}_A^\omega$. Without loss of generality assume that X and Z are indecomposable. Note that Z can not be simple projective. Thus we have a non-zero sink map $h : Y \longrightarrow Z$. In particular, $h \in \mathrm{rad}_A \backslash \mathrm{rad}_A^2$. For every $d \geq 1$ we have $f = f_d \cdots f_1$ for some homomorphisms $f_i \in \mathrm{rad}_A$. Since h is a sink map, we get $f_d = hf'_d$ for some homomorphism f'_d. This implies $f = hf'_d f_{d-1} \cdots f_1$ and $f'_d f_{d-1} \cdots f_1 \in \mathrm{rad}_A^{d-1}$. Since d can be arbitrary large, we get $\mathrm{rad}_A^\omega(X, Y) \neq 0$. Furthermore, $\mathrm{rad}_A^\omega(X, Y) = \mathrm{rad}_A^m(X, Y)$ for some $m \in \mathbb{N}$. Choose $d \geq m + 1$. Then $f = gh$ where $g = f'_m f_{m-1} \cdots f_1 \in \mathrm{rad}_A^\omega$. Dually, one can show that $f = g'h'$ where h' is the source map of Z and $g' \in \mathrm{rad}_A^\omega$. This finishes the proof.

Thus the transfinite power of rad_A would be just rad_A^ω if one uses the first or the second definition as described above.

Let A be the algebra from section 1.16. Then $\mathrm{rad}_A^{\omega 2}$, $\mathrm{rad}_A^{\omega 2}\mathrm{rad}_A^\omega$ and $\mathrm{rad}_A^\omega \mathrm{rad}_A^{\omega 2}$ are pairwise different.

A lower bound for the Krull-Gabriel dimension of a string algebra

In 1985 W. Geigle introduced the socalled Krull-Gabriel dimension $\mathrm{KGdim}(A)$ of an algebra A (see [Ge]).
However, there are only examples in the literature where the Krull-Gabriel dimension is 0 or 2. In his thesis W. Geigle the Krull-Gabriel dimension of tame hereditary algebras and showed that it is always equal to 2. Additionally, H. Krause proved that the Krull-Gabriel dimension of an algebra can not be equal to 1 (see [K3]). One might ask how this is related to the theorem of F.U. Coelho, E.N. Marcos, H.A. Merklen and A. Skowroński which we mentioned in the last chapter. Furthermore, it is not difficult to see that the Krull-Gabriel dimension of any wild algebra does not exist. There is the conjecture that the Krull-Gabriel dimension of an algebra exists if and only if the algebra is domestic. Recently, H. Krause proved the following theorem:

Theorem (H. Krause). *Let A be an algebra such that* $\mathrm{nil}(A) = \omega n + d$ *for some $n \geq 1$. Then* $\mathrm{KGdim}(A) \geq n$.

Proof. Listen [K2].

Theorem. *Let* $(1,1) \neq (n,d) \in \mathbb{N} \times \mathbb{N}$. *Then the domestic algebra* $A_{\omega n+d}$ *has Krull-Gabriel dimension at least* n.

Proof. Combine the previous theorem with the theorem in section 6.2.

Conjecture. *Let* A *be an algebra with* $\mathrm{nil}(A) = \omega n + d$ *where* $n \geq 1$. *Then* $\mathrm{KGdim}(A) = n + 1$.

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