Hecke algebras and Schur algebras of the symmetric group

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ABSTRACT. These notes give a fully self–contained introduction to the (modular) representation theory of the Iwahori–Hecke algebras and the q–Schur algebras of the symmetric groups. The central aim of this work is to give a concise, but complete, and an elegant, yet quick, treatment of the classification of the simple modules and of the blocks of these two important classes of algebras. Our approach is combinatorial in nature and is based upon the construction of (cellular) bases which are particularly well adapted to the representation theory.

Contents

Keep it simple; as simple as possible, but no simpler.
Albert Einstein

Introduction

Iwahori–Hecke algebras first appeared in the work of Iwahori [**34**] and now have applications in subjects ranging from the ordinary and modular representation theory of Coxeter groups and the groups of Lie type, to knot theory, quantum groups and statistical mechanics (see [**52**] and the references there). Schur algebras were introduced by Schur [**61**] who used them to classify the ordinary irreducible polynomial representations of $GL_n(\mathbb{C})$; their "quantum analogues," the q–Schur algebras, also determine the representation theory of the finite general linear groups in non–defining characteristic [**15**].

These notes aim to give a clear and concise treatment of the modular representation theory of the Iwahori–Hecke algebras of the symmetric groups and of the q–Schur algebras. Both of these theories were pioneered by Dipper and James in a series of landmark papers [**13**–**16**]. Taking full advantage of recent advances, these notes recast this theory with a primary goal of classifying and constructing the simple modules and the blocks of these algebras. Apart from some basic representation theory, these notes are entirely self–contained.

One of the central themes of these notes is to view the Iwahori–Hecke algebras and q–Schur algebras as *cellular algebras*, a class of algebras introduced by Graham and Lehrer [**26**]. This amounts to constructing a basis which is extremely well adapted to the representation theory of the algebra. For example, given a cellular basis one can immediately construct all of the simple modules of the algebra.

The cellular basis of the Iwahori–Hecke algebra which we concentrate upon is the so–called *Murphy basis* [**55**]; this is a very natural basis indexed by pairs of *standard tableaux*. Apart from being cellular, the Murphy basis has another marvelous property in that it is very easy to "lift" this basis to give a cellular basis for the q–Schur algebra, which is indexed by pairs of *semistandard tableaux*. In this way, one obtains a very clean and very elegant construction of the simple modules of both the Iwahori–Hecke algebras and the q–Schur algebras.

In order to classify the blocks of the our algebras we prove an analogue of *Jantzen's sum formula* for the *Weyl modules* of the q–Schur algebras. Key to our proof of this result are generalizations of the *Jucys–Murphy elements* from the representation theory of the symmetric groups. Using these elements we are able to construct orthogonal bases for the Weyl modules (when the q–Schur algebras are semisimple), which then allow us to compute the Gram determinant of the cellular basis of the Weyl modules. Essentially some highest weight theory then allows us to deduce the sum formula and so classify both the blocks of the q–Schur algebras and Iwahori–Hecke algebras.

Briefly, the structure of these notes is as follows. Chapter 1 begins by proving some basic facts about the symmetric group and its Iwahori–Hecke algebra $\mathscr{H}$. In fact, this is really a chapter Coxeter groups and general Iwahori–Hecke algebras in disguise because everything generalises to this more general context. Chapter 2 develops Graham and Lehrer's theory of cellular algebras, culminating with the construction of the simple modules in this setting.

In Chapter 3 these notes begin in earnest. The combinatorics of tableaux are introduced and Murphy's basis theorem is proved in the first two sections. Using Chapter 2 this yields the simple $\mathscr{H}$–modules. The third section uses the Jucys–Murphy elements to construct the q–analogue of Young's seminormal form for the Specht modules of $\mathscr{H}$ in the semisimple case. The fourth section classifies the simple $\mathscr{H}$–modules.

Chapter 4 shows how the Murphy basis of $\mathscr{H}$ can be "lifted" to give a cellular basis of the q–Schur algebra. This involves the introduction of semistandard tableaux which correspond to orbits of the standard tableaux which index Murphy's basis. The simple modules for the q–Schur algebras are constructed and classified using the results of Chapter 2. For free, we obtain the fact that the q–Schur algebras are quasi–hereditary.

Chapter 5 is devoted to proving the Jantzen–Schaper theorem and classifying the blocks of these algebras. Here we follow the paper of James and the author [**39**], using improvements from [**40**]. This is by far the most technical part of these notes. The key idea is to first construct an orthogonal basis of the Weyl modules of $\mathscr{S}(d,n)$ in a "modular system"; it is here that generalizations of the Jucys–Murphy elements play their most crucial role.

The most obvious omission from these notes is any mention of the beautiful result of Dipper and James [**15**] which says that the representation theory of the general linear groups in non–defining characteristic is completely determined by the q–Schur algebras. It would also have been good to include the Kleshchev–Brundan branching rules [**4**] and the work of Ariki [**2**], Grojnowski [**30**] and Lascoux, Leclerc and Thibon [**51**] connecting quantum groups and decomposition matrices of Iwahori–Hecke algebras over field of characteristic zero. These matters would have more than doubled the size of these notes so instead we refer the interested reader to the literature and, in time, to Brundan's forthcoming book [**5**].

These notes are based upon a series of lectures I gave at the Universität Bielefeld in 1997 and it is a pleasure to thank the members of the Sonderforschungsbereich 343, and in particular Steffen König and Claus Ringel, for their encouragement and support. At times these notes draw heavily upon my papers [**39**,**40**] with Gordon James, and our paper [**17**] with Richard Dipper, so in addition I would like to thank my coauthors for their contributions to this document.

Finally, I thank Jon Brundan and Steffen König for numerous suggestions and comments on an earlier version of this manuscript.

CHAPTER 1

The Iwahori–Hecke algebra of the Symmetric Group

In this chapter we begin by proving some fundamental results about the symmetric group and then define the associated Iwahori–Hecke algebra. Everything in this chapter, including the proofs, generalises to the case of arbitrary Coxeter groups and Iwahori–Hecke algebras.

1. The Symmetric group

Throughout these notes we fix an integer $n \geq 1$.

1.1 Let $\mathfrak{S}_n$ be the symmetric group acting on $1, 2, \dots, n$ from the *right*. For $i = 1, 2, \dots, n-1$ let s_i be the basic transposition $(i, i+1)$ and let S. As a Coxeter group, $\mathfrak{S}_n$ is generated by $s_1, s_2, \dots, s_{n-1}$, subject to the relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } i = 1, 2, \dots, n-1, \\ s_i s_j &= s_j s_i, && \text{for } 1 \leq i < j-1 \leq n-2, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, && \text{for } i = 1, 2, \dots, n-2. \end{aligned}$$

The reader is invited to prove that this does indeed give a presentation of $\mathfrak{S}_n$. (Hint: $\{1, s_{n-1}, s_{n-2}s_{n-1}, \dots, s_1 s_2 \dots s_{n-1}\}$ is a complete set of right coset representatives of $\mathfrak{S}_{n-1}$ in $\mathfrak{S}_n$.) The second and third relations are the **braid relations** of $\mathfrak{S}_n$.

Suppose that w is an element of $\mathfrak{S}_n$ and write $w = s_{i_1} \dots s_{i_k}$ for some $i_1, \dots, i_k$ in $\{1, 2, \dots, n-1\}$. If k is minimal then we define the **length**, $\ell(w)$, of w to be k and we will call both $s_{i_1} \dots s_{i_k}$ and $(i_1, \dots, i_k)$ a **reduced expression** for w.

The **sign representation** of $\mathfrak{S}_n$ is the one dimensional representation ε determined by $\varepsilon(s) = -1$ for all $s \in S$. Thus, $\varepsilon(w) = (-1)^{\ell(w)}$ for all $w \in \mathfrak{S}_n$. Consequently, if $w \in \mathfrak{S}_n$ and $s \in S$ then $\ell(sw) = \ell(w) \pm 1$.

We would like to know when $\ell(sw) = \ell(w) + 1$ and for this we need another description of the length function. For $w \in \mathfrak{S}_n$ let

$$N(w) = \{\, (j,k) \in \mathfrak{S}_n \mid 1 \leq j < k \leq n \text{ and } jw > kw \,\}.$$

So, for example, $N(1) = \varnothing$ and $N(s) = \{s\}$ for all $s \in S$.

Given sets A and B let $A \dot{+} B = (A \cup B) \setminus (A \cap B)$ be their symmetric difference.

1.2 Lemma (Dyer) *Suppose that $v, w \in \mathfrak{S}_n$. Then $N(vw) = N(v) \dot{+} vN(w)v^{-1}$.*

Proof. If $s \in S$ and $(j,k) \in \mathfrak{S}_n$ then $s(j,k)s = (js, ks)$ so $N(sw) = N(s) \dot{+} sN(w)s$ for any $w \in \mathfrak{S}_n$; so the Lemma holds when $\ell(v) = 1$. If $\ell(v) > 1$ then $v = su$ for some $s \in S$ and $u \in \mathfrak{S}_n$ and with $\ell(u) = \ell(v) - 1$. Therefore, by induction on the length of v,

$$\begin{aligned} N(vw) &= N(suw) = N(s) \dot{+} sN(uw)s = N(s) \dot{+} sN(u)s \dot{+} suN(w)u^{-1}s \\ &= N(v) \dot{+} vN(w)v^{-1} \end{aligned}$$

as required. □

For convenience let $T = \{(i,j) \in \mathfrak{S}_n\} = \bigcup_{w\in\mathfrak{S}_n} wSw^{-1}$. Then T is the set of reflections in $\mathfrak{S}_n$ (considering $\mathfrak{S}_n$ as a Coxeter group), and $N(w) \subseteq T$.

1.3 Proposition *Suppose that $w \in \mathfrak{S}_n$. Then*
(i) *$\ell(w) = |N(w)|$; and,*
(ii) *$N(w) = \{\, t \in T \mid \ell(tw) < \ell(w)\,\}$.*

Proof. Suppose that $s_{i_1}\dots s_{i_k}$ is a reduced expression for w and, for $a = 1,\dots,k$, let $t_a = s_{i_1}\dots s_{i_{a-1}}s_{i_a}s_{i_{a-1}}\dots s_{i_1}$. Then $t_a \in T$ for all a and, by Lemma 1.2, $N(w) = N(s_{i_1}\dots s_{i_k}) = \{t_1\}\dot{+}\cdots\dot{+}\{t_k\}$. We claim that $t_a \neq t_b$ for $a \neq b$. By way of contradiction, suppose that $t_a = t_b$ for some $a < b$. Then

$$\begin{aligned} w = t_a t_b w &= s_{i_1}\dots s_{i_{a-1}}s_{i_a}s_{i_{a-1}}\dots s_{i_1}s_{i_1}\dots s_{i_{b-1}}s_{i_b}s_{i_{b-1}}\dots s_{i_1}w \\ &= s_{i_1}\dots \widehat{s}_{i_a}\dots\widehat{s}_{i_b}\dots s_{i_k}, \end{aligned}$$

where $\widehat{s_j}$ indicates that the transposition s_j is omitted. However, this contradicts the assumption that w has length k so $t_1,\dots,t_k$ must all be distinct. Hence, $N(w) = \{t_1,\dots,t_k\}$ and $\ell(w) = |N(w)|$ proving (i).

Now $t_a w = s_{i_1}\dots\widehat{s}_{i_a}\dots s_{i_k}$ for $a = 1,\dots,k$ so $N(w) \subseteq \{\, t\in T \mid \ell(tw) < \ell(w)\,\}$. Conversely, by what we have already shown, if $t \in T$ then $t \in N(t)$. Furthermore, $N(tw) = N(t)\dot{+}tN(w)t$ by Lemma 1.2; so if $t \notin N(w)$ then $t \in N(tw)$ and $\ell(w) = \ell(ttw) < \ell(tw)$ by (i). This proves (ii). □

As a special case of the Proposition we have the following.

1.4 Corollary *Suppose that $w \in \mathfrak{S}_n$ and that $s_i \in S$. Then*

$$\ell(s_i w) = \begin{cases} \ell(w)+1, & \text{if } iw < (i+1)w, \\ \ell(w)-1, & \text{if } iw > (i-1)w. \end{cases}$$

1.5 Theorem (The Strong Exchange Condition) *Suppose that $s_{i_1},\dots,s_{i_k}$ are elements of S, $t \in T$ and that $\ell(ts_{i_1}\dots s_{i_k}) < \ell(s_{i_1}\dots s_{i_k})$. Then*

$$ts_{i_1}\dots s_{i_k} = s_{i_1}\dots\widehat{s}_{i_a}\dots s_{i_k}$$

for some a. Furthermore, $t = s_{i_1}\dots s_{i_{a-1}}s_{i_a}s_{i_{a-1}}\dots s_{i_1}$.

Proof. Let $w = s_{i_1}\dots s_{i_k}$ and, for $1 \le a \le k$, define t_a as in the proof of the Proposition 1.3. Then $N(w) = \{t_1\}\dot{+}\cdots\dot{+}\{t_k\}$ by Lemma 1.2 and $t \in N(w)$ by Proposition 1.3(ii). Therefore, $t = t_a$ for some a and the result follows. □

Note that in the statement of the strong exchange condition we do not require that $s_{i_1}\dots s_{i_k}$ be reduced. The exchange condition also has an equivalent right hand analogue.

1.6 Corollary *Suppose that $w \in \mathfrak{S}_n$ and $s \in S$. Then*
(i) *$\ell(sw) < \ell(w)$ if and only if w has a reduced expression beginning with s;*
(ii) *$\ell(ws) < \ell(w)$ if and only if w has a reduced expression ending with s.*

Proof. Because $\ell(sw) = \ell(w^{-1}s)$, parts (i) and (ii) are equivalent and it suffices to consider (i). Suppose then that $\ell(sw) < \ell(w)$ and let $s_{i_1}\dots s_{i_k}$ be a reduced expression for w. By the exchange condition there exists an integer a such that

$sw = s_{i_1} \dots \widehat{s}_{i_a} \dots s_{i_k}$. Therefore, $w = ss_{i_1} \dots \widehat{s}_{i_a} \dots s_{i_k}$ and this expression is reduced. The converse is clear. □

1.7 Corollary (The Deletion Condition) *Suppose that* $w \in \mathfrak{S}_n$, $\ell(w) < k$ *and that* $w = s_{i_1} \dots s_{i_k}$ *for some* $s_{i_j} \in S$.

(i) *There exist integers* $1 \le a < b \le k$ *such that* $w = s_{i_1} \dots \widehat{s}_{i_a} \dots \widehat{s}_{i_b} \dots s_{i_k}$.

(ii) *A reduced expression for* w *can be obtained by deleting an even number of the* s_{i_a} *from* $s_{i_1} \dots s_{i_k}$.

Proof. Since $\ell(w) < k$ there exists an integer a with $1 \le a < k$ such that $\ell(s_{i_a} \dots s_{i_k}) < \ell(s_{i_{a+1}} \dots s_{i_k})$. Hence, by the exchange condition $s_{i_a} \dots s_{i_k} = s_{i_{a+1}} \dots \widehat{s}_{i_b} \dots s_{i_k}$ for some b with $a < b \le k$. Thus, $w = s_{i_1} \dots \widehat{s}_{i_a} \dots \widehat{s}_{i_b} \dots s_{i_k}$ proving (i). Part (ii) is now immediate. □

We can now prove the main result we need from this section. Given two reduced expressions $(i_1, \dots, i_k)$ and $(j_1, \dots, j_k)$ write $(i_1, \dots, i_k) \sim_b (j_1, \dots, j_k)$ if one expression can be transformed into the other using only the braid relations. Thus, $\sim_b$ is the equivalence relation on k–tuples generated by $(\dots, i, j, \dots) \sim_b (\dots, j, i, \dots)$, when $|i - j| > 1$, and $(\dots, i, i+1, i, \dots) \sim_b (\dots, i+1, i, i+1, \dots)$.

1.8 Theorem (Matsumoto) *Suppose* $w \in \mathfrak{S}_n$ *and let* $(i_1, \dots, i_k)$ *and* $(j_1, \dots, j_k)$ *be two reduced expressions for* w. *Then* $(i_1, \dots, i_k) \sim_b (j_1, \dots, j_k)$.

Proof. We argue by induction on $k = \ell(w)$. If $k \le 1$ there is nothing to prove so suppose that $k > 1$. By assumption $s_{i_1} \dots s_{i_k} = w = s_{j_1} \dots s_{j_k}$ and these expressions are both reduced. Consequently, $s_{i_1} s_{j_1} \dots s_{j_k}$ is not a reduced expression and $s_{i_1} s_{j_1} \dots s_{j_k} = s_{j_1} \dots \widehat{s}_{j_a} \dots s_{j_k}$ for some integer a by the exchange condition. Therefore, (i) $s_{i_2} \dots s_{i_k} = s_{j_1} \dots \widehat{s}_{j_a} \dots s_{j_k}$ and (ii) $s_{i_1} s_{j_1} \dots s_{j_{a-1}} = s_{j_1} \dots s_{j_a}$ and all four of these expressions are all reduced.

If $a \ne k$ then all of the reduced expressions in (i) and (ii) have length strictly less than k. Therefore, using induction and (i) and (ii) in turn,

$$(i_1, i_2, \dots, i_k) \sim_b (i_1, j_1, \dots, j_{a-1}, j_{a+1}, \dots, j_k) \sim_b (j_1, \dots, j_a, j_{a+1}, \dots, j_k)$$

as required.

If $a = k$ only expressions in (i) have length less than k, so we deduce that $(i_2, \dots, i_k) \sim_b (j_1, \dots, j_{k-1})$; by symmetry we may also assume that $(j_2, \dots, j_k) \sim_b (i_1, \dots, i_{k-1})$. Hence, $(i_1, i_2, \dots, i_k) \sim_b (i_1, j_1, \dots, j_{k-1})$ and $(j_1, j_2, \dots, j_k) \sim_b (j_1, i_1, \dots, i_{k-1})$; so it suffices to show that $(i_1, j_1, \dots, j_{k-1}) \sim_b (j_1, i_1, \dots, i_{k-1})$. If $|i_1 - j_1| > 1$ then $(i_1, j_1) \sim_b (j_1, i_1)$ is a braid relation of length 2 so we need only show that $(i_2, \dots, i_{k-1}) \sim_b (j_2, \dots, j_{k-1})$ which follows by induction. If $|i_1 - j_1| = 1$ then by reapplying the argument of the first paragraph we are reduced to showing that $(i_1, j_1, i_1, i_2, \dots, i_{k-2}) \sim_b (j_1, i_1, j_1, j_2, \dots, j_{k-2})$. However, $(i_1, j_1, i_1) \sim_b (j_1, i_1, j_1)$ is a braid relation of length 3, so it is sufficient to show that $(i_2, \dots, i_{k-2}) \sim_b (j_2, \dots, j_{k-2})$ which again follows by induction. □

2. The Iwahori–Hecke algebra

1.9 Let R be a commutative integral domain with 1 and let q be an arbitrary element of R. The Iwahori–Hecke algebra $\mathscr{H} = \mathscr{H}_{R,q}(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ is the unital associative R–algebra with generators $T_1, T_2, \dots, T_{n-1}$ which are subject to the relations

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0, && \text{for } i = 1, 2, \dots, n-1, \\ T_i T_j &= T_j T_i, && \text{for } 1 \le i < j-1 \le n-2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && \text{for } i = 1, 2, \dots, n-2. \end{aligned}$$

Notice that when $q = 1$ the first relation reduces to $T_i^2 = 1$; thus, by (1.1), $\mathscr{H}$ is isomorphic to the group ring $R\mathfrak{S}_n$ of the symmetric group in this case. When $q \neq 1$ this may seem a strange algebra to consider but, in fact, $\mathscr{H}$ arises naturally in the representation theory of the general linear groups (see Note 4 on page 7).

Our first aim is to produce a basis of $\mathscr{H}$. Suppose that $w \in \mathfrak{S}_n$ and let $(i_1, \dots, i_k)$ be a reduced expression for w. We define

1.10 $$T_w = T_{i_1} \dots T_{i_k}.$$

By Theorem 1.8, T_w is independent of the choice of reduced expression for w. If w is the identity element of $\mathfrak{S}_n$ we identify T_w with 1, the identity element of R.

1.11 Theorem *The Iwahori–Hecke algebra $\mathscr{H} = \mathscr{H}_{R,q}(\mathfrak{S}_n)$ is free as an R–module with basis $\{ T_w \mid w \in \mathfrak{S}_n \}$. Moreover, the multiplication in $\mathscr{H}$ is determined by the formulae*

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w, & \text{otherwise.} \end{cases}$$

for all $w \in \mathfrak{S}_n$ and $s \in S$.

Proof. By Corollary 1.6(ii), if $w \in \mathfrak{S}_n$ and $s \in S$ then $\ell(ws) < \ell(w)$ if and only if w has a reduced expression ending in s; so the multiplication formulae follow from the definition of T_w and the relation $T_s^2 = (q-1)T_s + q$.

By the relations in $\mathscr{H}$ the set $\{ T_w \mid w \in \mathfrak{S}_n \}$ certainly spans $\mathscr{H}$. Thus, it remains to show that $\{ T_w \mid w \in \mathfrak{S}_n \}$ is linearly independent.

Let $\hat{q}$ be an indeterminate over $\mathbb{Z}$. Then R becomes a $\mathbb{Z}[\hat{q}]$–module via the homomorphism $\varphi : \mathbb{Z}[\hat{q}] \longrightarrow R$ determined by $\varphi(1) = 1$ and $\varphi(\hat{q}) = q$. The generators $T_i \otimes 1$ of the R–algebra $\mathscr{H}' = \mathscr{H}_{\mathbb{Z}[\hat{q}],\hat{q}}(\mathfrak{S}_n) \otimes_{\mathbb{Z}[\hat{q}]} R$ satisfy the relations in $\mathscr{H}$ so there is a surjection $\alpha : \mathscr{H} \twoheadrightarrow \mathscr{H}'$. Therefore, it is enough to show that $\mathscr{H}_{\mathbb{Z}[\hat{q}],\hat{q}}(\mathfrak{S}_n)$ is free as a $\mathbb{Z}[\hat{q}]$–module with basis $\{ T_w \mid w \in \mathfrak{S}_n \}$ because then $\mathscr{H}'$ is R–free with basis $\{ T_w \otimes 1 \mid w \in \mathfrak{S}_n \}$ and, consequently, α must be an isomorphism.

Suppose then that $\mathscr{H} = \mathscr{H}_{\mathbb{Z}[\hat{q}],\hat{q}}(\mathfrak{S}_n)$. If $\{ T_w \mid w \in \mathfrak{S}_n \}$ were not linearly independent then there would exist polynomials $a_w(\hat{q}) \in \mathbb{Z}[\hat{q}]$, not all zero, such that $\sum_w a_w(\hat{q}) T_w = 0$ in $\mathscr{H}$. Without loss of generality, the greatest common divisor of the polynomials $\{ a_w(\hat{q}) \mid w \in \mathfrak{S}_n \}$ is 1. Consider $\mathbb{Z}$ as a $\mathbb{Z}[\hat{q}]$–module by letting $\hat{q}$ act as 1. Then $\mathscr{H} \otimes_{\mathbb{Z}[\hat{q}]} \mathbb{Z} \cong \mathbb{Z}\mathfrak{S}_n$ and $\{ T_w \otimes 1 \mid w \in \mathfrak{S}_n \}$ maps to the natural basis $\{ w \mid w \in \mathfrak{S}_n \}$ of $\mathbb{Z}\mathfrak{S}_n$ under this isomorphism. Hence, we obtain the relation $\sum_w a_w(1) w = 0$ in $\mathbb{Z}\mathfrak{S}_n$; this forces $a_w(1) = 0$ for all $w \in \mathfrak{S}_n$. Therefore, for all $w \in \mathfrak{S}_n$ the polynomial $a_w(\hat{q})$ belongs to $(\hat{q} - 1)\mathbb{Z}[\hat{q}]$, contradicting our assumption that the $a_w(\hat{q})$ have greatest common divisor 1. Hence, $\{ T_w \mid w \in \mathfrak{S}_n \}$ is a basis of $\mathscr{H}_{\mathbb{Z}[\hat{q}],\hat{q}}(\mathfrak{S}_n)$ and the Theorem follows. □

Now that we have established that $\mathscr{H}$ is free over R, the argument from the second paragraph of the proof generalizes to yield the following result. (In the tensor product of the Corollary we consider R as an $\hat{R}$–module with $\hat{r} \in \hat{R}$ acting on R as $\varphi(\hat{r})$.)

1.12 Corollary *Suppose that $\varphi: \hat{R} \longrightarrow R$ is a ring homomorphism with $\varphi(\hat{q}) = q$ for some $\hat{q} \in \hat{R}$. Then $\mathscr{H}_{R,q}(\mathfrak{S}_n)$ and $\mathscr{H}_{\hat{R},\hat{q}}(\mathfrak{S}_n) \otimes_{\hat{R}} R$ are canonically isomorphic algebras.*

In the situation of Corollary 1.12, we say that $\mathscr{H}_{R,q}(\mathfrak{S}_n)$ is a specialization of $\mathscr{H}_{\hat{R},\hat{q}}(\mathfrak{S}_n)$; the tensor product map $\mathscr{H}_{\hat{R},\hat{q}}(\mathfrak{S}_n) \longrightarrow \mathscr{H}_{\hat{R},\hat{q}}(\mathfrak{S}_n)$ is called a specialization homomorphism.

Let $\mathcal{Z} = \mathbb{Z}[\hat{q}, \hat{q}^{-1}]$, where $\hat{q}$ is an indeterminate over $\mathbb{Z}$. Then $\mathscr{H}_{R,q}(\mathfrak{S}_n)$ is a specialization of $\mathscr{H}_{\mathcal{Z}} = \mathscr{H}_{\mathcal{Z},\hat{q}}(\mathfrak{S}_n)$ whenever q is a non–zero element of R. As we are most interested in the case where $q \neq 0$, we call $\mathscr{H}_{\mathcal{Z}}$ the generic Iwahori–Hecke algebra of $\mathfrak{S}_n$.

1.13 Corollary *The Iwahori–Hecke algebra has two one dimensional representations $1_{\mathscr{H}}$ and $\varepsilon_{\mathscr{H}}$ which are the R–linear maps determined by*

$$1_{\mathscr{H}}(T_w) = q^{\ell(w)} \qquad \text{and} \qquad \varepsilon_{\mathscr{H}}(T_w) = (-1)^{\ell(w)}$$

for all $w \in \mathfrak{S}_n$.

Proof. It is sufficient to check that these maps preserve the relations in $\mathscr{H}$ and this is clear. □

The representation $1_{\mathscr{H}}$ is the Iwahori–Hecke algebras analogue of the trivial representation of $\mathfrak{S}_n$ and that $\varepsilon_{\mathscr{H}}$ is its analogue of the sign representation.

Define an R–bilinear map $(\ ,\ ): \mathscr{H} \times \mathscr{H} \longrightarrow R$ by setting (h_1, h_2) equal to the coefficient of 1 in $h_1 h_2$ for all $h_1, h_2 \in \mathscr{H}$. In particular, this form is associative; that is, $(h_1 h_2, h_3) = (h_1, h_2 h_3)$ for all $h_1, h_2, h_3 \in \mathscr{H}$.

1.14 Proposition *Suppose that x and y are elements of $\mathfrak{S}_n$. Then*

$$(T_x, T_y) = \begin{cases} q^{\ell(x)}, & \text{if } x = y^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if q is invertible then $(\ ,\ )$ is a non–degenerate associative symmetric bilinear form.

Proof. Let $x, y \in \mathfrak{S}_n$. The Proposition will follow if we show that there exists an integer m and $a_z \in R$, for $z \in \mathfrak{S}_n$, such that $a_z = 0$ unless $\ell(z) \geq \ell(xy)$ and

$$T_x T_y = q^m T_{xy} + \sum_{z \in \mathfrak{S}_n} a_z T_z.$$

To prove this we argue by induction on the length of x. When $\ell(x) = 0$ then $x = 1$ and there is nothing to prove. If $\ell(x) > 0$ then there exists $s \in S$ such that $\ell(xs) < \ell(x)$. Then $T_x T_y = T_{xs}(T_s T_y)$ and the claim follows easily by induction using Theorem 1.11. □

Given any representation of $\mathscr{H}$, its character is defined in the usual way by taking traces. By Theorem 1.11 the characters of $\mathscr{H}$ are R–linear maps which are determined by their values on the basis elements T_w, for $w \in \mathfrak{S}_n$. Unlike in the case of finite groups, if χ is a character and x and y are conjugate elements of $\mathfrak{S}_n$ it is *not* necessarily true that $\chi(T_x) = \chi(T_y)$; for example, consider the trivial representation $1_{\mathscr{H}}$ of $\mathscr{H}$. Nonetheless, there is an analogue of the orthogonality relations when $\mathscr{H}$ is semisimple.

Let $W_n(\hat{q}) = \sum_{w\in\mathfrak{S}_n} \hat{q}^{\ell(w)}$ be the Poincaré polynomial of $\mathfrak{S}_n$; this polynomial arises naturally in the invariant theory of the symmetric group (see, for example, [**33**, §3.15]). Because $\{1, s_{n-1}, s_{n-1}s_{n-2}, \dots, s_{n-1}\dots s_1\}$ is a set of right coset representatives for $\mathfrak{S}_{n-1}$ in $\mathfrak{S}_n$, it follows that

$$W_n(\hat{q}) = (1+\hat{q})(1+\hat{q}+\hat{q}^2)\dots(1+\hat{q}+\dots+\hat{q}^{n-1});$$

thus, $W_n(\hat{q}) = [\hat{q}]_n^!$ in the notation of (3.28). Finally, note that if $\hat{q} = 1$ then $W_n(1) = n! = |\mathfrak{S}_n|$.

1.15 Corollary *Suppose that $\mathcal{K} = \mathbb{C}(\hat{q})$ and let $\mathscr{H}_{\mathcal{K}} = \mathscr{H}_{\mathcal{K},\hat{q}}(\mathfrak{S}_n)$. Then $\mathscr{H}_{\mathcal{K}}$ is semisimple. Moreover, if χ and ψ are irreducible characters of $\mathscr{H}_{\mathcal{K}}$ then*

$$\frac{1}{W_n(\hat{q})}\sum_{w\in\mathfrak{S}_n}\hat{q}^{-\ell(w)}\chi(T_w)\psi(T_{w^{-1}}) = \delta_{\chi\psi}D_\chi(\hat{q}),$$

for some rational function $D_\chi(\hat{q})$ in $\mathcal{K}$.

Proof. By Proposition 1.14, $\mathscr{H}_{\mathcal{K}}$ is a symmetric algebra with dual bases $\{T_w\}$ and $\{\hat{q}^{-\ell(w)}T_{w^{-1}}\}$, so the orthogonality relation will follow from the theory of symmetric algebras (see [**11**, §9A]) once we show that $\mathscr{H}_{\mathcal{K}}$ is semisimple.

Suppose that $\mathscr{H}_{\mathcal{K}}$ is not semisimple. Then we can find a non–zero element h in the radical of $\mathscr{H}_{\mathcal{K}}$. Without loss of generality we may assume that $h\in\mathscr{H}_{\mathcal{Z}}$ and $h\notin(\hat{q}-1)\mathscr{H}_{\mathcal{Z}}$. Let $\varphi:\mathscr{H}_{\mathcal{Z}}\longrightarrow\mathbb{C}\mathfrak{S}_n$ be the specialization homomorphism sending $\hat{q}$ to $1\in\mathbb{C}$. Then for any $a_{xy}\in\mathbb{C}$ we have

$$\sum_{x,y\in\mathfrak{S}_n} a_{xy}x\varphi(h)y = \varphi\Big(\sum_{x,y\in\mathfrak{S}_n} a_{xy}T_xhT_y\Big).$$

Now h is nilpotent so $\varphi(h)$ is also nilpotent; however, $\mathbb{C}\mathfrak{S}_n$ is semisimple so we must have $\varphi(h) = 0$. Therefore, $h\in\ker\varphi = (\hat{q}-1)\mathscr{H}_{\mathcal{Z}}$ which contradicts our assumptions on h. ◻

We shall see in Corollary 3.22 below that the Hecke algebra $\mathscr{H}_{\mathcal{K}}$ is split semisimple and that, as an algebra, it is isomorphic to $\mathcal{K}\mathfrak{S}_n$.

The rational functions $D_\chi(\hat{q})$ of Corollary 1.15 are the so–called *generic degrees* of $\mathfrak{S}_n$; they are all explicitly known (see [**6**, p. 446]) and turn out to be polynomials in $\mathbb{Z}[q]$. Therefore, given the remark of the last paragraph, the equation defining $D_\chi(\hat{q})$ in Corollary 1.15 can be evaluated at $\hat{q} = 1$ where it becomes one of the orthogonality relations for the ordinary irreducible characters if the symmetric group. It follows that $D_\chi(1)$ is the dimension of an irreducible complex representation of $\mathfrak{S}_n$. What is more, when q is a prime power the $D_\chi(q)$ give the dimensions of the principal series representations of $GL_n(q)$; see [**6**] and Note 4 below.

3. The 0–Hecke algebra

After this section we will assume that q is invertible. For completeness we now describe the simple $\mathscr{H}$–modules when $q = 0$.

1.16 Theorem (Norton) *Suppose that R is a field and that $q = 0$. Then every irreducible $\mathscr{H}$–module is one dimensional.*

Proof. Fix a total ordering $w_1 > w_2 > \cdots > w_{n!} = 1$ on $\mathfrak{S}_n$ such that $w_a > w_b$ whenever $\ell(w_a) > \ell(w_b)$. Define $\mathscr{H}_a$ to the R–module with basis $\{ T_{w_b} \mid b \le a \}$ and let $\mathscr{H}_0 = 0$. Then $0 = \mathscr{H}_0 \subset \mathscr{H}_1 \subset \mathscr{H}_2 \subset \cdots \subset \mathscr{H}_{n!} = \mathscr{H}$.

Now, if $w \in \mathfrak{S}_n$ and $s \in S$ then because $q = 0$

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) > \ell(w), \\ -T_w, & \text{if } \ell(ws) < \ell(w), \end{cases}$$

by Theorem 1.11. Consequently, for each $a \ge 1$ the R–module $\mathscr{H}_a$ is a right $\mathscr{H}$–module (in fact it is an ideal). Therefore, $\mathscr{H}_a/\mathscr{H}_{a+1}$ is a one dimensional $\mathscr{H}$–module; in particular, $\mathscr{H}_a/\mathscr{H}_{a+1}$ is simple. Since these modules give a filtration of $\mathscr{H}$, every irreducible $\mathscr{H}$–module is of the form $\mathscr{H}_a/\mathscr{H}_{a+1}$ for some a with $1 \le a \le n!$. The Theorem follows. $\square$

Norton [**57**] explicitly constructs the principal indecomposable modules of $\mathscr{H}$ when $q = 0$. She shows that, up to isomorphism, the irreducible representations of $\mathscr{H}$ are labelled by the subsets of S. If $I \subseteq S$ the corresponding irreducible representation $\varphi_I : \mathscr{H} \longrightarrow R$ is the unique homomorphism such that

$$\varphi_I(T_i) = \begin{cases} 1, & \text{if } i \in I, \\ 0, & \text{otherwise}, \end{cases}$$

for $i = 1, 2, \ldots, n-1$. Furthermore, the decomposition matrices and Cartan matrices of the 0–Hecke algebra of $\mathfrak{S}_n$ are all explicitly known [**7**] (in type A only).

Notes on Chapter 1

1. The elegant proof of the exchange condition given in Section 1 is due to Matthew Dyer [**22**, **23**]. Dyer calls the function N a *reflection cocycle* and he shows that a group W generated a set S of involutions is a Coxeter group if and only if W possesses a map N such that (i) $N(s) = \{s\}$ for all $s \in S$ and (ii) for all elements v and w in W, $N(vw) = N(v) \dot{+} vN(w)v^{-1}$; *cf.* Lemma 1.2.
2. Theorem 1.8 is a special case of Matsumoto's monid lemma [**11**, 64.20], which is really a result about the corresponding braid group. It is the key result of the first section because it shows that the T–basis is well–defined.
3. Theorem 1.11 is often taken as the definition of $\mathscr{H}$. In this case, it is necessary to prove that an algebra exists satisfying Theorem 1.11; this is done by working in the endomorphism ring of the R–module $\mathscr{H}$; see, for example, [**33**, Chapter 7].
4. Let $G = GL_n(q)$, where q is a prime power, and let B be the subgroup of upper triangular matrices in G (a *Borel subgroup*). Let $e_B = \frac{1}{|B|}\sum_{b\in B} b$; then e_B is idempotent and Iwahori [**34**] showed that $\mathscr{H}_{\mathbb{C},q} \cong e_B \mathbb{C}G e_B$. This is one of the main reasons why these algebras are interesting and important. (Note that $\mathfrak{S}_n$ is the Weyl group of $GL_n(q)$.) The principal series representations of G are just the irreducible constituents of $\mathbb{C}Ge_B$; this begins to explain why the dimensions of the principal series representations are given by specializing the generic degrees. Proofs and more complete details can be found in [**11**, §67-68].
5. The proof that $\mathscr{H}_{\mathbb{K}}$ is semisimple in Corollary 1.15 is due to Gyoja and Uno [**31**].
6. In the literature, a *Hecke algebra* is any algebra of the form $eAe \cong \operatorname{End}_A(eA)$ where A is an algebra and e is an idempotent in A. Such algebras were first studied by Hecke; this is why Iwahori called $\mathscr{H}$ a Hecke algebra.

7. Even though the characters of $\mathscr{H}_{\mathcal{K}}$ are not class functions, there is a well behaved notion of *character table* which is defined by the values of the irreducible characters on *minimal length* conjugacy class representatives. These characters were first calculated by Ram [**59**]; see also [**24**].

CHAPTER 2

Cellular algebras

Cellular algebras were introduced by Graham and Lehrer in [**25**,**26**]. Although it is possible to give a basis–free approach to the theory of cellular algebras [**50**], our perspective is that a cellular algebra is an algebra which has a basis which is particularly well–adapted to its representation theory.

This Chapter describes the simple modules and blocks of a cellular algebra; in Chapters 3 and 4 this theory is applied to the Iwahori–Hecke algebras of the symmetric groups and the q–Schur algebras.

1. Cellular bases

Let R be a commutative integral domain with 1 and A an associative unital R–algebra.

2.1 Suppose that $(\Lambda, \geq)$ is a (finite) poset and that for each $\lambda \in \Lambda$ we have a finite set $\mathcal{T}(\lambda)$ and elements $c^\lambda_{\mathfrak{s}\mathfrak{t}} \in A$ for all $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)$ such that

$$\mathscr{C} = \{\, c^\lambda_{\mathfrak{s}\mathfrak{t}} \mid \lambda \in \Lambda \text{ and } s, \mathfrak{t} \in \mathcal{T}(\lambda) \,\}$$

is a basis of A. For each $\lambda \in \Lambda$ let $\check{A}^\lambda$ be the R–submodule of A with basis $\{\, c^\mu_{\mathfrak{u}\mathfrak{v}} \mid \mu \in \Lambda, \mu > \lambda \text{ and } \mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\mu) \,\}$. The pair $(\mathscr{C}, \Lambda)$ is a cellular basis of A if

(i) the R–linear map $* : A \longrightarrow A$ determined by ${c^\lambda_{\mathfrak{s}\mathfrak{t}}}^* = c^\lambda_{\mathfrak{t}\mathfrak{s}}$, for all $\lambda \in \Lambda$ and all $\mathfrak{s}$ and $\mathfrak{t}$ in $\mathcal{T}(\lambda)$, is an algebra anti–isomorphism of A; and,

(ii) for any $\lambda \in \Lambda$, $\mathfrak{t} \in \mathcal{T}(\lambda)$ and $a \in A$ there exist $r_\mathfrak{v} \in R$ such that for all $\mathfrak{s} \in \mathcal{T}(\lambda)$

$$c^\lambda_{\mathfrak{s}\mathfrak{t}} a \equiv \sum_{\mathfrak{v} \in \mathcal{T}(\lambda)} r_\mathfrak{v} c^\lambda_{\mathfrak{s}\mathfrak{v}} \mod \check{A}^\lambda.$$

If A has a cellular basis we say that A is a cellular algebra.

It should be emphasized that a cellular algebra may have more than one cellular basis (up to an automorphism of A).

2.2 Examples (i) First an easy example which illustrates some of the properties of cellular algebras. Let $A = R[x]$, where x is in indeterminate over R, and take Λ to be the non–negative integers with their natural ordering. For each $n \in \Lambda$ let $\mathcal{T}(n) = \{\circ\}$ and set $c^n_{\circ\circ} = x^n$. It is easy to check that $\{\, x^n \mid n \geq 0 \,\}$ is a cellular basis of A. Here $\check{A}^n$ is the set of polynomials of degree greater than n and $\check{A}^n/\check{A}^{n+1} \cong R$ is simple for all n.

(ii) Let $\mathscr{H} = \mathscr{H}_{R,q}(\mathfrak{S}_3)$ and let $\Lambda = \{(3), (2,1), (1^3)\}$, the set of partitions of 3 ordered lexicographically. Let $\mathfrak{s} = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$, $\mathfrak{t} = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$, $\mathfrak{u} = \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ and $\mathfrak{v} = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$ and set

$\mathcal{T}(3)=\{\mathfrak{s}\}$, $\mathcal{T}(2,1)=\{\mathfrak{t},\mathfrak{u}\}$ and $\mathcal{T}(1^3)=\{\mathfrak{v}\}$. Then

$$\begin{array}{rclrcl} c^{(3)}_{\mathfrak{ss}} &=& \multicolumn{4}{l}{1+T_1+T_2+T_1T_2+T_2T_1+T_1T_2T_1} \\ c^{(2,1)}_{\mathfrak{tt}} &=& (1+T_1) & c^{(2,1)}_{\mathfrak{ut}} &=& T_2(1+T_1) \\ c^{(2,1)}_{\mathfrak{tu}} &=& (1+T_1)T_2 & c^{(2,1)}_{\mathfrak{uu}} &=& T_2(1+T_1)T_2 \\ & & \multicolumn{4}{l}{c^{(1^3)}_{\mathfrak{vv}}=1.} \end{array}$$

is a cellular basis of $\mathscr{H}$ (check!). Note that $\mathfrak{u}=\mathfrak{t}s_2$ and $c^{(2,1)}_{\mathfrak{tu}}=c^{(2,1)}_{\mathfrak{tt}}T_2$. We generalize this example in the next chapter.

Throughout this Chapter we assume that $\mathscr{C}=\{c^\lambda_{\mathfrak{st}}\}$ is a fixed cellular basis of the algebra A. We begin our study of cellular algebras by noting four easy consequences of the definition.

For $\lambda\in\Lambda$ let A^λ be the R–module with basis the set of $c^\mu_{\mathfrak{uv}}$ where $\mu\in\Lambda$, $\mu\ge\lambda$ and $\mathfrak{u},\mathfrak{v}\in\mathcal{T}(\mu)$. Thus, $\check{A}^\lambda\subset A^\lambda$ and $A^\lambda/\check{A}^\lambda$ has basis $c^\lambda_{\mathfrak{st}}+\check{A}^\lambda$ where $\mathfrak{s},\mathfrak{t}\in\mathcal{T}(\lambda)$.

2.3 Corollary *Let λ be an element of Λ.*

(i) *Suppose that $\mathfrak{s}\in\mathcal{T}(\lambda)$ and $a\in A$. Then for all $\mathfrak{t}\in\mathcal{T}(\lambda)$*

$$a^*c^\lambda_{\mathfrak{st}}=\sum_{\mathfrak{u}\in\mathcal{T}(\lambda)} r_{\mathfrak{u}}c^\lambda_{\mathfrak{ut}}\mod \check{A}^\lambda$$

where for each $\mathfrak{u}$, $r_{\mathfrak{u}}$ is the element of R determined by (2.1)(ii).

(ii) *The R–modules A^λ and $\check{A}^\lambda$ are two–sided ideals of A.*

(iii) *Suppose that $\mu,\lambda\in\Lambda$, $\lambda\not\ge\mu$ and $x\in A^\lambda$. Then $ax, xa\in\check{A}^\lambda$ for all $a\in A^\mu$.*

(iv) *Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are elements of $\mathcal{T}(\lambda)$. Then there exists an element $r_{\mathfrak{st}}$ of R such that for any $\mathfrak{u},\mathfrak{v}\in\mathcal{T}(\lambda)$*

$$c^\lambda_{\mathfrak{us}}c^\lambda_{\mathfrak{tv}}=r_{\mathfrak{st}}c^\lambda_{\mathfrak{uv}}\mod\check{A}^\lambda.$$

Proof. Applying the anti–isomorphism $*$ to (2.1)(ii) proves (i). The remaining statements now follow directly from part (i) and (2.1)(ii). □

Corollary 2.3(iv) is the key which unlocks the irreducible modules of a cellular algebra. For each $\lambda\in\Lambda$ the (right) cell module C^λ is the right A–module which is free as an R–module with basis $\{\,c^\lambda_{\mathfrak{t}}\mid \mathfrak{t}\in\mathcal{T}(\lambda)\,\}$ and where for each $a\in A$

2.4
$$c^\lambda_{\mathfrak{t}}a=\sum_{\mathfrak{v}\in\mathcal{T}(\lambda)} r_{\mathfrak{v}}c^\lambda_{\mathfrak{v}}$$

where $r_{\mathfrak{v}}$ is the element of R determined by (2.1)(ii). By (2.1)(ii) these definitions make C^λ into an A–module. Similarly, abusing notation define the left cell module $C^{\lambda*}$ to be the free R–module with basis $\{\,c^\lambda_{\mathfrak{t}}\mid\mathfrak{t}\in\mathcal{T}(\lambda)\,\}$ and A–action given by

2.5
$$a^*c^\lambda_{\mathfrak{t}}=\sum_{\mathfrak{v}\in\mathcal{T}(\lambda)} r_{\mathfrak{v}}c^\lambda_{\mathfrak{v}}$$

for all $a\in A$ and where, once again, $r_{\mathfrak{v}}$ is given by (2.1)(ii). Then the ideals $A^\lambda/\check{A}^\lambda$ and $C^{\lambda*}\otimes_R C^\lambda$ are canonically isomorphic via $c^\lambda_{\mathfrak{st}}+\check{A}^\lambda\longmapsto c^\lambda_{\mathfrak{s}}\otimes c^\lambda_{\mathfrak{t}}$ for $\mathfrak{s},\mathfrak{t}\in\mathcal{T}(\lambda)$.

By Corollary 2.3(iv) we can define a bilinear form $\langle\ ,\ \rangle\colon C^\lambda\times C^\lambda\longrightarrow R$ to be the unique bilinear map such that, for any $\mathfrak{s}$, $\mathfrak{t}\in\mathcal{T}(\lambda)$,

2.6
$$c^\lambda_{\mathfrak{us}}c^\lambda_{\mathfrak{tv}}=\langle c^\lambda_{\mathfrak{s}},c^\lambda_{\mathfrak{t}}\rangle c^\lambda_{\mathfrak{uv}}\mod\check{A}^\lambda,$$

where $\mathfrak{u}$ and $\mathfrak{v}$ are any elements of $\mathcal{T}(\lambda)$. The next result shows that $\langle\ ,\ \rangle$ is both symmetric and associative.

2.7 Proposition *Suppose that $\lambda\in\Lambda$ and let $x,y\in C^\lambda$. Then*
(i) $\langle x,y\rangle=\langle y,x\rangle$.
(ii) $\langle xa,y\rangle=\langle x,ya^*\rangle$ *for all* $a\in A$.
(iii) $xc^\lambda_{\mathfrak{u}\mathfrak{v}}=\langle x,c^\lambda_{\mathfrak{u}}\rangle c^\lambda_{\mathfrak{v}}$ *for all* $\mathfrak{u},\mathfrak{v}\in\mathcal{T}(\lambda)$.

Proof. Because $\langle\ ,\ \rangle$ is bilinear it is sufficient to consider the case where $x=c^\lambda_{\mathfrak{s}}$ and $y=c^\lambda_{\mathfrak{t}}$ for some $\mathfrak{s},\mathfrak{t}\in\mathcal{T}(\lambda)$. Let $\mathfrak{u}$ and $\mathfrak{v}$ be any elements of $\mathcal{T}(\lambda)$. Then, by definition,

$$\langle c^\lambda_{\mathfrak{s}},c^\lambda_{\mathfrak{t}}\rangle c^\lambda_{\mathfrak{u}\mathfrak{v}}\equiv c^\lambda_{\mathfrak{u}\mathfrak{s}}c^\lambda_{\mathfrak{t}\mathfrak{v}}=(c^\lambda_{\mathfrak{v}\mathfrak{t}}c^\lambda_{\mathfrak{s}\mathfrak{u}})^*\equiv(\langle c^\lambda_{\mathfrak{t}},c^\lambda_{\mathfrak{s}}\rangle c^\lambda_{\mathfrak{v}\mathfrak{u}})^*=\langle c^\lambda_{\mathfrak{t}},c^\lambda_{\mathfrak{s}}\rangle c^\lambda_{\mathfrak{u}\mathfrak{v}}\mod \check{A}^\lambda,$$

proving (i). Similarly for (ii),

$$\langle c^\lambda_{\mathfrak{s}}a,c^\lambda_{\mathfrak{t}}\rangle c^\lambda_{\mathfrak{u}\mathfrak{v}}\equiv(c^\lambda_{\mathfrak{u}\mathfrak{s}}a)c^\lambda_{\mathfrak{t}\mathfrak{v}}=c^\lambda_{\mathfrak{u}\mathfrak{s}}(ac^\lambda_{\mathfrak{t}\mathfrak{v}})\equiv\langle c^\lambda_{\mathfrak{s}},c^\lambda_{\mathfrak{t}}a^*\rangle c^\lambda_{\mathfrak{u}\mathfrak{v}}\mod \check{A}^\lambda.$$

Finally, when $x=c^\lambda_{\mathfrak{s}}$, part (iii) is precisely the definition of $\langle\ ,\ \rangle$. □

2.8 Let $\operatorname{rad}C^\lambda=\{\,x\in C^\lambda\mid\langle x,y\rangle=0\text{ for all }y\in C^\lambda\,\}$. Then by Proposition 2.7(ii), $\operatorname{rad}C^\lambda$ is an A–submodule of C^λ. Accordingly, we define $D^\lambda=C^\lambda/\operatorname{rad}C^\lambda$.

Although we have yet to do anything difficult, we can now exhibit some irreducible A–modules. Recall that the Jacobson radical of a module is the smallest submodule with a semisimple quotient; it is also the intersection of all of the maximal ideal in the module.

2.9 Proposition *Suppose that R is a field and let λ be any element of Λ such that $D^\lambda\ne0$.*
(i) *The right A–module D^λ is absolutely irreducible.*
(ii) *The Jacobson radical of C^λ is equal to* $\operatorname{rad}C^\lambda$.

Proof. Let x be any non–zero element of $C^\lambda\setminus\operatorname{rad}C^\lambda$. Then $\langle x,y\rangle\ne0$ for some $y\in C^\lambda$ and, because R is a field, we may assume that $\langle x,y\rangle=1$. Now $y=\sum_{\mathfrak{s}\in\mathcal{T}(\lambda)}r_{\mathfrak{s}}c^\lambda_{\mathfrak{s}}$ for some $r_{\mathfrak{s}}\in R$; so, for each $\mathfrak{t}\in\mathcal{T}(\lambda)$, define $y_{\mathfrak{t}}=\sum_{\mathfrak{s}\in\Lambda}r_{\mathfrak{s}}c^\lambda_{\mathfrak{s}\mathfrak{t}}\in A$. Then, by Proposition 2.7(iii),

$$xy_{\mathfrak{t}}=\sum_{\mathfrak{s}\in\mathcal{T}(\lambda)}r_{\mathfrak{s}}xc^\lambda_{\mathfrak{s}\mathfrak{t}}=\sum_{\mathfrak{s}\in\mathcal{T}(\lambda)}r_{\mathfrak{s}}\langle x,c^\lambda_{\mathfrak{s}}\rangle c^\lambda_{\mathfrak{t}}=\langle x,y\rangle c^\lambda_{\mathfrak{t}}=c^\lambda_{\mathfrak{t}}.$$

Consequently, x generates C^λ. This argument applies to any element of C^λ which does not belong to the radical; consequently, D^λ is irreducible and $\operatorname{rad}C^\lambda$ is the unique maximal proper submodule of C^λ, proving (ii). By the same argument, D^λ is irreducible for any extension field of R so in fact D^λ is absolutely irreducible. □

2.10 Proposition *Suppose that R is a field and let λ and μ be elements of Λ such that $D^\lambda\ne0$ and let M be a proper submodule of C^μ.*
(i) *If $\theta\colon C^\lambda\longrightarrow C^\mu/M$ is a non–zero homomorphism then $\lambda\ge\mu$.*
(ii) *Suppose that $\lambda=\mu$. Then* $\operatorname{Hom}_A(C^\lambda,C^\lambda/M)\cong R$.

Proof. Define elements x,y and $y_{\mathfrak{t}}=\sum_{\mathfrak{s}}r_{\mathfrak{s}}c^\lambda_{\mathfrak{s}\mathfrak{t}}$ as in the proof of Proposition 2.9, where $t\in\mathcal{T}(\lambda)$ and let $\theta\colon C^\lambda\longrightarrow C^\mu/M$ be an A–module homomorphism. Then there exists some $a_\theta\in C^\mu$ such that $\theta(x)=M+a_\theta$. Therefore, $\theta(c^\lambda_{\mathfrak{t}})=\theta(xy_{\mathfrak{t}})=\theta(x)y_{\mathfrak{t}}=M+a_\theta y_{\mathfrak{t}}$ for all $\mathfrak{t}\in\mathcal{T}(\lambda)$. By Corollary 2.3(iii) and (2.4), if $\lambda\not\ge\mu$ then $a_\theta y_{\mathfrak{t}}=0$ so $\theta(c^\lambda_{\mathfrak{t}})=0$ for all $\mathfrak{t}$. Consequently, if $\theta\ne0$ then $\lambda\ge\mu$, proving (i).

Now suppose that $\mu = \lambda$. Then by Proposition 2.7(iii) we find that

$$\theta(c_{\mathfrak{t}}^{\lambda}) = M + a_\theta y_{\mathfrak{t}} = M + \sum_{\mathfrak{s}\in\mathcal{T}(\lambda)} r_{\mathfrak{s}} a_\theta c_{\mathfrak{s}\mathfrak{t}}^{\lambda} = M + c_{\mathfrak{t}}^{\lambda}\langle a_\theta, y\rangle.$$

Hence, θ is just the natural projection $C^\lambda \to C^\lambda/M$ composed with multiplication by $\langle a_\theta, y\rangle$. □

2.11 Corollary *Suppose that R is a field and let λ and μ be elements of Λ such that $D^\lambda \neq 0$ and $D^\lambda \cong D^\mu$. Then $\lambda = \mu$.*

Proof. By assumption there exists a non–zero homomorphism $\theta : C^\lambda \longrightarrow D^\mu$; whence $\lambda \geq \mu$ by (i). By symmetry, $\mu \geq \lambda$. Thus $\lambda = \mu$ as required. □

2. Simple modules in a cellular algebra

In this section we show that every irreducible A–module is isomorphic to some D^μ, for some $\mu \in \Lambda$, and we describe the decomposition matrix of A. Throughout, we assume that the poset Λ is finite; thus A is a finite dimensional algebra.

2.12 Lemma *Suppose that λ is a minimal element of Λ. Then* $\operatorname{rad} C^\lambda = 0$*; consequently, $C^\lambda = D^\lambda$.*

Proof. By way of contradiction, suppose that $\operatorname{rad} C^\lambda \neq 0$; then there exists some $x \in C^\lambda$ such that $\langle x, y\rangle = 0$ for all $y \in C^\lambda$. Write $x = \sum_{\mathfrak{t}} r_{\mathfrak{t}} c_{\mathfrak{t}}^{\lambda}$ and consider $\hat{x} = \sum_{\mathfrak{t}} r_{\mathfrak{t}} c_{\mathfrak{t}\mathfrak{t}}^{\lambda}$. By Proposition 2.7(iii), if $\mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\lambda)$ then $\hat{x} c_{\mathfrak{u}\mathfrak{v}}^{\lambda} \in \check{A}^\lambda$. Therefore, by the minimality of λ and Corollary 2.3(iii), $\hat{x}a \in \check{A}^\lambda$ for all a in A. In particular, $\hat{x} = \hat{x}\cdot 1$ belongs to $\check{A}^\lambda$, yielding the promised contradiction. □

2.13 Theorem (Graham–Lehrer) *Suppose that R is a field and that Λ is finite. Then $\{\, D^\lambda \mid \lambda \in \Lambda$ and $D^\lambda \neq 0 \,\}$ is a complete set of pairwise inequivalent irreducible A–modules.*

Proof. By Proposition 2.9 if $D^\mu \neq 0$ then D^μ is irreducible and by Corollary 2.11 $D^\mu \ncong D^\lambda$ whenever $\lambda \in \Lambda$ and $\lambda \neq \mu$. It remains to show that every irreducible A–module is isomorphic to one of the non–zero modules D^μ with $\mu \in \Lambda$.

By the definition, any total ordering of Λ gives rise to a filtration of A in which each composition factor is isomorphic to a cell module of A. Consequently, to prove Theorem it suffices to show that if $\lambda \in \Lambda$ then every irreducible composition factor of C^λ is isomorphic to D^μ for some $\mu \in \Lambda$. To prove this we argue by induction on λ.

If λ is a minimal element of Λ then $C^\lambda = D^\lambda$ by Lemma 2.12 and there is nothing to prove.

Now suppose that λ is not minimal and let D be an irreducible composition factor of C^λ. Then either $D = D^\lambda$ or D is a composition factor of $\operatorname{rad} C^\lambda$. Let x be an element of $\operatorname{rad} C^\lambda$ and let $\check{A}_\lambda$ be the R–module spanned by

$$\{\, c_{\mathfrak{u}\mathfrak{v}}^{\nu} \mid \nu \in \Lambda, \lambda \not> \nu \text{ and } \mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\nu) \,\}.$$

Then $\check{A}_\lambda$ is a two–sided ideal of A, by (2.1)(i) and Corollary 2.3(i), and $x\check{A}_\lambda = 0$ by Corollary 2.3(iii). Therefore, any composition factor of $\operatorname{rad} C^\lambda$ is a composition factor of $A/\check{A}_\lambda$. However, by construction, $A/\check{A}_\lambda$ has a composition series with composition factors isomorphic to cell modules C^ν with $\nu \in \Lambda$ and $\lambda > \nu$. By induction, if $\lambda > \nu$ then every irreducible composition factor of C^ν is isomorphic to D^μ for some $\mu \in \Lambda$. This completes the proof. □

Let $\Lambda^+ = \{\,\mu \in \Lambda \mid D^\mu \neq 0\,\}$. Then $\mu \in \Lambda^+$ if and only if the bilinear form $\langle\ ,\ \rangle$ on C^μ is non–degenerate. In principle, Theorem 2.13 classifies the simple A–modules; however, in practice, it is often difficult to determine the set Λ^+.

If $\mu \in \Lambda^+$ and $\lambda \in \Lambda$ let $d_{\lambda\mu}$ be the composition multiplicity of the irreducible module D^μ in C^λ; by the Jordan–Hölder Theorem [**10**], $d_{\lambda\mu}$ is well–defined. The matrix $\mathbf{D} = (d_{\lambda\mu})$, where $\lambda \in \Lambda$ and $\mu \in \Lambda^+$, is the so–called decomposition matrix of A. The next result could also have been deduced during the proof of Theorem 2.13.

2.14 Corollary *Suppose that R is a field. Then the decomposition matrix $\mathbf{D}$ of A is unitriangular; that is, if $\mu \in \Lambda^+$ and $\lambda \in \Lambda$ then $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \geq \mu$.*

Proof. The decomposition number $d_{\lambda\mu}$ is non–zero if and only if there exists a submodule M of C^λ and a homomorphism $\theta\colon C^\mu \longrightarrow C^\lambda/M$ with the image of θ isomorphic to D^μ. Therefore, if $d_{\lambda\mu} \neq 0$ then $\lambda \geq \mu$ by Proposition 2.10(i). Furthermore, if $\lambda = \mu$ then $C^\lambda/\ker\theta$ is simple if and only if $\ker\theta = \operatorname{rad} C^\mu$ by Proposition 2.9(ii), so $d_{\mu\mu} = 1$. □

Next we describe the Cartan invariants of A. For each element $\lambda \in \Lambda^+$ we have a simple module D^λ and a projective indecomposable module P^λ, uniquely determined up to isomorphism by the property that it has simple head D^λ. In fact, since $D^{\lambda *} \otimes_R D^\lambda$ is a simple ideal we have $A \cong \bigoplus_{\lambda\in\Lambda^+} D^{\lambda *} \otimes_R P^\lambda$.

If $\mu \in \Lambda^+$ let $c_{\lambda\mu}$ be the composition multiplicity of D^μ in P^λ. Then $\mathbf{C} = (c_{\lambda\mu})$, where $\lambda, \mu \in \Lambda^+$, is the Cartan matrix of A.

In order to compute $\mathbf{C}$ we need two Lemmas and some more notation.

2.15 Lemma *Suppose that R is a field, $\lambda \in \Lambda^+$ and $\nu \in \Lambda$. Then*

$$d_{\nu\lambda} = \dim_R \operatorname{Hom}_A(P^\lambda, C^\nu) = \dim_R P^\lambda \otimes_A C^{\nu *}.$$

Proof. If $\nu \in \Lambda$ then $d_{\nu\lambda}$ is non–zero only if there exists a submodule M of C^ν together with a surjective homomorphism $\theta\colon M \longrightarrow D^\nu$. Since P^λ is projective, θ lifts uniquely to a homomorphism $\hat\theta\colon P^\lambda \longrightarrow M$ and the inclusion $M \hookrightarrow C^\nu$ produces an A–homomorphism $P^\lambda \longrightarrow C^\nu$. Thus, $d_{\nu\lambda} = \dim_R \operatorname{Hom}_A(P^\lambda, C^\nu)$ as required. □

A subset Γ of Λ a poset ideal if $\lambda \in \Gamma$ whenever $\mu \in \Gamma$ and $\lambda > \mu$. If Γ is a poset ideal let $A(\Gamma)$ be the R–submodule of A with basis $\{c^\mu_{\mathfrak{u}\mathfrak{v}}\}$ where $\mu \in \Gamma$ and $\mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\mu)$. Then $A(\Gamma)$ is a two–sided ideal of A by (2.1)(ii) and Corollary 2.3(i).

2.16 Lemma *Suppose that P is a projective A–module. Then P has a filtration $\varnothing = P_0 \subset \cdots \subset P_k = P$ with composition factors P_i/P_{i-1} isomorphic to the (non–zero) modules $P \otimes_A (C^{\mu *} \otimes_R C^\mu)$, with each $\mu \in \Lambda$ occurring exactly once.*

Proof. Let Γ be a poset ideal and suppose that $\Gamma \cup \{\mu\}$ is also an ideal for some $\mu \in \Lambda \setminus \Gamma$. Then $0 \to A(\Gamma) \to A(\Gamma \cup \{\mu\}) \to A(\Gamma)/A(\Gamma \cup \{\mu\}) \to 0$ is a short exact sequence of (A, A)–bimodules and $A(\Gamma \cup \{\mu\})/A(\Gamma) \cong C^{\mu *} \otimes_R C^\mu$. Because P is projective, we can tensor this sequence with P to obtain an exact sequence

$$0 \to P \otimes_A A(\Gamma) \to P \otimes_A A(\Gamma \cup \{\mu\}) \to P \otimes_A A(\Gamma \cup \{\mu\})/A(\Gamma) \to 0.$$

Consequently, $P \otimes_A A(\Gamma \cup \{\mu\})/P \otimes_A A(\Gamma) \cong P \otimes_A (C^{\mu*} \otimes_R C^\mu)$.

Now suppose that $\varnothing = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$ is a maximal chain of ideals in Λ. Then, for $i = 1, 2, \dots, k$, we have $\Gamma_i = \Gamma_{i-1} \cup \{\mu_i\}$ for some $\mu_i \in \Gamma$. Therefore, $0 = A(\Gamma_0) \hookrightarrow A(\Gamma_1) \hookrightarrow \cdots \hookrightarrow A(\Gamma_k) = A$ is a filtration of A. Tensoring with P gives a filtration of P with composition factors, by the last paragraph, precisely the non–zero modules $P \otimes_A (C^{\mu*} \otimes_R C^\mu)$ for $\mu \in \Lambda$. $\square$

Noting that $P \otimes_A (C^{\mu*} \otimes_R C^\mu) \cong (P \otimes_A C^{\mu*}) \otimes_R C^\mu \cong (\dim P \otimes_A C^{\mu*}) C^\mu$, we see that every projective A–module P has a cell module filtration; that is a filtration in which every composition factor is isomorphic to a cell module.

In the next result, $\mathbf{D}^t$ denotes the transpose of $\mathbf{D}$.

2.17 Theorem (Graham–Lehrer) *Suppose that R is a field and that Λ is finite. Then $\mathbf{C} = \mathbf{D}^t\mathbf{D}$. In particular, the Cartan matrix of A is symmetric.*

Proof. Let λ and μ be elements of Λ^+ and take $P = P^\lambda$. By Lemma 2.16, P^λ has a filtration with composition factors the non–zero modules $P^\lambda \otimes_A (C^{\nu*} \otimes_R C^\nu)$, and where each $\nu \in \Lambda$ occurs at most once. Therefore, by Lemma 2.15,

$$\begin{aligned} c_{\lambda\mu} = [P^\lambda : D^\mu] &= \sum_{\nu\in\Lambda} [(P^\lambda \otimes_A C^{\nu*}) \otimes_R C^\nu : D^\mu] \\ &= \sum_{\nu\in\Lambda} \dim_R(P^\lambda \otimes_A C^{\nu*})[C^\nu : D^\mu] \\ &= \sum_{\nu\in\Lambda} d_{\nu\lambda} d_{\nu\mu}. \end{aligned}$$

Hence, $\mathbf{C} = \mathbf{D}^t\mathbf{D}$ as claimed. $\square$

2.18 Corollary *Suppose that R is a field. Then the following are equivalent.*

(i) *A is (split) semisimple.*
(ii) *$C^\lambda = D^\lambda$ for all $\lambda \in \Lambda$.*
(iii) *$\operatorname{rad} C^\lambda = 0$ for all $\lambda \in \Lambda$.*
(iv) *$d_{\lambda\mu} = \delta_{\lambda\mu}$ for all λ and μ in Λ.*

Proof. First we note that if A is semisimple then R is a splitting field for A since each D^λ is absolutely irreducible by Proposition 2.9. Thus, the algebra A is semisimple if and only if it is split semisimple.

Now suppose that A is semisimple. Then $P^\lambda = D^\lambda$ for all $\lambda \in \Lambda^+$; consequently, $C^\lambda = D^\lambda$ by Theorem 2.17. Furthermore, because A is semisimple,

$$\dim_R A = \sum_{\lambda\in\Lambda^+} \dim_R(C^\lambda)^2 = \sum_{\lambda\in\Lambda^+} |\mathcal{T}(\lambda)|^2.$$

However, $\dim_R A = \sum_{\lambda\in\Lambda} |\mathcal{T}(\lambda)|^2$, so we must have $\Lambda = \Lambda^+$. Hence, (i) implies (ii).

Certainly (ii) and (iii) are both equivalent and they both imply (iv), so assume that (iv) holds. In particular, (iv) implies that $\Lambda = \Lambda^+$ and, using Theorem 2.17 again, that $P^\lambda = C^\lambda = D^\lambda$. Therefore, A is a direct sum of simple (projective) modules; that is to say it is semisimple. $\square$

The next result connects the theory of quasi–hereditary algebras and cellular algebras. Quasi–hereditary algebras are very a important class of algebras; however, as we do not need any results from their theory, we refer the interested reader to [**8**] for the definition.

2.19 Corollary *Suppose that R is a field. Then the following are equivalent.*
(i) $\Lambda = \Lambda^+$.
(ii) *The decomposition matrix* $\mathbf{D}$ *is a square unitriangular matrix.*
Moreover, if either of these conditions is satisfied then A is quasi–hereditary.

Proof. That (i) and (ii) are equivalent is clear from Corollary 2.14. If (i) holds then it follows directly from the definitions in [**8**] that A is quasi–hereditary. □

Implicitly, part (ii) plays a role in the proof of the Jantzen sum formula in Chapter 5; this criterion indicates that being quasi–hereditary is a non–degeneracy property.

Finally, we consider blocks of a cellular algebra. Suppose that R is a field (and that Λ is finite). Then A can be decomposed as a direct sum of indecomposable two–sided ideals; say $A = B_1 \oplus \dots \oplus B_k$. The ideals $B_1, \dots, B_k$ are uniquely determined and are called the block ideals of A. Furthermore, for each i, there exists a (unique) primitive central idempotent e_i such that $B_i = e_iA = e_iAe_i = Ae_i$. It follows that $1 = e_1 + \dots + e_k$ and that $e_ie_j = \delta_{ij}e_i$; thus, B_i is an R–algebra with identity e_i. Proofs of these general facts can be found in [**10**, Chapter VIII].

Two A–modules are said to belong to the same block if all of their irreducible composition factors lie in the same block. In particular, because $e_ie_j = \delta_{ij}$ for $1 \le i, j \le k$, a right A–module N belongs to the same block as B_i if and only if $Ne_i = N$ and $Ne_j = 0$ if $i \ne j$.

2.20 Lemma *Suppose that e and f are idempotents in A. Then* $\operatorname{Hom}_A(eA, fA)$ *and fAe are canonically isomorphic as R–modules. Moreover, if $e = f$ then* $\operatorname{End}_A(eA)$ *and eAe are isomorphic as algebras.*

Proof. If $\varphi \in \operatorname{Hom}_A(eA, fA)$ then φ is completely determined by $\varphi(e)$. Certainly, $\varphi(e) \in fA$ and, because e is idempotent, $\varphi(e) = \varphi(e^2) = \varphi(e)e \in fAe$. Hence, the map $\varphi \longmapsto \varphi(e)$ is an isomorphism.

Finally, suppose that $e = f$. For homomorphisms φ and ψ in $\operatorname{End}_A(eA)$, write $\varphi(e) = ea_\varphi e$ and $\psi(e) = ea_\psi e$ for some $a_\varphi, a_\psi \in A$. Then $\varphi(e)\psi(e) = (ea_\varphi e)(ea_\psi e) = (ea_\varphi e)a_\psi e = (\varphi\psi)(e)$. Hence, $\operatorname{End}_A(eA)$ and eAe are isomorphic algebras. □

In particular, if $1 \le i, j \le k$ then $\operatorname{Hom}_A(B_i, B_j) \cong e_jAe_i = e_je_iA = \delta_{ij}B_i$.

Say that λ and μ in Λ are linked if there exist $\lambda = \lambda_0, \lambda_1, \dots, \lambda_k = \mu$ in Λ such that $C^{\lambda_{i-1}}$ and C^{λ_i} have a common irreducible composition factor for $i = 1, 2, \dots, k$. This defines an equivalence relation on Λ; importantly, if $\mu \in \Lambda^+$ then $d_{\lambda\mu} \ne 0$ only if λ and μ are linked.

2.21 Theorem *Suppose that R is a field and that Λ is finite. Let λ and μ be elements of Λ. Then λ and μ are linked if and only if C^λ and C^μ are in the same block.*

Proof. As a right A–module each block B_i decomposes into a direct sum of principal indecomposable modules. This gives a partition $\Lambda^+ = \Lambda_1^+ \sqcup \dots \sqcup \Lambda_k^+$ of Λ^+ such that $B_i \cong \bigoplus_{\mu \in \Lambda_i^+} D^{\mu *} \otimes_R P^\mu$ for $i = 1, 2, \dots, k$. Furthermore, by Lemma 2.16, each indecomposable P^λ has a cell module filtration; thus, to prove the Theorem it suffices to show that $\lambda, \mu \in \Lambda^+$ are linked if and only if $\lambda, \mu \in \Lambda_i^+$ for some i.

Suppose first that λ and μ are in Λ^+ and that D^λ is a composition factor of P^μ. Then, as in the proof of Lemma 2.15, $\operatorname{Hom}_A(P^\lambda, P^\mu) \neq 0$. Consequently P^λ and P^μ must be contained in the same block since $\operatorname{Hom}_A(B_i, B_j) = 0$ if $i \neq j$. Hence, if λ and μ in Λ are linked then C^λ and C^μ are in the same block.

Conversely, suppose that there exists an integer i, with $1 \leq i \leq k$, and λ and μ in Λ_i^+ such that C^λ and C^μ are not linked. Then P^λ and P^μ have no composition factor in common so $\operatorname{Hom}_A(P^\lambda, P^\mu) = 0$. Let Λ_{i1}^+ be the set of $\nu \in \Lambda_i^+$ which are linked to λ and let $\Lambda_{i2}^+ = \Lambda_i^+ \setminus \Lambda_{i1}^+$. For $a = 1, 2$ let $B_{ia} = \bigoplus_{\nu \in \Lambda_{ia}^+} D^{\nu *} \otimes P^\nu$. Then, by Lemma 2.20 and the remark following it,

$$B_i \cong \operatorname{End}_A(B_i) = \operatorname{End}_A(B_{i1} \oplus B_{i2}) = \operatorname{End}_A(B_{i1}) \oplus \operatorname{End}_A(B_{i2}),$$

contradicting the fact that B_i is an indecomposable block ideal of A. $\square$

Notes on Chapter 2

1. All of the results in this Chapter are due to Graham and Lehrer and can be found in [**25**, **26**].
2. The introduction of cellular algebras was largely motivated by the Kazhdan–Lusztig [**46**] bases of the Iwahori–Hecke algebras of type **A**. Many more examples of cellular algebras can be found in [**25**, **26**]. For example, (generalized) Temperley-Lieb algebras, Jones' annular algebra, Ariki–Koike algebras, and the cyclotomic q–Schur algebras of [**17**] are all cellular.
3. A ring theoretic definition of cellular algebras has been König and Xi [**50**]; in particular, their approach is independent of a choice of basis. They have also investigated cellular algebras under Morita equivalence [**48**] and shown that the poset Λ is not, in any sense, an invariant of a cellular algebra [**49**].
4. If A is also a symmetric algebra then Graham [**25**, 4.12] has shown that the dual basis of a cellular basis of A is again cellular, providing that the trace form on A commutes with the $*$–automorphism.
5. Du and Rui [**20**] have proved a converse to Corollary 2.19; they showed that any quasi-hereditary algebra possessing a certain type of involution is cellular.

CHAPTER 3

The Modular representation theory of $\mathscr{H}$

In this chapter we construct Murphy's basis of the Iwahori–Hecke algebra $\mathscr{H}$. Murphy's papers [**55, 56**]appeared before the theory of cellular algebras: nevertheless, his basis is cellular.

For the rest of these notes we assume that q is an invertible element of R. In particular, this implies that T_i is invertible with $T_i^{-1} = q^{-1}(T_i - q + 1)$ for $i = 1, 2, \dots, n-1$. Consequently if $w \in \mathfrak{S}_n$ and $s_{i_1} \dots s_{i_k}$ is a reduced expression for w then $T_w^{-1} = T_{i_k}^{-1} \dots T_{i_1}^{-1}$.

1. The combinatorics of tableaux

Let m be a positive integer. A composition of m is a sequence $\mu = (\mu_1, \mu_2, \dots)$ of non–negative integers such that $|\mu| = \sum_i \mu_i = m$. The integers μ_i, $i \geq 1$, are the *parts* of μ; if $\mu_i = 0$ for $i > k$ we identify μ with $(\mu_1, \dots, \mu_k)$. A composition μ is a partition if $\mu_i \geq \mu_{i+1}$ for all $i \geq 1$.

The diagram of a composition μ is the subset

$$[\mu] = \{\, (i, j) \mid 1 \leq j \leq \mu_i \text{ and } i \geq 1 \,\}.$$

of $\mathbb{N} \times \mathbb{N}$. The elements of $[\mu]$ are called the nodes of μ; more generally, a node is any element of $\mathbb{N} \times \mathbb{N}$. It will be convenient to represent the diagram of μ as an array of boxes in the plane. For example, if $\mu = (2, 3)$ then $[\mu] = \begin{smallmatrix}\square&\square\\\square&\square&\square\end{smallmatrix}$.

If μ is a composition of n then a μ–tableau is a bijection $\mathsf{t} : [\mu] \longrightarrow \{1, 2, \dots, n\}$; we say that t has shaper μ and write $\operatorname{Shape}(\mathsf{t}) = \mu$. Notice that a tableau t corresponds to a labelled diagram; for example, if $\mu = (2, 3)$ then $\begin{smallmatrix}1&2\\3&4&5\end{smallmatrix}$ and $\begin{smallmatrix}4&2\\3&1&5\end{smallmatrix}$ are two μ–tableaux. Accordingly, we will speak of the rows and columns of tableaux.

We distinguish two particular types of μ–tableaux. A μ–tableau t is row standard if the entries in t increase from left to right in each row; t is standard if (i) μ is a partition and (ii) the entries in t increase from left to right in each row and from top to bottom in each column. If μ is a partition let $\operatorname{Std}(\mu)$ denote the set of standard μ–tableaux.

Of the two tableaux above, only the first is row standard; neither is standard because $(2, 3)$ is not a partition. If $\lambda = (3, 2)$ then $\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}$ is a standard λ–tableau.

Given any composition μ let t^μ be the row standard μ–tableau in which the integers $1, 2, \dots, n$ are entered in increasing order from left to right along the rows of $[\mu]$. For example, if $\mu = (2, 3)$ then $\mathsf{t}^\mu = \begin{smallmatrix}1&2\\3&4&5\end{smallmatrix}$.

If $\mu = (\mu_1, \dots, \mu_k)$ is a composition of n then the symmetric group $\mathfrak{S}_n$ acts from the right upon the set of μ–tableaux by permuting the entries inside a given tableau. For example, if $\mathsf{t} = \begin{smallmatrix}1&3\\2&4&5\end{smallmatrix}$ then $\mathsf{t}(2, 4, 5, 3) = \begin{smallmatrix}1&2\\4&5&3\end{smallmatrix}$.

Let $\mathfrak{S}_\mu$ be the subgroup of $\mathfrak{S}_n$ which stabilizes the rows of t^μ and let $\mathscr{H}(\mathfrak{S}_\mu)$ be the subalgebra of $\mathscr{H}$ spanned by $\{\, T_w \mid w \in \mathfrak{S}_\mu \,\}$. Then $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_k}$

and $\mathscr{H}(\mathfrak{S}_\mu)$ is isomorphic to $\mathscr{H}(\mathfrak{S}_{\mu_1})\times\cdots\times\mathscr{H}(\mathfrak{S}_{\mu_k})$, the Iwahori–Hecke of $\mathfrak{S}_\mu$. It follows that $\mathfrak{S}_\mu$ is generated by $S\cap\mathfrak{S}_\mu$ and that $\mathscr{H}(\mathfrak{S}_\mu)$ is generated by $\{T_s \mid s\in S\cap\mathfrak{S}_\mu\}$. Finally, we let $m_\mu=\sum_{w\in\mathfrak{S}_\mu}T_w$ and define M^μ to be the right ideal $m_\mu\mathscr{H}$.

3.1 Example Let $\mu=(2,3)$. Then $\mathfrak{t}^\mu=\begin{array}{|c|c|c}\hline 1&2\\ \hline 3&4&\multicolumn{1}{c|}{5}\\ \hline\end{array}$, $\mathfrak{S}_\mu=\mathfrak{S}_2\times\mathfrak{S}_3=\langle s_1,s_3,s_4\rangle$ and $m_\mu=(1+T_1)(1+T_3+T_4+T_3T_4+T_4T_3+T_3T_4T_3)$.

3.2 Lemma *Suppose that μ is a composition of n and let w be an element of $\mathfrak{S}_\mu$. Then $m_\mu T_w=q^{\ell(w)}m_\mu$.*

Proof. It suffices to show that if $s\in S\cap\mathfrak{S}_\mu$ then $m_\mu T_s=qm_\mu$. We have

$$\begin{aligned} m_\mu T_s &= \sum_{w\in\mathfrak{S}_\mu}T_wT_s=\sum_{\substack{w\in\mathfrak{S}_\mu\\ \ell(ws)>\ell(w)}}T_wT_s+\sum_{\substack{w\in\mathfrak{S}_\mu\\ \ell(ws)<\ell(w)}}T_wT_s\\ &=\sum_{\substack{w\in\mathfrak{S}_\mu\\ \ell(ws)>\ell(w)}}T_{ws}+\sum_{\substack{w\in\mathfrak{S}_\mu\\ \ell(ws)<\ell(w)}}\big((q-1)T_w+qT_{ws}\big)=qm_\mu,\end{aligned}$$

where the last line follows by the multiplication formulae of Theorem 1.11. □

Comparing Lemma 3.2 and Corollary 1.13 we see that $m_\mu\mathscr{H}(\mathfrak{S}_\mu)=Rm_\mu$ affords the trivial representation of $\mathscr{H}(\mathfrak{S}_\mu)$ and, moreover, that as an $\mathscr{H}$–module $M^\mu\cong m_\mu\mathscr{H}(\mathfrak{S}_\mu)\otimes_{\mathscr{H}(\mathfrak{S}_\mu)}\mathscr{H}$. Thus, M^μ is the Iwahori–Hecke algebra analogue of the induced representation $\operatorname{Ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_n}(1)$.

To give a basis of M^μ we need a set of coset representatives for $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$.

3.3 Proposition *Suppose that μ is a composition of n and let*

$$\mathscr{D}_\mu=\{\,d\in\mathfrak{S}_n\mid \mathfrak{t}^\mu d \text{ is row standard}\,\}.$$

Then $\mathscr{D}_\mu$ is a complete set of right coset representatives of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$. Moreover, if $w\in\mathfrak{S}_\mu$ and $d\in\mathscr{D}_\mu$ then $\ell(wd)=\ell(w)+\ell(d)$ and $T_{wd}=T_wT_d$.

Proof. If $\mathfrak{S}_\mu w=\mathfrak{S}_\mu v$ then the tableaux $\mathfrak{t}^\mu w$ and $\mathfrak{t}^\mu v$ agree up to a reordering of the entries in each row; therefore, $\mathscr{D}_\mu$ is a complete set of right coset representatives of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$.

It remains to show that if $w\in\mathfrak{S}_\mu$ and $d\in\mathscr{D}_\mu$ then $\ell(wd)=\ell(w)+\ell(d)$. First note that if $i<j$, where i and j are in the same row of $\mathfrak{t}^\mu$, then $id<jd$. In particular, if i and $i{+}1$ are in the same row of $\mathfrak{t}^\mu$ then $id<(i{+}1)d$ so $\ell(s_id)=1{+}\ell(d)$ whenever $s_i\in S\cap\mathfrak{S}_\mu$ by Corollary 1.4.

Now suppose that $w\in\mathfrak{S}_\mu$ and that $\ell(w)>1$. Then $w=s_iv$ and $\ell(w)=1{+}\ell(v)$ for some $s_i\in\mathfrak{S}_\mu\cap S$ and $v\in\mathfrak{S}_\mu$; therefore, $iv<(i+1)v$ by Corollary 1.4. Now iv and $(i+1)v$ belong to the same row of $\mathfrak{t}^\mu$, so $ivd<(i+1)vd$ by the last paragraph; whence, $\ell(s_ivd)=1+\ell(vd)$ by Corollary 1.4 again. Consequently, by induction on the length of v we find that

$$\ell(wd)=\ell(s_ivd)=1+\ell(vd)=1+\ell(v)+\ell(d)=\ell(w)+\ell(d)$$

as required. □

Thus each row standard μ–tableau corresponds to a right coset of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$. If $\mathfrak{t}$ is a row standard μ–tableau let $d(\mathfrak{t})$ be the unique element of $\mathfrak{S}_n$ such that $\mathfrak{t}=\mathfrak{t}^\mu d(\mathfrak{t})$. By Proposition 3.3 $d(\mathfrak{t})$ is the unique element of minimal length in the coset $\mathfrak{S}_\mu d(\mathfrak{t})$; such coset representatives are called distinguished coset representatives.

3.4 Corollary *Let μ be a composition of n. Then M^μ is a free R–module with basis $\{\, m_\mu T_{d(\mathfrak{t})} \mid \mathfrak{t}$ a row standard μ–tableau$\}$. Moreover, if $\mathfrak{t}$ is row standard and $\mathfrak{s} = \mathfrak{t}s_i$, for some $1 \le i < n$, then*

$$m_\mu T_{d(\mathfrak{t})} T_i = \begin{cases} q m_\mu T_{d(\mathfrak{t})}, & \text{if } \mathfrak{s} \text{ is not row standard,} \\ m_\mu T_{d(\mathfrak{s})}, & \text{if } \mathfrak{s} \text{ is row standard and } \ell(d(\mathfrak{s})) > \ell(d(\mathfrak{t})), \\ q m_\mu T_{d(\mathfrak{s})} + (q-1) m_\mu T_{d(\mathfrak{t})}, & \text{if } \mathfrak{s} \text{ is row standard and } \ell(d(\mathfrak{s})) < \ell(d(\mathfrak{t})). \end{cases}$$

Proof. That M^μ is a free R–module with the basis shown follows directly from Theorem 1.11, Lemma 3.2 and Proposition 3.3. It remains to verify the multiplication formulae.

Let $w = d(\mathfrak{t})s_i$; then $\mathfrak{s} = \mathfrak{t}^\mu w$. Now, $\mathfrak{s}$ is *not* row standard if and only if i and $i+1$ are in the same row of $\mathfrak{t}$, in which case w and $d(\mathfrak{t})$ are in the same right coset of $\mathfrak{S}_\mu$. Therefore, $w = sd(\mathfrak{t})$ for some $s \in \mathfrak{S}_\mu$ and $\ell(w) = \ell(d(\mathfrak{t})) + 1$ by Proposition 3.3; hence, $m_\mu T_{d(\mathfrak{t})} T_i = m_\mu T_s T_{d(\mathfrak{t})} = q m_\mu T_{d(\mathfrak{t})}$ by Lemma 3.2.

Finally, if i and $i+1$ are not in the same row of $\mathfrak{s}$ then $\mathfrak{s}$ is row standard; thus, $d(\mathfrak{s}) \in \mathscr{D}_\mu$ and the remaining statements follow from the multiplication formulae of Theorem 1.11. $\square$

3.5 To conclude this section we introduce two partial orders. Let μ and ν be compositions. Then $\mu \trianglerighteq \nu$ if

$$\sum_{j=1}^{i} \mu_j \ge \sum_{j=1}^{i} \nu_j \quad \text{for all} \quad i \ge 1.$$

If $\mu \trianglerighteq \nu$ we say that μ dominates ν. If $\mu \ne \nu$ we write $\mu \vartriangleright \nu$.

Next, given a row standard tableau $\mathfrak{t}$ and an integer $m \le n$ let $\mathfrak{t}{\downarrow}m$ be the tableau obtained from $\mathfrak{t}$ by deleting all of the entries greater than m. Note that $\mathfrak{t}{\downarrow}m$ is a row standard tableau for some composition of m. Given row standard μ–tableaux $\mathfrak{s}$ and $\mathfrak{t}$ write $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if $\operatorname{Shape}(\mathfrak{s}{\downarrow}m) \trianglerighteq \operatorname{Shape}(\mathfrak{t}{\downarrow}m)$ for $m = 1, 2 \ldots, n$. Similarly, we write $\mathfrak{s} \vartriangleright \mathfrak{t}$ when $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and $\mathfrak{s} \ne \mathfrak{t}$.

3.6 Example (i) Let λ be a partition of n and let $\mathfrak{t}$ be any row standard λ–tableau. Then $\mathfrak{t}^\lambda \trianglerighteq \mathfrak{t}$.

(ii) Let $\mathfrak{s} = \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$ and $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 2&3&5\\\hline 1&4\\\cline{1-2}\end{array}$. Then the following table shows that $\mathfrak{s} \vartriangleright \mathfrak{t}$.

m	1	2	3	4	5
$\mathfrak{s}{\downarrow}m$	$\begin{array}{\|c\|}\hline 1\\\hline \varnothing\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$
$\mathfrak{t}{\downarrow}m$	$\begin{array}{\|c\|}\hline \varnothing\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&3\\\hline 1&4\\\hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 2&3&5\\\hline 1&4\\\cline{1-2}\end{array}$

Suppose that $r_1 \ldots r_k$ is a not necessarily reduced expression for some element of $\mathfrak{S}_n$, where $r_1, \ldots, r_k \in S$. A subexpression of $r_1 \ldots r_k$ is any expression of the form $r_{i_1} \ldots r_{i_l}$ where $1 \le i_1 < \cdots < i_l \le k$.

Given two elements v and w in $\mathfrak{S}_n$ write $v \trianglerighteq w$ if v has a reduced expression which is a subexpression of some reduced expression of w. For example, $1 \trianglerighteq w$ for all $w \in \mathfrak{S}_n$. By the strong exchange condition (1.5), $v \trianglerighteq w$ if and only if there exist reflections $t_i, \ldots, t_k \in T$ such that $w = vt_1, \ldots t_k$ and $\ell(vt_1 \ldots t_{j-1}) < \ell(vt_1 \ldots t_j)$ for $j = 1, 2, \ldots, k$. (In fact, one can show that $v \trianglerighteq w$ if and only if there exist reflections $t_i, \dots, t_k \in T$ such that $w = vt_1, \dots t_k$ and $\ell(vt_1 \dots t_j) = \ell(v) + j$ for $j = 1, 2, \dots, k$.) In particular, this shows that $\trianglerighteq$ is a partial order on $\mathfrak{S}_n$.

The partial order $\trianglerighteq$ on $\mathfrak{S}_n$ is the ubiquitous Bruhat–Chevalley order. Perhaps surprisingly, it agrees with the dominance ordering on row standard μ–tableaux (when restricted to $\mathscr{D}_\mu$).

3.7 Proposition (Dipper–James) *Let μ be a composition of n and suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard μ–tableaux. Then $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if and only if $d(\mathfrak{s}) \trianglerighteq d(\mathfrak{t})$.*

Proof. First suppose that $s \triangleright \mathfrak{t}$ and choose r minimal such that row r of $\mathfrak{s}$ and row r of $\mathfrak{t}$ are different. Let a be the smallest number which is in row r of $\mathfrak{s}$ but not in row r of $\mathfrak{t}$; similarly, let b be the smallest number which is in row r of $\mathfrak{t}$ but not in row r of $\mathfrak{s}$. Then $a < b$ since $\mathfrak{s} \triangleright \mathfrak{t}$. Therefore, if $\mathfrak{v} = \mathfrak{t}(a, b)$ then $\mathfrak{v} \triangleright \mathfrak{t}$ and $d(\mathfrak{v}) \triangleright d(\mathfrak{t})$. Further, if a is in row t of $\mathfrak{t}$ and b is in row s of $\mathfrak{s}$ then $t > r$ and $t \geq s$; so, either $\mathfrak{v} = \mathfrak{s}$ or $\mathfrak{s} \triangleright \mathfrak{v}$. Therefore, by induction, $d(\mathfrak{s}) \trianglerighteq d(\mathfrak{v})$; so $d(\mathfrak{s}) \triangleright d(\mathfrak{t})$ as required.

To prove the converse it is sufficient to consider the case where $d(\mathfrak{t}) = d(\mathfrak{s})t$ for some $t \in T$ such that $\ell(d(\mathfrak{t})) = \ell(d(\mathfrak{s})) + 1$. However, in this case it is easy to see from the definitions that $\mathfrak{s} \triangleright \mathfrak{t}$. □

The Proposition has a technical Corollary which will be used many times below.

3.8 Corollary *Suppose that τ, $\mathfrak{g}$ and $\mathfrak{t}$ are row standard λ–tableaux such that $\tau \triangleright \mathfrak{g}$ and $d(\mathfrak{t}) = d(\mathfrak{g})w$ for some $w \in \mathfrak{S}_n$ with $\ell(d(\mathfrak{t})) = \ell(d(\mathfrak{g})) + \ell(w)$. Then there exist $r_\mathfrak{s} \in R$ such that $m_\lambda T_{d(\tau)} T_w = \sum_\mathfrak{s} r_\mathfrak{s} m_\mu T_{d(\mathfrak{s})}$ and where $r_\mathfrak{s} \neq 0$ only if $\mathfrak{s} \triangleright \mathfrak{t}$.*

Proof. It follows from Theorem 1.11 that there exist elements $r_x \in R$ such that $T_{d(\tau)} T_w = \sum r_x T_x$ where the sum is over elements $x \in \mathfrak{S}_n$ which have a reduced expression which is a subexpression of $d(\tau)w$; that is, $r_x \neq 0$ only if $x \trianglerighteq d(\tau)w$. Consequently, by Corollary 3.4, $m_\lambda T_{d(\tau)} T_w = \sum_\mathfrak{s} r_\mathfrak{s} m_\mu T_{d(\mathfrak{s})}$ where the sum is over row standard λ–tableaux $\mathfrak{s}$ such that $r_\mathfrak{s} \neq 0$ only if $d(s) \trianglerighteq d(\tau)w$. By assumption $d(\tau)$ is a proper subexpression of $d(\mathfrak{g})$ and $\ell(d(\mathfrak{g})w) = \ell(d(\mathfrak{g})) + \ell(w)$; so $d(\tau)w$ is a proper subexpression of $d(\mathfrak{t}) = d(\mathfrak{g})w$. Thus, if $r_\mathfrak{s} \neq 0$ then $\mathfrak{s} \trianglerighteq d(\tau)w \triangleright d(\mathfrak{t})$, giving the result. □

2. The Murphy basis

In Corollary 3.4 we saw that the permutation modules M^μ have a basis indexed by row standard tableaux; in this section we show that $\mathscr{H} = M^{(1^n)}$ has a cellular basis indexed by pairs of standard tableaux of the same shape.

Let $*$ be the R–linear antiautomorphism of $\mathscr{H}$ determined by $T_i^* = T_i$ for $i = 1, 2, \dots, n-1$. Then $T_w^* = T_{w^{-1}}$ for all $w \in \mathfrak{S}_n$.

Let μ be a composition of n and let $\mathfrak{s}$ and $\mathfrak{t}$ be row standard μ–tableaux. Define $m_{\mathfrak{s}\mathfrak{t}} i = T^*_{d(\mathfrak{s})} m_\mu T_{d(\mathfrak{t})}$ (*cf.* Example 2.2(ii)). In particular, $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$ for all row standard tableaux $\mathfrak{s}$ and $\mathfrak{t}$ of the same shape.

If $\mu = (1^n)$ then, as R–modules, $M^\mu = \mathscr{H}$ so the elements $\{m_{\mathfrak{s}\mathfrak{t}}\}$ certainly span $\mathscr{H}$ by Corollary 3.4. First we show that we need only those elements $m_{\mathfrak{s}\mathfrak{t}}$, where the shape of $\mathfrak{s}$ and $\mathfrak{t}$ is a partition (not just a composition).

3.9 Lemma *Let μ be a composition of n and let λ be the partition obtained from μ by reordering its parts. Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard μ–tableaux. Then*

$m_{\mathfrak{s}\mathfrak{t}}$ is an R–linear combination of terms of the form $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are row standard λ–tableaux.

Proof. Choose an element $d \in \mathfrak{S}_n$ such that $\mathfrak{S}_\lambda = d\mathfrak{S}_\mu d^{-1}$ and such that $\mathfrak{t}^\lambda d$ and $\mathfrak{t}^\mu d^{-1}$ are both row standard (d and d^{-1} permute the rows of $\mathfrak{t}^\lambda$ and $\mathfrak{t}^\mu$ respectively; see Example 3.10). Then $\mathfrak{S}_\lambda d = d\mathfrak{S}_\mu$, $d \in \mathscr{D}_\lambda$ and $d^{-1} \in \mathscr{D}_\mu$; so $m_\lambda T_d = T_d m_\mu$ by Proposition 3.3. Consequently, $m_\mu = T_d^{-1} m_\lambda T_d$ can be written as a linear combination of terms $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are row standard λ–tableaux by Corollary 3.4 and its left–handed analogue. The general case follows from this and Corollary 3.4 since $m_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})} m_\mu T_{d(\mathfrak{t})} = T^*_{d(\mathfrak{s})} T^*_d m_\mu T_d T_{d(\mathfrak{t})}$. □

3.10 Example Suppose that $\mu = (2,1,3,3)$. Then, using the notation of Lemma 3.9, $\lambda = (3,3,2,1)$ and

$$\mathfrak{t}^\mu = \begin{array}{lll} 1&2\\ 3\\ 4&5&6\\ 7&8&9 \end{array} \qquad \text{and} \qquad \mathfrak{t}^\lambda = \begin{array}{lll} 1&2&3\\ 4&5&6\\ 7&8\\ 9 \end{array}.$$

Thus we may take $d = (1,4,7)(2,5,8)(3,6,9)$, in which case

$$\mathfrak{t}^\mu d^{-1} = \begin{array}{lll} 7&8\\ 9\\ 1&2&3\\ 4&5&6 \end{array} \qquad \text{and} \qquad \mathfrak{t}^\lambda d = \begin{array}{lll} 4&5&6\\ 7&8&9\\ 1&2\\ 3 \end{array}.$$

Note, however, that d is not unique; we could also let $d = (1,7)(2,8)(3,9)$, then

$$\mathfrak{t}^\mu d^{-1} = \begin{array}{lll} 7&8\\ 9\\ 4&5&6\\ 1&2&3 \end{array} \qquad \text{and} \qquad \mathfrak{t}^\lambda d = \begin{array}{lll} 7&8&9\\ 4&5&6\\ 1&2\\ 3 \end{array}$$

are again row standard. However, for both choices of d we have $\mathfrak{S}_\lambda = d\mathfrak{S}_\mu d^{-1}$.

Let λ be a partition and suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard λ–tableaux. We need to show that $m_{\mathfrak{s}\mathfrak{t}}$ can be written as a linear combination of terms $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard tableaux of the same shape. We begin with a special case.

Suppose that $(i,j) \in [\lambda]$ where $\lambda_{i+1} \geq j$ and let $a = \mathfrak{t}^\lambda(i,j)$ and $b = \mathfrak{t}^\lambda(i+1,j)$. The (i,j)th Garnir tableau of shape λ is the row standard λ–tableau $\mathfrak{g} = \mathfrak{g}_{ij}$ in which all numbers less than a or greater than b occupy the same positions in $\mathfrak{t}^\lambda$ and $\mathfrak{g}$, and where the numbers $a, a+1, \dots, b$ are inserted in increasing order from left to right, first along row $i+1$ and then along row i, into the remaining positions.

3.11 Example Let $\lambda = (3,3)$ then $\mathfrak{g}_{11} = \begin{array}{lll} 2&3&4\\ 1&5&6 \end{array}$, $\mathfrak{g}_{12} = \begin{array}{lll} 1&4&5\\ 2&3&6 \end{array}$ and $\mathfrak{g}_{13} = \begin{array}{lll} 1&2&6\\ 3&4&5 \end{array}$. Let $\mathfrak{g} = \mathfrak{g}_{11}$ and $\nu = (4,2)$. In this case, the proof of the Lemma 3.12 below reduces to the (easily verified) coset identity

$$\begin{array}{lll} 2&3&4\\ 1&5&6 \end{array} + \begin{array}{lll} 1&3&4\\ 2&5&6 \end{array} + \begin{array}{lll} 1&2&4\\ 3&5&6 \end{array} + \begin{array}{lll} 1&2&3\\ 4&5&6 \end{array} = \begin{array}{llll} 1&2&3&6\\ 4&5 \end{array} + \begin{array}{llll} 1&2&3&5\\ 4&6 \end{array} + \begin{array}{llll} 1&2&3&4\\ 5&6 \end{array}$$

where, for this example, a tableau $\mathfrak{t}$ on the left hand side is identified with $m_\lambda T_{d(\mathfrak{t})}$ and a tableau $\mathfrak{v}$ on the right hand side is identified with $T^*_{d(\mathfrak{v})} m_\nu$.

For the next few Lemmas we need another ordering on the set of partitions. Given partitions λ and μ we write $\lambda > \mu$ if λ is greater than μ in the lexicographic ordering. That is, there exists an integer $j \geq 1$ such that $\lambda_i = \mu_i$ for $1 \leq i < j$ and $\lambda_j > \mu_j$. Similarly we write $\lambda \geq \mu$ if either $\lambda = \mu$ or $\lambda > \mu$. Note that if $\lambda \rhd \mu$ then $\lambda > \mu$.

3.12 Lemma *Let λ be a partition of n and suppose that $\mathfrak{g}$ is a Garnir tableau. Then there exists a partition μ of n such that $\mu > \lambda$ and*

$$m_{\mathfrak{t}^\lambda \mathfrak{g}} = h \;-\; \sum_{\substack{\tau\in\operatorname{Std}(\lambda)\\ \tau \rhd \mathfrak{g}}} m_{\mathfrak{t}^\lambda \tau}$$

where h is a linear combination of terms $m_{\mathfrak{u}\mathfrak{v}}$ such that $\mathfrak{u}$ and $\mathfrak{v}$ are row standard μ–tableaux.

Proof. Write $\lambda = (\lambda_1, \dots, \lambda_k)$ and suppose that $\mathfrak{g} = \mathfrak{g}_{ij}$ is the (i,j)th Garnir tableau. Set $\nu = (\lambda_1, \dots, \lambda_{i-1}, j-1, \lambda_i + 1, \lambda_{i+1} - j, \lambda_{i+2}, \dots, \lambda_k)$, a composition of n, and let $X = \{\, vw \mid v \in \mathfrak{S}_\lambda \text{ and } w \in \mathfrak{S}_\nu \,\}$. Then, by Proposition 3.3,

$$\sum_{w\in\mathfrak{S}_\nu\cap\mathscr{D}_\lambda} m_\lambda T_w = \sum_{x\in X} T_x = \sum_{v\in\mathfrak{S}_\lambda\cap\mathscr{D}_\nu} T_v^* m_\nu$$

(*cf.* Example 3.11). Now $\{\, \mathfrak{t}^\lambda w \mid w \in \mathfrak{S}_\nu \cap \mathscr{D}_\lambda \,\} = \{\mathfrak{g}\} \cup \{\, \tau \in \operatorname{Std}(\lambda) \mid \tau \rhd \mathfrak{g} \,\}$, so rearranging this equation we obtain

$$m_{\mathfrak{t}^\lambda \mathfrak{g}} = \sum_{d(\mathfrak{v})\in\mathfrak{S}_\lambda\cap\mathscr{D}_\nu} m_{\mathfrak{v}\mathfrak{t}^\nu} \;-\; \sum_{\substack{\tau\in\operatorname{Std}(\lambda)\\ \tau \rhd \mathfrak{g}}} m_{\mathfrak{t}^\lambda \tau}.$$

In order to complete the proof we need to show that $h = \sum_{\mathfrak{v}} m_{\mathfrak{v}\mathfrak{t}^\nu}$ has the required form. Let μ be the partition obtained from ν by reordering its parts. Then $\mu > \lambda$ so the result follows from Lemma 3.9. ◻

3.13 Lemma *Suppose that $\mathfrak{t}$ is a row standard λ–tableau which is not standard. Then there exists a Garnir tableau $\mathfrak{g}$ and $w \in \mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{g}w$ and $\ell(d(\mathfrak{t})) = \ell(d(\mathfrak{g})) + \ell(w)$.*

Proof. Because $\mathfrak{t}$ is not standard there exists a node (i,j) in $[\lambda]$ such that $\mathfrak{t}(i,j) > \mathfrak{t}(i+1,j)$. If $\mathfrak{t}$ is the (i,j)th Garnir tableau we may take $w = 1$ and there is nothing to prove; otherwise, there exists an $s \in S$ such that $\mathfrak{t}' = \mathfrak{t}s$ is row standard, $\mathfrak{t}'(i,j) > \mathfrak{t}'(i+1,j)$ and $\mathfrak{t}' \rhd \mathfrak{t}$. Let $\mathfrak{g}$ be the (i,j)th Garnir tableau; by induction we may assume that there exists a permutation $w' \in \mathfrak{S}_n$ such that $\mathfrak{t}' = \mathfrak{g}w'$ and $\ell(d(\mathfrak{t}')) = \ell(d(\mathfrak{g})) + \ell(w')$. Therefore, if we set $w = w's$ then $\mathfrak{t} = \mathfrak{g}w$ and

$$\ell(d(\mathfrak{t})) = \ell(d(\mathfrak{t}')) + 1 = \ell(d(\mathfrak{g})) + \ell(w') + 1 = \ell(d(\mathfrak{g})) + \ell(w)$$

as required. ◻

3.14 Lemma *Let λ be a partition of n and suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are row standard λ–tableaux. Then $m_{\mathfrak{s}\mathfrak{t}}$ can be written as a linear combination of elements $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableau for some partition μ of n such that either* (i) *$\mu > \lambda$, or* (ii) *$\mu = \lambda$ and $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and $\mathfrak{v} \trianglerighteq \mathfrak{t}$.*

Proof. First suppose that $\mathfrak{t}$ is row standard but not standard. By Lemma 3.13, $m_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{s}\mathfrak{g}} T_w$ for some Garnir tableau $\mathfrak{g}$ and some $w \in \mathfrak{S}_n$. Consequently, by Lemma 3.12 there exists an element $h = \sum r_{\mathfrak{u}\mathfrak{v}} m_{\mathfrak{u}\mathfrak{v}}$, where $\mathfrak{u}$ and $\mathfrak{v}$ are row standard μ–tableaux with $\mu > \lambda$, such that

$$m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})}^* m_{\mathfrak{t}^\lambda \mathfrak{g}} T_w = T_{d(\mathfrak{s})}^* \Big(h - \sum_{\tau \rhd \mathfrak{g}} m_{\mathfrak{t}^\lambda \tau} \Big) T_w = T_{d(\mathfrak{s})}^* h T_w - \sum_{\mathfrak{v} \rhd \mathfrak{t}} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}},$$

for some $r_{\mathfrak{v}} \in R$ and where the last equality uses Corollary 3.8. By Corollary 3.4 and induction on λ, $T^*_{d(\mathfrak{s})}hT_w$ can be written in the required form. Furthermore, by induction on $\mathfrak{t}$, the terms $m_{\mathfrak{s}\mathfrak{v}}$ can be written as a linear combination of terms $m_{\mathfrak{u}\mathfrak{v}'}$ where $\mathfrak{u}$ and $\mathfrak{v}'$ are μ–tableaux such that $\mu \trianglerighteq \lambda$ and $\mathfrak{u}$ is row standard and $\mathfrak{v}'$ is standard. Since $m_{\mathfrak{u}\mathfrak{v}'} = m^*_{\mathfrak{v}'\mathfrak{u}}$ we can repeat the argument to replace the left hand tableaux with standard tableaux of the required form, and so deduce the result.

□

We will often need an important special case of Lemma 3.14 which can be gleaned from a careful scrutiny of its proof.

3.15 Corollary *Suppose that* $\mathfrak{t}$ *is a standard* λ*–tableau such that* i *and* $i+1$ *are in the same column of* $\mathfrak{t}$ *for some* i*. Then, for any standard* λ*–tableau* $\mathfrak{s}$*,*

$$m_{\mathfrak{s}\mathfrak{t}}T_i = -m_{\mathfrak{s}\mathfrak{t}} + \sum r_{\mathfrak{u}\mathfrak{v}}m_{\mathfrak{u}\mathfrak{v}}$$

where $r_{\mathfrak{u}\mathfrak{v}} \in R$ *and the sum is over pairs* $(\mathfrak{u},\mathfrak{v})$ *of row standard* μ*–tableau such that either* (i) $\mu > \lambda$*, or* (ii) $\mu = \lambda$*,* $\mathfrak{u} = \mathfrak{s}$ *and* $\mathfrak{v} \rhd \mathfrak{t}$*.*

Proof. By assumption, $\mathfrak{t}s_i$ is not standard so we may write $\mathfrak{t}s_i = \mathfrak{g}w$ for some Garnir tableau $\mathfrak{g}$ and $w \in \mathfrak{S}_n$ by Lemma 3.13. Now, there is a unique standard tableau τ such that $\mathfrak{g} = \tau s$ for some $s \in S$; moreover, $\ell(d(\mathfrak{g})) = \ell(d(\tau)) + 1$ and if $\mathfrak{v}$ is any (standard) tableau with $\mathfrak{v} \rhd \mathfrak{g}$ then $\mathfrak{v} \trianglerighteq \tau$. Consequently, in this case Lemma 3.12 says that

$$m_{\mathfrak{s}\mathfrak{g}} = -m_{\mathfrak{s}\tau} + \sum r_{\mathfrak{u}\mathfrak{v}}m_{\mathfrak{u}\mathfrak{v}}$$

where $r_{\mathfrak{u}\mathfrak{v}} \in R$ and the sum is over pairs $(\mathfrak{u},\mathfrak{v})$ of standard μ–tableau such that either (i) $\mu > \lambda$, or (ii) $\mu = \lambda$, $\mathfrak{u} = \mathfrak{s}$ and $\mathfrak{v} \rhd \tau$. Finally, because $\ell(d(\mathfrak{t})) = \ell(d(\tau)) + \ell(w)$ and $\mathfrak{t} = \tau w$, by multiplying the equation we have obtained for $m_{\mathfrak{s}\mathfrak{g}}$ on the right by T_w and using Corollary 3.8 we obtain the result. □

Let $\mathcal{M} = \{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for some partition } \lambda \text{ of } n \,\}$. Then Lemma 3.9 and Lemma 3.14 show that $\mathcal{M}$ spans $\mathscr{H}$. By Theorem 1.11 we know that $\mathscr{H}$ is a free R–module of rank $n!$. However, $|\mathcal{M}| = \sum_\lambda |\mathrm{Std}(\lambda)|^2$ and it is well–known that this sum is equal to $n!$ (for example, use the Robinson–Schensted correspondence; see [**47**]). Thus, $\mathcal{M}$ is a basis of $\mathscr{H}$. We call $\mathcal{M}$ the Murphy basis of $\mathscr{H}$.

If we let Λ^{lex} be the set of partitions of n ordered lexicographically we have shown that $(\mathcal{M},\Lambda^{lex})$ is a cellular basis of $\mathscr{H}$. In order to prove the stronger result that the set of partitions can be ordered by dominance we need the following important Proposition.

If $h \in \mathscr{H}$ we may write $h = \sum r_{\mathfrak{s}\mathfrak{t}}m_{\mathfrak{s}\mathfrak{t}}$ where $r_{\mathfrak{s}\mathfrak{t}} \in R$ and $m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{M}$. We say that $m_{\mathfrak{s}\mathfrak{t}}$ is involved in h if $r_{\mathfrak{s}\mathfrak{t}} \neq 0$. Recall that $\mathscr{H}_{\mathcal{Z}} = \mathscr{H}_{\mathcal{Z},\hat q}$ where $\mathcal{Z} = \mathbb{Z}[\hat q, \hat q^{-1}]$.

3.16 Proposition *Let* i *be an integer with* $1 \le i < n$ *and suppose that* $h \in M^\mu$ *where* μ *is the composition of* n *such that* $m_\mu = (1+T_i)$*. Suppose that* $h = \sum r_{\mathfrak{s}\mathfrak{t}}m_{\mathfrak{s}\mathfrak{t}}$*, where* $r_{\mathfrak{s}\mathfrak{t}} \in R$ *and* $m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{M}$*. Then* $r_{\mathfrak{s}\mathfrak{t}} = 0$ *whenever* i *and* $i+1$ *belong to the same column of* $\mathfrak{s}$*.*

Proof. First suppose that $\mathscr{H} = \mathscr{H}_{\mathcal{Z}}$ and write $M^\mu_{\mathcal{Z}} = m_\mu\mathscr{H}_{\mathcal{Z}}$. Let λ be minimal in the lexicographic order such that $m_{\mathfrak{u}\mathfrak{v}}$ is involved in h for some λ–tableaux $\mathfrak{u}$ and $\mathfrak{v}$. Choose a pair $(\mathfrak{u},\mathfrak{v})$ of standard λ–tableaux such that (i) $m_{\mathfrak{u}\mathfrak{v}}$ is involved in h, and (ii) whenever $(\mathfrak{s},\mathfrak{t})$ is another pair of standard λ–tableaux such that $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and $\mathfrak{v} \trianglerighteq \mathfrak{t}$ then $m_{\mathfrak{s}\mathfrak{t}}$ is not involved in h.

Suppose that i and $i+1$ are in the same column of $\mathfrak{u}$. Then, by Corollary 3.15 and our choice of $(\mathfrak{u},\mathfrak{v})$,

$$T_i h = (h^*T_i)^* = -r_{\mathfrak{u}\mathfrak{v}}m_{\mathfrak{u}\mathfrak{v}} + \sum a_{\mathfrak{s}\mathfrak{t}}m_{\mathfrak{s}\mathfrak{t}}$$

for some $a_{\mathfrak{s}\mathfrak{t}} \in R$ such that whenever $\mathfrak{u} \trianglerighteq \mathfrak{s}$ and $\mathfrak{v} \trianglerighteq \mathfrak{t}$ then $a_{\mathfrak{s}\mathfrak{t}} = 0$. However, $T_i m_\mu = \hat{q} m_\mu$ by Lemma 3.2, so $T_i h = \hat{q}h$ and this forces $r_{\mathfrak{u}\mathfrak{v}} = 0$, a contradiction. Consequently, i and $i+1$ do not belong to the same column of $\mathfrak{u}$.

Next let $\mathfrak{u}' = \mathfrak{u}(i,i+1)$. If $\mathfrak{u}' \triangleright \mathfrak{u}$ then $T_i m_{\mathfrak{u}\mathfrak{v}} = m_{\mathfrak{u}'\mathfrak{v}}$ so $m_{\mathfrak{u}'\mathfrak{v}}$ is involved in h since $\hat{q}^{-1}T_i h = h$; however, this contradicts our assumption (ii) so, in fact, $\mathfrak{u} \triangleright \mathfrak{u}'$. Therefore, $m_{\mathfrak{u}\mathfrak{v}} + m_{\mathfrak{u}'\mathfrak{v}} = m_\mu m_{\mathfrak{u}\mathfrak{v}}$ is an element of $M^\mu_{\mathcal{Z}}$ and $h' = h - r_{\mathfrak{u}\mathfrak{v}}(m_{\mathfrak{u}\mathfrak{v}} + m_{\mathfrak{u}'\mathfrak{v}})$ is an element of $M^\mu_{\mathcal{Z}}$ which does not involve $m_{\mathfrak{u}\mathfrak{v}}$. Applying the same argument to h' it follows that i and $i+1$ are never in the same column of $\mathfrak{s}$ if $m_{\mathfrak{s}\mathfrak{t}}$ is involved in h. This completes the proof when $\mathscr{H} = \mathscr{H}_{\mathcal{Z}}$.

Now suppose that R is an arbitrary ring. By Corollary 3.4, $\{\, m_\mu T_d \mid d \in \mathscr{D}_\mu \,\}$ is a basis of M^μ. By what we have shown, inside $\mathscr{H}_{\mathcal{Z}}$ we can write each row standard basis element $m_\mu T_d$ as a $\mathcal{Z}$–linear combination of Murphy basis elements of the required form. The specialization homomorphism from $\mathscr{H}_{\mathcal{Z}}$ to $\mathscr{H}$ maps the Murphy basis of $\mathscr{H}_{\mathcal{Z}}$ to the Murphy basis of $\mathscr{H}$. It also maps the row standard basis of $M^\mu_{\mathcal{Z}}$ to that of M^μ. Hence, in $\mathscr{H}$ each basis element $m_\mu T_d$ can be written as an R–linear combination of Murphy basis elements of the required form. □

As in the last chapter, we define $\mathscr{H}^\lambda$ to be the R–module with basis

$$\left\{\, m_{\mathfrak{u}\mathfrak{v}} \;\middle|\; \begin{array}{c}\mathfrak{u},\mathfrak{v} \in \operatorname{Std}(\mu) \text{ for some partition}\\ \mu \text{ of } n \text{ such that } \mu \trianglerighteq \lambda\end{array} \right\}.$$

Similarly, we define $\check{\mathscr{H}}^\lambda$ to be the R–module with basis $m_{\mathfrak{s}\mathfrak{t}}$ where $\mathfrak{s}$ and $\mathfrak{t}$ are standard μ–tableaux and $\mu \triangleright \lambda$.

3.17 Corollary *Suppose that λ is a partition. Then $\check{\mathscr{H}}^\lambda$ and $\mathscr{H}^\lambda$ are two–sided ideals of $\mathscr{H}$.*

Proof. Let $\mathfrak{s}$ and $\mathfrak{t}$ be standard μ–tableaux and $w \in \mathfrak{S}_n$. Because $\check{\mathscr{H}}^\lambda = \sum_{\mu \triangleright \lambda} \mathscr{H}^\mu$, it is enough to show that $m_{\mathfrak{s}\mathfrak{t}}T_w \in \mathscr{H}^\lambda$. We prove this by induction on λ, the case $\lambda = (n)$ being a consequence of Lemma 3.2.

First note that by Lemma 3.14 there exist $r_{\mathfrak{u}\mathfrak{v}} \in R$ such that $m_{\mathfrak{t}^\lambda\mathfrak{t}}T_w = \sum r_{\mathfrak{u}\mathfrak{v}}m_{\mathfrak{u}\mathfrak{v}}$ where the sum is over pairs $(\mathfrak{u},\mathfrak{v})$ of standard μ–tableaux with $\mu \geq \lambda$. Suppose that $r_{\mathfrak{u}\mathfrak{v}} \neq 0$ for some μ–tableaux $\mathfrak{u}$ and $\mathfrak{v}$. By Proposition 3.16, if i and $i+1$ are in the same row of $\mathfrak{t}^\lambda$ then they cannot be in the same column of $\mathfrak{u}$. It follows that $1, 2 \dots, \lambda_1$ must all be in the first row of $\mathfrak{u}$; so $\mu_1 \geq \lambda_1$. Consideration of the second row shows that $\mu_1 + \mu_2 \geq \lambda_1 + \lambda_2$, and so on. Hence, $\mu \trianglerighteq \lambda$ whenever $r_{\mathfrak{u}\mathfrak{v}} \neq 0$.

Because $m_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})}m_{\mathfrak{t}^\lambda\mathfrak{t}}$, the general case follows by induction on λ using the left handed version of Lemma 3.14, together with Corollary 3.4. □

We can now prove the main result of this section. Let Λ be the set of partitions of n ordered by dominance. Recall that $* : \mathscr{H} \longrightarrow \mathscr{H}$ is the unique R–linear anti–isomorphism of $\mathscr{H}$ such that $T^*_w = T_{w^{-1}}$ for all $w \in \mathfrak{S}_n$.

3.18 The Murphy Basis Theorem (Murphy) *The Hecke algebra $\mathscr{H}$ is a free R–module with basis*

$$\mathcal{M} = \Big\{\, m_{\mathfrak{s}\mathfrak{t}} \;\Big|\; \mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\lambda) \text{ for some partition } \lambda \text{ of } n \,\Big\}.$$

Moreover,

(i) $m_{\mathfrak{s}\mathfrak{t}}^* = m_{\mathfrak{t}\mathfrak{s}}$ *for all* $\lambda \in \Lambda$ *and all* $\mathfrak{s},\mathfrak{t} \in \operatorname{Std}(\lambda)$.

(ii) *Suppose that* $h \in \mathscr{H}$ *and that* $\mathfrak{t} \in \operatorname{Std}(\lambda)$. *Then there exist* $r_{\mathfrak{v}} \in R$ *such that for all* $\mathfrak{s} \in \operatorname{Std}(\lambda)$

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \operatorname{Std}(\lambda)} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \mod \check{\mathscr{H}}^\lambda.$$

Consequently, $(\mathcal{M},\Lambda)$ *is a cellular basis of* $\mathscr{H}$.

Proof. We have already seen that $\mathcal{M}$ is a basis of $\mathscr{H}$ and, from the definitions, it is immediate that $m_{\mathfrak{s}\mathfrak{t}}^* = m_{\mathfrak{t}\mathfrak{s}}$. Hence, it remains to prove (ii).

Let N_λ be the R–module spanned by $\check{\mathscr{H}}^\lambda$ and $\{\, m_{\mathfrak{t}^\lambda\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\lambda)\,\}$. The argument of Corollary 3.17 shows that N_λ is a right ideal of $\mathscr{H}$. Therefore, for any standard tableau $\mathfrak{t} \in \operatorname{Std}(\lambda)$ and $h \in \mathscr{H}$ there exists $r_{\mathfrak{v}} \in R$ such that

$$m_{\mathfrak{t}^\lambda\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \operatorname{Std}(\lambda)} r_{\mathfrak{v}} m_{\mathfrak{t}^\lambda\mathfrak{v}} \mod \check{\mathscr{H}}^\lambda.$$

Multiplying this expression on the left by $T_{d(\mathfrak{s})}^*$ proves (ii) and hence the Theorem. □

For future use, we note the following strengthening of Corollary 3.15.

3.19 Corollary *Suppose* $\mathfrak{s}$ *and* $\mathfrak{t}$ *are standard* λ*–tableau such that* i *and* $i+1$ *are in the same column of* $\mathfrak{t}$ *for some* i. *Then, for all* $\mathfrak{s} \in \operatorname{Std}(\lambda)$,

$$m_{\mathfrak{s}\mathfrak{t}} T_i \equiv -m_{\mathfrak{s}\mathfrak{t}} + \sum_{v \rhd \mathfrak{t}} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \mod \check{A}^\lambda.$$

for some $r_{\mathfrak{v}} \in R$.

We can now use the theory of cellular algebras from Chapter 2 to construct all of the irreducible $\mathscr{H}$–modules. The cell modules of $\mathscr{H}$ are called Specht modules.

Let λ be a partition of n. Rather than the abstract definition of Chapter 2 we define the Specht module S^λ to be the right $\mathscr{H}$–module $(\check{\mathscr{H}}^\lambda + m_\lambda)\mathscr{H}$. Note that $\check{\mathscr{H}}^\lambda + m_\lambda$ is an element of the $\mathscr{H}$–module $\mathscr{H}/\check{\mathscr{H}}^\lambda$ so that S^λ is a submodule of $\mathscr{H}/\check{\mathscr{H}}^\lambda$.

Warning The Specht module S^λ, as we have defined it, is isomorphic to the dual of the Specht module which Dipper–James [**13**] index by λ', the partition which is conjugate to λ. Because of this, when comparing the results in these notes with those from the representation theory of the symmetric groups ($q=1$), or with the papers of Dipper and James, the Specht module S^λ must be replaced with $S^{\lambda'*}$. In the literature our Specht modules are sometimes called *dual* Specht modules.

Given a standard λ–tableau $\mathfrak{t}$ let $m_{\mathfrak{t}} = \check{\mathscr{H}}^\lambda + m_{\mathfrak{t}^\lambda\mathfrak{t}}$. Then Theorem 3.18 gives us the following basis of S^λ (and shows that S^λ is isomorphic to the cell module as defined in Chapter 2).

3.20 Proposition *Suppose that* λ *is a partition of* n. *Then the Specht module* S^λ *is a free* R*–module with basis* $\{\, m_{\mathfrak{t}} \mid \mathfrak{t} \in \operatorname{Std}(\lambda)\,\}$.

We call the basis of Proposition 3.20 the Murphy basis of S^λ. Translating the notation from (2.6), the bilinear form $\langle\ ,\ \rangle$ on S^λ is the unique R–bilinear map from $S^\lambda \times S^\lambda$ into R such that

$$\langle m_{\mathfrak{s}}, m_{\mathfrak{t}}\rangle m_\lambda \equiv m_{\mathfrak{t}^\lambda\mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^\lambda} \mod H^\lambda.$$

for all standard λ–tableaux $\mathfrak{s}$ and $\mathfrak{t}$. By Proposition 2.7, this form is associative and symmetric so $\operatorname{rad} S^\lambda = \{\, x \in S^\lambda \mid \langle x, y\rangle = 0 \text{ for all } y \in S^\lambda \,\}$ is an $\mathscr{H}$-submodule of S^λ. As in (2.8), for each partition λ of n we define D^λ to be the right $\mathscr{H}$–module $S^\lambda / \operatorname{rad} S^\lambda$. Thus by Theorem 2.13 we have the following result.

3.21 Theorem (Dipper–James) *Suppose that R is a field. Then*

$$\{\, D^\lambda \mid \lambda \textit{ a partition of } n \textit{ such that } D^\lambda \neq 0 \,\}$$

is a complete set of non–isomorphic irreducible $\mathscr{H}$–modules.

We will classify the partitions λ such that $D^\lambda \neq 0$ in Section 4. For now we note that by Corollary 2.18 and Corollary 1.15 we have the following.

Recall that $\mathscr{H}_{\mathcal{K}} = \mathscr{H}_{\mathbb{C}(\hat{q}),\hat{q}}(\mathfrak{S}_n)$.

3.22 Corollary (Hoefsmit) *The Iwahori–Hecke algebra $\mathscr{H}_{\mathcal{K}}$ is a split semisimple algebra and $\{\, S^\lambda \mid \lambda$ a partition of $n \,\}$ is a complete set of non–isomorphic irreducible $\mathscr{H}_{\mathcal{K}}$–modules.*

In particular, this shows that $\mathscr{H}_{\mathcal{K}}$ and $\mathcal{K}\mathfrak{S}_n$ are isomorphic as $\mathcal{K}$–algebras.

3. Specht modules and Jucys–Murphy elements

In this section we study the action of the following operators upon the Specht modules and use them to construct an orthogonal basis for S^λ. For $k = 1, 2, \dots, n$ let

$$L_k = q^{-1}T_{(k-1,k)} + q^{-2}L_{(k-2,k)} + \dots + q^{1-k}T_{(1,k)}.$$

The elements $L_1, \dots, L_n$ are the Jucys–Murphy elements of $\mathscr{H}$.

3.23 Example If $n = 4$ then the four Jucys–Murphy elements are $L_1 = 0$, $L_2 = q^{-1}T_{(1,2)}$, $L_3 = q^{-1}T_{(2,3)} + q^{-2}T_{(1,3)}$, and $L_4 = q^{-1}T_{(3,4)} + q^{-2}T_{(2,4)} + q^{-3}T_{(1,4)}$.

3.24 Proposition *Let i and k be integers with $1 \le i < n$ and $1 \le k \le n$.*

(i) *T_i and L_k commute if $i \neq k-1, k$.*
(ii) *$T_iL_{i+1} = (q-1)L_{i+1} + 1 + L_iT_i$ and $T_iL_i = T_iL_{i+1} - 1 - (q-1)L_{i+1}$.*
(iii) *T_i commutes with L_iL_{i+1} and $L_i + L_{i+1}$.*
(iv) *L_i and L_k commute.*

Proof. The proofs of all of these statements follow easily using the relations in $\mathscr{H}$, so we prove only (i). Because $L_1 = 0$ there is nothing to prove when $k = 1$, so assume that $k > 1$. Then, by definition, $L_k = q^{-1}(T_k + T_kL_{k-1}T_k)$, so if $|i-k| > 1$ then $T_iL_k = q^{-1}(T_iT_k + T_iT_kL_{k-1}T_k) = q^{-1}(T_kT_i + T_kT_iL_{k-1}T_k)$; so $T_iL_k = L_kT_i$ by induction. Finally, it remains to consider the case when $i = k+1$; however, this is easy because T_{k+1} commutes with $T_{(j,k)}$ for $j = 1, \dots, k$. □

3.25 Corollary *Any symmetric polynomial in $L_1, \dots, L_n$ belongs to the centre of $\mathscr{H}$.*

The Corollary indicates why the Jucys–Murphy elements are so useful. The idea is that $L_1, \dots, L_n$ generate a large abelian subalgebra of $\mathscr{H}$ — which is known to be maximal when $\mathscr{H}$ is semisimple — and consequently these elements can be used to distinguish between the blocks and irreducible representations of $\mathscr{H}$, much like the Cartan subalgebra of a Lie algebra.

In the Lemma below we write S^λ_R to indicate that the Specht module is defined over R. Recall that $\mathcal{K} = \mathbb{C}(\hat{q})$ and $\mathcal{Z} = \mathbb{Z}[\hat{q}, \hat{q}^{-1}]$.

3.26 Lemma *Let λ be a partition of n. Then there exists $\alpha \in R$ such that*

$$m_\mathfrak{t}(L_1 + \cdots + L_n) = \alpha m_\mathfrak{t}$$

for all standard λ–tableau $\mathfrak{t}$.

Proof. First suppose that $\mathscr{H} = \mathscr{H}_\mathcal{K}$. By Corollary 3.25, the symmetric polynomial $L_1 + \cdots + L_n$ is central in $\mathscr{H}$ and $S^\lambda_\mathcal{K}$ is an irreducible $\mathscr{H}_\mathcal{K}$–module by Corollary 3.22. Therefore, by Schur's Lemma $L_1 + \cdots + L_n$ acts as a scalar, say $\hat{\alpha}$, on $S^\lambda_\mathcal{K}$. In particular, if $\mathfrak{t}$ is a standard λ–tableau then $m_\mathfrak{t}(L_1 + \cdots + L_n) = \hat{\alpha} m_\mathfrak{t}$. By Corollary 3.4 the left hand side of this equation is contained in $\mathscr{H}_\mathcal{Z}$; consequently, $\hat{\alpha} \in \mathcal{Z}$.

Now suppose that R is an arbitrary ring. Then, as in Corollary 1.12, we can consider R as a $\mathcal{Z}$–module by specifying that $\hat{q}$ acts as multiplication by q on R. Then $S^\lambda_R \cong S^\lambda_\mathcal{Z} \otimes R$ as R–modules and the Murphy basis of $S^\lambda_\mathcal{Z}$ is mapped to the Murphy basis of S^λ_R by the specialization homomorphism. It follows that if $\hat{\alpha}$ acts on R as multiplication by $\alpha \in R$ then $m_\mathfrak{t}(L_1 + \ldots + L_n) = \alpha m_\mathfrak{t}$ for all $\mathfrak{t} \in \mathrm{Std}(\lambda)$. □

The precise value of α can be computed from Theorem 3.30 below; the next Lemma is the inductive step in proving this result.

3.27 Lemma *Let λ be a partition of n and let $\mathfrak{s}$ and $\mathfrak{t}$ be standard λ–tableaux such that $\mathfrak{s} \rhd \mathfrak{t}$ and $\mathfrak{t} = \mathfrak{s}(i, i+1)$ for some i with $1 \le i < n$. For $k = 1, 2, \ldots, n$, suppose that there exist $r_\mathfrak{s}(k) \in R$ such that*

$$m_\mathfrak{s} L_k = r_\mathfrak{s}(k) m_\mathfrak{s} + \sum_{\substack{\mathfrak{v} \in \mathrm{Std}(\lambda) \\ \mathfrak{v} \rhd \mathfrak{s}}} a_\mathfrak{v} m_\mathfrak{v}$$

for some $a_\mathfrak{v} \in R$. Then, for $k = 1, 2, \ldots, n$, there exist $b_\mathfrak{v} \in R$ such that

$$m_\mathfrak{t} L_k = r_\mathfrak{t}(k) m_\mathfrak{t} + \sum_{\substack{\mathfrak{v} \in \mathrm{Std}(\lambda) \\ \mathfrak{v} \rhd \mathfrak{t}}} b_\mathfrak{v} m_\mathfrak{v}$$

where $r_\mathfrak{t}(k) = r_\mathfrak{s}(k)$ if $k \ne i, i+1$, $r_\mathfrak{t}(i) = r_\mathfrak{s}(i+1)$ and $r_\mathfrak{t}(i+1) = r_\mathfrak{s}(i)$.

Proof. First we note that $m_\mathfrak{t} = m_\mathfrak{s} T_i$ by Theorem 1.11 because $\mathfrak{s} \rhd \mathfrak{t}$. Therefore, by Proposition 3.24(i), if $k \ne i, i+1$ then

$$m_\mathfrak{t} L_k = m_\mathfrak{s} T_i L_k = m_\mathfrak{s} L_k T_i = r_\mathfrak{s}(k) m_\mathfrak{s} T_i + \sum_{\substack{\mathfrak{v} \in \mathrm{Std}(\lambda) \\ \mathfrak{v} \rhd \mathfrak{s}}} a_\mathfrak{v} m_\mathfrak{v} T_i,$$

and the result follows by Corollary 3.8. If $k = i+1$ then, using Proposition 3.24(ii) and Corollary 3.8,

$$\begin{aligned} m_\mathfrak{t} L_{i+1} &= m_\mathfrak{s} T_i L_{i+1} = m_\mathfrak{s}\big\{L_i T_i + 1 + (q-1) L_{i+1}\big\} \\ &= r_\mathfrak{s}(i) m_\mathfrak{s} T_i + \big\{1 + (q-1) r_\mathfrak{s}(i+1)\big\} m_\mathfrak{s} + \sum_{\mathfrak{v} \rhd \mathfrak{s}} b_\mathfrak{v} m_\mathfrak{v} \\ &= r_\mathfrak{s}(i) m_\mathfrak{t} + \sum_{\mathfrak{v} \rhd \mathfrak{t}} b_\mathfrak{v} m_\mathfrak{v}. \end{aligned}$$

The case $k = i$ is similar. $\square$

Given an integer m let $[m]_q = 1 + q + \cdots + q^{m-1}$ if $m \geq 0$ and $[-m]_q = -q^{-m}[m]_q$ otherwise; then $[m]_q = \frac{q^m-1}{q-1}$ when $q \neq 1$. If $m \geq 0$ we also define $[m]_q^! = [1]_q[2]_q \dots [m]_q$. These are the so–called quantum integers and quantum factorials respectively; this is because $[m]_1 = m$ and $[m]_1^! = m!$ for all $m \in \mathbb{Z}$.

3.28 Define e to be the smallest positive integer such that $[e]_q = 0$; set $e = \infty$ if no such integer exists. In other words, either $q = 1$ and e is equal to the characteristic of R, or q is a primitive eth root of unity. As we shall see, the integer e behaves very much like the characteristic of the field in the representation theory of finite groups.

Suppose that $x = (i, j)$ is a node in $[\lambda]$. The e–residue of x is the integer $\operatorname{res}(x) = j - i \mod e$. If $\mathfrak{t}$ is a λ–tableau and k an integer with $1 \leq k \leq n$ then the e–residue of k in $\mathfrak{t}$ is $\operatorname{res}_\mathfrak{t}(k) = \operatorname{res}(x)$ where x is the unique node in $[\lambda]$ such that $\mathfrak{t}(x) = k$.

3.29 Example Suppose that $e = 3$, $\lambda = (4, 2)$ and $\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5\\\cline{1-2}\end{array}$. Then the e–residues in $[\lambda]$ are $\begin{array}{|c|c|c|c|}\hline 0&1&2&0\\\hline 2&0\\\cline{1-2}\end{array}$. So, for example, $[\operatorname{res}_\mathfrak{t}(3)]_q = [1]_q$ and $[\operatorname{res}_\mathfrak{t}(4)]_q = [4]_q$.

3.30 Theorem (Dipper–James) *Suppose that λ is a partition of n and let $\mathfrak{t}$ be a standard λ–tableau. Let k be any integer such that $1 \leq k \leq n$. Then there exist $a_\mathfrak{v} \in R$ such that*

$$m_\mathfrak{t} L_k = [\operatorname{res}_\mathfrak{t}(k)]_q m_\mathfrak{t} + \sum_{\substack{\mathfrak{v} \in \operatorname{Std}(\lambda)\\ \mathfrak{v} \rhd \mathfrak{t}}} a_\mathfrak{v} m_\mathfrak{v}.$$

Proof. By Lemma 3.27 it is sufficient to show that $m_{\mathfrak{t}^\lambda} L_k = [\operatorname{res}_{\mathfrak{t}^\lambda}(k)]_q m_{\mathfrak{t}^\lambda}$ for $k = 1, 2, \dots, n$. When $n = 1$ there is nothing to prove so by induction we can assume that the Theorem is known for smaller values of n. We now proceed by induction on k, the case $k = 1$ being trivial because $L_1 = 0$ and $[\operatorname{res}_\mathfrak{t}(1)]_q = [0]_q = 0$.

Suppose first that k is in row r of $\mathfrak{t}^\lambda$ where $\lambda_{r+1} \neq 0$. Let $\mu = (\lambda_1, \dots, \lambda_r)$. Then $|\mu| < n$ and $m_{\mathfrak{t}^\lambda} = \mathscr{H}^\lambda + m_\mu h \equiv m_\mu h$, where h is some element of the subalgebra of $\mathscr{H}$ generated by $T_{k+1}, \dots, T_{n-1}$. Consequently, $m_\mu L_k \in \mathscr{H}(\mathfrak{S}_{|\mu|})$ and h and L_k commute by Proposition 3.24(i). Hence, working modulo $\mathscr{H}^\lambda$ and arguing by induction on n,

$$m_{\mathfrak{t}^\lambda} L_k \equiv m_\mu h L_k = m_\mu L_k h \equiv [\operatorname{res}_{\mathfrak{t}^\mu}(k)]_q m_\mu h \equiv [\operatorname{res}_{\mathfrak{t}^\lambda}(k)]_q m_{\mathfrak{t}^\lambda}.$$

Thus we may assume that k is in the last row of $\mathfrak{t}^\lambda$.

Next, suppose that k is not in the first column of $\mathfrak{t}^\lambda$i and that we know the result for smaller k. Now s_{k-1} is an element of $\mathfrak{S}_\lambda$ so $m_\lambda T_{k-1} = q m_\lambda$ by Lemma 3.2; so,

$$\begin{aligned} m_{\mathfrak{t}^\lambda} L_k &= q^{-1} m_{\mathfrak{t}^\lambda}(T_{k-1} L_{k-1} T_{k-1} + T_{k-1}) = m_{\mathfrak{t}^\lambda}(1 + L_{k-1} T_{k-1}) \\ &= (1 + q[\operatorname{res}_{\mathfrak{t}^\lambda}(k-1)]_q) m_{\mathfrak{t}^\lambda} = [\operatorname{res}_{\mathfrak{t}^\lambda}(k)]_q m_{\mathfrak{t}^\lambda} \end{aligned}$$

by induction on k.

This reduces us to the case where k is the number in the last row and the first column of $\mathfrak{t}^\lambda$. Write $\lambda = (\lambda_1, \dots, \lambda_{r+1})$ where $\lambda_{r+1} > 0$. Then $m_\lambda = h m_{(\lambda_1, \dots, \lambda_r, 1)}$ for some $h \in \mathscr{H}(\mathfrak{S}_n)$ and $m_{(\lambda_1, \dots, \lambda_r, 1)} \in \mathscr{H}(\mathfrak{S}_k)$. Therefore, if $k < n$ we can again deduce the result by induction. Finally then, it remains then to consider the case where $k = n$ and $\lambda = (\lambda_1, \dots \lambda_r, 1)$.

Let i be the first number in row r of $\mathfrak{t}^\lambda$ and let $\mathfrak{t}$ be the standard λ–tableau $\mathfrak{t}^\lambda s_{n-1}\dots s_{i+1}$. Then the second last row of $\mathfrak{t}$ contains the numbers $i, i+2, \dots, n$ and the last row of $\mathfrak{t}$ consists of the single entry $i+1$.

So far we have shown that $m_{\mathfrak{t}^\lambda} L_k = [\operatorname{res}_{\mathfrak{t}^\lambda}(k)]_q m_{\mathfrak{t}^\lambda}$ for $1 \le k < n$. We also know that $m_{\mathfrak{t}^\lambda}(L_1+\dots+L_n) = \alpha m_{\mathfrak{t}^\lambda}$ for some $\alpha \in R$ by Lemma 3.26; consequently, $m_{\mathfrak{t}^\lambda} L_n = r_n m_{\mathfrak{t}^\lambda}$ for some $r_n \in R$. Therefore, $m_{\mathfrak{t}} L_i = [\operatorname{res}_{\mathfrak{t}^\lambda}(i)]_q m_{\mathfrak{t}} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} a_{\mathfrak{s}} m_{\mathfrak{s}}$ and $m_{\mathfrak{t}} L_{i+1} = r_n m_{\mathfrak{t}} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} b_{\mathfrak{s}} m_{\mathfrak{s}}$ by Lemma 3.27.

The key point now is that $\mathfrak{t}s_i$ is a Garnir tableau; because of this we can find elements $c_{\mathfrak{s}}$ in R such that $m_{\mathfrak{t}} T_i = -m_{\mathfrak{t}} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} c_{\mathfrak{s}} m_{\mathfrak{s}}$ by Corollary 3.19. This allows us to compute $m_{\mathfrak{t}} L_{i+1}$ in another way; namely, by Corollary 3.8 and Lemma 3.12 there exist $a'_{\mathfrak{s}}$ and $d_{\mathfrak{s}} \in R$, respectively, such that

$$\begin{aligned}
m_{\mathfrak{t}} L_{i+1} &= q^{-1} m_{\mathfrak{t}}(T_i L_i T_i + T_i)\\
&= -q^{-1}\Big\{ m_{\mathfrak{t}} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} c_{\mathfrak{s}} m_{\mathfrak{s}} \Big\}(L_i T_i + 1)\\
&= -q^{-1} m_{\mathfrak{t}}([\operatorname{res}_{\mathfrak{t}^\lambda}(i)]_q T_i + 1) + \sum_{\mathfrak{s}\rhd\mathfrak{t}} a'_{\mathfrak{s}} m_{\mathfrak{s}} T_i\\
&= q^{-1}([\operatorname{res}_{\mathfrak{t}^\lambda}(i)]_q - 1) m_{\mathfrak{t}} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} d_{\mathfrak{s}} m_{\mathfrak{s}}\\
&= [\operatorname{res}_{\mathfrak{t}^\lambda}(n)]_q m_{\mathfrak{t}} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} d_{\mathfrak{s}} m_{\mathfrak{s}}.
\end{aligned}$$

Consequently, $r_n = [\operatorname{res}_{\mathfrak{t}}(n)]_q$ as required. □

3.31 For the remainder of this section we assume that R is a field and that $e > n$ (see (3.28)). Let $\mathfrak{t}$ be a standard λ–tableau and define

$$F_{\mathfrak{t}} = \prod_{k=1}^{n} \prod_{\substack{\mathfrak{s}\in\operatorname{Std}(\lambda)\\ [\operatorname{res}_{\mathfrak{s}}(k)]_q \ne [\operatorname{res}_{\mathfrak{t}}(k)]_q}} \frac{L_k - [\operatorname{res}_{\mathfrak{s}}(k)]_q}{[\operatorname{res}_{\mathfrak{t}}(k)]_q - [\operatorname{res}_{\mathfrak{s}}(k)]_q}$$

and let $f_{\mathfrak{t}} = m_{\mathfrak{t}} F_{\mathfrak{t}}$.

Note that all of the factors in $F_{\mathfrak{t}}$ commute so there is no need to specify an ordering in the product. In practice many of the factors in the definition of $f_{\mathfrak{t}}$ are redundant; however, the above definition is convenient. The elements $F_{\mathfrak{t}}$ are defined whether or not $e < n$; it is the next Lemma where we need this assumption.

If λ is a partition then a node $x \in [\lambda]$ is removable if $[\lambda]\setminus\{x\}$ is the diagram of a partition.

3.32 Lemma *Suppose that R is a field, $e > n$ and let λ be a partition of n. Let $\mathfrak{s}$ and $\mathfrak{t}$ be standard λ–tableaux.*

(i) *Suppose that $[\operatorname{res}_{\mathfrak{s}}(k)]_q = [\operatorname{res}_{\mathfrak{t}}(k)]_q$ for $k = 1, \dots, n$. Then $\mathfrak{s} = \mathfrak{t}$.*

(ii) *Suppose that $[\operatorname{res}_{\mathfrak{s}}(k)]_q = [\operatorname{res}_{\mathfrak{t}}(k)]_q$ for $1 \le k \le n$ and $k \ne i, i+1$ for some i. Then either $\mathfrak{s} = \mathfrak{t}$ or $\mathfrak{s} = \mathfrak{t}(i, i+1)$.*

Proof. Note that because $e > n$ the nodes with the same e–residue occur in diagonal lines in the diagram of λ; consequently, distinct removable nodes have distinct e–residues. Therefore, by successively removing $n, n-1, \dots, 1$ from $\mathfrak{s}$ we see that $\mathfrak{t}$ is completely determined by the sequence $[\operatorname{res}_{\mathfrak{t}}(1)]_q, \dots, [\operatorname{res}_{\mathfrak{t}}(n)]_q$. Both parts of the Lemma now follow. □

3.33 Proposition *Suppose that R is a field and that $e > n$. Suppose that λ is a partition of n and let $\mathfrak{t}$ be a standard λ–tableau.*

(i) *There exist $a_\mathfrak{s} \in R$ such that $f_\mathfrak{t} = m_\mathfrak{t} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} a_\mathfrak{s} m_\mathfrak{s}$.*

(ii) *If $\mathfrak{s} \rhd \mathfrak{t}$ then $m_\mathfrak{s} F_\mathfrak{t} = 0$.*

(iii) *For $k = 1, \dots, n$, $f_\mathfrak{t} L_k = [\mathrm{res}_\mathfrak{t}(k)]_q f_\mathfrak{t}$.*

(iv) *If $\mathfrak{s} \in \mathrm{Std}(\lambda)$ then $f_\mathfrak{t} F_\mathfrak{s} = \delta_{\mathfrak{s}\mathfrak{t}} f_\mathfrak{t}$.*

Proof. Let $N = |\,\mathrm{Std}(\lambda)|$ and for any $\mathfrak{s} \in \mathrm{Std}(\lambda)$ let $f'_\mathfrak{s} = m_\mathfrak{s} F_\mathfrak{s}^N$. We will first prove analogues of (i)—(iv) for $f'_\mathfrak{s}$ and then show that $f_\mathfrak{s} = f'_\mathfrak{s}$.

We know from Theorem 3.30 that $m_\mathfrak{t} L_k = [\mathrm{res}_\mathfrak{t}(k)]_q m_\mathfrak{t} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} b_\mathfrak{s} m_\mathfrak{s}$, for some $b_\mathfrak{s} \in R$. Therefore, $f'_\mathfrak{t} = m_\mathfrak{t} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} a_\mathfrak{s} m_\mathfrak{s}$ for some $a_\mathfrak{s} \in R$. Call this statement (i)$'$.

Next suppose that $\mathfrak{s}$ is a standard λ–tableau such that $\mathfrak{s} \rhd \mathfrak{t}$. By Lemma 3.32(i) we can find an integer k such that $[\mathrm{res}_\mathfrak{t}(k)]_q \neq [\mathrm{res}_\mathfrak{s}(k)]_q$. Then $\big(L_k - [\mathrm{res}_\mathfrak{s}(k)]_q\big)$ is a factor of $F_\mathfrak{t}$ so $m_\mathfrak{s} F_\mathfrak{t} = \sum_{\mathfrak{v}\rhd\mathfrak{s}} c_\mathfrak{v} m_\mathfrak{v}$ for some $c_\mathfrak{v} \in R$ by Theorem 3.30. Now if $c_\mathfrak{v} \neq 0$ then $\mathfrak{v} \rhd \mathfrak{t}$ and, as before, we can find k' such that $[\mathrm{res}_\mathfrak{t}(k')]_q \neq [\mathrm{res}_\mathfrak{v}(k')]_q$; therefore, $m_\mathfrak{s} F_\mathfrak{t}^2 = \sum_{\mathfrak{v}'} c'_{\mathfrak{v}'} m'_\mathfrak{v}$ where $c'_{\mathfrak{v}'} \neq 0$ only if there exists $\mathfrak{v}$ with $\mathfrak{v}' \rhd \mathfrak{v} \rhd \mathfrak{t}$. Continuing in this way shows that $m_\mathfrak{s} F_\mathfrak{t}^N = 0$ whenever $\mathfrak{s} \rhd \mathfrak{t}$.

Next we claim that (iii)$'$ holds; that is, $f'_\mathfrak{t} L_k = [\mathrm{res}_\mathfrak{t}(k)]_q f'_\mathfrak{t}$ for all k with $1 \leq k \leq n$. This now follows because by the last paragraph and Theorem 3.30

$$f'_\mathfrak{t} L_k = m_\mathfrak{t} F_\mathfrak{t}^N L_k = m_\mathfrak{t} L_k F_\mathfrak{t}^N = \big([\mathrm{res}_\mathfrak{t}(k)]_q m_\mathfrak{t} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} b_\mathfrak{s} m_\mathfrak{s}\big) F_\mathfrak{t}^N = [\mathrm{res}_\mathfrak{t}(k)]_q f'_\mathfrak{t}.$$

Using (iii)$'$ and the definitions we discover that $f'_\mathfrak{t}$ is fixed by every factor of $F_\mathfrak{t}$; thus, $f'_\mathfrak{t} F_\mathfrak{t} = f'_\mathfrak{t}$. Furthermore, if $\mathfrak{s} \neq \mathfrak{t}$ then by Lemma 3.32(i) there exists an integer k such that $[\mathrm{res}_\mathfrak{s}(k)]_q \neq [\mathrm{res}_\mathfrak{t}(k)]_q$. However, (iii)$'$ shows that $\big(L_k - \mathrm{res}_\mathfrak{t}(k)\big)$ kills $f'_\mathfrak{t}$; so $f'_\mathfrak{t} F_\mathfrak{s} = 0$ because $\big(L_k - \mathrm{res}_\mathfrak{t}(k)\big)$ is a factor of $F_\mathfrak{s}$. Therefore, $f'_\mathfrak{t} F_\mathfrak{s} = \delta_{\mathfrak{s}\mathfrak{t}} f'_\mathfrak{t}$; call this statement (iv)$'$.

Finally we can show that $f_\mathfrak{t} = f'_\mathfrak{t}$. Suppose that $\mathfrak{s}$ is a standard λ–tableau such that $\mathfrak{s} \unrhd \mathfrak{t}$; then by (i)$'$ there exist $d_v \in R$ such that $m_\mathfrak{s} = f'_\mathfrak{s} + \sum_{\mathfrak{v}\rhd\mathfrak{s}} d_\mathfrak{v} f'_\mathfrak{v}$. Therefore, using (iv)$'$,

$$m_\mathfrak{s} F_\mathfrak{t} = \big(f'_\mathfrak{s} + \sum_{\mathfrak{v}\rhd\mathfrak{s}} d_\mathfrak{v} f'_\mathfrak{v}\big) F_\mathfrak{t} = \delta_{\mathfrak{s}\mathfrak{t}} f'_\mathfrak{t}.$$

In particular, $f_\mathfrak{t} = m_\mathfrak{t} F_\mathfrak{t} = f'_\mathfrak{t}$ and we have proved all that we set out to prove (note that the previous line includes (ii)). □

We can now prove the q–analogue of Young's seminormal form for $\mathscr{H}$.

3.34 Theorem (Dipper–James) *Suppose that R is a field and that $e > n$. Let λ be a partition of n.*

(i) *$\{\, f_\mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\,\}$ is an orthogonal basis of S^λ.*

(ii) *Let $\mathfrak{s}$ be a standard λ–tableau, $\mathfrak{t} = \mathfrak{s}(i, i+1)$ and $\rho = \mathrm{res}_\mathfrak{s}(i) - \mathrm{res}_\mathfrak{t}(i)$. Then*

$$f_\mathfrak{s} T_i = \begin{cases} q f_\mathfrak{s}, & \text{if } i \text{ and } i+1 \text{ are in the same row of } \mathfrak{s},\\ -f_\mathfrak{s}, & \text{if } i \text{ and } i+1 \text{ are in the same column of } \mathfrak{s}.\\ -\frac{1}{[\rho]_q} f_\mathfrak{s} + f_\mathfrak{t}, & \text{if } \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ and } \mathfrak{s} \rhd \mathfrak{t},\\ \frac{q^\rho}{[\rho]_q} f_\mathfrak{s} + \frac{[\rho+1]_q[\rho-1]_q}{[\rho]_q^2} f_\mathfrak{t}, & \text{if } \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ and } \mathfrak{t} \rhd \mathfrak{s}. \end{cases}$$

Proof. (i) By Proposition 3.33(i) the transition matrix between the standard basis of S^λ and $\{\, f_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\,\}$ is unitriangular so this set is a basis of S^λ. Moreover, if $\mathfrak{s}$ and t are standard λ–tableaux then

$$\langle f_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle = \langle f_{\mathfrak{s}}F_{\mathfrak{s}}, f_{\mathfrak{t}}\rangle = \langle f_{\mathfrak{s}}, f_{\mathfrak{t}}F_{\mathfrak{s}}^*\rangle = \langle f_{\mathfrak{s}}, f_{\mathfrak{t}}F_{\mathfrak{s}}\rangle = \delta_{\mathfrak{s}\mathfrak{t}}\langle f_{\mathfrak{s}}, f_{\mathfrak{s}}\rangle$$

by Proposition 2.7(ii) and Proposition 3.33(iv). Thus, $\{\, f_{\mathfrak{t}} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\,\}$ is an orthogonal basis of S^λ.

(ii) By part (i) we may write $f_{\mathfrak{s}}T_i = \sum_{\mathfrak{v}} c_{\mathfrak{v}} f_{\mathfrak{v}}$ for some $c_{\mathfrak{v}} \in R$ where $v \in \mathrm{Std}(\lambda)$. Let k be any integer such that $k \neq i, i+1$ (note that there is nothing to prove when $n \le 2$). Then T_i and L_k commute so $f_{\mathfrak{s}}T_iL_k = f_{\mathfrak{s}}L_kT_i = [\mathrm{res}_{\mathfrak{s}}(k)]_q f_{\mathfrak{s}}T_i$ by Proposition 3.33(iii). On the other hand,

$$f_{\mathfrak{s}}T_iL_k = \sum_{\mathfrak{v}} c_{\mathfrak{v}} f_{\mathfrak{v}} L_k = \sum_{\mathfrak{v}} c_{\mathfrak{v}}[\mathrm{res}_{\mathfrak{v}}(k)]_q f_{\mathfrak{v}}.$$

Equating coefficients, $c_{\mathfrak{v}}[\mathrm{res}_{\mathfrak{v}}(k)]_q = c_{\mathfrak{v}}[\mathrm{res}_{\mathfrak{s}}(k)]_q$ for all $k \neq i, i+1$. Therefore, by Lemma 3.32(ii), $c_{\mathfrak{v}} = 0$ unless $\mathfrak{v} = \mathfrak{s}$, or $\mathfrak{v} = \mathfrak{t}$ and $\mathfrak{t}$ is standard.

Suppose first that $\mathfrak{t}$ is not standard; then we have shown that $f_{\mathfrak{s}}T_i = c_{\mathfrak{s}} f_{\mathfrak{s}}$ for some $c_{\mathfrak{s}} \in R$. By Proposition 3.33(i), $f_{\mathfrak{s}} = m_{\mathfrak{s}} + \sum_{\mathfrak{v} \rhd \mathfrak{s}} a_{\mathfrak{v}} m_{\mathfrak{v}}$ for some $a_{\mathfrak{v}} \in R$. Because $\mathfrak{t}$ is not standard, either i and $i+1$ are in the same row of $\mathfrak{s}$ or they are in the same column. In the first case, $m_{\mathfrak{s}}T_i = qm_{\mathfrak{s}}$ by Corollary 3.4; in the second, $m_{\mathfrak{s}}T_i = -m_{\mathfrak{s}} + \sum_{v \rhd \mathfrak{s}} b_{\mathfrak{v}}$ for some $b_{\mathfrak{v}} \in R$ by Corollary 3.19. Therefore, by Proposition 3.33(i) again, $f_{\mathfrak{s}}T_i$ equals $qf_{\mathfrak{s}}$ or $-f_{\mathfrak{s}}$ respectively.

Now assume that $\mathfrak{t}$ is standard. We have shown that $f_{\mathfrak{s}}T_i = c_{\mathfrak{s}}f_{\mathfrak{s}} + c_{\mathfrak{t}}f_{\mathfrak{t}}$ for some $c_{\mathfrak{s}}, c_{\mathfrak{t}} \in R$. First suppose that $\mathfrak{s} \rhd \mathfrak{t}$. Then $m_{\mathfrak{s}}T_i = m_{\mathfrak{t}}$; so, $f_{\mathfrak{s}}T_i = m_{\mathfrak{t}} + \sum_{\mathfrak{u} \rhd \mathfrak{t}} b_{\mathfrak{u}} m_{\mathfrak{u}}$ for some $b_{\mathfrak{u}} \in R$. Therefore, as in the last paragraph, $c_{\mathfrak{t}} = 1$; that is, $f_{\mathfrak{s}}T_i = c_{\mathfrak{s}}f_{\mathfrak{s}} + f_{\mathfrak{t}}$. Now $\mathrm{res}_{\mathfrak{s}}(i) = \mathrm{res}_{\mathfrak{t}}(i+1)$ and $\mathrm{res}_{\mathfrak{s}}(i+1) = \mathrm{res}_{\mathfrak{t}}(i)$. Therefore, using Theorem 3.30 and Proposition 3.24(ii), we find that

$$\begin{aligned} c_{\mathfrak{s}}[\mathrm{res}_{\mathfrak{t}}(i)]_q f_{\mathfrak{s}} + [\mathrm{res}_{\mathfrak{s}}(i)] f_{\mathfrak{t}} &= \big(c_{\mathfrak{s}}f_{\mathfrak{s}} + f_{\mathfrak{t}}\big)L_{i+1} = f_{\mathfrak{s}}T_iL_{i+1} \\ &= f_{\mathfrak{s}}\big\{(q-1)L_{i+1} + 1 + L_iT_i\big\} \\ &= \big\{(q-1)[\mathrm{res}_{\mathfrak{t}}(i)]_q + 1 + c_{\mathfrak{s}}[\mathrm{res}_{\mathfrak{s}}(i)]_q\big\} f_{\mathfrak{s}} + [\mathrm{res}_{\mathfrak{s}}(i)]_q f_{\mathfrak{t}}. \end{aligned}$$

Consequently, remembering that $[m]_q = (q^m-1)/(q-1)$ (when $q \neq 1$),

$$c_{\mathfrak{s}} = \frac{(q-1)[\mathrm{res}_{\mathfrak{t}}(i)]_q + 1}{[\mathrm{res}_{\mathfrak{t}}(i)]_q - [\mathrm{res}_{\mathfrak{s}}(i)]_q} = q^{\mathrm{res}_{\mathfrak{t}}(i)}\frac{(q-1)}{q^{\mathrm{res}_{\mathfrak{t}}(i)} - q^{\mathrm{res}_{\mathfrak{s}}(i)}} = -\frac{1}{[\rho]_q}$$

as claimed.

The formula for $f_{\mathfrak{t}}T_i$ (or, equivalently, for $f_{\mathfrak{s}}T_i$ when $\mathfrak{t}$ is standard and $\mathfrak{t} \rhd \mathfrak{s}$) can now be obtained by expanding $f_{\mathfrak{s}}T_i^2 = f_{\mathfrak{t}}T_i - 1/[\rho]_q f_{\mathfrak{s}}T_i$. □

We will see in the next section that $\mathscr{H}$ is semisimple if and only if $e > n$; so the last few results have really been about the semisimple case. Even so, our main applications of these results will be in the case where $\mathscr{H}$ is not semisimple.

4. The irreducible $\mathscr{H}$–modules

In this section we give necessary and sufficient conditions for the module D^λ to be non–zero; by Theorem 3.21 this will classify the simple $\mathscr{H}$–modules. Throughout this section we assume that R is a field.

Recall that e is the smallest positive integer such that $[e]_q = 0$ (or $e = \infty$).

Suppose that $\lambda = (\lambda_1, \lambda_2, \dots)$. Then λ is **e–restricted** if $\lambda_i - \lambda_{i+1} < e$ for all $i \geq 1$. In addition, we say that a standard λ–tableau $\mathfrak{s}$ is **e–restricted** if whenever $\lambda \rhd \mu$ and $\mathfrak{t} \in \operatorname{Std}(\mu)$ then $[\operatorname{res}_{\mathfrak{s}}(k)]_q \neq [\operatorname{res}_{\mathfrak{t}}(k)]_q$ for some k. Let $\operatorname{Std}_e(\lambda)$ be the set of e–restricted tableaux.

We will show that λ is e–restricted if and only if there exists an e–restricted λ–tableau. The next two Lemmas prove one direction. Before the reader gets excited, we remark that the lower bound given for $\dim D^\lambda$ in the next Lemma is rarely sharp.

3.35 Lemma *Let λ be a partition of n. Then $\dim D^\lambda \geq |\operatorname{Std}_e(\lambda)|$.*

Proof. Let $\mathscr{L}$ be the subalgebra of $\mathscr{H}$ generated by $L_1, L_2, \dots, L_n$ and let ν be a partition of n. Suppose that $\mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_k$ are the standard ν–tableaux, ordered so that $i > j$ whenever $\mathfrak{t}_i \rhd \mathfrak{t}_j$. For $i = 0, 1 \dots, k$ let S_i^ν be the R–module spanned by $\{\, m_{\mathfrak{t}_j} \mid 1 \leq j \leq i \,\}$ and let $S^\nu(\mathfrak{t}_i) = S_i^\nu / S_{i-1}^\nu$. By Theorem 3.30 the R–modules S_i^ν and $S^\nu(\mathfrak{t}_i)$ are $\mathscr{L}$–modules and, up to isomorphism, $S_i^\nu(\mathfrak{t}_i)$ depends only upon the sequence $([\operatorname{res}_{\mathfrak{t}_i}(1)]_q, \dots, [\operatorname{res}_{\mathfrak{t}_i}(n)]_q])$. Thus, $S^\nu = S_1^\nu > S_2^\nu > \dots > S_k^\nu > 0$ is an $\mathscr{L}$–module composition series of S^ν with simple composition factors.

Now suppose that $\mathfrak{s}$ is an e–restricted λ–tableau and let μ be a partition of n with $D^\mu \neq 0$. By Corollary 2.14, D^μ is a composition factor of S^λ only if $\lambda \trianglerighteq \mu$; however, by assumption, the $\mathscr{L}$–module $S^\lambda(\mathfrak{s})$ is not an $\mathscr{L}$–composition factor of S^μ — and hence D^μ — if $\lambda \rhd \mu$. Therefore, $S^\lambda(\mathfrak{s})$ is an $\mathscr{L}$–composition factor of D^λ. This proves the Lemma. □

Next, we define the **ladder number** of a node $(i,j) \in \mathbb{N} \times \mathbb{N}$ to be the integer $l_e(i,j) = i - j + ej - e$. Thus, two nodes have the ladder number only if they have the same e–residue. For example, if $\lambda = (4,3)$ and $e = 3$ then the ladder numbers in $[\lambda]$ are $\begin{smallmatrix} 0&1&2&3 \\ 2&3&4 \end{smallmatrix}$ and we reduce these modulo 3 to obtain the e–residues $\begin{smallmatrix} 0&1&2&0 \\ 2&0&1 \end{smallmatrix}$.

Let λ be a partition of n. The **ladder tableau** $\mathfrak{l}_e^\lambda$ is defined inductively as follows. If $n = 1$ set $\mathfrak{l}_e^\lambda = \mathfrak{t}^\lambda$. When $n > 1$ let (i,j) be the unique node in $[\lambda]$ such that (i) $\mathfrak{l}_e(i,j) \geq \mathfrak{l}_e(k,l)$ for all $(k,l) \in [\lambda]$, and (ii), if $\mathfrak{l}_e(i,j) = \mathfrak{l}_e(k,l)$ then $i \leq k$. Let ν be the partition of $n-1$ with diagram $[\lambda] \setminus \{(i,j)\}$ and define $\mathfrak{l}_e^\lambda$ to be unique λ–tableau such that $\mathfrak{l}_e^\lambda \downarrow (n-1) = \mathfrak{l}_e^\nu$; that is,

$$\textbf{3.36} \qquad \mathfrak{l}_e^\lambda(k,l) = \begin{cases} \mathfrak{l}_e^\nu(k,l), & \text{if } (k,l) \neq (i,j), \\ n, & \text{if } (k,l) = (i,j). \end{cases}$$

By construction $\mathfrak{l}_e^\lambda$ is a standard λ–tableau.

For example, if $\lambda = (4,3)$ and $e = 3$, as above, then $\mathfrak{l}_e^\lambda = \begin{smallmatrix} 1&2&4&6 \\ 3&5&7 \end{smallmatrix}$.

3.37 Lemma *Suppose that λ is an e–restricted partition of n. Then the ladder tableau $\mathfrak{l}_e^\lambda$ is e–restricted.*

Proof. We prove this by induction on n, the result being clear when $n = 1$.

Let (i,j) and ν as in (3.36). We claim that the partition ν is e–restricted. If not then either (i) $\nu_i - \nu_{i+1} = e$, or (ii) $i > 1$ and $\nu_{i-1} - \nu_i = e$. The first case cannot happen because then $\lambda_i - \lambda_{i+1} > e$ which contradicts our assumption that λ is e–restricted. In the second scenario, row $(i-1)$ of λ would contain a node with higher ladder number contradicting our choice of (i,j). Hence, ν is e–restricted as claimed.

Let μ be a partition and suppose that $\lambda \trianglerighteq \mu$ and that there exists a μ–tableau $\mathfrak{t}$ with $[\operatorname{res}_{\mathfrak{t}}(k)]_q = [\operatorname{res}_{\mathfrak{l}_e^\lambda}(k)$ for $k = 1, \dots, n$. In particular, $[\operatorname{res}_{\mathfrak{t}}(n)]_q = [\operatorname{res}_{\mathfrak{l}_e^\lambda}(n)]_q$, so $\nu \trianglerighteq \operatorname{Shape}(\mathfrak{t}\downarrow(n-1))$ since (i,j) is the lowest node with e–residue $[\operatorname{res}_{\mathfrak{t}_e^\lambda}(n)]_q$ which may be added to $[\nu]$ to give the diagram of a partition. By induction on n the tableau $\mathfrak{t}_e^\nu$ is e–restricted, so this forces $\mathfrak{t}\downarrow(n-1)=\mathfrak{t}_e^\nu$. However, if $\mathfrak{t}(k,l)=n$ then $k\ge i$ because $\lambda\trianglerighteq\mu$. Therefore, $(k,l)=(i,j)$, $\mathfrak{t}=\mathfrak{t}_e^\lambda$ and $\lambda=\mu$; hence, $\mathfrak{t}_e^\lambda$ is e–restricted as required. □

Combining the last two results we see that if λ is e–restricted then D^λ is non–zero. To prove the converse we need some more notation.

If $\lambda=(\lambda_1,\dots,\lambda_k)$ is a partition and i an integer with $1\le i\le k$ let $\lambda^i=(\lambda_i,\lambda_{i+1},\dots,\lambda_k)$ and $\overline{\lambda}=(\lambda_1-\lambda_2,\lambda_2-\lambda_3,\dots)$. Let g_i be the greatest common divisor of $[\gamma]_q^!$, where γ runs over all partitions of λ_i such that $\gamma\rhd\overline{\lambda}^i$. Finally, we define $g_\lambda=g_1g_2\dots g_k$.

3.38 Lemma *Suppose that $\lambda=(\lambda_1,\dots,\lambda_k)$ is a partition of n and let $\mathfrak{s}$ and $\mathfrak{t}$ be standard λ–tableaux. Then g_λ is a factor of $\langle m_\mathfrak{s},m_\mathfrak{t}\rangle$.*

Proof. By definition, $\langle m_\mathfrak{s},m_\mathfrak{t}\rangle m_\lambda\equiv m_{\mathfrak{t}^\lambda\mathfrak{s}}m_{\mathfrak{t}\mathfrak{t}^\lambda}\mod\mathscr{H}^\lambda$; however, by Corollary 3.4, $m_{\mathfrak{t}^\lambda\mathfrak{s}}m_{\mathfrak{t}\mathfrak{t}^\lambda}$ is a linear combination of terms $m_\lambda T_{d(\mathfrak{v})}m_\lambda$ where $\mathfrak{v}$ is a row standard λ–tableau. So, changing notation, it is sufficient to show that if $\mathfrak{t}$ is a row standard λ–tableau then $m_\lambda T_{d(\mathfrak{t})}m_\lambda\equiv rg_\lambda m_\lambda$ for some $r\in R$.

Define $\mu=(\mu_1,\dots,\mu_k)$ to be the composition of λ_1 such that μ_i is equal to the number of entries in row i of $\mathfrak{t}$ which are less than or equal to λ_1. Abusing notation, write $m_\lambda=m_{\lambda_1}\dots m_{\lambda_k}$ where $\mathfrak{S}_\lambda=\mathfrak{S}_{\lambda_1}\times\dots\times\mathfrak{S}_{\lambda_k}$ and $m_{\lambda_i}=\sum_{x\in\mathfrak{S}_{\lambda_i}}T_x$ for $1\le i\le k$.

We first consider $m_{\mathfrak{t}^\lambda\mathfrak{t}}m_{\lambda_1}$. Let $\mathfrak{u}$ be the row standard λ–tableau which agrees with $\mathfrak{t}$ on all entries greater than λ_1 and where $1,2,\dots,\lambda_1$ are inserted in increasing order from left to right and then top to bottom into the remaining positions. Then $\mathfrak{t}=\mathfrak{u}w$ for some $w\in\mathfrak{S}_{\lambda_1}$ such that $\ell(d(\mathfrak{t}))=\ell(d(\mathfrak{u}))+\ell(w)$; so $m_{\mathfrak{t}^\lambda\mathfrak{t}}=m_{\mathfrak{t}^\lambda\mathfrak{u}}T_w$. Therefore, by Lemma 3.2 and Corollary 3.4,

$$m_{\mathfrak{t}^\lambda\mathfrak{t}}m_{\lambda_1}=m_{\mathfrak{t}^\lambda\mathfrak{u}}T_wm_{\lambda_1}=q^{\ell(w)}m_{\mathfrak{t}^\lambda\mathfrak{u}}m_{\lambda_1}=q^{\ell(w)}[\mu]_q^!m_{\mathfrak{t}^\lambda\mathfrak{u}}h$$

for some $h\in\mathscr{H}(\mathfrak{S}_{\lambda_1}$.Consequently, $m_{\mathfrak{t}^\lambda\mathfrak{u}}m_{\lambda_1}$ is an element of $\mathscr{H}m_\nu$ where ν is the composition $(\lambda_1,\lambda_1-\mu_1,\dots,\lambda_k-\mu_k)$.

Suppose that $\lambda\not\trianglerighteq\nu$. Then $\lambda\not\trianglerighteq\overline{\nu}$ where $\overline{\nu}$ is the partition obtained from ν by reordering its parts; therefore, $m_{\mathfrak{t}^\lambda\mathfrak{t}}m_{\lambda_1}\in\mathscr{H}^\lambda\cap\mathscr{H}m_\nu\subseteq\check{\mathscr{H}}^\lambda$ by Corollary 2.3(iii). Thus, $m_{\mathfrak{t}^\lambda\mathfrak{t}}m_{\lambda_1}\equiv 0\mod\check{\mathscr{H}}^\lambda$ and, in particular, our claim is correct in this case.

On the other hand, if $\lambda\trianglerighteq\nu$ then $(\lambda_1,\lambda_2,\dots,\lambda_k)\trianglerighteq(\lambda_1,\lambda_1-\mu_1,\dots,\lambda_1-\mu_k)$; equivalently, $\mu\trianglerighteq\overline{\lambda}^1=(\lambda_1-\lambda_2,\dots,\lambda_{k-1}-\lambda_k)$. Now, g_1 is a factor of $[\mu]_q^!$, so by Theorem 3.18 there exist $r_\mathfrak{v}\in R$ such that

$$m_{\mathfrak{t}^\lambda\mathfrak{t}}m_{\lambda_1}=q^{\ell(w)}[\mu]_q^!m_{\mathfrak{t}^\lambda\mathfrak{u}}h\equiv g_1\sum_{\mathfrak{v}\in\operatorname{Std}(\lambda)}r_\mathfrak{v}m_{\mathfrak{t}^\lambda v}\mod\check{\mathscr{H}}^\lambda.$$

Moreover, by the right handed version of Proposition 3.16, $r_\mathfrak{v}\ne 0$ only if the integers $1,2,\dots,\lambda_1$ all appear in the first row of $\mathfrak{v}$; that is to say that $r_\mathfrak{v}\ne0$ only if $\mathfrak{v}$ and $\mathfrak{t}^\lambda$ have the same first row. Repeating this argument we find that for $i=1,2\dots,k$ there exist $r_\mathfrak{v}\in R$ such that

$$m_\mathfrak{t}m_{\lambda_1}\dots m_{\lambda_i}=g_1\dots g_i\sum_{\mathfrak{v}\in\operatorname{Std}(\lambda)}r_\mathfrak{v}m_\mathfrak{v}$$

and $r_{\mathfrak{v}} \neq 0$ only if the tableaux $\mathfrak{v}$ and $\mathfrak{t}^\lambda$ agree on the first i rows. Taking $i = k$ proves the Lemma. □

Let $d_{\lambda\mu} = [S^\lambda : D^\mu]$ be the composition multiplicity of D^μ in S^λ. We can now prove the following key result of Dipper and James.

3.39 Theorem (Dipper–James) *Suppose that R is a field.*

(i) *$\{ D^\lambda \mid \lambda$ an e–restricted partition of $n \}$ is a complete set of non–isomorphic irreducible $\mathscr{H}$–modules.*

(ii) *Suppose that μ is an e–restricted partition of n and that λ is a partition of n. Then $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \trianglerighteq \mu$.*

Proof. Everything will follow from Theorem 2.13 and Corollary 2.14 once we show that $D^\lambda \neq 0$ if and only if λ is e–restricted. That $D^\lambda \neq 0$ when λ is e–restricted is immediate from Lemmas 3.37 and 3.35.

Conversely, if λ is not e–restricted then $\lambda_i - \lambda_{i+1} \geq e$ for some integer $i \geq 1$; consequently, $[e]^!_q$ is a factor of g_λ since $[e]^!_q$ divides g_i. However, $[e]^!_q = 0$ so g_λ must also be 0. Therefore, $\langle m_{\mathfrak{s}}, m_{\mathfrak{t}} \rangle = 0$ for any $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda)$ by Lemma 3.38; thus, $\operatorname{rad} S^\lambda = S^\lambda$ and $D^\lambda = 0$. □

In particular, we see that $S^\lambda = D^\lambda$ for all λ if and only if $e > n$. Hence, by Corollary 2.18 we have an easy way of determining whether or not $\mathscr{H}$ is semisimple. More precisely, we have the following.

3.40 Corollary *Suppose that R is a field. Then the following are equivalent.*

(i) *$\mathscr{H}$ is a split semisimple algebra;*

(ii) *$S^\lambda = D^\lambda$ for all partitions λ of n; and,*

(iii) *$e > n$.*

Note that e is equal to the characteristic of R in the case where $q = 1$. Consequently, as they must, these results agree with the corresponding results from the representation theory of the symmetric group (see [**36**] and Note 3 below).

Theorem 3.39 combined with Lemma 3.35 also gives us the following combinatorial result (a more direct proof can be extracted from [**35**, Theorem A]).

3.41 Corollary *Let λ be a partition of n. Then λ is e–restricted if and only if there exists an e–restricted λ–tableau.*

Notes on Chapter 3

1. The proof of Proposition 3.7 comes from [**13**]. Although this result must have been known for some time, this is the only place where I have seen it in the literature.
2. In his thesis [**32**], Hoefsmit classified the irreducible $\mathscr{H}_K$–modules by using Young's seminormal form (Theorem 3.34) as the definition of the Specht modules. Later Dipper and James [**13**] constructed the Specht modules as submodules of $\mathscr{H}$ and hence classified the irreducible $\mathscr{H}$–modules over an arbitrary field. The approach that we have taken here is based upon work of Murphy [**55**, **56**].
3. Let $\#$ be the R–linear automorphism of $\mathscr{H}$ determined by $T_w^\# = (-q)^{\ell(w)} T_{w^{-1}}^{-1}$ for all $w \in \mathfrak{S}_n$. Then $S^{\lambda\#}$ is isomorphic to the dual of Dipper and James' Specht module which is indexed by the partition conjugate to λ (see [**56**, Theorem 5.2] and the warning on page 25). This is also why the simple $\mathscr{H}$–modules

in Theorem 3.39 are indexed by e–restricted partitions rather than e–regular partitions (a partition is e–regular if its conjugate is e–restricted).

4. If λ is a partition let $n_\lambda = \sum_{w\in\mathfrak{S}_\lambda}(-q)^{-\ell(w)}T_w$ and for $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\lambda)$ define $n_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})}n_\lambda T_{d(\mathfrak{t})}$. Then
$$\mathcal{N} = \{\, n_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\lambda) \text{ for some partition } \lambda \text{ of } n\,\}$$
is a basis of $\mathscr{H}$. It is easy to check that $m_\lambda^{\#} = q^a n_\lambda$ for some integer a; since the automorphisms $*$ and $\#$ commute this shows that $(\mathcal{N},\Lambda)$ is a cellular basis.
5. In the representation theory of the symmetric groups the elements L_k were introduced independently by Jucys [**45**] and Murphy [**54**]. Dipper and James [**14**] first studied their analogues in the representation theory of $\mathscr{H}$.
6. Let $L'_k = q^{1-k}T_{k-1}\dots T_1T_1\dots T_{k-1}$ for $1\le k\le n$. An easy argument shows that $L'_k = (q-1)L_k + 1$; accordingly, many authors prefer to use these elements because they are easier to work with. Unfortunately, when $q = 1$, which is the case of the symmetric group, $L'_k = 1$ for all k. Rather that having to treat this important special case separately we use the elements L_k throughout.
7. Lemma 3.35 is essentially Proposition 5.9(i) of [**26**]. Lemma 3.37 is motivated by the work of Lascoux, Leclerc and Thibon [**51**] who, in turn, were inspired by James [**35**]. The argument of Lemma 3.38 is due to Murphy [**56**] (cf. [**13**, Theorem 6.8]).
8. The proof of Lemma 3.35 actually shows that $d_{\lambda\mu}\ge k$ where k is maximal such that $\operatorname{Std}(\lambda) = X_0\sqcup X_1\sqcup\dots\sqcup X_k$ (disjoint union) where for $i = 1,2\dots,k$ there exists a bijection $\theta_i : X_i \longrightarrow \operatorname{Std}_e(\mu)$ such that $[\operatorname{res}_{\mathfrak{t}}(k)]_q = [\operatorname{res}_{\theta_i(\mathfrak{t})}(k)]_q$ for all $\mathfrak{t}\in X_i$ and all k with $1\le k\le n$.
9. The rank of the Specht module S^λ is explicitly known (see [**36**]); it is equal to $n!/\prod h^\lambda_{ij}$ where the product if over the nodes $(i,j)\in[\lambda]$ and h^λ_{ij} is the (i,j)th hook length (see page 55). Except in some very special cases, the dimension of D^λ is not known.

CHAPTER 4

The q–Schur Algebra

The q–Schur algebra was introduced by Dipper and James [**15**] who showed that this algebra completely determines the decomposition matrices of the finite general linear groups in non–defining characteristic. The q–Schur algebras also have applications to the representation theory of the general linear groups over other fields. We do not explore these connections here. Instead, in this Chapter we 'lift' the Murphy basis of $\mathscr{H}$ to give a cellular basis of the q–Schur algebra.

1. Semistandard tableaux

4.1 Let$\Lambda(d,n)$ be a set of compositions of n with a most d non–zero parts. The q–Schur algebra is the endomorphism algebra

$$\mathscr{S}(d,n) = \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\lambda\in\Lambda(d,n)} M^\mu\Big).$$

Note that the q–Schur algebra, like $\mathscr{H}$, is dependent upon the choices of R and q.

By definition, $\mathscr{S}(d,n) \cong \bigoplus_{\mu,\nu\in\Lambda(d,n)} \operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$; accordingly, much of this Chapter is devoted to understanding the ideals M^μ and the homomorphisms between them. To this end we now introduce another kind a tableaux.

4.2 Let μ and ν be compositions of n. A μ–tableau of type indextableau!type ν is a map $\mathtt{T}\colon[\mu]\longrightarrow\mathbb{N}$ such that $\nu_i = \#\{\,x\in[\mu]\mid \mathtt{T}(x)=i\,\}$ for $i\ge 1$. Again we write $\operatorname{Shape}(\mathtt{T})=\mu$.

For example, if $\omega = (1^n)$ then the tableaux of Chapter 3 are precisely the tableaux of type ω. We continue to use lower case letters for tableaux of type ω; tableaux of arbitrary type will be denoted by upper case letters.

A μ–tableau $\mathtt{T}$ is row semistandard if the entries in each row of $\mathtt{T}$ are non–decreasing; $\mathtt{T}$ is semistandard if (i) μ is a partition, and (ii) $\mathtt{T}$ is row semistandard and the entries in each column of $\mathtt{T}$ are strictly increasing. Let $\mathcal{T}(\mu,\nu)$ be the set of row semistandard μ–tableaux of type ν and i $\mathcal{T}_0(\mu,\nu)$ be the set of semistandard μ–tableaux of type ν.

4.3 Examples Suppose that μ is a composition of n.

(i) Let$\mathtt{T}^\mu$ be the tableau of type μ such that $\mathtt{T}^\mu(ij)=i$ for all $(i,j)\in[\mu]$. Then $\mathtt{T}^\mu$ is the unique semistandard μ–tableau $\mathtt{T}^\mu$ of type μ. For example, if $\mu=(3,2,1)$ then $\mathtt{T}^\mu = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}$.

(ii) All of the semistandard tableaux of type λ can be obtained from $\mathtt{T}^\mu$ by 'moving nodes up' in $\mathtt{T}^\mu$. Continuing example (i), the other semistandard tableaux of type $(3,2,1)$ are $\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline\end{array}$, $\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&3\\\cline{1-2}\end{array}$, $\begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&2\\\cline{1-2}\end{array}$, $\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline 3\\\cline{1-1}\end{array}$, $\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\\hline 2\\\cline{1-1}\end{array}$ and $\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&2&2&3\\\hline\end{array}$.

Next we describe a way of converting tableaux of type ω into tableaux of type ν. Let t be a tableau of type ω and let ν be a composition. Then $\nu(\mathsf{t})$ the tableau of type ν obtained from $\mathfrak{t}$ by replacing each entry i in $\mathfrak{t}$ by r if i appears in row r of $\mathfrak{t}^\nu$.

4.4 Examples (i) If $\mathfrak{t}$ is a row standard μ–tableau then $\nu(\mathfrak{t})$ is row semistandard.

(ii) Suppose that $\mathfrak{t} = \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2&6\\\cline{1-2}\end{array}$ and let $\nu = (3,2,1)$. Then $\nu = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}$ so we have $\nu(\mathfrak{t}) = \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 1&3\\\cline{1-2}\end{array}$.

(iii) If μ is a composition then $\mathtt{T}^\mu = \mu(\mathfrak{t}^\mu)$.

Let μ and ν be compositions and suppose $d \in \mathscr{D}_\mu$. Then $\mathfrak{t}^\lambda d$ is row standard and the subgroup $d^{-1}\mathfrak{S}_\mu d$ is just the row stabilizer of $\mathfrak{t}^\lambda d$. Therefore, $d^{-1}\mathfrak{S}_\mu d \cap \mathfrak{S}_\nu$ is generated by $S \cap d^{-1}\mathfrak{S}_\mu d \cap \mathfrak{S}_\nu$ and so $d^{-1}\mathfrak{S}_\mu d \cap \mathfrak{S}_\nu = \mathfrak{S}_\nu$ for some composition ν of n. We write $\nu = \mu d \cap \nu$.

We can now generalize Proposition 3.3 to describe the $(\mathfrak{S}_\mu, \mathfrak{S}_\nu)$–double cosets.

4.5 Proposition *Suppose that μ and ν are compositions of n and let*

$$\mathscr{D}_{\mu\nu} = \left\{ d \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} \nu(\mathfrak{t}^\mu d) \text{ is row semistandard and } \mathfrak{t}^\mu d \trianglerighteq \mathfrak{t} \text{ whenever } \mathfrak{t} \\ \text{is a row standard tableau such that } \nu(\mathfrak{t}) = \nu(\mathfrak{t}^\mu d) \end{array} \right\}.$$

Then $\mathscr{D}_{\mu\nu}$ is a complete set of $(\mathfrak{S}_\mu, \mathfrak{S}_\nu)$–double coset representatives in $\mathfrak{S}_n$. In addition, the following hold.

(i) *Every element w of $\mathfrak{S}_n$ can be written uniquely in the form $w = vdu$ where $v \in \mathfrak{S}_\mu$, $d \in \mathscr{D}_{\mu\nu}$ and $u \in \mathscr{D}_{\mu d \cap \nu} \cap \mathfrak{S}_\nu$. Furthermore, $\ell(w) = \ell(v) + \ell(d) + \ell(u)$.*

(ii) *$\mathscr{D}_{\mu\nu} = \mathscr{D}_\mu \cap \mathscr{D}_\nu^{-1}$.*

(iii) *The map $d \longmapsto \nu(\mathfrak{t}^\mu d)$ gives a one–to–one correspondence between the elements of $\mathscr{D}_{\mu\nu}$ and the set of row semistandard μ–tableaux of type ν.*

Proof. By Proposition 3.3, the right cosets of $\mathfrak{S}_\mu$ are indexed by row standard μ–tableau. Therefore, the $(\mathfrak{S}_\mu, \mathfrak{S}_\nu)$–double cosets are indexed by $\mathfrak{S}_\nu$–orbits of row standard μ–tableaux. Furthermore, two row standard tableaux $\mathfrak{s}$ and $\mathfrak{t}$ are in the same $\mathfrak{S}_\nu$–orbit if and only if $\nu(\mathfrak{s}) = \nu(\mathfrak{t})$; that is, if d and d' are elements of $\mathscr{D}_\mu$ then $\mathfrak{S}_\mu d \mathfrak{S}_\nu = \mathfrak{S}_\mu d' \mathfrak{S}_\nu$ if and only if $\nu(\mathfrak{t}^\mu d) = \nu(\mathfrak{t}^\mu d')$.

Let $\mathcal{O}$ be a $\mathfrak{S}_\nu$–orbit of row standard μ–tableaux. Then $\mathcal{O}$ contains a unique tableau $\mathfrak{t}$ such that whenever i and $i+1$ are in the same row of $\mathfrak{t}^\mu$ then either i and $i+1$ are in the same row of $\mathfrak{t}$ or $\mathfrak{t} \rhd \mathfrak{t}(i, i+1)$. Let $d = d(\mathfrak{t})$. Then $\mathcal{O}$ is precisely the set of row standard tableau of the form $\mathfrak{t}^\mu du$ where $u \in \mathfrak{S}_\nu$ (and $du \in \mathscr{D}_\mu$). Now $\mathfrak{t}^\mu du = \mathfrak{t}u$ is row standard if and only if $iu < ju$ for all integers $i < j$ which are in the same row of $\mathfrak{t}^\mu d$. Therefore, $u \in \mathscr{D}_{\mu d \cap \nu}$ and $\ell(du) = \ell(d) + \ell(u)$ by Proposition 3.3. Hence, any w in the double coset $\mathfrak{S}_\mu d \mathfrak{S}_\nu$ is uniquely expressible in the form vdu for some $v \in \mathfrak{S}_\mu$ and $u \in \mathscr{D}_{\mu d \cap \nu}$. Moreover, by Proposition 3.3 again, $\ell(w) = \ell(v) + \ell(du)$ since $du \in \mathscr{D}_\mu$; thus, $\ell(w) = \ell(v) + \ell(d) + \ell(u)$, proving (i).

For (ii) note that by (i) if $d \in \mathscr{D}_{\mu\nu}$ then d is the unique element of minimal length in $\mathfrak{S}_\mu d \mathfrak{S}_\nu$. In particular, d is the unique element of minimal length in $\mathfrak{S}_\mu d$ and also in $d\mathfrak{S}_\nu$; so $\mathscr{D}_{\mu\nu} \subseteq \mathscr{D}_\mu \cap \mathscr{D}_\nu^{-1}$ by Proposition 3.3. Conversely, any element of $\mathscr{D}_\mu \cap \mathscr{D}_\nu^{-1}$ must be an element of minimal length in its double coset and hence belongs to $\mathscr{D}_{\mu\nu}$.

Finally, for (iii) if $\mathtt{T}$ is a row semistandard μ–tableau of type ν then we can easily find a row standard tableau $\mathfrak{t}$ such that $\nu(\mathfrak{t}) = \mathtt{T}$, so $\mathtt{T}$ corresponds to a $(\mathfrak{S}_\mu, \mathfrak{S}_\nu)$–double set. We have already observed the converse, so the proof is complete. □

The elements of $\mathscr{D}_{\mu\nu}$ are called distinguished double coset representatives of $\mathfrak{S}_n$.

Let $\mathtt{T}$ be a row semistandard μ–tableau of type ν and define first$(\mathtt{T})$ to be the unique row standard μ–tableau such that $\nu(\text{first}(\mathtt{T})) = \mathtt{T}$ and first$(\mathtt{T}) \trianglerighteq \mathfrak{t}$ whenever $\mathfrak{t}$ is a row standard tableau such that $\nu(\mathfrak{t}) = \mathtt{T}$. It is also easy to see that there exists a unique row standard tableau last$(\mathtt{T})$ such that $\nu(\text{last}(\mathtt{T})) = \mathtt{T}$ and $\mathfrak{t} \trianglerighteq \text{last}(\mathtt{T})$ whenever $\mathfrak{t}$ is another row standard tableau such that $\nu(\mathfrak{t}) = \mathtt{T}$.

4.6 Example Suppose that $\mu = (4, 4)$ and let $\mathtt{T}$ be the μ–tableau $\begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 2&3&3&4\\\hline\end{array}$. Then $\mathtt{T}$ is of type $\nu = (2, 1, 3, 2)$, first$(\mathtt{T}) = \begin{array}{|c|c|c|c|}\hline 1&2&4&7\\\hline 3&5&6&8\\\hline\end{array}$ and last$(\mathtt{T}) = \begin{array}{|c|c|c|c|}\hline 1&2&6&8\\\hline 3&4&5&7\\\hline\end{array}$. Let $d = d(\text{first}(\mathtt{T}))$; then $\mu d \cap \nu = (2, 1, 1, 2, 1, 1)$ since $\mathfrak{S}_{\mu d \cap \nu}$ is the intersection of $\mathfrak{S}_\nu$ with the row stabilizer of first$(\mathtt{T})$.

Let $d \in \mathscr{D}_{\mu\nu}$ and let $\mathtt{T} = \nu(\mathfrak{t}^\mu d)$ and $\mathtt{S} = \mu(\mathfrak{t}^\nu d^{-1})$. Then we have shown that $\mathtt{S}$ and $\mathtt{T}$ are both row semistandard and that

4.7
$$\sum_{\substack{y \in \mathscr{D}_\nu \\ \mu(\mathfrak{t}^\nu y) = \mathtt{S}}} T_y^* m_\nu = \sum_{w \in \mathfrak{S}_\mu d \mathfrak{S}_\nu} T_w = \sum_{\substack{x \in \mathscr{D}_\mu \\ \nu(\mathfrak{t}^\mu x) = \mathtt{T}}} m_\mu T_x .$$

The element d also determines an $\mathscr{H}$–module homomorphism $\varphi_{\mu\nu}^d : M^\nu \longrightarrow M^\mu$ given by
$$\varphi_{\mu\nu}^d(m_\nu h) = \sum_{w \in \mathfrak{S}_\mu d \mathfrak{S}_\nu} T_w h, \qquad \text{for all } h \in \mathscr{H}.$$
In fact, these elements give a basis of $\text{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$.

4.8 Theorem *Let μ and ν be compositions of n. Then* $\text{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ *is free as an R–module with basis $\{\, \varphi_{\mu\nu}^d \mid d \in \mathscr{D}_{\mu\nu} \,\}$.*

Proof. Suppose that $\varphi \in \text{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$. Then, for all $h \in \mathscr{H}$, we have that $\varphi(m_\nu h) = \varphi(m_\nu)h$; hence, φ is completely determined by $\varphi(m_\nu)$. By Corollary 3.4 there exist $r_d \in R$ such that
$$\varphi(m_\nu) = \sum_{d \in \mathscr{D}_\mu} r_d m_\mu T_d.$$
Choose $x \in \mathscr{D}_\mu$ of minimal length such that $r_x \neq 0$. If $x \notin \mathscr{D}_{\mu\nu}$ then by Proposition 4.5 there exists $s \in S \cap \mathfrak{S}_\nu$ such that $\ell(xs) = \ell(x) - 1$. Then $xs \in \mathscr{D}_\mu$ and
$$\varphi(m_\nu) = q^{-1}\varphi(m_\nu T_s) = q^{-1} \sum_{d \in \mathscr{D}_\nu} r_d m_\mu T_d T_s = r_x m_\mu T_{xs} + \sum_{d \in \mathscr{D}_\mu} b_d m_\mu T_d$$
for some $b_d \in R$ such that $b_{xs} = 0$. In particular, this shows that $m_\mu T_{xs}$ is involved in $\varphi(m_\nu)$ contrary to our assumption that x is an element of minimal length with $r_x \neq 0$. Therefore, $x \in \mathscr{D}_{\mu\nu}$ and, consequently, $\varphi' = \varphi - a_x\varphi$ belongs to $\text{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$. Furthermore, $\varphi'(m_\nu)$ is a linear combination of a terms of the form $m_\mu T_d$ where $d \in \mathscr{D}_\mu$ and $d \neq x$. In fact, the above argument shows that $m_\mu T_d$ is not involved in $\varphi'(m_\nu)$ whenever $\mathfrak{S}_\mu d \mathfrak{S}_\nu = \mathfrak{S}_\mu x \mathfrak{S}_\nu$. Hence, we can repeat this argument to write φ as linear combination of maps $\varphi_{\mu\nu}^x$ with $x \in \mathscr{D}_{\mu\nu}$ as required.

Finally, it remains to check that $\{\, \varphi_{\mu\nu}^d \mid d \in \mathscr{D}_{\mu\nu} \,\}$ is linearly independent. This follows because $\{\, \varphi_{\mu\nu}^d(m_\nu) \mid d \in \mathscr{D}_{\mu\nu} \,\}$ is a linearly independent subset of M^μ (being sums over distinct $(\mathfrak{S}_\mu, \mathfrak{S}_\nu)$–double cosets by (4.7)). ◻

4.9 Corollary *Let μ and ν be compositions of n. Then* $\text{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ *and $M^\mu \cap M^{\nu*}$ are canonically isomorphic as R–modules.*

Proof. Every homomorphism φ in $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ is determined by $\varphi(m_\nu)$ and, moreover, $\varphi(m_\nu) \in M^\mu \cap M^{\nu*}$ by Theorem 4.8. Conversely, very element of $M^\mu \cap M^{\nu*}$ determines a homomorphism in $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$. Consequently, the map $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu) \to M^\mu \cap M^{\nu*}$ given by $\varphi \longmapsto \varphi(m_\nu)$ is an isomorphism of R–modules. □

There are two points worth noting from Theorem 4.8. First, by Proposition 4.5 we could have indexed the basis $\{\, \varphi^d_{\mu\nu} \mid d \in \mathscr{D}_{\mu\nu} \,\}$ of $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ by row semistandard μ–tableau of type ν (or by row semistandard ν–tableaux of type μ). Thus, this basis is the analogue of the row standard basis of M^μ given by Corollary 3.4.

Secondly, by extending the maps $\varphi^d_{\mu\nu}$ to elements of $\mathscr{S}(d, n)$ we obtain a basis of the q–Schur algebra which is independent of R and q.

2. A Specht series for M^ν

In Theorem 4.8 we saw that $\operatorname{Hom}_{\mathscr{H}}(M^\mu, M^\lambda)$ has a basis indexed by row semistandard tableaux. As in Chapter 3, we want to replace this basis with a cellular basis which is indexed by pairs of semistandard tableaux. Rather than proving this directly, we first give another basis of M^μ which is compatible with the Murphy basis of $\mathscr{H}$.

Let λ be a partition of n and μ a composition of n. Let S be a semistandard λ–tableaux of type μ and let $\mathfrak{t}$ be a standard λ–tableau. We define

$$m_{\mathsf{S}\mathfrak{t}} == \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathsf{S}}} m_{\mathfrak{s}\mathfrak{t}}$$

and let $m_{\mathfrak{t}\mathsf{S}} \; = m^*_{\mathsf{S}\mathfrak{t}}$.

4.10 Theorem (Murphy) *Let μ be a composition of n. Then M^μ is free as an R–module with basis*

$$\Big\{\; m_{\mathsf{S}\mathfrak{t}} \;\Big|\; \mathsf{S} \in \mathcal{T}_0(\lambda,\mu),\ \mathfrak{t} \in \operatorname{Std}(\lambda) \textit{ for some partition } \lambda \textit{ of } n \;\Big\}.$$

Proof. Suppose that $\mathsf{S} \in \mathcal{T}_0(\lambda,\mu)$, $\mathfrak{t} \in \operatorname{Std}(\lambda)$ and let $d^{-1} = \operatorname{first}(\mathsf{S})$. By (4.7)

$$m_{\mathsf{S}\mathfrak{t}} = \sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\lambda)\\ \mu(\mathfrak{s})=\mathsf{S}}} T^*_{d(\mathfrak{s})} m_\lambda T_{d(\mathfrak{t})} = \sum_{w\in\mathfrak{S}_\mu d\mathfrak{S}_\lambda} T_w T_{d(\mathfrak{t})}.$$

In particular, $m_{\mathsf{S}\mathfrak{t}} \in M^\mu$.

Now suppose that h is an arbitrary non–zero element of M^μ. By Theorem 3.18 there exist $r_{\mathfrak{s}\mathfrak{t}} \in R$ such that $h = \sum r_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}}$ where $m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{M}$. Suppose that $m_{\mathfrak{s}\mathfrak{t}}$ is involved in h. By definition, $\mathsf{S} = \mu(\mathfrak{s})$ is always row standard; therefore, S is semistandard by Proposition 3.16. If $\mathfrak{s} \neq \operatorname{first}(\mathsf{S})$ then there exists $s_i \in \mathfrak{S}_\mu$ such that if $\mathfrak{s}' = \mathfrak{s}s_i$ then $\mu(\mathfrak{s}') = \mathsf{S}$ and $\mathfrak{s}' \rhd s$. Then $T_i m_{\mathfrak{s}\mathfrak{t}} = (q-1)m_{\mathfrak{s}\mathfrak{t}} + q m_{\mathfrak{s}'\mathfrak{t}}$ by Corollary 3.4 and $T_i h = qh$ by Lemma 3.2; hence, $m_{\mathfrak{s}'\mathfrak{t}}$ is also involved in h. Consequently, we may as well assume that $\mathfrak{s} = \operatorname{first}(S)$. Now let

$$h' = h - \sum_{\lambda\vdash n} \sum_{\mathsf{S}\in\mathcal{T}_0(\lambda,\mu)} \sum_{\substack{\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\lambda)\\ \mathfrak{s}=\operatorname{first}(\mathsf{S})}} r_{\mathfrak{s}\mathfrak{t}} m_{\mathsf{S}\mathfrak{t}}.$$

Then h' is an element of M^μ; however, h involves no term $m_{\mathfrak{s}\mathfrak{t}}$ with $\mathfrak{s} = \operatorname{first}(\mu(\mathfrak{s}))$. Therefore, $h' = 0$ and we have written h as a linear combination of the required form. □

This allows us to give a Specht series for M^μ; that is, a filtration of M^μ with each composition factor isomorphic to a Specht module.

4.11 Corollary *Let μ be a composition of n. Then, as an $\mathscr{H}$–module, M^μ has a filtration $M^\mu = M_1 > M_2 > \cdots > M_{k+1} = 0$ such that for $i = 1, 2, \dots, k$ there exists a partition λ_i of n with $M_i/M_{i+1} \cong S^{\lambda_i}$. Moreover, for each partition λ the number of λ_i equal to λ is the number of semistandard λ–tableaux of type μ.*

Proof. Let $\{\mathtt{S}_1, \dots, \mathtt{S}_k\}$ be the set of semistandard tableaux of type μ where $\mathtt{S}_i \in \mathcal{T}_0(\lambda_i, \mu)$ and $\lambda_i \trianglerighteq \lambda_j$ whenever $i > j$. For $i = 1, 2, \dots, k+1$ let M_i be the R–module of M^μ with basis $\{\, m_{\mathtt{S}_j\mathfrak{t}} \mid i \le j \le k \text{ and } \mathfrak{t} \in \operatorname{Std}(\lambda_j)\,\}$. Then

$$M^\mu = M_1 > M_2 > \cdots > M_k > M_{k+1} = 0$$

and by Theorem 3.18 each M_i is a right $\mathscr{H}$–module.

Fix an integer i with $1 \le i \le k$. Then $M_i \cap \check{\mathscr{H}}^{\lambda_i} \subseteq M_{i+1}$. Consequently, there is a well–defined homomorphism $\theta : S^{\lambda_i} \longrightarrow M_i/M_{i+1}$ of $\mathscr{H}$–modules given by $\theta(m_\mathfrak{t}) = m_{\mathtt{S}_i\mathfrak{t}} + M_{i+1}$ for all $\mathfrak{t} \in \operatorname{Std}(\lambda)$. The map θ is certainly surjective; moreover, both modules have the same rank, so θ is an isomorphism. □

The last result in this section requires some more notation. Let λ be a partition of n and let μ and ν be compositions of n. Suppose that $\mathtt{S} \in \mathcal{T}_0(\lambda, \mu)$ and $\mathtt{T} \in \mathcal{T}_0(\lambda, \nu)$. Now define

$$m_{\mathtt{ST}} = \sum_{\mathfrak{s},\mathfrak{t}} m_{\mathfrak{st}}$$

where the sum is over all pairs $(\mathfrak{s}, \mathfrak{t})$ of standard λ–tableaux such that $\mu(\mathfrak{s}) = \mathtt{S}$ and $\nu(\mathfrak{t}) = \mathtt{T}$. Then, by (4.7), $m_{\mathtt{ST}} \in M^\mu \cap M^{\nu *}$. This allows us to prove the following result whose importance derives from Corollary 4.9.

4.12 Corollary *Suppose that μ and ν are compositions of n. Then $M^\mu \cap M^{\nu*}$ is free as an R–module with basis*

$$\left\{\, m_{\mathtt{ST}} \;\middle|\; \begin{matrix} \mathtt{S} \in \mathcal{T}_0(\lambda, \mu) \text{ and } \mathtt{T} \in \mathcal{T}_0(\lambda, \nu) \\ \text{for some partition } \lambda \text{ of } n \end{matrix} \right\}.$$

Proof. Suppose that h is a non–zero element of $M^\mu \cap M^{\nu*}$. Then, with respect to the Murphy basis of $\mathscr{H}$ we may write $h = \sum r_{\mathfrak{st}} m_{\mathfrak{st}}$ for some $r_{\mathfrak{st}} \in R$. Now $h \in M^\mu$, so by Theorem 4.10 if $r_{\mathfrak{st}} \neq 0$ then $\mu(\mathfrak{s})$ is semistandard and $r_{\mathfrak{st}} = r_{\mathfrak{s}'\mathfrak{t}}$ whenever $\mu(\mathfrak{s}) = \mu(\mathfrak{s}')$. Similarly, because $h \in M^{\nu*}$ if $r_{\mathfrak{st}} \neq 0$ then $\nu(\mathfrak{t})$ is semistandard and $r_{\mathfrak{st}} = r_{\mathfrak{st}'}$ whenever $\nu(\mathfrak{t}) = \nu(\mathfrak{t}')$. Consequently, if $\mu(\mathfrak{s}) = \mu(\mathfrak{s}')$ and $\nu(\mathfrak{t}) = \nu(\mathfrak{t}')$ then $r_{\mathfrak{st}} = r_{\mathfrak{s}'\mathfrak{t}} = r_{\mathfrak{s}'\mathfrak{t}'}$. The result follows. □

3. The semistandard basis theorem

We are now ready to prove that the q–Schur algebras are cellular. In fact, we shall see that they are quasi–hereditary.

Recall that $\Lambda(d, n)$ is the set of compositions and let $\Lambda^+(d, n)$ be the set of partitions contained in $\Lambda(d, n)$. Suppose that $\lambda \in \Lambda^+(d, n)$, $\mu, \nu \in \Lambda(d, n)$ and that

$\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$ and $\mathtt{T} \in \mathcal{T}_0(\lambda,\nu)$. By Corollary 4.12, $m_{\mathtt{ST}} \in M^\mu \cap M^{\nu *}$ so we can define $\varphi_{\mathtt{ST}}$ to be the element of $\mathscr{S}(d,n)$ given by

4.13 $$\varphi_{\mathtt{ST}}(m_\alpha h) = \delta_{\alpha\mu} m_{\mathtt{ST}} h \text{ for all } \alpha \in \Lambda(d,n) \text{ and } h \in \mathscr{H}.$$

For any partition $\lambda \in \Lambda^+(d,n)$ let $\mathcal{T}_0(\lambda) = \bigcup_{\mu\in\Lambda(d,n)} \mathcal{T}_0(\lambda,\mu)$ and define $\check{\mathscr{S}}^\lambda(d,n)$ to be the R–submodule of $\mathscr{S}(d,n)$ spanned by the homomorphisms $\varphi_{\mathtt{ST}}$ where $\mathtt{S},\mathtt{T} \in \mathcal{T}_0(\alpha)$ for some $\alpha \in \Lambda^+(d,n)$ such that $\alpha \rhd \lambda$. Similarly, $\mathscr{S}^\lambda(d,n)$ is the R–submodule spanned by the maps $\varphi_{\mathtt{ST}}$ where $\mathtt{S},\mathtt{T} \in \mathcal{T}_0(\alpha)$ and $\alpha \trianglerighteq \lambda$.

4.14 The Semistandard Basis Theorem (R. Green) *The q–Schur algebra is free as an R–module with basis*

$$\Phi = \Big\{ \varphi_{\mathtt{ST}} \Bigm| \lambda \in \Lambda^+(d,n) \text{ and } \mathtt{S},\mathtt{T} \in \mathcal{T}_0(\lambda) \Big\}.$$

Moreover,

(i) *The R–linear map $* : \mathscr{S}(d,n) \longrightarrow \mathscr{S}(d,n)$ determined by $\varphi_{\mathtt{ST}}^* = \varphi_{\mathtt{TS}}$, for all $\mathtt{S},\mathtt{T} \in \mathcal{T}_0(\lambda)$ and all $\lambda \in \Lambda^+(d,n)$, is an anti–isomorphism of $\mathscr{S}(d,n)$.*

(ii) *Suppose that $\lambda \in \Lambda^+(d,n)$ and that $\mathtt{T}$ is a semistandard λ–tableau. Then for all $\varphi \in \mathscr{S}(d,n)$ there exist $r_{\mathtt{V}} \in R$ such that for all $\mathtt{S} \in \mathcal{T}_0(\lambda)$*

$$\varphi_{\mathtt{ST}}\varphi \equiv \sum_{\mathtt{V}\in\mathcal{T}_0(\lambda)} r_{\mathtt{V}}\varphi_{\mathtt{SV}} \mod \check{\mathscr{S}}^\lambda(d,n).$$

Consequently, $(\Phi, \Lambda^+(d,n))$ is a cellular basis of $\mathscr{S}(d,n)$.

Proof. Combining Corollary 4.9 and Corollary 4.12 shows that $\mathscr{S}(d,n)$ is free as an R–module with basis Φ. It remains to verify (i) and (ii).

Let λ be a partition in $\Lambda^+(d,n)$ and suppose that $\mathtt{S} \in \mathcal{T}_0(\lambda,\mu)$ and $\mathtt{T} \in \mathcal{T}_0(\lambda,\nu)$. Then $\varphi_{\mathtt{ST}}^*(m_\nu) = m_{\mathtt{ST}}^* = m_{\mathtt{TS}} = \big(\varphi_{\mathtt{TS}}(m_\mu)\big)^*$. Now suppose that $\alpha \in \Lambda^+(d,n)$, $\mathtt{U} \in \mathcal{T}_0(\alpha,\sigma)$ and $\mathtt{V} \in \mathcal{T}_0(\alpha,\tau)$. We may assume that $\nu = \sigma$ since otherwise $\varphi_{\mathtt{ST}}\varphi_{\mathtt{UV}} = 0$. Write $m_{\mathtt{ST}} = h_{\mathtt{S}} m_\nu$ and $m_{\mathtt{UV}} = h_{\mathtt{U}} m_\tau$ for $h_{\mathtt{S}}, h_{\mathtt{U}} \in \mathscr{H}$. Then

$$\begin{aligned}(\varphi_{\mathtt{ST}}\varphi_{\mathtt{UV}})^*(m_\mu) &= \big(\varphi_{\mathtt{ST}}\varphi_{\mathtt{UV}}(m_\tau)\big)^* = \big(h_{\mathtt{S}}h_{\mathtt{U}}m_\tau\big)^* = m_\tau h_{\mathtt{U}}^* h_{\mathtt{S}}^*\\ &= \varphi_{\mathtt{VU}}(m_\nu)h_{\mathtt{S}}^* \quad\; = \varphi_{\mathtt{VU}}(m_\nu h_{\mathtt{S}}^*) = \varphi_{\mathtt{VU}}\varphi_{\mathtt{TS}}(m_\mu).\end{aligned}$$

Hence, $(\varphi_{\mathtt{ST}}\varphi_{\mathtt{UV}})^* = \varphi_{\mathtt{UV}}^*\varphi_{\mathtt{ST}}^*$ and $*$ is an anti–isomorphism, proving (i).

Now consider (ii). We may assume that $\varphi \in \operatorname{Hom}_{\mathscr{H}}(M^\tau, M^\nu)$ for some $\tau \in \Lambda(d,n)$. By Corollary 4.9 there exists an $h \in \mathscr{H}$ such that $\varphi(m_\tau) = m_\nu h$; so, $\varphi_{\mathtt{ST}}\varphi(m_\tau) = m_{\mathtt{ST}}h$. Now $m_{\mathtt{ST}}h$ is an element of $M^\mu \cap M^{\tau *}$ so by Theorem 4.10 we may write $m_{\mathtt{ST}}h = \sum r_{\mathtt{UV}} m_{\mathtt{UV}}$ where $r_{\mathtt{UV}} \in R$ and the sum is over pairs $(\mathtt{U},\mathtt{V})$ of semistandard α–tableaux where $\alpha \in \Lambda^+(d,n)$. Therefore, by Theorem 3.18(ii), there exist $r_{\mathtt{V}}, r_{\mathtt{U}'\mathtt{V}'} \in R$ such that

$$m_{\mathtt{ST}}h = \sum_{\mathtt{V}\in\mathcal{T}_0(\lambda,\tau)} r_{\mathtt{V}} m_{\mathtt{SV}} + \sum_{\substack{\alpha\in\Lambda^+(d,n)\\ \alpha\rhd\lambda}} \sum_{\substack{\mathtt{U}'\in\mathcal{T}_0(\alpha,\mu)\\ \mathtt{V}'\in\mathcal{T}_0(\alpha,\tau)}} r_{\mathtt{U}'\mathtt{V}'} m_{\mathtt{U}'\mathtt{V}'}.$$

Hence, $\varphi_{\mathtt{ST}}\varphi \equiv \sum_{\mathtt{V}\in\mathcal{T}_0(\lambda,\tau)} r_{\mathtt{V}}\varphi_{\mathtt{SV}} \mod \check{\mathscr{S}}^\lambda(d,n)$, which proves (ii). ∎

We call Φ the semistandard basis of $\mathscr{S}(d,n)$.

Now that we have a cellular basis of the q–Schur algebra we can describe its irreducible representations over a field. Traditionally, the cell modules of $\mathscr{S}(d,n)$ are called Weyl modules.

Let $\lambda \in \Lambda^+(d,n)$ be a partition and recall that $\mathtt{T}^\lambda = \lambda(\mathfrak{t}^\lambda)$ is the unique semistandard λ–tableau of type λ. From the definitions one sees that $\varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}$ restricts to the identity map on M^λ.

The Weyl moduleW^λ is the right $\mathscr{S}(d,n)$–module $(\check{\mathscr{S}}^\lambda(d,n) + \varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda})\mathscr{S}(d,n)$. Thus, W^λ is a submodule of $\mathscr{S}(d,n)/\check{\mathscr{S}}^\lambda(d,n)$ and which, as an R–module, is free with basis $\{\, \check{\mathscr{S}}^\lambda(d,n) + \varphi_{\mathtt{T}^\lambda\mathtt{T}} \mid \mathtt{T} \in \mathcal{T}_0(\lambda)\,\}$.

There is another description of W^λ which is more convenient for our purposes. Given a semistandard λ–tableau $\mathtt{T}$ of type μ let $m_\mathtt{T} = \sum_\mathfrak{t} m_\mathfrak{t}$, where the sum is over those standard tableau $\mathfrak{t}$ such that $\mu(\mathfrak{t}) = \mathtt{T}$. Then $m_\mathtt{T} \in S^\lambda$. Define $\varphi_\mathtt{T}$ to be homomorphism in $\operatorname{Hom}_\mathscr{H}(M^\mu, S^\lambda)$ given by $\varphi_\mathtt{T}(m_\mu) = m_\mathtt{T}$. Note that $\operatorname{Hom}_\mathscr{H}(M^\mu, S^\lambda)$ becomes a right $\mathscr{S}(d,n)$–module under composition of maps.

4.15 Proposition *Let $\lambda \in \Lambda^+(d,n)$ be a partition of n. Then there is a canonical isomorphism of $\mathscr{S}(d,n)$–modules*

$$W^\lambda \cong \bigoplus_{\mu\in\Lambda(d,n)} \operatorname{Hom}_\mathscr{H}(M^\mu, S^\lambda) \quad \text{given by} \quad \check{\mathscr{S}}^\lambda(d,n) + \varphi_{\mathtt{T}^\lambda\mathtt{T}} \longmapsto \varphi_\mathtt{T},$$

for all $\mathtt{T} \in \mathcal{T}_0(\lambda)$.

Proof. Let $M = \bigoplus_{\nu\in\Lambda(d,n)} M^\nu$ and $\check{M}^\lambda = \bigoplus_{\mu\in\Lambda(d,n)} \check{\mathscr{H}}^\lambda \cap M^\mu$. Each homomorphism φ in $\check{\mathscr{S}}^\lambda(d,n)$ maps M into $\check{M}^\lambda$; so we may regard $\mathscr{S}(d,n)/\check{\mathscr{S}}^\lambda(d,n)$ as a set of maps from M into $M/\check{M}^\lambda$. In this way, if $\mathtt{T}$ is a semistandard λ–tableau then the homomorphism $\check{\mathscr{S}}^\lambda(d,n) + \varphi_{\mathtt{T}^\lambda\mathtt{T}}$ can be regarded as the map from M^μ into $M/\check{M}^\lambda$ which sends m_μ to $\check{M}^\mu + m_{\mathtt{T}^\lambda\mathtt{T}}$. By the third isomorphism theorem,

$$(\check{M}^\lambda + M^\lambda)/\check{M}^\lambda \cong M^\lambda/(\check{M}^\lambda \cap \mathscr{H}^\lambda) = M^\lambda/(\check{\mathscr{H}}^\lambda \cap M^\lambda) \cong (\check{\mathscr{H}}^\lambda + M^\lambda)/\check{\mathscr{H}}^\lambda.$$

However, $(\check{\mathscr{H}}^\lambda + M^\lambda)/\check{\mathscr{H}}^\lambda = S^\lambda$ so this shows that $\check{\mathscr{S}}^\lambda(d,n) + \varphi_{\mathtt{T}^\lambda\mathtt{T}}$ can be identified with the homomorphism $\varphi_\mathtt{T}$ as claimed. Consequently, the map given in the statement of the Proposition is an $\mathscr{S}(d,n)$–module isomorphism from W^λ into the R–module spanned by $\{\, \varphi_\mathtt{T} \mid \mathtt{T} \in \mathcal{T}_0(\lambda)\,\}$. Conversely, given $\varphi \in \operatorname{Hom}_\mathscr{H}(M^\mu, S^\lambda)$, the above identifications show that we can define an element $\hat{\varphi}$ of W^λ by requiring that $\hat{\varphi}(m_\mu h) = \varphi(m_\mu)h$ for all $h \in \mathscr{H}$. Hence, $\{\, \varphi_\mathtt{T} \mid \mathtt{T} \in \mathcal{T}_0(\lambda)\,\}$ must span the $\mathscr{S}(d,n)$–module $\bigoplus_\mu \operatorname{Hom}_\mathscr{H}(M^\mu, S^\lambda)$. The Proposition follows. □

Hereafter, we identify W^λ and $\bigoplus_\mu \operatorname{Hom}_\mathscr{H}(M^\mu, S^\lambda)$. In particular, we will call $\{\, \varphi_\mathtt{T} \mid \mathtt{T} \in \mathcal{T}_0(\lambda)\,\}$ the semistandard basis of W^λ.

As in Chapter 2, the Weyl module W^λ possesses an associative symmetric bilinear form which is completely determined by the equation

$$\langle \varphi_\mathtt{S}, \varphi_\mathtt{T}\rangle \varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda} \equiv \varphi_{\mathtt{T}^\lambda\mathtt{S}}\varphi_{\mathtt{T}\mathtt{T}^\lambda} \mod \check{\mathscr{S}}^\lambda(d,n)$$

for all $\mathtt{S}, \mathtt{T} \in \mathcal{T}_0(\lambda)$. In particular, $\langle \varphi_\mathtt{S}, \varphi_\mathtt{T}\rangle = 0$ unless $\mathtt{S}$ and $\mathtt{T}$ are semistandard tableaux of the same type.

Let $L^\lambda = W^\lambda/\operatorname{rad} W^\lambda$ where, once again, $\operatorname{rad} W^\lambda$ is the set of elements $x \in W^\lambda$ such that $\langle x, y\rangle = 0$ for all $y \in W^\lambda$.

4.16 Theorem (Dipper–James) *Suppose that R is a field.*

(i) *Let λ be a partition in $\Lambda^+(d,n)$. Then L^λ is an absolutely irreducible $\mathscr{S}(d,n)$–module.*

(ii) *$\{\, L^\lambda \mid \lambda \in \Lambda^+(d,n)\,\}$ is a complete set of non–isomorphic irreducible $\mathscr{S}(d,n)$–modules.*

Proof. We have already remarked that $\varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}$ is the identity map on M^λ. Therefore, we compute that

$$\langle\varphi_{\mathtt{T}^\lambda},\varphi_{\mathtt{T}^\lambda}\rangle\varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}\equiv\varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}\varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}=\varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}\mod\check{\mathscr{S}}^\lambda(d,n);$$

so $\langle\varphi_{\mathtt{T}^\lambda},\varphi_{\mathtt{T}^\lambda}\rangle=1$. In particular, $L^\lambda\neq 0$. Parts (i) and (ii) now follow from Theorem 2.13. □

4.17 Corollary (Parshall–Wang) *Suppose that R is a field. Then $\mathscr{S}(d,n)$ is quasi–hereditary.*

Proof. This is immediate from Theorem 4.16(ii) and Corollary 2.19. □

Given two partitions λ and μ in $\Lambda(d,n)$ let $d_{\lambda\mu}=[W^\lambda:L^\mu]$ be the composition multiplicity of the simple module L^μ in W^λ. Corollary 2.14 yields the next result.

4.18 Corollary *Suppose that R is a field and let λ and μ be partitions in $\Lambda^+(d,n)$. Then $d_{\mu\mu}=1$ and if $d_{\lambda\mu}>0$ then $\mu\rhd\lambda$.*

In Chapter 3 we defined $d_{\lambda\mu}$ to be the decomposition multiplicity $[S^\lambda:D^\mu]$; we close this chapter by investigating more closely the links between the Iwahori–Hecke algebras and the q–Schur algebras when $d\geq n$. In particular, we show that if μ is e–restricted then $[S^\lambda:D^\mu]=[W^\lambda:L^\mu]$; thus, the notation $d_{\lambda\mu}$ is consistent in this case. (If these two sets of decomposition numbers being equal worries you, reread the warning on page 25.)

For the remainder of this section we assume that $\omega\in\Lambda^+(d,n)$; or equivalently, that $d\geq n$. In this case, $\operatorname{Hom}_{\mathscr{H}}(\mathscr{H},\mathscr{H})$ is an R–submodule of $\mathscr{S}(d,n)$. Furthermore, there is a canonical isomorphism of R–modules $\mathscr{H}\cong\operatorname{Hom}_{\mathscr{H}}(\mathscr{H},\mathscr{H})$ given by $h\longmapsto\rho_h$ where ρ_h is the homomorphism given by left multiplication by h. In fact, $\mathscr{H}\cong\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\mathscr{S}(d,n)\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}$ since given standard λ–tableaux $\mathtt{S}\in\mathcal{T}_0(\lambda,\mu)$ and $\mathtt{T}\in\mathcal{T}_0(\lambda,\nu)$ we have that $\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\varphi_{\mathtt{ST}}\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}=\delta_{\omega\mu}\delta_{\nu\omega}\varphi_{\mathtt{ST}}$.

Noting that $\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}$ is idempotent in $\mathscr{S}(d,n)$, we can define a functor from the category of right $\mathscr{S}(d,n)$–modules to the category of right $\mathscr{H}$–modules; the Schur functor is the functor which sends a right $\mathscr{S}(d,n)$–module M to $M\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}$.

4.19 Proposition *Suppose that R is a field and that $d\geq n$. Let λ be a partition of n. Then, as right $\mathscr{H}$–modules,*
 (i) $W^\lambda\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\cong S^\lambda$;
 (ii) $L^\lambda\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\cong D^\lambda$.
Furthermore, if μ is an e–restricted partition of n then $[W^\lambda:L^\mu]=[S^\lambda:D^\mu]$.

Proof. Suppose that $\mathtt{T}$ is a standard λ–tableau of type μ. Then $\varphi_{\mathtt{T}}\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}=0$ unless $\mu=\omega$ and, if $\mu=\omega$, then $\mathtt{T}=\mathfrak{t}$ is a standard λ–tableau and $\varphi_{\mathtt{T}}\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}=\varphi_{\mathfrak{t}}$. Consequently, $W^\lambda\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}$ is free as an R–module with basis $\{\,\varphi_{\mathfrak{t}}\mid\mathfrak{t}\in\operatorname{Std}(\lambda)\,\}$ and the map $W^\lambda\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\to S^\lambda$ which sends $\varphi_{\mathfrak{t}}$ to $\varphi_{\mathfrak{t}}(m_\omega)=m_{\mathfrak{t}}$ is an isomorphism (cf. Proposition 4.15).

Part (ii) of the Proposition nows follows from Theorem 3.39(ii) and Corollary 4.18. In particular, this shows that if $L^\lambda\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\neq 0$ then λ is e–restricted.

Finally, let $W^\lambda=W^1>\cdots>W^k>0$ be a composition series for W^λ. Then $S^\lambda\cong W^1\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\geq\cdots\geq W^k\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\geq 0$ gives an $\mathscr{H}$–module filtration of S^λ with quotients $W^i\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}/W^{i+1}\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\cong(W^i/W^{i+1})\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}\cong L^{\mu_i}\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}$ for some partitions μ_i. By the last paragraph, $L^{\mu_i}\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}$ is simple when μ_i is e–restricted and zero otherwise. Therefore, by omitting the modules $W^i\varphi_{\mathtt{T}^\omega\mathtt{T}^\omega}$ such that $W^i\varphi_{\mathrm{T}^\omega\mathrm{T}^\omega} \cong W^{i+1}\varphi_{\mathrm{T}^\omega\mathrm{T}^\omega}$ we obtain a composition series for S^λ which has the required multiplicities. This proves (iii). □

Finally, let $M = \bigoplus_{\mu\in\Lambda(d,n)} M^\mu$. Then M is naturally a left $\mathscr{S}(d,n)$–module with action $\varphi\cdot m = \varphi(m)$ for all $\varphi\in\mathscr{S}(d,n)$ and all $m\in M$. Of course, M is also a right $\mathscr{H}$–module and it is clear that the actions of $\mathscr{S}(d,n)$ and $\mathscr{H}$ commute.

4.20 Theorem (The double centralizer property) *Suppose that $d\ge n$. Then*

$$\mathscr{S}(d,n)\cong \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\mu\in\Lambda(d,n)} M^\mu\Big) \quad\textit{and}\quad \mathscr{H}\cong \operatorname{End}_{\mathscr{S}(d,n)}\Big(\bigoplus_{\mu\in\Lambda(d,n)} M^\mu\Big).$$

Proof. The first isomorphism is just the definition of the q–Schur algebra, so there is nothing to prove here. For the second isomorphism, notice that by Corollary 4.9 for any composition μ we have $M^\mu = M^\mu\cap\mathscr{H}^*\cong \operatorname{Hom}(\mathscr{H},M^\mu)$. In fact, we claim that as $\mathscr{H}$–modules $\varphi_{\mathrm{T}^\mu\mathrm{T}^\mu}\mathscr{S}(d,n)\varphi_{\mathrm{T}^\omega\mathrm{T}^\omega}\cong M^\mu$. To see this first notice that by Theorem 4.14, the R–module $\varphi_{\mathrm{T}^\mu\mathrm{T}^\mu}\mathscr{S}(d,n)\varphi_{\mathrm{T}^\omega\mathrm{T}^\omega}$ has basis $\{\varphi_{\mathsf{St}}\}$ where $\mathsf{S}\in\mathcal{T}_0(\lambda,\mu)$ and $\mathsf{t}\in\operatorname{Std}(\lambda)$ for some $\lambda\in\Lambda^+(d,n)$. Hence, the R–linear map determined by $\varphi_{\mathsf{St}}\longmapsto\varphi_{\mathsf{St}}(m_\mu)$ is an isomorphism by Theorem 4.10. Consequently,

$$\bigoplus_{\mu\in\Lambda(d,n)} M^\mu \cong \bigoplus_{\mu\in\Lambda(d,n)} \operatorname{Hom}_{\mathscr{H}}(\mathscr{H},M^\mu)\cong \bigoplus_{\mu\in\Lambda(d,n)} \varphi_{\mathrm{T}^\mu\mathrm{T}^\mu}\mathscr{S}(d,n)\varphi_{\mathrm{T}^\omega\mathrm{T}^\omega}.$$

However, $\sum_\mu \varphi_{\mathrm{T}^\mu\mathrm{T}^\mu}$ is the identity element of the q–Schur algebra so, as an $\mathscr{S}(d,n)$–module, $\bigoplus_\mu M^\mu\cong\mathscr{S}(d,n)\varphi_{\mathrm{T}^\omega\mathrm{T}^\omega}$. Therefore, as algebras,

$$\operatorname{End}_{\mathscr{S}(d,n)}\Big(\bigoplus_\mu M^\mu\Big)\cong \operatorname{End}_{\mathscr{S}(d,n)}\Big(\mathscr{S}(d,n)\varphi_{\mathrm{T}^\omega\mathrm{T}^\omega}\Big)\cong \varphi_{\mathrm{T}^\omega\mathrm{T}^\omega}\mathscr{S}(d,n)\varphi_{\mathrm{T}^\omega\mathrm{T}^\omega};$$

the last isomorphism coming from Lemma 2.20. Since $\mathscr{H}$ and $\varphi_{\mathrm{T}^\omega\mathrm{T}^\omega}\mathscr{S}(d,n)\varphi_{\mathrm{T}^\omega\mathrm{T}^\omega}$ are canonically isomorphic, the Theorem follows. □

In fact, the argument shows that the isomorphisms in the Theorem are canonical.

Notes on Chapter 4

1. The q–Schur algebra was introduced by Dipper and James in [**15**] where they showed that it completely determined the representation theory of the finite general linear groups in non–defining characteristic. This mirrors the classical theory, for which the reader is referred to Green's excellent book [**27**].
2. Beilinson, Lusztig and MacPherson [**3**] have defined the q–Schur algebra using the geometry of the general linear groups (that these definitions coincide shown by Du [**19**]) and showed that there is a natural surjection from the quantum general linear group $U_q(gl_d)$ onto $\mathscr{S}(d,n)$. Latter Parshall and Wang [**58**] (see also Dipper and Donkin [**12**]) showed how to construct $U_q(gl_d)$ from the q–Schur algebras. For details consult Brundan [**5**] and Donkin [**18**].
3. A semistandard basis theorem for the Weyl modules of the q–Schur algebra was first proved by Dipper and James [**16**]; the treatment here is considerably easier and is based on [**17**].

4. In a different guise, the semistandard basis of the Schur algebra (i.e. the case $q = 1$), first appeared as J. A. Green's [**28**] codeterminant basis; shortly afterwards, Woodcock [**62**] gave a more direct proof. Richard Green [**29**] generalized Woodcocks argument to give a q–analogue of the codeterminant basis; it turns out that this basis is the semistandard basis of Theorem 4.14. The construction of the semistandard basis given here is based [**17**] and is very different from Green's; that the semistandard basis and the Murphy basis are related was first observed by Du and Scott [**21**].
5. In [**18**] Donkin parameterizes the irreducible modules of the 0–Schur algebra over a field and also gives an explicit character formula for the simple modules. As one might expect, the 0–Schur algebra is not quasi–hereditary; surprisingly, however, it is a quasi–Frobenius algebra (at least when $d \geq n$).
6. It is shown in [**10**, 61.2] that whenever A is a quasi–Frobenius algebra, $a \in A$ and J is an ideal of A then $\operatorname{Hom}_A(J, aA) \cong J \cap aA$. By Proposition 1.14, $\mathscr{H}$ is quasi–Frobenius so this gives another proof of Corollary 4.9. (See also [**17**, §5].)
7. Parshall and Wang [**58**] first showed that the q–Schur algebra is quasi–hereditary. In particular, this implies that the q–Schur algebras have finite global dimension; this dimension has been explicitly calculated by Donkin [**18**] when $d \geq n$.
8. The double centralizer property (4.20) is normally phrased in terms of actions of $\mathscr{S}(d, n)$ and $\mathscr{H}$ upon "q–tensor space"; see [**5**, **16**].

CHAPTER 5

The Jantzen–Schaper theorem

In this chapter we prove an analogue of the Jantzen sum formula for the q–Schur algebra. This is a strong result describing the structure of Weyl modules; however, its proof follows from the calculation of Gram determinants. As our main application of this result we classify the blocks of $\mathscr{H}$.

1. The Gram determinant of W^λ

Throughout this section we fix a partition $\lambda \in \Lambda^+(d,n)$ and a composition $\mu \in \Lambda(d,n)$. The μ–**weight space** of the Weyl module W^λ is $W^\lambda_\mu = W^\lambda \varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}$; it is the free R–submodule of $\mathscr{S}(d,n)$ with basis $\{\, \varphi_{\mathtt{T}} \mid \mathtt{T} \in \mathcal{T}_0(\lambda,\mu)\,\}$ and, by Proposition 4.15, $W^\lambda_\mu \cong \operatorname{Hom}_{\mathscr{H}}(M^\mu, S^\lambda)$. The aim of this section is to compute the **Gram determinant** $G^\lambda_\mu = \det\big(\langle \varphi_{\mathtt{S}}, \varphi_{\mathtt{T}}\rangle\big)$ of W^λ_μ, where and $\mathtt{S}$ and $\mathtt{T}$ run over $\mathcal{T}_0(\lambda,\mu)$ in some fixed order.

Suppose that k is an integer with $1 \le k \le d$ and let $x, x+1, \dots, y$ be the entries in row k of $\mathfrak{t}^\mu$. Proposition 3.24(iii) shows that

5.1 $$(L_x + L_{x+1} + \cdots L_y)m_\mu = m_\mu(L_x + L_{x+1} + \cdots L_y).$$

Hence, we can define a map $L^\mu_i : M^\mu \longrightarrow M^\mu$ by specifying that

$$L^\mu_k(m_\mu h) = (L_x + L_{x+1} + \cdots + L_y)m_\mu h,$$

for all $h \in \mathscr{H}$. Note that L^ω_k is just left multiplication by L_k, so these elements are a generalization of the Jucys–Murphy elements of Chapter 2. Furthermore, L^μ_k and L^μ_j commute for all j and k by Proposition 3.24(iv).

Let $\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)$. As in (3.5), if k is an integer let $\mathtt{T}\!\downarrow\! k$ be the tableau obtained from $\mathtt{T}$ by deleting all entries greater than k. Then, extending the dominance ordering on tableaux of type ω we write $\mathtt{S} \trianglerighteq \mathtt{T}$ if $\operatorname{Shape}(\mathtt{S}\!\downarrow\! k) \trianglerighteq \operatorname{Shape}(\mathtt{T}\!\downarrow\! k)$ for $k = 1, 2, \dots, n$. It follows from Proposition 3.3 and Proposition 4.5 that $\mathtt{S} \trianglerighteq \mathtt{T}$ if and only if $d(\operatorname{first}(\mathtt{S})) \trianglerighteq d(\operatorname{first}(\mathtt{T}))$.

Finally, let $[\operatorname{res}_{\mathtt{T}}(k)]_q = \displaystyle\sum_{x\in[\lambda]} [\operatorname{res}_{\mathtt{T}}(x)]_q$, where the sum is over those nodes x such that $\mathtt{T}(x) = k$ (see (3.28)).

5.2 Example Suppose that $e = 3$ and let $\lambda = (3,2)$ and $\mu = (2^2,1)$. Then $\mathcal{T}_0(\lambda,\mu) = \{\mathtt{S},\mathtt{T}\}$ where $\mathtt{S} = \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array}$ and $\mathtt{T} = \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2}\end{array}$. From the definitions, $\mathtt{S} \rhd \mathtt{T}$, $[\operatorname{res}_{\mathtt{S}}(1)]_q = [0]_q + [1]_q$, $[\operatorname{res}_{\mathtt{S}}(2)]_q = [2]_q + [2]_q$ and $[\operatorname{res}_{\mathtt{S}}(3)]_q = [0]_q$.

5.3 Theorem (James–Mathas) *Suppose that $\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)$ and let k be an integer with $1 \le k \le d$. Then*

$$\varphi_{\mathtt{T}} L^\mu_k = [\operatorname{res}_{\mathtt{T}}(k)]_q \varphi_{\mathtt{T}} + \sum_{\mathtt{S}\rhd\mathtt{T}} r_{\mathtt{S}}\varphi_{\mathtt{S}},$$

for some $r_{\mathtt{S}} \in R$.

Proof. Let $\mathfrak{t} = \text{last}(\mathtt{T})$; then $m_\mathtt{T} = m_\mathfrak{t} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} m_\mathfrak{s}$ where the sum is over those standard λ–tableaux such that $\mu(\mathfrak{s}) = \mathtt{T}$. Therefore, by Theorem 3.30,

$$\begin{aligned}\varphi_\mathtt{T} L_k^\mu(m_\mu) &= m_\mathtt{T}(L_x + \cdots + L_y)\\ &= \Big\{ m_\mathfrak{t} + \sum_{\mathfrak{s}\rhd\mathfrak{t}:\mu(\mathfrak{s})=\mathtt{T}} m_\mathfrak{s} \Big\}(L_x + \cdots + L_y)\\ &= [\text{res}_\mathtt{T}(k)]_q m_\mathfrak{t} + \sum_{\mathfrak{s}\rhd\mathfrak{t}} b_\mathfrak{s} m_\mathfrak{s},\end{aligned}$$

for some $b_\mathfrak{s} \in R$. However, $\varphi_{\mathtt{T}^\lambda\mathtt{T}} L_k^\mu(m_\mu) \in M^\lambda \cap M^{\mu*}$ so by Corollary 4.12 there exist $b_\mathtt{S} \in R$ such that

$$\varphi_\mathtt{T} L_k^\mu(m_\mu) = [\text{res}_\mathtt{T}(k)]_q m_\mathtt{T} + \sum_{\mathtt{S}\rhd\mathtt{T}} b_\mathtt{S} m_\mathtt{S}.$$

Hence, $\varphi_\mathtt{T} L_k^\mu = [\text{res}_\mathtt{T}(k)]_q \varphi_\mathtt{T} + \sum_{\mathtt{S}\rhd\mathtt{T}} b_\mathtt{S}\varphi_\mathtt{S}$ proving the Theorem. □

Recall that e is the smallest positive integer (or $e = \infty$) such that $[e]_q = 0$. Just as in section 3 of Chapter 3, until further notice we assume that R is a field and that $e > n$. An argument similar to that of Lemma 3.32 proves the following.

5.4 Lemma *Suppose that R is a field and that $e > n$. Let $\mathtt{S}$ and $\mathtt{T}$ be two semistandard λ–tableaux of type μ. Then $[\text{res}_\mathtt{S}(k)]_q \neq [\text{res}_\mathtt{T}(k)]_q$ for some integer k.*

Exactly as in (3.31), for any semistandard λ–tableau $\mathtt{T}$ of type μ let

$$F_\mathtt{T} = \prod_{k\geq 1} \prod_{\substack{\mathtt{S}\in\mathcal{T}_0(\lambda,\mu)\\ [\text{res}_\mathtt{S}(k)]_q\neq[\text{res}_\mathtt{T}(k)]_q}} \frac{L_k^\mu - [\text{res}_\mathtt{S}(k)]_q}{[\text{res}_\mathtt{S}(k)]_q - [\text{res}_\mathtt{T}(k)]_q}$$

and define $\psi_\mathtt{T} = \varphi_\mathtt{T} F_\mathtt{T}$.

Because of Theorem 5.3 and Lemma 5.4 the proof of Proposition 3.33 can now be repeated, word for word, to prove the following result.

5.5 Proposition *Suppose that R is a field and that $e > n$. Let $\mathtt{T}$ be a semistandard λ–tableau of type μ. Then*

(i) *There exist $a_\mathtt{S} \in R$ such that $\psi_\mathtt{T} = \varphi_\mathtt{T} + \sum_{\mathtt{S}\rhd\mathtt{T}} a_\mathtt{S}\varphi_\mathtt{S}$.*
(ii) *If $\mathtt{S} \rhd \mathtt{T}$ then $\varphi_\mathtt{S}\psi_\mathtt{T} = 0$.*
(iii) *For $k = 1, \dots, n$, $\psi_\mathtt{T} L_k = [\text{res}_\mathtt{T}(k)]_q \psi_\mathtt{T}$.*
(iv) *For any $\mathtt{S} \in \text{Std}(\lambda)$, $\psi_\mathtt{T} F_\mathtt{S} = \delta_{\mathtt{ST}}\psi_\mathtt{T}$.*

5.6 Corollary *$\{\, \psi_\mathtt{T} \mid \mathtt{T} \in \mathcal{T}_0(\lambda,\mu) \,\}$ is an orthogonal basis of W_μ^λ.*

Proof. That $\{\, \psi_\mathtt{T} \mid \mathtt{T} \in \mathcal{T}_0(\lambda,\mu) \,\}$ is a basis of W_μ^λ is immediate from part (i) of the Proposition. Using part (iv) we find that for any semistandard λ–tableaux $\mathtt{S}$ and $\mathtt{T}$ of type μ we have $\langle \psi_\mathtt{S}, \psi_\mathtt{T}\rangle = \langle \varphi_\mathtt{S} F_\mathtt{S}, \psi_\mathtt{T}\rangle = \langle \varphi_\mathtt{S}, \psi_\mathtt{T} F_\mathtt{S}\rangle = \delta_{\mathtt{ST}}\langle \varphi_\mathtt{S}, \psi_\mathtt{S}\rangle$, proving the Corollary. □

In order to compute the Gram determinant G_μ^λ of W_μ^λ we determine the inner products $\langle \psi_\mathtt{T}, \psi_\mathtt{T}\rangle$ for all $\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)$. For this we need some combinatorics.

Let $\mathtt{T}$ be a semistandard λ–tableau of type μ. For each node $(i,j) \in [\lambda]$ let

$$\Gamma_\mathtt{T}(i,j) = \{\, (k,l) \in [\lambda] \mid l < j, \mathtt{T}(k,l) < \mathtt{T}(i,j) \text{ and } \mathtt{T}(k',l) > \mathtt{T}(i,j) \text{ for all } k' > k \,\}$$

and let

$$\gamma_{\mathtt{T}}(q) = \prod_{(i,j)\in[\lambda]} \prod_{(k,l)\in\Gamma_{\mathtt{T}}(i,j)} \frac{[j-i+k-l+1]_q}{[j-i+k-l]_q}.$$

Most of the time we omit the q and simply write $\gamma_{\mathtt{T}}$.

Suppose that is $\mathfrak{t}$ is a standard λ–tableau and let $x=(i,j)$ and $y=(k,l)$. Then $\mathrm{res}_{\mathfrak{t}}(x)-\mathrm{res}_{\mathfrak{t}}(y) = j-i+l-k$ is the "axial distance" from x to y in the diagram of λ.

5.7 Example Suppose that $\lambda=(3,2)$ and let $\mathfrak{t} = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$, $\mathtt{S} = \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array}$ and $\mathtt{T} = \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2}\end{array}$. Then

$$\gamma_{\mathfrak{t}} = \frac{[2]_q}{[1]_q}\cdot\frac{[2]_q}{[1]_q}\frac{[4]_q}{[3]_q}\cdot\frac{[2]_q}{[1]_q}, \qquad \gamma_{\mathtt{S}} = \frac{[2]_q}{[1]_q}\frac{[2]_q}{[1]_q} \qquad\text{and}\qquad \gamma_{\mathtt{T}} = \frac{[3]_q}{[2]_q}\frac{[4]_q}{[3]_q}.$$

Notice that if $\mu=(2^2,1)$ then $\mathcal{T}_0(\lambda,\mu)=\{\mathtt{S},\mathtt{T}\}$ and $\gamma_{\mathtt{S}}\gamma_{\mathtt{T}} = [2]_q[4]_q$, so $\gamma_{\mathtt{S}}\gamma_{\mathtt{T}}\in\mathbb{Z}[q]$.

Recall that $[m]_q^! = [1]_q\dots[m]_q$. For any partition $\lambda=(\lambda_1,\dots,\lambda_k)$ we also let $[\lambda]_q^! = [\lambda_1]_q^!\dots[\lambda_k]_q^!$.

5.8 Lemma *Let $\lambda\in\Lambda^+(d,n)$. Then $\gamma_{\mathfrak{t}^\lambda} = [\lambda]_q^!$.*

Proof. When $n=1$ then $\lambda=(1)$ and $\gamma_{\mathfrak{t}^\lambda}=1$ so the Lemma is correct. Now suppose that $n>1$ and let $\lambda=(\lambda_1,\dots,\lambda_r)$, where $\lambda_r>0$, and let ν be the partition $(\lambda_1,\dots,\lambda_r-1)$. By induction, $\gamma_{\mathfrak{t}^\nu}=[\nu]_q^!$. Consequently,

$$\gamma_{\mathfrak{t}^\lambda} = \gamma_{\mathfrak{t}^\nu}\cdot\frac{[2]_q}{[1]_q}\cdots\frac{[\lambda_r]_q}{[\lambda_r-1]_q} = [\nu]_q^![\lambda_r]_q = [\lambda]_q^!,$$

completing the proof. □

5.9 Lemma *Suppose that $\mathfrak{s}$ and $\mathfrak{t}$ are standard λ–tableau such that $\mathfrak{s}\rhd\mathfrak{t}$ and $\mathfrak{s}=\mathfrak{t}(i,i+1)$ for some integer i with $1\le i<n$. Then $\gamma_{\mathfrak{t}} = q^{-1}(q+\alpha)(1-\alpha)\gamma_{\mathfrak{s}}$ where $\alpha^{-1} = [\mathrm{res}_{\mathfrak{s}}(i)-\mathrm{res}_{\mathfrak{t}}(i)]_q$.*

Proof. By definition, if $(a,b)\in[\lambda]$ then the sets $\Gamma_{\mathfrak{s}}(a,b)$ and $\Gamma_{\mathfrak{t}}(a,b)$ are equal unless $\mathfrak{s}(a,b)=i$. Furthermore, if $\mathfrak{s}(a,b)=i$ and $\mathfrak{s}(c,d)=i+1$ then $a>c$ and $d<b$ so $\Gamma_{\mathfrak{s}}(a,b)\cup\{(c-1,d)\} = \Gamma_{\mathfrak{t}}(a,b)\cup\{(c,d)\}$, where both unions are disjoint. It follows that

$$\gamma_{\mathfrak{t}} = \frac{[\mathrm{res}_{\mathfrak{s}}(i)-\mathrm{res}_{\mathfrak{t}}(i)+1]_q}{[\mathrm{res}_{\mathfrak{s}}(i)-\mathrm{res}_{\mathfrak{t}}(i)]_q}\times\frac{[\mathrm{res}_{\mathfrak{s}}(i)-\mathrm{res}_{\mathfrak{t}}(i)-1]_q}{[\mathrm{res}_{\mathfrak{s}}(i)-\mathrm{res}_{\mathfrak{t}}(i)]_q}\times\gamma_{\mathfrak{s}} = q^{-1}(q+\alpha)(1-\alpha)\gamma_{\mathfrak{s}}$$

as required. □

5.10 Proposition *Let $\mathfrak{t}$ be a standard λ–tableaux. Then $\langle\psi_{\mathfrak{t}},\psi_{\mathfrak{t}}\rangle = q^{\ell(d(\mathfrak{t}))}\gamma_{\mathfrak{t}}$.*

Proof. Suppose first that $\mathfrak{t}=\mathfrak{t}^\lambda$; then $\psi_{\mathfrak{t}^\lambda}=\varphi_{\mathfrak{t}^\lambda}$ by Proposition 5.5(i). By definition, $\langle\psi_{\mathfrak{t}^\lambda},\psi_{\mathfrak{t}^\lambda}\rangle\varphi_{\mathtt{T}^\lambda\mathtt{T}^\lambda}\equiv\varphi_{\mathtt{T}^\lambda\mathfrak{t}^\lambda}\varphi_{\mathfrak{t}^\lambda\mathtt{T}^\lambda} \mod \check{\mathscr{S}}^\lambda(d,n)$, and

$$\varphi_{\mathtt{T}^\lambda\mathfrak{t}^\lambda}\varphi_{\mathfrak{t}^\lambda\mathtt{T}^\lambda}(m_\lambda) = \varphi_{\mathtt{T}^\lambda\mathfrak{t}^\lambda}(m_\lambda) = \varphi_{\mathtt{T}^\lambda\mathfrak{t}^\lambda}(1)m_\lambda = m_\lambda m_\lambda = [\lambda]_q^! m_\lambda$$

by Lemma 3.2. Thus, $\langle\psi_{\mathfrak{t}^\lambda},\psi_{\mathfrak{t}^\lambda}\rangle = \gamma_{\mathfrak{t}^\lambda}$ by Lemma 5.8.

Now suppose that $\mathfrak{t}\ne\mathfrak{t}^\lambda$. Then there exists a standard λ–tableau $\mathfrak{s}$ such that $\mathfrak{s}\rhd\mathfrak{t}$ and $\mathfrak{s}=\mathfrak{t}(i,i+1)$ for some i. Furthermore, by Theorem 3.34, $f_{\mathfrak{s}}T_i = f_{\mathfrak{t}}-\alpha f_{\mathfrak{s}}$, where $\alpha^{-1} = [\mathrm{res}_{\mathfrak{s}}(i) - \mathrm{res}_{\mathfrak{t}}(i)]_q$. Therefore, $\psi_{\mathfrak{t}} = \psi_{\mathfrak{s}} T_i + \alpha\psi_{\mathfrak{s}}$ since $\psi_{\mathfrak{v}}(1) = f_{\mathfrak{v}}$ for all $v \in \mathrm{Std}(\lambda)$; consequently,

$$\begin{aligned}
\langle \psi_{\mathfrak{t}}, \psi_{\mathfrak{t}} \rangle &= \langle \psi_{\mathfrak{s}} T_i + \alpha\psi_{\mathfrak{s}}, \psi_{\mathfrak{s}} T_i + \alpha\psi_{\mathfrak{s}} \rangle \\
&= \langle \psi_{\mathfrak{s}} T_i, \psi_{\mathfrak{s}} T_i \rangle + 2\alpha \langle \psi_{\mathfrak{s}} T_i, \psi_{\mathfrak{s}} \rangle + \alpha^2 \langle \psi_{\mathfrak{s}}, \psi_{\mathfrak{s}} \rangle \\
&= \langle \psi_{\mathfrak{s}} T_i^2, \psi_{\mathfrak{s}} \rangle + 2\alpha \langle \psi_{\mathfrak{s}} T_i, \psi_{\mathfrak{s}} \rangle + \alpha^2 \langle \psi_{\mathfrak{s}}, \psi_{\mathfrak{s}} \rangle \\
&= (2\alpha + q - 1) \langle \psi_{\mathfrak{s}} T_i, \psi_{\mathfrak{s}} \rangle + (q + \alpha^2) \langle \psi_{\mathfrak{s}}, \psi_{\mathfrak{s}} \rangle \\
&= (2\alpha + q - 1) \langle \psi_{\mathfrak{t}} - \alpha\psi_{\mathfrak{s}}, \psi_{\mathfrak{s}} \rangle + (q + \alpha^2) \langle \psi_{\mathfrak{s}}, \psi_{\mathfrak{s}} \rangle \\
&= (q + \alpha)(1 - \alpha) \langle \psi_{\mathfrak{s}}, \psi_{\mathfrak{s}} \rangle,
\end{aligned}$$

where for the last equality we have used Corollary 5.6. However, by induction on dominance, $\langle \psi_{\mathfrak{s}}, \psi_{\mathfrak{s}} \rangle = q^{\ell(d(\mathfrak{s}))}\gamma_{\mathfrak{s}}$; so, by Lemma 5.9, $\langle \psi_{\mathfrak{t}}, \psi_{\mathfrak{t}} \rangle = q^{1+\ell(d(\mathfrak{s}))}\gamma_{\mathfrak{t}} = q^{\ell(d(\mathfrak{t}))}\gamma_{\mathfrak{t}}$ as required. □

5.11 Corollary *Suppose that* $\mathfrak{t}$ *is a standard* λ*–tableaux. Then* $\langle f_{\mathfrak{t}}, f_{\mathfrak{t}} \rangle = q^{\ell(d(\mathfrak{t}))}\gamma_{\mathfrak{t}}$. *Consequently, the Gram determinant of the Specht module* S^λ *is equal to*

$$G_\omega(\lambda) = \prod_{\mathfrak{t} \in \mathrm{Std}(\lambda)} q^{\ell(d(\mathfrak{t}))}\gamma_{\mathfrak{t}}.$$

Now we turn to tableaux of arbitrary type. To do this we need to compare $\varphi_{\mathtt{T}^\omega \mathtt{T}^\mu}\varphi_{\mathfrak{t}}$ and $\varphi_{\mathtt{T}}$ when $\mathtt{T} = \mu(\mathfrak{t})$; this will be accomplished by the next definition. Let $\mathfrak{t}$ be a standard λ–tableau such that $\mathtt{T} = \mu(\mathfrak{t})$ is semistandard. For each node $(i, j) \in [\lambda]$ let

$$P^\mu_{\mathfrak{t}}(i, j) = \{\, (k, l) \in [\lambda] \mid l < j,\ \mathfrak{t}(k, l) < \mathfrak{t}(i, j) \text{ and } \mathtt{T}(i, j) = \mathtt{T}(k, l) \,\}$$

and

$$\pi^\mu_{\mathfrak{t}} = \prod_{(i,j) \in [\lambda]} \prod_{(k,l) \in P^\mu_{\mathfrak{t}}(i,j)} \frac{[j - i + k - l + 1]_q}{[j - i + k - l]_q}.$$

5.12 Example Suppose that $\lambda = (3, 2)$ and $\mu = (2^2, 1)$. Then $\mathtt{T} = \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array}$ is a standard λ–tableau of type μ and $\mathfrak{s} = \mathrm{first}(\mathtt{T}) = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$ and $\mathfrak{t} = \mathrm{last}(\mathtt{T}) = \begin{array}{|c|c|c|}\hline 1&1&4\\\hline 3&5\\\cline{1-2}\end{array}$. Therefore,

$$\pi^\mu_{\mathfrak{s}} = \frac{[2]_q}{[1]_q} \quad \text{and} \quad \pi^\mu_{\mathfrak{t}} = \frac{[2]_q}{[1]_q}\frac{[4]_q}{[3]_q}.$$

Notice that $\gamma_{\mathfrak{t}} = \pi^\mu_{\mathfrak{t}}\gamma_{\mathtt{T}}$ by Example 5.7.

Let $d \in \mathscr{D}_{\lambda\mu}$ and recall from Proposition 4.5 that $\nu = \lambda d \cap \mu$ is the composition of n such that $\mathfrak{S}_\nu = d^{-1}\mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu$.

5.13 Lemma *Let* $\mathtt{T}$ *be a semistandard* λ*–tableau of type* μ.

(i) *Let* $\mathfrak{t} = \mathrm{first}(\mathtt{T})$ *and let* $d = d(\mathfrak{t})$. *Then* $\pi^\mu_{\mathfrak{t}} = [\nu]^!_q$ *where* $\nu = \lambda d \cap \mu$.

(ii) *Let* $\mathfrak{t} = \mathrm{last}(\mathtt{T})$. *Then* $\gamma_{\mathfrak{t}} = \pi^\mu_{\mathfrak{t}}\gamma_{\mathtt{T}}$.

Proof. Let $\mathfrak{t} = \mathrm{first}(\mathtt{T})$ and suppose that $(i, j) \in [\lambda]$. Then

$$P^\mu_{\mathfrak{t}}(i, j) = \{\, (k, l) \in [\lambda] \mid l < j \text{ and } \mathtt{T}(i, j) = \mathtt{T}(k, l) \,\}.$$

Therefore, if $p^\mu_{ij} = |P^\mu_{\mathfrak{t}}(i, j)|$ then

$$\prod_{(k,l) \in [\lambda]} \frac{[j - i + k - l + 1]_q}{[j - i + k - l]_q} = \frac{[2]_q}{[1]_q} \cdots \frac{[p^\mu_{ij} + 1]_q}{[p^\mu_{ij}]_q} = [p^\mu_{ij} + 1]_q.$$

Consequently, $\pi_{\mathfrak t}^\mu=\prod_{(i,j)\in[\lambda]}[p_{ij}^\mu+1]_q=\prod_{r\ge1}\prod_{k=1}^n\{c_{rk}(\mathfrak t)\}_q$ where $c_{rk}(\mathfrak t)$ is equal to the number of entries in row r of $\mathtt T$ which are equal to k. However, the parts of ν are precisely the numbers $c_{rk}(\mathfrak t)$ as r and k vary (cf. Example 4.6), so this proves part (i).

Part (ii) follows by comparing the definitions of $\gamma_{\mathfrak t}$ and $\pi_{\mathfrak t}^\mu$. □

5.14 Lemma *Suppose that $\mathfrak t$ is a standard λ–tableau, $\mathtt T=\mu(\mathfrak t)$ and $\mathfrak s=\mathrm{first}(\mathtt T)$.*

(i) *If $\mathtt T$ is semistandard then $\varphi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu}=q^{\ell(w)}\pi_{\mathfrak s}^\mu\varphi_{\mathtt T}$ where $\nu=\lambda d\cap\mu$ and w is the element of $\mathfrak S_\nu$ such that $d(\mathfrak t)=d(\mathfrak s)w$.*

(ii) *If $\mathtt T$ is not semistandard then there exist $c_{\mathtt S}\in R$ such that $\varphi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu}=\sum_{\mathtt S\rhd\mathtt T}c_{\mathtt S}\varphi_{\mathtt S}$.*

Proof. Applying the definitions we find that

$$\varphi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu}(m_\mu)=\varphi_{\mathfrak t}(m_\mu)=\varphi_{\mathfrak t}(1)m_\mu=m_{\mathfrak t}m_\mu.$$

Therefore, if $\mathtt T=\mu(\mathfrak t)$ is semistandard then $m_{\mathfrak s}m_\mu=[\nu]_q^!m_{\mathtt T}$ by Proposition 4.5(i) and Lemma 3.2; consequently, $m_{\mathfrak s}m_\mu=\pi_{\mathfrak s}^\mu\varphi_{\mathtt T}$ by Lemma 5.13(i). Moreover, $d(\mathfrak t)=d(\mathfrak s)w$, for some $w\in\mathfrak S_n$, and $m_{\mathfrak t}m_\mu=m_{\mathfrak s}T_wm_\mu=q^{\ell(w)}m_{\mathfrak s}m_\mu=q^{\ell(w)}\pi_{\mathfrak s}^\mu m_{\mathtt T}$ by Lemma 3.2. This proves (i).

When $\mathtt T$ is not semistandard the argument is a little bit ugly. First consider the case where $\mathscr H=\mathscr H_{\mathcal Z}$. Because $\mathtt T$ is not semistandard there exists an integer i such that $s_i\in\mathfrak S_\mu$ and i and $i+1$ are in the same column of $\mathfrak t$. As a result, by Corollary 3.19, there exist $r_{\mathfrak v}\in\mathcal Z$ such that

$$(1+\hat q)m_{\mathfrak t}m_\mu=m_{\mathfrak t}(1+T_i)m_\mu=\sum_{\mathfrak v\rhd\mathfrak t}r_{\mathfrak v}m_{\mathfrak v}m_\mu.$$

By induction on $\mathfrak v$ we can write $m_{\mathfrak v}m_\mu$ as a $\mathcal Z$–linear combination of terms $m_{\mathtt S}$ for which $\mathtt S\trianglerighteq\mu(\mathfrak v)\rhd\mathtt T$. Therefore, there exist $\hat c_{\mathtt S}\in\mathcal Z$ such that $(1+\hat q)m_{\mathfrak t}m_\mu=\sum_{\mathtt S}\hat c_{\mathtt S}m_{\mathfrak s}$ with $\hat c_{\mathtt S}\ne0$ only if $\mathtt S\rhd\mathtt T$. However, $\{\,m_{\mathtt S}\mid\mathtt S\in\mathcal T_0(\lambda,\mu)\,\}$ is a $\mathcal Z$–basis of $M_{\mathcal Z}^\lambda\cap M_{\mathcal Z}^{\mu*}$, and the structure constants lie in $\mathcal Z$, so it follows that each of $\hat c_{\mathtt S}$ is divisible by $(1+\hat q)$. Hence, we can divide by $1+\hat q$ and then specialize $\hat q$ to deduce the result. □

5.15 Lemma *Suppose that $\mathfrak s$ and $\mathfrak t$ are semistandard λ–tableaux of type μ such that $\mu(\mathfrak s)=\mu(\mathfrak t)$, $\mathfrak s\rhd\mathfrak t$ and $\mathfrak s=\mathfrak t(i,i+1)$ for some integer i. Then $\pi_{\mathfrak t}^\mu=(q+\alpha)\pi_{\mathfrak s}^\mu$ where $\alpha^{-1}=[\mathrm{res}_{\mathfrak s}(i)-\mathrm{res}_{\mathfrak t}(i)]_q$.*

Proof. Let (a,b) and (c,d) be the nodes in $[\lambda]$ such that $\mathfrak t(a,b)=i$ and $\mathfrak t(c,d)=i+1$; then $\mathfrak s(a,b)=i+1$ and $\mathfrak s(c,d)=i$ and $b<d$ since $\mathfrak s\rhd\mathfrak t$. Therefore, $P_{\mathfrak t}^\mu(a,b)=P_{\mathfrak s}^\mu(a,b)\cup\{(c,d)\}$ and $P_{\mathfrak t}^\mu(x)=P_{\mathfrak s}^\mu(x)$ for all nodes $x\ne(a,b)$ in $[\lambda]$. Hence, $\pi_{\mathfrak t}^\mu=[b-a+c-d+1]_q/[b-a+c-d]_q\pi_{\mathfrak s}^\mu=(q+\alpha)\pi_{\mathfrak s}^\mu$. □

5.16 Lemma *Suppose that $\mathfrak t$ is a standard λ–tableau such that $\mathtt T=\mu(\mathfrak t)$ is semistandard. Then $\psi_{\mathfrak t}\varphi_{\mathtt T^\mu\mathtt T^\omega}=\pi_{\mathfrak t}^\mu\psi_{\mathtt T}$.*

Proof. Suppose that $\mathfrak t$ is a standard λ–tableau such that $\mu(\mathfrak t)=\mathtt T$. We first show that $\psi_{\mathfrak t}\varphi_{\mathtt T^\mu\mathtt T^\omega}=r_{\mathfrak t}\varphi_{\mathtt T}$ for some $r_{\mathfrak t}\in R$. Let k be an integer with $1\le k\le d$ and let $x,x+1,\dots,y$ be the entries in row k of $\mathfrak t^\mu$. Then, by (5.1) and Proposition 5.5(iii),

$$\psi_{\mathfrak t}\varphi_{\mathtt T^\mu\mathtt T^\omega}L_k^\mu=\psi_{\mathfrak t}(L_x^\omega+L_{x+1}^\omega+\cdots+L_y^\omega)\varphi_{\mathtt T^\mu\mathtt T^\omega}=[\mathrm{res}_{\mathtt T}(k)]_q\psi_{\mathfrak t}\varphi_{\mathtt T^\mu\mathtt T^\omega}.$$

However, $\psi_{\mathfrak t}\varphi_{\mathtt T^\mu\mathtt T^\omega}L_k^\mu\in W_\mu^\lambda$ and $\{\,\varphi_{\mathtt S}\mid \mathtt S\in\mathcal T_0(\lambda,\mu)\,\}$ is a basis of W_μ^λ by Corollary 5.6; so it follows that there exist $r_{\mathfrak t}\in R$ such that $\psi_{\mathfrak t}\varphi_{\mathtt T^\mu\mathtt T^\omega}=r_{\mathfrak t}\psi_{\mathtt T}$ from Proposition 5.5(iv). It remains to show that $r_{\mathfrak t}=\pi_{\mathfrak t}^\mu$ for all $\mathfrak t$.

Suppose that $\mathfrak t=\operatorname{first}(\mathtt T)$. Then $\varphi_{\mathtt T}=\pi_{\mathfrak t}^\mu\varphi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu}$ by Lemma 5.13(i) and Lemma 5.14(i). Furthermore, by Proposition 5.5(i), there exist $a_{\mathfrak s}\in R$ such that $\psi_{\mathfrak t}=\varphi_{\mathfrak t}+\sum_{\mathfrak s\rhd\mathfrak t}a_{\mathfrak s}\varphi_{\mathfrak s}$. Therefore, by Lemma 5.14,

$$\psi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu}=\varphi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu}+\sum_{\mathfrak s\rhd\mathfrak t}a_{\mathfrak s}\varphi_{\mathfrak s}\varphi_{\mathtt T^\omega\mathtt T^\mu}=\pi_{\mathfrak t}^\mu\varphi_{\mathtt T}+\sum_{\mathtt S\rhd\mathtt T}b_{\mathtt S}\varphi_{\mathtt S}$$

for some $b_{\mathtt S}\in R$. However, $\psi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu}$ is a multiple of $\varphi_{\mathtt T}$ so $\psi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu}=\pi_{\mathfrak t}^\mu\varphi_{\mathtt T}$ by Proposition 5.5(i), and we are done in this case.

Finally suppose that $\mathfrak t\neq\operatorname{first}(\mathtt T)$. Then there exists a standard λ–tableau $\mathfrak s$ such that $\mu(\mathfrak s)=\mathtt T$, $\mathfrak s\rhd\mathfrak t$ and $\mathfrak s=\mathfrak t(i,i+1)$ for some integer i with $(i,i+1)\in\mathfrak S_\mu$. Now $\varphi_{\mathtt T^\omega\mathtt T^\mu}T_i=q\varphi_{\mathtt T^\omega\mathtt T^\mu}$ by Lemma 3.2 and $\psi_{\mathfrak t}=\psi_{\mathfrak s}T_i+\alpha\psi_{\mathfrak s}$, where $\alpha^{-1}=[\operatorname{res}_{\mathfrak s}(i)-\operatorname{res}_{\mathfrak t}(i)]_q$, by Theorem 3.34. Therefore, by induction,

$$\psi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu}=\psi_{\mathfrak s}(T_i+\alpha)\varphi_{\mathtt T^\omega\mathtt T^\mu}=(q+\alpha)\psi_{\mathfrak s}\varphi_{\mathtt T^\omega\mathtt T^\mu}=(q+\alpha)\pi_{\mathfrak s}^\mu\psi_{\mathtt T}=\pi_{\mathfrak t}^\mu\psi_{\mathtt T},$$

the last equality coming from Lemma 5.15. □

5.17 Lemma *Suppose that* $\mathtt T$ *is a semistandard* λ*–tableau of type* μ *and let* $\mathfrak t=\operatorname{last}(\mathtt T)$. *Then there exist* $b_{\mathfrak s}\in R$ *such that*

$$\psi_{\mathtt T}\varphi_{\mathtt T^\mu\mathtt T^\omega}=\psi_{\mathfrak t}+\sum_{\mathfrak s\rhd\mathfrak t}b_{\mathfrak s}\psi_{\mathfrak s}.$$

Proof. By Proposition 4.5, $m_{\mathtt T}=m_{\mathfrak t}+\sum m_{\mathfrak s}$ where the sum is over those standard λ–tableaux $\mathfrak s$ such that $\mathfrak s\rhd\mathfrak t$ and $\mu(\mathfrak s)=\mathtt T$. Therefore, by Proposition 5.5(i) there exist $a_{\mathtt S}\in R$ such that

$$\psi_{\mathtt T}\varphi_{\mathtt T^\mu\mathtt T^\omega}(1)=\psi_{\mathtt T}(m_\mu)=\varphi_{\mathtt T}(m_\mu)+\sum_{\mathtt S\rhd\mathtt T}a_{\mathtt S}\varphi_{\mathtt S}(m_\mu)=m_{\mathtt T}+\sum_{\mathtt S\rhd\mathtt T}a_{\mathtt S}m_{\mathtt S}=m_{\mathfrak t}+\sum_{\mathfrak s\rhd\mathfrak t}b'_{\mathfrak s}.$$

Hence, $\psi_{\mathtt T}\varphi_{\mathtt T^\mu\mathtt T^\omega}=\varphi_{\mathfrak t}+\sum_{\mathfrak s\rhd\mathfrak t}b'_{\mathfrak s}\varphi_{\mathfrak s}$. Now apply Proposition 5.5(i) once more. □

5.18 Theorem *Suppose that* R *is a field and that* $e>n$. *Let* $\mathtt T$ *be a semistandard* λ*–tableau of type* μ. *Then* $\langle\psi_{\mathtt T},\psi_{\mathtt T}\rangle=q^{\ell(d(\mathfrak t))}\gamma_{\mathtt T}$ *where* $\mathfrak t=\operatorname{last}(\mathtt T)$.

Proof. Let $\mathfrak t=\operatorname{last}(\mathtt T)$; then $\psi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu}=\pi_{\mathfrak t}^\mu\psi_{\mathtt T}$ by Lemma 5.17. Consequently,

$$\pi_{\mathfrak t}^\mu\langle\psi_{\mathtt T},\psi_{\mathtt T}\rangle=\langle\psi_{\mathfrak t}\varphi_{\mathtt T^\omega\mathtt T^\mu},\psi_{\mathtt T}\rangle=\langle\psi_{\mathfrak t},\psi_{\mathtt T}\varphi^*_{\mathtt T^\omega\mathtt T^\mu}\rangle=\langle\psi_{\mathfrak t},\psi_{\mathtt T}\varphi_{\mathtt T^\mu\mathtt T^\omega}\rangle,$$

the second last equality coming from Proposition 2.7(ii). By Lemma 5.17 there exist $b_{\mathfrak s}\in R$ such that $\psi_{\mathtt T}\varphi_{\mathtt T^\mu\mathtt T^\omega}=\psi_{\mathfrak t}+\sum_{\mathfrak s\rhd\mathfrak t}b_{\mathfrak s}\psi_{\mathfrak s}$, so

$$\pi_{\mathfrak t}^\mu\langle\psi_{\mathtt T},\psi_{\mathtt T}\rangle=\langle\psi_{\mathfrak t},\psi_{\mathtt T}\varphi_{\mathtt T^\mu\mathtt T^\omega}\rangle=\langle\psi_{\mathfrak t},\psi_{\mathfrak t}\rangle+\sum_{\mathfrak s\rhd\mathfrak t}b_{\mathfrak s}\langle\psi_{\mathfrak t},\psi_{\mathfrak s}\rangle=\langle\psi_{\mathfrak t},\psi_{\mathfrak t}\rangle=q^{\ell(d(\mathfrak t))}\gamma_{\mathfrak t},$$

by Corollary 5.6 and Proposition 5.10. However, $\pi_{\mathfrak t}^\mu\gamma_{\mathtt T}=\gamma_{\mathfrak t}$ by Lemma 5.13(ii) so we're done. □

Let $\hat q$ be an indeterminant over $\mathbb Z$ and define $g_\nu^\lambda(\hat q)=\prod_{\mathtt T}\gamma_{\mathtt T}(\hat q)$. For the remainder of this section we fix $N=\sum_{\mathtt T}\ell(d(\operatorname{last}(\mathtt T)))$ where the sum is over the semistandard λ–tableaux $\mathtt T$ of type μ.

5.19 Corollary *Suppose that R is an integral domain and let $\lambda \in \Lambda^+(d,n)$ and $\mu \in \Lambda(d,n)$. Then $G_\mu^\lambda = q^N g_\mu^\lambda$, considering g_ν^λ as an element of R.*

Proof. Suppose first that $R = \mathbb{Q}(\hat{q})$ where $\hat{q}$ is an indeterminant over $\mathbb{Q}$. By Proposition 5.5(i) the transition matrix between the semistandard basis of W_μ^λ and its orthogonal basis is unitriangular so by Theorem 5.18

$$G_\mu^\lambda(\hat{q}) = \hat{q}^N \prod_{\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)} \gamma_{\mathtt{T}}(\hat{q}) = \hat{q}^N g_\mu^\lambda(\hat{q}).$$

However, by definition, $G_\mu^\lambda(\hat{q}) \in \mathbb{Z}[\hat{q}]$ so it follows that $g_\mu^\lambda(\hat{q}) \in \mathbb{Z}[\hat{q}]$. (This is quite astounding given the definition of the rational functions $\gamma_{\mathtt{T}}(\hat{q})$!)

Now suppose that R is an arbitrary integral domain and let q be an invertible element of R. Consider $R[\hat{q}]$ as a $\mathbb{Z}[\hat{q}]$–module via the homomorphism $\pi : \mathbb{Z}[\hat{q}] \longrightarrow R$ which maps $\hat{q}$ to q. Then π sends an inner product $\langle \varphi_{\mathtt{S}}, \varphi_{\mathtt{T}} \rangle$ in $\mathscr{S}_{\mathbb{Z}[\hat{q}],\hat{q}}(d,n)$ to the corresponding inner product in $\mathscr{S}_{R,q}(d,n)$. Hence, $g_\mu^\lambda(q) = \pi\big(g_\mu^\lambda(\hat{q})\big)$ can be considered as an element of R and the Gram determinant $G_\mu^\lambda(q)$ of the R–module W_μ^λ is equal to $q^N g_\mu^\lambda(q)$. □

5.20 Example Let $\lambda = (3,2)$ and $\mu = (2^2,1)$. Then $\mathcal{T}_0(\lambda,\mu) = \{\mathtt{S},\mathtt{T}\}$ where $\mathtt{S} = \begin{smallmatrix} 1 & 1 & 2 \\ 2 & 3 \end{smallmatrix}$ and $\mathtt{T} = \begin{smallmatrix} 1 & 1 & 3 \\ 2 & 2 \end{smallmatrix}$. Therefore, $G_\mu^\lambda = q[2]_q[4]_q$ by Example 5.7.

Now that we have formula for G_μ^λ we pull some contortionists' tricks to rewrite the product g_μ^λ in a more usable form.

Let ν be a partition of n and let ν' be its conjugate partition ; that is, $\nu' = (\nu_1', \nu_2', \dots)$ where ν_i' is equal for the number of nodes in column i of $[\nu]$ for each $i \ge 1$.

For $1 \le i \le n$ let $\beta_i = n - i + \nu_i'$; then $\beta = (\beta_1, \dots, \beta_n)$ is the sequence of beta numbers for ν. Notice that $\beta_1 > \beta_2 > \dots > \beta_n \ge 0$. For example, if $\nu = (n)$ then $\nu' = (1^n)$ and $(n, n-1, \dots, 1)$ is the sequence of beta numbers for ν; if $\nu = (1^n)$ then $\nu' = (n)$ and $\beta = (2n-1, n-2, \dots, 1, 0)$.

5.21 Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be any sequence of integers and let $\overline{\alpha} = (\overline{\alpha}_1, \dots, \overline{\alpha}_n)$ be the sequence obtained by putting the terms in α into non–increasing order. Then $\overline{\alpha}$ is the sequence of beta numbers for some partition if and only if $\alpha_c \ge 0$ for $c = 1, \dots, n$ and $\alpha_b \ne \alpha_c$ if $b \ne c$.

Suppose that $\overline{\alpha}$ is a sequence of beta numbers for some partition $\nu \in \Lambda^+(d,n)$. Then there exists a unique element w in $\mathfrak{S}_n$ such that $\overline{\alpha}_c = \alpha_{cw}$ for $c = 1, \dots, n$. For any composition τ let $d_\tau(\alpha) = (-1)^{\ell(w)} |\mathcal{T}_0(\nu,\tau)|$; so $d_\tau(\alpha) = \pm \dim W_\tau^\nu$.

If $\overline{\alpha}$ is not the sequence of beta numbers for any partition in $\Lambda^+(d,n)$ then we set $d_\tau(\alpha) = 0$.

Now we return rewriting our formula for G_μ^λ. Let $\beta = (\beta_1, \dots, \beta_n)$ be the sequence of β numbers for λ and suppose that Z is the largest integer such that $\mu_Z \ne 0$. Let $z = \mu_Z$; then every tableau $\mathtt{T}$ in $\mathcal{T}_0(\lambda,\mu)$ has z exactly entries equal to Z and these occur at the feet of distinct columns, say columns $c_1 < c_1 < \dots < c_z$. Let $\mathtt{T}^C$ be the tableau obtained from $\mathtt{T}$ by deleting all of the entries labelled Z and let $\beta^C = (\beta_1^C, \dots, \beta_n^C)$ be sequence of integers given by

$$\beta_b^C = \begin{cases} \beta_b - 1, & \text{if } b \in C, \\ \beta_b, & \text{if } b \notin C, \end{cases}$$

where $C = \{c_1, \dots, c_z\}$. Let λ^C be the partition of $n - z$ whose sequence of beta numbers is β^C. Then $\mathtt{T}^C$ is a semistandard λ^C–tableau of type $\overline{\mu}$ where $\overline{\mu} = (\mu_1, \dots, \mu_{Z-1}, 0)$. Consequently, by the conventions of (5.21),

$$d_\mu(\beta) = |\mathcal{T}_0(\lambda, \mu)| = \sum_{\substack{C \subseteq \{1,2,\dots,n\} \\ |C|=z}} d_{\overline{\mu}}(\beta^C).$$

More generally, for any sequence $\alpha = (\alpha_1, \dots, \alpha_n)$ of integers we have

5.22
$$d_\mu(\alpha) = \sum_{\substack{C \subseteq \{1,2,\dots,n\} \\ |C|=z}} d_{\overline{\mu}}(\alpha^C),$$

where the sequence α^C is defined in exactly the same way as the sequence β^C.

For convenience let $\mathbf{n} = \{1, 2, \dots, n\}$. In the next Lemma, if $C \subseteq \mathbf{n}$ and β^C is not a sequence of beta numbers of a partition in $\Lambda^+(d, n)$ then we adopt that convention that $g_{\overline{\mu}}^{\lambda^C} = 1$ and $\mathcal{T}_0(\lambda^C, \overline{\mu}) = \varnothing$.

5.23 Lemma *Let $\beta = (\beta_1, \dots, \beta_n)$ be the sequence of beta numbers for λ. Then*

$$g_\mu^\lambda(\hat{q}) = B_2 B_3 \prod_{\substack{C \subseteq \mathbf{n} \\ |C|=z}} g_{\overline{\mu}}^{\lambda^C}(\hat{q})$$

where $B_2 = \displaystyle\prod_{h \geq 1} \prod_{\substack{C \subseteq \mathbf{n} \\ |C|=z-1}} \prod_{\substack{b,c \notin C \\ \beta_b^C - \beta_c^C + 1 = h}} [h]_{\hat{q}}^{-d_{\overline{\mu}}(\beta_1^C, \dots, \beta_b^C - h, \dots, \beta_c^C - 1 + h, \dots, \beta_n^C)}$

and $B_3 = \displaystyle\prod_{h \geq 1} \prod_{\substack{C \subseteq \mathbf{n} \\ |C|=z-1}} \prod_{\substack{b,c \notin C \\ \beta_b^C - \beta_c^C = h}} [h]_{\hat{q}}^{d_{\overline{\mu}}(\beta_1^C, \dots, \beta_b^C - 1 - h, \dots, \beta_c^C + h, \dots, \beta_n^C)}.$

Proof. By definition

$$g_\mu^\lambda(\hat{q}) = \prod_{\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)} \gamma_{\mathtt{T}}(\hat{q}) = \prod_{\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)} \prod_{(i,j) \in [\lambda]} \prod_{(k,l) \in P^\mu(i,j)} \frac{[j - i + k - l + 1]_{\hat{q}}}{[j - i + k - l]_{\hat{q}}}.$$

The discussion of (5.21) shows that deleting all of the nodes labelled Z from a tableau gives a bijective correspondence between the sets $\mathcal{T}_0(\lambda, \mu)$ and $\bigcup_C \mathcal{T}_0(\lambda^C, \overline{\mu})$, where C runs through the subsets of $\mathbf{n}$ of size z. Therefore,

$$\prod_{\mathtt{T} \in \mathcal{T}_0(\lambda,\mu)} \prod_{\substack{(i,j) \in [\lambda] \\ \mathtt{T}(i,j) \neq Z}} \prod_{(k,l) \in P^\mu(i,j)} \frac{[j - i + k - l + 1]_{\hat{q}}}{[j - i + k - l]_{\hat{q}}} = \prod_{\substack{C \subseteq \mathbf{n} \\ |C|=z}} g_{\overline{\mu}}^{\lambda^C}(\hat{q}).$$

Next suppose that $(i, j) \in [\lambda]$ and $\mathtt{T}(i, j) = Z$. Then (i, j) is at the foot of some column in $[\lambda]$ and if $(k, l) \in P^\mu(i, j)$ then $\mathtt{T}(k, l) < Z$ and $l < j$; therefore, $\beta_l - \beta_j =$

$j-i+k-l$. Hence,

$$\prod_{\substack{\mathtt{T}\in\mathcal{T}_0(\lambda,\mu)}}\ \prod_{\substack{(i,j)\in[\lambda]\\ \mathtt{T}(i,j)=Z}}\ \prod_{(k,l)\in P^\mu(i,j)} \frac{[j-i+k-l+1]_{\hat q}}{[j-i+k-l]_{\hat q}} = \prod_{\substack{C\subseteq\mathbf{n}\\ |C|=z}}\Bigg(\prod_{\substack{b<c\\ c\in C\\ b\notin C}} \frac{[\beta_b-\beta_c+1]_{\hat q}}{[\beta_b-\beta_c]_{\hat q}}\Bigg)^{d_{\overline{\mu}}(\beta^C)}$$

$$= \prod_{h\ge 1}\prod_{\substack{C\subseteq\mathbf{n}\\ |C|=z}} \frac{\displaystyle\prod_{\substack{b\notin C, c\in C\\ \beta_b-\beta_c+1=h}} [h]_{\hat q}^{d_{\overline{\mu}}(\beta_1^C,\dots,\beta_{b-1}^C,\beta_b,\beta_{b+1}^C,\dots,\beta_{c-1}^C,\beta_c-1,\beta_{c+1}^C,\dots,\beta_n^C)}}{\displaystyle\prod_{\substack{b\notin C, c\in C\\ \beta_b-\beta_c=h}} [h]_{\hat q}^{d_{\overline{\mu}}(\beta_1^C,\dots,\beta_{b-1}^C,\beta_b,\beta_{b+1}^C,\dots,\beta_{c-1}^C,\beta_c-1,\beta_{c+1}^C,\dots,\beta_n^C)}}$$

$$= \prod_{h\ge 1}\prod_{\substack{C\subseteq\mathbf{n}\\ |C|=z-1}} \frac{\displaystyle\prod_{\substack{b,c\notin C\\ \beta_b^C-\beta_c^C+1=h}} [h]_{\hat q}^{-d_{\overline{\mu}}(\beta_1^C,\dots,\beta_{b-1}^C,\beta_c^C-1,\beta_{b+1}^C,\dots,\beta_{c-1}^C,\beta_b^C,\beta_{c+1}^C,\dots,\beta_n^C)}}{\displaystyle\prod_{\substack{b,c\notin C\\ \beta_b^C-\beta_c^C=h}} [h]_{\hat q}^{-d_{\overline{\mu}}(\beta_1^C,\dots,\beta_{b-1}^C,\beta_c^C-1,\beta_{b+1}^C,\dots,\beta_{c-1}^C,\beta_b^C,\beta_{c+1}^C,\dots,\beta_n^C)}}$$

$$= \prod_{h\ge 1}\prod_{\substack{C\subseteq\mathbf{n}\\ |C|=z-1}} \frac{\displaystyle\prod_{\substack{b,c\notin C\\ \beta_b^C-\beta_c^C+1=h}} [h]_{\hat q}^{-d_{\overline{\mu}}(\beta_1^C,\dots,\beta_{b-1}^C,\beta_b^C-h,\beta_{b+1}^C,\dots,\beta_{c-1}^C,\beta_c^C-1+h,\beta_{c+1}^C,\dots,\beta_n^C)}}{\displaystyle\prod_{\substack{b,c\notin C\\ \beta_b^C-\beta_c^C=h}} [h]_{\hat q}^{-d_{\overline{\mu}}(\beta_1^C,\dots,\beta_{b-1}^C,\beta_b^C-1-h,\beta_{b+1}^C,\dots,\beta_{c-1}^C,\beta_c^C+h,\beta_{c+1}^C,\dots,\beta_n^C)}},$$

where the last equation follows using the relation $\beta_b^C-\beta_c^C+1=h$, in the numerator, and $\beta_b^C-\beta_c^C=h$, in the denominator. Combining the above equations proves the Lemma. □

In the next result (and similarly in the definition of B_2 and B_3 above), the only restriction on b and c is that $\beta_b<\beta_c+h$; in particular, we do not require that $b<c$ as might be inferred from the notation.

5.24 Proposition *Suppose that R is an integral domain. Let $\lambda\in\Lambda^+(d,n)$ and $\mu\in\Lambda(d,n)$ and suppose that $\beta=(\beta_1,\dots,\beta_n)$ is the sequence of beta numbers for λ. Then*

$$g_\mu^\lambda = \prod_{h\ge 1}\prod_{c\in\mathbf{n}}\prod_{\substack{b\in\mathbf{n}\\ \beta_b>\beta_c-h}} [h]_q^{-d_\mu(\beta_1,\dots,\beta_b+h,\dots,\beta_c-h,\dots,\beta_n)},$$

regarded as an element of R.

Proof. As in the proof of Corollary 5.19 it is sufficient to consider the case where $R=\mathbb{Z}[\hat q]$. We now argue by induction on n. When $n=1$ both g_μ^λ and the right hand side are equal to 1, so we may assume that $n>1$. By induction, $\prod_C g_{\overline{\mu}}^{\lambda^C} = G_1G_2G_3G_4$ where, in the product, C runs over all subsets of $\mathbf{n}$ which have z elements and

$$G_1 = \prod_{h\ge 1}\prod_{\substack{C\subseteq\mathbf{n}\\|C|=z}}\prod_{c\notin C}\prod_{\substack{b\in\mathbf{n}\setminus C\\ \beta_b^C>\beta_c^C-h}} [h]_{\hat q}^{-d_\mu(\beta_1^C,\dots,\beta_b^C+h,\dots,\beta_c^C-h,\dots,\beta_n^C)},$$

$$G_2 = \prod_{h\ge 1}\prod_{\substack{C\subseteq\mathbf{n}\\|C|=z-1}}\prod_{c\notin C}\prod_{\substack{b\notin C\\ \beta_b^C>\beta_c^C-1+h}} [h]_{\hat q}^{-d_\mu(\beta_1^C,\dots,\beta_b^C+h,\dots,\beta_c^C-1+h,\dots,\beta_n^C)},$$

$$G_3 = \prod_{h\ge 1}\prod_{\substack{C\subseteq\mathbf{n}\\|C|=z-1}}\prod_{c\notin C}\prod_{\substack{b\notin C\\ \beta_b^C+1>\beta_c^C-h}} [h]_{\hat q}^{-d_\mu(\beta_1^C,\dots,\beta_b^C+1-h,\dots,\beta_c^C-h,\dots,\beta_n^C)},$$

and

$$G_4 = \prod_{h\ge 1}\prod_{\substack{C\subseteq\mathbf{n}\\|C|=z-2}}\prod_{c\notin C}\prod_{\substack{b\notin C\\ \beta_b^C>\beta_c^C-h}} [h]_{\hat q}^{-d_\mu(\beta_1^C,\dots,\beta_b^C+1-h,\dots,\beta_c^C-h-1,\dots,\beta_n^C)}.$$

Let B_2 and B_3 be as in Lemma 5.23. Then

$$B_2G_2 = \prod_{h\ge 1}\prod_{\substack{C\subseteq\mathbf{n}\\|C|=z-1}}\prod_{c\notin C}\prod_{\substack{b\notin C\\ \beta_b^C>\beta_c^C-h}} [h]_{\hat q}^{-d_\mu(\beta_1^C,\dots,\beta_b^C+h,\dots,\beta_c^C-1+h,\dots,\beta_n^C)},$$

and

$$B_3G_3 = \prod_{h\ge 1}\prod_{\substack{C\subseteq\mathbf{n}\\|C|=z-1}}\prod_{c\notin C}\prod_{\substack{b\notin C\\ \beta_b^C>\beta_c^C-h}} [h]_{\hat q}^{-d_\mu(\beta_1^C,\dots,\beta_b^C+1-h,\dots,\beta_c^C-h,\dots,\beta_n^C)}.$$

By Lemma 5.23 again, $g_\mu^\lambda = G_1(B_2G_2)(B_3G_3)B_4$, so combining the last three equations and using (5.22) completes the proof. □

5.25 Example As in Example 5.20, let $\lambda=(3,2)$ and $\mu=(2^2,1)$. The sequence of beta numbers for λ is $\beta=(6,5,3,1,0)$ so

$$\begin{aligned} G_\mu^\lambda &= q[2]_q^{-d_\mu(4,7,3,1,0)}[3]_q^{-d_\mu(9,2,3,1,0)}[4]_q^{-d_\mu(2,5,7,1,0)-d_\mu(2,9,3,1,0)}\\ &= q[2]_q^{|\mathcal{T}_0((3,1^2),\mu)|}[3]_q^{|\mathcal{T}_0((1^5),\mu)|}[4]_q^{|\mathcal{T}_0((2^2,1),\mu)|-|\mathcal{T}_0((1^5),\mu)|}.\end{aligned}$$

Because $|\mathcal{T}_0((1^5),\mu)|=0$ and $|\mathcal{T}_0((2^2,1),\mu)|=|\mathcal{T}_0((3,1^2),\mu)|=1$, this agrees with the answer obtained for G_μ^λ in Example 5.20.

Proposition 5.24 gives a closed formula for G_μ^λ. A more combinatorial statement is possible; this requires a final piece of notation and a Lemma.

Let (a,b) be a node in the diagram of λ. Then (a,b)th **hook** of λ is the set

$$H_{ab}^\lambda = \{\,(a,c)\in[\lambda]\mid c\ge b\,\}\cup\{\,(d,b)\in[\lambda]\mid d\ge a\,\};$$

that is, (a,b)th hook is the set of nodes in $[\lambda]$ which are either in the same row and to the right of (a,b), or in the same column and below (a,b) – including (a,b). Let $h_{ab}^\lambda=|H_{ab}^\lambda|=\lambda_a+\lambda'_b-a-b+1$ be the (a,b)th **hook length**. The **leg length** of H_{ab}^λ is $l_{ab}^\lambda=\lambda'_b-a+1$; that is, l_{ab}^λ is equal to the number of nodes in H_{ab}^λ which are contained in column b of $[\lambda]$.

Similarly, the (a,b)th rim hook R^λ_{ab} is the collection of nodes along the rim of $[\lambda]$ from row a to column b. This is best illustrated by an example; let $\lambda=(6,4,3)$ then

$$H^\lambda_{12}=\begin{array}{|c|c|c|c|c|c|}\hline \ & \times & \times & \times & \times & \times \\ \hline \ & \times & \ & \ \\ \cline{1-4} \ & \times & \ \\ \cline{1-3}\end{array}\qquad\text{and}\qquad R^\lambda_{12}=\begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \times & \times & \times \\ \hline \ & \ & \times & \times \\ \cline{1-4} \ & \times & \times \\ \cline{1-3}\end{array}.$$

Thus, there is a one–to–one correspondence between hooks and rim hooks and $|R^\lambda_{ab}|=h^\lambda_{ab}$. Importantly, if λ is a partition of n then $[\lambda]\setminus R^\lambda_{ab}$ is the diagram of a partition of $n-h^\lambda_{ab}$. For example, when $\lambda=(6,4,3)$ then $[\lambda]\setminus R^\lambda_{12}=[(3,2,1)]$.

If $[\lambda]\setminus R^\lambda_{ab}=[\nu]$ we say that λ can be obtained from ν by wrapping on the rim hook R^λ_{ab}; alternatively, ν can be obtained from λ by unwrapping R^λ_{ab}.

5.26 Lemma *Let $\beta=(\beta_1,\dots,\beta_n)$ be the sequence of beta numbers for λ and let b and h be integers with $1\le b\le n$ and $h\ge 1$. Then there exists a composition τ such that $d_\tau(\beta_1,\dots,\beta_b-h,\dots,\beta_n)\ne 0$ only if $h=h^\lambda_{ab}$ for some $(a,b)\in[\lambda]$. In this case,*

$$d_\tau(\beta_1,\dots,\beta_b-h,\dots,\beta_n)=(-1)^{h-l^\lambda_{ab}}d_\tau(\alpha_1,\dots,\alpha_n)$$

where $(\alpha_1,\dots,\alpha_n)$ is the sequence of beta numbers for the partition ν which is obtained by unwrapping the rim hook R^λ_{ab} from λ.

Proof. There exists a composition τ such that $d_\tau(\beta_1,\dots,\beta_b-h,\dots,\beta_n)\ne 0$ only if $\beta_b-h\ne\beta_c$ for all integers c such that $b<c\le n$ and $\beta_b-h\ge 0$. Suppose that $\beta_b-h\ge 0$ and that there exists an integer c such that $b\le c\le n$ and $\beta_c>\beta_b-h>\beta_{c+1}$ (setting $\beta_{n+1}=-1$ if necessary). Then we can remove h nodes from along the rim of λ, starting with the last node in column b and ending in column c so as the leave the diagram of a partition ν. If the last node is in row a then $h=h^\lambda_{ab}$ and $[\nu]=[\lambda]\setminus R^\lambda_{ab}$. Finally, the sign attached to $d_\tau(\beta_1,\dots,\beta_b-h,\dots,\beta_n)$ is $(-1)^{b-c}=(-1)^{h-l}$ where $l=l^\lambda_{ab}$. □

5.27 Theorem (James–Mathas) *Suppose that R is an integral domain and let $\beta=(\beta_1,\dots,\beta_n)$ be the sequence of beta numbers for λ. Then*

$$G^\lambda_\mu=q^N\prod_{\substack{(a,b)\in[\lambda]\\(a,c)\in[\lambda]\\b<c}}\left(\frac{[h^\lambda_{ab}]_q}{[h^\lambda_{ac}]_q}\right)^{d_\mu(\beta_1,\dots,\beta_b+h^\lambda_{ab},\dots,\beta_c-h^\lambda_{ab},\dots,\beta_n)}.$$

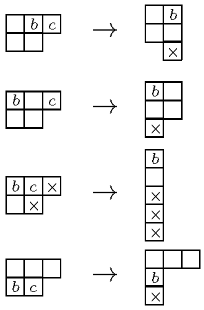
Proof. By Corollary 5.19, $G^\lambda_\mu=q^Ng^\lambda_\mu$; thus, by Proposition 5.24 and Lemma 5.26,

$$\begin{aligned}G^\lambda_\mu&=q^N\prod_{(a,b)\in[\lambda]}\prod_{\substack{c\in\mathbf{n}\\\beta_b>\beta_c-h^\lambda_{ab}}}[h^\lambda_{ab}]_q^{-d_\mu(\beta_1,\dots,\beta_b+h^\lambda_{ab},\dots,\beta_c-h^\lambda_{ab},\dots,\beta_n)}\\&=q^N\prod_{(a,b)\in[\lambda]}\prod_{\substack{b<c\in\mathbf{n}\\\beta_b>\beta_c-h^\lambda_{ab}}}[h^\lambda_{ab}]_q^{-d_\mu(\beta_1,\dots,\beta_b+h^\lambda_{ab},\dots,\beta_c-h^\lambda_{ab},\dots,\beta_n)}\\&\quad\times\prod_{(a,b)\in[\lambda]}\prod_{\substack{b>c\in\mathbf{n}\\\beta_b>\beta_c-h^\lambda_{ab}}}[h^\lambda_{ab}]_q^{-d_\mu(\beta_1,\dots,\beta_c-h^\lambda_{ab},\dots,\beta_b+h^\lambda_{ab},\dots,\beta_n)}\end{aligned}$$

Now if $c>b$ and $\beta_b>\beta_c-h^\lambda_{ab}$ then $-d_\mu(\beta_1,\dots,\beta_c-h^\lambda_{ab},\dots,\beta_b+h^\lambda_{ab},\dots,\beta_n)=d_\mu(\beta_1,\dots,\beta_b+h^\lambda_{ab},\dots,\beta_c-h^\lambda_{ab},\dots,\beta_n)$; moreover, by Lemma 5.26 this is equal to zero unless $(a,c)\in[\lambda]$ since $\beta_b-h^\lambda_{ab}=\beta_c-h^\lambda_{ac}$ if $(a,c)\in[\lambda]$. On the hand, if $c<b$ then $(a,c)\in[\lambda]$. Combining these facts proves the Theorem. □

By Lemma 5.26 the exponent $d_\mu(\beta_1,\dots,\beta_b+h^\lambda_{ab},\dots,\beta_c-h^\lambda_{ac},\dots,\beta_n)$ has a combinatorial interpretation which makes Theorem 5.27 easy to apply. Namely, suppose that the partition ν is obtained by unwrapping the (a,c)th rim hook R^λ_{ac} from $[\lambda]$ and then wrapping it back on in such a way that its lowest node is in column b; then, $d_\mu(\beta_1,\dots,\beta_b+h^\lambda_{ab},\dots,\beta_c-h^\lambda_{ac},\dots,\beta_n)=(-1)^{l^\lambda_{ab}+l^\nu_{ac}}|\mathcal{T}_0(\nu,\mu)|$.

5.28 Example Suppose that $\lambda=(3,2)$. In the diagrams below, using the notation of Theorem 5.27, we label the nodes (a,b) and (a,c) by b and c respectively and we mark the other nodes in the various rim hooks with a cross; in the right hand column we give the contribution of each pair of nodes to G^λ_μ.



contributing 1

contributing $\left(\frac{[4]_q}{[1]_q}\right)^{|\mathcal{T}_0((2^2,1),\mu)|}$

contributing $\left(\frac{[4]_q}{[3]_q}\right)^{-|\mathcal{T}_0((1^5),\mu)|}$

contributing $\left(\frac{[2]_q}{[1]_q}\right)^{|\mathcal{T}_0((3,1^2),\mu)|}$

Therefore, $G^\lambda_\mu=[2]_q^{|\mathcal{T}_0((3,1^2),\mu)|}[3]_q^{|\mathcal{T}_0((1^5),\mu)|}[4]_q^{|\mathcal{T}_0((2^2,1),\mu)|-|\mathcal{T}_0((1^5),\mu)|}$; agreeing with the answer obtained in Example 5.25.

2. The Jantzen sum formula

So far this Chapter has been devoted to calculating determinants. Using a technique due to Jantzen we now transform these results into strong statements about the structure of Weyl modules and Specht modules.

Throughout this section R will be a principal ideal domain and $\mathfrak{p}$ a prime in R. Let $\mathbb{F}=R/\mathfrak{p}R$ be the residue field of $(R,\mathfrak{p})$ and $\nu_\mathfrak{p}:R^\times\longrightarrow\mathbb{N}$ the $\mathfrak{p}$–adic valuation map; that is, if $r\in R$ and $r\neq 0$ then $\nu_\mathfrak{p}(r)=i$ if i is the largest integer such that r is an element of $\mathfrak{p}^iR$.

Suppose that M is a free finite dimensional R–module which is equipped with a symmetric bilinear form $\langle\ ,\ \rangle:M\times M\longrightarrow R$. For each integer $k\geq 1$ let

$$M(k)=\{\,e\in M\mid \langle e,f\rangle\in\mathfrak{p}^kR \text{ for all } f\in M\,\},$$

and let $M_\mathbb{F}(k)=M\otimes_R\mathbb{F}$ be the reduction of $M(k)$ modulo $\mathfrak{p}$. Because $M_\mathbb{F}=M\otimes_R\mathbb{F}$ is a finite dimensional $\mathbb{F}$–module, there exists an integer $z\geq 1$ such that $M_\mathbb{F}(k)=0$ whenever $k\geq z$. The filtration

$$M_\mathbb{F}=M_\mathbb{F}(0)\geq M_\mathbb{F}(1)\geq\cdots\geq M_\mathbb{F}(z)>0$$

is so–called Jantzen filtration of $M_\mathbb{F}$. Observe that $M_\mathbb{F}(1)=\operatorname{rad}M_\mathbb{F}$.

Let $e_1,\dots,e_d$ be a basis for M and let $G=\det(\langle e_i,e_j\rangle)$ be the Gram determinant of the bilinear form with respect to this basis. Note that G belongs to R.

5.29 Lemma (Jantzen) *Let R be a principal ideal domain and suppose that the bilinear form $\langle\ ,\ \rangle$ is non–degenerate. Then $\nu_\mathfrak{p}(G)=\sum_{i>0}\dim_\mathbb{F}M_\mathbb{F}(i)$.*

Proof. Because R is a principal ideal domain, we can find a basis $\{f_1,\dots,f_d\}$ of M such that there exist $r_i\in R^\times$ with $\langle e_i,f_j\rangle=\delta_{ij}r_i$ for $1\le i,j\le d$. Furthermore, the Gram determinant $G=\prod_{i=1}^z\langle e_i,f_i\rangle=\prod_{i=1}^d r_i$, so $\nu_\mathfrak{p}(G)=\sum_{i=1}^d\nu_\mathfrak{p}(r_i)$.

Now, $m=\sum_i a_if_i$, for $a_i\in R$, is an element of $M(k)$ if and only if a_ir_i is an element of $\mathfrak{p}^kR$ for $i=1,\dots,d$; that is, $\nu_\mathfrak{p}(a_ir_i)\ge k$ for all i. Therefore, $M(k)$ is the free R–submodule of M with basis

$$\{\,f_i\mid\nu_\mathfrak{p}(r_i)\ge k\,\}\cup\{\,\mathfrak{p}^{k-\nu_\mathfrak{p}(r_i)}f_i\mid\nu_\mathfrak{p}(r_i)<k\,\}.$$

Accordingly, $M_\mathbb{F}(k)$ has basis $\{\,f_i\otimes 1\mid\nu_\mathfrak{p}(r_i)\ge k\,\}$ and so

$$\sum_{k>0}\dim_F M_\mathbb{F}(k)=\sum_{i=1}^z\nu_\mathfrak{p}(r_i)=\nu_\mathfrak{p}(G),$$

as we wished to show. □

We are going to apply this Lemma in the case where $M=W_R^\lambda\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}$; before we can do this however, we need some preparation.

Let $\hat q$ be an element of R such that $[n]^!_{\hat q}\ne 0$; by Theorem 5.27 this ensures that $G^\lambda_\mu(\hat q)\ne 0$ for all $\lambda\in\Lambda^+(d,n)$ and $\mu\in\Lambda(d,n)$. Let $q=\hat q+\mathfrak{p}R$ and consider $\mathbb{F}$ as an R–module in the natural way. Then $\mathscr{S}_{\mathbb{F},q}(d,n)\cong\mathscr{S}_{R,\hat q}(d,n)\otimes_R\mathbb{F}$.

Suppose that $M_\mathbb{F}$ is a right $\mathscr{S}_{\mathbb{F},q}(d,n)$–module and let $\{\,a_\nu\mid\nu\in\Lambda^+(d,n)\,\}$ be a set of integers. Write $M_\mathbb{F}\longleftrightarrow\sum_\nu a_\nu W_\mathbb{F}^\nu$ if

$$M_\mathbb{F}\oplus\bigoplus_{a_\nu<0}(-a_\nu)W_\mathbb{F}^\nu\qquad\text{and}\qquad\bigoplus_{a_\nu>0}a_\nu W_\mathbb{F}^\nu$$

have the same composition factors. Thus, $M_\mathbb{F}$ and the virtual module $\sum_\nu a_\nu W_\mathbb{F}^\nu$ have the same images in the Grothendieck group of $\mathscr{S}_{\mathbb{F},q}(d,n)$.

5.30 Lemma *Let $M_\mathbb{F}$ be a right $\mathscr{S}_{\mathbb{F},q}(d,n)$–module and suppose that there exist integers $\{\,a_\nu\mid\nu\in\Lambda^+(d,n)\,\}$ such that*

$$\dim_\mathbb{F} M_\mathbb{F}\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}=\sum_{\nu\in\Lambda^+(d,n)}a_\nu\dim W_\mathbb{F}^\nu\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}$$

for all $\mu\in\Lambda^+(d,n)$. Then $M_\mathbb{F}\longleftrightarrow\sum_\nu a_\nu W_\mathbb{F}^\nu$.

Proof. By Theorem 4.16 there exist integers c_ν such that $M_\mathbb{F}=\sum_\nu c_\nu W_\mathbb{F}^\nu$. Therefore,

$$0=\sum_{\nu\in\Lambda^+(d,n)}(c_\nu-a_\nu)\dim_\mathbb{F}W_\mathbb{F}^\nu\varphi_{\mathtt{T}^\mu\mathtt{T}^\mu}=\sum_{\nu\in\Lambda^+(d,n)}(c_\nu-a_\nu)\cdot|\mathcal{T}_0(\nu,\mu)|.$$

However, $\mathcal{T}_0(\nu,\nu)=\{\mathtt{T}^\nu\}$ and $\mathcal{T}_0(\nu,\mu)=\varnothing$ unless $\nu\trianglerighteq\mu$ so the matrix $\big(|\mathcal{T}_0(\nu,\mu)|\big)$ is unitriangular, where $\nu,\mu\in\Lambda^+(d,n)$. Hence, the equations above have a unique solution; namely, $c_\nu=a_\nu$ for all $\nu\in\Lambda^+(d,n)$. □

Given a sequence $\alpha=(\alpha_1,\dots,\alpha_n)$ of integers recall that $\overline{\alpha}$ is the sequence obtained by reordering the terms in α so that they are in non–increasing order. Let $w\in\mathfrak{S}_n$ be a permutation such that $\overline{\alpha}_i=\alpha_{iw}$. If $\overline{\alpha}$ is the sequence of beta numbers for a partition $\mu\in\Lambda^+(d,n)$ then w is uniquely determined (see 5.21) and we define $W_\mathbb{F}^\alpha$ to be the (virtual) $\mathscr{S}_{\mathbb{F},q}(d,n)$–module $(-1)^{\ell(w)}W_\mathbb{F}^\mu$. Similarly, let

$S_{\mathbb{F}}^{\alpha} = (-1)^{\ell(w)} S_{\mathbb{F}}^{\mu}$, a virtual $\mathscr{H}_{\mathbb{F},q}$–module. If $\overline{\alpha}$ is not the sequence of beta numbers for any partition in $\Lambda^+(d,n)$ we set $W_{\mathbb{F}}^{\alpha} = 0$ and $S_{\mathbb{F}}^{\alpha} = 0$.

5.31 The Jantzen Sum Formula ([1, 39]) *Suppose that* $[n]_{\hat{q}}^! \neq 0$ *in* R *and let* λ *be a partition in* $\Lambda^+(d,n)$. *Then*

$$\sum_{i>0} W_{\mathbb{F}}^{\lambda}(i) \longleftrightarrow \sum_{\substack{(a,b)\in[\lambda]\\(a,c)\in[\lambda]\\b<c}} \Big(\nu_{\mathfrak{p}}([h_{ab}^{\lambda}]_{\hat{q}}) - \nu_{\mathfrak{p}}([h_{ac}^{\lambda}]_{\hat{q}})\Big) W_{\mathbb{F}}^{(\beta_1,\dots,\beta_b+h_{ab}^{\lambda},\dots,\beta_c-h_{ac}^{\lambda},\dots,\beta_n)}.$$

Proof. Let $\mu \in \Lambda^+(d,n)$ and consider the R–module $W_{\mu}^{\lambda} = W_R^{\lambda}\varphi_{\mathrm{T}^{\mu}\mathrm{T}^{\mu}}$. Let $\beta = (\beta_1,\dots,\beta_n)$ be the sequence of beta numbers for λ. Now $[n]_{\hat{q}}^!$ is non–zero, so $G_{\mu}^{\lambda} \neq 0$. Therefore, by Jantzen's Lemma (5.29), $\nu_{\mathfrak{p}}(G_{\mu}^{\lambda}) = \sum_{i>0} \dim_{\mathbb{F}} W_{\mathbb{F}}^{\lambda}\varphi_{\mathrm{T}^{\mu}\mathrm{T}^{\mu}}$. On the other hand, by Theorem 5.27,

$$\begin{aligned}\nu_{\mathfrak{p}}(G_{\mu}^{\lambda}) &= \sum_{\substack{(a,b)\in[\lambda]\\(a,c)\in[\lambda]\\b<c}} \Big(\nu_{\mathfrak{p}}([h_{ab}^{\lambda}]_{\hat{q}}) - \nu_{\mathfrak{p}}([h_{ac}^{\lambda}]_{\hat{q}})\Big) d_{\mu}(\beta_1,\dots,\beta_b+h_{ab}^{\lambda},\dots,\beta_c-h_{ac}^{\lambda},\dots,\beta_n)\\ &= \sum_{\substack{(a,b)\in[\lambda]\\(a,c)\in[\lambda]\\b<c}} \Big(\nu_{\mathfrak{p}}([h_{ab}^{\lambda}]_{\hat{q}}) - \nu_{\mathfrak{p}}([h_{ac}^{\lambda}]_{\hat{q}})\Big) \dim W_{\mathbb{F}}^{(\beta_1,\dots,\beta_b+h_{ab}^{\lambda},\dots,\beta_c-h_{ac}^{\lambda},\dots,\beta_n)}\varphi_{\mathrm{T}^{\mu}\mathrm{T}^{\mu}}.\end{aligned}$$

The last equation holds for all partitions $\mu \in \Lambda^+(d,n)$, so the Theorem now follows by applying Lemma 5.30. □

The Jantzen Sum formula is important because $W_{\mathbb{F}}^{\lambda}(1) = \operatorname{rad} W_{\mathbb{F}}^{\lambda}$; consequently, if $\mu \neq \lambda$ then $L_{\mathbb{F}}^{\mu}$ is a composition factor of $W_{\mathbb{F}}^{\lambda}$ if and only if $L_{\mathbb{F}}^{\mu}$ is a composition factor of $\sum_{i>0} W_{\mathbb{F}}^{\lambda}(i)$. In general, the composition multiplicity of $L_{\mathbb{F}}^{\mu}$ in $\sum_{i>0} W_{\mathbb{F}}^{\lambda}(i)$ will be much higher than the composition multiplicity of $L_{\mathbb{F}}^{\mu}$ in $W_{\mathbb{F}}^{\lambda}$; however, this multiplicity is an upper bound on $d_{\lambda\mu}$.

Applying the Schur functor of Proposition 4.19 gives the analogue of the sum formula for $\mathscr{H}_{\mathbb{F},q}$.

5.32 Corollary (The q–Schaper Theorem) *Suppose that* $[n]_{\hat{q}}^! \neq 0$ *in* R *and let* λ *be a partition in* $\Lambda^+(d,n)$. *Then*

$$\sum_{i>0} S_{\mathbb{F}}^{\lambda}(i) \longleftrightarrow \sum_{\substack{(a,b)\in[\lambda]\\(a,c)\in[\lambda]\\b<c}} \Big(\nu_{\mathfrak{p}}([h_{ab}^{\lambda}]_{\hat{q}}) - \nu_{\mathfrak{p}}([h_{ac}^{\lambda}]_{\hat{q}})\Big) S_{\mathbb{F}}^{(\beta_1,\dots,\beta_b+h_{ab}^{\lambda},\dots,\beta_c-h_{ac}^{\lambda},\dots,\beta_n)}.$$

5.33 Example Suppose that $R = \mathbb{C}[\hat{q}]$ and that $\mathfrak{p} = [m]_{\hat{q}}$ is a cyclotomic polynomial in R for some integer $m \geq 2$. If $\lambda = (3,2)$ then, by Example 5.28,

$$\sum_{i>0} W_{\mathbb{F}}^{\lambda}(i) \longleftrightarrow \nu_{\mathfrak{p}}([2]_{\hat{q}}) W_{\mathbb{F}}^{(3,1^2)} + \Big(\nu_{\mathfrak{p}}([3]_{\hat{q}}) - \nu_{\mathfrak{p}}([4]_{\hat{q}})\Big) W_{\mathbb{F}}^{(1^5)} + \nu_{\mathfrak{p}}([4]_{\hat{q}}) W^{(2^2,1)}.$$

Now, $\nu_{\mathfrak{p}}([k]_{\hat{q}}) = 1$ if m divides k and $\nu_{\mathfrak{p}}([k]_{\hat{q}}) = 0$ otherwise. Recall that e is the smallest positive integer such that $[e]_q = 0$ (in $\mathbb{F}$); therefore, $W_{\mathbb{F}}^{\lambda}$ is irreducible (that is $W_{\mathbb{F}}^{\lambda} = L_{\mathbb{F}}^{\lambda}$), if and only if $e > 4$.

Next assume that $\mathfrak{p} = 1 + \hat{q}$; then $\mathbb{F} = \mathbb{C}$ and $q = -1$, so $e = 2$. Therefore, $\nu_\mathfrak{p}([2]_{\hat{q}}) = \nu_\mathfrak{p}([4]_{\hat{q}}) = 1$ and $\nu_\mathfrak{p}([3]_{\hat{q}}) = 0$. Consequently,

$$\sum_{i>0} W_\mathbb{F}^{(3,2)}(i) \longleftrightarrow W_\mathbb{F}^{(3,1^2)} + W_\mathbb{F}^{(2^2,1)} - W_\mathbb{F}^{(1^5)}.$$

Further applications of the sum formula shows that $W_\mathbb{F}^{(1^5)} = L_\mathbb{F}^{(1^5)}$, $W_\mathbb{F}^{(2^2,1)} = L_\mathbb{F}^{(2^2,1)}$ and $W_\mathbb{F}^{(3,1^2)} \longleftrightarrow L_\mathbb{F}^{(3,1^2)} + L_\mathbb{F}^{(1^5)} + L^{(2^2,1)}$. The composition factors of $W_\mathbb{F}^{(3,1^2)}$ are completely determined here because the multiplicities given by the sum formula are 0 or 1. Combining these facts shows that $\sum_{i>0} W_\mathbb{F}^{(3,2)} \longleftrightarrow L_\mathbb{F}^{(3,1^2)} + 2L_\mathbb{F}^{(2^2,1)}$; consequently, $d_{(3,2)(3,1^2)} = 1$ and $d_{(3,2)(2^2,1)}$ is either 1 or 2.

Next consider the Specht module $S_\mathbb{F}^\lambda$; as before we have

$$\sum_{i>0} S_\mathbb{F}^\lambda(i) \longleftrightarrow \nu_\mathfrak{p}([2]_{\hat{q}}) S_\mathbb{F}^{(3,1^2)} + \Big(\nu_\mathfrak{p}([3]_{\hat{q}}) - \nu_\mathfrak{p}([4]_{\hat{q}})\Big) S_\mathbb{F}^{(1^5)} + \nu_\mathfrak{p}([4]_{\hat{q}}) S^{(2^2,1)}.$$

Suppose once more that $\mathfrak{p} = 1 + q$. A calculation similar to the above shows that $\sum_{i>0} S_\mathbb{F}^{(3,2)}(i) \longleftrightarrow 2D_\mathbb{F}^{(2^2,1)}$. Therefore, $S_\mathbb{F}^{(3,2)} \cong D_\mathbb{F}^{(2^2,1)}$ by counting dimensions; hence, $d_{(3,2)(2^2,1)} = 1$ by Proposition 4.19(iii).

At this point we have calculated all but the last two rows of the decomposition matrix of $\mathscr{S}_{\mathbb{C},-1}(d,5)$ where $d \geq 5$. Two further applications of the sum formula give the remaining entries; the full matrix is as follows.

1^5	**1**						
$2,1^3$	.	**1**					
$2^2,1$	.	.	**1**				
$3,1^2$	**1**	.	**1**	1			
$3,2$	.	.	**1**	1	1		
$4,1$	.	**1**	.	.	.	1	
5	**1**	.	.	1	.	.	1

By Proposition 4.19, the submatrix of bold faced entries is the decomposition matrix of $\mathscr{H}_{\mathbb{C},-1}$.

Notice that difference between these calculations for Weyl and Specht modules arises because $D_\mathbb{F}^\mu = 0$ whenever the partition μ is not e–restricted (in particular, the partition $(3,2)$ is not 2–restricted).

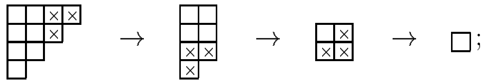
3. The blocks of $\mathscr{S}(d,n)$ and $\mathscr{H}$

Throughout this section we assume that $d \geq n$. In this case, we use the sum formula (5.31) to classify the blocks of the q–Schur algebras $\mathscr{S}(d,n)$ and, from this, classify the blocks of $\mathscr{H}$. In order to classify the blocks we first define the e–core of a partition.

Recall the definition of rim hook from the previous section. We call a rim hook R_{ij}^λ an *e–rim hook* if it is a rim hook of length e; that is if $l_{ij}^\lambda = e$.

Let λ be a partition of n. If the diagram of λ contains an e–rim hook let λ^1 be the partition of $n-e$ obtained by unwrapping this rim hook. Similarly, let λ^2 be the partition obtained from λ^1 by unwrapping an e–rim hook, if possible. Continuing in this way we obtain a sequence of partitions $\lambda = \lambda^0, \lambda^1, \dots, \lambda^w$ with $w \geq 0$ and such that the diagram of λ^w contains no e–rim hook and the partition λ^i is obtained from λ^{i-1} by unwrapping an e–rim hook for $i = 1, \dots, w$. The partition λ^w is the e–core of λ; it is a partition of $n - we$.

For example, if $\lambda = (4,3,2,1)$ then one sequence of partitions leading to the 3–core of λ is the following.



so the 3–core of λ is the partition (1). It is not at all clear that the e–core does not depend upon the order in which the rim hooks are removed. The next Lemma shows that this is not the case.

5.34 Lemma *The e–core of a partition is well defined.*

Proof. Let $\beta = (\beta_1, \dots, \beta_n)$ be the sequence of beta numbers for the partition λ. Think of the beta numbers as being arranged on an *abacus* with e–runners, where the bead positions on the abacus are numbered from left to right as $0, 1, \dots, e-1$ in the first row, $e, e+1, \dots, 2e-1$ in the second row and so on. For example, if $e = 3$ and $\lambda = (4,3,2,1)$ as above then $\beta = (13, 11, 9, 7, 5, 4, 3, 2, 1, 0)$ is the sequence of beta numbers for λ and they are represented on an abacus as follows.

$$\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\cdot & \bullet & \cdot \\
\bullet & \cdot & \bullet \\
\cdot & \bullet & \cdot \\
\cdot & \cdot & \cdot
\end{array}$$

By Lemma 5.26, unwrapping an e–hook from λ corresponds to decreasing some beta number β_i by e; this is the same as moving a bead one row higher in some column of the abacus. Consequently, the e–core of λ corresponds to pushing all of the beads up to their highest possible positions in each column of the abacus. In particular, the e–core is independent of the sequence of rim hooks removed. □

5.35 Lemma *Let M be an $\mathscr{S}_{\mathbb{F},q}(d,n)$–module and suppose that all of the composition factors of M belong to the same block and that $M \longleftrightarrow \sum_\mu a_\mu W_{\mathbb{F}}^\mu$ for some integers a_μ, $\mu \in \Lambda^+(d,n)$. Then $W_{\mathbb{F}}^\mu$ and M lie in the same block whenever $a_\mu \neq 0$.*

Proof. By Lemma 5.30 a virtual module $\sum_\mu a_\mu W_{\mathbb{F}}^\mu \longleftrightarrow 0$ if and only if $a_\mu = 0$ for all $\mu \in \Lambda^+(d,n)$. Hence, the Lemma follows by applying a suitable sum of block idempotents. □

5.36 Theorem *Suppose that $\mathbb{F}$ is a field and that $d \geq n$. Then the Weyl modules $W_{\mathbb{F}}^\lambda$ and $W_{\mathbb{F}}^\mu$ belong to the same block if and only if λ and μ have the same e–core.*

Proof. By Theorem 5.31 the composition factors of $W_{\mathbb{F}}^\lambda$, apart from $L_{\mathbb{F}}^\lambda$ itself, are of the form $L_{\mathbb{F}}^\mu$ where the partition μ is obtained by unwrapping a rim hook from λ and wrapping it back on lower down in the resulting diagram. Suppose that $W_{\mathbb{F}}^\mu$ appears with non–zero coefficient in the Jantzen sum formula for $W_{\mathbb{F}}^\lambda$. Then there exist nodes (a,b) and (a,c) in the diagram of λ such that $\nu_{\mathfrak{p}}([h_{ab}^\lambda]_{\hat{q}}) \neq \nu_{\mathfrak{p}}([h_{ac}^\lambda]_{\hat{q}})$ and μ can be obtained from λ by unwrapping the rim hook R_{ac}^λ and wrapping it back on lower down. Now, if h is an integer and $\nu_{\mathfrak{p}}([h]_{\hat{q}}) \neq 0$ then $[h]_{\hat{q}} \in \mathfrak{p}^i R$; therefore, $[h]_q = 0$ in $\mathbb{F}$ and, consequently, e divides h. It follows that e must divide at least one of h^λ_{ab} and h^λ_{ac}. If $e|h^\lambda_{ac}$ then μ is obtained from λ by moving a number of e–hooks and so μ and λ have the same e–core. If e does not divide h^λ_{ac} then e divides h^λ_{ab} and it is easy to see that μ can be obtained from λ by "sliding" the (a,b)–rim hook; thus again, λ and μ have the same e–core. Consequently, if $W^\lambda_{\mathbb{F}}$ and $W^\mu_{\mathbb{F}}$ are in the same block then λ and μ have the same e–core.

Now suppose that ρ is an e–core of some partition of n; then $|\rho| = n - ew$ for some $w \geq 0$. Let $\gamma = (\gamma_1, \gamma_2, \dots)$ be the partition of r for which $\gamma_1 = \rho_1 + we$ and $\gamma_i = \rho_i$ for $i > 1$. Then γ has e–core ρ.

Assume that $\mu \vdash r$ and that μ has e–core ρ. To prove the theorem it suffices to show that $W^\mu_{\mathbb{F}}$ and $W^\gamma_{\mathbb{F}}$ are in the same block. Suppose that $\mu \neq \gamma$. Then, for some (a,b) the rim hook R^μ_{ab} of $[\mu]$ has length e and its lowest node is in a row later than the first row. Remove this rim hook from $[\mu]$ and add it back at the end of the first row of $[\mu]$, to obtain the diagram $[\lambda]$, say. Then $W^\mu_{\mathbb{F}}$ appears with non–zero multiplicity in $\sum_{i>0} W^\lambda_{\mathbb{F}}(i)$ by Theorem 5.31; therefore, $W^\lambda_{\mathbb{F}}$ and $W^\mu_{\mathbb{F}}$ are in the same block by Lemma 5.35. This proves the Theorem because, by induction, $W^\lambda_{\mathbb{F}}$ and $W^\gamma_{\mathbb{F}}$ are already in the same block. □

We can now classify the blocks of $\mathscr{H}$. At first sight, it is possible that there could exist Weyl modules $W^\lambda_{\mathbb{F}}$ and $W^\mu_{\mathbb{F}}$ which are in the same block of $\mathscr{S}(d,n)$ and such that the Specht modules $S^\lambda_{\mathbb{F}}$ and $S^\mu_{\mathbb{F}}$ are in different blocks of $\mathscr{H}$; for example, the Weyl modules $W^\lambda_{\mathbb{F}}$ and $W^\mu_{\mathbb{F}}$ could be linked via an irreducible module L^ν in $\mathscr{S}(d,n)$ where ν is partition which is not e–restricted. The next result shows that this does not happen.

5.37 Corollary (Dipper–James, James–Mathas) *Suppose that λ and μ are partitions of r. Then the $\mathscr{H}_{\mathbb{F}}(\mathfrak{S}_r)$–modules $S^\lambda_{\mathbb{F}}$ and $S^\mu_{\mathbb{F}}$ belong to the same block if and only if λ and μ have the same e–core.*

Proof. By Proposition 4.19 and Theorem 5.36, $S^\lambda_{\mathbb{F}}$ and $S^\mu_{\mathbb{F}}$ are in the same block only if λ and μ have the same e–core. Thus, to complete the proof it suffices to show that $\mathscr{S}(d,n)$ has at least as many blocks as $\mathscr{H}$.

Let $M = \bigoplus_\mu M^\mu$; then $\mathscr{S}(d,n) \cong \operatorname{End}_{\mathscr{H}}(M)$. Write $\mathscr{H} = B_1 \oplus \cdots \oplus B_k$ as a direct sum of block ideals. Since M contains $\mathscr{H}$ as a submodule, $MB_i \neq 0$ for $i = 1, 2, \dots, k$; moreover, if $i \neq j$ then $MB_i \cap MB_j = 0$. Therefore, $M = M\mathscr{H} = M(B_1 \oplus \cdots \oplus B_k) = MB_1 \oplus \cdots \oplus MB_k$ and so

$$\operatorname{End}_{\mathscr{H}}(M) = \operatorname{End}_{\mathscr{H}}(MB_1 \oplus \cdots \oplus MB_k) = \operatorname{End}_{B_1}(MB_1) \oplus \cdots \oplus \operatorname{End}_{B_k}(MB_k),$$

since if $i \neq j$ then $\operatorname{Hom}_{\mathscr{H}}(B_i, B_j) = 0$ because B_i and B_j have no common irreducible consituents by Theorem 2.21. Therefore, the number of block ideals contained in $\mathscr{S}(d,n)$ is at least as great the number of blocks in $\mathscr{H}$. The Corollary follows. □

Suppose that λ is a partition of n with e–core ρ. Then ρ is a partition of k for some $k \leq n$ and $k + we$ for some $w \geq 0$. We say that λ is a partition of e**–weight** w.

If λ has e–weight 0 then λ is an e–core and λ is the only partition in its block; thus, $P^\lambda_{\mathbb{F}} = W^\lambda_{\mathbb{F}} = L^\lambda_{\mathbb{F}}$ is irreducible and projective. The structure of the Weyl modules indexed by partitions of e–weight 1 are also easy to describe.

5.38 Proposition (James) *Suppose that $n = m + e$ and let ρ be a partition of m which is an e–core. Then there exist precisely e partitions $\lambda^1 \rhd \lambda^2 \rhd \cdots \rhd \lambda^e$ of n*

whose e–core is ρ. Furthermore, if $\mu \neq \lambda^k$ is a partition of n then $d_{\mu\lambda^k} = \delta_{\mu\lambda^{k+1}}$ for $k = 1, \dots, e-1$.

Proof. The enumeration of the weight 1 partitions is easy (for example, think of the abaci of Lemma 5.34). Hence, for $k = 1, 2, \dots, e$

$$\sum_{i>0} W_{\mathbb{F}}^{\lambda^k}(i) \longleftrightarrow \sum_{j>k} (-1)^{j-k+1} W_{\mathbb{F}}^{\lambda^j}$$

by the Jantzen sum formula (5.31). In particular, $W_{\mathbb{F}}^{\lambda^e} = L_{\mathbb{F}}^{\lambda^e}$ is reducible. If $1 \leq k < e$ it now follows easily by induction that $W_{\mathbb{F}}^{\lambda^k} \longleftrightarrow L_{\mathbb{F}}^{\lambda^k} + L_{\mathbb{F}}^{\lambda^{k+1}}$ as we wished to show. □

4. Irreducible Specht modules and Weyl modules

In the last section of these notes we classify the irreducible Weyl modules and the irreducible Specht modules which are indexed by e–restricted partitions. The first two results depend upon the choice of Jantzen filtration; our final result will depend only upon the field and on q.

As in Section 2 we fix a principal ideal domain R, an invertible element $\hat{q} \in R$ and a prime $\mathfrak{p}$ in R and let $\mathbb{F} = R/\mathfrak{p}$ and $q = \hat{q} + \mathfrak{p}R$.

5.39 Theorem *Suppose that $\lambda \in \Lambda^+(d,n)$ and that $[n]^!_{\hat{q}} \neq 0$ in R. Then the Weyl module $W_{\mathbb{F}}^{\lambda}$ is irreducible if and only if $\nu_{\mathfrak{p}}([h_{ab}^{\lambda}]_{\hat{q}}) = \nu_{\mathfrak{p}}([h_{ac}^{\lambda}]_{\hat{q}})$ for all (a,b) and (a,c) in $[\lambda]$.*

Proof. If $\nu_{\mathfrak{p}}([h_{ab}^{\lambda}]_{\hat{q}}) = \nu_{\mathfrak{p}}([h_{ac}^{\lambda}]_{\hat{q}})$ for all (a,b) and $(a,c) \in [\lambda]$ then $\sum_{i>0} W_{\mathbb{F}}^{\lambda}(i) = 0$ by Theorem 5.31; in particular, Lemma 5.30 now shows that $\operatorname{rad} W_{\mathbb{F}}^{\lambda} = 0$, so $W_{\mathbb{F}}^{\lambda}$ is irreducible.

Conversely, suppose that $\nu_{\mathfrak{p}}([h_{ab}^{\lambda}]_{\hat{q}}) \neq \nu_{\mathfrak{p}}([h_{ac}^{\lambda}]_{\hat{q}})$ for some nodes (a,b) and (a,c) in $[\lambda]$. Choose a, b and c so that the partition ν whose sequence of beta numbers is $(\beta_1, \dots, \beta_b + h_{ab}^{\lambda}, \dots, \beta_c - h_{ac}^{\lambda}, \dots, \beta_n)$ is minimal in the dominance ordering among all the partitions appearing on the right hand side of the equation in Theorem 5.31. Then $\lambda \rhd \nu$ and $L_{\mathbb{F}}^{\nu}$ is a composition factor of $\sum_{i>0} W_{\mathbb{F}}^{\lambda}(i)$. Hence, $L_{\mathbb{F}}^{\nu}$ is a composition factor of $W_{\mathbb{F}}^{\lambda}$ and $W_{\mathbb{F}}^{\lambda}$ is not irreducible. □

The irreduciblity of the Specht modules is more subtle; indeed, the classification of the irreducible Specht modules even in the case of the symmetric groups is an open problem (except in characteristic 2; see [**41**]). However, as the next result shows, the answer is known when λ is e–restricted.

5.40 Theorem *Suppose that $[n]^!_{\hat{q}} \neq 0$ in R and let λ be an e–restricted partition of n. Then $S_{\mathbb{F}}^{\lambda}$ is an irreducible $\mathscr{H}$–module if and only if $\nu_{\mathfrak{p}}([h_{ab}]_{\hat{q}}) = \nu_{\mathfrak{p}}([h_{ac}]_{\hat{q}})$ for all nodes (a,b) and (a,c) in $[\lambda]$.*

Proof. First suppose that $S_{\mathbb{F}}^{\lambda} = D_{\mathbb{F}}^{\lambda}$. Then the Gram matrix of the $S_{\mathbb{F}}^{\lambda}$ with respect to its standard basis $\{m_{\mathfrak{t}}\}$ is non–zero. By Corollary 5.11 and Theorem 5.27 this determinant is equal to

$$G_{\omega}^{\lambda} = q^N \prod_{\substack{(a,b)\in[\lambda] \\ (a,c)\in[\lambda] \\ b<c}} \left(\frac{[h_{ab}^{\lambda}]_q}{[h_{ac}^{\lambda}]_q} \right)^{d_{\omega}(\beta_1, \dots, \beta_b + h_{ab}^{\lambda}, \dots, \beta_c - h_{ab}^{\lambda}, \dots, \beta_n)},$$

for some $N \in \mathbb{N}$. Suppose that $d \geq n$. Then $\lambda \in \Lambda^+(d,n)$ and the $\mathscr{S}_{\mathbb{F},q}(d,n)$–module $W^\lambda_{\mathbb{F}}$ is irreducible if and only if its Gram matrix G^λ is non–zero. By Theorem 5.27 again,

$$G^\lambda = \prod_{\mu\in\Lambda(d,n)} G^\lambda_\mu = q^M \prod_{\mu\in\Lambda(d,n)} \prod_{\substack{(a,b)\in[\lambda]\\(a,c)\in[\lambda]\\b<c}} \left(\frac{[h^\lambda_{ab}]_q}{[h^\lambda_{ac}]_q}\right)^{d_\mu(\beta_1,\dots,\beta_b+h^\lambda_{ab},\dots,\beta_c-h^\lambda_{ab},\dots,\beta_n)}$$

for some $M \in \mathbb{N}$. Moreover, $d_\mu(\alpha_1,\dots,\alpha_n)$ is non–zero only if $d_\omega(\alpha_1,\dots,\alpha_n) \neq 0$ for any partition μ. Therefore, if $S^\lambda_{\mathbb{F}} = D^\lambda_{\mathbb{F}}$ then $W^\lambda_{\mathbb{F}}$ is irreducible; whence, $\nu_{\mathfrak{p}}([h_{ab}]_{\hat q}) = \nu_{\mathfrak{p}}([h_{ac}]_{\hat q})$ for all nodes (a,b) and (a,c) in $[\lambda]$ by Theorem 5.39.

The converse is much easier because $S^\lambda_{\mathbb{F}} \longleftrightarrow D^\lambda_{\mathbb{F}} + \operatorname{rad} S^\lambda_{\mathbb{F}}$ and $D^\lambda_{\mathbb{F}} \neq 0$ since λ is e–restricted. Therefore, $S^\lambda_{\mathbb{F}} = D^\lambda_{\mathbb{F}}$ if $\nu_{\mathfrak{p}}([h_{ab}]_{\hat q}) = \nu_{\mathfrak{p}}([h_{ac}]_{\hat q})$ for all nodes (a,b) and (a,c) in $[\lambda]$ by Corollary 5.32. □

The last two results depended upon the choice of R, $\hat q$ and $\mathfrak{p}$, and not directly upon $\mathbb{F}$ and q; our final Theorem removes this restriction.

Fix a field $\mathbb{F}$ and a non–zero element q in $\mathbb{F}$. Let $\hat q$ be an determinant over $\mathbb{F}$ and let $R = \mathbb{F}[\hat q]$ and $\mathfrak{p} = \hat q - q$. Then R is a principal ideal domain, $\mathfrak{p}$ is a prime and $[n]^!_{\hat q} \neq 0$. Furthermore, $\mathbb{F} \cong R/\mathfrak{p}R$ and $\mathscr{S}_{\mathbb{F},q}(d,n) \cong \mathscr{S}_{R,\hat q}(d,n) \otimes_R \mathbb{F}$ so we are in a situation where all of the results of this section apply.

Recall that e is the smallest positive integer such that $[e]_q = 0$. Notice that $[k]_q = 0$ if and only if $[k]_{\hat q} \in \mathfrak{p}R$. If $\mathbb{F}$ is a field of characteristic 0 set $p = \infty$; otherwise, let p be the characteristic of $\mathbb{F}$. Define ν_p to be the usual p–adic valuation on $\mathbb{N}$. (If $p = \infty$ then, by convention, $\nu_p(k) = 0$ for all $k \in \mathbb{N}$.)

5.41 Lemma *Let a and b be integers with $1 \leq a \leq n$ and $1 \leq b \leq n$. Then $\nu_{\mathfrak{p}}([a]_{\hat q}) = \nu_{\mathfrak{p}}([b]_{\hat q})$ if and only if either*
(i) *$e \nmid a$ and $e \nmid b$; or,*
(ii) *$e \mid a$ and $e \mid b$ and $\nu_p(a) = \nu_p(b)$.*

Proof. We have already observed that e is the smallest positive integer such that $\mathfrak{p}$ divides $[e]_{\hat q}$. Therefore, if $k \in \mathbb{N}$ then $\mathfrak{p}$ divides $[k]_{\hat q}$ if and only if e divides k. It follows that $\nu_{\mathfrak{p}}([a]_{\hat q}) = \nu_{\mathfrak{p}}([b]_{\hat q})$ only if either (i) $e \nmid a$ and $e \nmid b$, or (ii) $e \mid a$ and $e \mid b$.

If $e \nmid a$ and $e \nmid b$ then $\nu_{\mathfrak{p}}([a]_{\hat q}) = \nu_{\mathfrak{p}}([b]_{\hat q}) = 0$ and there is nothing to prove. Hence we may assume that $e \mid a$ and $e \mid b$.

First consider the case where $p = \infty$. If $k \in \mathbb{N}$ then $[ke]_{\hat q} = [e]_{\hat q}[k]_{\hat q^e}$ and $[k]_{\hat q^e} \equiv k1_R \mod \mathfrak{p}$ since $\mathfrak{p}$ divides $\hat q^e - 1$. Therefore, $\nu_{\mathfrak{p}}([ke]_{\hat q}) = \nu_{\mathfrak{p}}([e]_{\hat q})$ since $\mathfrak{p} \nmid k1_R$. Hence, $\nu_{\mathfrak{p}}([a]_{\hat q}) = \nu_{\mathfrak{p}}([b]_{\hat q}) = \nu_{\mathfrak{p}}([e]_{\hat q})$ and the Lemma holds in this case.

Now consider the case where $\mathbb{F}$ is a field of positive characteristic. Write $a = eup^x$ and $b = evp^y$ where p does not divide u or v. Then

$$\begin{aligned}
\nu_{\mathfrak{p}}([a]_{\hat q}) = \nu_{\mathfrak{p}}([b]_{\hat q}) &\iff \nu_{\mathfrak{p}}(\hat q^a - 1_R) = \nu_{\mathfrak{p}}(\hat q^b - 1_R)\\
&\iff \nu_{\mathfrak{p}}\big((\hat q^{ep^x} - 1_R)[u]_{\hat q^{ep^x}}\big) = \nu_{\mathfrak{p}}\big((\hat q^{ep^y} - 1_R)[v]_{\hat q^{ep^y}}\big)\\
&\iff \nu_{\mathfrak{p}}(\hat q^{ep^x} - 1_R) = \nu_{\mathfrak{p}}(\hat q^{ep^y} - 1_R),
\end{aligned}$$

the last line following because $[u]_{\hat q^{ep^x}} \equiv u1_R \mod \mathfrak{p}$, since $\mathfrak{p}$ divides $\hat q^e - 1_R$, and $u1_R \not\equiv 0 \mod \mathfrak{p}$ (and similarly for $[v]_{\hat q^{ep^y}}$). Now, if $y > x$ then $\hat q^{ep^y} - 1_R = (\hat q^{ep^x} - 1_R)[p^{y-x}]_{\hat q^{ep^x}}$. However, $[p^{y-x}]_{\hat q^{ep^x}} \equiv p^{y-x}1_R \equiv 0 \mod \mathfrak{p}$, so this forces

$\nu_{\mathfrak{p}}(\hat{q}^{ep^y}-1_R) > \nu_{\mathfrak{p}}(\hat{q}^{ep^x}-1_R)$. Therefore,

$$\nu_{\mathfrak{p}}([a]_{\hat{q}}) = \nu_{\mathfrak{p}}([b]_{\hat{q}}) \iff \nu_p(a) = \nu_p(b),$$

completing the proof of the Lemma. □

Given partitions $\sigma = (\sigma_1, \sigma_2, \dots)$ and $\tau = (\tau_1, \tau_2, \dots)$ and $m \in \mathbb{N}$, we define $\sigma+\tau$ to be the partition $(\sigma_1+\tau_1, \sigma_2+\tau_2, \dots)$ and $m\tau$ to be the partition $(m\tau_1, m\tau_2, \dots)$. Recall that λ' is the partition which is conjugate to λ.

5.42 Theorem (James–Mathas) *Suppose that $\mathbb{F}$ is a field of characteristic p and let λ be an e–regular partition of r. Then the $S^\lambda_{\mathbb{F}}$ is irreducible if and only if there exist partitions $\sigma^{(1)}, \sigma^{(2)}, \dots$ and $\tau^{(1)}, \tau^{(2)}, \dots$ such that*

(i) *$\lambda' = \sigma^{(1)} + e\tau^{(1)}$, $\sigma^{(1)}$ is an e–core, and $\sigma^{(1)}_i = \sigma^{(1)}_{i-1} + e - 1$ for all i with $\tau^{(1)}_i \neq 0$; and*

(ii) *for all $j \geq 1$, $\tau^{(j)} = \sigma^{(j+1)} + p\tau^{(j+1)}$, $\sigma^{(j+1)}$ is a p–core, and $\sigma^{(j+1)}_i = \sigma^{(j+1)}_{i-1} + p - 1$ for all i with $\tau^{(j+1)}_i \neq 0$.*

Proof. If $e = \infty$, then $S^\lambda_{\mathbb{F}}$ is always irreducible; in this case take $\sigma^{(1)} = \lambda$ and let the partitions $\sigma^{(2)}, \dots, \tau^{(1)}, \tau^{(2)}, \dots$ be partitions of 0. Thus, we may assume that $e \neq \infty$.

By Theorem 5.40 and Lemma 5.41, $S^\lambda_{\mathbb{F}}$ is irreducible if and only if for all nodes (a,b) and (a,c) in $[\lambda]$ either $e \nmid h^\lambda_{ab}$ and $e \nmid h^\lambda_{ac}$, or $e \mid h^\lambda_{ab}$ and $e \mid h^\lambda_{ac}$ and $\nu_p(h^\lambda_{ab}) = \nu_p(h^\lambda_{ac})$. We show that this condition is equivalent to (i) and (ii) above.

Draw the diagram $[\lambda']$ and circle those nodes which have hook lengths divisible by e, as in Example 5.43 below. The condition that for all (a,b) and $(a,c) \in [\lambda]$ either $e \nmid h^\lambda_{ab}$ and $e \nmid h^\lambda_{ac}$, or $e \mid h^\lambda_{ab}$ and $e \mid h^\lambda_{ac}$ amounts to saying that in each column of $[\lambda']$ either no node is circled or every node is circled. By removing an e–rim hook from the rightmost column which contains a circled node and using induction this is equivalent to (i). Assume therefore that condition (i) holds.

Suppose that the (a,b)–node of $[\lambda]$ is circled (ie. e divides h^λ_{ab}), and that (a,b) lies in the r–th column of circled nodes in $[\lambda']$. Then $h^\lambda_{ab} = eh^{\tau^{(1)}}_{rb}$. Therefore, $\nu_p(h^\lambda_{ab}) = \nu_p(h^\lambda_{ac})$ for all c with $(a,c) \in [\lambda]$ if and only if $\nu_p(h^{\tau^{(1)}}_{rb}) = \nu_p(h^{\tau^{(1)}}_{rc})$ for all c with $(r,c) \in [\tau^{(1)}]$. Replace λ by $\tau^{(1)\prime}$, e by p and iterate this argument to complete the proof. □

5.43 Example Suppose that $\lambda = (4, 3^3, 2^5, 1^2)$ and $e = 3$. We illustrate $[\lambda']$, marking those nodes in the e–core by $\times$ and circling those nodes which have hook lengths divisible by e.

$$[\lambda'] = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline \times & \otimes & \times & \times & \times & & \odot & & & & \\ \hline \times & \otimes & \times & & & & \odot & & & \multicolumn{2}{c}{} \\ \cline{1-9} \times & \odot & & & \multicolumn{7}{c}{} \\ \cline{1-4} \times & \multicolumn{10}{c}{} \\ \cline{1-1} \end{array}$$

Therefore, by Theorem 5.40, $\mathsf{S}^\lambda_{\mathbb{F}}$ is irreducible when $\mathbb{F}$ is a field of characteristic 0. In this case, the decomposition of λ' given by Theorem 5.42 is $\lambda' = (5, 3, 1^2) + 3(2^2, 1)$. If $\mathbb{F}$ is a field of characteristic 3 then the analogous diagram for $\tau^{(1)}$ is $\begin{array}{|c|c|c|}\hline \times & \odot & \\ \hline \times & & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$; so $S^\lambda_{\mathbb{F}}$ is not irreducible when $p = 3$.

Notes on Chapter 5

1. The beta numbers used in the proof of Theorem 5.27 were introduced by Littlewood [**53**]. The abaci of Lemma 5.34 are due to James [**37**].

2. The combinatorics underpinning the proof of Theorem 5.27 were introduced by James and Murphy [**42**] (see also [**54**]); they consider the corresponding result in the representation theory of the symmetric group (ie. when $q = 1$ and $\mu = \omega$). The extension of this argument to the q–Schur algebras is due to James and Mathas [**39**]; we have basically followed their argument adding some improvements from [**40**].
3. The Weyl modules W^λ of the q–Schur algebra can also viewed as Weyl modules for the quantum general linear groups (see, for example, [**12**, Theorem 3.3.2]). Using a geometric argument Andersen, Polo and Wen [**1**] proved a sum formula for the Weyl modules of quantum groups; Theorem 5.31 can be deduced from their result using the argument of Schaper's thesis [**60**].
4. Jantzen [**43**] proved his sum formula first for the Weyl modules of the general linear groups and later [**44**] extended the sum formula to Weyl modules of other types. In Jantzen's work the sum formula is expressed in terms of the dot action of an affine Weyl group upon the weight lattice. Starting with Jantzen's result, and guided by [**42**], Schaper [**60**] deduced the combinatorial statement of Corollary 5.32 in the case of the symmetric groups.
5. Necessary and sufficient conditions for two Specht modules of $\mathscr{H}$ to belong to the same block were proved in [**14**] and [**39**] respectively.
6. In the representation theory of the symmetric group, Corollary 5.37 is known as the Nakayama conjecture (even though it has been proved for a long time). The elegant idea of using the double centralizer property to classify the blocks of $\mathscr{H}$ is due to König.
7. Theorem 5.36 classifies the blocks for the q–Schur algebras when $d \geq n$; when $d < n$ the blocks of $\mathscr{S}(d, n)$ have been classified by Cox [**9**].
8. Proposition 5.38 first appeared in [**38**]. In this paper James also gives a series of "rules" which partially determine the decomposition matrices of the q–Schur algebras together with tables of decomposition numbers when $n \leq 10$.

Notational Index

Chapter 1.

1.1	$\mathfrak{S}_n$	The symmetric group on $\{1, 2, \dots, n\}$
	s_i	The simple transposition $(i, i+1)$
	S	The set $\{s_1, \dots, s_{n-1}\}$ of Coxeter generators for $\mathfrak{S}_n$
	$\ell(w)$	The length of $w \in \mathfrak{S}_n$
	ε	The sign representation of $\mathfrak{S}_n$
	T	The set of reflections in $\mathfrak{S}_n$
1.2	$\mathscr{H} = \mathscr{H}_{R,q}(\mathfrak{S}_n)$	The Iwahori–Hecke algebra of $\mathfrak{S}_n$
	T_i, T_s	The generators $T_1, \dots, T_{n-1}$ of $\mathscr{H}$
	T_w	A standard basis element of $\mathscr{H}$, $w \in \mathfrak{S}_n$
	$\mathcal{Z}$	$\mathbb{Z}[\hat{q}, \hat{q}^{-1}]$, where $\hat{q}$ is an indeterminate over $\mathbb{Z}$
	$\mathscr{H}_{\mathcal{Z}}$	The generic Iwahori–Hecke algebra $\mathscr{H}_{\mathcal{Z},\hat{q}}(\mathfrak{S}_n)$

Chapter 2.

2.1	$(\Lambda, \geq)$	The poset indexing the cellular basis
	$\mathcal{T}(\lambda)$	An indexing set for the cellular basis
	$\breve{A}^\lambda$	The span of the $c^\mu_{\mathfrak{u}\mathfrak{v}}$ where $\mu > \lambda$
	$*$	The automorphism of A given by ${c^\lambda_{\mathfrak{s}\mathfrak{t}}}^* = c^\lambda_{\mathfrak{t}\mathfrak{s}}$
	A^λ	The span of the $c^\mu_{\mathfrak{u}\mathfrak{v}}$ where $\mu \geq \lambda$
	C^λ	A (right) cell module
	$C^{\lambda *}$	A left cell module
	D^λ	$C^\lambda / \operatorname{rad} C^\lambda$
2.2	$d_{\lambda\mu}$	The composition multiplicity of D^μ in C^λ

Chapter 3.

3.1	$[\mu]$	The diagram $\{\, (i,j) \mid 1 \leq j \leq \mu_i \text{ and } i \geq 1 \,\}$ of μ
	$\operatorname{Shape}(\mathfrak{t})$	The shape of the tableau $\mathfrak{t}$
	$\mathfrak{s}, \mathfrak{t}, \dots$	(Standard) tableau (of type ω)
	$\operatorname{Std}(\mu)$	The set of standard μ–tableaux
	$\mathfrak{t}^\mu$	The standard μ-tableau with $\{1, 2, \dots, n\}$ entered along its rows.
	$\mathfrak{S}_\mu$	The Young subgroup $\mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots$

	$\mathscr{H}(\mathfrak{S}_\mu)$	The Iwahori–Hecke algebra of $\mathfrak{S}_n$
	m_μ	$\sum_{w\in\mathfrak{S}_\mu} T_w$
	M^μ	The "permutation module" $m_\mu\mathscr{H}$
	$\mathscr{D}_\mu$	The distinguished coset representatives of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$
	$d(\mathfrak{t})$	The element of $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}^\mu d(\mathfrak{t})$
	$\trianglerighteq, \triangleright$	The dominance order on the set of compositions
	$\mathfrak{t}{\downarrow}m$	The subtableau of $\mathfrak{t}$ containing the entries $1, 2, \dots, m$
3.2	$*$	The R–linear antiautomorphism of $\mathscr{H}$ determined by $T_w^* = T_{w^{-1}}$ for all $w \in \mathfrak{S}_n$
	$m_{\mathfrak{s}\mathfrak{t}}$	$T^*_{d(\mathfrak{s})} m_\mu T_{d(\mathfrak{t})}$
	$\mathcal{M}$	The Murphy basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ of $\mathscr{H}$
	$\mathscr{H}^\lambda$	The ideal of $\mathscr{H}$ with basis the set of $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableau with $\mu \trianglerighteq \lambda$
	$\check{\mathscr{H}}^\lambda$	The ideal of $\mathscr{H}$ with basis the set of $m_{\mathfrak{u}\mathfrak{v}}$ where $\mathfrak{u}$ and $\mathfrak{v}$ are standard μ–tableau with $\mu \triangleright \lambda$
	Λ	The set of partitions of n
	S^λ	The Specht module indexed by λ
	$m_\mathfrak{t}$	A Murphy basis element of a Specht module
	D^λ	A simple $\mathscr{H}$–module; $D^\lambda = S^\lambda / \operatorname{rad} S^\lambda$
3.3	L_k	$q^{-1}T_{(k-1,k)} + q^{-2}L_{(k-2,k)} + \dots + q^{1-k}T_{(1,k)}$
	$[m]_q$	The "quantum integer" $1 + q + \dots + q^{m-1}$
	$[m]^!_q$	The "quantum factorial" $[1]_q[2]_q \dots [m]_q$
	e	The smallest positive integer such that $[e]_q = 0$
	$\operatorname{res}(x)$	The e–residue of the node x
	$f_\mathfrak{t}$	An orthogonal basis element of a simple Specht module
3.4	$d_{\lambda\mu} = [S^\lambda : D^\mu]$	The composition multiplicity of D^μ in S^λ

Chapter 4.

4.1	$\Lambda(d, n)$	The set of compositions of n with at most d parts
	$\mathscr{S}(d, n)$	The q–Schur algebra
	ω	The partition (1^n)
	$\mathtt{S}, \mathtt{T}, \dots$	(Semistandard) tableaux of type μ
	$\mathcal{T}_0(\mu, \nu)$	The set of semistandard μ–tableaux of type ν
	$\mathtt{T}^\mu$	The unique semistandard μ-tableau of type μ
	$\nu(\mathfrak{t})$	The tableau of type ν obtained from $\mathfrak{t}$ by replacing each entry in $\mathfrak{t}$ by its row index in $\mathfrak{t}^\nu$
	$\mu d \cap \nu$	The composition ν such that $\mathfrak{S}_\nu = d^{-1}\mathfrak{S}_\mu d \cap \mathfrak{S}_\nu$
	$\mathscr{D}_{\mu\nu}$	The set of distinguished $(\mathfrak{S}_\mu, \mathfrak{S}_\nu)$–double coset representatives
	$\operatorname{first}(\mathtt{T})$	See page 38

	$\mathrm{last}(\mathtt{T})$	See page 38
4.2	$m_{\mathtt{S}\mathfrak{t}}$	$\sum_{\mathfrak{s}} m_{\mathfrak{s}\mathfrak{t}}$ where $\mu(\mathfrak{s}) = \mathtt{S}$
	$m_{\mathtt{ST}}$	$\sum_{\mathfrak{s},\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}}$ where $\mu(s) = \mathtt{S}$ and $\nu(\mathfrak{t}) = \mathtt{T}$
4.3	$\Lambda^+(d,n)$	The set of partitions in $\Lambda(d,n)$
	$\varphi_{\mathtt{ST}}$	The map in $\mathrm{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ given by $\varphi_{\mathtt{ST}}(m_\nu) = m_{\mathtt{ST}}$
	$\mathcal{T}_0(\lambda)$	$\bigcup_{\mu\in\Lambda(d,n)} \mathcal{T}_0(\lambda,\mu)$
	$\check{\mathscr{S}}^\lambda(d,n)$	The ideal of $\mathscr{S}(d,n)$ with basis the $\varphi_{\mathtt{ST}}$ where $\mathtt{S},\mathtt{T} \in \mathcal{T}_0(\alpha)$ where $\alpha \rhd \lambda$
	$\mathscr{S}^\lambda(d,n)$	The ideal of $\mathscr{S}(d,n)$ with basis the $\varphi_{\mathtt{ST}}$ where $\mathtt{S},\mathtt{T} \in \mathcal{T}_0(\alpha)$ where $\alpha \trianglerighteq \lambda$
	$*$	The automorphism of $\mathscr{S}(d,n)$ determined by $\varphi^*_{\mathtt{ST}} = \varphi_{\mathtt{TS}}$
	W^λ	The Weyl module indexed by λ
	$m_{\mathtt{T}}$	$\sum_{\mathfrak{t}} m_{\mathfrak{t}}$ where $\mu(\mathfrak{t}) = \mathtt{T}$
	$\varphi_{\mathtt{T}}$	The map in $\mathrm{Hom}_{\mathscr{H}}(M^\mu, S^\lambda)$ given by $\varphi_{\mathtt{T}}(m_\mu) = m_{\mathtt{T}}$
	L^λ	$W^\lambda / \mathrm{rad}\, W^\lambda$
	$d_{\lambda\mu}$	The composition multiplicity of L^μ in W^λ

Chapter 5.

5.1	W^λ_μ	The μ–weight space of W^λ
	G^λ_μ	The Gram determinant of W^λ_μ
	L^μ_i	The element of $\mathrm{End}_{\mathscr{H}}(M^\mu)$ given by $L^\mu_k(m_\mu) = m_\mu(L_x + L_{x+1} + \cdots + L_y)$ where $x,\dots,y$ are the entries in row i of $\mathfrak{t}^\mu$
	$[\mathrm{res}_{\mathtt{T}}(k)]_q$	$\sum_{x:\mathtt{T}(x)=k} [\mathrm{res}_{\mathtt{T}}(x)]_q$
	$\gamma_{\mathtt{T}}(q)$	See page 48.
	$[\lambda]^!_q$	$[\lambda_1]^!_q \dots [\lambda_k]^!_q$
	$\pi^\mu_{\mathfrak{t}}$	See page 49
	$g^\lambda_\nu(\hat{q})$	$\prod_{\mathtt{T}} \gamma_{\mathtt{T}}(\hat{q})$
	N	$\sum_{\mathtt{T}} \ell(d(\mathrm{last}(\mathtt{T})))$
	α,β	Sequences of integers (beta numbers)
	$d_\tau(\alpha)$	See (5.21)
	Z	The largest integer with $\mu_Z \neq 0$
	$\overline{\mu}$	The composition $(\mu_1,\dots,\mu_{Z-1})$
	z	μ_Z
	β^C	A sequence of beta numbers; see page 53
	$\mathbf{n}$	$\{1,2,\dots,n\}$
	l^λ_{ab}	The leg length of the $a,b)$th hook in λ
5.2	$W^\alpha_{\mathbb{F}}$	The (virtual) Weyl module corresponding to the sequence α of beta numbers

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Index