An Example of an Arithmetic 3–Manifold With Cubic Ground Field

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Introduction

In this note we work out some technical details which are typically faced in the context of hyperbolic lattices in dimension three. We take a specific example of a group given by an abstract presentation and construct a (cocompact) lattice representation in the group of isometries of hyperbolic three space.

The group is

$$\Pi_\infty = \langle \xi, \eta, \mu | \mu\xi\mu^{-1} = \eta\xi, \mu\eta\mu^{-1} = (\eta\xi)^{-5}\eta \rangle;$$

it is well known to be the fundamental group of a three dimensional manifold M_∞ which is hyperbolic (see [5]). This manifold fibers over the circle, the fiber being a once punctured torus; it may be discribed in the following way: Let T_0 a topological torus with a small open disc removed. We select a pair $\{\xi, \eta\}$ of generators of $\pi_1(T_0)$ such that

$$\lambda = \eta^{-1}\xi^{-1}\eta\xi$$

is represented by a loop around the hole. We take a homeomorphism

$$f : T_0 \longrightarrow T_0$$

which is the identity close to the hole and which, on $\pi_1(T_0)$, induces the isomorphism

$$f_* : (\xi, \eta) \longrightarrow (\eta\xi, (\eta\xi)^{-5}\eta).$$

Then M is defined to be

$$M = (T_0 \times [0, 1]) / \sim$$

where the equivalence relation is

$$(p, 0) \sim (f(p), 1) \qquad \text{for all } p \in T_0.$$

A peripheral system (see [3]) is composed by λ and μ. We define

$$\Pi = \langle \xi, \eta, \mu | \mu\xi\mu^{-1} = \eta\xi, \mu\eta\mu^{-1} = (\eta\xi)^{-5}\eta, \mu^2 = (\eta^{-1}\xi^{-1}\eta\xi)^{-3} \rangle$$

as a quotient of Π_∞. This group is the fundamental group of a closed oriented three-manifold which comes as result of a specific Dehn surgery performed on M_∞. It is hyperbolic, compare [5]. We explicitly exhibit a cocompact lattice $\tilde{\Gamma}$ in $Iso^+(\mathbf{H}^3)$ which as a group is isomorphic to $\Pi = \pi_1(M)$. This lattice turns out to be arithmetic. We are in the position to present a fundamental domain for Γ which allows to compute the volume of $M \cong \Gamma \setminus \mathbf{H}^3$.

It should be noted that actually a certain lattice $\tilde{\Gamma}$ in $Iso^+(\mathbf{H}^3)$ will be presented which is easier to handle than Γ. This $\tilde{\Gamma}$ will be an index two extension of Γ.

It turns out that when doing matrix calculations in $GL(2,\mathbf{C})$ it makes no sense to insist on determinant 1. So matrix identities are when checking group relators most frequently understood as identities modulo the center of $GL(2,\mathbf{C})$.

It is natural to try to apply the general formula for the volume of an arithmetic manifold which is proven in [2]. Also in that case it seems to be indispensable to have a fundamental domain at one's disposal. In addition one would have to estimate Dedekind's ζ-function for a certain cubic number field at the argument $s = 2$ with high accuracy. We use Lobačevskij's function, instead.

A quaternion algebra

The ground field of the quaternion algebra which will contain a lattice commensurable to Γ is cubic and defined over the rationals by

$$t^3 - t - 1 = 0.$$

We denote the abstract field $\mathbf{Q}(t)$ by K. K has one real and one pair of complex conjugate embeddings into $\mathbf{C}$. The discriminant of K is -23; any root of the above polynomial generates the absolute class field over $\mathbf{Q}(\sqrt{-23})$, see [4]. This field is Galois over $\mathbf{Q}$ with Galois group dihedral of order 6. If t is any root the two others are

$$-\frac{1}{2}t \pm \frac{4+9t-6t^2}{2\sqrt{-23}}$$

If we choose t to be the real root this is the explicit decomposition of the nonreal ones into real and imaginary part. The ring $\mathbf{o}$ of integers in K is easily identified to $\mathbf{Z} + \mathbf{Z}t + \mathbf{Z}t^2$. Note that t is a unit in the ring of integers of K; it is easy to see that t is not a square in K.

For computations in geometrical context we prefer to work with

$$x = 1 - \frac{1}{t}.$$

Its minimal equation over $\mathbf{Q}$ is

(1)
$$x^3 - 4x^2 + 5x - 1 = 0.$$

Note that $x, x-1$, and $x-2$ are units; we have

$$x = -(x-1)^5,$$

$$x-2 = -(x-1)^{-2}.$$

We note that $x-4$ satisfies

$$(x-4)^3 + 8(x-4)^2 + 21(x-4) + 19 = 0,$$

so $x-4$ is a prime number in $\mathbf{K}$ with norm 19, actually the only one up to associates: The ideal (19) decomposes into the product of two prime ideals, namely the principal ideal $(x-4)$ and the ideal $(19, x^2+5)$ which has norm 19^2. The numerical values of the roots of equation (1) are

$$1.87744 \pm .744862i,$$

$$.245122.$$

The geometry described below works with

$$x = 1.87744 + .744862i.$$

Let $\mathbf{A}$ be the quaternion algebra with groundfield $\mathbf{K}$ which ramifies at the real place and at $x-4$ but nowhere else. The group of elements with reduced norm 1 in some maximal order of $\mathbf{A}$ is an arithmetic group with faithful cocompact lattice representation in $SL(2,\mathbf{C})$. The lattice to be constructed will be commensurable to such a group.

We choose a splitting field for $\mathbf{A}$ by adjoining an element α to $\mathbf{K}$ which satisfies

$$(\alpha+1)(\alpha^{-1}+1) = x.$$

If $\mathbf{K}$ is embedded into $\mathbf{C}$ such that $Im(x) > 0$ then α and α^{-1}, as complex numbers have absolute values which are inverse to each other. In the geometrical construction below we understand α to have absolute value smaller than 1. This means numerically

$$\alpha = -.039862 - .693124i.$$

We show that

$$\mathbf{A} \cong \{\begin{pmatrix} a+b\alpha & (x-1)(c+d\alpha^{-1}) \\ c+d\alpha & a+b\alpha^{-1} \end{pmatrix} | a,b,c,d \in \mathbf{K}\}.$$

The $\mathbf{K}$-algebra of $2\times 2-$matrices written down here at once is seen to be a skewfield: $x-1$ cannot be a norm from $\mathbf{K}(\alpha)$ to $\mathbf{K}$. If it were we had

$$x-1 = (a+b\alpha)(a+b\alpha^{-1}) = a^2 + (x-2)ab + b^2$$

with $a, b \in \mathbf{K}$. But if $\mathbf{K}$ is given its real embedding into the complex numbers then the discriminant $x(x-4)$ of this binary quadratic form is negative, so the form is positive definite. On the other hand $x - 1 < 0$.

Let

$$X = \begin{pmatrix} 1+\alpha & x-1 \\ 1 & 1+\alpha^{-1} \end{pmatrix},$$

$$Y = \begin{pmatrix} \frac{y}{x}(1+\alpha^{-1}) & 1-y+\alpha \\ \frac{1-y+\alpha^{-1}}{x-1} & \frac{y}{x}(1+\alpha \end{pmatrix}$$

with $y = x\frac{x^2-x-1}{x-1} \in \mathbf{K}$. These matrices are elements of the skewfield. We have

$$trX = x, trY = y, trYX = x, detX = 1, detY = 1.$$

So

$$\mathbf{O} := \mathbf{oE} + \mathbf{o}X + \mathbf{o}Y + \mathbf{o}YX$$

is contained in some maximal order of this skewfield. We compute its discriminant

$$det \begin{pmatrix} tr\mathbf{E} & trX^{-1} & trY^{-1} & tr(YX)^{-1} \\ trX & trXX^{-1} & trXY^{-1} & trX(YX)^{-1} \\ trY & trYX^{-1} & trYY^{-1} & trY(YX)^{-1} \\ trYX & trYXX^{-1} & trYXY^{-1} & trYX(YX)^{-1} \end{pmatrix}$$

$$= det \begin{pmatrix} 2 & x & y & x \\ x & 2 & xy-x & y \\ y & xy-x & 2 & x \\ x & y & x & 2 \end{pmatrix} = (x-4)^2.$$

So $\mathbf{O}$ is a maximal order, the discriminant being shown to be the square of a prime. At the same time we have shown that the skewfield equals the quaternion algebra $\mathbf{A}$.

Comments. The matrices X and Y are the result of certain canonical trace computations: If $X, Y \in SL(2, \mathbf{C})$ with $x = trX, y = trY, z = trYX$ are such that

$$trX = trYX,$$

$$trY = tr(YX)^{-5}Y$$

then, by direct calculation one has

$$z = x, \qquad (x-1)y = x(x^2 - x - 1).$$

(See also [1].) Let $l = trY^{-1}X^{-1}YX$, then

$$2 + l = x^2 + y^2 + z^2 - xyz,$$

so

(1)

$$(x-1)^2 l = x^4 - 3x^3 + x^2 + 4x - 2.$$

A matrix $M \in GL(2, \mathbf{C})$ to satisfy

$$M^{-2} = L^3$$

(with $L = Y^{-1}X^{-1}YX$) means

$$M^{-1} \approx (l+1)L - \mathbf{E}$$

where $\approx$ means equality up to a nonzero complex factor. We want this matrix to conjugate X into YX (and Y into $(YX)^{-5}Y$). This, e.g., means

$$XM^{-1}X^{-1} = M^{-1}Y,$$

or, equivalently,

$$X((l+1)L - \mathbf{E})X^{-1} = ((l+1)L - \mathbf{E})Y.$$

We take traces and get

(2)

$$l(l+1) - 2 = ly = lx\frac{x^2 - x - 1}{x - 1}.$$

We eliminate l from (1) and (2) and get the following algebraic equation for x:

$$x(x-1)(x^2+x-1)(x^3-4x^2+5x-1) = 0.$$

The roots of the first three factors are without relevance for geometry. We take, as mentioned above, the root x of the fourth factor which has positive imaginary part. It determines

$$y = trY = x\frac{x^2 - x - 1}{x - 1},$$

$$z = trYX = x,$$

so determines the pair of matrices $X, Y \in SL(2, \mathbf{C})$ up to common conjugacy.

A fundamental domain of $\tilde{\Gamma}$

We draw a compact hyperbolic polyhedron $\mathbf{F}$ which will, by Poincaré's criterion, be the fundamental domain of a lattice $\tilde{\Gamma}$. It will be defined to be the common exterior of the isometric spheres of 22 specific elements from $\mathbf{O}$ and their inverses, intersected with a shell lying between two concentric hemispheres. These latter ones actually are the isometric spheres of two involutions:

$$I_1 = \begin{pmatrix} 0 & (x-1)\alpha \\ \alpha^{-1} & 0 \end{pmatrix} \qquad (inner \quad sphere),$$

$$I_2 = \begin{pmatrix} 0 & \frac{1+\alpha}{(x-1)^2} \\ \frac{1+\alpha^{-1}}{(x-1)^3} & 0 \end{pmatrix} \qquad (outer \quad sphere).$$

Let

$$D_1 = \begin{pmatrix} x-1 & -(x-1)^2\alpha \\ -(x-1)\alpha^{-1} & x-1 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} (x-1)^3 & -\frac{1+\alpha}{x-1} \\ -\frac{1+\alpha^{-1}}{(x-1)^2} & (x-1)^3 \end{pmatrix}.$$

Note that D_1 and I_1 commute as well as D_2 and I_2 do.

The elements whose isometric spheres contribute to the boundary of $\mathbf{F}$ are expressed in terms of I_1, I_2, D_1, and D_2; we use as an abbreviation

$$C \sim I_2 I_1 \sim \begin{pmatrix} \frac{1+\alpha^{-1}}{(x-1)^2} & 0 \\ 0 & \frac{1+\alpha}{(x-1)^2} \end{pmatrix}.$$

There are 11 such elements with determinant unity and 13 with determinant $-(x-1)$.

Matrices with determinant 1:

$$I_2 = \begin{pmatrix} 0 & \frac{1+\alpha}{(x-1)^2} \\ \frac{1+\alpha^{-1}}{(x-1)^3} & 0 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} x-1 & -(x-1)^2\alpha \\ -(x-1)\alpha^{-1} & x-1 \end{pmatrix},$$

$$D_3 \sim I_1 C D_1 \sim \begin{pmatrix} \frac{1+\alpha}{x-1} & -\frac{\alpha(1+\alpha)}{x-1} \\ -\frac{\alpha^{-1}(1+\alpha^{-1})}{(x-1)^2} & \frac{1+\alpha^{-1}}{x-1} \end{pmatrix},$$

$$D_4 \sim C D_1^{-1} C \sim \begin{pmatrix} (x-1)\alpha^{-1} & (x-1)^2\alpha \\ (x-1)\alpha^{-1} & (x-1)\alpha \end{pmatrix},$$

$$I_1 D_2 \sim \begin{pmatrix} -\frac{1+\alpha}{(x-1)^2} & (x-1)^3\alpha \\ (x-1)^2\alpha^{-1} & -\frac{1+\alpha^{-1}}{(x-1)^2} \end{pmatrix},$$

$$D_1 I_2 \sim \begin{pmatrix} \frac{1+\alpha}{x-1} & -\frac{1+\alpha}{x-1} \\ -\frac{1+\alpha^{-1}}{(x-1)^2} & \frac{1+\alpha^{-1}}{x-1} \end{pmatrix},$$

$$D_5 \sim I_2 D_1 I_2 \sim \begin{pmatrix} x-1 & -(x-1)^2 \\ -(x-1) & x-1 \end{pmatrix},$$

$$D_2 I_1 \sim \begin{pmatrix} \frac{1+\alpha^{-1}}{(x-1)^2} & -(x-1)^3\alpha \\ -(x-1)^2\alpha^{-1} & \frac{1+\alpha}{(x-1)^2} \end{pmatrix},$$

$$D_9 \sim C D_1 C \sim \begin{pmatrix} (x-1)\alpha^{-1} & -(x-1)^2\alpha \\ -(x-1)\alpha^{-1} & (x-1)\alpha \end{pmatrix},$$

$$D_{10} \sim I_1 D_1 C \sim \begin{pmatrix} \frac{1+\alpha^{-1}}{x-1} & -\frac{\alpha(1+\alpha)}{x-1} \\ -\frac{\alpha^{-1}(1+\alpha^{-1})}{(x-1)^2} & \frac{1+\alpha}{x-1} \end{pmatrix},$$

$$I_2 D_1 \sim \begin{pmatrix} \frac{1+\alpha^{-1}}{x-1} & -\frac{1+\alpha}{x-1} \\ -\frac{1+\alpha^{-1}}{(x-1)^2} & \frac{1+\alpha}{x-1} \end{pmatrix},$$

Matrices with determinant $-(x-1)$:

$$I_1 \sim \begin{pmatrix} 0 & (x-1)\alpha \\ \alpha^{-1} & 0 \end{pmatrix},$$

$$I_1 D_1 \sim \begin{pmatrix} (x-1)^2 & -(x-1)^2\alpha \\ -(x-1)\alpha^{-1} & (x-1)^2 \end{pmatrix},$$

$$C D_1 \sim \begin{pmatrix} \frac{1+\alpha^{-1}}{x-1} & -(1+\alpha) \\ -\frac{1+\alpha^{-1}}{x-1} & \frac{1+\alpha}{x-1} \end{pmatrix},$$

$$C D_1 I_2 \sim \begin{pmatrix} (x-1)^2 & -(x-1)^2 \\ -(x-1) & (x-1)^2 \end{pmatrix},$$

$$D_1C^{-1} \sim \begin{pmatrix} \frac{1+\alpha}{x-1} & -(1+\alpha) \\ -\frac{1+\alpha^{-1}}{x-1} & \frac{1+\alpha^{-1}}{x-1} \end{pmatrix},$$

$$D_2 \sim \begin{pmatrix} (x-1)^3 & -\frac{1+\alpha}{x-1} \\ -\frac{1+\alpha^{-1}}{(x-1)^2} & (x-1)^3 \end{pmatrix},$$

$$I_2D_2 \sim \begin{pmatrix} x-1 & 1+\alpha \\ 1+\alpha^{-1} & x-1 \end{pmatrix},$$

$$D_6 \sim CD_2I_1 \sim \begin{pmatrix} (x-1)\alpha^{-1} & (x-1)(1+\alpha) \\ 1+\alpha^{-1} & (x-1)\alpha \end{pmatrix},$$

$$I_1D_2I_1 \sim \begin{pmatrix} (x-1)^3 & -\frac{\alpha(1+\alpha)}{x-1} \\ -\frac{\alpha^{-1}(1+\alpha^{-1})}{(x-1)^2} & (x-1)^3 \end{pmatrix},$$

$$D_7 \sim CD_2C^{-1} \sim \begin{pmatrix} (x-1)^3 & -\frac{1+\alpha^{-1}}{x-1} \\ -\frac{1+\alpha}{(x-1)^2} & (x-1)^3 \end{pmatrix},$$

$$D_8 \sim D_1C \sim \begin{pmatrix} \frac{1+\alpha^{-1}}{x-1} & -\alpha(1+\alpha) \\ -\frac{\alpha^{-1}(1+\alpha^{-1})}{x-1} & \frac{1+\alpha}{x-1} \end{pmatrix},$$

$$D_{11} \sim CD_2^{-1}I_1 \sim \begin{pmatrix} (x-1)\alpha^{-1} & -(x-1)(1+\alpha) \\ -(1+\alpha^{-1}) & (x-1)\alpha \end{pmatrix},$$

$$D_{12} \sim D_1^{-1}C \sim \begin{pmatrix} \frac{1+\alpha^{-1}}{x-1} & \alpha(1+\alpha) \\ \frac{\alpha^{-1}(1+\alpha^{-1})}{x-1} & \frac{1+\alpha}{x-1} \end{pmatrix}.$$

The drawing of the polyhedron **F** is based on the computation of 88 points in hyperbolic space which will be the vertices of **F**. Each vertex will be a triple intersection point of isometric spheres. The identification among the isometric spheres creates, among the vertices, a grouping into 21 quadruples and two pairs. The vertices in the same group are given the same label which is an integer l, $1 \leq l \leq 23$, plus an index; see the following table. If A, B, C are any elements or inverses of elements from the above list we write $A \cap B \cap C$ for the point which lies on the isometric spheres of A, B, and C (in case it exists and is unique). We shall draw a vertical projection of **F** to the complex plane, only. We note the spacial coordinates of the various vertices in the list below. The formula for the z-coordinate is taken from [1]: Assume

$$S_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in SL_2(\mathbf{C})$$

for $j = 1, 2, 3$ to possess isometric spheres (which means $c_1 c_2 c_3 \neq 0$) which intersect in a (unique) point $(z, r) \in \mathbf{H}^3$. Then

$$z = \frac{w}{iv}$$

where

$$w = det \begin{pmatrix} c_1\bar{c}_1 & \bar{c}_1 d_1 & d_1\bar{d}_1 - 1 \\ c_2\bar{c}_2 & \bar{c}_2 d_2 & d_2\bar{d}_2 - 1 \\ c_3\bar{c}_3 & \bar{c}_3 d_3 & d_3\bar{d}_3 - 1 \end{pmatrix},$$

$$iv = det \begin{pmatrix} c_1\bar{c}_1 & c_1\bar{d}_1 & \bar{c}_1 d_1 \\ c_2\bar{c}_2 & c_2\bar{d}_2 & \bar{c}_2 d_2 \\ c_3\bar{c}_3 & c_3\bar{d}_3 & \bar{c}_3 d_3 \end{pmatrix}.$$

The r-coordinate is computed using elementary geometry. It is constant in equivalence classes, due to the fact that the points lie on isometric spheres.

The numerical material.

label	point	z	r
1_1	$D_1 \cap I_1 D_1 \cap I_1$	.149845-.494049i	.536879
1_2	$D_1^{-1} \cap (I_1 D_1)^{-1} \cap I_1$	-.149845+.494049i	
2_1	$I_1 D_1 \cap C D_1 \cap I_1$	.29218-.386649i	.565606
2_2	$I_2 D_1 \cap D_1 \cap I_1$	-.028169-.483811i	
2_3	$(I_1 D_!)^{-1} \cap D_1^{-1} \cap I_2$	-.168645+.671363i	
2_4	$(I_2 D_1)^{-1} \cap (C D_1)^{-1} \cap 1 I_2$	-.637495-.269757i	

label	point	z	r
3_1	$CD_1 \cap D_2^{-1} \cap I_1$	.375048-.391415i	.510793
3_2	$(I_1D_2)^{-1} \cap I_2d_1 \cap I_1$	-.094428-.533807i	
3_3	$D_9^{-1} \cap I_1D_2 \cap I_1$	-.69252+.091676i	
3_4	$(I_2D_1)^{-1} \cap D_9 \cap (CD_1)^{-1}$	-.555592-.256281i	
4_1	$D_2^{-1} \cap (D_2I_1)^{-1} \cap I_1$	.50514-.211207i	.504977
4_2	$(I_1D_2I_1)^{-1} \cap (I_1D_2)^{-1} \cap I_1$	-.302707-.456227i	
4_3	$I_1D_2I_1 \cap D_2I_1 \cap I_1$	.302707+.456227i	
4_4	$D_2 \cap I_1D_2 \cap I_1$	-.50514+.211207i	
5_1	$(D_2I_1)^{-1} \cap D_{12} \cap I_1$	.550859-.062944i	.497362
5_2	$(I_2D_1)^{-1} \cap (I_1D_2I_1)^{-1}$	-.423083-.358341i	
5_3	$D_2I_1 \cap I_1D_2I_1 \cap D_7^{-1}$	.226914+.591606i	
5_4	$D_7 \cap I_2D_1 \cap D_{12}^{-1}$	-.111956-.700143i	
6_1	$D_{12} \cap CD_1I_2 \cap I_1$	.560252+.018449i	.490463
6_2	$D_9 \cap (I_2D_1)^{-1} \cap I_1$	-.476107-.295878i	
6_3	$(CD_1I_2)^{-1} \cap I_1D_2 \cap D_9^{-1}$	-.649139+.014192i	
6_4	$D_{12}^{-1} \cap I_2D_1 \cap (I_1D_2)^{-1}$	-.120595-.618666i	
7_1	$CD_1I_2 \cap D_1C^{-1} \cap I_1$	.515344+.07135i	.533016
7_2	$D_{10} \cap D_9 \cap I_1$	-.468152-.226944i	
7_3	$D_{10}^{-1} \cap (D_1C^{-1})^{-1} \cap I_2$	-.43457+.571071i	
7_4	$(CD_1I_2)^{-1} \cap D_9^{-1} \cap I_2$	-.71761+.002925i	
8_1	$D_1c^{-1} \cap D_3^{-1} \cap I_1$	.515344+.156334i	.514542
8_2	$(CD_1)^{-1} \cap D_{10} \cap I_1$	-.515344-.156334i	
8_3	$D_3 \cap (I_2D_2)^{-1} \cap CD_1$	425612-.486591i	
8_4	$I_2D_2 \cap (D_1C^{-1})^{-1} \cap D_{10}^{-1}$	-.425612+.486591i	
9_1	$D_3^{-1} \cap D_4 \cap I_1$	.468152+.226944i	.533016
9_2	$(CD_1I_2)^{-1} \cap (CD_1)^{-1} \cap I_1$	-.515344-.07135i	
9_3	$D_3 \cap CD_1 \cap I_2$	43457-.571071i	
9_4	$CD_1I_2 \cap D_4^{-1} \cap I_2$	.71761-.002925i	
10_1	$D_4 \cap D_1I_2 \cap I_1$	.476107+.295878i	.490463
10_2	$D_8 \cap (CD_1I_2)^{-1} \cap I_1$	-.560252-.018440i	
10_3	$(D_2I_1)^{-1} \cap D_4^{-1} \cap CD_1I_2$	.649139-.014192i	
10_4	$D_2I_1 \cap D_8^{-1} \cap (D_1I_2)^{-1}$	.120595+.618666i	
11_1	$D_1I_2 \cap I_1D_2I_1 \cap I_1$	.423083+.358341i	.497362
11_2	$I_1D_2 \cap D_8 \cap I_1$	-.550859+.062944i	
11_3	$D_8^{-1} \cap D_7^{-1} \cap (D_1I_2)^{-1}$	.111956+.700143i	
11_4	$D_7 \cap (I_1D_2)^{-1} \cap (I_1D_2I_1)^{-1}$	-.226914-.591606i	

label	point	z	r
12_1	$D_2I_1 \cap (D_1I_2)^{-1} \cap I_1$	.094428+.533807i	.510793
12_2	$(D_1C^{-1})^{-1} \cap D_2 \cap I_1$	-.375048+.391415i	
12_3	$D_2^{-1} \cap D_4^{-1} \cap (D_2I_1)^{-1}$	.69252-.091676i	
12_4	$D_1C^{-1} \cap D_1I_2 \cap D_4$	.555592+.256281i	
13_1	$(D_1I_2)^{-1} \cap D_1^{-1} \cap I_1$	.028169+.483811i	.565606
13_2	$(I_1D_1)^{-1} \cap (D_1C^{-1})^{-1} \cap I_1$	-.29218+.386649i	
13_3	$D_1I_2 \cap D_1C^{-1} \cap I_2$	.637495+.269757i	
13_4	$I_1D_1 \cap D_1 \cap I_2$	.168645-.671363i	
14_1	$D_2^{-1} \cap (I_2D_2)^{-1} \cap I_2$	.577257-.287567i	.619017
14_2	$D_2 \cap I_2D_2 \cap I_2$	-.577237+287567i	
15_1	$I_2D_2 \cap D_{10}^{-1} \cap I_2$	-.506567+.516745i	.524828
15_2	$D_9^{-1} \cap D_2 \cap I_2$	-.71761+.093119i	
15_3	$(I_2D_2)^{-1} \cap D_{21}^{-1} \cap CD_1$	.375048-.416547i	
15_4	$D_9 \cap D_{10} \cap (CD_1)^{-1}$	-.558242-.231289i	
16_1	$D_1C^{-1} \cap CD_1I_2 \cap I_2$	.637495+.091449i	.619933
16_2	$CD_1 \cap I_1D_1 \cap I_2$	.31098-.563963i	
16_3	$(D_1C^{-1})^{-1} \cap (I_1D_1)^{-1} \cap I_2$	-.31098+.563963i	
16_4	$(CD_1)^{-1} \cap (CD_1I_2)^{-1} \cap I_2$	-.637495-.091449	
17_1	$D_5 \cap D_1I_2 \cap I_2$	.522101+.405691i	.601585
17_2	$D_1 \cap I_2D_1 \cap I_2$	-.009369-.661125	
17_3	$D_1^{-1} \cap (D_1I_2)^{-1} \cap I_2$	.009369+.661125	
17_4	$D_5^{-1} \cap (I_2D_1)^{-1} \cap I_2$	-.522101-.405691i	
18_1	$(D_1I_2)^{-1} \cap D_7^{-1} \cap I_2$	.096616+.726958i	.511152
18_2	$D_{11} \cap D_5^{-1} \cap I_2$	-.482417-.439379i	
18_3	$D_1I_2 \cap D_5 \cap I_1D_2I_1$	.414253+.387945i	
18_4	$D_{11}^{-1} \cap D_7 \cap (I_1D_2I_1)^{-1}$	-.254566-.577831i	
19_1	$I_1D_2I_1 \cap D_6 \cap I_2$	.418477+.553586i	.56347
19_2	$D_7 \cap D_{11}^{-1} \cap I_2$	-.189842-.667489i	
19_3	$D_7^{-1} \cap D_6^{-1} \cap I_2$	.189842+.667489i	
19_4	$(I_1D_2I_1)^{-1} \cap D_{11} \cap I_2$	-.418477-.553586i	
20_1	$D_6^{-1} \cap I_1D_2I_1 \cap I_2$	.316926+.636164i	.542156
20_2	$D_{11}^{-1} \cap (I_1D_2I_1)^{-1} \cap I_2$	-.316926-.636164i	
20_3	$D_5 \cap D_6 \cap I_1D_2I_1$	.482417+.439379i	
20_4	$D_5^{-1} \cap D_{11} \cap (I_1D_2I_1)^{-1}$	-.482417-.439379i	

label	point	z	r
21_1	$D_6 \cap D_5 \cap I_2$	.522101+.514988i	.511153
21_2	$I_2 D_1 \cap D_7 \cap I_2$	-.096616-.726958i	
21_3	$I_1 D_2 I_1 \cap D_6^{-1} \cap D_7^{-1}$	.254566+.577831i	
21_4	$(I_1 D_2 I_1)^{-1} \cap (I_2 D_1)^{-1} \cap D_5^{-1}$	-.414253-.387945i	
22_1	$D_4^{-1} \cap D_2^{-1} \cap I_2$	.71761-.093119i	.524828
22_2	$(I_2 D_2)^{-1} \cap D_3 \cap I_2$	.506567-.516745i	
22_3	$D_1 C^{-1} \cap D_4 \cap D_3^{-1}$	.558242+.231289i	
22_4	$D_2 \cap (D_1 C^{-1})^{-1} \cap I_2 D_2$	-.375048+.416547i	
23_1	$(D_2 I_1)^{-1} \cap C D_1 I_2 \cap D_{12}$	.602889-.002792i	.484225
23_2	$D_2 I_1 \cap D_7^{-1} \cap D_8^{-1}$	.163427+.639511i	
23_3	$(C D_1 I_2)^{-1} \cap D_8 \cap I_1 D_2$	-.602889+.002792i	
23_4	$D_7 \cap D_{12}^{-1} \cap (I_1 D_2)^{-1}$	-.163427-.639511i	

We compute center M and "complex" radius CR of all isometric spheres which partly show up on the boundary of $\mathbf{F}$. If $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbf{C})$ such that $c \neq 0$ we have $M = -\frac{d}{c}$ for the center, $CR = \frac{\sqrt{detS}}{c}$ for the complex radius and $R = |CR|$ for the ordinary (Euclidian) radius of the isometric sphere. We abbreviate $\rho = \sqrt{-(x-1)}$.

S	M	CR	R
I_2	0	$\frac{(x-1)^3}{1+\alpha^{-1}}$ =.398606+.80012i	.893912
D_1	α =-.039862-.693124i	$-\frac{\alpha}{x-1}$ =.416133+.436684i	.603207
D_1^{-1}	$-\alpha$ =.039862+.693124i	$\frac{\alpha}{x-1}$ =-.416133-.436684i	.603207
D_3	$(x-1)\alpha$ =.481306-.637866i	$-\alpha\frac{(x-1)^2}{1+\alpha^{-1}}$ =-.183526+.507021i	.539214
D_3^{-1}	$-(x-1)\alpha^2$ =.307192-.761122i	$\alpha\frac{(x-1)^2}{1+\alpha^{-1}}$ =.183526-.507021i	.539214
D_4	$-\alpha^2$ =.478833-.055259i	$\frac{\alpha}{x-1}$ =-.416133-.436684i	.603207
D_4^{-1}	1	$-\frac{\alpha}{x-1}$ =.416133+.436684i	.603207
$I_1 D_2$	$\frac{1+\alpha}{(x-1)^4}$ =-644833+.198843i	$\frac{\alpha}{(x-1)^2}$ =-.521167-.055259i	.524089
$(I_1 D_2)^{-1}$	$-\frac{\alpha(1+\alpha)}{(x-1)^2}$ =-.163527-.439023i	$-\frac{\alpha}{(x-1)^2}$ =.521167+.055259i	.524089

S	M	CR	R
D_1I_2	$x-1$ =.877439+.744862i	$-\frac{(x-1)^2}{1+\alpha^{-1}}$ =-.713912-.305839i	.776664
$(D_1I_2)^{-1}$	$-(x-1)\alpha$ =-.481306+.637866i	$\frac{(x-1)^2}{1+\alpha^{-1}}$ =.713912+.305839i	.776664
D_5	1	$-\frac{1}{x-1}$ =-.662359+.56228i	.868837
D_5^{-1}	-1	$\frac{1}{x-1}$ =.662359-.56228i	.868837
D_2I_1	$\frac{\alpha(1+\alpha)}{(x-1)^4}$ =.163527+.439023i	$-\frac{\alpha}{(x-1)^2}$ =.521167+.055259i	.524089
$(D_2I_1)^{-1}$	$-\frac{1+\alpha}{(x-1)^4}$ =.644833-.198843i	$\frac{\alpha}{(x-1)^2}$ =-.521167-.055259i	.524089
D_9	α^2 =-.039862-.693124i	$-\frac{\alpha}{x-1}$ =.416133+.436684i	.603207
D_9^{-1}	-1	$\frac{\alpha}{x-1}$ =-.416133-.436684i	.603207
D_{10}	$(x-1)\alpha^2$ =-.461306-.308178i	$-(x-1)^2\frac{\alpha}{1+\alpha^{-1}}$ =-.183526+.502021i	.539214
D_{10}^{-1}	$-(x-1)\alpha$ =-.481306+.637866i	$(x-1)^2\frac{\alpha}{1+\alpha^{-1}}$ =.183526-.507021i	.539214
I_2D_1	$(x-1)\alpha$ =.481306-.637866i	$-\frac{(x-1)^2}{1+\alpha^{-1}}$ =-.713912-.305839i	.776664
$(I_2D_1)^{-1}$	$-(x-1)$ =-.877439-.744862i	$\frac{(x-1)^2}{1+\alpha^{-1}}$ =.713912+.305839i	.776664
I_1	0	$\rho\alpha$ =.71277+.216183i	.744833
I_1D_1	$(x-1)\alpha$ =.481306-.637866i	$-\frac{\rho\alpha}{x-1}$ =-.593665+.257585i	.647139
$(I_1D_1)^{-1}$	$-(x-1)\alpha$ =-.481306+.637866i	$\frac{\rho\alpha}{x-1}$ =.593665-.257585i	.647139
CD_1	α =-.039862-.693124i	$-\frac{\rho(x-1)}{1+\alpha^{-1}}$ =.038218-.72293i	.72394
$(CD_1)^{-1}$	-1	$\frac{\rho(x-1)}{1+\alpha^{-1}}$ =-.038218+.72293i	.72394
CD_1I_2	$x-1$ =.877439+.744862i	$-\frac{\rho}{x-1}$ =-.321308-.874985i	.932114
$(CD_1I_2)^{-1}$	$-(x-1)$ =-.877439-.744862i	$\frac{\rho}{x-1}$ =.321308+.874985i	.932114
D_1C^{-1}	1	$-\frac{\rho(x-1)}{1+\alpha^{-1}}$ =.038218-.72293i	.72394
$(D_1C^{-1})^{-1}$	$-\alpha$ =.939862+.693124i	$\frac{\rho(x-1)}{1+\alpha^{-1}}$ =-.038218+.72293i	.72394

S	M	CR	R
D_2	$-(1+\alpha)$ =-.960138+.693124i	$-\frac{\rho(x-1)^2}{1+\alpha^{-1}}$ =.572017-.60586i	.833229
D_2^{-1}	$1+\alpha$ =.960138-.693124i	$\frac{\rho(x-1)^2}{1+\alpha^{-1}}$ =-.572017+.60586i	.833229
I_2D_2	$-\frac{x-1}{1+\alpha^{-1}}$ =-.644833+.198843i	$\frac{\rho}{1+\alpha^{-1}}$ =.381175+.500329i	.628986
$(I_2D_2)^{-1}$	$\frac{x-1}{1+\alpha^{-1}}$ =.644833-.198843i	$-\frac{\rho}{1+\alpha^{-1}}$ =-.381175-.500329i	.628986
D_6	$-\frac{(x-1)\alpha}{1+\alpha^{-1}}$ =.163527+.439023i	$\frac{\rho}{1+\alpha^{-1}}$ =.381175+.500329	.628986
D_6^{-1}	$\frac{x-1}{1+\alpha}$ =.232606+.943705i	$-\frac{\rho}{1+\alpha^{-1}}$ =-.381175-.500329i	.628986
$I_1D_2I_1$	$-\alpha(1+\alpha)$ =.518694+.637866i	$-\frac{\rho\alpha(x-1)^2}{1+\alpha^{-1}}$ =-.442738-.372329i	.578485
$(I_1D_2I_1)^{-1}$	$\alpha(1+\alpha)$ =-.518694-.637866i	$\frac{\rho\alpha(x-1)^2}{1+\alpha^{-1}}$ =.442738+.372329i	.578435
D_7	$-(1+\alpha^{-1}$ =-.917301-1.43799i	$-\frac{\rho(x-1)^2}{1+\alpha}$ =.823912+.872657i	1.20015
D_7^{-1}	$1+\alpha^{-1}$ =.917301+1.43799i	$\frac{\rho(x-1)^2}{1+\alpha}$ =-.823912-.872657i	1.20015
D_8	α^2 =-.478833+.055259i	$-\frac{\rho\alpha(x-1)}{1+\alpha^{-1}}$ =-.502604+.002327i	.502609
D_8^{-1}	$-(x-1)\alpha$ =-.481306+.637866i	$\frac{\rho\alpha(x-1)}{1+\alpha^{-1}}$ =.502604-.002327i	.502609
D_{11}	$\frac{(x-1)\alpha}{1+\alpha^{-1}}$ =-.163527-.439023i	$-\frac{rho}{1+\alpha^{-1}}$ =-.381175-.500329i	.628986
D_{11}^{-1}	$-\frac{x-1}{1+\alpha}$ =-.232606-.943705i	$\frac{\rho}{1+\alpha^{-1}}$ =.381175+.500329i	.628986
D_{12}	$-(1+\alpha)\alpha^2$ =.478833-.055259i	$\frac{\rho\alpha(x-1)}{1+\alpha^{-1}}$ =.502604-.002327i	.502609
D_{12}^{-1}	α =-.039862-.693124i	$-\frac{\rho\alpha(x-1)}{1+\alpha^{-1}}$ =-.502604+.002327i	.502609

Observe that the fundamental domain **F** for $\tilde{\Gamma}$ is not at all convex. E.g. the isometric spheres of D_4^{-1}, D_5, and D_1C^{-1} are concentric with common center 1; nevertheless all three of them contribute to the boundary of **F**.

A presentation of $\tilde{\Gamma}$ and Γ

We now follow the identification among the onedimensional simplices on the boundary of **F**, induced by the matching of the twodimensional pieces. We show that the spacial angles add up to the correct value which is 2π in all but two cases and π in the exceptional two cases. This is done in the following way: There are exactly 134 edges on the boundary of $\mathbf{F}$. Two of them play the rôle of axes of involutions, and there we have the spacial angle π. The remaining 132 edges fall into 44 triples. Each triple consists of (parts of) intersections of isometric spheres

$$A^{-1} \cap B, B^{-1} \cap C, C^{-1} \cap A$$

with A, B, C some matrix triple from the system of 24 geometric generators and their inverses listed above. (Here again we denote the isometric sphere of a matrix by the matrix symbol itself.) If

$$CBA = identity \quad in \quad PSL(2, \mathbf{C})$$

is true for all such triples the angle conditions are satisfied and, at the same time, a presentation of $\tilde{\Gamma}$ is achieved.

We have, except in the first two cases, the following triples:

	triple	relator
1		I_1^2
2		I_2^2
3	$D_1 \cap I_2, I_2 \cap D_1I_2, (D_1I_2)^{-1} \cap D_1^{-1}$	$(D_1I_2)I_2D_1^{-1}$
4	$D_1 \cap I_1D_1, (I_1D_1)^{-1} \cap I_1, I_1 \cap D_1^{-1}$	$I_1(I_1D_1)D_1^{-1}$
5	$D_1 \cap I_1, I_1 \cap I_1D_1, (I_1D_1)^{-1} \cap D_1^{-1}$	$(I_1D_1)I_1D_1^{-1}$
6	$D_1 \cap I_2D_1, (I_2D_1)^{-1} \cap I_2, I_2 \cap D_1^{-1}$	$I_2(I_2D_1)D_1^{-1}$
7	$I_1D_1 \cap I_2, I_2 \cap D_1C^{-1}, (D_1C^{-1})^{-1} \cap (I_1D_1)^{-1}$	$(D_1C^{-1})I_2(I_1D_1)^{-1}$
8	$I_1D_1 \cap CD_1, (CD_1)^{-1} \cap I_2, I_2 \cap (I_1D_1)^{-1}$	$I_2(CD_1)(I_1D_1)^{-1}$
9	$CD_1 \cap I_2, I_2 \cap CD_1I_2, (CD_1I_2)^{-1} \cap (CD_1)^{-1}$	$(CD_1I_2)I_2(CD_1)^{-1}$
10	$CD_1 \cap D_3, D_3^{-1} \cap I_1, I_1 \cap (CD_1)^{-1}$	$I_1D_3(CD_1)^{-1}$
11	$CD_1 \cap (I_2D_2)^{-1}, I_2D_2 \cap D_{10}^{-1}, D_{10} \cap (CD_1)^{-1}$	$D_{10}^{-1}(I_2D_2)^{-1}(CD_1)^{-1}$
12	$CD_1 \cap D_2^{-1}, D_2 \cap D_9^{-1}, D_9 \cap (CD_1)^{-1}$	$D_9^{-1}D_2^{-1}(CD_1)^{-1}$
13	$CD_1 \cap I_1, I_1 \cap I_2D_1, (I_2D_1)^{-1} \cap (CD_1)^{-1}$	$(I_2D_1)I_1(CD_1)^{-1}$
14	$D_3 \cap I_2, I_2 \cap D_4^{-1}, D_4 \cap D_3^{-1}$	$D_4^{-1}I_2D_3^{-1}$
15	$D_3 \cap (I_2D_2)^{-1}, I_2D_2 \cap (D_1C^{-1})^{-1}, D_1C^{-1} \cap D_3^{-1}$	$(D_1C^{-1})^{-1}(I_2D_2)^{-1}D_3^{-1}$
16	$(I_2D_2)^{-1} \cap I_2, I_2 \cap D_2^{-1}, D_2 \cap I_2D_2$	$D_2^{-1}I_2(I_2D_2)$
17	$(I_2D_2)^{-1} \cap D_2^{-1}, D_2 \cap I_2, I_2 \cap I_2D_2$	$I_2D_2^{-1}(I_2D_2)$
18	$D_2^{-1} \cap D_4^{-1}, D_4 \cap D_1C^{-1}, (D_1C^{-1})^{-1} \cap D_2$	$(D_1C^{-1})D_4^{-1}D_2$
19	$D_2^{-1} \cap (D_2I_1)^{-1}, D_2I_1 \cap I_1, I_1 \cap D_2$	$I_1(D_2I_1)^{-1}D_2$
20	$D_2^{-1} \cap I_1, I_1 \cap (I_1D_2)^{-1}, I_1D_2 \cap D_2$	$(I_1D_2)^{-1}I_1D_2$
21	$(D_2I_1)^{-1} \cap D_4^{-1}, D_4 \cap D_1I_2, (D_1I_2)^{-1} \cap D_2I_1$	$(D_1I_2)D_4^{-1}(D_2I_1)$
22	$(D_2I_1)^{-1} \cap CD_1I_2, (CD_1I_2)^{-1} \cap D_8, D_8^{-1} \cap D_2I_1$	$D_8(CD_1I_2)(D_2I_1)$
23	$(D_2I_1)^{-1} \cap D_{12}, D_{12}^{-1} \cap D_7, D_7^{-1} \cap D_2I_1$	$D_7D_{12}(D_2I_1)$
24	$(D_2I_1)^{-1} \cap I_1, I_1 \cap (I_1D_2I_1)^{-1}, I_1D_2I_1 \cap D_2I_1$	$(I_1D_2I_1)^{-1}I_1(D_2I_1)$

	triple	relator
25	$D_4^{-1} \cap CD_1I_2, (CD_1I_2)^{-1} \cap I_1, I_1 \cap D_4$	$I_1(CD_1I_2)D_4$
26	$CD_1I_2 \cap D_1C^{-1}, (D_1C^{-1})^{-1} \cap I_2, I_2 \cap (CD_1I_2)^{-1}$	$I_2(D_1C^{-1})(CD_1I_2)^{-1}$
27	$CD_1I_2 \cap I_1, I_1 \cap D_9, D_9^{-1} \cap (CD_1I_2)^{-1}$	$D_9I_1(CD_1I_2)^{-1}$
28	$CD_1I_2 \cap D_{12}, D_{12}^{-1} \cap (I_1D_2)^{-1}, I_1D_2 \cap (CD_1I_2)^{-1}$	$(I_1D_2)^{-1}D_{12}(CD_1I_2)^{-1}$
29	$D_{12} \cap I_1, I_1 \cap (I_2D_1)^{-1}, I_2D_1 \cap D_{12}^{-1}$	$(I_2D_1)^{-1}I_1(D_1^{-1}C)^{-1}$
30	$D_1C^{-1} \cap D_1I_2, (D_1I_2)^{-1} \cap I_1, I_1 \cap (D_1C^{-1})^{-1}$	$I_1(D_1I_2)(D_1C^{-1})^{-1}$
31	$D_1C^{-1} \cap I_1, I_1 \cap D_{10}, D_{10}^{-1} \cap (D_1C^{-1})^{-1}$	$D_{10}I_1(D_1C^{-1})^{-1}$
32	$D_1I_2 \cap D_5, D_5^{-1} \cap I_2, I_2 \cap (D_1I_2)^{-1}$	$I_2D_5(D_1I_2)^{-1}$
33	$D_1I_2 \cap I_1D_2I_1, (I_1D_2I_1)^{-1} \cap D_7, D_7^{-1} \cap (D_1I_2)^{-1}$	$D_7(I_1D_2I_1)(D_1I_2)^{-1}$
34	$D_1I_2 \cap I_1, I_1 \cap D_8, D_8^{-1} \cap (D_1I_2)^{-1}$	$D_8I_1(D_1I_2)^{-1}$
35	$D_5 \cap I_2, I_2 \cap I_2D_1, (I_2D_1)^{-1} \cap D_5^{-1}$	$(I_2D_1)I_2D_5^{-1}$
36	$D_5 \cap D_6, D_6^{-1} \cap I_1D_2I_1, (I_1D_2I_1)^{-1} \cap D_5^{-1}$	$(I_1D_2I_1)D_6D_5^{-1}$
37	$D_5 \cap I_1D_2I_1, (I_1D_2I_1)^{-1} \cap D_{11}^{-1}, D_{11} \cap D_5^{-1}$	$D_{11}^{-1}(I_1D_2I_1)D_5^{-1}$
38	$D_6 \cap I_2, I_2 \cap D_7, D_7^{-1} \cap D_6^{-1}$	$D_7I_2D_6^{-1}$
39	$D_6 \cap I_1D_2I_1, (I_1D_2I_1)^{-1} \cap I_2, I_2 \cap D_6^{-1}$	$I_2(I_1D_2I_1)D_6^{-1}$
40	$I_1D_2I_1 \cap I_2, I_2 \cap D_{11}^{-1}, D_{11} \cap (I_1D_2I_1)^{-1}$	$D_{11}^{-1}I_2(I_1D_2I_1)^{-1}$
41	$I_1D_2I_1 \cap D_7^{-1}, D_7 \cap I_2D_1, (I_2D_1)^{-1} \cap (I_1D_2I_1)^{-1}$	$(I_2D_1)D_7^{-1}(I_1D_2I_1)^{-1}$
42	$I_1D_2I_1 \cap I_1, I_1 \cap I_1D_2, (I_1D_2)^{-1} \cap (I_1D_2I_1)^{-1}$	$(I_1D_2)I_1(I_1D_2I_1)^{-1}$
43	$D_7^{-1} \cap I_2, I_2 \cap D_{11}, D_{11}^{-1} \cap D_7$	$D_{11}I_2D_7$
44	$D_7^{-1} \cap D_8^{-1}, D_8 \cap I_1D_2, (I_1D_2)^{-1} \cap D_7$	$(I_1D_2)D_8^{-1}D_7$
45	$D_{10}^{-1} \cap I_2, I_2 \cap D_9^{-1}, D_9 \cap D_{10}$	$D_9^{-1}I_2D_{10}$
46	$I_1D_2 \cap D_9^{-1}, D_9 \cap (I_2D_1)^{-1}, I_2D_1 \cap (I_1D_2)^{-1}$	$(I_2D_1)^{-1}D_9^{-1}(I_1D_2)^{-1}$

We get several relations among the matrices I_1, I_2, D_1, D_2 which are checked to be true by explicit computation; we use $C \sim I_2I_1$:

$$I_1^2 = id$$

from case 1,

$$I_2^2 = id$$

from case 2,

$$I_1D_1 = D_1I_1$$

from case 5,

$$I_2D_2 = D_2I_2$$

from case 17.

Making use of this the rest of the relations reduces to identities which are simply equivalent to the definitions of the various matrices $D_3, D_4, ..., D_{12}$ plus three nontrivial relations, namely

$$D_2^{-1} \sim CD_1C^2D_1$$

from cases $11, 12, 22, 28, 46$,

$$D_2 \sim CD_1^{-1}C^2D_1^{-1}$$

from cases $15, 18, 21, 26$,

$$I_1 \sim CD_2C^{-1}D_1^{-1}CD_2$$

from cases $23, 33, 36, 37, 41, 42$.

So we have the following

Theorem. The hyperbolic polyhedron $\mathbf{F}$ is the fundamental domain of the lattice $\tilde{\Gamma}$ with generators I_1, I_2, C, D_1, and D_2 and relators

$$I_1^2 = 1,$$

$$I_2^2 = 1,$$

$$C = I_2 I_1,$$

$$I_1 D_1 = D_1 I_1,$$

$$I_2 D_2 = D_2 I_2,$$

$$D_2^{-1} = CD_1C^2D_1,$$

$$D_2 = CD_1^{-1}C^2D_1^{-1},$$

$$I_1 = CD_2C^{-1}D_1^{-1}CD_2.$$

Obviously, the number of generators of $\tilde{\Gamma}$ may be reduced to two: C and $D = D_1$ would suffice. But it is doubtful whether this would lead to a simplification.

One notices that the lattice $\tilde{\Gamma}$ may be enlarged by

$$I_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

the result again being a lattice. We have

$$I_0 I_1 = -I_1 I_0,$$

$$I_0 I_2 = -I_2 I_0,$$

$$I_0 D_1 = D_1^{-1} I_0,$$

$$I_0 D_2 = D_2^{-1} I_0.$$

We check that the relators listed in the above theorem allow conjugation by I_0, as well. This is evident for all but the very last relator:

Lemma. $I_1 = CD_2^{-1}C^{-1}D_1CD_2^{-1}$, to be understood as an identity in $PSL(2, \mathbf{C})$.

Proof. We have

$$I_1 = CD_2C^{-1}D_1^{-1}CD_2$$

$$= I_1^{-1} = D_2^{-1}C^{-1}D_1CD_2^{-1}C^{-1}$$
$$= D_1C^{-2}D_1C^{-2}D_1CD_2^{-1}C^{-1}$$

where we have replaced D_2^{-1} by $D_1C^{-2}D_1C^{-1}$, once. As I_1 and D_1 commute we have

$$I_1 = C^{-2}D_1C^{-2}D_1CD_2^{-1}C^{-1}D_1$$
$$= C^{-1}D_1C^{-2}D_1CD_2^{-1}C^{-1}D_1C$$

from $CI_1 = I_1C^{-1}$. We have

$$C^{-1}D_1C^{-2}D_1 = I_1(CD_1C^2D_1)I_1 = I_1D_2^{-1}I_1,$$

so

$$I_1 = I_1D_2^{-1}I_1CD_2^{-1}C^{-1}D_1C$$

which means

$$D_2 = I_1CD_2^{-1}C^{-1}D_1C,$$

or, equivalently,

$$I_1 = CD_2^{-1}C^{-1}D_1CD_2^{-1}.$$

So the lemma is proven.

The fundamental domain constructed above shows this rotational symmetry.

Proposition. $\tilde{\Gamma}^{ab} \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/10\mathbf{Z}$; more precisely:

$$\tilde{\Gamma}^{ab} \cong < \bar{I}_1, \bar{D}_1 | 2\bar{I}_1 = 10\bar{D}_1 = 0 >,$$

$$\bar{I}_2 = 5\bar{D}_1,$$

$$\bar{D}_2 = 3\bar{D}_1 + \bar{I}_1,$$

$$\bar{C} = 5\bar{D}_1 + \bar{I}_1.$$

Here $\bar{S}$ for $S \in \tilde{\Gamma}$ means $S mod[\tilde{\Gamma}, \tilde{\Gamma}]$.

The proof is easily read off from the presentation of $\tilde{\Gamma}$.

We are now in the position to define the lattice Γ:

Definition. Take the endomorphism of $\tilde{\Gamma}^{ab}$ which sends both $\bar{I}_1$ and $\bar{D}_1$ to $\bar{I}_1$, and let

$$\varphi : \tilde{\Gamma} \longrightarrow \tilde{\Gamma}^{ab}$$

the composition of the canonical mapping from $\tilde{\Gamma}$ onto $\tilde{\Gamma}^{ab}$ with this endomorphism. Then

$$\Gamma := ker\varphi.$$

Obviously, Γ has index 2 in $\tilde{\Gamma}$. A glance at the fundamental domain of $\tilde{\Gamma}$ shows that Γ contains no torsion: There are only two conjugacy classes of torsion elements in $\tilde{\Gamma}$, namely those of I_1 and I_2, but

$$\varphi(I_1) = \varphi(I_2) = \bar{I}_1.$$

We create, using standard routine from combinatorial group theory, a presentation of Γ:

Proposition. Let $A = D_2, B = I_1 D_1$. Then Γ is generated by A, B, and C and defined by the following relators:

$$ACBC^{-2}B = 1,$$

$$ABC^2BC^{-1} = 1,$$

$$AC^{-2}B^{-1}C^2B^{-1}CB^{-1}C = 1.$$

Theorem. Γ may be given the following presentation: generators:

$$X, Y, M,$$

relators:

$$MXM^{-1} = YX,$$

$$MYM^{-1} = (YX)^{-5}Y,$$

$$M^2(Y^{-1}X^{-1}YX)^3 = 1.$$

Proof. Let

$$X = C^{-2}A^{-1}C^2B,$$

$$Y = (C^{-2}A^{-1}C^2B)^{-3}B^{-1}C,$$

$$M = C^3.$$

Then

$$A = C^3Y^{-1}X^{-4}C^{-2},$$

$$B = CY^{-1}X^{-3},$$

$$C = MY^{-1}X^{-1}YX.$$

So X, Y, and M generate Γ. We translate the relators among A, B, and C into relators among X, Y, and M:

(1)
$$X^3YCX^3YX^4YC^{-3} = 1,$$

(2)
$$X^3YC^{-3}X^3YCX^4YC^{-2} = 1,$$

(3)
$$C^3Y^{-1}X^{-4}C^{-4}X^3YCX^3YX^3Y = 1.$$

We use (1) in order to single out C:

(4)
$$C = Y^{-1}X^{-3}C^3Y^{-1}X^{-4}Y^{-1}X^{-3}.$$

Equation (2) contains the first power of C exactly once. If we replace this by its expression (4) in terme of X, Y, and C^3 we get

(5)
$$C^2Y^{-1}X^{-1}YX = 1.$$

If we do the same to equation (3) we get

(6)
$$C^3Y^{-1}X^{-1}YX^4YC^{-3}X^2Y = 1.$$

This is

(7)
$$CC^2Y^{-1}X^{-1}YXX^3YC^{-3}X^2Y = 1$$

and by virtue of (5):

(8)
$$CX^3YC^{-3}X^2Y = 1.$$

We again use (4) and get, now with M instead of C^3:

(9)
$$MY^{-1}X^{-4}M^{-1} = X.$$

This allows to shorten (3):

(10)
$$XC^{-1}X^3YCX^3YX^3Y = 1.$$

Here we use (4) twice in order to replace C:

(11)
$$C^3Y^{-1}X^{-1}YX^4YX^4YC^{-3} = Y^{-1}X^{-3}.$$

Equation (9) allows considerable simplification:

(12)
$$MY^{-1}X^{-1}YM^{-1} = Y^{-1}X^{-1}.$$

This may be rewritten as

(13)
$$MY^{-1}X^{-4}X^3YM^{-1} = Y^{-1}X^{-1},$$

so

(14)
$$MX^3YM^{-1} = X^{-1}Y^{-1}X^{-1},$$

again by (9). Combination of (9) and (14) now gives

(15)
$$MXM^{-1} = YX$$

and

(16)
$$MYM^{-1} = (YX)^{-5}Y.$$

We cube the equation (5) and get, together with (15) and (16) the relators among X, Y, and M listed in the theorem. Conversely, the relators defining Γ in terms of generators A, B, and C are consequences of the relators among X, Y, and M. This is a matter of straightforward computation.

We add a presentation of Γ based on the generators C and X:

Proposition.
$$\Gamma \cong \langle C, X | XC^3X^4C^3X = C^3XC^3, C^3XC^{-1}XC^3 = XC^3X \rangle.$$

The proof is elementary.

Although this presentation may not be very practical it, at least, allows to quickly read off the homology of Γ:
$$2\bar{C} = 5\bar{X} = 0.$$

So we have

Proposition. Γ^{ab} is cyclic of order 10.

Computation of the Volume of M

Let T be a hyperbolic tetrahedron in $\mathbf{H}^3$ with exactly one vertex on the boundary of $\mathbf{H}^3$, which we transfer to $r = \infty$. Then three of the four faces of T are euclidean planes; the projection of T onto $\mathbf{C}$ is a euclidean triangle with vertices z_1, z_2, z_3; the fourth face of T is a hyperbolic plane that is a euclidean sphere with center $z_0 \in \mathbf{C}$. $T = T(S, z_1, z_2, z_3)$ is completely determined by the data S, z_1, z_2, z_3. Fig. 2 shows the projection of S and T onto $\mathbf{C}$ as well as α and β, two of the six angles between pairs of faces of T. Let γ be the angle between S and the hyperbolic plane through z_2, z_3, ∞ (cf. fig. 1).

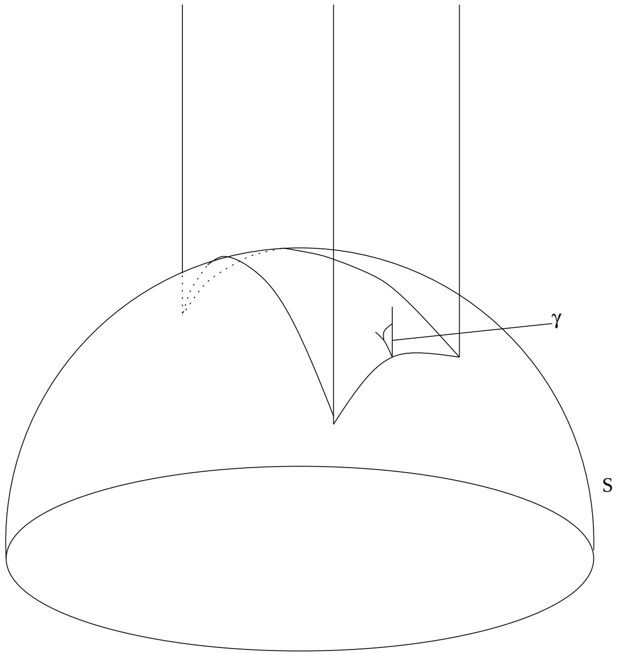

Figure 1: Hyperbolic tetrahedron with vertex at ∞ and opposite face

Let $z_1 = z_0$, $\beta = \pi/2$. Then the volume of T is given by

$$V(T) = \frac{1}{4}\left[\mathcal{L}(\frac{\pi}{2} - \alpha + \gamma) + \mathcal{L}(\frac{\pi}{2} - \alpha - \gamma) + 2\mathcal{L}(\alpha)\right] \tag{1}$$

(cf. [6], chap. 7.1); the function $\mathcal{L}$ is the Lobachevsky–function

$$\mathcal{L}(\theta) := -\int_0^\theta \log(|2\sin u|)\mathrm{d}u.$$

Now let $z_1 = z_0$, α, β arbitrary. Then by the above formula

$$\begin{aligned} V(T) &= \frac{1}{4}\Big[\mathcal{L}(\frac{\pi}{2} - \alpha + \gamma) + \mathcal{L}(\frac{\pi}{2} - \alpha - \gamma) + 2\mathcal{L}(\alpha) \\ &\quad + \mathcal{L}(\frac{\pi}{2} - \beta + \gamma) + \mathcal{L}(\frac{\pi}{2} - \beta - \gamma) + 2\mathcal{L}(\beta)\Big]. \end{aligned}$$

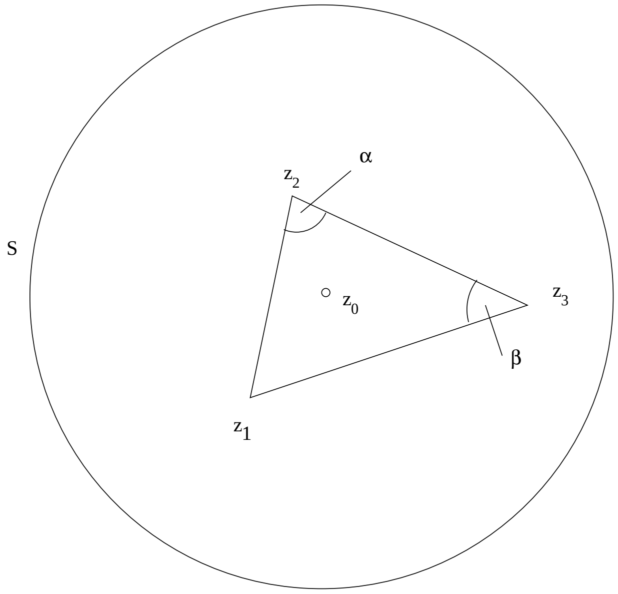

Figure 2: Projection of tetrahedron and isometric sphere

Now let z_1, z_2, z_3 be arbitrary points in $\mathbf{C}$ in the interior of the projection of S. Then the volume of T is a weighted sum of volumes of the above type:

$$\begin{aligned} V(T(S, z_1, z_2, z_3)) &= \epsilon_1 V(T(S, z_0, z_2, z_3)) \\ &\quad + \epsilon_2 V(T(S, z_0, z_1, z_3)) \\ &\quad + \epsilon_3 V(T(S, z_0, z_1, z_2)), \end{aligned}$$

where $\epsilon_i = +/-1$ are signs determined by:

$\epsilon_i = +1$, iff z_0 and z_i are on the same side of the euclidean line through the remaining two points (i.e. iff the scalar products of $z_0 - z_j$ and $z_i - z_j$ with the normal vector $e^{i\pi/2}(z_k - z_j)$ $(i \neq j \neq k \neq i)$ have the same sign). If z_0 lies on the segment between the two remaining points, then ϵ_i doesn't matter, since the corresponding tetrahedron is degenerate.

Given S, z_1, z_2, z_3, the angles α, β, γ are easily computed:

$$\alpha = \arccos \frac{(z_3 - z_2)\overline{(z_0 - z_2)}}{|z_3 - z_2||z_0 - z_2|},$$

$$\beta = \arccos \frac{(z_2 - z_3)\overline{(z_0 - z_3)}}{|z_2 - z_3||z_0 - z_3|},$$

$$\gamma = \arccos \frac{(z_2 - z_0)\overline{(z_3 - z_0)} - \overline{(z_2 - z_0)}(z_3 - z_0)}{2R(S)|z_2 - z_3|},$$

where $R(S)$ is the euclidean radius of S.

So the volume of a hyperbolic polyhedron with one vertex at $r = \infty$ and the remaining vertices on one and the same hyperbolic plane S is the sum of volumes of tetrahedra of the above type.

The following figure shows the projection of the fundamental domain $\mathbf{F}$ of $\tilde{\Gamma}$ onto $\mathbf{C}$. $\mathbf{F}$ is bounded by isometric spheres of group elements of $\tilde{\Gamma}$, as indicated in the figure. The lines and vertices are the projections of intersecting lines and triplepoints of these isometric spheres. In the euclidean sense, $\mathbf{F}$ is bounded from above by the isometric sphere of I_2 and bounded from below by all the other spheres. So the volume of $\mathbf{F}$ is the sum of the volumes of the polyhedra above the lower spheres minus the volume of the polyhedron above I_2.

The following list gives the computed result. We computed all needed volumes for half of the fundamental domain. This explains the "2" in the last result. The names are self–explanatory. Within the computation of the Lobachevsky–function the numerical integration algorithm of Mathematica was used. We include a complete commented listing of the used algorithms.

volinvD1	0.04719510988978218
volinvD1I2	0.02965980923145807
volinvD8	0.003790561224624151
volinvD7	0.0081291472103686
volD2I1	0.03437480178756513
volinvD6	0.007979438304947739
volI1D2I1	0.0571924268879434
volD6	0.007979438304947934
volD5	0.007084006725649822
volD1I2	0.02965980923145782
volD4	0.007559630567222156
volinvD3	0.005936934666564642
volD1invC	0.03044613590417197
volCD1I2	0.02052618944190808
volD12	0.003790561224624151
volinvD4	0.00755963056722106
volinvD2I1	0.03437480178756558
volinvD2	0.04553351915169055
volinvI2D2	0.03275390848395124
volD3	0.00593693466656442
volCD1	0.03044613590417179
volI1D1	0.04038076032689237
volI1	0.5544788288965777
volI2	0.6997047541794377

$$\text{total volume}(\mathbf{F}) = 2\left(\text{-volI2} + \sum(\text{all others})\right) = 0.7061275284168657.$$

Listing of Used Algorithms

```
(* ******************************************************************
*                                                                  *
* Set of Algorithms volumen.m for the Computation of Volume        *
* of Hyperbolic Manifolds With Cusp                                *
*                                                                  *
****************************************************************** *)

Print["'Hilfe' fuer Hilfe."];
Hilfe="WasIst[f]=Hilfe zur Funktion f. Fuer folgende Funktionen ist
       Hilfe erhaeltlich: R, M, rv, volumen, tp, lob. Folgende weitere
       Funktionen stehen zur Verfuegung: wa, wb, wg, v, selbeSeite. Eine
       Ueberpruefung auf Falscheingaben erfolgt generell nicht!";

(* ******************************************************************
* Function M (center of isometric sphere)                          *
*                                                                  *
* Input:      MA: 2X2 complex matrix                               *
* Condition:  none                                                 *
*                                                                  *
* Output:     M: complex number                                    *
* Condition:  M is the center of the isometr. sphere of MA         *
****************************************************************** *)
M[MA_]:=-MA[[2,2]]/MA[[2,1]];

WasIst[M]="M[S]=Mittelpunkt der isom.Sph. der Matrix S.";

* ******************************************************************
* Function R (radius of isometric sphere)                          *
*                                                                  *
* Input:     RA: 2x2 complex matrix                                *
* Condition: none                                                  *
*                                                                  *
* Output:    R: real number                                        *
* Condition: R is the radius of hte isom. sphere of MA             *
****************************************************************** *)
R[RA_]:=Abs[Sqrt[Det[RA]]/RA[[2,1]]];

WasIst[R]="R[S]=Radius der isom.Sph. der Matrix S.";

(* ******************************************************************
* Function tp (triple-point of three isom. spheres)                *
*                                                                  *
* Input:     tA, tB, tC: 2x2 complex matrices                      *
* Condition: none                                                  *
```

```
*                                                                      *
* Output:    z: complex number or ComplexInfinity                      *
* Condition: If det(tA)=..=det(tC)=1, then [if isom. spheres of        *
*            tA,tB,tC intersect, then z=projection(point of            *
*            intersection), else ComplexInfinity]                      *
********************************************************************** *)
tp[tA_,tB_,tC_]:=If[
   (Abs[M[tA]-M[tB]]<R[tA]+R[tB])&&
   (Abs[M[tA]-M[tC]]<R[tA]+R[tC])&&
   (Abs[M[tB]-M[tC]]<R[tB]+R[tC]),
   Det[{{tA[[2,1]] Conjugate[tA[[2,1]]],Conjugate[tA[[2,1]]] tA[[2,2]],
tA[[2,2]] Conjugate[tA[[2,2]]]-1},
       {tB[[2,1]] Conjugate[tB[[2,1]]],Conjugate[tB[[2,1]]] tB[[2,2]],
tB[[2,2]] Conjugate[tB[[2,2]]]-1},
       {tC[[2,1]] Conjugate[tC[[2,1]]],Conjugate[tC[[2,1]]] tC[[2,2]],
tC[[2,2]] Conjugate[tC[[2,2]]]-1}}]/Det[
      {{tA[[2,1]] Conjugate[tA[[2,1]]], tA[[2,1]] Conjugate[tA[[2,2]]],
        Conjugate[tA[[2,1]]] tA[[2,2]]},
       {tB[[2,1]] Conjugate[tB[[2,1]]], tB[[2,1]] Conjugate[tB[[2,2]]],
        Conjugate[tB[[2,1]]] tB[[2,2]]},
       {tC[[2,1]] Conjugate[tC[[2,1]]], tC[[2,1]] Conjugate[tC[[2,2]]],
        Conjugate[tC[[2,1]]] tC[[2,2]]}}],
   ComplexInfinity];
WasIst[tp]="tp[A,B,C]=Tripelpunkt der drei Matrizen A,B,C. ComplexInfinity,
            falls dieser nicht existiert. (Keine Ueberpruefung von
            Falscheingaben!)";

***********************************************************************
* Function wa (angle alpha, cf. figures in sect. 4.7)                 *
*                                                                     *
* Input:      wS: 2x2 complex matrix                                  *
*             wz1, wz2: complex numbers                               *
* Condition:  none                                                    *
*                                                                     *
* Output:     wa: real number                                         *
* Condition:  wa=angle at wz1 of triangle M[wS],wz1,wz2               *
*             (cf. sect. 4.7)                                         *
********************************************************************** *)
wa[wS_,wz1_,wz2_]:=Abs[N[ArcCos[(Re[wz2-wz1] Re[M[wS]-wz1]+
                 Im[wz2-wz1] Im[M[wS]-wz1])/
                (Abs[wz2-wz1] Abs[M[wS]-wz1])]]];

***********************************************************************
* Function wb (angle beta, cf. figures in sect. 4.7)                 *
*                                                                     *
* Input:      wS: 2x2 complex matrix                                  *
*             wz1, wz2: complex numbers                               *
```

```
* Condition:  none                                                  *
*                                                                   *
* Output:     wb: real number                                       *
* Condition:  wb=angle at wz2 of triangle M[wS],wz1,wz2             *
*             (cf. sect. 4.7)                                       *
******************************************************************* *)
wb[wS_,wz1_,wz2_]:=Abs[N[ArcCos[(Re[wz1-wz2] Re[M[wS]-wz2]+
            Im[wz1-wz2] Im[M[wS]-wz2])/
(Abs[wz1-wz2] Abs[M[wS]-wz2])]]];

*********************************************************************
* Function wg (angle gamma, cf. figures in sect. 4.7)               *
*                                                                   *
* Input:      wS: 2x2 complex matrix                                *
*             wz1, wz2: complex numbers                             *
* Condition:  none                                                  *
*                                                                   *
* Output:     wg: real number                                       *
* Condition:  If det[wS]=1 and if wz1,wz2 in interior of proj. of   *
*             isom. sphere of wS, then:                             *
*             wg=angle gamma of tetrahedron determined by projected *
*             vertices M[wS],wz1,wz2                                *
*             (cf. sect. 4.7)                                       *
******************************************************************* *)
wg[wS_,wz1_,wz2_]:=Abs[N[ArcCos[Abs[(wz1-M[wS]) Conjugate[(wz2-M[wS])]-
             Conjugate[(wz1-M[wS])] (wz2-M[wS])]/
            (2 R[wS] Abs[wz1-wz2])]]];

(* *****************************************************************
* Function selbeSeite                                               *
*                                                                   *
* Input:     sM, sz1, sz2, sz3: complex numbers                     *
* Condition: none                                                   *
*                                                                   *
* Output:    selbeSeite: +/-1 (sign)                                *
* Condition: If sM and sz1 are on the same side of the eucl. line   *
*            through sz2, sz3, i.e. if scalar product of sM-sz2 and *
*            sz1-sz2 respectively with normal vector                *
*            (-Im[sz3-sz2],Re[sz3-sz2]) has same sign, then         *
*            selbeSeite=+1, else selbeSeite=-1                      *
******************************************************************* *)
selbeSeite[sM_,sz1_,sz2_,sz3_]:=If[
(-Re[sM -sz2] Im[sz3-sz2]+Im[sM -sz2] Re[sz3-sz2])*
(-Re[sz1-sz2] Im[sz3-sz2]+Im[sz1-sz2] Re[sz3-sz2])>0,
+1,-1];

(* *****************************************************************
```

```
* Function rv (volume of tetrahedron with vertex M[matrix])         *
*                                                                    *
* Input:     rM: 2x2 complex matrix                                  *
*            rz1, rz2: complex numbers                               *
* Condition: none                                                    *
*                                                                    *
* Output:    rv: real number                                         *
* Condition: If det[rM]=1 and if rz1,rz2 in project. of isom.sphere  *
*            of rM, then rv = volume(tetrahedron with vertex infty,  *
*            projected vertices M[rM],rz1,rz2)                       *
*            (cf. sect. 4.7 for formula)                             *
******************************************************************** *)
rv[rM_,rz1_,rz2_]:=1/4(2 lob[wa[rM,rz1,rz2]]+
       2 lob[wb[rM,rz1,rz2]]-
 lob[wa[rM,rz1,rz2]+wg[rM,rz1,rz2]-Pi/2]-
 lob[wa[rM,rz1,rz2]-wg[rM,rz1,rz2]-Pi/2]-
 lob[wb[rM,rz1,rz2]+wg[rM,rz1,rz2]-Pi/2]-
 lob[wb[rM,rz1,rz2]-wg[rM,rz1,rz2]-Pi/2]);

WasIst[rv]="rv[M,z1,z2]=hyp. Volumen des Tetraeders ueber der isom.Sph.
            von M, geg. durch die drei Punkte M[M], z1,z2. (Keine
            Uberpruefung von Falscheingaben!)";

(* ******************************************************************
* Function lob (Lobachevsky-function)                                *
*                                                                    *
* Input:      y: real number                                         *
* Condition:  none                                                   *
*                                                                    *
* Output:     lob[y]: real number                                    *
* Condition:  lob[y] = Lobachevsky-function[y], computed with        *
*             numerical integration (cf. sect. 4.7 for formula)      *
******************************************************************** *)
lob[y_]:=NIntegrate[-Log[Abs[2 Sin[u]]],{u,0,y}];

WasIst[lob]="lob[y]=Integral von 0 bis y ueber -log(|2 sin u|) du.
             (Lobacevskij-Funktion; numerisch integriert.) (y reell >0.
             Keine Ueberpruefung von Falscheingaben!)";

(* ******************************************************************
* Function v (volume of tetrahedron)                                 *
*                                                                    *
* Input:     vM: 2x2 complex matrix                                  *
*            vz1,vz2,vz3: complex numbers                            *
* Condition: none                                                    *
*                                                                    *
* Output:    v: real number                                          *
```

```
* Condition: If det[vM]=1 and vz1,vz2,vz3 in proj. of isom. sphere of *
*            vM, then                                                  *
*            v = volume of tetrahedron with vertex infty, projected    *
*            vertices vz1,vz2,vz3, face on isom.sphere of vM           *
******************************************************************** *)
v[vM_,vz1_,vz2_,vz3_]:=selbeSeite[M[vM],vz1,vz2,vz3] rv[vM,vz2,vz3]+
       selbeSeite[M[vM],vz2,vz1,vz3] rv[vM,vz1,vz3]+
         selbeSeite[M[vM],vz3,vz2,vz1] rv[vM,vz2,vz1];

(* *********************************************************************
* Function volumen (volume of certain polyeder)                        *
*                                                                      *
* Input:     volS: 2x2 complex matrix                                  *
*            vollist: (ordered) list of complex numbers                *
* Condition: none                                                      *
*                                                                      *
* Output:    volumen: real number                                      *
* Condition: If det[volS]=1 and vollist has at least three elements,   *
*            all of which are in interior of proj. of isom. sphere of *
*            volS, and elements of vollist are clockwise or counter-   *
*            clockwise ordered vertices of polygon on isom. sph. of    *
*            volS, then                                                *
*            volumen = volume of polyeder with vertex infty, other     *
*            vertices on isom. sphere of volS with proj. given by      *
*            vollist                                                   *
******************************************************************** *)
volumen[volS_,vollist_]:=Sum[
v[volS,vollist[[1]],vollist[[i-1]],vollist[[i]]],
{i,3,Length[vollist]}];

WasIst[volumen]="volumen[S,{z1,z2,...,zn}]=Volumen des hyp. Polyeders
                 ueber der isom.Sph. von S, geg. durch die Punkte
                 z1,...,zn. (Erforderlich: Det[S]=1, zi liegen im
                 Inneren des isom.Kreises von S, n>2, zi begrenzen ein
                 konvexes Polygon. Keine Ueberpruefung von Falscheingaben!)";

(* *********************************************************************
* End of volumen.m                                                     *
******************************************************************** *)
```

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