Contents

1 Introduction

1.1 Background

William Thurston wrote on page 1.1 of his famous lecture notes[1] that "topology and geometry, in dimension up through 3, are very intricately related. (...) It is not totally unreasonable to hope that eventually all 3–manifolds will be understood in a systematic way." This strategy has been very fruitful for the study of surfaces. In dimension 3, however, the main conjecture, Thurston's hyperbolization conjecture, that every compact, irreducible, atoroidal 3–manifold with infinite fundamental group is hyperbolic, has so far been proved only under additional hypotheses, e. g. for compact, atoroidal, Haken 3–manifolds[2].

If the 3–manifold M fibers over the circle, the monodromy being a pseudo–Anosov diffeomorphism of the fiber, then the geometrization conjecture can be proved. The proof, however, fills books[3] and is far from being easily readable. Moreover, it is non–constructive, i. e. you get no idea of how the manifold looks like. Thurston wrote[4]: "If you draw pictures of a pseudo–Anosov diffeomorphism, you can understand geometrically what it *does* (...)" However, he gave no example. In fact, in contrast to the very complicated and elaborated theory, only few examples can be found in the literature. There is a series of examples in [Casson/Bleiler 88], §5, for a set of automorphisms of the closed surface of genus 2. Otal provides two examples[5], one is a fibration with the punctured torus, the other a closed compact fibered manifold. We present examples of a hyperbolic 3–manifold that fibers over S^1 with a noncompact surface of genus $g > 1$.

There is a method, based on old results about the representability of groups in $\mathrm{SL}_2(\mathbf{C})$ and working in purely algebraic terms, that allows to decide, whether a given fibered manifold M^3 is hyperbolic. This method starts from the known fundamental group and is totally constructive, i. e. in the end you have constructed a fundamental domain for M in $\mathbf{H^3}$. It has been used by Helling and others for widespread investigations on torus–fibrations. The adaptation of this method to manifolds, the fiber of which is a once–punctured surface of genus $g = 2$, has been the outset of the present work.

1.2 Outline and Summary of Results

The aim has been to present series of hyperbolic 3–manifolds, fibering over S^1 with a once–punctured surface of genus 2, and to state general results about the connection of the algebraic presentation of the monodromy f and the property

[1] [Thurston 82].
[2] See [Canary]; [Kapovich] for a proof; [Otal 96] for a discussion of the hypotheses.
[3] [Otal 96]; [Fathi/Laudenbach/Poénaru 79].
[4] [Thurston 88], p. 431.
[5] [Otal 96], chap. 9.

of being pseudo–Anosov. This aim could not yet be achieved. It turned out that two major difficulties had to be overcome.

First, the set of algebraic equations which are used to decide on representability of the fundamental group $\pi_1(M^3)$ in $\mathrm{SL}_2(\mathbf{C})$ are high–degree equations in 14 variables. They are far too complicated to be solved by hand. The built–in (numarical as well as algebraical) automatic solution algorithms of computer algebra systems like e. g. `Mathematica` and `Maple` have failed as well even in simple cases.

Second, in case of fibers of genus $g > 1$ reducible diffeomorphisms f can in general not be distinguished algebraically from pseudo–Anosov diffeomorphisms. There seems to be no practicable criterion up to now.

These two difficulties have been overcome in single cases. Moreover, it is not totally unreasonable to hope that in the near future we will be able to present elaborated tools, highly adapted to the present context, as well as their results, (one or more) series of hyperbolic manifolds, the fiber of which is a once–punctured surface of genus 2.

We now give an outline of the following sectionss.

In section 2, the fibering 3–manifold $M_g(f)$ and its fundamental group are constructed. The presentation of the fundamental group is discussed.

In section 3 we discuss several criteria for hyperbolicity of $M_g(f)$, depending on the genus g and the number of punctures of the fiber.

Section 4 deals with technical conditions in form of algebraic equations for faithful, discrete, cofinite representability of groups in $\mathrm{SL}_2(\mathbf{C})$ and consequences thereof. We include a subsection on computation of the volume of $M_g(f)$, once a fundamental domain has been constructed.

The adapted algebraic method for constructing a representation of $\pi_1(M_g(f))$ in $\mathrm{SL}_2(\mathbf{C})$ is described in full detail in section 5. Using the computer is inevitable in order to solve the high–degree algebraic equations in 14 variables that have been described in section 4.

It is most difficult to find automorphisms that are pseudo–Anosov in case of a fiber with $g > 1$. We illustrate this in section 6.

Two examples (one of a hyperbolic, one of a non–hyperbolic $M_g(f)$) are presented in section 7. These examples shed some light on the question, why examples are seldom to be found.

In section 8 we collect further results—mostly negative, but one further hyperbolic manifold—about several (relatively easy to compute) automorphisms f.

Section 9 summarizes preliminary considerations about fibrations with a twice–punctured surface. We present 3 negative results for the twice–punctured torus.

Work has not stopped. In section 10 we set some milestones that seem to be within reach for the nearer or further future.

We now summarize the main results of the present work.

- There are hyperbolic 3–manifolds that fiber over S^1 with a punctured surface of genus $g > 1$; we present two examples in detail for $g = 2$ with one puncture.

No one was really in doubt about this; but:

- Those manifolds do not lie around openly, in fact they are hard to find, whereas one might expect something else according to the fact that hyperbolic geometry is the typical geometry.

The reason for this is:

- The homology of the diffeomorphism f yields no criterion for hyperbolicity in the case $g > 1$, number of punctures > 0.
- The action of f on (all) 0–homologous curves, which cannot be seen algebraically, is the crucial point.

Nonetheless:

- The method of finding fixed points under f on the character variety of the fiber, which has been very successful for torus fibrations, can with the aid of a computer be applied to the case $g > 1$ as well.
- We present a set of computer algorithms, which are tools for the interactive construction of hyperbolic 3–manifolds. These algorithms work algebraically/non–numerically as far as possible. They are easily adaptable to other cases.

2 Orientable 3–Manifolds Fibering Over the Circle With a Surface of Genus g

2.1 Construction and Fundamental Group

Let $S_g^{(r)}$ be an orientable compact surface of genus $g > 0$, punctured r times, i. e. with r disjoint open disks removed ($r > 0$). The fundamental group of $S_g^{(r)}$ is determined by g and r:

$$\begin{aligned}\pi_1(S_g^{(r)}) &= \langle a_1, a_2, \ldots, a_{2g-1}, a_{2g}, \lambda_1, \lambda_2, \ldots, \lambda_r | \\ &\quad \Pi_{i=1}^{g}[a_{2i-1}, a_{2i}]\,\Pi_{j=1}^{r}\lambda_j = 1\rangle,\end{aligned}$$

where a_i represent the loops around the handles of $S_g^{(r)}$, whereas λ_j represent the boundary components, i. e. the loops around the removed disks. If $r > 0$, then $\pi_1(S_g^{(r)})$ is a non–abelian free group with $2g + r - 1$ generators:

$$\pi_1(S_g^{(r)}) \cong \langle a_1, \ldots, a_{2g}, \lambda_2, \ldots, \lambda_r\rangle.$$

Let f be a homeomorphism of $S_g^{(r)}$; let I be the closed unit interval in $\mathbf{R}$. We define

$$M_g(f) := S_g^{(r)} \times I/(x,0) \sim (f(x),1) \quad \forall x \in S_g^{(r)},$$

omitting the "r" in the name, since it will always be clear from the context—it can be read off the definition of f anyway and will in most cases be equal to 1. $M_g(f)$ is a compact 3-manifold with boundary, the latter consisting of $r \geq 0$ tori.

The homeomorphism f is determined up to homotopy (i. e., according to the theorem of Baer, up to isotopy) by the induced action f_* on $\pi_1(S_g^{(r)})$. Hence $M_g(f)$ is determined up to topological equivalence by f_*. For that reason we will mostly omit the "$*$".

The fundamental group of $M_g(f)$ can immediately be read off fig. 2.1 (the cylinders represent the boundary tori). Let μ be a continuous path $[0,1] \to$

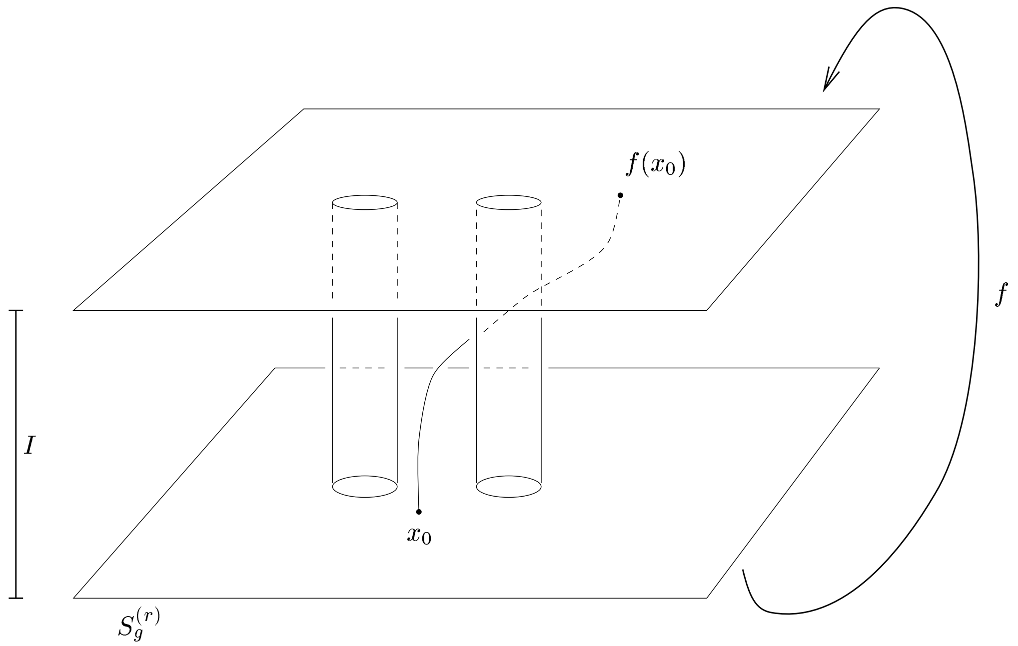

Figure 1: Construction of $M_g(f)$.

$S_g^{(r)} \times I$, connecting the base point x_0 of $\pi_1(S_g^{(r)})$ below with its image under f above, i. e.

$$\mu(0) = x_0,\ \mu(1) = f(x_0).$$

Then the image of μ in $M_g(f)$ (called μ again) is a closed loop. So we have

$$\pi_1(M_g(f)) \cong \langle \pi_1(S_g^{(r)}), \mu | \\ \mu\omega\mu^{-1} = f(\omega) \quad \forall \omega \in \pi_1(S_g^{(r)})\rangle.$$

Sometimes we use the relations $\mu^{-1}\omega\mu = f(\omega)$ instead.

2.2 The Case of a Once–Punctured Surface

Now let $r = 1$. This is the interesting case in the present work. Let $\lambda_1 = \lambda \in \pi_1(S_g^{(1)})$ represent the loop around the removed disk. Boundary components of $S_g^{(r)}$ are mapped to boundary components by f, therefore $f_*(\lambda)$ is conjugate to λ or λ^{-1}. Since $M_g(f)$ is supposed to be orientable, the second case is ruled out. Choosing the base point x_0 for $\pi_1(S_g^{(1)})$ on the boundary, we may assume

$$f_*(\lambda) = \lambda.$$

Let $\mathrm{Aut}(\pi_1(S_g^{(1)}), \lambda) < \mathrm{Aut}(\pi_1(S_g^{(1)}))$ be the subgroup of all λ–preserving automorphisms of $\pi_1(S_g^{(1)})$. From now on we will omit the "$*$"; in the case $r = 1$ we will consider only $f \in \mathrm{Aut}(\pi_1(S_g^{(1)}), \lambda)$.

The interior of $M_g(f)$ has one topological end. If $M_g(f)$ is a complete oriented hyperbolic manifold, this end is an ϵ–end, which is homeomorphic either to a tube $(\overline{D^2} \times S^1)$ or a cusp $(T^2 \times [0,\infty))$. Since we know that $\partial M_g(f)$ is a torus, the second case is the only remaining possibility.[6] So if $M_g(f)$ is a complete oriented hyperbolic manifold, there is a fundamental domain in $\mathbf{H}^3$ extending towards a point P on the sphere at infinity; since

$$\mu\lambda\mu^{-1} = f(\lambda) = \lambda$$

and μ as well as λ represent arbitrarily short loops around the boundary, μ and λ are in this case represented by a commuting pair of parabolic isometries of $\mathbf{H}^3$ fixing P.

2.3 The Case $g = 2$

Now let $g = 2$. Then the fundamental group of $S_2^{(1)}$ is free with four generators:

$$\pi_1(S_2^{(1)}) \cong \Phi_4 = \langle a_1, a_2, a_3, a_4\rangle.$$

However, in order to specify $f \in \mathrm{Aut}(\pi_1(S_2^{(1)}), \lambda)$, where λ is the boundary loop, it is necessary to specify λ itself in terms of the generators. The choice

$$\lambda = [a_4, a_3]\,[a_2, a_1]$$

[6] [Benedetti/Petronio 91], p. 143 ff.

mentioned above is not the only possible choice; every image of λ under an automorphism of Φ_4 can be taken as a starting point. The convenient choice of λ has some influence on the simplicity of the (algebraic) representation of (a topologically given) f.

Geometrically, the process of transforming λ can be traced; see fig. 2.2 as an example. Let $\mathcal{F}$ be the sketch of a fundamental domain for $S_2^{(1)}$ in $\mathbf{H}^2$. (λ

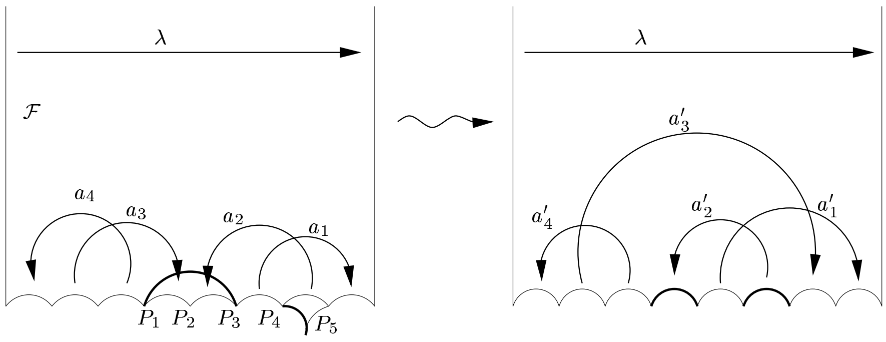

Figure 2:

and a_i are identified with the corresponding covering transformations.) Cutting off a triangle between P_1 and P_3 and pasting it via a_2^{-1} between P_4 and P_5 leads to a new fundamental domain with new identifications of pairs of sides. The depicted process corresponds to an algebraic Tietze transformation other than renaming and inverting:

$$\begin{array}{rcl}
a_1 & \mapsto & a_1' = a_1 \\
a_2 & \mapsto & a_2' = a_2 \\
a_3 & \mapsto & a_3' = a_2^{-1} a_3 \\
a_4 & \mapsto & a_4' = a_4 \\
\lambda & \mapsto & (a_4')^{-1}(a_3')^{-1}(a_2')^{-1} a_4' a_2' a_3' (a_2')^{-1}(a_1')^{-1} a_2' a_1'.
\end{array}$$

Two representations of λ have been used throughout, namely

$$\lambda = [a_4, a_3]\,[a_2, a_1]$$

and

$$\lambda = a\,b\,c\,d\,a^{-1} b^{-1} c^{-1} d^{-1};$$

in the latter case the generators of Φ_4 have been named differently in order to distinguish the two cases. So every f acting on a, b, c, d is supposed to leave $\lambda = abcda^{-1}b^{-1}c^{-1}d^{-1}$ invariant, every f acting on a_1, a_2, a_3, a_4, however, leaves $\lambda = [a_4, a_3]\,[a_2, a_1]$ invariant.

The mostly used transformation between a_1, a_2, a_3, a_4 and a, b, c, d is given by

$$\begin{array}{rclcrcl} a_1 &=& c^{-1}d^{-1} & \qquad & a &=& a_4^{-1} \\ a_2 &=& c^{-1}a^{-1}b^{-1} & & b &=& a_3^{-1}a_4 \\ a_3 &=& a^{-1}b^{-1} & & c &=& a_3a_2^{-1} \\ a_4 &=& a^{-1}, & & d &=& a_1^{-1}a_2a_3^{-1}. \end{array}$$

This transformation is of course not unique. You can actually guess another one by regrouping $a\,b\,c\,d\,a^{-1}b^{-1}c^{-1}d^{-1}$ and inserting the needed "1"s:

$$a\,b\,c\,d\,a^{-1}b^{-1}c^{-1}d^{-1} = \underbrace{ab}_{a_4^{-1}}\ \underbrace{cda^{-1}}_{a_3^{-1}}\ \underbrace{b^{-1}a^{-1}}_{a_4}\ \underbrace{ad^{-1}c^{-1}}_{a_3}\ \underbrace{cd}_{a_2^{-1}a_1^{-1}}\ \underbrace{c^{-1}d^{-1}}_{a_2a_1},$$

i. e.:

$$\begin{array}{rclcrcl} a_1 &=& d^{-1} & \qquad & a &=& a_3a_2^{-1}a_1^{-1} \\ a_2 &=& c^{-1} & & b &=& a_1a_2a_3^{-1}a_4^{-1} \\ a_3 &=& ad^{-1}c^{-1} & & c &=& a_2^{-1} \\ a_4 &=& b^{-1}a^{-1}, & & d &=& a_1^{-1}. \end{array}$$

3 Criteria for Hyperbolicity

3.1 Criterion: f is Pseudo–Anosov

On the theoretical level, there is a known condition for f in order to ensure hyperbolicity of $M_g(f)$:

Theorem. (Thurston) Let $S_g^{(r)}$ be a hyperbolic surface; let f be a homeomorphism of $S_g^{(r)}$ that is pseudo–Anosov. Then $M_g(f)$ is hyperbolic.

A homeomorphism f of $S_g^{(r)}$ is called *pseudo–Anosov*, if f has one (and therefore two) irrational fixed points in the projectified set of measured geodesic laminations in $S_g^{(r)}$, i. e. in the boundary of Thurston's compactification of Teichmüller space of $\pi_1(S_g^{(r)})$. A *geodesic lamination* in $S_g^{(r)}$ is a compact disjoint union of simple geodesics in $S_g^{(r)}$ (as geometric space). A *measured geodesic lamination* is a (geodesic lamination together with a) transverse measure on the given lamination, i. e. a measure on compact paths in $S_g^{(r)}$ generalizing a weighted counting of intersections of the path with the given geodesic lamination. We will not go into details here[7]; the reported facts may suffice to indicate that this criterion is not very easy to understand. Given a particular f in terms of generators of $\pi_1(S_g^{(r)})$, you cannot easily read off, whether f

[7] See e.g. [Otal 96].

is pseudo–Anosov. Furthermore, the criterion is non–constructive in the sense that it yields no idea of a representation of $\pi_1(S_g^{(r)})$ in (P)$\mathrm{SL}_2(\mathbf{C})$ nor a picture of a fundamental domain for $M_g(f)$.

On the level of f acting on $S_g^{(r)}$, the term pseudo–Anosov might best be understood by considering what it is *not*. The homeomorphism is pseudo–Anosov, if it is neither periodic nor reducible. It is *periodic*, if there is a $k > 0$ such that f^k is isotopic to the identity. Since the action of f on $\pi_1(S_g^{(r)})$ depends only on the isotopy (= homotopy) class of f, we then have $f^k = \mathrm{id}$ on $\pi_1(S_g^{(r)})$. The homeomorphism f is called *reducible*, if f leaves invariant a *multicurve*, i. e. a finite disjoint union of essential simple closed pairwise non–homotopic curves.

The following is a motivation for studying nonperiodic irreducible homeomorphisms. For every periodic f with f^k isotopic id, $M_g(f^k)$ is not atoroidal, since every generating element a_i of $\pi_1(S_g^{(r)})$ gives rise to an embedding of a torus $a_i \times I/\sim$ with injected fundamental group that is not boundary–parallel. However, $M_g(f)$ is supposed to be hyperbolic, therefore $M_g(f^k)$ is hyperbolic as well and hence atoroidal. If f is nonperiodic reducible, leaving, say, the union of $C_1, C_2, \ldots, C_k$ (C_i pairwise disjoint simple closed curves) invariant, mapping C_i onto C_{i+1} and C_k back to C_1 again, then the concatenation of the cylinders $C_i \times I$ projects onto a torus in $M_g(f)$, i. e. the image of

$$\begin{array}{rcl} S^1 \times [0,1] & \to & M_g(f), \text{ where} \\ S^1 \times [\frac{i}{k}, \frac{i+1}{k}] \supset S^1 \times \{t\} & \mapsto & C_i \times \{k(t-i)\} \end{array}$$

in $M_g(f)$ is a homeomorphic image of the torus. So $M_g(f)$ is not atoroidal, hence not hyperbolic.

3.2 The Closed Case

Let $S_g = S_g^{(0)}$ be closed. The automorphism $f : \pi_1(S_g) \to \pi_1(S_g)$ induces an automorphism $\bar{f} : H_1(S_g) \to H_1(S_g)$ of the first homology group of S_g. Let A_f denote the matrix of $\bar{f}$ with respect to some basis of $H_1(S_g)$. We will call this matrix the *Anosov–monodromy* of f. Let $\chi_{A_f}(t)$ denote the characteristic polynomial of A_f. Then the following theorem holds:

Theorem. If $\chi_{A_f}(t)$ is irreducible over $\mathbf{Z}$, has no roots of unity as zeros and is not a polynomial in a nontrivial power t^n of t, then f is irreducible and non–periodic (and therefore pseudo–Anosov, if S_g is hyperbolic).

Proof. If f is periodic, then $A_f^k = \mathbf{1}$ for some $k > 0$. Therefore all eigenvalues of A_f are k–th roots of unity.

If f is non–periodic reducible, there is a multicurve which is left invariant by f. There are two cases to consider.

Case 1: There is an invariant multicurve containing a simple essential closed curve C, which does not separate S_g. Then C is not null–homologous and there is a $k > 0$ such that $f^k(C) = C$. Thus $A_f^k(\bar{C}) = \bar{C}$, where $\bar{C}$ is the image of C in $H_1(S_g)$. Therefore A_f has a k–th root of unity as eigenvalue.

Case 2: Every component of every invariant multicurve separates S_g. (Then every such component is null–homologous.) Let C be a component cutting off a once–punctured surface $S_{g'}$ of a certain genus g' which does not contain any other component of the C–containing invariant multicurve; C is essential, so $g' > 0, g' < g$. There is a minimal $k > 1$ such that $f^k(S_{g'}) = S_{g'}$. Since

$$S_g = S_{g'} \cup f(S_{g'}) \cup \ldots \cup f^{k-1}(S_{g'}) \cup \text{ remainder of } S_g,$$

we have

$$H_1(S_g) \cong H_1(S_{g'}) \oplus \ldots \oplus H_1(f^{k-1}(S_{g'})) \oplus H_1(\overline{\text{remainder}}),$$

where $\overline{\text{remainder}}$ denotes the remainder with all punctures from the removed $f^j(S_{g'})$ capped. Since $\bar{f}$ permutes the $H_1(f^i(S_{g'}))$ cyclically, there is a basis for $H_1(S_g)$ such that A_f has the form

$$A_f = \left.\begin{pmatrix} \mathbf{0} & \ldots & & \mathbf{0} & \mathbf{B} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \ldots & & \mathbf{0} & \vdots \\ \mathbf{0} & \ddots & & & \vdots & \\ \vdots & & & & & \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ldots & & & \mathbf{0} & \mathbf{C} \end{pmatrix} \quad \right\} k$$

Hence $\chi_{A_f}(t) = \det(\mathbf{B} - t^k\mathbf{1})\det(\mathbf{C} - t\mathbf{1})$. If χ_{A_f} is irreducible, the second factor is equal to 1; the first factor is a polynomial in $t^k \neq t$, since $k \neq 1$. □

The proof of this theorem has been adapted from Casson and Bleiler in order to point out the importance of null–homologous simple closed curves in S_g. In the case of a once–punctured surface of genus $g > 1$ the null–homologous curves represent the main difficulty; see section 6.

3.3 The Case of the Once–Punctured Torus

The case of a 3–manifold fibering over the circle with a once–punctured torus $S_1^{(1)}$ has most thoroughly been investigated by Helling, Alestalo and Koch[8].

Let $\Phi_2 = \langle \xi, \eta \rangle$ denote the fundamental group of the once–punctured torus. Let $f : S_1^{(1)} \to S_1^{(1)}$ be a homeomorphism leaving $\lambda = [\xi, \eta]$ invariant. There is a sufficient condition for f to ensure hyperbolicity of $M_1(f)$, resulting from the fact that in a hyperbolic surface $S_g^{(r)}$ every simple closed curve, which is represented by a hyperbolic element in $\pi_1(S_g^{(r)})$ is freely homotopic to a unique simple closed geodesic[9].

[8] See [Helling 96], [Helling/Alestalo 97], [Koch].

[9] See e.g. [Casson/Bleiler 88], p. 24; the proof can easily be adapted to the nonclosed case.

Theorem. If $\chi_{A_f}(t)$ has no roots of unity as zeros, then $M_1(f)$ is hyperbolic.

Proof. Let $C \subset S_1^{(1)}$ be a null–homologous simple closed curve. Then C bounds a once–punctured surface $S_{g'}$ of genus g', $0 \leq g' \leq g = 1$. So $S_{g'}^{(1)}$ is a disk or a once–punctured torus. In the first case, $C = 1$ in $\pi_1(S_1^{(1)})$. In the second case there is a representation of $\pi_1(S_1^{(1)})$ such that $C = \lambda = [\xi, \eta]$; hence in this representation of $\pi_1(M_1(f))$ in $\mathrm{SL}_2(\mathbf{C})$ C is represented by a parabolic element; therefore C is not homotopic to a simple closed geodesic in $S_1^{(1)}$ and cannot be (homotopic to) a component of an invariant multicurve. Therefore the case of invariant multicurves containing null–homologous simple closed curves is excluded.

However, the existence of not null–homologous simple closed curves invariant under f^k for some $k > 0$ can easily be checked by computing $\chi_{A_f}(t)$. If $\chi_{A_f}(t)$ has no root of unity as zero, A^k has no eigenvalue 1 for all $k \in \mathbf{N} \setminus 0$, so no not null–homologous element of $\pi_1(S_1^{(1)})$ is mapped to a conjugate of its own by f. Therefore f is neither periodic nor reducible, hence pseudo–Anosov. □

The above–mentioned condition for the characteristic polynomial χ_{A_f} of the Anosov–monodromy A_f of f is necessary as well, as has been shown by Helling[10]. Now A_f is (represented by) a $\mathrm{SL}_2(\mathbf{Z})$–matrix, since the orientation–preserving f, acting on $\pi_1(S_1^{(1)})$, is invertible. There are only five degree–2–polynomials, which are characteristic polynomials of $\mathrm{SL}_2(\mathbf{Z})$–matrices (i. e. which have constant coefficient $= +1$) and have roots of unity as zeros, namely $t^2 \pm t + 1$, $t^2 \pm 2t + 1$, and $t^2 + 1$. These polynomials are characterized by their only nontrivial mid coefficient which is the trace of the Anosov–monodromy of f. Hence the criterion for hyperbolicity can be rephrased in the following way:

Theorem. Let $f : S_1^{(1)} \to S_1^{(1)}$ be a homeomorphism leaving λ invariant. $M_1(f)$ is hyperbolic, if and only if the characteristic polynomial of the Anosov–monodromy of f has no root of unity as zero, i. e. if and only if the Anosov–monodromy of f is represented by a hyperbolic matrix (see fig. 3).

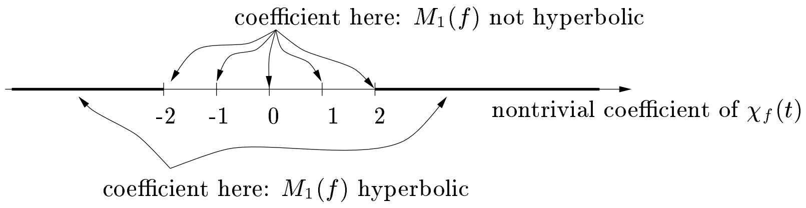

Figure 3: Representation of the only non–trivial coefficient of $\chi_{A_f}(t)$

[10]Unpublished manuscript.

Again this case reveals the importance of the null–homologous simple closed curves in $S_g^{(r)}$.

3.4 The Case of a Once–Punctured Surface of Genus $g > 1$

Consider $f : S_g^{(1)} \to S_g^{(1)}$, $g > 1$. Suppose $\chi_{A_f}(t)$ has no root of unity as zero. Unfortunately, this is neither a sufficient nor a necessary criterion for hyperbolicity of $M_g(f)$.

It is not sufficient, because the action of f on the first homology group does not see the null–homologous curves. There might be an invariant multicurve consisting of simple closed null–homologous geodesics only. In fact, there are many homeomorphisms of $S_2^{(1)}$, e. g., that satisfy this criterion, but do not lead to a hyperbolic $M_2(f)$ (e. g. the homeomorphisms $f_1\, f_2^{-1}\, f_3\, f_4^{-1}$, $g_1 \bar{f}_1^{-2}$, and several others on $S_2^{(1)}$ described in section 10, subsection 7.2, and section 8.2, respectively).

The criterion is not necessary either; the homeomorphism $g_1\, \bar{f}_1^{-1}$ on $S_2^{(1)}$, described in subsection 8.1 leads to a hyperbolic manifold $M_2(g_1\, \bar{f}_1^{-1})$; however, $\chi_{A_{g_1\, \bar{f}_1^{-1}}}(t) = (t^2 - t + 1)^2$ has sixth roots of unity as zeros.

Nevertheless, nonperiodicity of f and the nonexistence of not null–homologous simple closed curves, that are invariant under f^k for certain k, can be judged by a short look at $\chi_{A_f}(t)$. For the case $g = 2$ this can be done most conveniently by using the following argument (cf. fig. 4).

The characteristic polynomial $\chi_{A_f}(t)$ of a homeomorphism $f : S_2^{(1)} \to S_2^{(1)}$ is given by

$$\chi_{A_f}(t) = t^4 + \sigma t^3 + \tau t^2 + \sigma t + 1 \in \mathbf{Z}[t].$$

Therefore, it can be represented by a point $(\sigma, \tau) \in \mathbf{Z^2}$. The Polynomial $\chi_{A_f}(t)$ has a primitive n–th root of unity as zero, if it is divisible by the n–th cyclotomic polynomial ϕ_n. There are four cyclotomic polynomials of degree 4 (determined by the equation $\varphi(n) = 4$, where $\varphi(n)$ is the Euler function): $\phi_8 = t^4 + 1$, $\phi_{12} = t^4 - t^2 + 1$, $\phi_5 = t^4 + t^3 + t^2 + t + 1$, and $\phi_{10} = t^4 - t^3 + t^2 - t + 1$. These polynomials are represented by the blobs in fig. 4.

There are two cyclotomic polynomials of degree 1, namely $\phi_1 = t - 1$ and $\phi_2 = t + 1$. Suppose $\chi_{A_f}(t)$ is divisible by ϕ_1. Performing a polynomial division yields the condition $\tau + 2\sigma + 2 = 0$. So if χ_{A_f} is divisible by $t - 1$, it is represented by a point (σ, τ) satisfying $\tau = -2\sigma - 2$. Similarly, if χ_{A_f} is divisible by $t + 1$, it is represented by a point (σ, τ) satisfying $\tau = 2\sigma - 2$.

There are three cyclotomic polynomials of degree 2: $\phi_4 = t^2 + 1$, $\phi_3 = t^2 + t + 1$, and $\phi_6 = t^2 - t + 1$. If χ_{A_f} is divisible by these polynomials, it is represented by a point satisfying $\tau = 2$, $\tau = -\sigma + 1$, and $\tau = \sigma + 1$, respectively.

Any χ_{A_f}, which is represented by (σ, τ) not on the lines and blobs drawn in fig. 4, has no root of unity as zero.

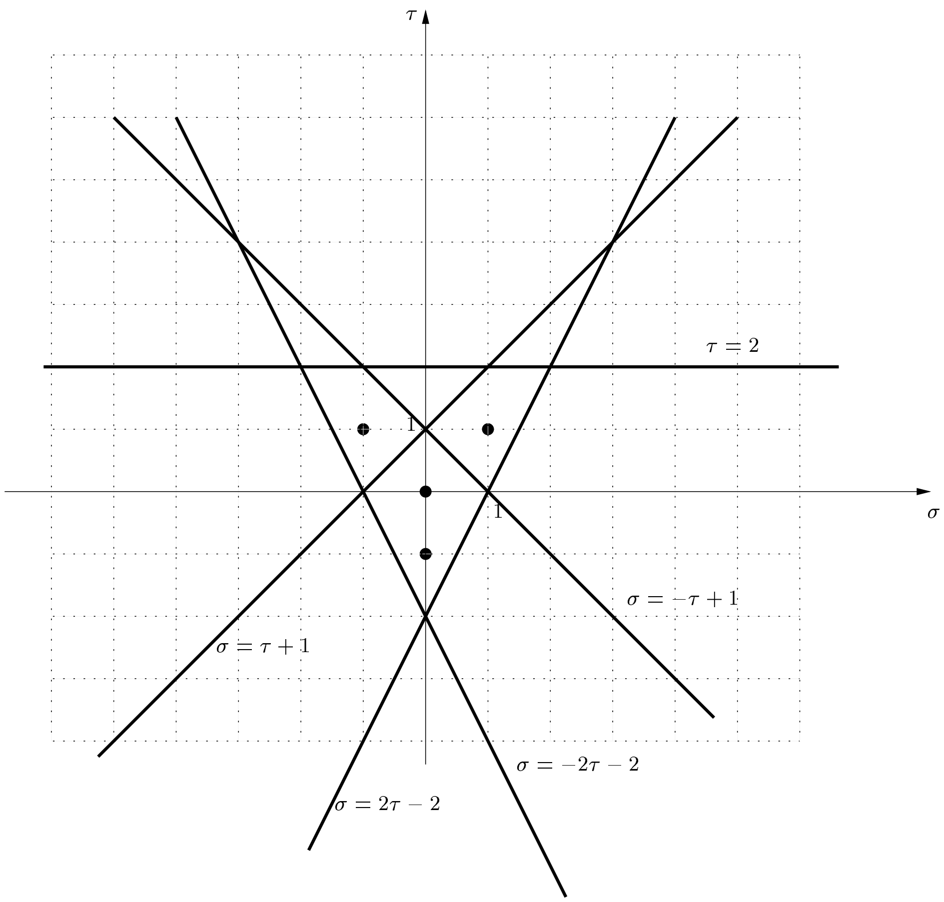

Figure 4: $\chi_{A_f}(t)$ represented as point in $\mathbf{Z^2}$

A sufficient criterion for hyperbolicity of $M_g(f)$ may be formulated as follows.

There is a 1–1–correspondence between free homotopy–classes of essential closed curves in $S_g^{(1)}$ and conjugacy classes of hyperbolic elements of $\pi_1(S_g^{(1)})$. Suppose $C_1 \cup C_2 \cup \ldots \cup C_k$ (k minimal) is an invariant multicurve, where C_1 is a null–homologous curve cutting off a surface $S_{g'}$ $(0 < g' < g)$, which does not contain any other C_i. Then $f^i(S_{g'}) \cap S_{g'} = \emptyset \; \forall i = 1, \ldots, k-1$ (for otherwise $f^i(C_1) \cap C_1 \neq \emptyset$), whereas $f^k(S_{g'}) = S_{g'}$. By putting the base point onto C_1, we may assume that the fundamental group $\pi_1(S_g)$ is given by

$$\langle a_1, \ldots a_{2g'}, a_{2g'+1}, \ldots, a_{2g} \rangle,$$

satisfying

$$[a_{2g'}, a_{2g'-1}] \dots [a_2, a_1] = C_1.$$

Since C_1 cuts off at least one handle, and since there are at most $g-1$ possible images of $S_{g'}$ under f up to suitable representation of $\pi_1(S_g^{(1)})$, we have the following condition:

Theorem. If χ_{A_f} has no root of unity as zero, and if no product of commutators

$$[a_{2g'}, a_{2g'-1}] \dots [a_2, a_1] \in \pi_1(S_g^{(1)}),\ 1 \le g' < g,$$

is mapped to a conjugate of its own by $f^{g!}$, then f is nonperiodic irreducible and therefore pseudo–Anosov.

This criterion is easy to understand; however, it is not easy to apply. In general, the problem of finding products of commutators that are mapped to conjugates of their own is related to the solution of the conjugacy problem.

4 Technical Prerequesites

We are interested in the question, whether $M_g(f)$ is hyperbolic, i. e. whether there is a faithful, discrete, cofinite representation of $\pi_1(M_g(f))$ in $\mathrm{PSL}_2(\mathbf{C}) = \mathrm{Iso}^+(\mathbf{H}^3)$. In this section a set of technical preliminaries is summarized.

4.1 Representability of a Group G in $\mathrm{SL}_2(\mathrm{C})$

Theorem 4.1.[11] Let G be a group, $s : G \to \mathbf{C}$ a function. There is a unique semisimple conjugacy class of representations $[\rho] : G \to \mathrm{SL}_2(\mathbf{C})$ with $\mathrm{tr}\,[\rho] = s$, if and only if s satisfies the following three conditions:

(i) $s(e) = 2$, where e is the neutral element of G;

(ii) $s(g^2) - s^2(g) + 2 = 0 \quad \forall g \in G$;

(iii) $s(g_1 g_2 g_3) + s(g_3 g_2 g_1) - s(g_1)s(g_2 g_3) - s(g_2)s(g_1 g_3) - s(g_3)s(g_1 g_2)$
$+ s(g_1)s(g_2)s(g_3) = 0 \quad \forall g_1, g_2, g_3 \in G$.

If G is finitely generated, $G = \langle a_1, \dots, a_d \mid R_1, \dots \rangle$, then the values of s are determined once you have fixed the values of $s(a_i), s(a_i a_j)\,(i < j)$, and $s(a_i a_j a_k)\,(i < j < k)$. However, these values cannot be prescribed arbitrarily, since the relations $R_1, \dots$ have to be satisfied. In our case, G is the fundamental group of $M_g(f)$. For $S_g^{(1)}$ we know

$$G = \pi_1(M_g(f)) = \langle a_1, \dots, a_{2g}, \mu \mid \mu a_i \mu^{-1} = f(a_i)\,(i = 1, \dots, 2g) \rangle,$$

whence we get the *system of trace equations*

$$s(\omega) = s(f(\omega)) \quad \forall \omega \in \pi_1(M_g(f)). \tag{1}$$

[11] See e.g. [Helling 74], S. 496.

For convenience we introduce the variables x, y, z:

$$\begin{aligned} x_a &:= s(a), && (2)\\ y_{ab} &:= 2s(ab) - x_a x_b, && (3)\\ z_{abc} &:= s(abc) - s(cba) && (4) \end{aligned}$$

for arbitrary elements $a, b, c \in G$. In case of $a, b, c = a_i, a_j, a_k$ we write x_i, y_{ij}, z_{ijk} for short. We have the following

Theorem 4.2.[12] Let G be a finitely generated free group: $G = \langle a_1, \ldots, a_d \rangle$. Let x_i $(1 \le i \le d)$, y_{ij} $(1 \le i \le j \le d)$, and z_{ijk} $(1 \le i < j < k \le d)$ be complex numbers. Then there is a unique character $s : G \to \mathbf{C}$ (i. e. a function satisfying the three equations of theorem 4.1) such that

$s(a_i) = x_i$ $(1 \le i \le d)$,

$s(a_i a_j) = \frac{1}{2}(x_i x_j + y_{ij})$ $(1 \le i \le j \le d)$, and

$s(a_i a_j a_k) = \frac{1}{2}(2z_{ijk} + x_i y_{jk} + x_j y_{ik} + x_k y_{ij} + x_i x_j x_k)$ $(1 \le i < j < k \le d)$,

if and only if the numbers x_i, y_{ij}, z_{ijk} satisfy

$$\begin{aligned} 4z_{i_1 i_2 i_3} z_{j_1 j_2 j_3} + \det(y_{i_\nu j_\mu})_{\nu,\mu=1,\ldots,3} &= 0 \quad \text{and} && (5)\\ y_{ii} &= x_i^2 - 4. && (6) \end{aligned}$$

The set of equations 5 will be called *determinant equations*. If s satisfies in addition the relations $s(R_i) = s(e) = 2$, then there is a representation of $\langle a_1, \ldots, a_d \,|\, R_1, \ldots \rangle$ in $\mathrm{SL}_2(\mathbf{C})$, and of course vice versa.

The set of solutions x_i, y_{ij}, z_{ijk} of the determinant equations and trace equations is called the *character variety* of $\pi_1(M_g(f))$.

4.2 Representability of $\pi_1(M_g(f))$ as Cofinite Lattice in $\mathrm{PSL}_2(\mathbf{C})$

Our aim is to find hyperbolic manifolds fibering over the circle with fiber equal to $S_g^{(1)}$. So we are interested in a discrete, cofinite, faithful representation of $\pi_1(M_g(f))$ in $\mathrm{PSL}_2(\mathbf{C})$.

Now if

$$\rho : G \to \mathrm{SL}_2(\mathbf{C})$$

is a faithful representation and if G does not contain an element of order 2 (which is true in our case of a torsion–free fundamental group of a manifold), the representation ρ projects to a faithful representation $G \to \mathrm{PSL}_2(\mathbf{C})$. If, conversely, $\bar\rho : G \to \mathrm{PSL}_2(\mathbf{C})$ is a representation of a group $G = \langle a_i, i \in I | R_1, \ldots, R_d \rangle$, which contains no element of order 2, defined by

$$\bar\rho(a_i) = \bar A_i = \{A_i, -A_i\},$$

[12] See [Helling 96], p. 4.

then the representation can be lifted to a representation

$$\rho : G \to \mathrm{SL}_2(\mathbf{C})$$

by adjusting the signs of $\rho(a_i) = +/-A_i$ such that $R_1 = R_2 = \ldots = R_d = +\mathbf{1}$.[13] So we find every faithful representation of $\pi_1(M_g(f))$ in $\mathrm{PSL}_2(\mathbf{C})$ by finding the representations in $\mathrm{SL}_2(\mathbf{C})$.

In order to ensure faithfulness, the values $x_i = 0, +1, -1$ are excluded. For by Caley–Hamilton we know

$$A_i^2 - \mathrm{tr}(A_i)A_i + \mathbf{1} = \mathbf{0} \quad \forall A_i \in \mathrm{SL}_2(\mathbf{C});$$

now $\mathrm{tr}(A_i) = 0$ entails $A_i^2 = -\mathbf{1}$, $\mathrm{tr}(A_i) = 1$ entails $A_i^3 = -\mathbf{1}$, and from $\mathrm{tr}(A_i) = -1$ follows $A_i^3 = \mathbf{1}$, so A_i is a torsion element in each case.

The values $x_i = +/-2$ are excluded as well. For let x_i be equal to $+2$ or -2. Then $A_i = \rho(a_i)$ is parabolic. A_i must not commute with $M = \rho(\mu)$, since otherwise we have either a new relation $A_i = M^k L^n$ $(L = \rho(\lambda))$ or $\langle A_i, M, L\rangle$ is not discrete. Since M, L generate the single cusp, there is no element Y in $\rho(\pi_1(M_g(f)))$ commuting with A_i. Put the fixed point of A_i to ∞. Then the fundamental domain of $M_g(f)$ contains the intersection of a horoball centered at ∞ with the portion of $\mathbf{H}^3$ between two hyperbolic planes $E_1, E_2 = A_i(E_1)$ through ∞ and so has infinite volume.

The element $L = \rho(\lambda)$ has to be parabolic, so we have the additional equation

$$s(\lambda) = +/-2. \tag{7}$$

This singles out a 0–dimensional set of solutions in the character variety.

4.3 Trace Calculus and Coordinate Calculus

We now provide a set of simple, useful formulae for the calculation of traces $s(\omega)$, where ω is a long product, resulting from the properties of the character s listed in theorem 4.1. Let G be any group and s a character on G.

Lemma 4.3. For all $a, b, c, d \in G$ we have

(i) $s(a^2b) = s(a)s(ab) - s(b)$;

(ii)

$$\begin{aligned} s(abcd) = \tfrac{1}{2} \quad & [s(a)s(bcd) + s(b)s(acd) + s(c)s(abd) + s(d)s(abc) \\ & +s(ab)s(cd) - s(ac)s(bd) + s(ad)s(bc) \\ & -s(a)s(b)s(cd) - s(a)s(d)s(bc) \\ & -s(b)s(c)s(ad) - s(c)s(d)s(ab) \\ & +s(a)s(b)s(c)s(d)]; \end{aligned}$$

[13] [Culler 86].

(iii) $s(cacb) = s(ca)s(cb) + s(ab) - s(a)s(b)$;

(iv) $s(a) = s(a^{-1})$;

(v) $s(c^{-1}a) = s(c)s(a) - s(ca)$.

Proof. All equations follow by direct calculation using the above–mentioned properties of s (cf. theorem 4.1). To show (i), calculate $2s(aab) = s(aab) + s(baa)$. To show (ii), calculate $2s(abcd) = s(ab(cd)) + s(ba(cd)) - s(ac(db)) - s(ca(db)) + s(ad(bc)) + s(da(bc))$, using the third property three times. Equation (iii) is a consequence of (ii). Equation (iv) is a consequence of (i) by calculating $s(a^{-1}a^{-1}a)$. Equation (v) follows from (iii) by calculating $s(c^{-1}cc^{-1}a)$. □

Lemma 4.4. From Lemma 4.3 we get by a tiring, but direct calculation:

$$\begin{aligned} s(a\,b\,c\,d\,a^{-1}b^{-1}c^{-1}d^{-1}) \quad = \quad & s(dcba)s(abcd) + s(a)^2 + s(b)^2 + s(c)^2 + s(d)^2 \\ & -s(a)^2 s(c)^2 - s(b)^2 s(d)^2 \\ & -s(a)s(b)s(ab) + s(a)s(c)s(ac) - s(a)s(d)s(ad) \\ & -s(b)s(c)s(bc) + s(b)s(d)s(bd) - s(c)s(d)s(cd) \\ & -s(a)s(d)s(ab)s(bd) - s(a)s(d)s(ac)s(cd) \\ & -s(a)s(d)s(ab)s(bc)s(cd) \\ & +s(b)s(d)s(bc)s(cd) + s(a)s(c)s(ab)s(bc) \\ & +s(a)^2 s(c)s(d)s(cd) + s(a)s(b)s(d)^2 s(ab) - 2. \end{aligned}$$

We note the following obvious rules for the coordiantes x_i, y_{ij}, z_{ijk}:

Lemma 4.5. For all $a, b, c, d \in G$ we have:

$y_{ab} = y_{ba}$,

$y_{aa} = x_a^2 - 4$,

$y_{ab^{-1}} = -y_{ab}$,

$z_{aab} = z_{aba} = z_{baa} = 0$,

$z_{abc^{-1}} = -z_{abc}$,

$z_{abc} = z_{bca} = -z_{cba} = 2s(abc) - s(a)s(bc) - s(b)s(ac) - s(c)s(ab) + s(a)s(b)s(c)$,

$s(abcd) - s(dcba) = \frac{1}{2}[x_a z_{bcd} + x_b z_{acd} + x_c z_{abd} + x_d z_{abc}]$.

4.4 Further Equations as Consequences from the Representability in $SL_2(C)$

Lemma 4.6. Let $A^0, B^0, C^0 \in \mathbf{C}^{2\times 2}$ be three trace–0–matrices, let $\det A^0 \neq 0$. Then the following implication holds:

$$z_{A^0B^0C^0} = \mathrm{tr}(A^0(B^0C^0 - C^0B^0)) = 0 \Rightarrow A^0, B^0, C^0 \text{ are linearly dependent.}$$

Proof. The three matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are a basis for the linear subspace of trace–0–matrices in $\mathbf{C}^{2\times 2}$. Since $\det A^0 \neq 0$ and $\chi_{A^0}(t) = t^2 + \det A^0$, A^0 is diagonalizable; since constant factors can w.l.o.g. be dropped, we may assume $A^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By linearity of the trace and the rules of lemma 4.5, we may further assume

$$B^0 = \beta_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{pmatrix}, \; C^0 = \begin{pmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{pmatrix},$$

i. e., B^0, C^0 have component 0 in the direction of A^0. From $z_{A^0B^0C^0} = 0$ follows

$$\beta_1\gamma_2 - \gamma_1\beta_2 = 0.$$

Case 1: $\beta_1 \neq 0 \neq \gamma_1$. Then $\frac{\beta_2}{\beta_1} = \frac{\gamma_2}{\gamma_1}$, therefore B^0, C^0 are linearly dependent.
Case 2: $\beta_1 = 0 = \gamma_1$. Then B^0, C^0 are linearly dependent. □

Lemma 4.7. Let $A, B, C, D \in SL_2(\mathbf{C})$. If two of the four variables z_{ABC}, z_{ABD}, z_{ACD}, z_{BCD} are equal to 0, then all four of them are equal to 0.

Proof. By direct calculation $z_{ABC} = z_{A^0B^0C^0}$. If the trace–0–parts satisfy $A^0, \ldots, D^0 \in \mathrm{Span}\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$, then the statement is trivial. If A^0, B^0 are linearly independent, $\det(A^0) \neq 0$, then we know from Lemma 4.6:

$$z_{ABC} = 0 \Rightarrow C^0 \in \mathrm{Span}\{A^0, B^0\},$$
$$z_{ABD} = 0 \Rightarrow D^0 \in \mathrm{Span}\{A^0, B^0\};$$

.

Therefore $A, B, C, D \in \mathrm{Span}\{\mathbf{1}, A^0, B^0\}$. Let $C = \gamma\mathbf{1} + \gamma_1 A^0 + \gamma_2 B^0, D = \delta\mathbf{1} + \delta_1 A^0 + \delta_2 B^0$. Then $z_{ACD} = z_{BCD} = 0$ by inserting and calculating. □

The determinant equations

$$4z_{i_1i_2i_3}z_{j_1j_2j_3} + \det(y_{i_\nu j_\mu})_{\nu,\mu=1,2,3}$$

can be written in matrix form. Let a_1, a_2, a_3, a_4 be four group elements; let us write

$$z_{\hat{1}} = z_{234}, z_{\hat{2}} = z_{134}, z_{\hat{3}} = z_{124}, z_{\hat{4}} = z_{123}$$

for short. Let

$$Y := \begin{pmatrix} y_{11} & \cdots & y_{14} \\ \vdots & \ddots & \vdots \\ y_{14} & \cdots & y_{44} \end{pmatrix},$$

and let

$$\tilde{Y} := \left((-1)^{i+j} \det Y'_{ij}\right)_{i,j=1,\ldots,4}$$

be the complementary matrix of Y. Let $\underline{z} := (z_{\hat{1}}, -z_{\hat{2}}, z_{\hat{3}}, -z_{\hat{4}})$ and

$$Z := \underline{z}^t \cdot \underline{z}.$$

Then the determinant equations 5 read

$$4Z + \tilde{Y} = \mathbf{0}.$$

Now Z has rank 0 or 1, so $\det \tilde{Y} = 0$, whence

$$\det Y = 0. \tag{8}$$

From this follows by Laplace's theorem

$$\begin{aligned} \sum_j y_{ij}(-1)^{i+j} \det Y'_{ij} &= 0 \;\; \forall i = 1, \ldots, 4 & (9) \\ \Leftrightarrow \sum_j y_{ij}(-1)^{i+j} z_{\hat{i}} z_{\hat{j}} &= 0 \;\; \forall i = 1, \ldots, 4. & (10) \end{aligned}$$

If $z_{\hat{i}} \neq 0$, we have

$$y_{i1} z_{\hat{1}} - y_{i2} z_{\hat{2}} + y_{i3} z_{\hat{3}} - y_{i4} z_{\hat{4}} = 0. \tag{11}$$

If all four $z = 0$, equations 11 are trivial. If anyone of the z is $\neq 0$, at least three of them are by Lemma 4.6. Suppose $z_{\hat{1}} = 0$, whereas the other $z_{\hat{j}} \neq 0$. But since $\operatorname{rk} Y \leq 3$, the first row of Y is a linear combination of the three other rows, so

$$y_{11} z_{\hat{1}} - y_{12} z_{\hat{2}} + y_{13} z_{\hat{3}} - y_{14} z_{\hat{4}}$$

is a linear combination of

$$y_{i1} z_{\hat{1}} - y_{i2} z_{\hat{2}} + y_{i3} z_{\hat{3}} - y_{i4} z_{\hat{4}}, \; i = 2, 3, 4,$$

each of which is equal to 0.

So in any case the set of equations 11 (the so–called *5–identities*) is valid for $i = 1, 2, 3, 4$.

4.5 Miscellaneous Consequences

Let $G = \langle a, b, c, d \rangle$, $\lambda = a\,b\,c\,d\,a^{-1}b^{-1}c^{-1}d^{-1}$.

Lemma 4.8. In the closed case, $\lambda = e$, whence $z_{abc} = z_{abd} = z_{acd} = z_{bcd} = 0$.

Proof. From
$$a\,b\,c\,d\,a^{-1}b^{-1}c^{-1}d^{-1} = e$$
we have
$$s(c^{-1}b^{-1}a^{-1}) = s(c^{-1}b^{-1}a^{-1}\lambda) = s(d\,a^{-1}b^{-1}c^{-1}d^{-1}) = s(a^{-1}b^{-1}c^{-1})$$
$$\Rightarrow z_{abc} = 0;$$
$$s(dcb) = s(\lambda dcb) = s(bcd)$$
$$\Rightarrow z_{bcd} = 0.$$
From Lemma 4.7 follows $z_{acd} = z_{abd} = 0$. □

Lemma 4.9. Let A, B, C, D, and L represent a, b, c, d, and λ, respectively. Let $s(a\,b\,c\,d) = s(d\,c\,b\,a)$. In the nonclosed case, either $\mathrm{tr}L = -2$ or $ABCD$ and $DCBA$ cannot be hyperbolic (and so have to be parabolic).

Proof. $L \neq +/-\mathbf{1}$, since we are in the nonclosed case. Let $X = DCBA, Y = ABCD, L = \begin{pmatrix} k & l \\ 0 & k \end{pmatrix}$ with $k^2 = 1$. Assume X is hyperbolic; w. l. o. g. $X = \begin{pmatrix} \mathrm{tr}X & -\frac{1}{\mathrm{tr}X} \\ \mathrm{tr}X & 0 \end{pmatrix}$. Let $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & \mathrm{tr}X - y_1 \end{pmatrix}$. From
$$Y\,X^{-1} = \begin{pmatrix} -y_2\mathrm{tr}X & \frac{y_1}{\mathrm{tr}X} + y_2\mathrm{tr}X \\ -(\mathrm{tr}X)^2 + y_1\mathrm{tr}X & \frac{y_3}{\mathrm{tr}X} + (\mathrm{tr}X)^2 - y_1\mathrm{tr}X \end{pmatrix} = \begin{pmatrix} k & l \\ 0 & k \end{pmatrix} = L$$
and $\mathrm{tr}X \neq 0$ we get
$$y_1 = \mathrm{tr}X,\; y_2 = \frac{-k}{\mathrm{tr}X},$$
which implies: $y_3 = k\mathrm{tr}X$. Now if $\mathrm{tr}L = +2$, then $k = 1$, whence $X = Y$ and $L = \mathbf{1}$, a contradiction. Therefore $\mathrm{tr}L = -2$. □

4.6 Discreteness

Solutions found so far for the x_i, y_{ij}, z_{ijk} do not necessarily lead to a hyperbolic $M_g(f)$, since it might be that the corresponding representation of $\pi_1(M_g(f))$ in $\mathrm{SL}_2(\mathbf{C})$ is not discrete. The following lemma of Shimizu[14] is an easy–to–apply criterion:

Lemma 4.10. If $M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ is a parabolic and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a hyperbolic element of $G \leq \mathrm{SL}_2(\mathbf{C})$ satisfying
$$\frac{1}{|c|} > |m|,$$

[14] See e. g. [Leutbecher 67], Lemma 2.1.

then G is not discrete.

Proof. The sequence $A_1 = A, A_{i+1} = [A_i, M]$ has isometric spheres with radii tending towards ∞. Therefore the interior of the Fordian fundamental domain is empty. □

4.7 Computation of the Volume of M

Let T be a hyperbolic tetrahedron in $\mathbf{H}^3$ with exactly one vertex on the boundary of $\mathbf{H}^3$, which we transfer to $r = \infty$ (cf. fig. 5). Then three of the

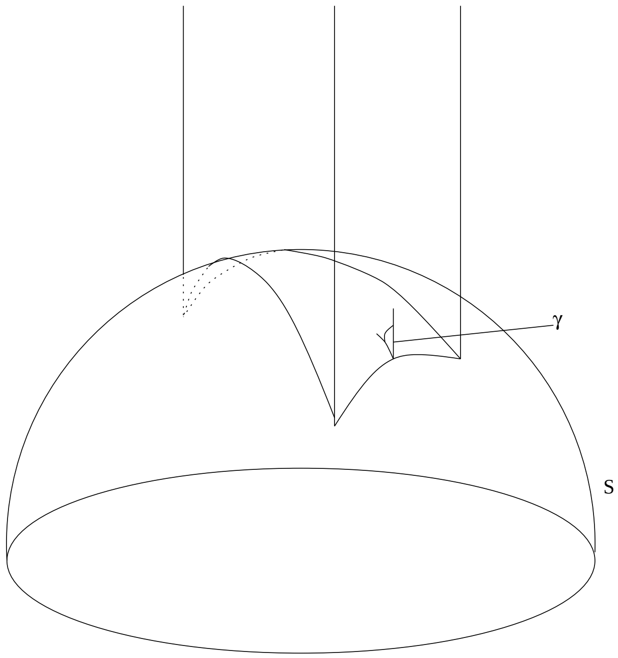

Figure 5: Hyperbolic tetrahedron with vertex at ∞ and opposite face

four faces of T are euclidean planes; the projection of T onto $\mathbf{C}$ is a euclidean triangle with vertices z_1, z_2, z_3; the fourth face of T is a hyperbolic plane that is a euclidean sphere with center $z_0 \in \mathbf{C}$. $T = T(S, z_1, z_2, z_3)$ is completely determined by the data S, z_1, z_2, z_3. Fig. 6 shows the projection of S and T onto $\mathbf{C}$ as well as α and β, two of the six angles between pairs of faces of T. Let γ be the angle between S and the hyperbolic plane through z_2, z_3, ∞ (cf.

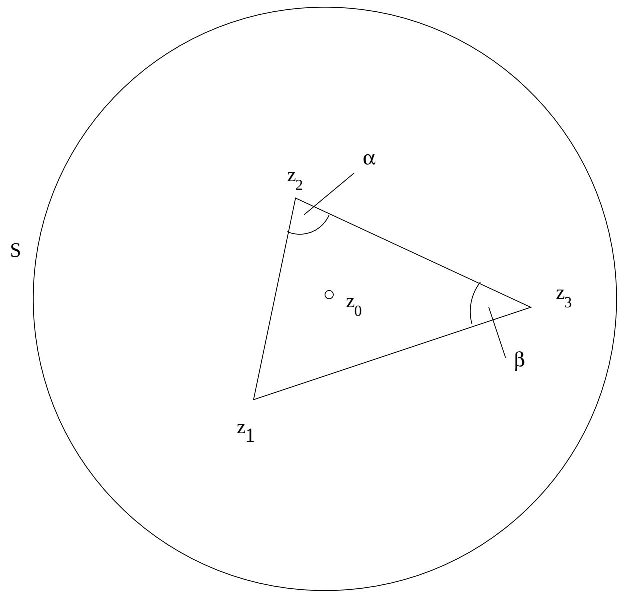

Figure 6: Projection of tetrahedron and isometric sphere

fig. 5).

Let $z_1 = z_0$, $\beta = \pi/2$. Then the volume of T is given by[15]

$$V(T) = \frac{1}{4}\left[\mathcal{L}(\frac{\pi}{2} - \alpha + \gamma) + \mathcal{L}(\frac{\pi}{2} - \alpha - \gamma) + 2\mathcal{L}(\alpha)\right]; \qquad (12)$$

the function $\mathcal{L}$ is the Lobachevsky–function

$$\mathcal{L}(\theta) := -\int_0^\theta \log(|2\sin u|)\mathrm{d}u.$$

Now let $z_1 = z_0$, α, β arbitrary. Then by the above formula

$$\begin{aligned} V(T) &= \frac{1}{4}\left[\mathcal{L}(\frac{\pi}{2} - \alpha + \gamma) + \mathcal{L}(\frac{\pi}{2} - \alpha - \gamma) + 2\mathcal{L}(\alpha)\right. \\ &\quad \left. + \mathcal{L}(\frac{\pi}{2} - \beta + \gamma) + \mathcal{L}(\frac{\pi}{2} - \beta - \gamma) + 2\mathcal{L}(\beta)\right]. \end{aligned}$$

Now let z_1, z_2, z_3 be arbitrary points in $\mathbf{C}$ in the interior of the projection of S. Then the volume of T is a weighted sum of volumes of the above type:

$$\begin{aligned} V(T(S, z_1, z_2, z_3)) &= \epsilon_1 V(T(S, z_0, z_2, z_3)) \\ &\quad + \epsilon_2 V(T(S, z_0, z_1, z_3)) \\ &\quad + \epsilon_3 V(T(S, z_0, z_1, z_2)), \end{aligned}$$

[15] [Thurston 82], chap. 7.1.

where $\epsilon_i = +/-1$ are signs determined by:

$\epsilon_i = +1$, iff z_0 and z_i are on the same side of the euclidean line through the remaining two points (i.e. iff the scalar products of $z_0 - z_j$ and $z_i - z_j$ with the normal vector $e^{i\pi/2}(z_k - z_j)$ $(i \neq j \neq k \neq i)$ have the same sign). If z_0 lies on the segment between the two remaining points, then ϵ_i doesn't matter, since the corresponding tetrahedron is degenerate.

Given S, z_1, z_2, z_3, the angles α, β, γ are given by

$$\alpha = \arccos \frac{(z_3 - z_2)\overline{(z_0 - z_2)}}{|z_3 - z_2||z_0 - z_2|},$$

$$\beta = \arccos \frac{(z_2 - z_3)\overline{(z_0 - z_3)}}{|z_2 - z_3||z_0 - z_3|},$$

$$\gamma = \arccos \frac{(z_2 - z_0)\overline{(z_3 - z_0)} - \overline{(z_2 - z_0)}(z_3 - z_0)}{2R(S)|z_2 - z_3|},$$

where $R(S)$ is the euclidean radius of S.

So the volume of a hyperbolic polyhedron with one vertex at $r = \infty$ and the remaining vertices on one and the same hyperbolic plane S is the sum of volumes of tetrahedra of the above type. Any hyperbolic polyhedron can be represented as set difference of polyhedra of the above type.

5 Given f, How to Find the Hyperbolic Structure on $M_2(f)$?

Now let $g = 2, r = 1$,

$$\pi_1(M_2(f)) = \langle a, \ldots, d, \mu \,|\, \mu\omega\mu^{-1} = f(\omega) \quad \forall \omega \in \Phi_4 \rangle$$

or

$$\pi_1(M_2(f)) = \langle a_1, \ldots, a_4, \mu \,|\, \mu\omega\mu^{-1} = f(\omega) \quad \forall \omega \in \Phi_4 \rangle,$$

where f is given in terms of the generators of Φ_4.

Assume $M_2(f)$ is hyperbolic. Then there is a solution x_i, y_{ij}, z_{ijk} of the following system of equations

(i) $\mathrm{tr}(\lambda) = +/-2,$ (cf. 7)

(ii) $4z_{\hat{i}}z_{\hat{j}} + \det Y'_{ij} = 0,$ (cf. 5)

(iii) $\det Y = 0,$ (cf. 8)

(iv) $\sum_{j=1}^{4}(-1)^j y_{ij} z_{\hat{j}} = 0 \ \forall i = 1, \ldots, 4,$ (cf. 11)

(v) $s(\omega) = s(f(\omega)) \ \forall \omega \in \Phi_4,$ (cf. 1), satisfying the additional condition

$$\text{None of the } x_i \text{ equals } 0, +/-1, +/-2.$$

(Solutions with $x_i = 0, \pm 1, \pm 2$ will be called trivial.)

We now describe the complete procedure of finding the solutions x_i, y_{ij}, z_{ijk} (if they exist), the representing matrices $M, L, A_1, \ldots, A_4$ or $A, \ldots, D$ respectively in $\mathrm{SL}_2(\mathbf{C})$ and the Fordian fundamental domain (if the image of $\pi_1(M_2(f))$ in $\mathrm{SL}_2(\mathbf{C})$ is discrete). We use a set of computer algorithms written in `Mathematica`. For syntax and specifications see [Wolfram 96]. All major algorithms are collected in the appendix as commented listings.

a) The System of Equations. All equations are treated as polynomial expressions, which are supposed to be equal to 0. Equations (i) through (iv) above are independent of f. They are represented as polynomial expressions $e1$ $(\mathrm{tr}(\lambda)-/+2), e2, \ldots, e11$ (determinant equations), $e12$ $(\det Y)$, and $e13, \ldots, e16$ (5–identities). They are formulated in terms of the coordiantes x_i, y_{ij}, z_{ijk} as well as the trace values $s(i), s(ij), s(ijk)$ and s(higher order products); this is done by the algorithm `gleichng.m`. The trace values are automatically reduced to the coordinates; trace values of higher order products are automatically reduced to expressions in $s(i), s(ij), s(ijk)$ by using formula (ii) of lemma 4.3. This is done by the algorithm `reduktio.m`.

The f–dependent trace equations (v) give rise to the polynomial expressions $s(a_i) - s(f(a_i)), s(a_ia_j) - s(f(a_ia_j)), s(a_ia_ja_k) - s(f(a_ia_ja_k))$ and—if helpful—s(higher order product) $-$ $s(f$(higher order product)). They come under the name $e17, e18, \ldots$ and are read in from an additional file `f.m`. If necessary, these equations are simplified by hand in advance, using lemma 4.3.

b) Simplification of the Polynomial Expressions. The algorithms `rules.m`, `apply.m`, and `zurueck.m` are interactively used to replace monomial expressions, mainly in order to reduce the total degree of the polynomials. Using `ersatz` as a replacement for `zuersetzen`, every monomial $p \cdot$ `zuersetzen` is automatically replaced by $p \cdot$ `ersatz`.

c) Elimination. We search for simple substitutions in order to eliminate all but two of the variables x_i, y_{ij}, z_{ijk}. Factors $x_i, (x_i \pm 1), (x_i \pm 2)$ are cancelled without mentioning, since they do not lead to a hyperbolic $M_2(f)$.

Up to now, we have not found a systematic procedure for b) and c), leading to the solution on the shortest path. Finding a solution or—more difficult—showing that no interesting solution exists, has so far been one of the major problems.

d) Numerical Evaluation of the Remaining Equations. If the system of equations is solvable at all, we normally retain two equations $ei_1 = 0, ei_2 = 0$ in two variables, all other variables being rational expressions in them. Computing the two resultants of ei_1, ei_2 with respect to the remaining variables, solving numerically and finding matching pairs (using `numerloes.m`, part 1) yields a 0–dimensional set S of solutions. If one of the remaining variables is x_i, every pair of solutions with $-2 \leq x_i \leq 2$ is ignored. If the two remaining variables are x_i and x_j (which has been the case so far), every solution with $x_i, x_j \in \mathbf{R}$ is ignored, since otherwise we obtain a real, therefore not cofinite representation.

e) Numerical Calculation of the Matrices. Let x_4 be one of the remaining variables. We define

$$M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, L = \begin{pmatrix} \pm 1 & l \\ 0 & \pm 1 \end{pmatrix}, A_4 = \begin{pmatrix} x_4 & -\frac{1}{x_4} \\ x_4 & 0 \end{pmatrix},$$

which is possible, since A_4 has to be hyperbolic. The other matrices can then be calculated by using the definition of f and the found dependencies between the x_i, y_{ij}, z_{ijk}. For $f = g_1$ (see subsection 7.1) this is done by the algorithm `g1numer.m`. (A_i are called X_i there.)

f) Discreteness. Up to now, there may be several solutions for M, L, A_i. At most one of them (and its complex conjugate) is discrete. By the lemma of Shimizu the group

$$G = \langle A_i, M \mid MA_iM^{-1} = f(A_i) \rangle \leq \mathrm{SL}_2(\mathbf{C})$$

is not discrete, if an element $B = (b_{rs})$ can be found, satisfying

$$\frac{1}{|b_{21}|} > |m|.$$

So at most one solution A_i, M, L remains.

g) Search for Isometric Spheres of Large Radius The Fordian fundamental domain of G is bounded from below by isometric spheres of matrices in G, which have large radius. We calculate the radii of isometric spheres of words in A_i up to a fixed length, using the algorithm `g1radien.m`. All words with radius greater than a fixed cutoff are printed, including radius and center of the isometric sphere.

h) The Fordian Fundamental Domain. This domain is best presented as projection onto $\mathbf{C}$, seen from ∞. Having found group elements S with isometric spheres with large radius, we draw the circles of intersection of these spheres with $\mathbf{C}$ (as well as those of conjugates $M^r L^p S L^{-p} M^{-r}$). By hand we draw the projections of intersection lines of isometric spheres and the resulting tiling of $\mathbf{C}$. Now the generating elements G_i of G and the relations between them can be read off the drawing.

i) Representing A_i by G_i and Proof of Isomorphy of Fundamental Groups. In order to check whether the found fundamental domain is correct, we represent the generating elements A_i of Φ_4 as words in G_i and vice versa. Then the relations $MA_iM^{-1} = f(A_i)$ can be checked using the relations between the G_i read off the drawing, and vice versa. If every relation $MA_iM^{-1} = f(A_i)$ can be reproduced from the relations between the G_i, the groups

$$\langle G_i, M, L \mid ML = LM, R_j(G_i)\rangle$$

and

$$\langle A_i, M \mid MA_iM^{-1} = f(A_i)\rangle$$

are isomorphic.

j) Volume. The volume of the fundamental domain is computed using the algorithm `volumen.m`. See subsection 4.7.

6 How to Find Suitable Automorphisms?

6.1 Facts About Aut(Φ_{2g}, λ)

We will first work in the representation

$$\lambda = \Pi_{i=1}^{g}[a_{2i-1}, a_{2i}].$$

We cite two theorems from [Magnus/Karass/Solitar 66][16]:

Theorem 6.1. All automorphisms α of the group

$$\pi_1(S_g^{(0)}) = \langle \Phi_{2g} \mid \Pi_{i=1}^{g}[a_{2i-1}, a_{2i}] = 1\rangle$$

can be represented as free isomorphisms acting on the generators a_i. If α maps a_i onto a_i', then

$$\alpha(\lambda) = \Pi[a'_{2i-1}, a'_{2i}]$$

is freely equal to a word

$$T\lambda^{\epsilon}T^{-1},$$

where T is a word in the a_i and $\epsilon = \pm 1$.

Theorem 6.2. The group of proper (i. e. $\epsilon = +1$) automorphisms of $\pi_1(S_2^{(0)})$ is generated by the inner automorphisms and

$$\alpha_1 : \begin{array}{ccc} a_1 & \mapsto & a_2^{-1} \\ a_2 & \mapsto & a_1 a_4^{-1} \\ a_3 & \mapsto & a_4^{-1} \\ a_4 & \mapsto & a_3 a_2^{-1} \end{array}, \quad \bar{f}_4 : \begin{array}{ccc} a_1 & \mapsto & a_1 \\ a_2 & \mapsto & a_2 \\ a_3 & \mapsto & a_3 \\ a_4 & \mapsto & a_3 a_4 \end{array}.$$

[16] Pages 176–177, Th. N10 and N12, notation adapted.

First of all we may as well substitute α_1 by the automorphism

$$\bar\alpha_1: \begin{array}{lcl} a_1 & \mapsto & a_4^{-1}a_2^{-1}a_4 \\ a_2 & \mapsto & a_4^{-1}a_2^{-1}a_1a_4^{-1}a_2a_4 \\ a_3 & \mapsto & a_4^{-1}a_2^{-1}a_4^{-1}a_2a_4 \\ a_4 & \mapsto & a_4^{-1}a_2^{-1}a_3a_4, \end{array}$$

which is α_1 followed by conjugation with a_2a_4. However, $\bar\alpha_1$ leaves λ invariant.

Now every automorphism of $\pi_1(S_2^{(1)})$, i. e. every $\varphi \in \mathrm{Aut}(\Phi_4,\lambda)$ induces an automorphism of $\pi_1(S_2^{(0)})$. Therefore every such φ is a word in $\bar\alpha_1$, $\bar f_4$, and inner automorphisms.

On the other hand, let $\psi \in \mathrm{Aut}(\pi_1(S_2^{(0)}))$ be a proper automorphism. Then ψ maps λ onto a conjugate of its own:

$$\lambda \mapsto \psi(\lambda) = T^{-1}\lambda T.$$

Now ψ is a word in $\bar\alpha_1, \bar f_4$, and inner automorphisms. The latter may be commuted to the left; so we may assume

$$\psi = [T'] \circ W(\bar\alpha_1, \bar f_4),$$

where $[T']$ denotes conjugation by T'. Therefore every proper automorphism of the closed surface $S_2^{(0)}$ that leaves λ invariant is a word in $\bar\alpha_1$ and $\bar f_4$, followed by a conjugation with a power of λ. Hence $\mathrm{Aut}(\Phi_4,\lambda)$ is generated by $\bar\alpha_1, \bar f_4$, and $[\lambda]$.

Neither $\bar f_4$ nor $\bar\alpha_1$ is pseudo–Anosov (e. g. $\bar\alpha_1^6$ maps a_1 to a conjugate of its own).

Now we turn to the representation $\lambda = a\,b\,c\,d\,a^{-1}b^{-1}c^{-1}d^{-1}$. The automorphism group of π_1 of the closed surface $S_2^{(0)}$ is generated by[17] the inner automorphisms and

$$f_1: \begin{array}{lcl} a & \mapsto & a \\ b & \mapsto & ba \\ c & \mapsto & a^{-1}c \\ d & \mapsto & da \end{array}, \quad \beta: \begin{array}{lcl} a & \mapsto & a^{-1}b^{-1}c^{-1}d^{-1}b^{-1} \\ b & \mapsto & c^{-1}a\,b\,c\,d \\ c & \mapsto & a^{-1}b^{-1}c^{-1}d^{-1}d^{-1} \\ d & \mapsto & a\,b\,c\,d \end{array}.$$

The automorphism β may as well be substituted by

$$\hat\beta: \begin{array}{lcl} a & \mapsto & b^{-1}x^{-1} \\ b & \mapsto & x\,c \\ c & \mapsto & d^{-1}x^{-1} \\ d & \mapsto & x \end{array},$$

where the abbreviation $x = d\,c\,b\,a$ is used. This automorphism leaves λ invariant.

[17]See [Bergau/Mennicke 60], p. 426, Satz 2; notation adapted.

We introduce the automorphisms

$$\tau : \begin{array}{lcl} a & \mapsto & b^{-1} \\ b & \mapsto & c^{-1} \\ c & \mapsto & d^{-1} \\ d & \mapsto & a \end{array} \quad \text{and } \hat{\alpha} := [y^{-1}] \circ \tau \circ f_1 : \begin{array}{lcl} a & \mapsto & y\,a^{-1}b^{-1}\,y^{-1} \\ b & \mapsto & y\,c^{-1}a\,y^{-1} \\ c & \mapsto & y\,a^{-1}d^{-1}\,y^{-1} \\ d & \mapsto & y\,a\,y^{-1}, \end{array}$$

($y = a\,b\,c\,d$), the latter leaving λ invariant. Hence $\mathrm{Aut}(\pi_1(S_2^{(0)}))$ is generated by $\hat{\beta}, \hat{\alpha}, \tau$, and the inner automorphisms. The subgroup

$$\langle \hat{\alpha}, \hat{\beta} \rangle < \mathrm{Aut}(\Phi_4, \lambda)$$

has been the starting point of our investigations on the hyperbolicity of 3–manifolds fibering over the circle with $S_2^{(1)}$.

The automorphism f_1 is obviously not pseudo–Anosov. Neither $\hat{\alpha}$ nor $\hat{\beta}$ is pseudo–Anosov; $\hat{\alpha}^5$ and $\hat{\beta}^6$ are inner automorphisms. See subsection 8.3 for some further negative results.

6.2 How to Find an Automorphism of $S_2^{(1)}$ that is Nonperiodic Irreducible?

Up to now, we have not been succesful in answering this question algebraically. So we returned to topological imagination.

As mentioned above, the null–homologous curves represent the main problem. Now every null–homologous simple closed curve cuts off a disc, a punctured torus or a punctered surface of genus 2, i. e. is null–homotopic, is an interesting curve or is freely homotopic to λ, respectively. The second case is the important one in the present context. Now two non–homotopic, “canonical”[18] such curves γ_1, γ_2 are shown in figure 7: The goal is to find a homeomorphism

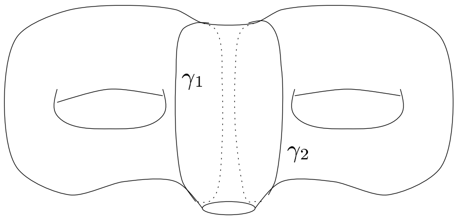

Figure 7: Two “canonical” null–homologous curves.

f of $S_2^{(1)}$ that destroys both of these curves, i. e. such that $f^n(\gamma_i)$ is not freely homotopic to γ_i for every $n \in \mathbf{Z} \setminus 0$.

[18] in the sense of [Stillwell 80], p. 197.

The automorphism shown in figure 6.2 obviously destroys both of these canonical curves in a nonperiodic way. From the figure we obtain:

$$\begin{aligned} blue &\mapsto blue \\ orange &\mapsto blue^{-1}\ orange\ blue \\ red &\mapsto blue^{-1}\ orange^{-1}\ blue\ red\ orange\ blue \\ green &\mapsto green\ blue^{-1}\ orange\ blue. \end{aligned}$$

(The black curve is given by $black = [orange^{-1}, red^{-1}]$.)

In this representation, $\lambda = blue\ green\ blue^{-1}\ black\ green^{-1}$ is left invariant. We identify

$$\begin{aligned} blue &\leftrightarrow a_2^{-1} \\ green &\leftrightarrow a_1 \\ orange &\leftrightarrow a_3^{-1} \\ red &\leftrightarrow a_4^{-1} \end{aligned}$$

and perform a conjugation with $blue\ green\ blue^{-1}$, obtaining

$$g_0 : \begin{array}{lcl} a_1 & \mapsto & a_2 a_3^{-1} a_2^{-1} a_1 \\ a_2 & \mapsto & a_2 a_3^{-1} a_2 a_3 a_2^{-1} \\ a_3 & \mapsto & a_2 a_3 a_2^{-1} \\ a_4 & \mapsto & a_2 a_4, \end{array} \qquad g_0^{-1} : \begin{array}{lcl} a_1 & \mapsto & a_3 a_1 \\ a_2 & \mapsto & a_3 a_2 a_3^{-1} \\ a_3 & \mapsto & a_3 a_2^{-1} a_3 a_2 a_3^{-1} \\ a_4 & \mapsto & a_3 a_2^{-1} a_3^{-1} a_4. \end{array}$$

Of course, g_0 is reducible: a_2 and a_3 are mapped onto conjugates of their own. Performing some Dehn twists, however, immediately yields automorphisms that are to complicated to solve the system of trace equations by means of our algorithms.

During a first performance of the procedure shown in figure 6.2, however, we arrived—by a lucky mistake—at the following automorphism g_1, which by chance turned out to lead to a hyperbolic $M_2(g_1)$:

$$g_1 : \begin{array}{lcl} a_1 & \mapsto & a_4^{-1} a_2 a_1 \\ a_2 & \mapsto & a_4^{-1} a_1^{-1} a_4 \\ a_3 & \mapsto & a_4^{-1} a_1 a_4^{-1} a_1^{-1} a_4 \\ a_4 & \mapsto & a_4^{-1} a_1 a_3 a_4. \end{array}$$

See subsection 7.1 for a complete solution for $M_2(g_1)$; see subsection 8.1 for a solution for $M_2(g_1 \circ \bar{f}_1^{-1})$.

Following page: Figure 6.2: Topological automorphism g_0. The base point is the intersection of *blue* and *green*.

7 Two Examples

7.1 $M_2(g_1)$ is Hyperbolic

The automorphism g_1 was given in subsection 6.2 by:

$$g_1 : \begin{array}{lcl} a_1 & \mapsto & a_4^{-1}a_2a_1 \\ a_2 & \mapsto & a_4^{-1}a_1^{-1}a_4 \\ a_3 & \mapsto & a_4^{-1}a_1a_4^{-1}a_1^{-1}a_4 \\ a_4 & \mapsto & a_4^{-1}a_1a_3a_4, \end{array}$$

$$g_1^{-1} : \begin{array}{lcl} a_1 & \mapsto & a_2a_3^{-1}a_2^{-1}a_3a_2^{-1} \\ a_2 & \mapsto & a_2a_3^{-1}a_2^{-1}a_1a_2a_3^{-1}a_2a_3a_2^{-1} \\ a_3 & \mapsto & a_2a_3^{-1}a_4a_2a_3a_2^{-1} \\ a_4 & \mapsto & a_2a_3^{-1}a_2^{-1}. \end{array}$$

This automorphism of $S_2^{(1)}$ leaves $\lambda = [a_4, a_3][a_2, a_1]$ invariant. It has Anosov–monodromy

$$A_{g_1} = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

and charactieristic polynomial

$$\chi_{A_{g_1}}(t) = t^4 - t^3 + 3t^2 - t + 1,$$

which has no root of unity as zero.

Now the trace equations $s(\omega) = s(g_1(\omega))$ are evaluated. For simplicity we refer to $s(a_i) = s(g_1(a_i))$ as equation i, to $s(a_ia_j^{-1}) = s(g_1(a_ia_j^{-1}))$ as equation ij^{-1} etc.

From equations 2 and 3 we have

$$x_2 = x_1,\ x_4 = x_3.$$

From 3, 4, and 24:

$$x_4 = x_3 = s(a_1a_3) = s(a_2a_4) \Rightarrow y_{13} = y_{24};$$

from 23:

$$s(a_2a_3) = s(a_1a_4) \Rightarrow y_{23} = y_{14}.$$

Equations 123^{-1} and 1 imply

$$s(a_1a_2a_3^{-1}) = s(a_4^{-1}a_3a_1),$$

which by lemma 4.5 entails

$$z_{124} = -z_{123}.$$

From 234 and $4^{-1}3$ we obtain in the same way

$$z_{234} = -z_{134}.$$

Now take a look at the algorithm `g1.m` in the appendix. Elements a_1, a_2, a_3, a_4 are called `a,b,c,d` there. The expressions $e17, \ldots, e41$ result from the equations indicated as comments.

Solving the system $e1 = e2 = \ldots = e41 = 0$ means exploring the ideal of polynomials generated by the ei. Since the solution is tricky and some ways are dead ends (even using a computer system), we will list the main steps of substitutions. Denominators and factors $x_i, x_i \pm 1, x_i \pm 2$ are cancelled without mentioning.

Many of the equations turn out to be redundant, $e19$ is one of them. So first of all we define $e19 = e2 + e3$. This expression factorizes, one factor being $y_{12} - x_1^2 + 4$. This factor cannot be equal to 0, as can be checked seperately. Now perform the following steps:

step	*using expression*	*comment*
replace x_1x_3	$e18$	reduction of total degree
solve for z_{123}	$e21$	
solve for z_{134}	$e22$	
replace x_1x_3	$e18$	reduction of total degree
replace $y_{12}y_{34}$	$e19$	separation of yy–products
replace $y_{12}y_{24}$	$e25$	separation of yy–products
replace x_1x_3	$e18$	reduction of total degree
solve for y_{24}	$e18$	
replace $y_{12}y_{23}$	$e25$	separation of yy–products
replace y_{23}^2	$e36$	separation of yy–products
replace $y_{12}y_{34}$	$e19$	separation of yy–products
solve for y_{34}	$e13$	
solve for y_{12}	$e26$	
replace $x_3y_{23}^2$	$e36$	reduction of total degree
replace $x_1^2y_{23}^2$	$e25$	reduction of total degree
replace $x_3^3y_{23}$	$e15$	reduction of total degree

Now $e19$ can be solved for y_{23} under the assumption that the occurring denominator is not 0 (this leads to the desired solution and is thereby justified).

Note that from the fifth step in the list the separation of yy–products, not the reduction of total degree, is the main operation.

Equation $e2$ is by now the polynomial gcd of the remaining expressions $e2, \ldots, e41$. All variables are rational expressions in x_1 and x_3.

The resultants of $e1$ and $e2$ with respect to x_1 and x_3 are (up to unimportant factors and powers):

$$\begin{aligned} f &= (x_1^7 + 7x_1^6 - 2x_1^5 - 13x_1^4 - 14x_1^3 - 18x_1^2 + 5x_1 + 5) \cdot \\ &\quad \cdot (x_1^8 + 3x_1^7 - 12x_1^6 - 48x_1^5 + 28x_1^4 + 202x_1^3 + 44x_1^2 - 271x_1 - 163) \\ &=: f_1 \cdot f_2; \end{aligned}$$

$$g = (x_3^5 + 5x_3^4 + 6x_3^3 - 7x_3^2 - 14x_3 - 4) \cdot$$

$$
\begin{aligned}
& \cdot(x_3^7 - 24x_3^6 - 105x_3^5 + 49x_3^4 + 559x_3^3 + 376x_3^2 - 460x_3 - 397)\cdot \\
& \cdot(x_3^8 - 10x_3^7 + 16x_3^6 + 138x_3^5 - 567x_3^4 + 420x_3^3 + 833x_3^2 - 714x_3 - 641) \\
=: \ & g_1 \cdot g_2 \cdot g_3.
\end{aligned}
$$

Part 1 of the algorithm `numerloes.m` now produces a list S of pairs of numerically found solutions (x_1, x_3) that satisfy $e1$ and $e2$. Discarding all pairs of real solutions leaves four solutions plus complex conjugates:

```
S={{x[1] -> -2.71973833140336 - 1.124382960689727*I,
    x[3] -> 4.006060353828278 + 0.4113234830578347*I},
   {x[1] -> -2.71973833140336 + 1.124382960689727*I,
    x[3] -> 4.006060353828278 - 0.4113234830578347*I},
   {x[1] -> -1.622458264073988 - 0.1867151609093862*I,
    x[3] -> -0.7765035223987667 + 0.0878383780479993*I},
   {x[1] -> -1.622458264073988 + 0.1867151609093862*I,
    x[3] -> -0.7765035223987667 - 0.0878383780479993*I},
   {x[1] -> -0.2329985889499337 - 1.099316038670357*I,
    x[3] -> -1.649591126994625 + 0.1444286296689254*I},
   {x[1] -> -0.2329985889499337 + 1.099316038670357*I,
    x[3] -> -1.649591126994625 - 0.1444286296689254*I},
   {x[1] -> 1.714156174534332 - 0.2525884759172021*I,
    x[3] -> 2.500831808822981 - 0.2339277674764923*I},
   {x[1] -> 1.714156174534332 + 0.2525884759172021*I,
    x[3] -> 2.500831808822981 + 0.2339277674764923*I}}.
```

We now choose

$$X_4 = \begin{pmatrix} x_4 & -\frac{1}{x_4} \\ x_4 & 0 \end{pmatrix}, M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, X_2 = \begin{pmatrix} b_1 & b_2 \\ b_3 & x_1 - b_1 \end{pmatrix}.$$

The definde matrices have to satify the trace equations which have been used above to find solutions for x_i, y_{ij}, z_{ijk}. This condition is used to explicitely calculate the entries of X_1, X_2, X_3. The definition of X_2 implies $\mathrm{tr}X_2 = x_1$. From $MX_2M^{-1} = f(X_2)$ we get

$$X_1 = X_4 M X_2^{-1} M^{-1} X_4^{-1} \Rightarrow \mathrm{tr}X_1 = x_1;$$

from the above equation and $MX_4M^{-1} = f(X_4)$ follows

$$X_3 = X_4 M X_2 X_4 M^{-1} X_4^{-1} \Rightarrow \mathrm{tr}X_3 = \mathrm{tr}(X_2X_4).$$

The equation $\mathrm{tr}(X_1X_3) = \mathrm{tr}X_3$ is automatically satisfied.

The entries b_1, b_2, b_3 can easily be found by using equations

$$\mathrm{tr}(X_2X_4) = x_3, \mathrm{tr}(X_1X_4) = s(a_1a_4), \text{ and } \mathrm{tr}(X_3X_4) = s(a_3a_4).$$

We now solve a set of some trace equations for m. This is done by algorithms `g1numer.m` and `numerloes.m`, part 2. Every tripel (x_1, x_3, m) found gives rise to a representation of $\pi_1(M_2(g_1))$ in $\mathrm{SL}_2(\mathbf{C})$.

Using `numerloes.m`, part 3, we single out the only discrete one (plus complex conjugate). The remaining triple is

```
{m -> -1.346916629434901 + 0.1213911342966917*I,
 x[1] -> -2.71973833140336 - 1.124382960689727*I,
 x[3] -> 4.0060603538283 + 0.4113234830579139*I},
```

`x[1]` and `x[3]` being zeros of f_2 and g_3, respectively. The resulting matrices are

$$X_1 = \begin{pmatrix} 0.64606 - 3.42893i & -3.66175 + 2.58026i \\ 0.81277 - 2.98567i & -3.36580 + 2.30454i \end{pmatrix},$$

$$X_2 = \begin{pmatrix} -9.82676 - 5.25567i & 10.36409 + 5.08812i \\ -6.79477 - 4.18529i & 7.10702 + 4.13128i \end{pmatrix},$$

$$X_3 = \begin{pmatrix} -8.71461 + 1.30695i & 9.15623 - 1.46294i \\ -12.20330 + 0.71838i & 12.72068 - 0.89562i \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 4.00606 + 0.41132i & -0.24702 + 0.02536i \\ 4.00606 + 0.41132i & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & -1.34692 + 0.12139i \\ 0 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} -1 & 3.24789 + 2.49015i \\ 0 & -1 \end{pmatrix}.$$

There is no proof of correctness for the used software, so we checked numerically all original expressions $e1, \ldots, e41$ by inserting the traces of X_i and products thereof.

The discriminants of f_2 and g_3 contain the three identical large prime factors 83, 191, and 3709, which suggests that the zeros of both are not completely independent. In fact, x_1 is an element of $\mathbf{Q}(x_3)$, namely

$$\begin{aligned} x_1 = \frac{1}{3^2 \cdot 21001} \quad \cdot \quad & \big(31 \cdot 223 \cdot x_3^7 + 2 \cdot 67 \cdot 107 \cdot x_3^6 - 2^3 \cdot 11 \cdot 17 \cdot 61 \cdot x_3^5 \\ & -2 \cdot 127081 \cdot x_3^4 + 2 \cdot 97 \cdot 1801 \cdot x_3^3 + 3 \cdot 11^2 \cdot 3121 \cdot x_3^2 \\ & -5 \cdot 53231 \cdot x_3 - 2^2 \cdot 197 \cdot 1801\big) . \end{aligned}$$

This result has been found using `Maple`.

Using `g1radien.m`, we now calculate a list of matrices with large radii of their isometric spheres (see table 1). It turned out that 0.47 is the smallest radius of importance for the fundamental domain (see below). The five indices in column 2 refer to matrix $X[i_1]X[i_2]X[i_3]X[i_4]X[i_5]$, where $X[0] = \mathbf{1}, X[1] = X_1, X[2] = X_1^{-1}, \ldots, X[7] = X_4, X[8] = X_4^{-1}$. See below for the occuring G_i.

This set of matrices consists of conjugacy classes under conjugation by M and L. The projection of their isometric spheres onto $\mathbf{C} = \partial\mathbf{H^3} \setminus \infty$ is a regular pattern of intersecting circles. Every line segment joining two intersection points of intersecting circles is the projection (onto $\mathbf{C}$) of the hyperbolic line of intersection of the corresponding isometric spheres (i. e. hyperbolic planes). So the projection of the hyperbolic polygons bounding the fundamental domain from below can be read off this pattern of circles (see fig. 8).

Table 1: Large isometric spheres for g_1 up to products of length 5.

no.	$i_1i_2i_3i_4i_5$	radius	center	remark
1	00027	$R = 0.52$	$C = -0.37 - 0.16\,i$	G_2L^{-1}
2	00036	$R = 0.52$	$C = 0.98 - 0.29\,i$	
3	00037	$R = 0.78$	$C = -1.35 - 0.77\,i$	
4	00054	$R = 0.52$	$C = 1.67 + 0.67\,i$	
5	00081	$R = 0.52$	$C = 0.32 + 0.79\,i$	inverse of no. 1
6	00084	$R = 0.78$	$C = 1.76 - 0.23\,i$	
7	00231	$R = 0.69$	$C = -0.39 + 1.4\,i$	
8	00241	$R = 0.69$	$C = 1.35 + 0.33\,i$	
9	00254	$R = 0.69$	$C = 2.7 + 0.21\,i$	inv. of no. 11
10	00257	$R = 0.52$	$C = -1.91 - 0.59\,i$	
11	00361	$R = 0.69$	$C = 0.96 + 1.28\,i$	$M^{-2}G_5$
12	00857	$R = 0.78$	$C = -2.7 - 0.65\,i$	
13	00861	$R = 0.52$	$C = 0.64 + 0.94\,i$	
14	00867	$R = 0.78$	$C = 0.41 - 0.11\,i$	
15	02231	$R = 0.47$	$C = -1.38 + 1.22\,i$	
16	02411	$R = 0.47$	$C = 1. + 0.64\,i$	
17	02454	$R = 0.47$	$C = 2.34 + 0.51\,i$	$G_{11}M^2$
18	03631	$R = 0.47$	$C = -0.03 + 1.1\,i$	inv. of no. 17
19	03637	$R = 0.52$	$C = -0.57 - 0.72\,i$	$M^{-1}G_4L^{-1}$
20	08454	$R = 0.52$	$C = 1.99 + 0.82\,i$	inv. of no. 19
21	24131	$R = 0.69$	$C = 0.01 + 0.45\,i$	
22	24136	$R = 0.78$	$C = 0.41 - 0.11\,i$	
23	24137	$R = 0.78$	$C = -0.93 + 0.01\,i$	$M^2G_1L^{-1}$
24	24154	$R = 0.51$	$C = 2. + 0.48\,i$	inv. of no. 28
25	24157	$R = 0.52$	$C = -1.71 - 0.04\,i$	
26	24231	$R = 0.69$	$C = -1.73 + 1.52\,i$	
27	26857	$R = 0.69$	$C = -4.98 - 0.97\,i$	
28	36231	$R = 0.51$	$C = -1.04 + 1.25\,i$	$M^{-1}G_7M^{-1}$
29	36857	$R = 0.69$	$C = -3.63 - 1.09\,i$	
30	54231	$R = 0.78$	$C = 0.55 + 1.84\,i$	
31	84231	$R = 0.78$	$C = -0.79 + 1.96\,i$	inv. of no. 23
32	86231	$R = 0.52$	$C = -1.02 + 0.91\,i$	
33	86751	$R = 0.69$	$C = 0.01 + 0.45\,i$	
34	86754	$R = 0.69$	$C = 1.35 + 0.33\,i$	
35	86757	$R = 0.78$	$C = -0.93 + 0.01\,i$	
36	86857	$R = 0.78$	$C = -4.04 - 0.53\,i$	

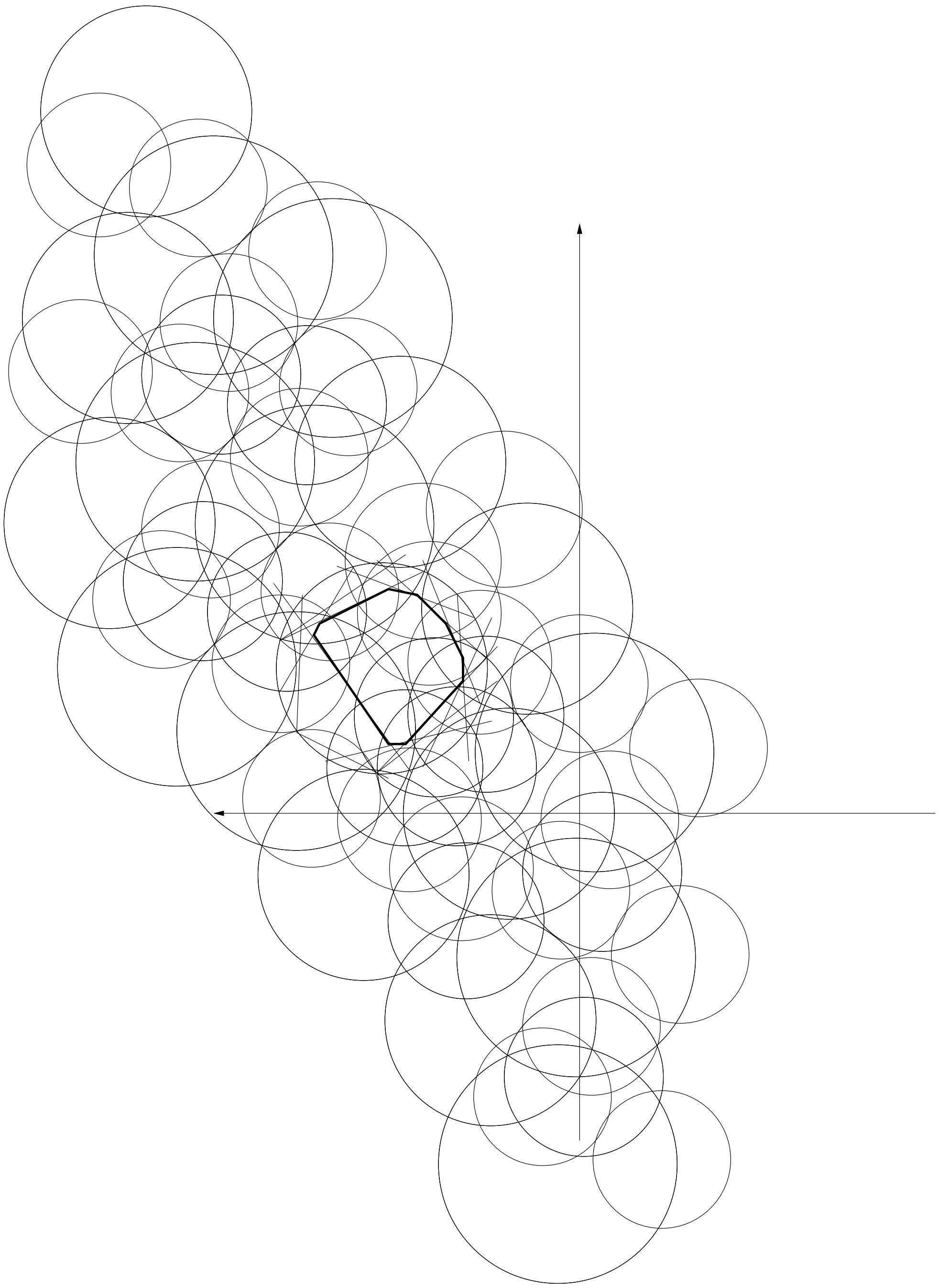

Figure 8: Large isometric spheres for elements of $\pi_1(M_2(g_1))$.

As a result we get a tiling of $\mathbf{C}$, where every tile is (the projection of) a boundary–polygon of the Fordian fundamental domain (see fig. 9 and 10). Twelve of the tiles have been selected as representatives of the conjugacy classes. The corresponding group elements $G_1, G_2, G_4, G_5, G_7, G_{11}$ (see remarks in table 1) can be used as generating elements for $\pi_1(M_2(g_1))$, as we will show now.

The elements G_i are given by

$$\begin{aligned} G_1 &= M^{-2}X_1^{-1}X_2^{-1}X_1X_2X_4L \\ G_2 &= X_1^{-1}X_4L \\ G_4 &= MX_2X_3^{-1}X_2X_4L \\ G_5 &= M^2X_2X_3^{-1}X_1 \\ G_7 &= MX_2X_3^{-1}X_1^{-1}X_2X_1M \\ G_{11} &= X_1^{-1}X_2^{-1}X_3X_2^{-1}M^{-2} \end{aligned}$$

From this follows:

$$\begin{aligned} X_1 &= G_5^{-1}G_{11}^{-1}M^2G_1G_2^{-1}G_{11}G_5, \\ X_2 &= G_5^{-1}G_{11}^{-1}M^2G_1G_2^{-1}X_1^{-1}, \\ X_3 &= X_1G_5^{-1}M^2X_2, \\ X_4 &= X_1G_2L^{-1}. \end{aligned}$$

No smaller isometric sphere contributes to the boundary of the fundamental domain, as can be checked by calculating the euclidean height of the vertices above $\mathbf{C}$ in fig. 9.

The parallelogram shown in fig. 9 through 12 indicates the operation of conjugation by M (= translation to the left) and L^{-1} (= translation towards the upper right corner). From fig. 9 we read off the following relations:

no.	*relator* $= 1$	*edge no.*
R_1:	$G_1L^{-1}MG_5MG_1$	1
R_2:	$MG_2^{-1}G_4G_1$	2
R_3:	$L^{-1}MG_7G_2M^{-1}G_1$	3
R_4:	$L^{-1}G_5G_4M^{-1}G_1$	4
R_5:	$G_2^{-1}G_5G_1$	5
R_6:	$G_4^{-1}G_7MG_1$	6
R_7:	$G_{11}MG_5^{-1}MG_7$	7
R_8:	$MG_5^{-1}G_{11}MG_7$	8
R_9:	$G_5M^{-1}G_5M^{-1}G_{11}$	9
R_{10}:	$G_5^{-1}LG_2^{-1}G_7$	10
R_{11}:	$G_4M^{-1}G_5^{-1}G_7$	11

(For the edge numbering cf. fig. 11 and 12.)

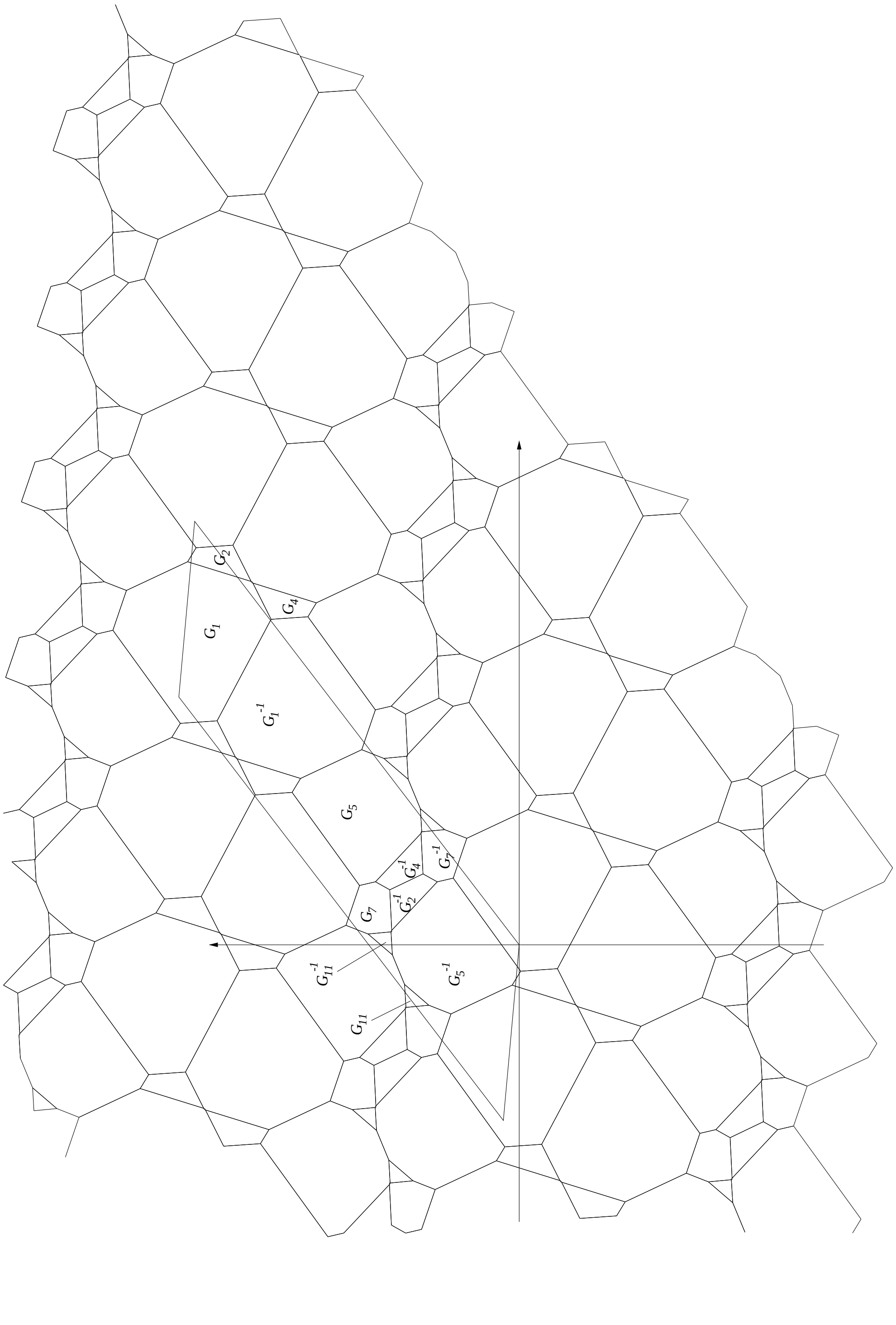

Figure 9: Projected boundary of fundamental domain of $M_2(g_1)$.

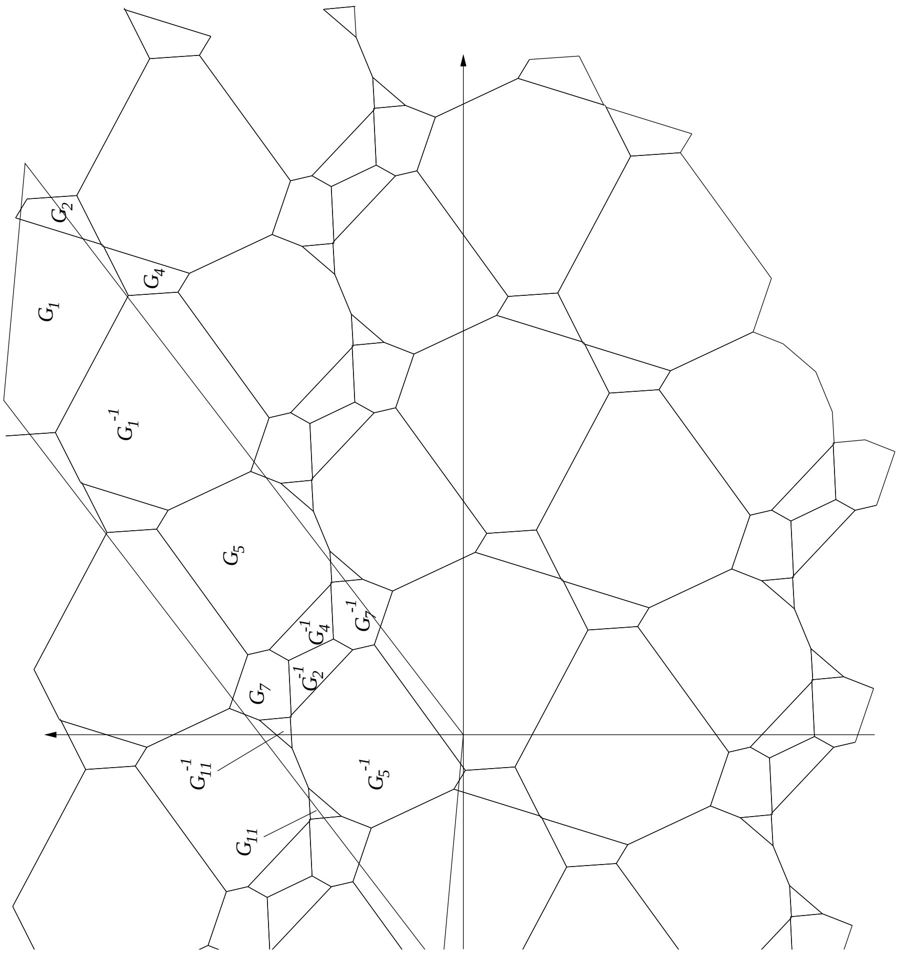

Figure 10: Projected boundary, enlarged.

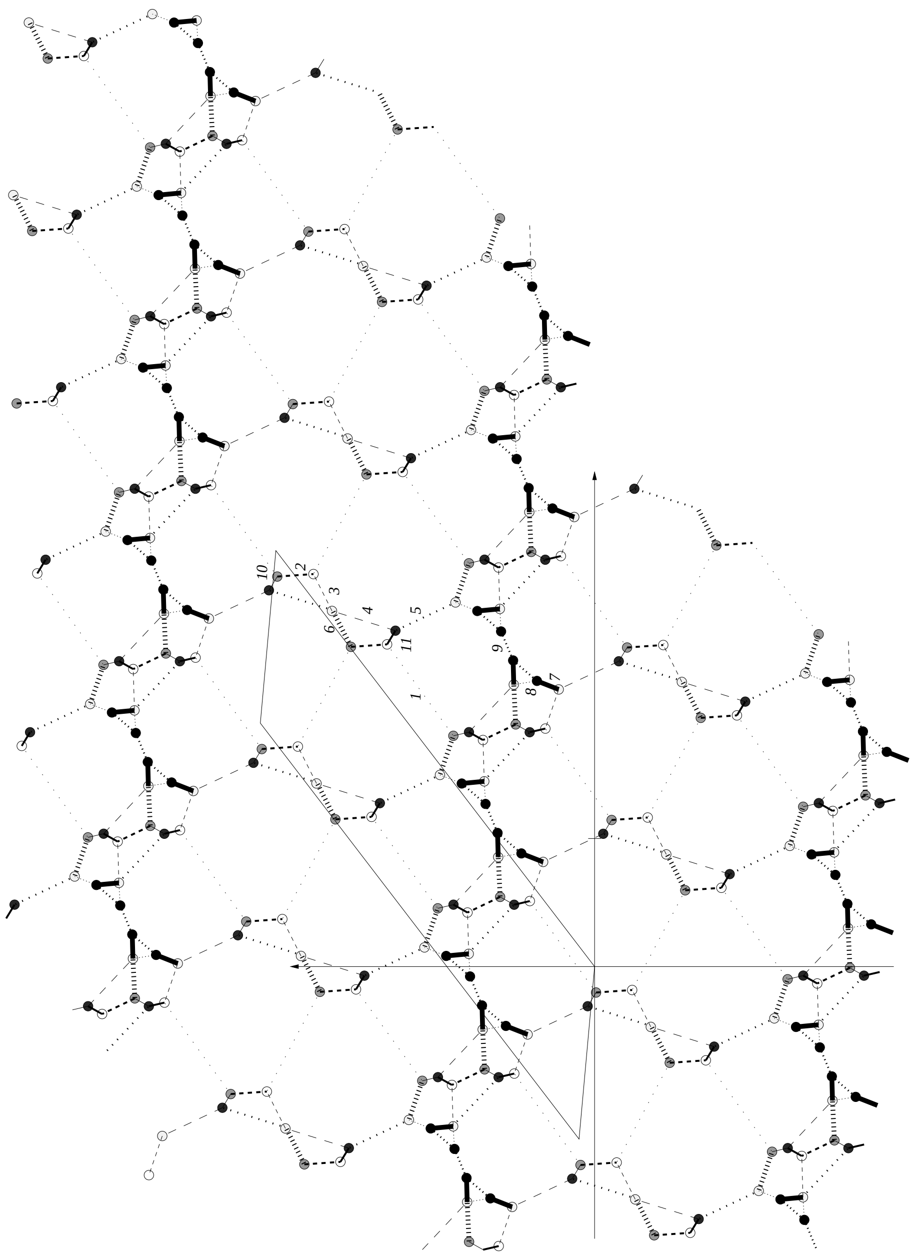

Figure 11: Identification of edges.

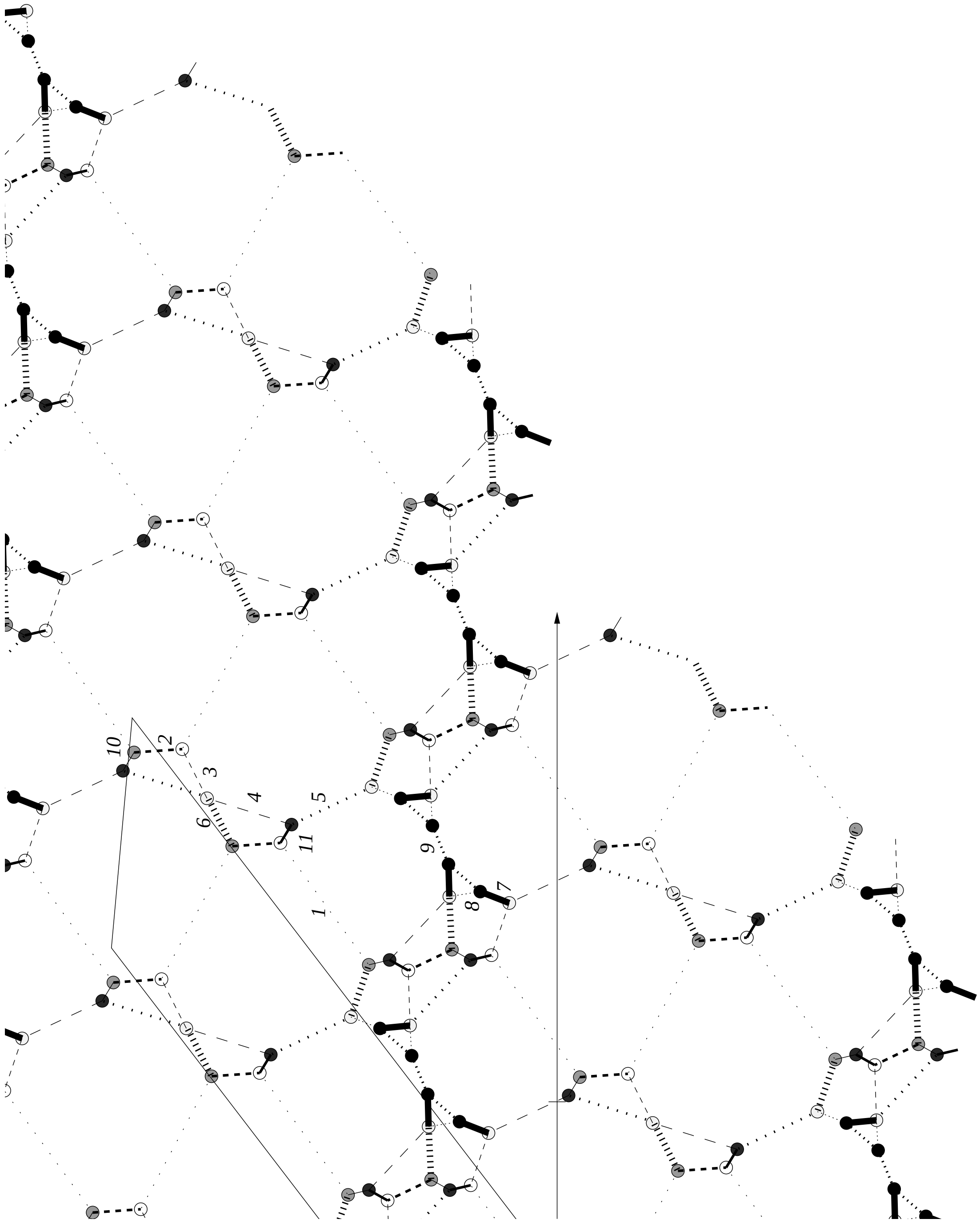

Figure 12: Identification of edges, enlarged.

In order to prove that $M_2(g_1)$ is hyperbolic, it remains to show that the groups

$$\pi_1(M_2(g_1)) = \langle X_i, M | M X_i M^{-1} = f(X_i) \rangle$$

and

$$\langle G_i, M, L | R_i = 1, ML = LM, L = [X_4, X_3][X_2, X_1] \rangle$$

are indeed isomorphic.

Proof. It remains to show that the relations R_i are satisfied in the first representation and that the relations $MX_iM^{-1} = f(X_i)$ are satisfied in the second representation.

Take R_{10} as an example:

$$\begin{aligned}
& G_5^{-1} L G_2^{-1} G_7 \\
= \quad & X_1^{-1} X_3 X_2^{-1} M^{-1} \quad \underbrace{M^{-1} X_4^{-1} X_1}_{\substack{\text{commute using} \\ M^{-1}X_i = f^{-1}(X_i)M^{-1}}} \quad M X_2 X_3^{-1} X_1^{-1} X_2 X_1 M \\
= \quad & X_1^{-1} X_3 X_2^{-1} \underbrace{M^{-1} X_1^{-1} X_2 X_1}_{\text{dito}} M \\
= \quad & 1.
\end{aligned}$$

The other relations R_i have been checked numerically.

On the other hand, take $f(X_1)MX_1^{-1}M^{-1}$ as an example:

$$\begin{aligned}
& \underbrace{L G_2^{-1}}_{R_{10}: L G_2^{-1} G_7 G_5^{-1}} \quad G_5^{-1} G_{11}^{-1} M G_5^{-1} G_{11}^{-1} G_2 G_1^{-1} M^{-2} \quad \underbrace{G_{11} G_5 M^{-1}}_{R_9: G_5 M^{-1} G_{11} G_5 M^{-1}} \\
= \quad & G_5 G_7^{-1} \; \underbrace{G_5^{-1} G_{11}^{-1} M G_5^{-1}}_{R_9: G_5^{-1} G_{11}^{-1} M G_5^{-1} M} \; G_{11}^{-1} G_2 G_1^{-1} M^{-1} G_5^{-1} \\
= \quad & G_5 \quad \underbrace{G_7^{-1} M^{-1} G_{11}^{-1}}_{R_8: G_5 M^{-1} G_7^{-1} M^{-1} G_{11}^{-1}} \quad G_2 G_1^{-1} M^{-1} G_5^{-1} \\
= \quad & G_5 M \; \underbrace{G_5^{-1} G_2 G_1^{-1}}_{R_5: G_5^{-1} G_2 G_1^{-1}} \; M^{-1} G_5^{-1} \\
= \quad & 1.
\end{aligned}$$

The other relations have been checked numerically. □

Figures 11 and 12 now give a complete picture of the fundamental domain as well as group of $M_2(g_1)$. Edges and vertices are marked according to their identification under $\pi_1(M_2(g_1))$. There are eleven classes of edges. Identification of faces corresponds to euclidean congruence of the tiles. A fundamental domain is given by the (euclidean) column above the twelve isometric spheres of G_i and G_i^{-1}.

The volume of $M_2(g_1)$ has been calculated using algorithm `volumen.m`. It is

$$\mathrm{vol}(M_2(g_1)) = 5.2126\ldots.$$

Remarks.

- There are "quadruple points", i. e. vertices in the boundary of the fundamental domain, where four isometric spheres intersect.
- None of the X_i contributes to the boundary of the fundamental domain, as opposed to manifolds fibering with the once–punctured torus.
- The algorithm `numerloes.m`, part 3, does of course not prove discreteness, nor is there a guarantee to find out non–discreteness within reasonable time. Discreteness is shown by proving isomorphy of the two representations of π_1, as done above.

  We had luck in the present case; see however subsection 8.2 for a less comfortable example.
- The tiles do not seem to have a "standard form" like in the case of manifolds fibering with $S_1^{(1)}$.

As far as we know, $M_2(g_1)$ is the first known example of a 3–manifold fibering over the circle with a noncompact surface of genus $g > 1$.

7.2 $\pi_1(M_2(g_1\bar{f}_1^{-2}))$ is Not Discrete

The automorphism $g_1 f_1^{-2}$ is given by

$$g_1\bar{f}_1^{-2}: \begin{array}{lcl} a_1 & \mapsto & a_4^{-1}a_1^2a_2a_1 \\ a_2 & \mapsto & a_4^{-1}a_1^{-1}a_4 \\ a_3 & \mapsto & a_4^{-1}a_1a_4^{-1}a_1^{-1}a_4 \\ a_4 & \mapsto & a_4^{-1}a_1a_3a_4. \end{array}$$

The Anosov–monodromy has characteristic polynomial $\chi_{g_1\bar{f}_1^{-2}} = 1 - 3t + 3t^2 - 3t + t^4$, which has no root of unity as zero.

Now $\pi_1(M_2(g_1\bar{f}_1^{-2}))$ admits a torsionfree $\mathrm{SL}_2(\mathbf{C})$–representation, which is, however, not discrete. We merely state the results which have been found along the same procedure as described above. (It seems to be most important to solve the system of equations in exactly the reported order, since otherwise the expressions become too complicated.)

All variables are rational expressions in x_1 and x_3, which are roots of the polynomials

$$f(x_1) = x_1^5 - 3x_1^4 + 28x_1^3 - 95x_1^2 + 203x_1 - 163$$

and

$$g(x_3) = x_3^5 - 15x_3^4 + 88x_3^3 - 251x_3^2 + 347x_3 - 187.$$

The matrices are given by

$$X_1 = \begin{pmatrix} 2.44061 + 1.95471i & -1.02430 - 0.16225i \\ 3.43733 + 1.85355i & -1.04524 - 0.16929i \end{pmatrix},$$

$$X_2 = \begin{pmatrix} -1.18863 + 1.20488i & 1.21285 - 1.09668i \\ -3.15817 - 0.85761i & 2.58401 + 0.58054i \end{pmatrix},$$

$$X_3 = \begin{pmatrix} -0.15745 + 0.59850i & 0.70765 - 0.64665i \\ -1.73818 + 0.11911i & 1.95110 - 0.25765i \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 1.79365 + 0.34085i & -0.53809 + 0.10226i \\ 1.79365 + 0.34085i & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & -0.03179 - 0.34085i \\ 0 & 1 \end{pmatrix},$$

where the entry $m = M_{1,2}$ has been found solving a subset of the trace equations, as in the case of g_1,

$$L = \begin{pmatrix} -1 & -1.59013 - 0.71642i \\ 0 & -1 \end{pmatrix}.$$

This group is not discrete, however, since the radii of the isometric spheres of X_3 and X_4 are greater than $|m| = |M_{1,2}|$.

8 Further Results

8.1 $M_2(g_1\bar{f}_1^{-1})$ is Hyperbolic

Let

$$\bar{f}_1 : a_1 \mapsto a_2 a_1, \text{ all other } a_i \text{ invariant}$$

be a Dehn twist. Then

$$g_1\bar{f}_1^{-1} : \begin{array}{lcl} a_1 & \mapsto & a_4^{-1}a_1a_2a_1 \\ a_2 & \mapsto & a_4^{-1}a_1^{-1}a_4 \\ a_3 & \mapsto & a_4^{-1}a_1a_4^{-1}a_1^{-1}a_4 \\ a_4 & \mapsto & a_4^{-1}a_1a_3a_4 \end{array}$$

has Anosov-monodromy

$$A_{g_1\bar{f}_1^{-1}} = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

with characteristic polynomial

$$\chi_{g_1\bar{f}_1^{-1}}(t) = (t^2 - t + 1)^2,$$

which has roots of unity as zeros.

The system of trace equations can be solved. We merely state the results; the procedure is the same as for $M_2(g_1)$.

We obtain

$$X_1 = \begin{pmatrix} 1.69549 - 3.45329i & -5.52089 + 2.03089i \\ 1.77384 - 3.12833i & -5.15001 + 1.82201i \end{pmatrix},$$

$$X_2 = \begin{pmatrix} -11.18067 - 4.82140i & 11.731445 + 4.70533i \\ -7.43459 - 3.23374i & 7.72615 + 3.19013i \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 10.34384 - 0.62389i & -10.79601 + 0.73753i \\ 13.99015 - 0.16567i & -14.51104 + 0.29520i \end{pmatrix},$$

$$X_4 = \begin{pmatrix} -4.16720 - 0.32868i & 0.23849 - 0.018810i \\ -4.16720 - 0.32868i & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & -1.47697 + 0.03762i \\ 0 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} -1 & 1.71464 + 3.14042i \\ 0 & -1 \end{pmatrix}.$$

The tiling of $\mathbf{C}$ by projected polygons on large isometric spheres is shown in figure 13; four matrices, $G_1, \ldots, G_4$, plus M and L generate the fundamental group. The isometric spheres belonging to G_i and G_i^{-1} are also shown in figure 13. The matrices G_i are given by:

$$\begin{aligned} G_1 &= M^2 X_2 X_4, \\ G_2 &= M X_1^{-1} X_2 X_1, \\ G_3 &= X_1^{-1} X_4, \\ G_4 &= M^{-2} X_1^{-1} X_2^{-1} X_1 X_3 X_2^{-1} M^{-2}, \end{aligned}$$

implying

$$\begin{aligned} X_1 &= M^{-2} G_1 G_3^{-1} G_2^{-1} M, \\ X_2 &= X_1 M^{-1} G_2 X_1^{-1}, \\ X_3 &= X_1^{-1} X_2 X_1 M G_4 M^2 X_2, \\ X_4 &= X_1 G_3. \end{aligned}$$

From the figure we obtain the following relations (each determining an equivalence class of edges):

$$\begin{aligned} R_1 &: \quad G_2 M G_4 G_2 M \\ R_2 &: \quad G_2 M L^{-1} G_1^{-1} G_3^{-1} \\ R_3 &: \quad M G_3 G_2 M^2 G_4 \\ R_4 &: \quad L^{-1} M G_1^{-2} M G_2 M \\ R_5 &: \quad M^{-1} G_1^{-1} G_2 M^2 G_3 \\ R_6 &: \quad M^{-2} L G_2^{-1} M^{-1} G_4^{-1} G_3 \end{aligned}$$

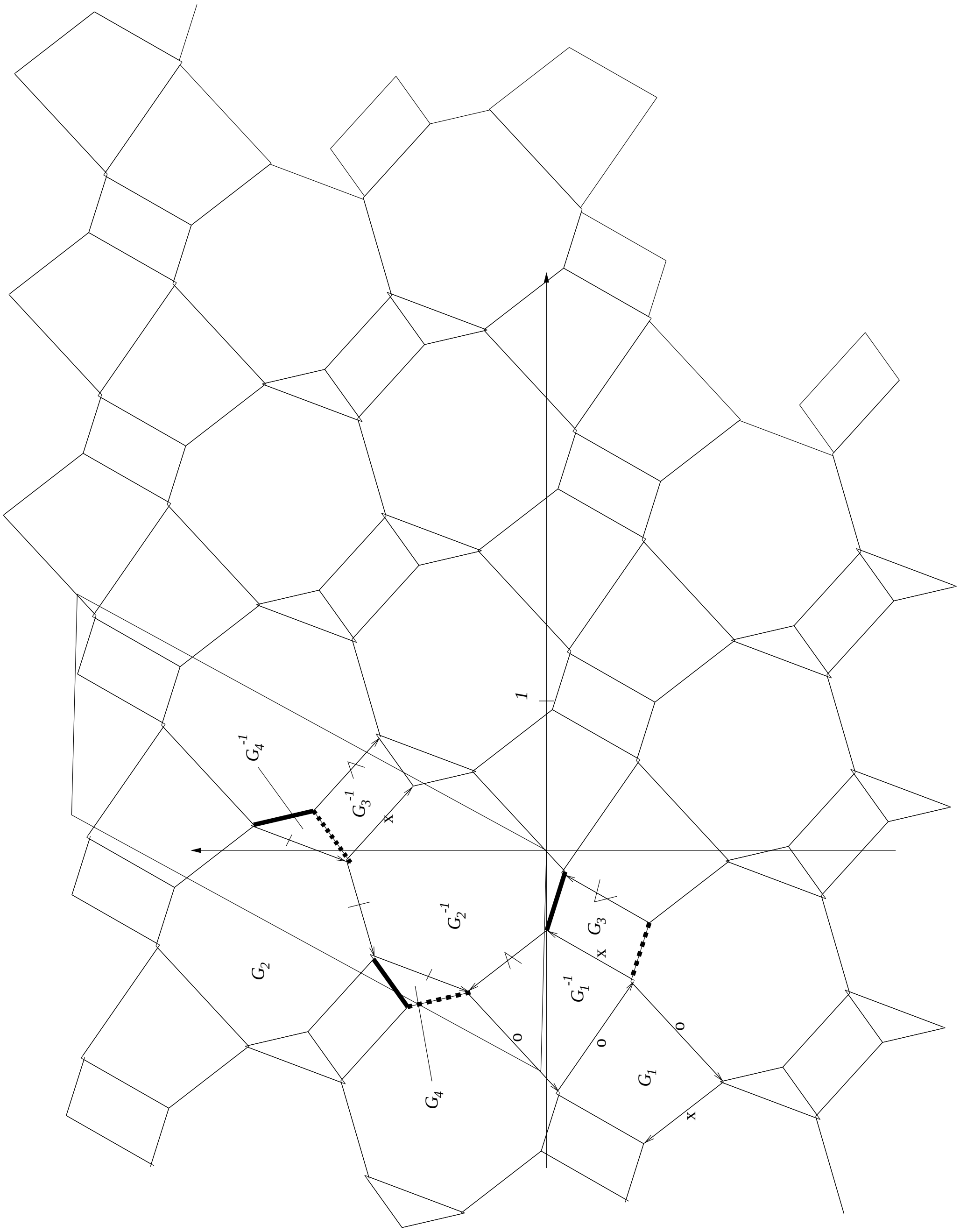

Figure 13: Fundamental domain and edge identification for $M_2(g_1 f_1^{-1})$.

The groups

$$\langle X_1, X_2, X_3, X_4, M \mid M^{-1}X_iM = f(X_i)\rangle$$

and

$$\langle G_i, M, L \mid [M, L] = 1, R_i = 1, L = [X_4, X_3][X_2, X_1]\rangle$$

are isomorphic.

There are quadruple points as in the case of $M_2(g_1)$.

8.2 $M_2(\hat\alpha f_1^2)$: an Open Case

The automorphism $\hat\alpha f_1^2$ is given by

$$\hat\alpha f_1^2 : \begin{array}{lcl} a & \mapsto & y\, a^{-1}b^{-1}\, y^{-1} \\ b & \mapsto & y\, c^{-1}b^{-1}a^{-1}b^{-1}\, y^{-1} \\ c & \mapsto & y\, babd^{-1}\, y^{-1} \\ d & \mapsto & y\, b^{-1}a^{-1}b^{-1}\, y^{-1}, \end{array}$$

$$(\hat\alpha f_1^2)^{-1} : \begin{array}{lcl} a & \mapsto & x\, a^{-2}d\, x^{-1} \\ b & \mapsto & x\, d^{-1}a\, x^{-1} \\ c & \mapsto & x\, b^{-1}d\, x^{-1} \\ d & \mapsto & x\, d^{-1}c^{-1}\, x^{-1}, \end{array}$$

where we introduced the abbreviations $x = d\,c\,b\,a$ and $y = a\,b\,c\,d$. This automorphism has Anosov–monodromy

$$A_{\hat\alpha f_1^2} = \begin{pmatrix} -1 & -1 & 1 & -1 \\ -1 & -2 & 2 & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and characteristic polynomial

$$\chi_{\hat\alpha f_1^2}(t) = t^4 + 3t^3 + 3t^2 + 3t + 1,$$

which has no root of unity as zero.

From the trace equations we deduce the following results:

$$\begin{array}{llcl} 1: & s(ab) & = & x_a \\ 23, 34: & s(bc) & = & s(cd) = x_d \\ 2^{-1}4: & s(bd) & = & x_bx_d - x_c \\ 1^{-1}4: & s(ad) & = & x_ax_d - x_b \\ 13: & s(ac) & = & x_c \\ 3: & x_c & = & \frac{x_b - x_ax_d}{1 - x_a} \\ 4: & x_d & = & x_ax_b - x_a, \end{array}$$

hence

$$y_{ab} = x_a - x_d, \quad y_{ac} = 2x_c - x_a x_c, \quad y_{ad} = -2x_b + x_a x_d,$$
$$y_{bc} = 2x_d - x_b x_c, \quad y_{bd} = x_b x_d - 2x_c, \quad y_{cd} = 2x_d - x_c x_d.$$

Further:

$$\begin{array}{lrcl} 123: & z_{abc} & = & s(x) - s(y) \\ 1^{-1}34, 124^{-1}: & z_{abd} & = & z_{acd} = z_{abc} \\ 234: & z_{abc} & = & -(1+x_a) z_{bcd} \end{array}$$

Let $A, \ldots, D, X, Y$ be the matrices representing $a, \ldots, d, x, y$ (if they exist). Let

$$M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, L = \begin{pmatrix} \pm 1 & -l \\ 0 & \pm 1 \end{pmatrix}.$$

We define

$$G := M^{-1} Y, H := MX.$$

Then

$$D = GH, C = (G^2 H^2)^{-1}, B = G^3 H^3, A = G^4 H^4$$

as a consequence of the relations $MAM^{-1} = \hat\alpha f_1^2(A)$ etc. The relation $L = YX^{-1}$ transforms into

$$G^4 H^3 G^4 = (H^4 G^3 H^4)^{-1}.$$

Starting at this point, further calculations have been made using the algorithms `gleichng.m`, `rules.m`, `apply.m`. Four cases have to be considered: $\operatorname{tr}\lambda = \pm 2, z_{bcd} = / \neq 0$. In none of these cases is X parabolic, as can be checked separately by investigating implications of the equation

$$s(dcba)^2 - 4 = 0.$$

Therefore we choose

$$X = \begin{pmatrix} s(x) & -\frac{1}{s(x)} \\ s(x) & 0 \end{pmatrix}.$$

Case 1: $\operatorname{tr}\boldsymbol{\lambda} = -2$, $z_{bcd} \neq 0$. Then

$$L = \begin{pmatrix} -1 & -l \\ 0 & -1 \end{pmatrix},$$

whence

$$Y = LX = \begin{pmatrix} -s(x)(1+l) & \frac{1}{s(x)} \\ -s(x) & 0 \end{pmatrix},$$

$$\operatorname{tr} Y = -s(x)(1+l) = s(x) + (1+x_a) z_{bcd}$$

$$\Leftrightarrow l = \frac{2s(x) + (1+x_a) z_{bcd}}{-s(x)}$$

by trace equations 123 and 234. A complete reduction of the system of trace equations leaves

$$\begin{aligned} e1 &= (x_a^4 - x_a^3 - 2x_a^2 + 2x_a + 1)(x_a^5 + 3x_a^4 + 2x_a^3 - 3x_a^2 - 3x_a + 1) \\ &=: f_1 f_2, \end{aligned}$$

all other x_i being rational expressions in x_a; the remaining second variable is z_{bcd}. Now

$$s(x) = \frac{f_2 + z_{bcd}(-2 + x_a^2 + f_2)}{2(1 - x_a)(1 + x_a)^3},$$

as has been found using `reduktio.m`. If $f_2 = 0$, we get

$$s(x) = \frac{(x_a^2 - 2)z_{bcd}}{2(1 - x_a)(1 + x_a)^3} \Rightarrow l = \frac{-f_2}{(x_a - 1)(x_a + 1)^2} = 0,$$

which is impossible.

Therefore $f_1 = 0$, leaving a single non–real solution (plus complex conjugate) for x_a.

From $D = GH$ we find

$$x_d = \mathrm{tr}(GH) = s(x)(1 + m)(s(x)m + s(y)) + 2. \tag{13}$$

Inserting the complex root of f_1 for x_a and solving the remaining equation $e2$ for z_{bcd} and equation 13 for m numerically yields representations of $\pi_1(M_2(\hat{\alpha} f_1^2))$ in $\mathrm{SL}_2(\mathbf{C})$. None of them is discrete; either B or AD^{-1} have isometric spheres with radius $> |m|$.

Case 2: tr $\boldsymbol{\lambda = +2}$, $\boldsymbol{z_{bcd} \neq 0}$. The same procedure as in case 1 applies; $e1$, the equation $\mathrm{tr}\lambda = 2$, becomes

$$e1 = (x_a^4 + 3x_a^3 + 2x_a^2 - 2x_a - 3)(x_a^5 + x_a^4 - 4x_a^3 - 3x_a^2 + 3x_a + 1);$$

the second factor has zeros in the interval $(-2, +2)$ and can be ignored. The first factor has one complex root (plus complex conjugate), which is substituted numerically for x_a. As in case 1, $e2$ and the equation $x_d - \mathrm{tr}(GH) = 0$ are solved numerically for z_{bcd} and m. In any case a matrix with radius of isometric sphere $> |m|$ can be found (try one of H^2A, $AD^{-1}BH^2$, $CG^{-1}C$).

These results can be seen in another way as well. We will prove now by elementary substitutions that the fundamental group of $M_2(\hat{\alpha} f_1^2)$ is generated by two elements with only one relator. The group is given by

$$\begin{aligned} \langle A, B, C, D, M \quad | \quad & R_1 : MAM^{-1} = ABCDA^{-1}B^{-1}(ABCD)^{-1} \\ & R_2 : MBM^{-1} = ABCD(BABC)^{-1}(ABCD)^{-1} \\ & R_3 : MCM^{-1} = ABCDBABD^{-1}(ABCD)^{-1} \\ & R_4 : MDM^{-1} = ABCD(BAB)^{-1}(ABCD)^{-1} \\ & R_5 : MABCD(DCBA)^{-1}M^{-1} = ABCD(DCBA)^{-1} \rangle, \end{aligned}$$

where R_5 is a consequence of $R_1, \ldots, R_4$. We introduce new names as usual, obtaining

$$\begin{aligned} \cong \langle A,B,C,D,M,X,Y,G,H \quad | \quad & R_1,\ldots,R_5, \\ & R_6 : X = DCBA, R_7 : Y = ABCD, \\ & R_8 : MXM^{-1} = B^{-1}A^{-1}B^{-1}Y^{-1}, \\ & R_9 : G = M^{-1}Y, R_{10} : H = MX\rangle \end{aligned}$$

(R_8 is a consequence of $R_1, \ldots, R_4$)

$$\begin{aligned} \cong \langle A,B,C,D,M,X,Y,G,H \quad | \quad & R_4 : D = (M^{-1}Y)(MX), \\ & R_3 : C = (X^{-1}M^{-1})^2(Y^{-1}M)^2, \\ & R_2 : B = (M^{-1}Y)^3(MX)^3, \\ & R_1 : A = (M^{-1}Y)^4(MX)^4, R_5, \\ & R_6 : X = (M^{-1}Y)(X^{-1}M^{-1})(M^{-1}Y) \cdot \\ & \cdot(MX)^3(M^{-1}Y)^4(MX)^4, \\ & R_7 : Y = (M^{-1}Y)^4(MX)^4(M^{-1}Y)^3 \cdot \\ & \cdot(MX)(Y^{-1}M)(MX), \\ & R_8 : MXM^{-1} = ((M^{-1}Y)^3(MX)^3(M^{-1}Y)^4 \cdot \\ & \cdot(MX)^4(M^{-1}Y)^3(MX)^3)^{-1}Y^{-1}, \\ & R_9 : Y = MG, R_{10} : X = M^{-1}H\rangle. \end{aligned}$$

We now may omit A, B, C, D, X,and Y and their defining relators $R_1, \ldots, R_4$, R_9, R_{10} and combine R_9 with R_7 and R_{10} with R_8 and obtain

$$\begin{aligned} \cong \langle M,G,H \quad | \quad & R_5 : MGH^{-1} = GH^{-1}M, \\ & R_6 : M^{-1}H = GH^{-1}GH^3G^4H^4, \\ & R_7 : MG = G^4H^4G^3HG^{-1}H, \\ & R_8 : HG = H^{-3}G^{-3}H^{-4}G^{-4}H^{-3}G^{-3}\rangle. \end{aligned}$$

Now M can be written in terms of G and H, using R_7:

$$M = G^3H^4G^3HG^{-1}HG$$

and can therefore be omitted. Inserting this into the other relations yields the desired representation with two generators and one relator:

$$\cong \langle G,H \quad | \quad R_8 : H^4G^3H^4 = (G^4H^3G^4)^{-1}\rangle.$$

The geometry comes in by postulating that $L = MGH^{-1}M$ and M as given above be represented by parabolic matrices.

Now since every element of this group is a word in G and H, all z–coordinates have to be zero in order to obtain a $\mathrm{SL}_2(\mathbf{C})$–representation.

We have included this proof in detail, since the present case turned out to be one of the most difficult cases studied.

Case 3 and 4: tr $\boldsymbol{\lambda} = +/-2$, $z_{bcd} = 0$. As mentioned above, X and Y may not be parabolic. So from lemma 4.9 we know that the case tr$L = +2$ is excluded. So tr$L = -2$. A solution of the system of equations can then be found. Discreteness remains to check. There is a hint, however, that $M_2(\hat{\alpha} f_1^2)$ is not reducible. According to the theorem of subsection 3.4, $(\hat{\alpha} f_1^2)^2$ maps a commutator onto its conjugate, if it is reducible. We have computed the images of all commutators of words of length ≤ 4; no one was mapped to a conjugate of its own.

8.3 Collection of (Mostly Negative) Results

In this subsection we shortly report on several automorphisms φ that have been investigated, mostly with the result that $M_2(\varphi)$ is not hyperbolic.

First we work in the representation $\lambda = [a_4, a_3][a_2, a_1]$. The automorphism $\bar{f}_4$ has already been defined; let $\bar{f}_i\,(i = 1, 2, 3)$ be the other three Dehn twists that leave λ invariant, i. e.

$$\bar{f}_1 : a_1 \mapsto a_2 a_1, \quad \bar{f}_2 : a_2 \mapsto a_1 a_2, \quad \bar{f}_3 : a_3 \mapsto a_4 a_3,$$

all other generating elements being unchanged. Then the whole subgroup

$$\langle \bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4 \rangle < \mathrm{Aut}(\Phi_4, \lambda)$$

consists of automorphisms that are not pseudo–Anosov, since the "canonical" 0–homologous simple closed curve $[a_4, a_3]$ is left invariant by all of them.

The same is true for

$$\bar{\alpha}_1^2 \bar{f}_4^m \quad \text{and} \quad \bar{\alpha}_1^3 \bar{f}_4^m \quad \forall m \in \mathbf{Z};$$

the third power of the first maps a_2 to a conjugate of its own; the second leaves $[a_4, a_3]$ invariant.

Since $\alpha_1^3 : (a_1, a_2, a_3, a_4) \mapsto (a_3, a_4, a_1, a_2)$, $\bar{\alpha}_1^3$ is a permutation, followed by a conjugation. So $\bar{\alpha}_1 \bar{f}_4^m\,(m \in \mathbf{Z})$ remains to investigate.

If $m = \pm 1$ (and 0, of course), then the system of equations for $\bar{\alpha}_1 \bar{f}_4^m$ has trivial solutions only. In the cases $m = \pm 2, \pm 3$, nontrivial solutions have been found (i. e., there is a $\mathrm{SL}_2(\mathbf{C})$–representation of $\pi_1(M_2(\bar{\alpha} \bar{f}_4^m)))$ which is a candidate; discreteness remains to check. A general result for arbitrary m is not yet found. These results are up to now totally computer–dependent.

The automorphisms $g_1 \bar{f}_1^{-n}$ have been investigated for $n = 2, \ldots, 6$ and $n = -1, \ldots, -18$. None of them leads to a hyperbolic manifold. These results are computer results; however, the solution suggests that in general $M_2(g_1 \bar{f}_1^{-n})$ is not hyperbolic except for $n = 0, 1$. A proof has not yet been found, due to the fact that the trace equations are too complicated.

The automorphisms $g_1 \bar{f}_i^{\pm 1}$ have been investigated as well for $i = 2, 3, 4$. There are no nontrivial solutions for $i = 2, 3$ and for $i = 4$ with negative exponent. For $g_1 \bar{f}_4$, a solution of the system of equations has been found. Discreteness of the corresponding representation of the fundamental group remains to check.

We now turn to the representation $\lambda = a\,b\,c\,d\,a^{-1}\,b^{-1}\,c^{-1}\,d^{-1}$.

The subgroup $\langle\hat{\alpha},\hat{\beta}\rangle < \mathrm{Aut}(\Phi_4,\lambda)$ has been the starting point for all investigations.

We considered automorphisms of the form $\hat{\beta}^j\hat{\alpha}^i$. Since $\beta^6 = \alpha^5 = \mathrm{id}$, we may restrict ourselves to $j = 0,\ldots,5, i = 0,\ldots,4$. Of these, only two—$\hat{\beta}\hat{\alpha}^3$ and $\hat{\beta}^5\hat{\alpha}^2$—have Anosov–monodromies without roots of unity as eigenvalues. Both of them have been investigated with the computer; the corresponding manifolds are not hyperbolic. Only three of the others have been investigated—$\hat{\alpha},\hat{\beta}$, and $\hat{\beta}\hat{\alpha}$. None of them leads to a hyperbolic manifold either. (Referring to the Anosov–monodromy is not a good argument, though, as we see from the previous sections.)

9 The Case of a Twice–Punctured Surface

In this chapter we report some basic thoughts about hyperbolicity of manifolds with a twice–punctured surface as fiber. The major goal was to check the software and to find out whether there are quick solutions. The latter is not the case, however. So the following three sections merely have to be viewed as a starting point for further research.

9.1 Preliminary Considerations

Let $S_g^{(2)}$ be the surface of genus $g > 0$ with two punctures. Then the fundamental group is given by

$$\begin{aligned}\pi_1(S_g^{(2)}) &= \langle a_1,\ldots,a_{2g},\kappa_1,\kappa_2 \mid \kappa_2 = \Pi_{i=1}^g[a_{2i-1},a_{2i}]\,\kappa_1\rangle \\ &\cong \langle a_1,\ldots,a_{2g},\kappa\rangle = \Phi_{2g+1}.\end{aligned}$$

A homeomorphism f of $S_g^{(2)}$ maps boundary components to boundary components and hence acts on κ_1,κ_2 in the following way:

$$\begin{array}{rcl}\kappa_1 & \mapsto & \omega_1^{-1}\,\kappa_{i_1}^{\epsilon_{i_1}}\,\omega_1, \\ \kappa_2 & \mapsto & \omega_2^{-1}\,\kappa_{i_2}^{\epsilon_{i_2}}\,\omega_2,\end{array}$$

where (i_1,i_2) is a permutation of $(1,2)$ and ϵ_{i_j} are signs.

Now if $M_2(f)$ is hyperbolic, then $M_2(f^4)$ is, since $M_2(f^4)$ is a 4–fold covering of $M_2(f)$ and $\pi_1(M_2(f^4)) < \pi_1(M_2(f))$. So we restrict ourselves to the case

$$\begin{array}{rcl}\kappa_1 & \mapsto & \omega_1^{-1}\,\kappa_1\,\omega_1, \\ \kappa_2 & \mapsto & \omega_2^{-1}\,\kappa_2\,\omega_2.\end{array}$$

Further we may assume that the base point x_0 is on κ_2, whence

$$\kappa_2 \mapsto \kappa_2.$$

As a consequence of the relation in $\pi_1(S_g^{(2)})$ we get

$$f(\lambda\kappa) = \lambda\kappa$$

(where $\lambda = \Pi[a_{2i-1}, a_{2i}]$ as above and $\kappa_1 = \kappa$), or

$$f(\lambda) = \lambda\kappa f^{-1}(\kappa) = \lambda\kappa\omega^{-1}\kappa^{-1}\omega.$$

This condition replaces the former $f(\lambda) = \lambda$.

Now every automorphism $\varphi \in \mathrm{Aut}(\Phi_{2g}, \lambda)$ can be extended to an automorphism $\tilde{\varphi}$ of Φ_{2g+1} that satisfies $\tilde{\varphi}(\lambda\kappa) = \lambda\kappa$ by putting $\tilde{\varphi}(\kappa) = \kappa$.

9.2 Consequences for the Matrices

$M_2(f)$ has two tori as boundary, so if $M_2(f)$ is hyperbolic, it has two cusps. Without loss of generality we may assume that κ is represented by a parabolic matrix K with fixed point 0:

$$K = \begin{pmatrix} k/2 & 0 \\ r & k/2 \end{pmatrix}, \quad k^2 = 4.$$

As above we define

$$M = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

Let

$$\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the matrix representing ω. Then in $\pi_1(M_2(f))$ we have

$$M^{-1}KM = \pm\Omega^{-1}K\Omega.$$

If we assume "$-$" on the right hand side, then from

$$0 = KM\Omega^{-1} + M\Omega^{-1}K = \begin{pmatrix} k(d-cm) + r(am-b) & k(am-b) \\ -ck + (a+d-cm)r & ak + (am-b)r \end{pmatrix}$$

we obtain $b = am \Rightarrow d = cm, a = 0 \Rightarrow c = 0 \Rightarrow \Omega = \mathbf{0}$, a contradiction.

Hence

$$M^{-1}KM = \Omega^{-1}K\Omega \Leftrightarrow [\Omega M^{-1}, K] = \mathbf{1}.$$

This is possible only if ΩM^{-1} is of the form $\begin{pmatrix} \pm 1 & 0 \\ s & \pm 1 \end{pmatrix}$. Therefore

$$\begin{pmatrix} a & b-am \\ c & d-cm \end{pmatrix} = \Omega M^{-1} = \begin{pmatrix} \pm 1 & 0 \\ s & \pm 1 \end{pmatrix},$$

implying $b = am, b \neq 0 \neq a$. If $\Omega \neq \pm\mathbf{1}$, then $c \neq 0$. ΩM^{-1} represents the meridian of the second boundary torus resulting from κ.

From $\det(\Omega) = 1$ we obtain $d = a + cm$, hence

$$\Omega = \begin{pmatrix} a & am \\ c & a+cm \end{pmatrix}, \Omega M^{-1} = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix},$$

$$\mathrm{tr}(\Omega) = 2a + cm, \quad a^2 = 4.$$

9.3 The Case of the Twice Punctured Torus

Two automorphisms of $\pi_1(S_1^{(2)})$ that are not extensions of elements of $\mathrm{Aut}(\Phi_2, \lambda)$ are given by

$$\varphi_1 : \begin{array}{lcl} \xi & \mapsto & \xi^{-1}\kappa\eta\xi \\ \eta & \mapsto & \xi^{-1}\kappa\xi^{-1}\kappa^{-1}\xi \\ \kappa & \mapsto & \xi^{-1}\kappa\xi \end{array}, \quad \varphi_1^{-1} : \begin{array}{lcl} \xi & \mapsto & \kappa^{-1}\eta^{-1}\kappa \\ \eta & \mapsto & \kappa^{-1}\eta^{-1}\xi\kappa^{-1}\eta\kappa \\ \kappa & \mapsto & \kappa^{-1}\eta^{-1}\kappa\eta\kappa \end{array},$$

$$\varphi_2 : \begin{array}{lcl} \xi & \mapsto & \eta\kappa\xi \\ \eta & \mapsto & \eta\kappa\eta\kappa^{-1}\eta^{-1} \\ \kappa & \mapsto & \eta\kappa\eta^{-1} \end{array}, \quad \varphi_2^{-1} : \begin{array}{lcl} \xi & \mapsto & \kappa^{-1}\eta^{-1}\xi \\ \eta & \mapsto & \kappa^{-1}\eta\kappa \\ \kappa & \mapsto & \kappa^{-1}\eta^{-1}\kappa\eta\kappa \end{array}.$$

(Here $\lambda = \xi^{-1}\eta^{-1}\xi\eta$.)

The automorphism φ_1 admits totally real solutions only (up to trivial solutions), as can be seen in the following way:

From eq. 2 we obtain $x_\xi = x_\eta$. We know $x_\kappa = \pm 2$. From eq. 1 follows $x_\xi = s(\kappa\eta) \Rightarrow y_{\eta\kappa} = 2x_\xi - x_\xi x_\eta = 0$ or $4x_\xi$. Eq. 23 implies $y_{\xi\kappa} = -y_{\eta\kappa}$. Eq. 12 $\Rightarrow z_{\xi\eta\kappa} = 1/2(2x_\xi^2 - x_\xi^2 x_\kappa - 2y_{\xi\eta} - x_\kappa y_{\xi\eta})$. The remaining equations 13 and 123 yield results already known. The remaining equations are

$$\mathrm{tr}(\lambda\kappa) = \pm 2 \Rightarrow \mathrm{tr}(\lambda\kappa)^2 - 4 = 0$$

$$\begin{aligned} \Leftrightarrow 0 \;=\; & (-8 - 8x_\xi^2 + x_\xi^4 + 4x_\kappa + 2x_\xi^2 y_{\xi\eta} + y_{\xi\eta}^2) \cdot \\ & (+8 - 8x_\xi^2 + x_\xi^4 + 4x_\kappa + 2x_\xi^2 y_{\xi\eta} + y_{\xi\eta}^2) \end{aligned}$$

and

$$4z_{\xi\eta\kappa}^2 + \det Y = 0$$

$$\Leftrightarrow 0 = 16x_\xi^2 - 2x_\xi^4 - 8x_\xi^2 x_\kappa + x_\xi^4 x_\kappa - 4x_\xi^2 y_{\xi\eta} + 2x_\xi^2 x_\kappa y_{\xi\eta} + 2y_{\xi\eta}^2 + x_\kappa y_{\xi\eta}^2.$$

Letting $x_\kappa = +2$ entails $y_{\xi\eta} = 0$ from the second equation and $x_\xi^2 = 8$ from the first. Letting $x_\kappa = -2$ entails $y_{\xi\eta} = (8 - x_\xi^2)/2$ from the second equation and hence $x_\xi^2 = 8$ or 16 from the first.

The automorphisms φ_2 and $\varphi_2 \circ \varphi_1$ do not admit nontrivial solutions at all, as can be seen equally.

10 Glimpses

During the preparation of the present work the investigations have already made some progress. We formulate some aims/directions of further research that we will try to achieve in the future.

Construction of New Hyperbolic Manifolds The trace values have already been found in several cases. We have solved the set of equations for the automorphisms $\bar{\alpha}_1 \bar{f}_1^m$ for $m = \pm 2, \pm 3$. These candidates will be checked for discreteness next. The solution will probably be computable for general m. This may in the end allow to completely investigate all automorphisms in $\mathrm{Aut}(\Phi_4, \lambda = [a_4, a_3][a_2, a_1])$.

In the representation $\lambda = a\,b\,c\,d\,a^{-1}b^{-1}c^{-1}d^{-1}$ we introduce f_2, f_3, f_4 as analogues to f_1:

$$f_2 : (a, b, c, d) \mapsto (ab, b, b^{-1}c, db),$$
$$f_3 : (a, b, c, d) \mapsto (ac, c^{-1}b, c, dc),$$
$$f_4 : (a, b, c, d) \mapsto (ad, d^{-1}b, cd, d).$$

The series $f_1 f_2^{-1} f_3 f_4^{-n}$ has already been investigated. Most probably the last resultants can be computed for general $n > 2$, using Čebyšev polynomials. This may lead to a second series of hyperbolic manifolds.

Some general results might possibly be derived from these series, such as statements about the trace fields, the importance of the homology, the series of volumes, combinatorial properties of the pairing of faces of the fundamental domain as well as different presentations of the fundamental groups, best ways of solving the system of equations.

There may as well be guiding hints for the investigation of the case $g > 2$.

The case $g = 1, r > 2$, as proposed in subsection 9.3, will be traced further. The group of automorphisms

$$\{\varphi : \Phi_{2+1} \to \Phi_{2+1} \,|\, \varphi(\lambda) = \lambda[\kappa^{-1}, \omega] \text{ for some } \omega\} < \mathrm{Aut}(\Phi_{2+1})$$

might possibly be written down explicitely and surveyed in a systematic way.

We will try to generalize found results to the case of general $g > 1, r \geq 1$.

Adaptation and Improvement of the Algorithms Our rather stoneaged algorithms listed in the appendix have already been improved. Due to this improvement are the proposed new candidates of the previous paragraph. Further improvement is in progress in several directions:

- We will hopefully be able to present an automatized version of replacement for the reduction of total degree.
- Several new interactive–to–use tools for simplifications will be added.

- We will shortly present a fully automatized core that generates and partly simplifies all determinant– and trace–equations and 5–identities once a genus g, a number of punctures r and an automorphism f of $\pi_1(S_g^{(r)})$ are specified.

- We will as well present a fully automatized construction of fundamental domains, including algebraic and numeric presentation of $\pi_1(M^3) < \mathrm{SL}_2(\mathbf{C})$, once the traces $s(i), s(ij), s(ijk)$ are given.

The idea of providing tools for the interactive solution of high degree systems of equations rather than fully automatized solution algorithms is most important for a successful improvement.

Note that the simplification tools as well as parts of the equation generating algorithms can be used in a much wider context, since the conditions for (discrete) representability of a group in $\mathrm{SL}_2(\mathbf{C})$ are independent of the present context of fibering manifolds. Every finitely presented group can be checked for discrete, torsionfree representations in $\mathrm{SL}_2(\mathbf{C})$ using our set of algorithms.

All algorithms operate completely symbolically up to computation of the last resultants; only then numerical algorithms are applied.

Distant Aims There are some distant aims that play the role of leading ideas behind our work. We try to formulate them carefully; however, up to now there are only vague ideas how to achieve them.

Our algorithms might be applied to arbitrary points on the character variety of the fiber. So the action of a diffeomorphism of the fiber on the space of all representations might be traced explicitely. This might lead to some ideas how to represent Teichmüller space or the space of quasi–Fuchsian deformations of the fundamental group of the fiber. In the end there might be a deeper understanding of Thurston's compactification of Teichmüller space.

A large class of examples may in the end lead to the formulation of a criterion for hyperbolicity that can be read off the algebraic presentation of the monodromy f directly. This would then be a generalization of the characteristic–polynomial–criterion that works for fibrations with closed surfaces or the once punctured torus.

There might be connections between the numbers g and r or the monodromy f and the trace fields of the found manifolds that can be made explicit.

Appendix

In this appendix we give commented listings of those algorithms that have been at the core of the investigations. Every listing contains a complete specification and (if necessary) further comments. Since the syntax of `Mathematica` is quite clear and all algorithms are well–structured, we did not include extra formal proofs of correctness. We do not know a proof of correctness for the basic software, anyway. (Note that in the set of algorithms `volumen.m` the input conditions are not checked, but are neither automatically satisfied. So this may produce wrong results when not used properly. Nothing depends on this, however.)


```
(* *************************************************************
*                                                              *
* Specification of the algorithm gleichng.m                    *
*                                                              *
* Input:     set of relations between x[i], y[ij], z[ijk],     *
*            s[i], s[ij], s[ijk], s[ij..l] as specified in     *
*            the algorithm reduktio.m                          *
*            optional: file f.m, specifying the trace equations *
*            for a given f: S_2^(1) -> S_2^(1)                 *
*                                                              *
* Condition: none                                              *
*                                                              *
* Output:    e1,..,e16, optionally e17, e18, ...: polynomial   *
*            expressions in x[i], y[ij], z[ijk]                *
*                                                              *
* Condition: every ei is supposed to be 0;                     *
*            see comments below                                *
*                                                              *
************************************************************* *)

<<reduktio.m

(* * def. of matrix Y * *)
Y:={{y[a,a],y[a,b],y[a,c],y[a,d]},{y[a,b],y[b,b],y[b,c],y[b,d]},
    {y[a,c],y[b,c],y[c,c],y[c,d]},{y[a,d],y[b,d],y[c,d],y[d,d]}};


(* optional: input of file f.m
<<eingabe.dir/g1.m
*)
```

```
(* * def. of complementary matrix, ignoring the sign (-1)^(i+j) * *)
ytaa:=Det[Y[[{2,3,4},{2,3,4}]]];
ytab:=Det[Y[[{2,3,4},{1,3,4}]]];
ytac:=Det[Y[[{2,3,4},{1,2,4}]]];
ytad:=Det[Y[[{2,3,4},{1,2,3}]]];
ytbb:=Det[Y[[{1,3,4},{1,3,4}]]];
ytbc:=Det[Y[[{1,3,4},{1,2,4}]]];
ytbd:=Det[Y[[{1,3,4},{1,2,3}]]];
ytcc:=Det[Y[[{1,2,4},{1,2,4}]]];
ytcd:=Det[Y[[{1,2,4},{1,2,3}]]];
ytdd:=Det[Y[[{1,2,3},{1,2,3}]]];

Ytilde:={{ytaa,ytab,ytac,ytad},
         {ytab,ytbb,ytbc,ytbd},
         {ytac,ytbc,ytcc,ytcd},
         {ytad,ytbd,ytcd,ytdd}};


(* * def. of tr(lambda) in the representation a, b, c, d; cf. lemma 4.4 * *)
lambda:=(s[{d,c,b,a}]*s[{a,b,c,d}]
         +x[a]^2+x[b]^2+x[c]^2+x[d]^2-x[a]^2 x[c]^2-x[b]^2 x[d]^2
         -x[a] x[b] s[a,b]+x[a] x[c] s[a,c]
         -x[a] x[d] s[a,d]-x[b] x[c] s[b,c]+x[b] x[d] s[b,d]-x[c] x[d] s[c,d]
         -x[a] x[d] s[a,b] s[b,d]-x[a] x[d] s[a,c] s[c,d]
         -x[a] x[d] s[a,b] s[b,c] s[c,d]
         +x[b] x[d] s[b,c] s[c,d] +x[a] x[c] s[a,b] s[b,c]
         +x[a]^2 x[c] x[d] s[c,d]+x[a] x[b] x[d]^2 s[a,b])-2;


(* condition for trace of lambda in the representation a, b, c, d *)
e1:=lambda-2;

(* determinant equations; cf. section 4.4 *)
Z:={{z[b,c,d],z[a,c,d],z[a,b,d],z[a,b,c]}};
v:=4*Transpose[Z].Z+Ytilde;

e2:= v[[1,1]];
e3:= v[[1,2]];
e4:= v[[1,3]];
e5:= v[[1,4]];
e6:= v[[2,2]];
e7:= v[[2,3]];
e8:= v[[2,4]];
e9:= v[[3,3]];
e10:=v[[3,4]];
```

```
e11:=v[[4,4]];

(* det(Y)=0; cf. eq. 4.6 *)
e12:=Det[Y];

(* 5-identities; cf. eq. 4.9 *)
Z2:={{z[b,c,d]},{-z[a,c,d]},{z[a,b,d]},{-z[a,b,c]}};
v2:=Y.Z2;

e13:=v2[[1]][[1]];
e14:=v2[[2]][[1]];
e15:=v2[[3]][[1]];
e16:=v2[[4]][[1]];

(* def. of tr(lambda) in the representation a1, a2, a3, a4, in which case
lambda = [a4,a3] [a2, a1]; this result has been found using reduktio.m *)
kommutatorprodukt=(64 - 32*x[a]^2 - 32*x[b]^2 + 8*x[a]^2*x[b]^2 -
     32*x[c]^2 + 16*x[a]^2*x[c]^2 + 16*x[b]^2*x[c]^2 -
     4*x[a]^2*x[b]^2*x[c]^2 - 32*x[d]^2 + 16*x[a]^2*x[d]^2 +
     16*x[b]^2*x[d]^2 - 4*x[a]^2*x[b]^2*x[d]^2 + 8*x[c]^2*x[d]^2 -
     4*x[a]^2*x[c]^2*x[d]^2 - 4*x[b]^2*x[c]^2*x[d]^2 +
     x[a]^2*x[b]^2*x[c]^2*x[d]^2 - 8*y[a, b]^2 + 4*x[c]^2*y[a, b]^2 +
     4*x[d]^2*y[a, b]^2 - x[c]^2*x[d]^2*y[a, b]^2 + 16*x[a]*x[c]*y[a, c] -
     4*x[a]*x[b]^2*x[c]*y[a, c] - 4*x[a]*x[c]*x[d]^2*y[a, c] +
     x[a]*x[b]^2*x[c]*x[d]^2*y[a, c] - 4*x[b]*x[c]*y[a, b]*y[a, c] +
     x[b]*x[c]*x[d]^2*y[a, b]*y[a, c] - 16*x[a]*x[d]*y[a, d] +
     4*x[a]*x[b]^2*x[d]*y[a, d] + 4*x[a]*x[c]^2*x[d]*y[a, d] -
     x[a]*x[b]^2*x[c]^2*x[d]*y[a, d] + 4*x[b]*x[d]*y[a, b]*y[a, d] -
     x[b]*x[c]^2*x[d]*y[a, b]*y[a, d] - 16*x[b]*x[c]*y[b, c] +
     4*x[a]^2*x[b]*x[c]*y[b, c] + 4*x[b]*x[c]*x[d]^2*y[b, c] -
     x[a]^2*x[b]*x[c]*x[d]^2*y[b, c] + 4*x[a]*x[c]*y[a, b]*y[b, c] -
     x[a]*x[c]*x[d]^2*y[a, b]*y[b, c] +
     x[a]*x[b]*x[c]*x[d]*y[a, d]*y[b, c] +
     x[c]*x[d]*y[a, b]*y[a, d]*y[b, c] + 16*x[b]*x[d]*y[b, d] -
     4*x[a]^2*x[b]*x[d]*y[b, d] - 4*x[b]*x[c]^2*x[d]*y[b, d] +
     x[a]^2*x[b]*x[c]^2*x[d]*y[b, d] - 4*x[a]*x[d]*y[a, b]*y[b, d] +
     x[a]*x[c]^2*x[d]*y[a, b]*y[b, d] -
     x[a]*x[b]*x[c]*x[d]*y[a, c]*y[b, d] -
     x[c]*x[d]*y[a, b]*y[a, c]*y[b, d] - 4*x[a]*x[d]*y[a, c]*y[c, d] +
     x[a]*x[b]^2*x[d]*y[a, c]*y[c, d] + x[b]*x[d]*y[a, b]*y[a, c]*y[c, d] +
     4*x[a]*x[c]*y[a, d]*y[c, d] - x[a]*x[b]^2*x[c]*y[a, d]*y[c, d] -
     x[b]*x[c]*y[a, b]*y[a, d]*y[c, d] + 4*x[b]*x[d]*y[b, c]*y[c, d] -
     x[a]^2*x[b]*x[d]*y[b, c]*y[c, d] - x[a]*x[d]*y[a, b]*y[b, c]*y[c, d] +
     x[a]*x[b]*y[a, d]*y[b, c]*y[c, d] + y[a, b]*y[a, d]*y[b, c]*y[c, d] -
     4*x[b]*x[c]*y[b, d]*y[c, d] + x[a]^2*x[b]*x[c]*y[b, d]*y[c, d] +
     x[a]*x[c]*y[a, b]*y[b, d]*y[c, d] - x[a]*x[b]*y[a, c]*y[b, d]*y[c, d] -
```

```
    y[a, b]*y[a, c]*y[b, d]*y[c, d] - 8*y[c, d]^2 + 4*x[a]^2*y[c, d]^2 +
    4*x[b]^2*y[c, d]^2 - x[a]^2*x[b]^2*y[c, d]^2 + y[a, b]^2*y[c, d]^2 -
    8*x[a]*x[b]*x[c]*z[a, b, c] + 2*x[a]*x[b]*x[c]*x[d]^2*z[a, b, c] -
    8*x[c]*y[a, b]*z[a, b, c] + 2*x[c]*x[d]^2*y[a, b]*z[a, b, c] -
    x[b]*x[c]*x[d]*y[a, d]*z[a, b, c] -
    2*x[a]*x[c]*x[d]*y[b, d]*z[a, b, c] +
    2*x[a]*x[b]*x[d]*y[c, d]*z[a, b, c] +
    2*x[d]*y[a, b]*y[c, d]*z[a, b, c] - x[b]*y[a, d]*y[c, d]*z[a, b, c] -
    2*x[a]*y[b, d]*y[c, d]*z[a, b, c] + 8*x[a]*x[b]*x[d]*z[a, b, d] -
    2*x[a]*x[b]*x[c]^2*x[d]*z[a, b, d] + 8*x[d]*y[a, b]*z[a, b, d] -
    2*x[c]^2*x[d]*y[a, b]*z[a, b, d] + x[b]*x[c]*x[d]*y[a, c]*z[a, b, d] +
    2*x[a]*x[c]*x[d]*y[b, c]*z[a, b, d] -
    2*x[a]*x[b]*x[c]*y[c, d]*z[a, b, d] -
    2*x[c]*y[a, b]*y[c, d]*z[a, b, d] + x[b]*y[a, c]*y[c, d]*z[a, b, d] +
    2*x[a]*y[b, c]*y[c, d]*z[a, b, d] + x[b]*x[c]*x[d]*y[a, b]*z[a, c, d] +
    x[b]*y[a, b]*y[c, d]*z[a, c, d] + 4*x[b]*x[c]*x[d]*z[b, c, d] -
    x[a]^2*x[b]*x[c]*x[d]*z[b, c, d] + 4*x[b]*y[c, d]*z[b, c, d] -
    x[a]^2*x[b]*y[c, d]*z[b, c, d])/32;

(* ****************************************************************
* End of gleichng.m                                               *
**************************************************************** *)


(* **************************************************************
*                                                              *
* Specification of the algorithm reduktio.m                    *
*                                                              *
* Input:     none                                              *
* Condition: none                                              *
*                                                              *
* Output:    set of relations between variables                *
*            x[i], y[ij], z[ijk], s[i], s[ij], s[ijk], s[ij..l]  *
* Condition: the relations are given by sections 4.1, 4.3, 4.4   *
*                                                              *
************************************************************** *)

(* * cf. lemma 4.3.iv * *)
x[i_^-1]:=             x[i];

(* * cf. lemma 4.5 * *)
Attributes[y]:={Orderless};
y[a_,a_]:=             x[a]^2-4;
z[i_,i_,j_]:=          0;
z[i_,j_,i_]:=          0;
```

```
z[j_,i_,i_]:=           0;

(* * reduction of s-expressions to x[i], y[ij], z[ijk] * *)
(* * cf. eq. 4.2 resp. theorem 4.2; cyclicity of s[.] * *)
s[i_,j_]:=              1/2*(y[i,j]+x[i]x[j]);
s[i_,j_^-1]:=           x[i]x[j]-s[i,j];
s[j_^-1,i_]:=           s[i,j^-1];
s[i_^-1,j_^-1]:=        s[i,j];
s[i_,j_,k_]:=           1/2*Signature[{i,j,k}]*
                         z[Sort[{i,j,k}][[1]],Sort[{i,j,k}][[2]],Sort[{i,j,k}][[3]]]+
                         1/4(x[i]y[j,k]+x[j]y[i,k]+x[k]y[i,j]+x[i]x[j]x[k]);
s[i_^-1,j_,k_]:=        x[i]s[j,k]-s[i,j,k];
s[j_,k_,i_^-1]:=        s[i^-1,j,k];
s[k_,i_^-1,j_]:=        s[i^-1,j,k];
s[i_^-1,j_^-1,k_]:=     s[k^-1,j,i];
s[j_^-1,k_,i_^-1]:=     s[k^-1,j,i];
s[k_,i_^-1,j_^-1]:=     s[k^-1,j,i];
s[i_^-1,j_^-1,k_^-1]:=  s[k,j,i];

(* * handling of higher order products via s[list] * *)
s[{i_}]:=               x[i];
s[{i_,j_}]:=            s[i,j];

(* * cf. 4.3.ii * *)
s[l_]:=                 If[Length[l]>3,
                        1/2(s[{l[[1]]}]*s[Join[{l[[2]]},{l[[3]]},Drop[l,3]]]
                           +s[{l[[2]]}]*s[Join[{l[[1]]},{l[[3]]},Drop[l,3]]]
                           +s[{l[[3]]}]*s[Join[{l[[1]]},{l[[2]]},Drop[l,3]]]
                           +s[Drop[l,3]]*s[{l[[1]],l[[2]],l[[3]]}]
                           +s[{l[[1]],l[[2]]}]*s[Join[{l[[3]]},Drop[l,3]]]
                           -s[{l[[1]],l[[3]]}]*s[Join[{l[[2]]},Drop[l,3]]]
                           +s[Join[{l[[1]]},Drop[l,3]]]*s[{l[[2]],l[[3]]}]
                           -s[{l[[1]]}]*s[{l[[2]]}]*s[Join[{l[[3]]},Drop[l,3]]]
                           -s[{l[[1]]}]*s[Drop[l,3]]*s[{l[[2]],l[[3]]}]
                           -s[{l[[2]]}]*s[{l[[3]]}]*s[Join[{l[[1]]},Drop[l,3]]]
                           -s[{l[[3]]}]*s[Drop[l,3]]*s[{l[[1]],l[[2]]}]
                           +s[{l[[1]]}]*s[{l[[2]]}]*s[{l[[3]]}]*s[Drop[l,3]]),
                        (* else *)
                        s[l[[1]],l[[2]],l[[3]]]
                        ];


(* *************************************************************
* End of reduktio.m                                            *
************************************************************* *)
```

```
(* ******************************************************************
*                                                                   *
* algorithm g1.m                                                    *
*                                                                   *
* Input:      set of relations between variables x[i], y[ij], z[ijk],  *
*             s[i], s[ij], s[ijk], as specified in the algorithm    *
*             reduktio.m                                            *
* Condition:  none                                                  *
*                                                                   *
* Output:     set of further relations between x[i],..z[ijk];       *
*             e17,..,e41,e1: polynomial expressions                 *
* Condition:  e17=e18=..=e41=e1=0 according to the trace equations  *
*             resulting from automorphismus g1 (see comments below) *
*                                                                   *
****************************************************************** *)

(* * see section 8.1 * *)
x[b]=x[a];
x[d]=x[c];

y[a,c]=y[b,d];
y[a,d]=y[b,c];

z[a,b,d]=-z[a,b,c];
z[b,c,d]=-z[a,c,d];

(* expression --                              -- resulting from eqn. *)
e17=-x[a]+s[d^-1,b,a];                                  (* 1         *)
e18=-x[c]+s[a,c];                                       (* 4         *)
e19=-x[c]+s[b,d];                                       (* 24        *)
e20=-s[d,b,a]+x[c]s[a,b]-x[a];                          (* 1         *)
e21=-s[a,b,c]+x[c]s[a,b]-x[a];                          (* 123^(-1) *)
e22=-s[d,c,a]+x[c]^2-s[c,d];                            (* 34        *)
e23=-s[b,c,d]+x[c]^2-s[c,d];                            (* 234       *)
e24=-s[b,c]+s[a,d];                                     (* 23        *)
e25=-s[a,b]s[a,d]+s[a,b]+x[a](x[c]s[a,b]-x[a])-x[c];    (* 12        *)
e26=-s[a,d]+(x[a]x[c]-s[a,d])s[c,b,a]-s[d,c,b];         (* 14        *)
e27=0;
e28=0;
e29=-s[a,c,d]+(x[a]x[c]-s[a,d])s[a,b,d]-s[a,d];         (* 134       *)
e30=-s[c,b,a]+x[a]x[c]-x[a]^2x[c]+x[a]s[a,d]+x[a];      (* 321       *)
e31=-s[d,c,b]+s[a,d]x[c]-x[a]s[d,c,a]+s[c,d];           (* 432       *)
e32=0;
e33=-s[a,b]+s[{b,a,d^-1,a^-1}];                         (* 12        *)
e34=-s[a,c]+s[{b,a,d^-1,a,d^-1,a^-1}];                  (* 13        *)
e35=-s[a,d]+s[{b,a,d^-1,a,c}];                          (* 14        *)
```

```
e36=-s[a,b,d]+s[{b,a,d^-1,c}];                              (* 124      *)
e37=0;
e38=-s[c,b^-1,a]+s[a,d^-1]s[b,a,d^-1]-x[b];                 (* 32^(-1)1 *)
e39=-s[d,b,a]+s[{a,c,a^-1,b,a,d^-1}];                       (* 421      *)
e40=-s[d^-1,c,a]+s[{c^-1,d^-1,a^-1,b,a,d^-1}];              (* 4^(-1)31 *)
e41=-s[d^-1,c,b]+s[{c^-1,d^-1,a^-1,a^-1}];                  (* 4^(-1)32 *)
(* e27,e28,e32,e37: former equations that turned out to be redundant  *)


(* * trace(lambda)=-2, lambda=[a4,a3][a2,a1] * *)
e1=kommutatorprodukt+2;

(* *****************************************************************
* end of g1.m                                                      *
****************************************************************** *)


(* ************************************************************
*                                                             *
* Specification of rules.m                                    *
*                                                             *
* Input:     x1, x2, ersatz: polynomial expressions           *
*            zuersetzen: monomial expression                  *
* Condition: x1, x2 are monomials                             *
*                                                             *
* Output:    rules: set of replacement rules                  *
* Condition: rules={x1^i x2^j zuersetzen -> x1^i x2^j ersatz |    *
*                   i,j = 0,..,10}                            *
*                                                             *
************************************************************* *)

rules={};
For[i=0,i<10,i++,
   For[j=0,j<10,j++,
       rules=Union[rules,
                  {Factor[x1^i x2^j zuersetzen] ->
                   Expand[x1^i x2^j ersatz]}
                 ]
      ]
  ];

(* ************************************************************
* End of rules.m                                              *
************************************************************* *)
```

```
(* ***********************************************************
*                                                           *
* Specification of the Algorithm apply                      *
*                                                           *
* Input:     e1,..,e41: polynomial expressions,             *
*            rules: set of replacement rules                *
*            p zuersetzen -> p ersatz (p: pol. expr.)       *
*                                                           *
* Condition: none                                           *
*                                                           *
* Output:    f1,..,f41: polynomial expressions              *
*                                                           *
* Condition: fi=ei/.rules (fi = ei with every expression    *
*            p zuersetzen replaced by p ersatz)             *
*                                                           *
*********************************************************** *)

f1=Factor[Factor[e1//.rules]//.rules];
f2=Factor[Factor[e2//.rules]//.rules];
 .
 .
 .
f39=Factor[Factor[e39//.rules]//.rules];
f40=Factor[Factor[e40//.rules]//.rules];
f41=Factor[Factor[e41//.rules]//.rules];

(* *******************************************************************
*                                                                   *
* Remark: This algorithm can be used for tentative substitutions. In    *
*         order to not destroy former results, the performed replace-   *
*         ments are stored in the fi. Via the algorithm zurueck.m they  *
*         can be shifted back to the ei.                            *
*                                                                   *
******************************************************************* *
* End of apply.m                                                    *
******************************************************************* *)


(* *******************************************************************
*                                                                   *
* algorithm numerloes.m                                             *
*                                                                   *
*********************************************************************

*********************************************************************
```

```
* part 1                                                            *
*                                                                   *
* Input:      e1, e2: polynomial expressions in two variables,      *
*             f: pol. expr. in first variable,                      *
*             g: pol. expr. in second variable                      *
* Condition:  f=Resultant[e1,e2] w.resp.t. first variable,          *
*             g=Resultant[e1,e2] w.resp.t. second variable          *
*                                                                   *
* Output:     S: set of replacement rules                           *
*             {{first variable -> numerical value1[i],              *
*               second variable -> numerical  value2[i]} | i=1,2,..}  *
* Condition:  value1[i] solves f, value2[i] solves g,               *
*             pair solves e1 and e2. (Numerical solution up to      *
*             accuracy given by eps)                                *
*                                                                   *
*********************************************************************
* Remark:     Find matching pairs of solutions of resultants and    *
*             remaining equations (cf. chapter 5, section 8.1)      *
******************************************************************** *)

S1=NSolve[f==0];
S2=NSolve[g==0];
S={};
eps=10^-4;
For[ii=1,ii<Length[S1]+1,ii++,
  For[jj=1,jj<Length[S2]+1,jj++,
    If[Abs[(e1/.S1[[ii]]/.S2[[jj]])]<eps &&
       Abs[(e2/.S1[[ii]]/.S2[[jj]])]<eps,
        S=Union[S,{{S1[[ii]][[1]],S2[[jj]][[1]]}}];
        Print["Loesung gef."];
    ]
  ]
];

(*********************************************************************
* part 2                                                            *
*                                                                   *
* Input:     f1,f2,f3,f4: polynomial expressions                    *
*            S: set of replacement rules for all but one variables  *
*            occuring in the fi                                     *
* Condition: none                                                   *
*                                                                   *
* Output:    S2: set of replacement rules for all variables occ. in fi *
* Condition: fi/.S2[j]=0 for all j=1,..,Length[S2]                  *
*            (Numerical solution up to accuracy eps)                *
*                                                                   *
```

```
***********************************************************************
* Remark:    Given a solution for the variables x[i], y[ij], z[ijk],   *
*            find solution for euclidean translation distance m of     *
*            matrix M, such that all relations are satisfied           *
*********************************************************************** *)

(*
eps=10^-4;
S2={};

For[ii=1,ii<Length[S]+1,ii++,
  S1=NSolve[(f1/.S[[ii]])==0];
  Print[S1];
  For[jj=1,jj<Length[S1]+1,jj++,
    If[ Abs[f2/.S[[ii]]/.S1[[jj]]]<eps &&
        Abs[f3/.S[[ii]]/.S1[[jj]]]<eps &&
        Abs[f4/.S[[ii]]/.S1[[jj]]]<eps,
        S2=Union[S2,{{S[[ii]][[1]],S[[ii]][[2]],S1[[jj]][[1]]}}];
        Print["Loesung gefunden"]
    ]
  ]
];
*)

(* ********************************************************************
* part 3                                                               *
*                                                                      *
* Input:      X1,..,X4: det-1 2x2 matrices                             *
*             S: set of replacement rules                              *
* Condition:  replacement rules in S replace variables occuring in     *
*             Xi and variable m                                        *
*                                                                      *
* Output:     S2: subset of S                                          *
* Condition:  under replacement S2[i], all Xj have R[Xj]<|m|           *
*             (numerical up to accuracy eps)                           *
*                                                                      *
***********************************************************************
* Remark:     Given solutions from part 2; discard all solutions       *
*             leading to a non-discrete representation (cf. sect. 4.?) *
*********************************************************************** *)

(*
S2={};
R[X_]:=1/Abs[X[[2,1]]];                 (* def. of radius of isom.sph.*)
For[ii=1,ii<Length[S]+1,ii++,
   r1=R[X1/.S[[ii]]];
```

```
   r2=R[X2/.S[[ii]]];
   r3=R[X3/.S[[ii]]];
   r4=R[X4/.S[[ii]]];
   mabs=Abs[m/.S[[ii]]];
   If[r1<mabs && r2<mabs && r3<mabs && r4<mabs,
     S2=Union[S2,{S[[ii]]}]
   ]
];
*)

(* ****************************************************************
* End of numerloes.m                                              *
**************************************************************** *)


(* *******************************************************
*                                                        *
* Specification of algorithm g1numer.m                   *
*                                                        *
* Input:      None                                       *
* Condition:  None                                       *
*                                                        *
* Output:     MM,mm,X4,x4,X2,x2,X3,x3,X1,x1: det-1 2x2 matri- *
*             ces;                                       *
*             f1,f2,f3,f4: pol. expr. in x[i],y[ij],z[ijk],m; *
*             set of relations between x[i],y[ij],z[ijk] as   *
*             specified by algorithms reduktio.m, g1.m;  *
*             S: set of numerical solutions of system of *
*             trace equations for g1                     *
* Condition:  Matrices are defined using knowledge about g1;  *
*             fi=0 are some trace euqations of g1;       *
*             see comments below                         *
*                                                        *
**********************************************************
* Remark:     provide general setup in order to calculate *
*             matrices Xi and M numerically, once numerical *
*             solutions of trace equations are known     *
******************************************************* *)

MM={{1,m},{0,1}};                (* M: parabolic with euclid. *)
mm=Inverse[MM];                  (* transl.dist. m            *)

X4={{x[c],-1/x[c]},{x[c],0}};    (* Choose X4 s.t. isom. sph. *)
x4=Inverse[X4];                  (* centd. at 0,radius |x[4]| *)
```

```
X2={{b1,b2},{b3,x[a]-b1}};       (* X2 the other independ.    *)
x2=Inverse[X2]*Det[X2];          (* matrix                    *)

X3=X4.MM.X2.X4.mm.x4;            (* X3 and ...                *)
x3=Inverse[X3]*Det[X3];

X1=X4.MM.x2.mm.x4;               (* ... X1 depend on   X2, X4,*)
x1=Inverse[X1]*Det[X1];          (* M via relations           *)

tr[X_]:=X[[1,1]]+X[[2,2]];       (* def. of trace function    *)

b3=(b1+b2-1)x[c]^2;              (* trX2.X4=x[3]              *)
                                 (* solve for entries of X2   *)
b2=(b2/.Solve[tr[X1.X4]-s[a,d]==0,b2][[1]]); (* from obvious  *)
b1=(b1/.Solve[tr[X3.X4]-s[c,d]==0,b1][[1]]); (* trace relat.  *)
Print["Eintraege von X2 errechnet"];

f1=Factor[Det[X2]-1];
f2=tr[X1.X4]-tr[X2.X3];          (* f2,..,f4 result from trace*)
f3=tr[x4.X2.X1]-x[a];            (* equations; see algor. g1.m*)
f4=tr[X1.X2.x3]-x[a];

<<reduktio.m;<<g1.dat;           (* read in known relations;  *)
Print["Daten eingelesen"];       (* g1.dat contains former    *)
                                 (* results like rational expr*)
                                 (* of variables in x[1],x[3];*)
                                 (* f,g: resultants of last   *)
                                 (* eqn.s;                    *)
                                 (* S: set of pairs of solut. *)
                                 (* found with numerloes.m,   *)
                                 (* part 1                    *)
(* ***********************************************************
* End of g1numer.m                                           *
*********************************************************** *)


(* ***************************************************************
*                                                               *
* algorithm g1radien.m                                          *
*                                                               *
* Input:     X1,X2,X3,X4 in SL(2,C), numerically given          *
* Condition: none                                               *
*                                                               *
* Output:    List of records (h,i,j,k,l,R,M),                   *
* Condition: X[h].X[i].X[j].X[k].X[l] has radius R of isom.sph. *
```

```
*            > predefined constant 0.469 and center M (def. of    *
*            X[nu] see below)                                     *
*                                                                 *
*******************************************************************
* Remark:    Xi: generating matrices for fund.group of M(2,g1);   *
*            products in Xi and their inverses up to length 5     *
*            investigated; predefined radius 0.469 = least impor- *
*            tant radius, as has been found during an earlier     *
*            computation of this algorithm; trivial products      *
*            such as X[i].X[i]^(-1) excluded                      *
****************************************************************** *)

"">>out;
<<ergebnis.dir/g1_2.dat;              (* read in X1,..,X4         *)
x1=Inverse[X1];
x2=Inverse[X2];
x3=Inverse[X3];
x4=Inverse[X4];

R[S_]:=If[Abs[S[[2,1]]]>0.001,        (* def. of radius...        *)
          N[Round[Round[1/Abs[S[[2,1]]]*100]]/100],
          0
       ];
M[S_]:=N[Round[Round[-S[[2,2]]/S[[2,1]]*100]]/100];
                                      (* ...and center of isom.  *)
                                      (* sph., up to 2 decimals  *)
X[0]=IdentityMatrix[2];               (* def. of matrices X[nu]  *)
X[1]=X1;
X[2]=x1;
X[3]=X2;
X[4]=x2;
X[5]=X3;
X[6]=x3;
X[7]=X4;
X[8]=x4;

SS={{1,2},{3,4},{5,6},{7,8},{2,1},{4,3},{6,5},{8,7}};

For[h=0,h<9,h++,
  For[i=0,i<9,i++,
    For[j=0,j<9,j++,
      For[k=0,k<9,k++,
        For[l=1,l<9,l++,
          H=X[h].X[i].X[j].X[k].X[l];
          r=R[H];
          If[r>0.469 &&            (* if radius big enough and ...   *)
```

```
             Not[j!=0 && k==0] && Not[i!=0 && j k==0] &&
             Not[h!=0 && i j k==0] &&
             Intersection[SS,{{h,i},{k,l},{j,k},{i,j}}]=={},
                                   (* ...if product is not trivial...*)
             SequenceForm[h,i,j,k,l,"    ","R=",r,"    ","M=",M[H]]>>>out
          ];
        ]
      ]
    ]
  ]
];
(* *************************************************************
* End of g1radien.m                                             *
************************************************************** *)


(* ****************************************************************
*                                                                 *
* Set of Algorithms volumen.m for the Computation of Volume       *
* of Hyperbolic Manifolds With Cusp                               *
*                                                                 *
***************************************************************** *)

Print["'Hilfe' fuer Hilfe."];
Hilfe="WasIst[f]=Hilfe zur Funktion f. Fuer folgende Funktionen ist
       Hilfe erhaeltlich: R, M, rv, volumen, tp, lob. Folgende weitere
       Funktionen stehen zur Verfuegung: wa, wb, wg, v, selbeSeite. Eine
       Ueberpruefung auf Falscheingaben erfolgt generell nicht!";

(* ****************************************************************
* Function M (center of isometric sphere)                         *
*                                                                 *
* Input:      MA: 2X2 complex matrix                              *
* Condition:  none                                                *
*                                                                 *
* Output:     M: complex number                                   *
* Condition:  M is the center of the isometr. sphere of MA        *
***************************************************************** *)
M[MA_]:=-MA[[2,2]]/MA[[2,1]];

WasIst[M]="M[S]=Mittelpunkt der isom.Sph. der Matrix S.";

* ****************************************************************
* Function R (radius of isometric sphere)                         *
*                                                                 *
```

```
* Input:     RA: 2x2 complex matrix                                  *
* Condition: none                                                    *
*                                                                    *
* Output:    R: real number                                          *
* Condition: R is the radius of hte isom. sphere of MA               *
****************************************************************** *)
R[RA_]:=Abs[Sqrt[Det[RA]]/RA[[2,1]]];

WasIst[R]="R[S]=Radius der isom.Sph. der Matrix S.";

(* ******************************************************************
* Function tp (triple-point of three isom. spheres)                  *
*                                                                    *
* Input:     tA, tB, tC: 2x2 complex matrices                        *
* Condition: none                                                    *
*                                                                    *
* Output:    z: complex number or ComplexInfinity                    *
* Condition: If det(tA)=..=det(tC)=1, then [if isom. spheres of      *
*            tA,tB,tC intersect, then z=projection(point of          *
*            intersection), else ComplexInfinity]                    *
****************************************************************** *)
tp[tA_,tB_,tC_]:=If[
   (Abs[M[tA]-M[tB]]<R[tA]+R[tB])&&
   (Abs[M[tA]-M[tC]]<R[tA]+R[tC])&&
   (Abs[M[tB]-M[tC]]<R[tB]+R[tC]),
   Det[{{tA[[2,1]] Conjugate[tA[[2,1]]],Conjugate[tA[[2,1]]] tA[[2,2]],
tA[[2,2]] Conjugate[tA[[2,2]]]-1},
       {tB[[2,1]] Conjugate[tB[[2,1]]],Conjugate[tB[[2,1]]] tB[[2,2]],
tB[[2,2]] Conjugate[tB[[2,2]]]-1},
       {tC[[2,1]] Conjugate[tC[[2,1]]],Conjugate[tC[[2,1]]] tC[[2,2]],
tC[[2,2]] Conjugate[tC[[2,2]]]-1}}]/Det[
      {{tA[[2,1]] Conjugate[tA[[2,1]]], tA[[2,1]] Conjugate[tA[[2,2]]],
        Conjugate[tA[[2,1]]] tA[[2,2]]},
       {tB[[2,1]] Conjugate[tB[[2,1]]], tB[[2,1]] Conjugate[tB[[2,2]]],
        Conjugate[tB[[2,1]]] tB[[2,2]]},
       {tC[[2,1]] Conjugate[tC[[2,1]]], tC[[2,1]] Conjugate[tC[[2,2]]],
        Conjugate[tC[[2,1]]] tC[[2,2]]}}],
   ComplexInfinity];
WasIst[tp]="tp[A,B,C]=Tripelpunkt der drei Matrizen A,B,C. ComplexInfinity,
            falls dieser nicht existiert. (Keine Ueberpruefung von
            Falscheingaben!)";

********************************************************************
* Function wa (angle alpha, cf. figures in sect. 4.7)                *
*                                                                    *
* Input:      wS: 2x2 complex matrix                                 *
```

```
*             wz1, wz2: complex numbers                          *
* Condition:  none                                               *
*                                                                *
* Output:     wa: real number                                    *
* Condition:  wa=angle at wz1 of triangle M[wS],wz1,wz2          *
*             (cf. sect. 4.7)                                    *
***************************************************************** *)
wa[wS_,wz1_,wz2_]:=Abs[N[ArcCos[(Re[wz2-wz1] Re[M[wS]-wz1]+
                 Im[wz2-wz1] Im[M[wS]-wz1])/
                (Abs[wz2-wz1] Abs[M[wS]-wz1])]]];

******************************************************************
* Function wb (angle beta, cf. figures in sect. 4.7)            *
*                                                                *
* Input:      wS: 2x2 complex matrix                             *
*             wz1, wz2: complex numbers                          *
* Condition:  none                                               *
*                                                                *
* Output:     wb: real number                                    *
* Condition:  wb=angle at wz2 of triangle M[wS],wz1,wz2          *
*             (cf. sect. 4.7)                                    *
***************************************************************** *)
wb[wS_,wz1_,wz2_]:=Abs[N[ArcCos[(Re[wz1-wz2] Re[M[wS]-wz2]+
            Im[wz1-wz2] Im[M[wS]-wz2])/
(Abs[wz1-wz2] Abs[M[wS]-wz2])]]];

******************************************************************
* Function wg (angle gamma, cf. figures in sect. 4.7)            *
*                                                                *
* Input:      wS: 2x2 complex matrix                             *
*             wz1, wz2: complex numbers                          *
* Condition:  none                                               *
*                                                                *
* Output:     wg: real number                                    *
* Condition:  If det[wS]=1 and if wz1,wz2 in interior of proj. of    *
*             isom. sphere of wS, then:                          *
*             wg=angle gamma of tetrahedron determined by projected  *
*             vertices M[wS],wz1,wz2                             *
*             (cf. sect. 4.7)                                    *
***************************************************************** *)
wg[wS_,wz1_,wz2_]:=Abs[N[ArcCos[Abs[(wz1-M[wS]) Conjugate[(wz2-M[wS])]-
             Conjugate[(wz1-M[wS])] (wz2-M[wS])]/
            (2 R[wS] Abs[wz1-wz2])]]];

(* ***************************************************************
* Function selbeSeite                                            *
```

```
*                                                                  *
* Input:     sM, sz1, sz2, sz3: complex numbers                    *
* Condition: none                                                  *
*                                                                  *
* Output:    selbeSeite: +/-1 (sign)                               *
* Condition: If sM and sz1 are on the same side of the eucl. line  *
*            through sz2, sz3, i.e. if scalar product of sM-sz2 and *
*            sz1-sz2 respectively with normal vector               *
*            (-Im[sz3-sz2],Re[sz3-sz2]) has same sign, then        *
*            selbeSeite=+1, else selbeSeite=-1                     *
******************************************************************* *)
selbeSeite[sM_,sz1_,sz2_,sz3_]:=If[
(-Re[sM -sz2] Im[sz3-sz2]+Im[sM -sz2] Re[sz3-sz2])*
(-Re[sz1-sz2] Im[sz3-sz2]+Im[sz1-sz2] Re[sz3-sz2])>0,
+1,-1];

(* *****************************************************************
* Function rv (volume of tetrahedron with vertex M[matrix])        *
*                                                                  *
* Input:     rM: 2x2 complex matrix                                *
*            rz1, rz2: complex numbers                             *
* Condition: none                                                  *
*                                                                  *
* Output:    rv: real number                                       *
* Condition: If det[rM]=1 and if rz1,rz2 in project. of isom.sphere *
*            of rM, then rv = volume(tetrahedron with vertex infty, *
*            projected vertices M[rM],rz1,rz2)                     *
*            (cf. sect. 4.7 for formula)                           *
******************************************************************* *)
rv[rM_,rz1_,rz2_]:=1/4(2 lob[wa[rM,rz1,rz2]]+
       2 lob[wb[rM,rz1,rz2]]-
 lob[wa[rM,rz1,rz2]+wg[rM,rz1,rz2]-Pi/2]-
 lob[wa[rM,rz1,rz2]-wg[rM,rz1,rz2]-Pi/2]-
 lob[wb[rM,rz1,rz2]+wg[rM,rz1,rz2]-Pi/2]-
 lob[wb[rM,rz1,rz2]-wg[rM,rz1,rz2]-Pi/2]);

WasIst[rv]="rv[M,z1,z2]=hyp. Volumen des Tetraeders ueber der isom.Sph.
            von M, geg. durch die drei Punkte M[M], z1,z2. (Keine
            Uberpruefung von Falscheingaben!)";

(* *****************************************************************
* Function lob (Lobachevsky-function)                              *
*                                                                  *
* Input:      y: real number                                       *
* Condition:  none                                                 *
*                                                                  *
```

```
* Output:     lob[y]: real number                                 *
* Condition:  lob[y] = Lobachevsky-function[y], computed with      *
*             numerical integration (cf. sect. 4.7 for formula)   *
****************************************************************** *)
lob[y_]:=NIntegrate[-Log[Abs[2 Sin[u]]],{u,0,y}];

WasIst[lob]="lob[y]=Integral von 0 bis y ueber -log(|2 sin u|) du.
             (Lobacevskij-Funktion; numerisch integriert.) (y reell >0.
             Keine Ueberpruefung von Falscheingaben!)";

(* ******************************************************************
* Function v (volume of tetrahedron)                                *
*                                                                   *
* Input:     vM: 2x2 complex matrix                                 *
*            vz1,vz2,vz3: complex numbers                           *
* Condition: none                                                   *
*                                                                   *
* Output:    v: real number                                         *
* Condition: If det[vM]=1 and vz1,vz2,vz3 in proj. of isom. sphere of *
*            vM, then                                               *
*            v = volume of tetrahedron with vertex infty, projected *
*            vertices vz1,vz2,vz3, face on isom.sphere of vM        *
****************************************************************** *)
v[vM_,vz1_,vz2_,vz3_]:=selbeSeite[M[vM],vz1,vz2,vz3] rv[vM,vz2,vz3]+
       selbeSeite[M[vM],vz2,vz1,vz3] rv[vM,vz1,vz3]+
         selbeSeite[M[vM],vz3,vz2,vz1] rv[vM,vz2,vz1];

(* ******************************************************************
* Function volumen (volume of certain polyeder)                     *
*                                                                   *
* Input:     volS: 2x2 complex matrix                               *
*            vollist: (ordered) list of complex numbers             *
* Condition: none                                                   *
*                                                                   *
* Output:    volumen: real number                                   *
* Condition: If det[volS]=1 and vollist has at least three elements, *
*            all of which are in interior of proj. of isom. sphere of *
*            volS, and elements of vollist are clockwise or counter- *
*            clockwise ordered vertices of polygon on isom. sph. of  *
*            volS, then                                             *
*            volumen = volume of polyeder with vertex infty, other  *
*            vertices on isom. sphere of volS with proj. given by   *
*            vollist                                                *
****************************************************************** *)
volumen[volS_,vollist_]:=Sum[
v[volS,vollist[[1]],vollist[[i-1]],vollist[[i]]],
```

```
{i,3,Length[vollist]}];

WasIst[volumen]="volumen[S,{z1,z2,...,zn}]=Volumen des hyp. Polyeders
                 ueber der isom.Sph. von S, geg. durch die Punkte
                 z1,...,zn. (Erforderlich: Det[S]=1, zi liegen im
                 Inneren des isom.Kreises von S, n>2, zi begrenzen ein
                 konvexes Polygon. Keine Ueberpruefung von Falscheingaben!)";

(* *****************************************************************
* End of volumen.m                                                 *
****************************************************************** *)
```

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