Divisibility Properties of Functions Counting Subgroups of Finite Index in Virtually Free Groups

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Contents

Introduction

A group Γ is called *virtually free* if it contains a free subgroup of finite index. The purpose of this thesis is to investigate certain arithmetic functions counting subgroups of finite index in finitely generated virtually free groups with regard to their divisibility properties. More precisely, we will be concerned with two arithmetic functions $b_\Gamma(\lambda)$ and $s_\Gamma(n)$ associated with such a group Γ, counting torsion-free respectively all subgroups of given finite index in Γ.

The idea to associate with a group Γ some type of counting function, and to investigate the connection between arithmetic properties of this function and the algebraic structure of Γ is not new. A famous example of this philosophy is given by the theory of *word growth* initiated by Milnor, and investigated by Milnor, Wolf, Bass, Gromov, Tits, and others; cf. [Mi68a], [Mi68b], [W68], [Ba72], [G81], [T81]. In [Seg85] Segal studies for a finitely generated solvable group Γ of finite subgroup rank the growth of the index $[\Gamma : \Gamma^n]$ as a function of n, where Γ^n denotes the subgroup of Γ generated by all n-th powers of elements in Γ. The theory of *subgroup growth*, to which this thesis contributes, was initiated by Grunewald and Segal for finitely generated nilpotent groups, and - in a different direction - by Müller for finitely generated virtually free groups; cf. [GSS], [M89], [M91], [M92], [M96], [M97]. Since then it has become an active area of research with the most exciting developments centering either on "small groups" like nilpotent or - more generally - solvable groups of finite subgroup rank or on "large groups" such as virtually free or Fuchsian groups. Some of the major developments concerning the function $s_\Gamma(n)$ are described in Lubotzky's Galway notes [L93] and [L95].

As it turns out, subgroup counting functions tend to exhibit a highly complex behavior, and connections with the algebraic structure reveal themselves better after some type of stratification or smoothing operation has been performed. Again, this observation is not new: in the theory of word growth the most successful approach has been to link not the counting function itself but rather its type of growth with the group structure. For functions counting finite index subgroups in virtually free groups, asymptotic methods have often proved very successful; cf. [M92], [M96], [M97]. Here, the idea is to deform functions of complicated or erratic behavior within their asymptotic class to obtain functions of smoother variation, more likely to lend themselves to stable connections with the algebraic structure of the underlying group. Another possible stratification is the search for recognizable divisibility patterns modulo a prime p, and it is this idea which motivates the present thesis.

In Chapter 2 we investigate the function $b_\Gamma(\lambda)$ with regard to its divisibility properties. Let Γ be a finitely generated virtually free group and m_Γ denotes the least common multiple of the orders of finite subgroups in Γ. If Γ is infinite then a positive integer ι is the index of a free subgroup in Γ if and only if ι is divisible by m_Γ [M89, (24) iv)]. The total information on the number of subgroups of finite index in Γ is concentrated in the constant m_Γ and the function $b_\Gamma(\lambda) : \mathbb{N} \to \mathbb{N}$,

$$b_\Gamma(\lambda) = \text{number of } \textit{free subgroups} \text{ of index } \lambda m_\Gamma \text{ in } \Gamma.$$

Müller [M91, Prop. 6] states for the parity of $b_\Gamma(\lambda)$ for finitely generated virtually free groups of free rank $\mu(\Gamma) = 2$, i.e., the rank of a free subgroup of index m_Γ in Γ, that for some groups $b_\Gamma(\lambda)$ is odd if and only if λ is of the form $\lambda = 2^n - 1,\ n \geq 1$, and for all other cases $b_\Gamma(\lambda)$ is constant modulo 2. We investigate systematically the divisibility patterns of the function $b_\Gamma(\lambda)$ for groups of rank $\mu(\Gamma) \leq 3$ and primes $p \leq 13$. Some groups, for example the classical modular group $\Gamma = PSL(2, \mathbb{Z})$ and the special linear group $\Gamma = SL(2, \mathbb{Z})$, show an interesting divisibility pattern for $b_\Gamma(\lambda)$ modulo 3. One of our results for groups of free rank $\mu(\Gamma) = 2$ states:

Theorem: *Let Γ be a free product $\Gamma = G_1 \underset{S}{*} G_2$ of finite groups G_i with amalgamated subgroup S of cardinality $|S| \equiv 1 \pmod 3$ or $|S| \equiv 2 \pmod 3$ and indices $\{[G_1 : S], [G_2 : S]\} = \{2, 3\}$ or $[G_1 : S] = [G_2 : S] = 3$. Then $b_\Gamma(\lambda) \equiv 1 \pmod 3$ if and only if λ has a representation as*

$$\begin{aligned}
&\lambda = c \cdot 3^6 - k \cdot 3^3 - l \ \textit{ or} \\
&\lambda = (2 \cdot 3^n + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l \\
&\textit{where } n \geq 1,\ c \in \{1, 3\},\ I \subseteq [n-1], \\
&k \textit{ chosen maximal in } K = \{0, 2, 6, 8, 18, 20, 24, 26\} \\
&\textit{and } l \in L_1 = \{0, 4, 6, 10, 18, 22, 24, 28\}.
\end{aligned} \qquad (*)$$

$b_\Gamma(\lambda) \equiv 2 \pmod 3$ *if and only if λ is of the form $(*)$ with an $l \in L_2 = \{2, 8, 20, 26\}$.*

The divisibility patterns found for the other classes of groups of free rank 2 and primes $p \leq 13$ are either constant or show periodicity modulo p.

For the groups of free rank $\mu(\Gamma) = 3$ we have a similar occurrence of divisibility patterns. We detect one type of groups which shows a special behavior modulo 2.

Theorem: *Let Γ be of the type $\tau(\Gamma) = (2k; 0, ..., 0, 3_k, -1_{2k})$, where $k \equiv 1 \pmod 2$. Then $b_\Gamma(\lambda) \equiv 1 \pmod 2$ if and only if λ is of the form*

$$\begin{aligned}
&\lambda = 2^k \quad k \geq 0 \quad \text{or} \\
&\lambda = 2^k + 2^{i_1} + ... + 2^{i_m} \quad m \in \{1, ..., \lfloor \frac{k}{2} \rfloor\}
\end{aligned}$$

with $k - i_1 \geq 2$ and $i_{j-1} - i_j \geq 2$ for all $j \in \{2, ..., m\}$.

Again, like in the case of groups of free rank $\mu(\Gamma) = 2$, $b_\Gamma(\lambda)$ is either constant or periodic modulo p for the remaining types of groups.

Furthermore, we state for virtually free groups of free rank $\mu(\Gamma) \leq 3$ and all primes p necessary and sufficient conditions for $b_\Gamma(\lambda)$ to be constant modulo p. Additionally, for one specific class of groups we are able to describe the divisibility patterns of $s_\Gamma(n)$ for $p = 2, 3$ and arbitrary rank.

In dealing with the function $s_\Gamma(n)$ denoting *all subgroups* of index n in Γ - the main topic of Chapter 3 - we concentrate on free products of cyclic groups of the form $\Gamma = C_p * C_p$ and $\Gamma = C_2 * C_p$ for odd primes p. The latter groups are sometimes called the Hecke groups. For certain groups, mostly in the case when Γ is free or the classical modular group $\Gamma = PSL(2,\mathbb{Z})$, some results have been obtained on the divisibility of $s_\Gamma(n)$. The earliest one is stated by Stothers [Sto78] on the parity of the subgroups of the modular group: For $\Gamma = C_2 * C_3$ the $s_\Gamma(n)$ are odd if and only if n is of the form $2^k - 3$ or $2(2^k - 3)$, $k \geq 2$, besides the statement that for $\Gamma = C_2 * C_2$ we get $s_\Gamma(2n) = s_\Gamma(2n+1) = 2n+1 \equiv 1 \pmod 2$. Müller [M89] has found that the special linear group $\Gamma = SL(2,\mathbb{Z}) \cong C_4 \underset{C_2}{*} C_6$ shows the same parity pattern as the modular group plus the case $s_\Gamma(n)$ is odd if $n = 4$. For a free group F_p of rank p, where p is prime, $s_\Gamma(n)$ is odd for all n. This can easily be obtained by induction from a recurrence for the number of subgroups in the free group, known as Hall's formula [H49]. Grady and Newman [GN92] proved for the free products $\Gamma = C_p * C_p$, $p > 3$, $s_\Gamma(n)$ can be calculated as a linear recurrence modulo p.

Investigating such groups Γ we extend the list of results by three further theorems. For $\Gamma = C_3 * C_3$ we obtain the following divisibility patterns for $p = 2$ and $p = 3$:

Theorem: *Let $\Gamma = C_3 * C_3$. Then for the number of subgroups $s_\Gamma(n)$ of given index n in Γ we have $s_\Gamma(n) \equiv 1 \pmod 2$ if and only if n has the form $n = 2^k - 3$ for $k \geq 2$.*

Theorem: *For $\Gamma = C_3 * C_3$ we have*

$$s_\Gamma(n) \equiv \begin{cases} 0 \pmod 3, & \text{if } n \equiv 2, 6 \pmod 8 \\ 1 \pmod 3, & \text{if } n \equiv 0, 1, 3 \pmod 8 \\ 2 \pmod 3, & \text{if } n \equiv 4, 5, 7 \pmod 8. \end{cases}$$

This result on the parity, which is of a similar type as the one from Stothers, is not included in the statement of Grady and Newman and furthermore we state that $s_\Gamma(n)$ is periodic modulo 3 for this class of groups.

For Γ of the form $\Gamma = C_5 * C_5$ we observe for the first time the phenomenon that n has to be of fractional form, for $s_\Gamma(n) \equiv 1 \pmod 2$ to hold:

Theorem: *Let $\Gamma = C_5 * C_5$. Then $s_\Gamma(n)$ is odd if and only if n is of the form*

$$n = \frac{2 \cdot 4^k - 5}{3}$$

with $k \geq 1$.

In the case $\Gamma = C_2 * C_5$ we state a conjecture concerning the parity of $s_\Gamma(n)$ to the effect that $s_\Gamma(n)$ is odd if and only if n is of the form $n = \frac{2^{2k+1}-5}{3}$ or $n = \frac{2\,(2^{2k+1}-5)}{3}$ with $k \geq 1$.

The problem of counting finite index subgroups in a group Γ is related to the enumeration of Γ-actions on finite sets, i.e., the permutation representations of Γ of finite degree. This connection was first studied by Hall and Radò [HR48], resulting in a formula, known as Hall's formula, for the number of subgroups of given finite index in a finitely generated free group [H49]. Later, Dey [D65] generalized this result to free products, and Wohlfahrt [Wo77] to arbitrary groups: Let $|Hom(\Gamma, S_n)|$ be the number of homomorphisms of Γ into S_n. Then the $s_\Gamma(n)$ satisfy the recurrence formula

$$s_\Gamma(n) = \frac{|Hom(\Gamma, S_n)|}{(n-1)!} - \sum_{k=1}^{n-1} \frac{|Hom(\Gamma, S_{n-k})|}{(n-k)!}\, s_\Gamma(k)$$

with $s_\Gamma(1) = 1$. Considering the torsion-free subgroups of given index in Γ there exists also a transformation formula which relates the function $b_\Gamma(\lambda)$ to another arithmetic function $a_\Gamma(\lambda)$. For the definition of this function we consider torsion-free Γ-actions on a set X, i.e., a Γ-action on X which is free when restricted to finite subgroups. For a finite set X it is necessary and sufficient that $|X|$ is divisibility by m_Γ. For a positive integer μ we define the positive rational number $a_\Gamma(\mu)$ by the condition that

$$\begin{aligned}(\mu m_\Gamma)!\, a_\Gamma(\mu) = {} & \text{number of torsion-free } \Gamma\text{-actions} \\ & \text{on a set with } \mu m_\Gamma \text{ elements,}\end{aligned}$$

where $a_\Gamma(0) = 1$. The arithmetic functions a_Γ and b_Γ associated with a finitely generated virtually free group Γ satisfy the transformation formula [M91, Cor. 1]

$$\sum_{\mu=1}^{\lambda} a_\Gamma(\lambda - \mu)\, b_\Gamma(\mu) \;=\; m_\Gamma\, \lambda\, a_\Gamma(\lambda), \qquad \lambda \geq 1.$$

Both formulas can be derived as special cases from a general counting principle called the Exponential Principle for labeled combinatorial structures [DM97] and we refer to Chapter 1 for more detailed information.

The recurrences for $s_\Gamma(n)$ respectively for $b_\Gamma(\lambda)$ given by such transformation formulas are not particularly suitable for the investigation of divisibility properties since their use automatically involves the discussion of divisibility properties modulo high prime powers. Instead, our aim is to obtain recurrences solely depending on previous values of the subgroup counting functions. It can be shown that the generating function for the sequence $a_\Gamma(\lambda) = \frac{|Hom(\Gamma, S_{\lambda m_\Gamma})|}{(\lambda m_\Gamma)!}$ satisfies a homogeneous linear differential equation which can be calculated explicitly from the *type* $\tau(\Gamma)$, a structural invariant of Γ, introduced by Müller; cf. [M91, Prop. 5]. This differential equation can be transformed into a differential equation of generalized Riccati-type for the generating function for $b_\Gamma(\lambda)$.

Comparing coefficients we then obtain a recurrence for $b_\Gamma(\lambda)$ which is of the desired type and whose coefficients are often (but not always) integral. If integral, these coefficients lend themselves rather conveniently to an inductive proof of divisibility properties extracted from previous calculations.

The strategy for the investigation of divisibility properties for $s_\Gamma(n)$ is analogous, with one significant difference: there is no general differential equation with coefficients known for the function $|Hom(\Gamma, S_n)|$. We can apply only the transformation formula and use the generating functions for $s_\Gamma(n)$ and $|Hom(\Gamma, S_n)|$. For $\Gamma = G_1 * G_2$ with finite G_i we have $|Hom(\Gamma, S_n)| = |Hom(G_1, S_n)| \cdot |Hom(G_2, S_n)|$ and it is not difficult to describe $|Hom(G, S_n)|$ for a finite group G by a linear recurrence with polynomial coefficients:

$$|Hom(G, S_{n+1})| = \sum_{d|m} \binom{n}{n-d+1} (d-1)!\; |Hom(G, S_{n-d+1})|\; s_G(d),$$

where $m = |G|$, $n \geq 0$ and $|Hom(G, S_0)| = 1$.

The problem is then, given two arithmetic functions say f_n, g_n satisfying a linear recurrence with polynomial coefficients, to obtain a recurrence for the pointwise (Hadamard) product $h_n = f_n \cdot g_n$. In order to solve this problem there is an algorithm available [SZ92]. However, we have not been able to treat the Hadamard product for our special cases of linear recurrences using this algorithm. By a thorough analysis of the recurrences and occurring substitutions we provide more efficient procedures *(Hadamard product $C_p * C_p$, Hadamard product $C_2 * C_p$)* for two special cases of recurrences associated with groups of the form $\Gamma = C_p * C_p$ and $\Gamma = C_2 * C_p$. With this recurrences for $|Hom(\Gamma, S_n)|$ we then derive a differential equation for $|Hom(\Gamma, S_n)|$ and, via the transformation of generating functions and comparing coefficients, a recurrence for $s_\Gamma(n)$. With such a recursive description of $s_\Gamma(n)$ in hand we can proceed to an inductive discussion of divisibility properties extracted from massive computer calculations. This strategy fails in case of $\Gamma = C_2 * C_5$ because of an even coefficient at the "wrong" place. Hence, we could not complete our parity considerations for this group and only state a conjecture based on computer calculations:

Conjecture: *Let $\Gamma = C_2 * C_5$. Then $s_\Gamma(n) \equiv 1 \pmod 2$ if and only if $n = \frac{2^{2k+1}-5}{3}$ or $n = \frac{2\,(2^{2k+1}-5)}{3}$ where $k \geq 1$.*

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1 Background and Preliminaries

In this chapter we introduce *finitely generated virtually free groups* Γ and describe the characterization of such groups as fundamental groups of certain graphs of groups. With each such group Γ a function $b_\Gamma(\lambda)$ is associated which counts the free subgroups of given finite index in Γ. We present two recursive descriptions for $b_\Gamma(\lambda)$ which follow from a more general counting principle, and we describe a structural invariant of Γ, the *type* $\tau(\Gamma)$, which is closely related to $b_\Gamma(\lambda)$. Our main references are Serre's book [Ser80] (for the graph-theoretical definitions) and [M91].

1.1 Graphs

Definition 1.1 *A* ***graph*** $\boldsymbol{Y = (V, E, ^-, \partial_0, \partial_1)}$ *consists of a set of vertices $V = V(Y)$ and a set of (directed) edges $E = E(Y)$, together with three mappings $^-, \partial_0, \partial_1$:*
$^- : E \to E$ *is a fixed-point-free involution reversing the orientation of edges;*
$\partial_0, \partial_1 : E \to V$ *assign to an edge $e \in E$ the origin $\partial_0(e)$, respectively the terminus $\partial_1(e)$.*

A *geometric edge* is a pair $\{e, \bar{e}\}$, and an *orientation* $E_+(Y)$ consists of a choice of exactly one edge in each pair $\{e, \bar{e}\}$. We define a *path* p in Y as a sequence of edges $p = e_1\ e_2 ... e_n$, such that $\partial_1(e_i) = \partial_0(e_{i+1})$ for $1 \leq i < n$ with origin $\partial_0(p) = \partial_0(e_1)$ and terminus $\partial_1(p) = \partial_1(e_n)$. Further, the *length of* p is $l(p) = n$. A path p is *closed* if $\partial_0(p) = \partial_1(p)$, and $p = e_1\ e_2 ... e_n$ is called *reduced* if $e_{i+1} \neq \bar{e}_i$ for $1 \leq i < n$. A *circuit* is a non-empty, reduced and closed path. For each vertex $v \in V(Y)$ there exists the *empty path* $\wedge_v$ based at v, a closed path of length $l(\wedge_v) = 0$, starting and ending at v.

Definition 1.2 *A graph Y is* ***connected*** *if each two vertices $v_1, v_2 \in V(Y)$ can be joined by a path. A connected non-empty graph Y is a* ***tree*** *if for any two vertices $v_1, v_2 \in V(Y)$ there exists precisely one reduced path p in Y, beginning in v_1 and ending in v_2, or, equivalently if Y contains no circuits.*

The existence of a *maximal tree* T in a non-empty connected graph Y follows from Zorn's Lemma applied to the set of all subtrees of Y, and it is easy to see that a maximal tree $T \leq Y$ is a spanning tree, i.e., $V(T) = V(Y)$.
Let Y be a connected graph. Two paths p, q in Y can be multiplied in the obvious sense if $\partial_1(p) = \partial_0(q)$. Let $\mathcal{P}(Y)$ denote the set of all paths in Y. Two paths $p, q \in \mathcal{P}(Y)$ are called *homotopic* ($p \simeq q$), if p can be transformed into q by a sequence of elementary transformations of the form

a) insertion of a pair $e\bar{e}$

b) deletion of a pair $e\bar{e}$

This relation is an equivalence relation on $\mathcal{P}(Y)$, and the multiplication of paths induces a partial multiplication on $\mathcal{P}(Y)/\!\simeq$, which is associative (where defined). Since origin and terminus of a path remain unaltered under elementary transformations, we can define ∂_0 and ∂_1 on homotopy classes. Two classes $[p], [q] \in \mathcal{P}(Y)/\!\simeq$ can be multiplied whenever $\partial_1([p]) = \partial_0([q])$, in which case $[p][q] := [pq]$, independently of the chosen representatives p, q. The classes $[\wedge_v]$ are one-sided (partial) *neutral elements*, $[\wedge_{\partial_0(p)}][p] = [p] = [p][\wedge_{\partial_1(p)}]$. For a path $p = e_1...e_n$ its *formal inverse* p^{-1} is defined as $p^{-1} := \bar{e}_n\bar{e}_{n-1}...\bar{e}_1$, and so $[p][p^{-1}] = [\wedge_{\partial_0(p)}]$ and $[p^{-1}][p] = [\wedge_{\partial_1(p)}]$. $\mathcal{P}(Y)/\!\simeq$ is called the *fundamental groupoid* of Y.

Definition 1.3 *Fix a vertex $v_0 \in V(Y)$. Then the set*

$$\pi_1(Y, v_0) := \{\ [p] \in \mathcal{P}(Y)/\!\simeq :\ \partial_0([p]) = v_0 = \partial_1([p])\ \}$$

*is the **fundamental group of $\boldsymbol{Y}$ based at $\boldsymbol{v_0}$**.*

$\pi_1(Y, v_0)$ forms a group under the multiplication in $\mathcal{P}(Y)/\!\simeq$; its identity element is $[\wedge_{v_0}]$ and we have $[p]^{-1} = [p^{-1}]$. Two fundamental groups $\pi_1(Y, v_0)$ and $\pi_1(Y, v_1)$ based at vertices v_0, respectively at v_1, are isomorphic since they can be conjugated into each other within the fundamental groupoid using a path joining v_0 to v_1. Up to isomorphism, the fundamental group is an invariant of the connected graph Y itself, denoted by $\pi_1(Y)$. In fact, $\pi_1(Y)$ is a free group and a basis can be constructed explicitly as follows: Choose a base point v_0, an orientation $E_+(Y)$, and a spanning tree $T \leq Y$. For a vertex $v \in V(Y)$ let p_v denote the unique reduced path in T joining v_0 to v. Then

$$\mathcal{S} = \left\{\ [p_{\partial_0(e)}]\ e\ [p^{-1}_{\partial_1(e)}]\ :\ e \in E_+(Y) - E(T)\ \right\}$$

is a set of free generators for $\pi_1(Y, v_0)$.

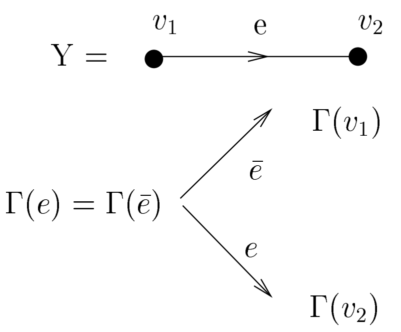
1.2 Graphs of Groups

Definition 1.4 *A **graph of groups $\boldsymbol{(\Gamma(-), Y)}$** consists of a connected non-empty graph Y, a group $\Gamma(v)$ for each vertex $v \in V(Y)$, a group $\Gamma(e)$ for every edge $e \in E(Y)$, and monomorphisms $\Gamma(e) \to \Gamma(\partial_1(e))$ for every edge e, which we denote by $a \mapsto a^e$. We require further that $\Gamma(e) = \Gamma(\bar{e})$ for every $e \in E(Y)$.*

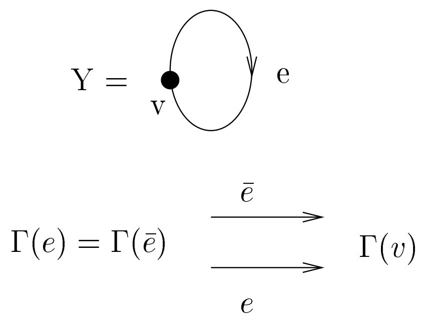
Example 1:

a) If $\Gamma(v) = 1$ for all $v \in V(Y)$ then $(\Gamma(-), Y)$ can be identified with the graph Y itself.

b) Segment of groups



c) Loop of groups



We define the *path group* $\pi(\Gamma(-), Y)$ of a graph of groups $(\Gamma(-), Y)$ as the group

$$\pi(\Gamma(-), Y) = \left(\mathop{*}_{v \in V(Y)} \Gamma(v) \; * \; F(E(Y)) \right) \; / \, N,$$

where $F(E(Y))$ is the free group with basis $E(Y)$ and the normal subgroup N is generated by the elements $e\bar{e}$ and $ea^e e^{-1}(a^{\bar{e}})^{-1}$ for $e \in E(Y)$ and $a \in \Gamma(e)$. Hence, $\pi(\Gamma(-), Y)$ is generated by the groups $\Gamma(v)$ and the elements $e \in E(Y)$, subject to the relations $\bar{e} = e^{-1}$ and $ea^e e^{-1} = a^{\bar{e}}$ for $e \in E(Y)$ and $a \in \Gamma(e)$.

Let $p = e_1, ..., e_n$ be a path in Y with a sequence of vertices $v_0, ..., v_n$. A *path of type* p in $(\Gamma(-), Y)$ is a pair $\underline{p} = (p, \underline{\gamma})$ where $\underline{\gamma} = (\gamma_0, \gamma_1, ..., \gamma_n)$ is a sequence of elements $\gamma_i \in \Gamma(v_i)$. The element

$$|\underline{p}| := \gamma_0 e_1 \gamma_1 ... \gamma_{n-1} e_n \gamma_n \quad \in \pi(\Gamma(-), Y)$$

is called *associated* with the path $\underline{p} = (p, \underline{\gamma})$.

We recall that the definition of the fundamental group $\pi_1(Y)$ is - up to isomorphism - independent of the choice of a vertex $v \in V(Y)$ and the calculation of $\pi_1(Y)$ can be done alternatively by using the unique reduced paths in a maximal tree $T \leq Y$. These two concepts are also applicable to graphs of groups. In fact, since ordinary graphs are a special case of graphs of groups (Example 1 a)) the following definition together with Proposition 1.1 generalize these two concepts.

Definition 1.5

i) Let v_0 be a fixed vertex of Y. The set

$$\pi_1(\Gamma(-), Y, v_0) = \{|\underline{p}| \in \pi(\Gamma(-), Y) : \partial_0(\underline{p}) = v_0 = \partial_1(\underline{p})\}$$

is a subgroup of $\pi(\Gamma(-), Y)$ called the ***fundamental group of*** $\boldsymbol{(\Gamma(-), Y)}$ ***at*** $\boldsymbol{v_0}$.

ii) *Let $T \leq Y$ be a maximal tree. The* ***fundamental group*** $\boldsymbol{\pi_1(\Gamma(-), Y, T)}$ ***of*** $\boldsymbol{(\Gamma(-), Y)}$ ***at*** $\boldsymbol{T}$ *is the quotient of $\pi(\Gamma(-), Y)$ by the normal subgroup generated by $E(T)$. If σ_e denotes the image of the edge e in $\pi_1(\Gamma(-), Y, T)$ then $\pi_1(\Gamma(-), Y, T)$ is generated by the groups $\Gamma(v)$ $(v \in V(Y))$ and the elements σ_e $(e \in E(Y))$ subject to the relations:*

$$\sigma_e a^e \sigma_e^{-1} = a^{\bar{e}}, \quad \sigma_{\bar{e}} = \sigma_e^{-1} \quad \textit{for } e \in E(Y),\ a \in \Gamma(e)$$
$$\sigma_e = 1 \quad \textit{for } e \in E(T).$$

Example 2:

a) $\Gamma(v) = 1\ \forall v$. Then $\pi_1(\Gamma(-), Y, v_0) = \pi_1(Y)$ is the fundamental group of Y in the ordinary sense, $\pi_1(\Gamma(-), Y, T)$ the usual combinatorial description for it.

b) For the segment of groups 1a) $\pi_1(\Gamma(-), Y, Y)$ is generated by $\Gamma(v_1)$ and $\Gamma(v_2)$ subject to the relations $a^e = a^{\bar{e}}$ for $a \in \Gamma(e)$, i.e., $\pi_1(\Gamma(-), Y, Y) = \Gamma(v_1) \underset{\Gamma(e)}{*} \Gamma(v_2)$ is the free product of $\Gamma(v_1)$ and $\Gamma(v_2)$ amalgamated along $\Gamma(e)$.

c) Let Y be a loop of groups 1b) and set $\Gamma(e) = A$. The group $\pi_1(\Gamma(-), Y, v) = \pi(\Gamma(-), Y)$ is generated by the group $\Gamma(v)$ and an element $\sigma = \sigma_e$ subject to the relations
$$\sigma a^e \sigma^{-1} = a^{\bar{e}} \quad (a \in A).$$
$\pi_1(\Gamma(-), Y, v)$ is an HNN-extension with base group $\Gamma(v)$, stable letter σ, and associated subgroups A^e and $A^{\bar{e}}$.

The following result shows that $\pi_1(\Gamma(-), Y, v_0)$ and $\pi_1(\Gamma(-), Y, T)$ are independent of the choice of the base point v_0, respectively the maximal tree T and coincide up to canonical isomorphism.

Proposition 1.6 ([Ser80, I. §5, Prop. 20]) *Let $(\Gamma(-), Y)$ be a graph of groups, $v_0 \in V(Y)$ a vertex and $T \leq Y$ a maximal tree. The canonical projection $q : \pi(\Gamma(-), Y) \to \pi_1(\Gamma(-), Y, T)$ restricts to an isomorphism of $\pi_1(\Gamma(-), Y, v_0)$ onto $\pi_1(\Gamma(-), Y, T)$.*

1.3 Virtually free groups

Definition 1.7 *A group is called* ***virtually free*** *if it contains a free subgroup of finite index.*

More generally, a group is said to be *virtually-$\mathcal{P}$*, where $\mathcal{P}$ denotes some group property, if it contains a subgroup of finite index with property $\mathcal{P}$.

The following important result characterizes finitely generated virtually free groups as the fundamental groups of finite graphs of finite groups.

Theorem 1.8 *Let Γ be a group. The following two assertions on Γ are equivalent:*

i) *Γ is finitely generated and virtually free.*

ii) *Γ is the fundamental group of a finite graph of finite groups.*

The implication $i) \Rightarrow ii)$ was first proved in [KPS73] and generalizes the fundamental result of Stallings to the effect that every torsion-free group which is virtually free of finite rank is in fact free. The implication ii) $\Rightarrow$ i) can be found in [Ser80]; cf. also [DD89].

Definition 1.9 *Let Γ be a finitely generated virtually free group. Each finite graph of finite groups $(\Gamma(-), Y)$ with $\Gamma \cong \pi_1(\Gamma(-), Y)$ is called a* ***decomposition of Γ***.

If Γ is a group and Γ' a torsion-free subgroup of Γ then every finite subgroup $A \leq \Gamma$ acts freely by left multiplication on the set Γ/Γ' of left cosets. Hence, if Γ is virtually torsion-free then the orders of the finite subgroups in Γ are bounded, i.e.,

$$m_\Gamma := lcm\,\{|A| : A \leq \Gamma,\ |A| < \infty\}$$

is well defined. Moreover, for every torsion-free subgroup $\Gamma' \leq \Gamma$ of finite index we have $m_\Gamma \mid [\Gamma : \Gamma']$.

As a consequence of Theorem 1.8 we have the following.

Proposition 1.10 *Let Γ be a finitely generated virtually free group, and let $(\Gamma(-), Y)$ be a decomposition of Γ.*

i) *Every torsion-free subgroup of Γ is free.*

ii) *The set of torsion elements of Γ is*

$$tor(\Gamma) = \bigcup_{x \in \Gamma} \ \bigcup_{v \in V(Y)} x\ \Gamma(v)\ x^{-1}$$

iii) $m_\Gamma = lcm\,\{|\Gamma(v)| : v \in V(Y)\}$

iv) *Let k be a positive integer. There exists a torsion-free subgroup $\Gamma' \leq \Gamma$ with index $[\Gamma : \Gamma'] = k$ if and only if $k \equiv 0 \pmod{m_\Gamma}$.*

Proof:

i) A torsion-free subgroup of $\Gamma \cong \pi_1(\Gamma(-), Y)$ acts freely on the tree associated with the graph of groups $(\Gamma(-), Y))$, hence it is free.

ii) Let $t \in \Gamma$ be a torsion element. The cyclic group $< t >$ has as a finite group a fixpoint under the induced action on the tree canonically associated with $(\Gamma(-), Y)$ in the sense of Bass and Serre. Hence, $t \in x\Gamma(v)x^{-1}$, for some $v \in V(Y)$, $x \in \Gamma$.

iii) $\mathrm{lcm}\,\{|\Gamma(v)| : v \in V(Y)\} \mid m_\Gamma$ is clear by the definition of m_Γ. Is $A \leq \Gamma$ an arbitrary finite subgroup, then A has a fixpoint when acting on a tree, i.e., $A \leq \Gamma(v)$, hence, $|A| \mid |\Gamma(v)| \mid \mathrm{lcm}\,\{|\Gamma(v)| : v \in V(Y)\}$. This implies $m_\Gamma \mid \mathrm{lcm}\,\{|\Gamma(v)| : v \in V(Y)\}$.

iv) Let n be an arbitrary multiple of the orders $|\Gamma(v)|$ $(v \in V(Y))$. Then there exists a permutation representation ([Ser80, II. §2, 2.6, Lem. 10]) $\tau : \Gamma \to S_n$, which induces a free action of every stabilizer $\Gamma(v)$. For the subgroup $H = stab_\tau(1)$ we have $1 \leq [\Gamma : H] = |\Gamma \cdot 1| \leq n$. H is torsion-free, since because of i) a torsion element $h \in tor(H)$ lies only in one of the stabilizers $\Gamma(v)$ and for $h \in H \cap \Gamma(v)$ for a $v \in V(Y)$ we have $1 = \tau(h)(1) = \tau|_{\Gamma(v)}\,(h)(1)$ hence $h = 1$.
Let $n = lcm\{\ |\Gamma(v)|\ :\ v \in V(Y)\ \}$. By i) and ii) and the definition of m_Γ H is a free subgroup of index m_Γ. Since a finitely generated infinite free subgroup contains subgroups of every finite index the assertion follows.

An important invariant of Γ is the Euler characteristic $\chi(\Gamma)$. The *rank* of a group Γ $rk(\Gamma)$ is defined as the minimum number of generators of Γ.

Definition 1.11 *The* ***Euler characteristic*** $\boldsymbol{\chi(\Gamma)}$ *of a finitely generated virtually free group* Γ *is defined by choosing a free subgroup* Γ' *with index* $[\Gamma : \Gamma'] < \infty$ *and setting*

$$\chi(\Gamma) := \frac{1 - rk(\Gamma')}{[\Gamma : \Gamma']}. \tag{1.1}$$

In order to see that the Euler characteristic $\chi(\Gamma)$ is well defined we use Schreier's index formula.

Theorem 1.12 ([Sch27])

a) Every subgroup of a free group is free.

b) If Γ *is free of finite rank and* Γ' *is a subgroup of finite index in* Γ *then*

$$rk(\Gamma') = [\Gamma : \Gamma']\,(rk(\Gamma) - 1) + 1 \tag{1.2}$$

Let Γ_0 and Γ_1 be two free subgroups of finite index in Γ. Their intersection $\Gamma_{01} = \Gamma_0 \cap \Gamma_1$ is also free of finite index in Γ, and by 1.2, we have

$$[\Gamma_0 : \Gamma_{01}]\,(rk(\Gamma_0) - 1) = rk(\Gamma_{01}) - 1 = [\Gamma_1 : \Gamma_{01}]\,(rk(\Gamma_1) - 1).$$

It follows that

$$\frac{1-rk(\Gamma_0)}{[\Gamma:\Gamma_0]} = \frac{1-rk(\Gamma_{01})}{[\Gamma_0:\Gamma_{01}][\Gamma:\Gamma_0]} = \frac{1-rk(\Gamma_{01})}{[\Gamma_1:\Gamma_{01}][\Gamma:\Gamma_1]} = \frac{1-rk(\Gamma_1)}{[\Gamma:\Gamma_1]}.$$

This shows that the (rational) Euler characteristic $\chi(\Gamma)$ of Γ is indeed well defined. The following result describes a more convenient way of calculating the Euler characteristic $\chi(\Gamma)$ in terms of a decomposition of Γ.

Proposition 1.13 ([DD89, p. 104–105] or [Ser80, II. §2 2.9])
The Euler characteristic of a finitely generated virtually free group Γ can be calculated using a decomposition $(\Gamma(-), Y)$ of Γ via

$$\chi(\Gamma) = \sum_{v\in V(Y)} \frac{1}{|\Gamma(v)|} - \sum_{e\in E_+(Y)} \frac{1}{|\Gamma(e)|}. \tag{1.3}$$

Example 3:

a) Let Γ be the classical modular group $\Gamma = PSL(2,\mathbb{Z}) \cong C_2 * C_3$.
Then we have $\chi(\Gamma) = \frac{1}{2} + \frac{1}{3} - 1 = -\frac{1}{6}$.

b) For $\Gamma = SL(2,\mathbb{Z}) \cong C_4 \underset{C_2}{*} C_6$ we calculate $\chi(\Gamma) = \frac{1}{4} + \frac{1}{6} - \frac{1}{2} = -\frac{1}{12}$.

Definition 1.14 *The **free rank $\mu(\Gamma)$** of a finitely generated virtually free group Γ is the rank of a free subgroup Γ' of index $[\Gamma:\Gamma'] = m_\Gamma$.*

By Proposition 1.2.iv) a finitely generated virtually free group Γ in particular contains at least one free subgroup of index m_Γ and by the definition of the Euler characteristic, $\chi(\Gamma)$ and the free rank $\mu(\Gamma)$ are related via

$$\mu(\Gamma) + m_\Gamma \cdot \chi(\Gamma) = 1, \tag{1.4}$$

which shows in particular that the free rank $\mu(\Gamma)$ is well defined.

1.4 The functions $b_\Gamma(\lambda)$ and $a_\Gamma(\lambda)$

The total information on the number of free subgroups of finite index in Γ is concentrated in the constant m_Γ and the function $b_\Gamma(\lambda) : \mathbb{N} \to \mathbb{N}$,

$$b_\Gamma(\lambda) = \text{number of free subgroups of index } \lambda m_\Gamma \text{ in } \Gamma.$$

This function plays the central role in the next chapter where we investigate $b_\Gamma(\lambda)$ with regard to its divisibility properties. We will make use of a transformation formula for $b_\Gamma(\lambda)$ which we introduce in a more general setup.

Definition 1.15 *Let Γ be a group, and $\Sigma \subseteq \Gamma$ is a normal set with $1 \notin \Sigma$. For a positive integer n we define*

$$s_\Gamma^\Sigma(n) := |\,\{\Gamma' : \Gamma' \leq \Gamma,\ [\Gamma : \Gamma'] = n,\ \Gamma' \cap \Sigma = \varnothing\}\,| \tag{1.5}$$

Example 4:

a) $\Sigma = \varnothing$. Then $(s_\Gamma^\Sigma(n))_{n=1,2,\ldots}$ counts *all subgroups* of given index in Γ. This function will be of importance in Chapter 3.

b) $\Sigma = tor(\Gamma)\setminus\{1\}$ where $tor(\Gamma)$ is the torsion set of a finitely generated virtually free group Γ. Here, $(s_\Gamma^\Sigma(n))_{n=1,2,\ldots}$ counts the *free subgroups* of given index in Γ.

Definition 1.16 *For a set $S \subseteq \mathbb{N}$ and a set Ω, $\boldsymbol{Hom_\Sigma^S(\Gamma, S(\Omega))}$ denotes the set of all Γ-actions τ on Ω with the following properties:*

i) τ induces a fixed-point-free action of the elements of Σ on Ω.

ii) The lengths of the orbits into which Ω decomposes under τ are contained in S.

*The elements of $Hom_\Sigma^S(\Gamma, S(\Omega))$ are called (Σ, S)-admissible Γ-actions on Ω. A triple (Γ, Σ, S) is termed **admissible** if $|Hom_\Sigma^S(\Gamma, S(\Omega))| < \infty$ for all $n \geq 0$.*

Remark:
If Γ is finitely generated or of finite subgroup rank then (Γ, Σ, S) is admissible for each normal subset $\Sigma \subseteq \Gamma\setminus\{1\}$ and every set $S \subseteq \mathbb{N}$. It is easily seen, that the admissibility of (Γ, Σ, S) implies that $s_\Gamma^\Sigma(n)$ is finite for all $n \in S$.

Definition 1.17 *Let $\Sigma \subseteq \Gamma$ be a normal set with $1 \notin \Sigma$, $S \subseteq \mathbb{N}$ and $T \subseteq \mathbb{N}_0$. $Hom_\Sigma^{S,T}(\Gamma, S_n)$ denotes the set of those (Σ, S)- admissible Γ-actions on the standard n-set $[n]$ whose number of orbits is in a given set T.*

The following theorem relates the subgroup counting functions associated with a finitely generated virtually free group Γ to the enumeration of Γ-actions on finite sets. This formula is an application of a general counting principle called the Exponential Principle for labeled combinatorial structures [DM97].

Theorem 1.18 ([DM97, Prop. 1]) *Let (Γ, Σ, S) be an admissible triple and $T \subseteq \mathbb{N}_0$ a set of non-negative integers. Then we have*

$$\sum_{n\geq 0} \frac{|Hom_\Sigma^{S,T}(\Gamma, S_n)|}{n!}\, z^n = e_T\left(\sum_{n\in S} \frac{s_\Gamma^\Sigma(n)}{n}\, z^n\right), \tag{1.6}$$

where $e_T(z) := \sum\limits_{t\in T} \frac{z^t}{t!}$.

The special case

$$\sum_{n=0}^{\infty} \frac{|Hom_{\Sigma}(\Gamma, S_n)|}{n!}\, z^n = \exp\left(\sum_{n=1}^{\infty} \frac{s_{\Gamma}^{\Sigma}(n)}{n}\, z^n\right) \tag{1.7}$$

where $S = \mathbb{N}$ and $T = \mathbb{N}_0$ will suffice for our investigations.

For $\Sigma = \varnothing$ and $\Gamma = G$ a finite group we obtain from 1.7

$$\sum_{n=0}^{\infty} \frac{|Hom(G, S_n)|}{n!}\, z^n = \exp\left(\sum_{d|m} \frac{s_G(d)}{d}\, z^d\right), \quad m = |G|. \tag{1.8}$$

Comparing coefficients in 1.7 gives that $\left(s_{\Gamma}^{\Sigma}(n)\right)_{n=1,2,\ldots}$ can be recursively calculated in terms of the sequence $h_{\Gamma}^{\Sigma}(n) := \frac{|Hom_{\Sigma}(\Gamma,S_n)|}{n!}$ via

$$s_{\Gamma}^{\Sigma}(n) = n \cdot h_{\Gamma}^{\Sigma}(n) - \sum_{k=1}^{n-1} h_{\Gamma}^{\Sigma}(n-k)\, s_{\Gamma}^{\Sigma}(k) \tag{1.9}$$

with

$$s_{\Gamma}^{\Sigma}(1) = \begin{cases} 1, & \Sigma = \varnothing \\ 0, & \Sigma \neq \varnothing. \end{cases}$$

For $\Sigma = \varnothing$ formula 1.9 relates $s_{\Gamma}(n)$, the number of all subgroups of index n in a finitely generated group Γ, to the number $h_{\Gamma}(n) := \frac{|Hom(\Gamma,S_n)|}{n!}$.

Having introduced this more general transformation formula 1.9 we come back to virtually free groups. Let Γ be finitely generated virtually free and $\Sigma = tor(\Gamma)\backslash\{1\}$. Then $Hom_{\Sigma}(\Gamma, S_n)$ consists of the *torsion-free* Γ*-actions* on $[n]$, i.e., those Γ-actions on $[n]$ which are free when restricted to finite subgroups. Since every torsion-free subgroup of Γ is free we have $s_{\Gamma}^{\Sigma}(\lambda m_{\Gamma}) = b_{\Gamma}(\lambda)$. Put $a_{\Gamma}(\lambda) := h_{\Gamma}^{\Sigma}(\lambda m_{\Gamma}) = \frac{|Hom_{\Sigma}(\Gamma,S_{\lambda m_{\Gamma}})|}{(\lambda m_{\Gamma})!}$ and $a_{\Gamma}(0) = 1$ and using the fact that $s_{\Gamma}^{\Sigma}(n)$ vanishes for $m_{\Gamma} \nmid n$ we obtain from formula 1.9:

Corollary 1.19 ([M91, Cor. 1]) *The arithmetic functions a_{Γ} and b_{Γ} associated with a finitely generated virtually free group Γ satisfy the transformation formula*

$$\sum_{\mu=1}^{\lambda} a_{\Gamma}(\lambda-\mu)\, b_{\Gamma}(\mu) \;=\; m_{\Gamma}\, \lambda\, a_{\Gamma}(\lambda), \qquad \lambda \geq 1. \tag{1.10}$$

In order to use this transformation formula for determining $b_{\Gamma}(\lambda)$ we need to know $a_{\Gamma}(\lambda)$ explicitly.

Proposition 1.20 ([M91, Prop. 3]) *Let Γ be a finitely generated virtually free group, $(\Gamma(-), Y)$ a decomposition of Γ, and $E_+(Y)$ an orientation of the graph Y. Then we have*

$$a_\Gamma(\lambda) = \frac{\prod_{e \in E_+(Y)} \left\{ (\lambda m_\Gamma / |\Gamma(e)|)! \ |\Gamma(e)|^{\lambda m_\Gamma / |\Gamma(e)|} \right\}}{\prod_{v \in V(Y)} \left\{ (\lambda m_\Gamma / |\Gamma(v)|)! \ |\Gamma(v)|^{\lambda m_\Gamma / |\Gamma(v)|} \right\}}, \qquad \lambda \geq 0. \tag{1.11}$$

Formula 1.10 together with Proposition 1.20 provides a recursive description of $b_\Gamma(\lambda)$ in terms of the sequence $a_\Gamma(\lambda)$.

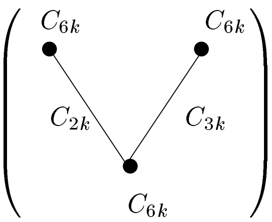
1.5 The type $\boldsymbol{\tau}(\Gamma)$

Inspecting formula 1.11 we observe that $a_\Gamma(\lambda)$ depends only on the graph Y and the order of the vertex and edge groups. The internal structure of the groups $\Gamma(v), \Gamma(e)$ has no relevance for $a_\Gamma(\lambda)$. In particular, the combinatorial datum (m_Γ, b_Γ) is not sufficient to determine the group Γ up to isomorphism. In order to achieve the precise content of the structural information contained in the pair (m_Γ, b_Γ) we need the following definition.

Definition 1.21 *The **type $\boldsymbol{\tau}(\Gamma)$** of a finitely generated virtually free group $\Gamma \cong \pi_1(\Gamma(-), Y)$ is defined as the tuple $(m_\Gamma; \zeta_1(\Gamma), ..., \zeta_k(\Gamma), ..., \zeta_{m_\Gamma}(\Gamma))$, where the $\zeta_k(\Gamma)$ are integers indexed by the divisors of m_Γ, given by*

$$\zeta_k(\Gamma) = |\{e \in E_+(Y) : |\Gamma(e)| \ \Big| \ k \}| - |\{v \in V(Y) : |\Gamma(v)| \ \Big| \ k\}|. \tag{1.12}$$

Example 5:

$\Gamma \cong \pi_1$ 

We have $\mu(\Gamma) = 3, \quad m_\Gamma = 6k, \quad \tau(\Gamma) = (6k; 0, ..., 0, 1_{2k}, 1_{3k}, -1_{6k})$. Here the corresponding divisor of m_Γ is denoted by a subscript.

Definition 1.22 *Two finitely generated virtually free groups Γ_1, Γ_2 are called equivalent, $\Gamma_1 \sim \Gamma_2$, if $(m_{\Gamma_1}, b_{\Gamma_1}) = (m_{\Gamma_2}, b_{\Gamma_2})$, i.e., if Γ_1 and Γ_2 contain the same number of free subgroups for each finite index.*

The precise relation between the number of free subgroups in a finitely generated virtually free group Γ and the type $\tau(\Gamma)$ is given by the following theorem.

Theorem 1.23 ([M91, Thm. 2]) *The type $\tau(\Gamma)$ is an invariant of the group Γ, i.e., $\tau(\Gamma)$ is independent of a particular decomposition of Γ. We have $\Gamma_1 \sim \Gamma_2$ if and only if $\tau(\Gamma_1) = \tau(\Gamma_2)$.*

The Euler characteristic $\chi(\Gamma)$ can be expressed in terms of the type $\tau(\Gamma)$ via

$$\chi(\Gamma) = -m_\Gamma^{-1} \sum_{k|m_\Gamma} \phi(\frac{m_\Gamma}{k})\zeta_k(\Gamma), \tag{1.13}$$

where ϕ is Euler's totient function. Indeed,

$$\begin{aligned}
&\sum_{\kappa|m_\Gamma} \phi(m_\Gamma/\kappa)\zeta_\kappa(\Gamma) = \\
&\sum_{1\leq k\leq m_\Gamma} \left(|\{e \in E : |\Gamma(e)|\Big|k\}| - |\{v \in V : |\Gamma(v)|\Big|k\}| \right) = \\
&\sum_{e\in E} \frac{m_\Gamma}{|\Gamma(e)|} - \sum_{v\in V} \frac{m_\Gamma}{|\Gamma(v)|} = -m_\Gamma\, \chi(\Gamma)\,,
\end{aligned}$$

the last equality coming from the fact that the Euler characteristic $\chi(\Gamma)$ in the sense of Wall coincides with the equivariant Euler characteristic $\chi_X(\Gamma)$ relative to the tree X associated with Γ in the sense of Bass and Serre [B82, Chap. IX, Prop. 7.3].

By 1.4 and 1.13, the free rank $\mu(\Gamma)$ can be expressed as

$$\mu(\Gamma) = 1 + \sum_{k|m_\Gamma} \phi(\frac{m_\Gamma}{k})\zeta_k(\Gamma) \tag{1.14}$$

An important consequence of Theorem 1.23 and 1.13 is the following Corollary:

Corollary 1.24 ([M91, Cor. 3]) *Equivalent groups have the same (rational) Euler characteristic.*

We now collect some further observations, which will be of use in Chapter 2.

Proposition 1.25 ([M91, Lem. 2]) *We have $\zeta_k \geq 0$ for $k < m_\Gamma$ and $\zeta_{m_\Gamma} \geq -1$ with equality occurring in the latter inequality if and only if Y is a tree.*

Proposition 1.26 ([M91, Lem. 3]) *Let $e_1 \in E(Y)$ be an edge such that $\partial_0(e_1) \neq \partial_1(e_1)$ and $|\Gamma(\partial_0(e_1))| \neq |\Gamma(e_1)| \neq |\Gamma(\partial_1(e_1))|$. Then $\zeta_{|\Gamma(e_1)|} > 0$.*

Proposition 1.27 ([M91, Prop. 4]) *Each set of isomorphism classes of finitely generated virtually free groups Γ with prescribed constant m_Γ and Euler characteristic $\chi(\Gamma)$ is finite.*

Theorem 1.28 ([M91, Thm. 3]) *An equivalence class of finitely generated virtually free groups contains only a finite number of non-isomorphic groups.*

This can be deduced from Theorem 1.23, Corollary 1.24 and Proposition 1.27

1.6 An alternative recursive description of $b_\Gamma(\lambda)$

Theorem 1.18 relates the generating functions $\alpha_\Gamma(z) = \sum_{\lambda=0}^{\infty} a_\Gamma(\lambda) z^\lambda$ and $\beta_\Gamma(z) = \sum_{\lambda=0}^{\infty} b_\Gamma(\lambda+1) z^\lambda$ via the transformation formula

$$\beta_\Gamma(z) = m_\Gamma \, \frac{d}{dz} \, (\ln \, \alpha_\Gamma(z)).$$

Moreover, $\alpha_\Gamma(z)$ satisfies a homogeneous linear differential equation in terms of the type $\tau(\Gamma)$.

Theorem 1.29 ([M91, Prop. 5])
Let Γ be a virtually free group of type $\tau(\Gamma) = (m_\Gamma;\, \zeta_1(\Gamma), ..., \zeta_k(\Gamma), ..., \zeta_{m_\Gamma}(\Gamma))$. The generating function $\alpha_\Gamma(z)$ satisfies a homogeneous linear differential equation

$$A_0 \, \alpha_\Gamma(z) + (A_1 z - m_\Gamma) \, \alpha_\Gamma'(z) + \sum_{\nu=2}^{\mu(\Gamma)} A_\nu \, z^\nu \alpha_\Gamma^{(\nu)}(z) = 0 \tag{1.15}$$

of order $\mu(\Gamma)$ with integral coefficients

$$A_\nu = \frac{1}{\nu!} \, \sum_{i=0}^{\nu} (-1)^{\nu-i} \binom{\nu}{i} \, m_\Gamma \, (i+1) \prod_{k \mid m_\Gamma} \prod_{\substack{1 \leq l \leq m_\Gamma \\ (m_\Gamma, l) = k}} (i \, m_\Gamma + l)^{\zeta_k} \tag{1.16}$$

for $0 \leq \nu \leq \mu(\Gamma)$.

Recall Bell's formula for the higher derivatives of combined functions:

$$\frac{d^\nu}{dz^\nu} f(g(z)) = \sum_{\lambda \,\vdash\, \nu} \frac{\nu!}{\lambda_1!...\lambda_\nu!} \left[\prod_{i=1}^{\nu} \left(\frac{g^{(i)}(z)}{i!} \right)^{\lambda_i} \right] f^{(l(\lambda))}(g(z)) \tag{1.17}$$

where λ runs through the partitions of ν, i.e., the ν-tuples $\lambda = (\lambda_1, ..., \lambda_\nu)$ with $\lambda_i \geq 0$ and $\sum i\, \lambda_i = \nu$; further $l(\lambda) = \lambda_1 + ... + \lambda_\nu$ denotes the length of the partition λ.

Using 1.17 with $f(t) = e^t$ and $g(z) = \int m_\Gamma^{-1}\, \beta_\Gamma(z)\, dz$ we find that

$$\alpha_\Gamma^{(\nu)}(z) = \nu!\; \alpha_\Gamma(z) \sum_{l \geq 1} \sum_{\substack{\nu_1, ..., \nu_l > 0 \\ \sum \nu_j = \nu}} m_\Gamma^{-l}\, \frac{1}{l!} \prod_{j=1}^{l} \frac{\beta_\Gamma^{(\nu_j - 1)}(z)}{\nu_j!}, \qquad \nu \geq 1,$$

implying the differential equation

$$\beta_\Gamma(z) = A_0 + \sum_{\nu=1}^{\mu(\Gamma)} \sum_{l=1}^{\nu} \sum_{\substack{\nu_1, ..., \nu_l > 0 \\ \sum \nu_j = \nu}} \binom{\nu}{\nu_1, ..., \nu_l} A_\nu\, m_\Gamma^{-l}\, \frac{1}{l!}\, z^\nu \prod_{j=1}^{l} \beta_\Gamma^{(\nu_j - 1)}(z).$$

for $\beta_\Gamma(z)$. By comparing coefficients we deduce the following recursion for $b_\Gamma(\lambda)$:

Proposition 1.30 ([M91a])

$$b_\Gamma(1) = A_0$$

$$b_\Gamma(\lambda+1) = \sum_{l \geq 1} \sum_{\substack{\lambda_1, ..., \lambda_l \geq 1 \\ \sum \lambda_j = \lambda}} \left(\sum_{\substack{\nu \\ l \leq \nu \leq \mu(\Gamma)}} \sum_{\substack{\nu_1, ..., \nu_l \geq 0 \\ \sum \nu_j = \nu - l}} \binom{\nu}{\nu_1 + 1, ..., \nu_l + 1} A_\nu \cdot \right.$$
$$\left. \prod_{j=1}^{l} \left[\binom{\lambda_j - 1}{\nu_j} \nu_j!\, (m_\Gamma\, j)^{-1} \right] \right) \cdot \prod_{j=1}^{l} b_\Gamma(\lambda_j) \tag{1.18}$$

for $\lambda \geq 1$, with A_ν as defined in 1.16.

2 Divisibility properties of the function $b_\Gamma(\lambda)$

In this chapter we investigate finitely generated virtually free groups Γ of free rank $\mu(\Gamma) \leq 3$, with regard to divisibility properties of the number $b_\Gamma(\lambda)$ of free subgroups of index λm_Γ in Γ. First, we classify these groups by types $\tau(\Gamma)$ occurring for different ranks $\mu(\Gamma) = 1, 2, 3$. Then we calculate the recursion formulas for $b_\Gamma(\lambda)$ using Proposition 1.30. Evaluating those recursion formulas for each occurring type and primes $p \leq 13$, we observe that some groups, for example the classical modular group $\Gamma = PSL(2, \mathbb{Z})$ and the special linear group $\Gamma = SL(2, \mathbb{Z})$, show a rather interesting divisibility pattern for $b_\Gamma(\lambda)$, and we dedicate those "special" groups separate theorems. The divisibility patterns of $b_\Gamma(\lambda)$ found for the other types of groups are either constant modulo p or have a periodicity modulo p.

2.1 The case $\mu(\Gamma) = 1$

We have two possible types $\tau(\Gamma)$ of finitely generated virtually free groups Γ with free rank $\mu(\Gamma) = 1$.

Proposition 2.1 ([Mü91a, Prop. 4.3]) *Let Γ be a finitely generated virtually free group of rank $\mu(\Gamma) = 1$. The type $\tau(\Gamma)$ belongs to one of the following two forms:*

i) $\tau(\Gamma) = (n; 0, ..., 0_n)$

ii) $\tau(\Gamma) = (2k; 0, ..., 0, 1_k, -1_{2k})$

For the group realization of those types we get:

Proposition 2.2 ([Mü89, (49)]) *A virtually infinite-cyclic group Γ belongs to one of the two classes:*

i) Γ contains a finite normal subset H with infinite-cyclic factor group Γ/H.

ii) Γ is a free product $\Gamma = G_1 \underset{S}{*} G_2$ of finite groups G_i with amalgamated subgroup S and indices $[G_1 : S] = [G_2 : S] = 2$.

Determining the recursion formulas for $b_\Gamma(\lambda)$ using formula 1.16 and Proposition 1.30 gives:

Proposition 2.3 ([Mü89, (50)]) *For a virtually free infinite-cyclic group Γ the function $b_\Gamma(\lambda)$ is constant:*

in case i) $b_\Gamma(\lambda) = n$

in case ii) $b_\Gamma(\lambda) = k$.

Hence, the divisibility property of $b_\Gamma(\lambda)$ depends solely on the one for m_Γ.

2.2 The case $\mu(\Gamma) = 2$

The possible types $\tau(\Gamma)$ for finitely generated virtually free groups of rank $\mu(\Gamma) = 2$ are:

Proposition 2.4 ([Mü91a, Prop. 4.3]) *Let Γ be a finitely generated virtually free group of rank $\mu(\Gamma) = 2$. The type $\tau(\Gamma)$ belongs to one of the following six classes:*

i) $\tau(\Gamma) = (n; 0, ..., 0, 1_n)$

ii) $\tau(\Gamma) = (2k; 0, ..., 0, 1_k, 0_{2k})$

iii) $\tau(\Gamma) = (2k; 0, ..., 0, 2_k, -1_{2k})$

iv) $\tau(\Gamma) = (3k; 0, ..., 0, 1_k, 0, ..., 0, -1_{3k})$

v) $\tau(\Gamma) = (4k; 0, ..., 0, 1_k, 0, ..., 0, -1_{4k})$

vi) $\tau(\Gamma) = (6k; 0, ..., 0, 1_k, 0, ..., 0, -1_{6k})$

Using this Proposition we can describe the finitely generated virtually free groups of rank $\mu(\Gamma) = 2$ as follows:

Proposition 2.5 ([Mü89, (51)]) *A finitely generated virtually free group of rank $\mu(\Gamma) = 2$ belongs to one of the following five classes:*

i) Let Γ be a free product $\Gamma = G_1 \underset{S}{*} G_2$ of two finite groups G_i with an amalgamated subgroup S, whose indices $[G_i : S]$ satisfy one of the following conditions:

i_1) $\{[G_1 : S], [G_2 : S]\} = \{2, 3\}$

i_2) $\{[G_1 : S], [G_2 : S]\} = \{2, 4\}$

i_3) $[G_1 : S] = 3 = [G_2 : S]$

ii) Let Γ be of the form $\Gamma = (G_1 \underset{G_{12}}{*} G_2) \underset{G_{23}}{*} G_3$ with finite groups G_i and indices $[G_1 : G_{12}] = 2 = [G_2 : G_{12}]$ and $[G_2 : G_{23}] = 2 = [G_3 : G_{23}]$

iii) Γ is an HNN-extension $\Gamma = G_1 \underset{S,\sigma}{*}$ with finite base group G_1 and index $[G_1 : S] = 2$

iv) Γ is a free product $\Gamma = G_1 \underset{S}{*} G_2$, where G_1 contains a finite normal subset $H_1 \supset S$ with infinite-cyclic factor group, and G_2 is finite with $[H_1 : S] = 2 = [G_2 : S]$

v) Let Γ contain a finite normal subset H with factor group $\Gamma/H \cong F_2$, free of rank 2.

We present the different groups of rank $\mu(\Gamma) = 2$ together with their recursions for $b_\Gamma(\lambda)$ accordingly to formula 1.16 and Proposition 1.30 and investigate the divisibility properties of $b_\Gamma(\lambda)$ for primes $p \leq 13$. In the cases $p = 2$ and $p = 3$ we observe for some groups more interesting divisibility conditions, which we exhibit as separate theorems.

Theorem 2.6 ([Mü91, Prop. 6]) *Suppose that Γ belongs to one of the classes i_1), i_2) or ii), where $|S| \equiv 1 \pmod 2$, respectively $|G_{12}| \equiv 1 \pmod 2$. Then $b_\Gamma(\lambda)$ is odd if and only if λ is of the form $\lambda = 2^n - 1$ with an integer $n \geq 1$.*

Now we consider the case $p = 3$. Here the groups of class i_1) (e.g. $\Gamma = PSL(2,\mathbb{Z}) \cong C_2 * C_3$ and $\Gamma = SL(2,\mathbb{Z}) \cong C_4 \underset{C_2}{*} C_6$) again show an interesting behavior, together with the class i_3).

Theorem 2.7 *Let Γ be a free product $\Gamma = G_1 \underset{S}{*} G_2$ of finite groups G_i with amalgamated subgroup S of cardinality $|S| \equiv 1 \pmod 3$ or $|S| \equiv 2 \pmod 3$ and indices $\{[G_1 : S], [G_2 : S]\} = \{2, 3\}$ or $[G_1 : S] = [G_2 : S] = 3$. Then $b_\Gamma(\lambda) \equiv 1 \pmod 3$ if and only if λ has a representation as*

$$\begin{aligned}
&\lambda = c \cdot 3^6 - k \cdot 3^3 - l \quad \textit{or} \\
&\lambda = (2 \cdot 3^n + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l \\
&\textit{where } n \geq 1,\ c \in \{1, 3\},\ I \subseteq [n-1], \\
&k \textit{ chosen maximal in } K = \{0, 2, 6, 8, 18, 20, 24, 26\} \\
&\textit{and } l \in L_1 = \{0, 4, 6, 10, 18, 22, 24, 28\}.
\end{aligned} \tag{$*$}$$

$b_\Gamma(\lambda) \equiv 2 \pmod 3$ if and only if λ is of the form $(*)$ with an $l \in L_2 = \{2, 8, 20, 26\}$.

Proof:
$b_\Gamma(\lambda + 1)$ is given by the recurrence calculated from formula 1.16 and Proposition 1.30

$$b_\Gamma(\lambda + 1) = (\lambda + 1) \cdot m_\Gamma \cdot b_\Gamma(\lambda) + \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) \cdot b_\Gamma(\lambda - \mu),$$

starting with $b_\Gamma(1) = (m_\Gamma - |S|)|S| = 5|S|^2$, where $m_\Gamma = 6|S|$, respectively $m_\Gamma = 3|S|$ for $[G_1 : s] = [G_2 : S] = 3$. We get $b_\Gamma(1) \equiv 2 \pmod 3$ for both $|S| \equiv 1 \pmod 3$ and $|S| \equiv 2 \pmod 3$.

Since $m_\Gamma \equiv 0 \pmod 3$ it is sufficient for the divisibility of $b_\Gamma(\lambda + 1)$ to consider

$$b_\Gamma(\lambda + 1) = \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) \cdot b_\Gamma(\lambda - \mu).$$

We prove the congruences

$$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 3 & \text{if} \quad \lambda \text{ is of form } (*) \text{ with } l \in L_1, \\ 2 \pmod 3 & \text{if} \quad \lambda \text{ is of form } (*) \text{ with } l \in L_2, \\ 0 \pmod 3 & \text{else} \end{cases} \tag{2.1}$$

by induction on λ.

To begin, we show by calculations for $\lambda = 1, ..., 6 \cdot 3^6 - 2 = (2 \cdot 3 + 1) \cdot 3^6 - 28 \cdot 3^3 - 28 - 1$ the assertion is true and observe a pattern of the L_1, L_2 numbers which is characteristic for all following blocks given by the form $(*)$.

$L_1 : b_\Gamma(\lambda) \equiv 1(3)$		53	161	215	485	539	647	701
$L_2 : b_\Gamma(\lambda) \equiv 2(3)$	1	55	163	217	487	541	649	703
$L_1 : b_\Gamma(\lambda) \equiv 1(3)$	3	57	165	219	489	543	651	705
$L_1 : b_\Gamma(\lambda) \equiv 1(3)$	5	59	167	221	491	545	653	707
$L_2 : b_\Gamma(\lambda) \equiv 2(3)$	7	61	169	223	493	547	655	709
$L_1 : b_\Gamma(\lambda) \equiv 1(3)$	9	63	171	225	495	549	657	711
$7 \times$ $b_\Gamma(\lambda) \equiv 0(3)$								
$L_1 : b_\Gamma(\lambda) \equiv 1(3)$	17	71	179	233	503	557	665	719
$L_2 : b_\Gamma(\lambda) \equiv 2(3)$	19	73	181	235	505	559	667	721
$L_1 : b_\Gamma(\lambda) \equiv 1(3)$	21	75	183	237	507	561	669	723
$L_1 : b_\Gamma(\lambda) \equiv 1(3)$	23	77	185	239	509	563	671	725
$L_2 : b_\Gamma(\lambda) \equiv 2(3)$	25	79	187	241	511	565	673	727
$L_1 : b_\Gamma(\lambda) \equiv 1(3)$	27	81	189	243	513	567	675	729
number of $b_\Gamma(\lambda) \equiv 0(3)$	25	79	25	241	25	79	25	727

Figure 1: First block with L_1, L_2 pattern

Suppose that the assertion is true for all $\lambda \leq t$. We distinguish two cases for $t+1$

a) $t+1$ is of form $(*)$ with $l \in L_1$ or $l \in L_2$

b) $t+1$ otherwise

Case a) $t+1$ is of form $(*)$ with $n \geq 1,\ l \in L_1$ or $l \in L_2$

$$t+1 = (2 \cdot 3^n + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l, \quad I \subseteq [n-1],\ c \in \{1, 3\},\ k \in K$$

Unless $I = \varnothing$ there exists an $i_{\max} \in I$, s.t. $\forall i \in I \;\; i_{\max} > i$. We rewrite $t+1$ as

$$t+1 = (2 \cdot 3^n + 2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l,$$

where $I \subseteq [i_{\max} - 1]$ for $i_{\max} > 1$ and $I = \varnothing$ for $i_{\max} = 1$.

With

$$\begin{aligned} B_1 &= (2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l - 2, \quad I \subseteq [i_{\max} - 1] \\ B_2 &= 2 \cdot 3^n \cdot 3^6 \end{aligned}$$

we get for $b_\Gamma(t+1)$

$$\begin{aligned} b_\Gamma(t+1) &= \sum_{\mu=1}^{t-1} b_\Gamma(\mu) \cdot b_\Gamma(t-\mu) \\ &= \sum_{\mu=1}^{B_1} b_\Gamma(\mu) \cdot b_\Gamma(t-\mu) && (\Sigma_1) \\ &\quad + \sum_{\mu=B_1+1}^{B_2} b_\Gamma(\mu) \cdot b_\Gamma(t-\mu) && (\Sigma_2) \\ &\quad + \sum_{\mu=B_2+1}^{t-1} b_\Gamma(\mu) \cdot b_\Gamma(t-\mu) && (\Sigma_3) \end{aligned} \tag{2.2}$$

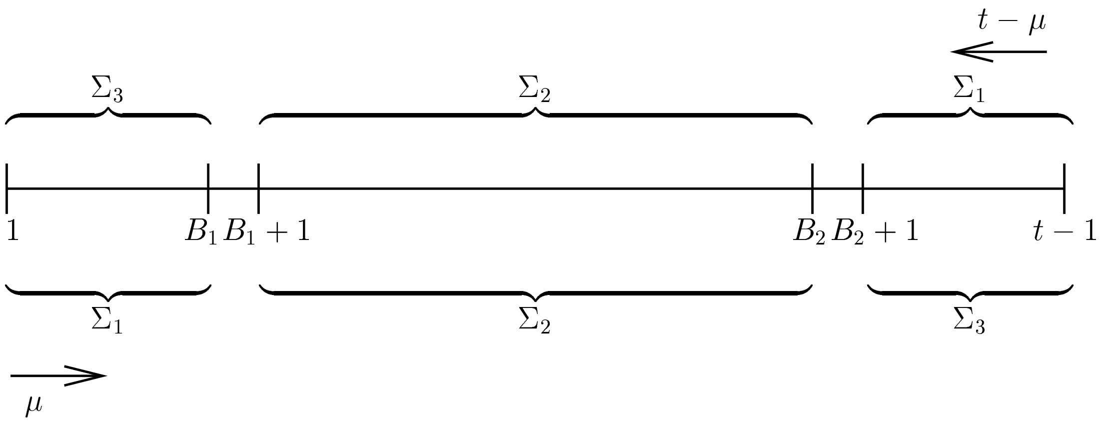


Figure 2: Sum Splitting

Observation 1:
Except for the cases $\nexists i_{\max}$, $I = \varnothing$, $c = 1$, $k = 26$, $l \in \{26, 28\}$ we observe that for $r = 1, ..., (2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l - 1$ and $I \subseteq [i_{\max} - 1]$, or $I = \varnothing$

$$b_\Gamma(2 \cdot 3^n \cdot 3^6 + r) \equiv b_\Gamma(r) \pmod 3 \tag{2.3}$$

We consider the two cases:
Case i) r is representable as an L_1- or L_2-number

$$r = (2 \cdot 3^{n'} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l, \quad I \subseteq [n' - 1],\ n' < n,\ k \in K,\ l \in L_1 \cup L_2,$$

then

$$\begin{aligned} 2 \cdot 3^n \cdot 3^6 + r &= (2 \cdot 3^n + 2 \cdot 3^{n'} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l, \quad I \subseteq [n' - 1] \\ &= (2 \cdot 3^n + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l, \quad I \subseteq [n' - 1] \subset [n - 1] \end{aligned}$$

fulfills condition $(*)$. So by the induction hypothesis

$$b_\Gamma(2 \cdot 3^n \cdot 3^6 + r) \equiv b_\Gamma(r) \pmod 3 \equiv \begin{cases} 1 \pmod 3, & l \in L_1 \\ 2 \pmod 3, & l \in L_2 \end{cases}$$

Case ii) r is not an L_1- or L_2-number

a) $2 \mid r$
Since $2 \mid 2 \cdot 3^n \cdot 3^6$ also $2 \mid 2 \cdot 3^n \cdot 3^6 + r$, but all L_1- and L_2-numbers are odd, hence by our assumption $b_\Gamma(2 \cdot 3^n \cdot 3^6 + r) \equiv b_\Gamma(r) \equiv 0 \pmod 3$.

b) $2 \nmid r$
Assume that $2 \cdot 3^n \cdot 3^6 + r$ is an L_1- or L_2-number and $2 \mid 2 \cdot 3^n \cdot 3^6$. Then it has a representation as

$$(2 \cdot 3^n + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l, \quad l \in L_1 \cup L_2,\ k \in K,\ I \subseteq [n - 1]$$

and

$$\begin{aligned} r &= (2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l \\ &= (2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l \end{aligned}$$

with an $i_{\max} < n$, $I \subseteq [i_{\max} - 1]$ or $\nexists i_{\max}$ and $I = \varnothing$. So r is a L_1- or L_2-number, a contradiction.

Applying the congruence argument (Observation 1) we get for (Σ_1) and by the symmetry of μ and $t-\mu$ also for (Σ_3) with

$$t^* = (2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l - 1$$

since

$$t + 1 = (2 \cdot 3^n + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l,$$

$$\begin{aligned} \sum_{\mu=1}^{B_1} b_\Gamma(\mu) \cdot b_\Gamma(t-\mu) &\equiv \sum_{\mu=1}^{B_1} b_\Gamma(\mu) \cdot b_\Gamma(t^*-\mu) \pmod 3 \\ &\equiv b_\Gamma((2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l) \pmod 3, \end{aligned} \tag{2.4}$$

and by induction hypothesis the value of $b_\Gamma((2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l)$ is congruent $1 \pmod 3$ or $2 \pmod 3$, depending on $l \in L_1$ or $l \in L_2$.

Analyzing (Σ_2) in $b_\Gamma(t+1)$ we calculate the first two products.

$$b_\Gamma((2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l - 1) \cdot b_\Gamma(2 \cdot 3^n \cdot 3^6) \equiv 0 \pmod 3$$

since $2 \mid 2 \cdot 3^n \cdot 3^6$ and by induction hypothesis $b_\Gamma(2 \cdot 3^n \cdot 3^6) \equiv 0 \pmod 3$.

The value of

$$b_\Gamma((2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l) \cdot b_\Gamma(2 \cdot 3^n \cdot 3^6 - 1) \tag{2.5}$$

depends on l since $2 \cdot 3^n \cdot 3^6 - 1 = (2 \cdot 3^n + 1) \cdot 3^6 - 26 \cdot 3^3 - 28$ is of form $(*)$ with $c = 1,\ k = 26,\ l = 28$ so $b_\Gamma(2 \cdot 3^n \cdot 3^6 - 1) \equiv 1 \pmod 3$.

Again by the symmetry of μ and $t-\mu$ we get the same for the last two products in (Σ_2). With 2.4 and 2.5 we have so far $b_\Gamma(t+1)$ partially calculated as

$$\begin{aligned} 2 \cdot \left(b_\Gamma((2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l) \cdot (1 + b_\Gamma(2 \cdot 3^n \cdot 3^6 - 1)) \right) \\ \equiv \begin{cases} 1 \pmod 3, & l \in L_1 \\ 2 \pmod 3, & l \in L_2 \end{cases} \end{aligned} \tag{2.6}$$

Regarding the remaining products in $\sum\limits_{\mu=B_1+3}^{B_2-2} b_\Gamma(\mu)b_\Gamma(t-\mu)$ we use

Observation 2:

$$b_\Gamma(z) \equiv 0 \pmod 3 \qquad \forall z \text{ with } 3^n \cdot 3^6 < z < 2 \cdot 3^n \cdot 3^6 - 1, \tag{2.7}$$

since $3^n \cdot 3^6 = (2 \cdot 3^{n-1} + 2 \cdot \sum\limits_{i=1}^{n-2} 3^i + 3) \cdot 3^6$ is for $n-1$ by the induction hypothesis the largest number representable with an $l \in L_1 \cup L_2$ ($I = [n-2],\ c = 3,\ k, l = 0$) and $2 \cdot 3^n \cdot 3^6 - 1 = (2 \cdot 3^n + 1) \cdot 3^6 - 26 \cdot 3^3 - 28$ is for n the least number representable with an $l \in L_1 \cup L_2$ ($\nexists i_{\max},\ I = \varnothing,\ c = 1,\ k = 26,\ l = 28$).

Further, the difference

$$3^n \cdot 3^6 - ((2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l) \quad < 3^n \cdot 3^6 - 2$$
$$I \subseteq [i_{\max} - 1],\ i_{\max} \le n-1, \tag{2.8}$$

except for the cases where $\nexists i_{\max},\ c = 1,\ k = 26,\ l \in \{26, 28\}$. So, in $\sum\limits_{\mu=B_1+3}^{B_2-2} b_\Gamma(\mu)b_\Gamma(t-\mu)$ we always get products of the form $b_\Gamma(\mu)b_\Gamma(t-\mu)$ where at least one factor is congruent $0 \pmod 3$ because of 2.7, hence the whole sum vanishes modulo 3.
It remains to consider the exceptional cases $\nexists i_{\max},\ c = 1,\ k = 26,\ l \in \{26, 28\}$.
For $l = 26 \in L_2$ we have

$$\begin{aligned} t+1 &= (2 \cdot 3^n + 1) \cdot 3^6 - 26 \cdot 3^3 - 26 \\ &= 2 \cdot 3^n \cdot 3^6 + 1. \end{aligned}$$

With $t - 1 = 2 \cdot 3^n \cdot 3^6 - 1$ we get

$$\begin{aligned} b_\Gamma(t+1) &= \sum_{\mu=1}^{2\cdot 3^n \cdot 3^6 - 1} b_\Gamma(\mu)b_\Gamma(t-\mu) \qquad\qquad (\text{reduced}(\Sigma_2)) \\ &= 2 \cdot b_\Gamma(1) \cdot b_\Gamma(2 \cdot 3^n \cdot 3^6 - 1) + \sum_{\mu=2}^{2\cdot 3^n \cdot 3^6 - 2} b_\Gamma(\mu)b_\Gamma(t-\mu), \end{aligned}$$

where by induction hypothesis $2 \cdot b_\Gamma(1) \cdot b_\Gamma(2 \cdot 3^n \cdot 3^6 - 1) \equiv 2 \cdot 2 \cdot 1 \pmod 3 \equiv 1 \pmod 3$.

Using that for $2 \nmid t+1$

$$b_\Gamma(t+1) = 2 \cdot \sum_{\mu=1}^{\frac{t}{2}-1} b_\Gamma(\mu)b_\Gamma(t-\mu) \quad + b_\Gamma(\frac{t}{2})^2, \tag{2.9}$$

and $\frac{t}{2} = 3^n \cdot 3^6$ and by induction hypothesis $b_\Gamma(3^n \cdot 3^6)^2 \equiv 1 \pmod 3$ and by 2.7 is the sum congruent $0 \pmod 3$ hence $b_\Gamma(2 \cdot 3^n \cdot 3^6 + 1) \equiv 2 \pmod 3$ as required.

For $l = 28 \in L_1$ we have

$$\begin{aligned} t+1 &= (2 \cdot 3^n + 1) \cdot 3^6 - 26 \cdot 3^3 - 28 \\ &= 2 \cdot 3^n \cdot 3^6 - 1, \\ t-1 &= 2 \cdot 3^n \cdot 3^6 - 3 \end{aligned}$$

$$b_\Gamma(t+1) = \sum_{\mu=1}^{2\cdot 3^n \cdot 3^6 - 3} b_\Gamma(\mu) b_\Gamma(t-\mu).$$

Further $\frac{t}{2} = 3^n \cdot 3^6 - 1$ and since $2 \mid \frac{t}{2}$ we have by our induction hypothesis $b_\Gamma(\frac{t}{2}) \equiv 0 \pmod 3$. By 2.7 and 2.9 we need only to calculate $2 \cdot b_\Gamma(3^n \cdot 3^6 - 2) \cdot b_\Gamma(3^n \cdot 3^6) \equiv 1 \pmod 3$ since $3^n \cdot 3^6 - 2 = (2 \cdot 3^{n-1} + 2 \cdot \sum\limits_{i=1}^{n-2} 3^i + 3) - 2$ is an L_2-number and similarly $3^n \cdot 3^6$ an L_1-number.

Case b) $t+1$ is not of form $(*)$. We consider two cases:

i) $2 \mid t+1,\ t+1$ not of form $(*)$

ii) $2 \nmid t+1,\ t+1$ not of form $(*)$

Case i) $2 \mid t+1,\ t+1$ not of form $(*)$
Since $2 \mid t-1$ we have in $b_\Gamma(t+1) = \sum\limits_{\mu=1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu)$ for every product $b_\Gamma(\mu) b_\Gamma(t-\mu)$ either $2 \mid \mu$ or $2 \mid t-\mu$ hence by induction hypothesis $b_\Gamma(\mu) \equiv 0 \pmod 3$ or $b_\Gamma(t-\mu) \equiv 0 \pmod 3$ and this results to $b_\Gamma(t+1) \equiv 0 \pmod 3$.

Case ii) $t+1$ not of form $(*)$ and $2 \nmid t+1$
Then there exists a tuple (n, I, c, k, l) as in $(*)$ such that $(2 \cdot 3^n + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l \quad I \subseteq [n-1]$ is the largest L_1- or L_2-number less than $t+1$. So we set

$$t+1 = (2 \cdot 3^n + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l + s' \qquad (*_n)$$

The congruence argument of Observation 1 holds up to $r = (2 \cdot \sum_{i=1}^{n-1} 3^i + 3) \cdot 3^6 = 3^n \cdot 3^6$. In fact, because of 2.7 it can be extended to

$$b_\Gamma(2 \cdot 3^n \cdot 3^6 + r') \equiv b_\Gamma(r') \pmod 3 \equiv 0 \pmod 3 \tag{2.10}$$

for all r' with $3^n \cdot 3^6 + 1 \leq r' \leq 2 \cdot 3^n \cdot 3^6 - 2$ and we get

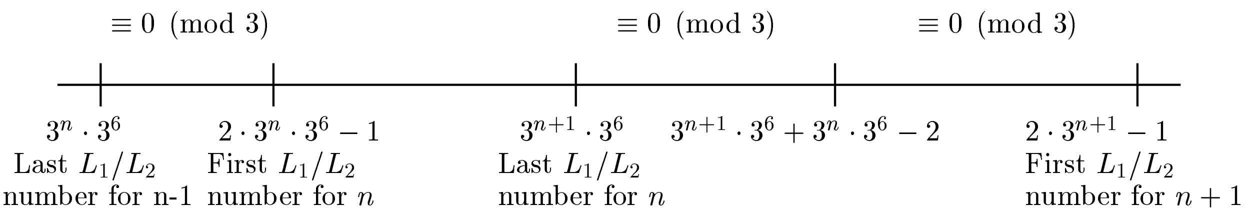
$$b_\Gamma(2 \cdot 3^n \cdot 3^6 + r) \equiv b_\Gamma(r) \pmod 3 \tag{2.11}$$

for all $1 \leq r \leq 2 \cdot 3^n \cdot 3^6 - 2$.

By the observation we consider two cases with maximal (n, I, c, k, l) for s', $2 \mid s'$:

I) $2 \cdot 3^n \cdot 3^6 - 1 < t + 1 < 3^{n+1} \cdot 3^6 + 3^n \cdot 3^6 - 2$ and $t+1$ not an L_1- or L_2-number. $(2 \leq s' < 3^n \cdot 3^6 - 2, \quad 2 \mid s')$

II) $3^{n+1} \cdot 3^6 + 3^n \cdot 3^6 - 1 \leq t + 1 < 2 \cdot 3^{n+1} \cdot 3^6 - 2)$ and $t+1$ not an L_1- or L_2-number. $(3^n \cdot 3^6 - 1 \leq s' < 3^{n+1} \cdot 3^6 - 2, 2 \mid s')$



Case I) $\quad 2 \leq s' < 3^n \cdot 3^6 - 2 \quad 2 \mid s'$ and $t+1$ not an L_1- or L_2-number.

$$t + 1 = (2 \cdot 3^n + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l + s'$$

with maximal (n, I, c, k, l) as in $(*_n)$. Then there exists $i_{\max}$, s.t.

$$t+1 = (2 \cdot 3^n + 2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l + s', \quad I \in [\imath_{\max} - 1] \text{ or } \nexists i_{\max} \text{ and } I = \varnothing$$

So

$$\begin{aligned} b_\Gamma(t+1) &= \sum_{\mu=1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &= \sum_{\mu=1}^{B_1} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=B_1+1}^{B_2} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=B_2+1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \end{aligned}$$

where

$$B_1 = (2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l - 2 + s'$$
$$B_2 = 2 \cdot 3^n \cdot 3^6$$

Since $s' < 3^n \cdot 3^6 - 2$ we apply 2.4 and 2.11 and get with $t^* = (2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l - 1 + s'$

$$\begin{aligned}
\sum_{\mu=1}^{B_1} b_\Gamma(\mu) b_\Gamma(t-\mu) &= \sum_{\mu=1}^{B_1} b_\Gamma(\mu) b_\Gamma(t^*-\mu) \\
&= b_\Gamma((2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l + s') \\
&\equiv 0 \pmod 3,
\end{aligned}$$

because $(2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l + s'$ is with the maximal chosen $(i_{\max}, I, c, k, l)$ and s' not of form $(*)$.

Analyzing $\sum\limits_{\mu=B_1+1}^{B_2} b_\Gamma(\mu) b_\Gamma(t-\mu)$ we calculate the first 2 products:

$$b_\Gamma((2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l + s' - 1) \cdot b_\Gamma(2 \cdot 3^n \cdot 3^6) \equiv 0 \pmod 3$$

since $2 \mid 2 \cdot 3^n \cdot 3^6$, and by induction hypothesis

$$b_\Gamma((2 \cdot 3^{i_{\max}} + 2 \cdot \sum_{i \in I} 3^i + c) \cdot 3^6 - k \cdot 3^3 - l + s') \cdot b_\Gamma(2 \cdot 3^n \cdot 3^6 - 1) \equiv 0 \pmod 3$$

by the same maximal $(i_{\max}, I, c, k, l)$ and s'.

The remaining products are congruent $0 \pmod 3$ by 2.7 and 2.8, so for $2 \le s' < 3^n \cdot 3^6 - 2$ we have $b_\Gamma(t+1) \equiv 0 \pmod 3$.

Case II) $3^n \cdot 3^6 - 1 \le s' < 3^{n+1} \cdot 3^6 - 2$ and $t+1$ not an L_1- or L_2-number.
Since we have chosen (n, I, c, k, l) maximal as in $(*_n)$ we rewrite

$$t + 1 = 3^{n+1} \cdot 3^6 + s'$$

or

$$t + 1 = 3^{n+1} \cdot 3^6 + 3^n \cdot 3^6 - 1 + s \quad \text{with } 0 \le s \le 2 \cdot 3^n \cdot 3^6 - 2,\ 2 \mid s.$$

In order to apply the congruence arguments 2.3 and 2.11 we consider two cases:

Case II_1) $\quad 3^n \cdot 3^6 + 3 \le s \le 2 \cdot 3^n \cdot 3^6 - 2$

$$\begin{aligned} b_\Gamma(t+1) &= \sum_{\mu=1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &= \sum_{\mu=1}^{2\cdot 3^n \cdot 3^6} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=2\cdot 3^n\cdot 3^6+1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu), \end{aligned}$$

where $\sum_{\mu=1}^{2\cdot 3^n\cdot 3^6} b_\Gamma(\mu) b_\Gamma(t-\mu) \equiv 0 \pmod 3$ since $t-\mu > 3^{n+1}\cdot 3^6$ for $1 \le \mu \le 2\cdot 3^n \cdot 3^6$ and so by 2.7 $b_\Gamma(t-\mu) \equiv 0 \pmod 3$. Further, $\sum_{\mu=2\cdot 3^n\cdot 3^6+1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \equiv 0 \pmod 3$ since for $\mu = 2\cdot 3^n \cdot 3^6 + 1$, we have $3^{n+1}\cdot 3^6 \le t-\mu \le 3^{n+1}\cdot 3^6 + 3^n \cdot 3^6 - 1$ and by 2.3 and 2.11

$$\begin{aligned} \sum_{\mu=2\cdot 3^n\cdot 3^6+1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) &= \sum_{\mu=1}^{t-1-2\cdot 3^n\cdot 3^6} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv b_\Gamma(t+1-2\cdot 3^n\cdot 3^6) \pmod 3 \\ &\equiv b_\Gamma(2\cdot 3^n \cdot 3^6 + s - 1) \pmod 3 \\ &\equiv b_\Gamma(3^{n+1}\cdot 3^6 + s') \pmod 3 \quad (s \ge 3^n\cdot 3^6 + 3) \\ &\equiv 0 \pmod 3 \qquad (2 \le s' \le 3^n \cdot 3^6 - 3). \end{aligned}$$

Case II_2) $\quad 0 \le s \le 3^n\cdot 3^6 + 1 \quad 2 \mid s$
Then we have

$$t = 3^{n+1}\cdot 3^6 + 3^n \cdot 3^6 - 2 + s.$$

Since $t-\mu \leq 3^{n+1}\cdot 3^6$ for $\mu > 2\cdot 3^n\cdot 3^6,\ 4\leq s\leq 3^n\cdot 3^6+1$ we apply 2.7 and get

$$\begin{aligned}
b_\Gamma(t+1) &= \sum_{\mu=1}^{t-1} b_\Gamma(\mu)b_\Gamma(t-\mu)\\
&= \sum_{\mu=1}^{3^n\cdot 3^6} b_\Gamma(\mu)b_\Gamma(t-\mu)\\
&+ \sum_{\mu=3^n\cdot 3^6+1}^{2\cdot 3^n\cdot 3^6-2} b_\Gamma(\mu)b_\Gamma(t-\mu)\\
&+ \sum_{\mu=2\cdot 3^n\cdot 3^6-1}^{2\cdot 3^n\cdot 3^6} b_\Gamma(\mu)b_\Gamma(t-\mu)\\
&+ \sum_{\mu=2\cdot 3^n\cdot 3^6+1}^{t-(2\cdot 3^n\cdot 3^6+1)} b_\Gamma(\mu)b_\Gamma(t-\mu)\\
&+ \sum_{\mu=2\cdot 3^n\cdot 3^6+s-2}^{2\cdot 3^n\cdot 3^6+s} b_\Gamma(\mu)b_\Gamma(t-\mu)\\
&+ \sum_{\mu=2\cdot 3^n\cdot 3^6+s+1}^{t-1} b_\Gamma(\mu)b_\Gamma(t-\mu)
\end{aligned}$$

where $\sum\limits_{\mu=1}^{3^n\cdot 3^6} b_\Gamma(\mu)b_\Gamma(t-\mu)\equiv 0 \pmod 3$ for $4\leq s\leq 3^n\cdot 3^6+1$ since $t-\mu > 3^n\cdot 3^6$ for $1\leq \mu\leq 3^n\cdot 3^6,\ s\geq 4$ and so by 2.7 $b_\Gamma(t-\mu)\equiv 0 \pmod 3$, hence the whole sum.
$\sum\limits_{3^n\cdot 3^6+1}^{2\cdot 3^n\cdot 3^6-2} b_\Gamma(\mu)b_\Gamma(t-\mu)\equiv 0 \pmod 3$ for $4\leq s\leq 3^n\cdot 3^6+1$ since by 2.7 $b_\Gamma(\mu)\equiv 0 \pmod 3$ for $3^n\cdot 3^6+1\leq \mu\leq 2\cdot 3^n\cdot 3^6-2$.

Calculating $\sum\limits_{2\cdot 3^n\cdot 3^6-1}^{2\cdot 3^n\cdot 3^6} b_\Gamma(\mu)b_\Gamma(t-\mu)$ and by the symmetry of μ and $t-\mu$ also for $\sum\limits_{2\cdot 3^n\cdot 3^6+s-2}^{2\cdot 3^n\cdot 3^6+s} b_\Gamma(\mu)b_\Gamma(t-\mu)$:

With $\mu = 2\cdot 3^n\cdot 3^6-1 = (2\cdot 3^n\cdot 3^6+1)\cdot 3^6 - 26\cdot 3^3 - 28$ where $c=1,\ k=26,\ l=28\in L_1,\ b_\Gamma(2\cdot 3^n\cdot 3^6-1)\equiv 1 \pmod 3$ by induction hypothesis, and also for $\mu = 2\cdot 3^n\cdot 3^6+s-1$ in $t-\mu = 2\cdot 3^n\cdot 3^6-1$. $2\mid 2\cdot 3^n\cdot 3^6$ and $2\mid 2\cdot 3^n\cdot 3^6+s-2$ and $2\mid 2\cdot 3^n\cdot 3^6+s$ since

$2 \mid s$ and so all other remaining products in the sums are congruent $0 \pmod 3$. We get

$$2 \cdot b_\Gamma(2 \cdot 3^n \cdot 3^6 - 1) \cdot b_\Gamma(2 \cdot 3^n \cdot 3^6 + s - 1) \equiv \begin{cases} 2\ (3) & \text{if s-1 of form } (*) \text{ with } l \in L_1 \\ 1\ (3) & \text{if s-1 of form } (*) \text{ with } l \in L_2 \\ 0\ (3) & \text{if s-1 not of form } (*) \end{cases} \tag{2.12}$$

For $\sum\limits_{2\cdot 3^n \cdot 3^6+1}^{t-(2\cdot 3^n\cdot 3^6+1)} b_\Gamma(\mu) b_\Gamma(t-\mu)$ applies 2.7 since $t - 2 \cdot 3^n \cdot 3^6 - 1 = 2 \cdot 3^n \cdot 3^6 + s - 3$ and with $s \leq 3^n \cdot 3^6 + 1,\ t - \mu < 3^{n+1} \cdot 3^6$. With 2.3 and 2.4 we get with $t^* = s - 2$

$$\begin{aligned} \sum_{2\cdot 3^n\cdot 3^6+1}^{2\cdot 3^n\cdot 3^6+s-3} b_\Gamma(\mu) b_\Gamma(t-\mu) &\equiv \sum_{\mu=1}^{s-3} b_\Gamma(\mu) b_\Gamma(t^* - \mu) \\ &\equiv b_\Gamma(s-1) \pmod 3 \end{aligned} \tag{2.13}$$

For $b_\Gamma(s-1)$ we consider 3 cases and apply our induction hypothesis:

a) $s - 1$ of form $(*)$ with $l \in L_1$ then $b_\Gamma(s-1) \equiv 1 \pmod 3$

b) $s - 1$ of form $(*)$ with $l \in L_2$ then $b_\Gamma(s-1) \equiv 2 \pmod 3$

c) $s - 1$ not of form $(*)$ then $b_\Gamma(s-1) \equiv 0 \pmod 3$.

Finally, calculating

$\sum\limits_{2\cdot 3^n\cdot 3^6+s+1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu)$ gives since $3^n \cdot 3^6 + 1 \leq t - \mu \leq 2 \cdot 3^n \cdot 3^6 - 3$ or $\mu > 3^{n+1} \cdot 3^6$ and by 2.7 $b_\Gamma(t-\mu) \equiv 0 \pmod 3$.

So in total, we have to add 2.12 and 2.13 which gives always $b_\Gamma(t+1) \equiv 0 \pmod 3$.

It remains to consider the exceptional cases $s \in \{0, 2\}$.

$s = 0$
Using 2.9 we get for

$$t + 1 = 4 \cdot 3^n \cdot 3^6 - 1, \quad \frac{t}{2} = 2 \cdot 3^n \cdot 3^6 - 1$$

$$
\begin{aligned}
b_\Gamma(t+1) = 2 \cdot \sum_{\mu=1}^{3^n\cdot 3^6-3} & b_\Gamma(\mu) b_\Gamma(t-\mu) \qquad (\equiv 0 \pmod 3,\ 3^{n+1}\cdot 3^6 < t-\mu < t-1) \\
&+ 2 \cdot b_\Gamma(3^n\cdot 3^6 - 2) b_\Gamma(3^{n+1}\cdot 3^6) \qquad (\equiv 1 \pmod 3) \\
&+ 2 \cdot b_\Gamma(3^n\cdot 3^6 - 1) b_\Gamma(3^{n+1}\cdot 3^6 - 1) \qquad (\equiv 0 \pmod 3,\ 2 \mid 3^n\cdot 3^6 - 1,\ 2 \mid 3^{n+1}\cdot 3^6 - 1) \\
&+ 2 \cdot b_\Gamma(3^n\cdot 3^6) b_\Gamma(3^{n+1}\cdot 3^6 - 2) \qquad (\equiv 1 \pmod 3) \\
&+ \sum_{\mu=3^n\cdot 3^6+1}^{2\cdot 3^n\cdot 3^6-2} b_\Gamma(\mu) b_\Gamma(t-\mu) \qquad (\equiv 0 \pmod 3,\ \text{by } 2.7) \\
&+ b_\Gamma(2\cdot 3^n\cdot 3^6 - 1)^2 \qquad (\equiv 1 \pmod 3) \\
&\equiv 0 \pmod 3
\end{aligned}
$$

$s = 2$

$$
t+1 = 4\cdot 3^n \cdot 3^6 + 1, \quad \frac{t}{2} = 2\cdot 3^n\cdot 3^6
$$

$$
\begin{aligned}
b_\Gamma(t+1) = 2 \cdot \sum_{\mu=1}^{3^n\cdot 3^6-1} & b_\Gamma(\mu) b_\Gamma(t-\mu) \qquad (\equiv 0 \pmod 3,\ 3^{n+1}\cdot 3^6 < t-\mu < t-1) \\
&+ 2 \cdot b_\Gamma(3^n\cdot 3^6) b_\Gamma(3^{n+1}\cdot 3^6) \qquad (\equiv 2 \pmod 3) \\
&+ 2 \cdot \sum_{\mu=3^n\cdot 3^6+1}^{2\cdot 3^n\cdot 3^6-2} b_\Gamma(\mu) b_\Gamma(t-\mu) \qquad (\equiv 0 \pmod 3,\ \text{by } 2.7) \\
&+ 2 \cdot b_\Gamma(2\cdot 3^n\cdot 3^6 - 1) b_\Gamma(2\cdot 3^n\cdot 3^6 + 1) \qquad (\equiv 1 \pmod 3) \\
&+ b_\Gamma(2\cdot 3^n\cdot 3^6 - 1)^2 \qquad (\equiv 0 \pmod 3,\ 2 \mid 2\cdot 3^n\cdot 3^6) \\
&\equiv 0 \pmod 3
\end{aligned}
$$

□

In the rest of this section we investigate each type $\tau(\Gamma)$ with regard to divisibility properties modulo primes $p \leq 13$. The results are presented in form of a table ordered by the primes p. In order to give a better overview, the divisibility patterns found in the previous theorems are included again in the theorem for the specific groups.

Theorem 2.8 *Let Γ be a free product $\Gamma = G_1 \underset{S}{*} G_2$ of finite groups G_i with amalgamated subgroup S and indices $\{[G_1 : S], [G_2 : S]\} = \{2, 3\}$. Then we have for primes $p \leq 13$ and $|S| \equiv 1 \pmod p$ the following divisibility conditions for $b_\Gamma(\lambda)$:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 2 & \text{if } \lambda = 2^n - 1 \\ 0 \pmod 2 & \text{else} \end{cases}$
3	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 3 & \text{if} \quad \lambda \text{ is of form } (*) \text{ with } l \in L_1, \\ 2 \pmod 3 & \text{if} \quad \lambda \text{ is of form } (*) \text{ with } l \in L_2, \\ 0 \pmod 3 & \text{else} \end{cases}$
5	$b_\Gamma(\lambda) \equiv 0 \pmod 5 \quad \forall \lambda \geq 1$
7	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 7 & \text{if } \lambda \equiv 0 \pmod 6 \\ 2 \pmod 7 & \text{if } \lambda \equiv 4 \pmod 6 \\ 3 \pmod 7 & \text{if } \lambda \equiv 5 \pmod 6 \\ 4 \pmod 7 & \text{if } \lambda \equiv 2 \pmod 6 \\ 5 \pmod 7 & \text{if } \lambda \equiv 1 \pmod 6 \\ 6 \pmod 7 & \text{if } \lambda \equiv 3 \pmod 6 \end{cases}$
11	$b_\Gamma(\lambda) \equiv 5 \pmod{11} \quad \forall \lambda \geq 1$
13	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod{13} & \text{if } \lambda \equiv 3, 9 \pmod{12} \\ 1 \pmod{13} & \text{if } \lambda \equiv 0 \pmod{12} \\ 2 \pmod{13} & \text{if } \lambda \equiv 4 \pmod{12} \\ 5 \pmod{13} & \text{if } \lambda \equiv 1, 8 \pmod{12} \\ 6 \pmod{13} & \text{if } \lambda \equiv 11 \pmod{12} \\ 7 \pmod{13} & \text{if } \lambda \equiv 5 \pmod{12} \\ 8 \pmod{13} & \text{if } \lambda \equiv 2, 7 \pmod{12} \\ 11 \pmod{13} & \text{if } \lambda \equiv 10 \pmod{12} \\ 12 \pmod{13} & \text{if } \lambda \equiv 6 \pmod{12} \end{cases}$

Proof:
The recurrence formula for $b_\Gamma(\lambda)$ using this type of groups can be calculated from formula 1.16 and Proposition 1.30. We get

$$\begin{aligned} b_\Gamma(1) &= (m_\Gamma - |S|)|S| \\ b_\Gamma(\lambda+1) &= (\lambda+1)\cdot m_\Gamma \cdot b_\Gamma(\lambda) + \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) b_\Gamma(\lambda-\mu), \qquad \lambda \geq 1. \end{aligned}$$

Case $p = 2$:
Theorem 2.6.

Case $p = 3$:
Theorem 2.7.

Case $p = 5$:
Induction on λ. $b_\Gamma(1) = 5 \cdot |S|^2 \equiv 0 \pmod 5$. We suppose $b_\Gamma(\lambda) \equiv 0 \pmod 5$ for all $1 \leq \lambda \leq t$. The recursion formula states for $t+1$

$$\begin{aligned} b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv 0 \cdot (t+1) \cdot m_\Gamma \cdot 0 + (t-1)\cdot 0 \pmod 5 \\ &\equiv 0 \pmod 5. \end{aligned}$$

Case $p = 7$:
Induction on λ. $b_\Gamma(1) = 5 \cdot |S|^2 \equiv 5 \pmod 7$. We suppose $b_\Gamma(\lambda)$ is of the values stated in the theorem for all $\lambda \leq t$.

Let $t+1 = 6k-s$ with an $s \in \{0, ..., 5\}$ and $k \geq 1$. Regarding the products $b_\Gamma(\mu)b_\Gamma(t-\mu)$, we have by induction hypothesis for the values of $b_\Gamma(\lambda)$ $(\lambda = 1, ..., t)$ iterated sequences of $(5, 4, 6, 2, 3, 1)$ and by symmetry the sequence $(1, 3, 2, 6, 4, 5)$ for $\lambda = t, ..., 1$. The start value $(b_\Gamma(\lambda-1))$ depends on the congruence class of λ modulo 6. Calculating $b_\Gamma(t+1) = (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{t-1} b_\Gamma(\mu)b_\Gamma(t-\mu)$ we get always $k-1$ blocks B_1 of 6 products $b_\Gamma(\mu)b_\Gamma(t-\mu)$ with the same values plus an initial block B_0 with a shifted sequence, which is given by s in the representation of $t+1$. The results of B_0, B_1 can be obtained from the following tables:

B_0							
$(t-1)$...	$b_\Gamma(t-1)$...	$b_\Gamma(\mu),\ \mu=1,...,6-s$					
6k	1	5					
6k-1	3	4	5				
6k-2	2	6	4	5			
6k-3	6	2	6	4	5		
6k-4	4	3	2	6	4	5	
6k-5	5	1	3	2	6	4	5
$\sum b_\Gamma(\mu)b_\Gamma(t-\mu) \pmod 7$		2	5	5	6	5	4

B_1							
$(t-\mu)$	$b_\Gamma(t-\mu)$	$b_\Gamma(\mu),\ \mu=6+1,...,t-1$					
6(k-1)	1	5	1	3	2	6	4
6(k-1)-1	3	4	5	1	3	2	6
6(k-1)-2	2	6	4	5	1	3	2
6(k-1)-3	6	2	6	4	5	1	3
6(k-1)-4	4	3	2	6	4	5	1
6(k-1)-5	5	1	3	2	6	4	5
$\sum b_\Gamma(\mu)b_\Gamma(t-\mu) \pmod 7$		2	6	4	5	1	3

We have to consider the 6 cases $s \in \{0,...,5\},\ k \geq 1$:

Splitting the sum $\sum_{\mu=1}^{t-\mu} b_\Gamma(\mu)b_\Gamma(t-\mu)$ into the initial block B_0 and $(k-1)$ times block B_1 gives:

$s=0, \quad t+1=6k, \quad t=6k-1, \quad t-1=6k-2, \quad b_\Gamma(t)\equiv 3 \pmod 7$

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{4} b_\Gamma(\mu)b_\Gamma(t-\mu) + \sum_{\mu=5}^{t-1} b_\Gamma(\mu)b_\Gamma(t-\mu)\\
&\equiv 6k\cdot 6\cdot 3 + 5 + (k-1)\cdot 4 \pmod 7\\
&\equiv ((k-1)\cdot 6+6)\cdot 6\cdot 3 + (k-1)\cdot 4 + 5 \pmod 7\\
&\equiv (k-1)\cdot(3+4) + 3 + 5 \pmod 7\\
&\equiv 1 \pmod 7
\end{aligned}$$

$s = 1, \quad t+1 = 6k-1, \quad t = 6k-2, \quad t-1 = 6k-3, \quad b_\Gamma(t) \equiv 2 \pmod 7$

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{3} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=4}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv (6k-1)\cdot 6 \cdot 2 + 6 + (k-1)\cdot 5 \pmod 7 \\
&\equiv ((k-1)\cdot 6 + 5)\cdot 5 + (k-1)\cdot 5 + 6 \pmod 7 \\
&\equiv (k-1)\cdot(2+5) + 4 + 6 \pmod 7 \\
&\equiv 3 \pmod 7
\end{aligned}$$

$s = 2, \quad t+1 = 6k-2, \quad t = 6k-3, \quad t-1 = 6k-4, \quad b_\Gamma(t) \equiv 6 \pmod 7$

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{2} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=3}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv (6k-2)\cdot 1 + 5 + (k-1)\cdot 1 \pmod 7 \\
&\equiv ((k-1)\cdot 6 + 4)\cdot 1 + (k-1)\cdot 1 + 5 \pmod 7 \\
&\equiv (k-1)\cdot(6+1) + 4 + 5 \pmod 7 \\
&\equiv 2 \pmod 7
\end{aligned}$$

$s = 3, \quad t+1 = 6k-3, \quad t = 6k-4, \quad t-1 = 6k-5, \quad b_\Gamma(t) \equiv 4 \pmod 7$

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{1} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=2}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv (6k-3)\cdot 3 + 4 + (k-1)\cdot 3 \pmod 7 \\
&\equiv ((k-1)\cdot 6 + 3)\cdot 3 + (k-1)\cdot 3 + 4 \pmod 7 \\
&\equiv (k-1)\cdot(4+3) + 2 + 4 \pmod 7 \\
&\equiv 6 \pmod 7
\end{aligned}$$

$s = 4, \quad t+1 = 6k-4, \quad t = 6k-5, \quad t-1 = 6(k-1), \quad b_\Gamma(t) \equiv 5 \pmod 7$

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv (6k-4)\cdot 2 + (k-1)\cdot 2 \pmod 7 \\
&\equiv ((k-1)\cdot 6 + 2)\cdot 2 + (k-1)\cdot 2 \pmod 7 \\
&\equiv (k-1)\cdot(5+2) + 4 \pmod 7 \\
&\equiv 4 \pmod 7
\end{aligned}$$

$s = 5, \quad t+1 = 6k-5, \quad t = 6(k-1), \quad t-1 = 6(k-1)-1, \quad b_\Gamma(t) \equiv 1 \pmod 7$

$$\begin{aligned} b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{5} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=6}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv (6k-5)\cdot 6 + 5 + (k-2)\cdot 6 \pmod 7 \\ &\equiv ((k-2)\cdot 6 + 7)\cdot 6 + (k-2)\cdot 6 + 5 \pmod 7 \\ &\equiv (k-2)\cdot(1+6) + 5 \pmod 7 \\ &\equiv 5 \pmod 7. \end{aligned}$$

Case $p = 11$
Induction on λ. $b_\Gamma(1) = 5\cdot|S|^2 \equiv 5 \pmod{11}$. We suppose $b_\Gamma(\lambda) \equiv 5 \pmod{11}$ for all $1 \le \lambda \le t$. The recursion formula calculates for $t+1$ to

$$\begin{aligned} b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv (t+1)\cdot 8 + (t-1)\cdot 3 \pmod{11} \\ &\equiv (t-1)\cdot 8 + (t-1)\cdot 3 + 16 \pmod{11} \\ &\equiv (t-1)\cdot(8+3) + 5 \pmod{11} \\ &\equiv 5 \pmod{11} \end{aligned}$$

Case $p = 13$
Induction on λ. $b_\Gamma(1) = 5\cdot|S|^2 \equiv 5 \pmod{13}$. We suppose $b_\Gamma(\lambda)$ is of the values stated in the theorem for all $\lambda \le t$.

Let $t+1 = 12k - s$ with an $s \in \{0, ..., 11\}$ and $k \ge 1$.

Using the same argumentation of the shifted sequences as in the case $p = 7$ we calculate B_0 and B_1 for the different values of s.

B_0													
$(t-1)...$	$b_\Gamma(t-1)...$	$b_\Gamma(\mu),\ \mu=1,...,12-s$											
12k	1	5											
12k-1	6	8	5										
12k-2	11	0	8	5									
12k-3	0	2	0	8	5								
12k-4	5	7	2	0	8	5							
12k-5	8	12	7	2	0	8	5						
12k-6	12	8	12	7	2	0	8	5					
12k-7	7	5	8	12	7	2	0	8	5				
12k-8	2	0	5	8	12	7	2	0	8	5			
12k-9	0	11	0	5	8	12	7	2	0	8	5		
12k-10	8	6	11	0	5	8	12	7	2	0	8	5	
12k-11	5	1	6	11	0	5	8	12	7	2	0	8	5
$\sum b_\Gamma(\mu)b_\Gamma(t-\mu) \pmod{13}$		4	5	11	8	11	3	11	11	7	12	2	12

B_1													
$(t-\mu)$	$b_\Gamma(t-\mu)$	$b_\Gamma(\mu),\ \mu=12-s+1,...,t-1$											
12(k-1)	1	5	1	6	11	0	5	8	12	7	2	0	8
12(k-1)-1	6	8	5	1	6	11	0	5	8	12	7	2	0
12(k-1)-2	11	0	8	5	1	6	11	0	5	8	12	7	2
12(k-1)-3	0	2	0	8	5	1	6	11	0	5	8	12	7
12(k-1)-4	5	7	2	0	8	5	1	6	11	0	5	8	12
12(k-1)-5	8	12	7	2	0	8	5	1	6	11	0	5	8
12(k-1)-6	12	8	12	7	2	0	8	5	1	6	11	0	5
12(k-1)-7	7	5	8	12	7	2	0	8	5	1	6	11	0
12(k-1)-8	2	0	5	8	12	7	2	0	8	5	1	6	11
12(k-1)-9	0	11	0	5	8	12	7	2	0	8	5	1	6
12(k-1)-10	8	6	11	0	5	8	12	7	2	0	8	5	1
12(k-1)-11	5	1	6	11	0	5	8	12	7	2	0	8	5
$\sum b_\Gamma(\mu)b_\Gamma(t-\mu) \pmod{13}$		4	6	10	1	0	4	9	7	3	12	0	9

We have to consider the 12 cases $s \in \{0,...,11\}$ and calculate $b_\Gamma(\lambda+1)$ using the partial sums B_0 and $(k-1)$ times B_1:

$s = 0, \quad t+1 = 12k, \quad t = 12k-1, \quad t-1 = 12k-2, \quad b_\Gamma(t) \equiv 6 \pmod{13}$

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{10} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=11}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv 12k \cdot 6 \cdot 6 + 11 + (k-1)\cdot 10 \pmod{13} \\
&\equiv (k-1)\cdot(3+10) + 3 + 11 \pmod{13} \\
&\equiv 1 \pmod{13}
\end{aligned}$$

$s = 1, \quad t+1 = 12k-1, \quad t = 12k-2, \quad t-1 = 12k-3, \quad b_\Gamma(t) \equiv 11 \pmod{13}$

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{9} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=10}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv (12k-1) \cdot 6 \cdot 11 + 8 + (k-1)\cdot 1 \pmod{13} \\
&\equiv (k-1)\cdot(12+1) + 11 + 8 \pmod{13} \\
&\equiv 6 \pmod{13}
\end{aligned}$$

$s = 2, \quad t+1 = 12k-2, \quad t = 12k-3, \quad t-1 = 12k-4, \quad b_\Gamma(t) \equiv 0 \pmod{13}$

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{8} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=9}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv 11 + (k-1)\cdot 0 \pmod{13} \\
&\equiv 11 \pmod{13}
\end{aligned}$$

$s = 3, \quad t+1 = 12k-3, \quad t = 12k-4, \quad t-1 = 12k-5, \quad b_\Gamma(t) \equiv 5 \pmod{13}$

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{7} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=8}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv (12k-3) \cdot 4 + 3 + (k-1)\cdot 4 \pmod{13} \\
&\equiv (k-1)\cdot(9+4) + 13 \pmod{13} \\
&\equiv 0 \pmod{13}
\end{aligned}$$

$s = 4, \quad t+1 = 12k-4, \quad t = 12k-5, \quad t-1 = 12k-6, \quad b_\Gamma(t) \equiv 8 \pmod{13}$

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{6} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=7}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv (12k-4) \cdot 9 + 11 + (k-1)\cdot 9 \pmod{13} \\
&\equiv (k-1)\cdot(4+9) + 18 \pmod{13} \\
&\equiv 5 \pmod{13}
\end{aligned}$$

$s = 5, \quad t+1 = 12k-5, \quad t = 12k-6, \quad t-1 = 12k-7, \quad b_\Gamma(t) \equiv 12 \pmod{13}$

$$\begin{aligned} b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{5} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=6}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv (12k-5)\cdot 7 + 11 + (k-1)\cdot 7 \pmod{13} \\ &\equiv (k-1)\cdot(6+7) + 21 \pmod{13} \\ &\equiv 8 \pmod{13} \end{aligned}$$

$s = 6, \quad t+1 = 12k-6, \quad t = 12k-7, \quad t-1 = 12k-8, \quad b_\Gamma(t) \equiv 7 \pmod{13}$

$$\begin{aligned} b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{4} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=5}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv (12k-6)\cdot 3 + 7 + (k-1)\cdot 3 \pmod{13} \\ &\equiv (k-1)\cdot(10+3) + 12 \pmod{13} \\ &\equiv 12 \pmod{13} \end{aligned}$$

$s = 7, \quad t+1 = 12k-7, \quad t = 12k-8, \quad t-1 = 12k-9, \quad b_\Gamma(t) \equiv 2 \pmod{13}$

$$\begin{aligned} b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{3} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=4}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv (12k-7)\cdot 12 + 12 + (k-1)\cdot 12 \pmod{13} \\ &\equiv (k-1)\cdot(1+12) + 20 \pmod{13} \\ &\equiv 7 \pmod{13} \end{aligned}$$

$s = 8, \quad t+1 = 12k-8, \quad t = 12k-9, \quad t-1 = 12k-10, \quad b_\Gamma(t) \equiv 0 \pmod{13}$

$$\begin{aligned} b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{2} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=3}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv 0 + 2 + (k-1)\cdot 0 \pmod{13} \\ &\equiv 2 \pmod{13} \end{aligned}$$

$s = 9, \quad t+1 = 12k-9, \quad t = 12k-10, \quad t-1 = 12k-11, \quad b_\Gamma(t) \equiv 8 \pmod{13}$

$$\begin{aligned} b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{1} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=2}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv (12k-9)\cdot 9 + 12 + (k-1)\cdot 9 \pmod{13} \\ &\equiv (k-1)\cdot(4+9) + 13 \pmod{13} \\ &\equiv 0 \pmod{13} \end{aligned}$$

$s = 10, \quad t+1 = 12k-10, \quad t = 12k-11, \quad t-1 = 12(k-1), \quad b_\Gamma(t) \equiv 5 \pmod{13}$

$$\begin{aligned} b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv (12k-10)\cdot 4 + (k-1)\cdot 4 \pmod{13} \\ &\equiv (k-1)\cdot(9+4) + 8 \pmod{13} \\ &\equiv 8 \pmod{13} \end{aligned}$$

$s = 11, \quad t+1 = 12k-11, \quad t = 12(k-1), \quad t-1 = 12(k-1)-1, \quad b_\Gamma(t) \equiv 1 \pmod{13}$

$$\begin{aligned} b_\Gamma(t+1) &= (t+1)\cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{11} b_\Gamma(\mu) b_\Gamma(t-\mu) + \sum_{\mu=12}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\ &\equiv (12k-11)\cdot 6 + 5 + (k-2)\cdot 6 \pmod{13} \\ &\equiv (k-2)\cdot(7+6) + 5 \pmod{13} \\ &\equiv 5 \pmod{13}. \end{aligned}$$

$\square$

Theorem 2.9 *Let Γ be a free product $\Gamma = G_1 \underset{S}{*} G_2$ of finite groups G_i with amalgamated subgroup S and indices $[G_1 : S] = [G_2 : S] = 3$. We get for primes $p \leq 11$ and $|S| \equiv 1 \pmod p$ the following divisibility properties for $b_\Gamma(\lambda)$:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 2 & \text{if } \lambda = 2^n - 1 \\ 0 \pmod 2 & \text{else} \end{cases}$
3	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 3 & \text{if} \quad \lambda \text{ is of form } (*) \text{ with } l \in L_1, \\ 2 \pmod 3 & \text{if} \quad \lambda \text{ is of form } (*) \text{ with } l \in L_2, \\ 0 \pmod 3 & \text{else} \end{cases}$
5	$b_\Gamma(\lambda) \equiv 2 \pmod 5 \quad \forall \lambda \geq 1$
7	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod 7 & \text{if } \lambda \equiv 3 \pmod 6 \\ 1 \pmod 7 & \text{if } \lambda \equiv 0 \pmod 6 \\ 2 \pmod 7 & \text{if } \lambda \equiv 1 \pmod 6 \\ 3 \pmod 7 & \text{if } \lambda \equiv 5 \pmod 6 \\ 5 \pmod 7 & \text{if } \lambda \equiv 2 \pmod 6 \\ 6 \pmod 7 & \text{if } \lambda \equiv 4 \pmod 6 \end{cases}$
11	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod{11} & \text{if } \lambda \equiv 5, 10, 12, 20, 25, 27 \pmod{30} \\ 1 \pmod{11} & \text{if } \lambda \equiv 2, 22, 24 \pmod{30} \\ 2 \pmod{11} & \text{if } \lambda \equiv 19, 21, 29 \pmod{30} \\ 5 \pmod{11} & \text{if } \lambda \equiv 0, 1, 3, 8, 11, 28 \pmod{30} \\ 6 \pmod{11} & \text{if } \lambda \equiv 4, 6, 14 \pmod{30} \\ 9 \pmod{11} & \text{if } \lambda \equiv 13, 15, 16, 18, 23, 26 \pmod{30} \\ 10 \pmod{11} & \text{if } \lambda \equiv 7, 9, 17 \pmod{30} \end{cases}$

Proof:
The recursion formula for $b_\Gamma(\lambda)$ can be obtained from formula 1.16 and Proposition 1.30:

$$b_\Gamma(1) = (m_\Gamma - |S|)|S|$$
$$b_\Gamma(\lambda + 1) = (\lambda + 1) \cdot m_\Gamma \cdot b_\Gamma(\lambda) + \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) b_\Gamma(\lambda - \mu), \qquad \lambda \geq 1.$$

The cases $p = 2$ and $p = 3$ are stated in Theorem 2.6 and Theorem 2.7. The proofs of the remaining cases follow the same arguments used in Theorem 2.8, first determining the appropriate blocks B_0 and B_1 and calculation of $b_\Gamma(t+1)$ with distinguishing of the different values for s. $\square$

Theorem 2.10 *Let Γ be a free product $\Gamma = G_1 \underset{S}{*} G_2$ of finite groups G_i with amalgamated subgroup S and indices $\{[G_1 : S], [G_2 : S]\} = \{2, 4\}$. For primes $p \leq 11$ and $|S| \equiv 1 \pmod p$ we get the following divisibility properties for $b_\Gamma(\lambda)$:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 2 & \text{if } \lambda = 2^n - 1 \\ 0 \pmod 2 & \text{else} \end{cases}$
3	$b_\Gamma(\lambda) \equiv 0 \pmod 3 \quad \forall \lambda \geq 1$
5	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 5 & \text{if } \lambda \equiv 0 \pmod 4 \\ 2 \pmod 5 & \text{if } \lambda \equiv 3 \pmod 4 \\ 3 \pmod 5 & \text{if } \lambda \equiv 1 \pmod 4 \\ 4 \pmod 5 & \text{if } \lambda \equiv 2 \pmod 4 \end{cases}$
7	$b_\Gamma(\lambda) \equiv 3 \pmod 7 \quad \forall \lambda \geq 1$
11	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod{11} & \text{if } \lambda \equiv 3, 9, 15, 21, 27, 33, 39, 45, 51, 57 \pmod{60} \\ 1 \pmod{11} & \text{if } \lambda \equiv 4, 36, 37, 44, 47 \pmod{60} \\ 2 \pmod{11} & \text{if } \lambda \equiv 2, 5, 12, 54, 55 \pmod{60} \\ 3 \pmod{11} & \text{if } \lambda \equiv 0, 1, 8, 11, 28 \pmod{60} \\ 4 \pmod{11} & \text{if } \lambda \equiv 12, 13, 20, 23, 40 \pmod{60} \\ 5 \pmod{11} & \text{if } \lambda \equiv 16, 48, 49, 56, 59 \pmod{60} \\ 6 \pmod{11} & \text{if } \lambda \equiv 18, 19, 26, 29, 46 \pmod{60} \\ 7 \pmod{11} & \text{if } \lambda \equiv 10, 42, 43, 50, 53 \pmod{60} \\ 8 \pmod{11} & \text{if } \lambda \equiv 30, 31, 38, 41, 58 \pmod{60} \\ 9 \pmod{11} & \text{if } \lambda \equiv 24, 25, 32, 35, 52 \pmod{60} \\ 10 \pmod{11} & \text{if } \lambda \equiv 6, 7, 14, 17, 34 \pmod{60} \end{cases}$

Proof:
Also for this class of groups we derive from formula 1.16 and Proposition 1.30

$$\begin{aligned} b_\Gamma(1) &= (m_\Gamma - |S|)|S| \\ b_\Gamma(\lambda+1) &= (\lambda+1)\cdot m_\Gamma \cdot b_\Gamma(\lambda) + \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) b_\Gamma(\lambda-\mu), \qquad \lambda \geq 1. \end{aligned}$$

In the case $p = 2$ applies Theorem 2.6 and all other cases are proved by induction using the same ideas as in the previous two theorems. $\square$

Theorem 2.11 *Let Γ be of the form $\Gamma = (G_1 \underset{G_{12}}{*} G_2) \underset{G_{23}}{*} G_3$ with finite groups G_i and indices $[G_1 : G_{12}] = 2 = [G_2 : G_{12}]$ and $[G_2 : G_{23}] = 2 = [G_3 : G_{23}]$. We get the following divisibility conditions for $b_\Gamma(\lambda)$ with primes $p \leq 7$ and $|G_{12}| = |G_{23}| \equiv 1 \pmod p$:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 2 & \text{if } \lambda = 2^n - 1 \\ 0 \pmod 2 & \text{else} \end{cases}$
3	$b_\Gamma(\lambda) \equiv 1 \pmod 3 \quad \forall \lambda \geq 1$
5	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod 5 & \text{if } \lambda \equiv 3, 9, 15, 21 \pmod{24} \\ 1 \pmod 5 & \text{if } \lambda \equiv 0, 1, 5, 14, 22 \pmod{24} \\ 2 \pmod 5 & \text{if } \lambda \equiv 8, 16, 18, 19, 23 \pmod{24} \\ 3 \pmod 5 & \text{if } \lambda \equiv 10, 6, 7, 11, 20 \pmod{24} \\ 4 \pmod 5 & \text{if } \lambda \equiv 2, 10, 12, 13, 17 \pmod{24} \end{cases}$
7	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod 7 & \text{if } \lambda \equiv 11, 13 \pmod{16} \\ 1 \pmod 7 & \text{if } \lambda \equiv 0, 7 \pmod{16} \\ 2 \pmod 7 & \text{if } \lambda \equiv 8 \pmod{16} \\ 3 \pmod 7 & \text{if } \lambda \equiv 9, 15 \pmod{16} \\ 4 \pmod 7 & \text{if } \lambda \equiv 2, 3, 5, 14 \pmod{16} \\ 5 \pmod 7 & \text{if } \lambda \equiv 4, 12 \pmod{16} \\ 6 \pmod 7 & \text{if } \lambda \equiv 6, 10 \pmod{16} \end{cases}$

Proof:
Applying formula 1.16 and Proposition 1.30 we get for $b_\Gamma(\lambda)$ the recurrence:

$$b_\Gamma(1) = \left(\frac{m_\Gamma}{2}\right)^2$$
$$b_\Gamma(\lambda+1) = (\lambda+1)\cdot m_\Gamma \cdot b_\Gamma(\lambda) + \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) b_\Gamma(\lambda-\mu), \qquad \lambda \geq 1.$$

The case $p = 2$ is the statement of Theorem 2.6 and the other cases can be proved inductively using the methods of Theorem 2.8. $\square$

Theorem 2.12 *Let Γ belong to the following two classes:*

a) Γ is an HNN-extension $\Gamma = G_1 \underset{S,\sigma}{*}$ with finite base group G_1 and index $[G_1 : S] = 2$.

b) Γ is a free product $\Gamma = G_1 \underset{S}{*} G_2$ where G_1 contains a finite normal subset $H_1 \supset S$ with infinite-cyclic factor group, and G_2 is finite with $[H_1 : S] = 2 = [G_2 : S]$.

For primes $p \leq 7$ and $|S| \equiv 1 \pmod p$ we get the following divisibility conditions for $b_\Gamma(\lambda)$:

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv 0 \pmod 2 \quad \forall \lambda \geq 1$
3	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 3 & \text{if } 2 \mid \lambda \\ 2 \pmod 3 & \text{if } 2 \nmid \lambda \end{cases}$
5	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod 5 & \text{if } \lambda \equiv 2 \pmod 4 \\ 1 \pmod 5 & \text{if } \lambda \equiv 0 \pmod 4 \\ 2 \pmod 5 & \text{if } \lambda \equiv 1 \pmod 4 \\ 4 \pmod 5 & \text{if } \lambda \equiv 3 \pmod 4 \end{cases}$
7	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod 7 & \text{if } \lambda \equiv 15, 16, 17, 14, 35, 36 \pmod{38} \\ 1 \pmod 7 & \text{if } \lambda \equiv 0, 8, 9, 10, 12, 18, 23, 25, 32, 34 \pmod{38} \\ 2 \pmod 7 & \text{if } \lambda \equiv 1, 7, 11 \pmod{38} \\ 3 \pmod 7 & \text{if } \lambda \equiv 2, 14, 22 \pmod{38} \\ 4 \pmod 7 & \text{if } \lambda \equiv 3, 21, 33 \pmod{38} \\ 5 \pmod 7 & \text{if } \lambda \equiv 20, 26, 30 \pmod{38} \\ 6 \pmod 7 & \text{if } \lambda \equiv 4, 6, 13, 15, 19, 27, 28, 29, 31, 37 \pmod{38} \end{cases}$

Proof:
For both classes a) and b) we calculate by formula 1.16 and Proposition 1.30 the same recursive formula even with the same initial value $b_\Gamma(1)$.

$$b_\Gamma(1) = \frac{m_\Gamma^2}{2}$$

$$b_\Gamma(\lambda + 1) = \frac{2\lambda + 3}{2} \cdot m_\Gamma \cdot b_\Gamma(\lambda) + \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) b_\Gamma(\lambda - \mu), \qquad \lambda \geq 1,$$

where $b_\Gamma(\lambda+1)$ is integral because of $m_\Gamma = 2|S|$. We obtain the results by applying the same arguments as used in Theorem 2.8 but avoid here the calculations. □

Theorem 2.13 *Let Γ contain a finite normal subset H with factor group $\Gamma/H \cong F_2$, free of rank 2. Then we have for primes $p \leq 5$ and $|S| \equiv 1 \pmod p$:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv 1 \pmod 2 \quad \forall \lambda \geq 1$
3	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod 3 & \text{if } \lambda \equiv 2,6 \pmod 8 \\ 1 \pmod 3 & \text{if } \lambda \equiv 0,1,3 \pmod 8 \\ 2 \pmod 3 & \text{if } \lambda \equiv 4,5,7 \pmod 8 \end{cases}$
5	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod 5 & \text{if } \lambda \equiv 9,12,19,24,33,39,41,42,45,47,49,52,58,59,60 \\ & \quad \pmod{62} \\ 1 \pmod 5 & \text{if } \lambda \equiv 0,1,4,5,17,20,23,25,34,38,44,46,53 \pmod{62} \\ 2 \pmod 5 & \text{if } \lambda \equiv 6,11,21,26,27,29,30,43,55 \pmod{62} \\ 3 \pmod 5 & \text{if } \lambda \equiv 2,3,7,8,10,13,14,15,16,18,28,31,35,40,50,51 \\ & \quad \pmod{62} \\ 4 \pmod 7 & \text{if } \lambda \equiv 22,32,36,37,48,54,56,57,61 \pmod{62} \end{cases}$

Proof:

Again applying formula 1.16 and Proposition 1.30 we calculate

$$b_\Gamma(1) = m_\Gamma^2$$
$$b_\Gamma(\lambda+1) = (\lambda+2)\cdot m_\Gamma \cdot b_\Gamma(\lambda) + \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) b_\Gamma(\lambda-\mu) \qquad \lambda \geq 1$$

and use the same arguments for the inductions like in the previous theorems.

□

2.3 The case $\mu(\Gamma) = 3$

In order to determine the possible types $\tau(\Gamma)$ for finitely generated virtually free groups Γ of rank $\mu(\Gamma) = 3$ we use the relation (1.4) between the Euler characteristic and the free rank and the fact that the free rank can be expressed in terms of the type $\tau(\Gamma)$ via formula (1.14). First, we need the following properties of Euler's function $\phi(n)$.

Lemma 2.14 *For Euler's function $\phi(n)$ we have:*

i) $\phi(n) = 1$ *if* $n \in \{1, 2\}$

ii) $\phi(n) = 2$ *if* $n \in \{3, 4, 6\}$

iii) $\phi(n) = 4$ *if* $n \in \{5, 8, 10, 12\}$

iv) $2 \mid \phi(n)$ *for all* $n \geq 2$.

Proof:
With $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_m^{\alpha_m}$ and primes p_i Euler's function $\phi(n)$ is given by

$$\phi(n) = p_1^{\alpha_1 - 1} \cdot (p_1 - 1) \cdot ... \cdot p_m^{\alpha_m - 1} \cdot (p_m - 1), \tag{2.14}$$

and we get as the only possible combinations of p_i, α_i for the different values of $\phi(n)$:

i) $\phi(n) = 1$ for n = 1 by definition, and for $p_1 = 2,\ \alpha_1 = 1$.

ii) $\phi(n) = 2$, i.e. $p_i \leq 3$ for

a) $p_1 = 2,\ \alpha_1 = 2$
b) $p_1 = 3,\ \alpha_1 = 1$
c) $p_1 = 2,\ \alpha_1 = 1,\ p_2 = 3,\ \alpha_2 = 1$

iii) $\phi(n) = 4$, i.e. $p_i \leq 5$ for

a) $p_1 = 2,\ \alpha_1 = 3$
b) $p_1 = 5,\ \alpha_1 = 1$
c) $p_1 = 2,\ \alpha_1 = 2,\ p_2 = 3,\ \alpha_2 = 1$
d) $p_1 = 2,\ \alpha_1 = 1,\ p_2 = 5,\ \alpha_2 = 1$

iv) $2 \mid \phi(n)$ for all $n \geq 2$
We have

$$\phi(n) = n \cdot \prod_{p \mid n} (1 - \frac{1}{p}). \tag{2.15}$$

Case a) $2 \mid n$ then by 2.15 $2 \mid \phi(n)$

Case b) $2 \nmid n$

Then $2 \nmid p$ for all p. Fix one p and the factor in 2.15 rewrites to $(1 - \frac{1}{p}) = \frac{p-1}{p}$ and $2 \mid p-1$. Since $p \mid n$ also $2 \mid \frac{n \cdot (p-1)}{p}$, hence $2 \mid \phi(n)$.

□

Knowing now all values for $\phi(n)$ which can occur in (1.14), we are able to exhibit the possible types $\tau(\Gamma)$ for finitely generated virtually free groups of rank $\mu(\Gamma) = 3$.

Lemma 2.15 *Let Γ be a finitely generated virtually free group of rank $\mu(\Gamma) = 3$. The type $\tau(\Gamma)$ belongs to one of the following ten classes:*

i) $\tau(\Gamma) = (n; 0, ..., 0, 2_n)$

ii) $\tau(\Gamma) = (2k; 0, ..., 0, 2_k, 0_{2k})$

iii) $\tau(\Gamma) = (2k; 0, ..., 0, 3_k, -1_{2k})$

iv) $\tau(\Gamma) = (2k; 0, ..., 0, 1_k, 1_{2k})$

v) $\tau(\Gamma) = (3k; 0, ..., 0, 1_k, 0_{3k})$

vi) $\tau(\Gamma) = (4k; 0, ..., 0, 1_k, 0, ..., 0_{4k})$

vii) $\tau(\Gamma) = (6k; 0, ..., 0, 1_k, 0, ..., 0_{6k})$

viii) $\tau(\Gamma) = (4k; 0, ..., 0, 1_k, 1_{2k}, -1_{4k})$

ix) $\tau(\Gamma) = (6k; 0, ..., 0, 1_k, 0, ..., 0, 1_{3k}, -1_{6k})$

x) $\tau(\Gamma) = (6k; 0, ..., 0, 1_{2k}, 1_{3k}, -1_{6k})$

Proof:
Formula 1.14 states for finitely generated virtually free groups of rank $\mu(\Gamma) = 3$

$$\sum_{k \mid m_\Gamma} \phi(\frac{m_\Gamma}{k}) \cdot \zeta_k = 2. \tag{2.16}$$

With Proposition 1.25 and $-1 \leq \zeta_{m_\Gamma} \leq 2$, $\phi(\frac{m_\Gamma}{m_\Gamma}) = 1$ we get

$$\sum_{\substack{k \mid m_\Gamma \\ k \neq m_\Gamma}} \phi(\frac{m_\Gamma}{k}) \cdot \zeta_k \leq 3 \qquad (0 \leq \zeta_k \leq 3).$$

Hence, for all divisors k, $\phi(\frac{m_\Gamma}{k}) \leq 3$ and by Lemma 2.3.iv) $\phi(\frac{m_\Gamma}{k}) \leq 2$. So we determine all integers n, such that $\phi(n) \leq 2$ and get then m_Γ by k ($k \geq 1$) multiples of n. Lemma 2.3.i),ii) gives the only possible values of n as $n \in \{1, 2, 3, 4, 6\}$, hence
$m_\Gamma \in \{n, 2k, 3k, 4k, 6k\},\ k \geq 1$.

It is left to calculate the combinations of the ζ_k for different m_Γ in order to fulfill 2.16. We fix $\zeta_{m_\Gamma} \in \{-1, 0, 1, 2\}$ and calculate the $\zeta_k' s$.

Case $m_\Gamma = n$:
With $\phi(\frac{n}{n}) = 1$ we get $\zeta_{m_\Gamma} = 2$ and
$\tau(\Gamma) = (n; 0, ..., 0, 2_n)$ (i).

Case $m_\Gamma = 2k$:
For the divisors d of m_Γ we have $d \in \{k, 2k\}$ and $\phi(\frac{2k}{k}) = 1 = \phi(\frac{2k}{2k})$.

$\zeta_{2k} = -1$: Then $\zeta_k = 3,\ \zeta_{2k} = -1,\ \tau(\Gamma) = (2k; 0, ..., 0, 3_k, -1_{2k})$ (iii)

$\zeta_{2k} = 0$: Then $\zeta_k = 2,\ \tau(\Gamma) = (2k; 0, ..., 0, 2_k, 0_{2k})$ (ii)

$\zeta_{2k} = 1$: Then $\zeta_k = 1,\ \tau(\Gamma) = (2k; 0, ..., 0, 1_k, 1_{2k})$ (iv)

$\zeta_{2k} = 2$: Then $\zeta_k = 0$, equivalent to case $m_\Gamma = n$ (i)

Case $m_\Gamma = 3k$:
For the divisors d of m_Γ we have $d \in \{k, 3k\}$ and $\phi(\frac{3k}{k}) = 2,\ \phi(\frac{3k}{3k}) = 1$.

$\zeta_{3k} = -1$: Since $\phi(\frac{3k}{k}) = 2$ not possible.

$\zeta_{3k} = 0$: Then $\zeta_k = 1,\ \tau(\Gamma) = (3k; 0, ..., 0, 1_k, 0_{3k})$ (v)

$\zeta_{3k} = 1$: Since $\phi(\frac{3k}{k}) = 2$ not possible.

$\zeta_{3k} = 2$: Then $\zeta_k = 0$, equivalent to case $m_\Gamma = n$ (i)

Case $m_\Gamma = 4k$:
For the divisors d of m_Γ we have $d \in \{k, 2k, 4k\}$ and $\phi(\frac{4k}{k}) = 2,\ \phi(\frac{4k}{2k}) = 1 = \phi(\frac{4k}{4k})$.

$\zeta_{4k} = -1$: Then

$\zeta_k = 1,\ \zeta_{2k} = 1,\ \tau(\Gamma) = (4k; 0, ..., 0, 1_k, 1_{2k}, -1_{4k})$ (viii)

$\zeta_k = 0,\ \zeta_{2k} = 3$, equivalent to case iii) in Lemma 2.15.

$\zeta_{4k} = 0$: Then

$\zeta_k = 1,\ \zeta_{2k} = 0,\ \tau(\Gamma) = (4k; 0, ..., 0, 1_k, 0, ..., 0_{4k})$ (vi)

$\zeta_k = 0,\ \zeta_{2k} = 2$, equivalent to case ii) in Lemma 2.15.

$\zeta_{4k} = 1$: Then $\zeta_k = 0,\ \zeta_{2k} = 1$, equivalent to case iv) in Lemma 2.15.

$\zeta_{4k} = 2$: Then $\zeta_k = 0, \zeta_{2k} = 0$, equivalent to case $m_\Gamma = n$ (i)

Case $m_\Gamma = 6k$:
For the divisors d of m_Γ we have $d \in \{k, 2k, 3k, 6k\}$ and $\phi(\frac{6k}{k}) = 2 = \phi(\frac{6k}{2k}),\ \phi(\frac{6k}{3k}) = 1 = \phi(\frac{6k}{6k})$.

$\zeta_{6k} = -1$: Then

$\zeta_k = 1,\ \zeta_{2k} = 0,\ \zeta_{3k} = 1,\ \tau(\Gamma) = (6k; 0, ..., 0, 1_k, 0, ..., 0, 1_{3k}, -1_{6k})$ (ix)

$\zeta_k = 0,\ \zeta_{2k} = 1,\ \zeta_{3k} = 1,\ \tau(\Gamma) = (6k; 0, ..., 0, 1_{2k}, 1_{3k}, -1_{6k})$ (x)

$\zeta_k = 0,\ \zeta_{2k} = 0,\ \zeta_{3k} = 3$, equivalent to case iii) in Lemma 2.15.

$\zeta_{6k} = 0$: Then

$\zeta_k = 1,\ \zeta_{2k} = 0,\ \zeta_{3k} = 0,\ \tau(\Gamma) = (6k; 0, ..., 0, 1_k, 0, ..., 0_{6k})$ (vii)

$\zeta_k = 0,\ \zeta_{2k} = 1,\ \zeta_{3k} = 0$, equivalent to case v) in Lemma 2.15.

$\zeta_k = 0,\ \zeta_{2k} = 0,\ \zeta_{3k} = 2$, equivalent to case ii) in Lemma 2.15.

$\zeta_{6k} = 1$: Then $\zeta_k = 0,\ \zeta_{2k} = 0,\ \zeta_{3k} = 1$, equivalent to case iv) in Lemma 2.15.

$\zeta_{6k} = 2$: Then $\zeta_k = 0, \zeta_{2k} = 0,\ \zeta_{3k} = 0$, equivalent to case $m_\Gamma = n$ (i)

We detected 10 possibilities for $\tau(\Gamma)$ as stated in the Lemma. With regard to their realization we give the following examples:

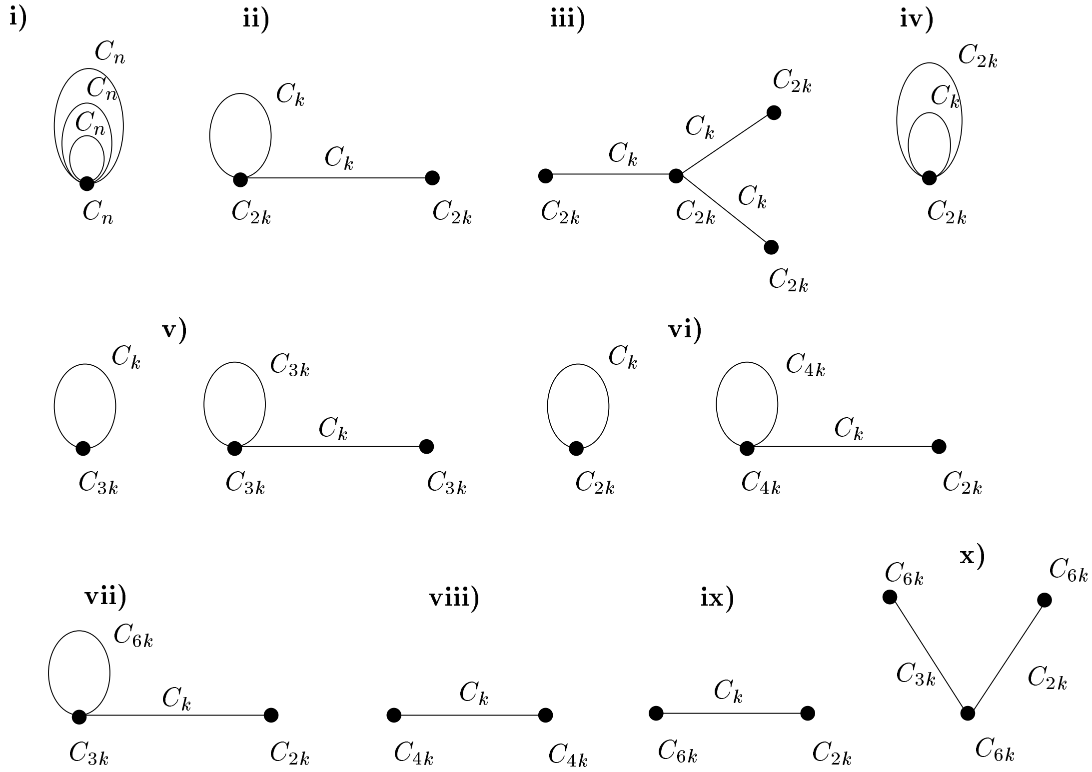


Figure 3: Graphs of groups realizing the types (i) – (x) in Lemma 2.15

$\square$

For the investigation of the divisibility conditions for $b_\Gamma(\lambda)$ we need a recursive description of $b_\Gamma(\lambda)$ for every type $\tau(\Gamma)$.

Proposition 2.16 *Let Γ be a finitely generated virtually free group of rank $\mu(\Gamma) = 3$. Then we have the following recursions for $b_\Gamma(\lambda)$:*

i) $\quad \tau(\Gamma) = (n; 0, ..., 0, 2_n)$ *then*
$m_\Gamma = n,\ b_\Gamma(1) = n^3$

$$\begin{aligned} b_\Gamma(\lambda+1) = n^2 \cdot ((\lambda+1)(\lambda+2)+1) \cdot b_\Gamma(\lambda) \\ + 3n \cdot (\frac{\lambda+2}{2}) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ + \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3) \end{aligned} \tag{2.17}$$

ii) $\quad \tau(\Gamma) = (2k; 0, ..., 0, 2_k, 0_{2k})$ *then*
$m_\Gamma = 2k,\ b_\Gamma(1) = m_\Gamma k^2$

$$\begin{aligned} b_\Gamma(\lambda+1) &= \frac{1}{4} m_\Gamma^2 \cdot (\lambda(4\lambda+8)+5) \cdot b_\Gamma(\lambda) \\ &+ \frac{1}{2} m_\Gamma \cdot (3\lambda+4) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ &+ \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3) \end{aligned} \tag{2.18}$$

iii) $\quad \tau(\Gamma) = (2k; 0, ..., 0, 3_k, -1_{2k})$ *then*
$m_\Gamma = 2k,\ b_\Gamma(1) = \left(\frac{m_\Gamma}{2}\right)^3$

$$\begin{aligned} b_\Gamma(\lambda+1) &= \frac{1}{4} m_\Gamma^2 \cdot (\lambda(4\lambda+6)+3) \cdot b_\Gamma(\lambda) \\ &+ \frac{3}{2} m_\Gamma \cdot (\lambda+1) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ &+ \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3) \end{aligned} \tag{2.19}$$

iv) $\quad \tau(\Gamma) = (2k; 0, ..., 0, 1_k, 1_{2k})$ *then*
$m_\Gamma = 2k,\ b_\Gamma(1) = \frac{m_\Gamma^3}{2}$

$$\begin{aligned} b_\Gamma(\lambda+1) &= \frac{1}{2} m_\Gamma^2 \cdot (\lambda(2\lambda+5)+4) \cdot b_\Gamma(\lambda) \\ &+ \frac{1}{2} m_\Gamma \cdot (3\lambda+5) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ &+ \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3) \end{aligned} \tag{2.20}$$

v) $\quad \tau(\Gamma) = (3k; 0, ..., 0, 1_k, 0_{3k})$ *then*
$m_\Gamma = 3k,\ b_\Gamma(1) = (m_\Gamma - k) \cdot 3k^2$

$$\begin{aligned} b_\Gamma(\lambda+1) &= \frac{1}{9} m_\Gamma^2 \cdot (9\lambda(\lambda+2)+11) \cdot b_\Gamma(\lambda) \\ &+ \frac{1}{2} m_\Gamma \cdot (9\lambda+4) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ &+ \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3) \end{aligned} \tag{2.21}$$

vi) $\tau(\Gamma) = (4k; 0, ..., 0, 1_k, 0, ..., 0_{4k})$ *then*
$m_\Gamma = 4k,\ b_\Gamma(1) = 12 \cdot k^3$

$$\begin{aligned} b_\Gamma(\lambda+1) = &\frac{1}{16} m_\Gamma^2 \cdot (16\lambda(\lambda+2)+19) \cdot b_\Gamma(\lambda) \\ &+ \frac{1}{2} m_\Gamma \cdot (3\lambda+4) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ &+ \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3) \end{aligned} \tag{2.22}$$

vii) $\tau(\Gamma) = (6k; 0, ..., 0, 1_k, 0, ..., 0_{6k})$ *then*
$m_\Gamma = 6k,\ b_\Gamma(1) = m_\Gamma \cdot 5k^2$

$$\begin{aligned} b_\Gamma(\lambda+1) = &\frac{1}{36} m_\Gamma^2 \cdot (36\lambda(\lambda+2)+41) \cdot b_\Gamma(\lambda) \\ &+ \frac{m_\Gamma}{2} \cdot (3\lambda+4) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ &+ \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3) \end{aligned} \tag{2.23}$$

viii) $\tau(\Gamma) = (4k; 0, ..., 0, 1_k, 1_{2k}, -1_{4k})$ *then*
$m_\Gamma = 4k,\ b_\Gamma(1) = (m_\Gamma - k) \cdot 2k^2$

$$\begin{aligned} b_\Gamma(\lambda+1) = &\frac{1}{16} m_\Gamma^2 \cdot (8\lambda(2\lambda+3)+11) \cdot b_\Gamma(\lambda) \\ &+ \frac{3}{2} m_\Gamma \cdot (\lambda+1) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ &+ \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3) \end{aligned} \tag{2.24}$$

ix) $\tau(\Gamma) = (6k; 0, ..., 0, 1_k, 0, ..., 0, 1_{3k}, -1_{6k})$ *then*
$m_\Gamma = 6k,\ b_\Gamma(1) = (2m_\Gamma + 3k) \cdot k^2$

$$\begin{aligned} b_\Gamma(\lambda+1) = &\frac{1}{36} m_\Gamma^2 \cdot (9\lambda(4\lambda+6)+23) \cdot b_\Gamma(\lambda) \\ &+ \frac{3}{2} m_\Gamma \cdot (\lambda+1) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ &+ \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3) \end{aligned} \tag{2.25}$$

x) $\quad \tau(\Gamma) = (6k; 0, ..., 0, 1_{2k}, 1_{3k}, -1_{6k})$ *then*
$m_\Gamma = 6k,\ b_\Gamma(1) = 4m_\Gamma \cdot k^2$

$$\begin{aligned} b_\Gamma(\lambda+1) = & \frac{1}{18} m_\Gamma^2 \cdot (9\lambda(2\lambda+3)+13) \cdot b_\Gamma(\lambda) \\ & + \frac{3}{2} m_\Gamma \cdot (\lambda+1) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ & + \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3) \end{aligned} \tag{2.26}$$

Proof:
Apply Lemma 2.15 and calculate the recursion formulas for the different types $\tau(\Gamma)$ using formula formula 1.16 together with Proposition 1.30. □

Investigating the divisibility conditions for $b_\Gamma(\lambda)$ we detect, like in the case of virtually free groups of rank 2, a class which shows a more interesting divisibility condition. The proof again uses an additive structure in the representation of λ.

Theorem 2.17 *Let Γ be of the type $\tau(\Gamma) = (2k; 0, ..., 0, 3_k, -1_{2k})$, where $k \equiv 1 \pmod 2$. Then $b_\Gamma(\lambda) \equiv 1 \pmod 2$ if and only if λ is of the form*

$$\begin{aligned} & \lambda = 2^k \quad k \geq 0 \quad or \\ & \lambda = 2^k + 2^{i_1} + ... + 2^{i_m} \quad m \in \{1, ..., \lfloor \frac{k}{2} \rfloor\} \end{aligned} \tag{$**$}$$

with $k - i_1 \geq 2$ and $i_{j-1} - i_j \geq 2$ for all $j \in \{2, ..., m\}$.

Proof:
We consider the recursion for those groups

$$b_\Gamma(1) = \left(\frac{m_\Gamma}{2}\right)^3$$

$$\begin{aligned} b_\Gamma(\lambda+1) = & \frac{1}{4} m_\Gamma^2 \cdot (\lambda(4\lambda+6)+3) \cdot b_\Gamma(\lambda) \\ & + \frac{3}{2} m_\Gamma \cdot (\lambda+1) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ & + \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3), \qquad \lambda \geq 1, \end{aligned} \tag{2.27}$$

and with $m_\Gamma = 2k,\ k \equiv 1 \pmod 2$, we rewrite

$$\begin{aligned} b_\Gamma(\lambda+1) = ((\lambda(4\lambda+6)+3)\cdot b_\Gamma(\lambda) \\ + 3\cdot(\lambda+1)\cdot \sum_{\substack{\lambda_1,\lambda_2\geq 1\\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1)b_\Gamma(\lambda_2) \\ + \sum_{\substack{\lambda_1,\lambda_2,\lambda_3\geq 1\\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1)b_\Gamma(\lambda_2)b_\Gamma(\lambda_3). \end{aligned} \tag{2.28}$$

We use induction on λ. For $\lambda = 1 = 2^0$ we obtain $b_\Gamma(1) \equiv 1 \pmod 2$, $b_\Gamma(2) \equiv 1 \pmod 2$, $b_\Gamma(3) \equiv 0 \pmod 2$, and $b_\Gamma(4) \equiv 1 \pmod 2$. We suppose the assertion is true for all $\lambda \leq t$ and consider for $t+1$ four cases:

$a_1)$ $2 \mid t+1$ and $t+1$ is of form $(**)$

$a_2)$ $2 \mid t+1$ and $t+1$ is not of form $(**)$

$b_1)$ $2 \nmid t+1$ and $t+1$ is of form $(**)$

$b_2)$ $2 \nmid t+1$ and $t+1$ is not of form $(**)$

Case a_1)
Since $2 \mid t+1$ we have $2 \mid 3\cdot(t+1)\cdot \sum_{\substack{t_1,t_2\geq 1\\ t_1+t_2=t}} b_\Gamma(t_1)b_\Gamma(t_2)$ and hence the recursion reduces to

$$b_\Gamma(t+1) \equiv 3\cdot b_\Gamma(t) + \sum_{\substack{t_1,t_2,t_3\geq 1\\ t_1+t_2+t_3=t}} b_\Gamma(t_1)b_\Gamma(t_2)b_\Gamma(t_3) \pmod 2 \tag{2.29}$$

Calculating $\sum_{\substack{t_1,t_2,t_3\geq 1\\ t_1+t_2+t_3=t}} b_\Gamma(t_1)b_\Gamma(t_2)b_\Gamma(t_3)$ we fix w.l.o.g. $t_1 \in \{1,...,t-2\}$ and investigate the partitions of t as $\sum_{i=0}^{t-3} b_\Gamma(t-2-i)\cdot \sum_{\mu=1}^{i+1} b_\Gamma(\mu)b_\Gamma(i+2-\mu)$. For the second sum we have
Case i) $2 \nmid i \Leftrightarrow 2 \mid i+1$

$$\begin{aligned} \sum_{\mu=1}^{i+1} b_\Gamma(\mu)b_\Gamma(i+2-\mu) &= 2\cdot \sum_{\mu=1}^{\frac{i+1}{2}} b_\Gamma(\mu)b_\Gamma(i+2-\mu) \\ &\equiv 0 \pmod 2 \end{aligned}$$

Case ii) $2 \mid i \Leftrightarrow 2 \nmid i+1$

$$\begin{aligned}\sum_{\mu=1}^{i+1} b_\Gamma(\mu) b_\Gamma(i+2-\mu) &= 2 \cdot \sum_{\mu=1}^{\frac{i}{2}} b_\Gamma(\mu) b_\Gamma(i+2-\mu) \; + (b_\Gamma(\frac{i}{2}+1))^2 \\ &\equiv (b_\Gamma(\frac{i}{2}+1))^2 \pmod 2 \\ &\equiv b_\Gamma(\frac{i}{2}+1) \pmod 2.\end{aligned}$$

So, it remains to calculate

$$\begin{aligned} b_\Gamma(t+1) &= 3 \cdot b_\Gamma(t) + \sum_{\substack{0 \le i \le t-3 \\ i \equiv 0(2)}} b_\Gamma(t-2-i) \cdot b_\Gamma(\frac{i}{2}+1) \\ &= 3 \cdot b_\Gamma(t) + \sum_{l=0}^{\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1). \end{aligned} \tag{2.30}$$

We have $t+1 = 2^k$ or $t+1 = 2^k + 2^{i_1} + ... + 2^{i_m}$, where $k \ge 1$, $1 \le m \le \lfloor \frac{k}{2} \rfloor$, with $k - i_1 \ge 2$ and $i_{j-1} - i_j \ge 2$ for all $j \in \{2, ..., m\}$.

Observation 1:
$b_\Gamma(2^k + r) \equiv b_\Gamma(r) \pmod 2$ for $r = 1, ..., t^*$ where $t^* = 2^{i_1} + ... + 2^{i_m} - 1$. This follows from the additive structure of the λ of form $(**)$ and our induction assumption. Hence, modulo 2

$$\begin{aligned} b_\Gamma(t+1) &= 3 \cdot b_\Gamma(t) + \sum_{l=0}^{\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\ &= 3 \cdot b_\Gamma(t) + \sum_{l=0}^{\frac{t^*-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) + \sum_{l=\frac{t^*-3}{2}+1}^{\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\ &\equiv 3 \cdot b_\Gamma(t^*) + \sum_{l=0}^{\frac{t^*-3}{2}} b_\Gamma(t^*-2(l+1)) \cdot b_\Gamma(l+1) + \sum_{l=\frac{t^*-3}{2}+1}^{\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\ &\equiv 3 \cdot b_\Gamma(t^*+1) + \sum_{l=\frac{t^*-3}{2}+1}^{\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\ &\equiv 1 + \sum_{l=\frac{t^*-3}{2}+1}^{\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \pmod 2. \end{aligned} \tag{2.31}$$

by the Observation 1 (congruence argument) and the induction hypothesis.

It remains to calculate

$$
\begin{aligned}
\sum_{l=\frac{t^*-3}{2}+1}^{\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) = & \sum_{l=\frac{t^*-1}{2}}^{2^{k-2}-2} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) && (\Sigma_1) \\
& + b_\Gamma(2^{k-1}+t^*-1) \cdot b_\Gamma(2^{k-2}) && (P_1) \\
& + \sum_{l=2^{k-2}}^{2^{k-2}+\frac{t^*-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) && (\Sigma_2) \\
& + \sum_{l=2^{k-2}+\frac{t^*-1}{2}}^{2^{k-1}-2} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) && (\Sigma_3) \quad (2.32) \\
& + b_\Gamma(t^*-1) \cdot b_\Gamma(2^{k-1}) && (P_2) \\
& + \sum_{l=2^{k-1}}^{2^{k-1}+\frac{t^*-3}{2}=\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) && (\Sigma_4) \\
& \equiv 0 \pmod 2
\end{aligned}
$$

We analyze the different sums separately:
(Σ_1) :
$\sum_{l=\frac{t^*-1}{2}}^{2^{k-2}-2} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \equiv 0 \pmod 2$
The structure of the values for $t-2(l+1)$ and our induction hypothesis state that $b_\Gamma(t-2(l+1)) \equiv 0 \pmod 2$ for $t-2(l+1) = 2^k-1, ..., 2^{k-1}+2^{k-3}+2^{k-5}+...+2^0+1$ where the last value t' for $t-2(l+1)$ is of one of the forms
i) $2 \mid k$

$$t' = 2^{k-1}+2^{k-3}+2^{k-5}+...+2^3+2^1$$

followed by $(2^k-1 = \sum_{i=0}^{k-1} 2^i)$

$$2^{k-2}+2^{k-4}+2^{k-6}+...+2^2+2^0$$

entries for which is by our induction hypothesis $b_\Gamma \equiv 0 \pmod 2$.

ii) $2 \nmid k$. Then we have for t'

$$t' = 2^{k-1}+2^{k-3}+2^{k-5}+...+2^2+2^0$$

followed by

$$2^{k-2}+2^{k-4}+2^{k-6}+...+2^3+2^1$$

entries for which is by our induction hypothesis $b_\Gamma \equiv 0 \pmod 2$. Hence, in the sum are the first

i) $\left\lceil \frac{2^{k-2}+2^{k-4}+2^{k-6}+...+2^2+2^0}{2} \right\rceil = 2^{k-3} + 2^{k-5} + ... + 2^3 + 2^1 + 1$ or

ii) $\frac{2^{k-2}+2^{k-4}+2^{k-6}+...+2^3+2^1}{2} = 2^{k-3} + 2^{k-5} + ... + 2^2 + 2^0$ products congruent $0 \pmod 2$.

We are now at the positions

i)

$$\begin{aligned} l+1 &= 2^{k-3} + 2^{k-5} + ... + 2^3 + 2^1 + 1 + \frac{t^*-1}{2} \\ t-2(l+1) &= 2^{k-1} + 2^{k-3} + 2^{k-5} + ... + 2^3 + 2^1 + 1 \end{aligned} \tag{2.33}$$

or *ii)*

$$\begin{aligned} l+1 &= 2^{k-3} + 2^{k-5} + ... + 2^2 + 2^0 + \frac{t^*-1}{2} \\ t-2(l+1) &= 2^{k-1} + 2^{k-3} + 2^{k-5} + ... + 2^2 + 2^0 + 2 \end{aligned} \tag{2.34}$$

depending on t'.

With $\frac{t^*-1}{2} \le 2^{k-3} + 2^{k-5} + ... + 2^3 + 2^1$ we consider two cases respecting the form $(**)$.

Case 1) $1 \le \frac{t^*-1}{2} \le 2^{k-4} + 2^{k-6} + ... + 2^2 + 2^0$
Case 2) $2^{k-3} \le \frac{t^*-1}{2} \le 2^{k-3} + 2^{k-5} + ... + 2^3 + 2^1$

Case 1) $1 \le \frac{t^*-1}{2} \le 2^{k-4} + 2^{k-6} + ... + 2^2 + 2^0$
The next $2^{k-2} - 1 - (l+1)$ values for $b_\Gamma(l+1)$ are by our induction hypothesis congruent $0 \pmod 2$ and we have now as position $t - 2(l+1) = 2^{k-1} + t^* + 1$.

Case 2) $2^{k-3} \le \frac{t^*-1}{2} \le 2^{k-3} + 2^{k-5} + ... + 2^3 + 2^1$
Here, $2^{k-3}+2^{k-5}+...+2^3+2^1+1+\frac{t^*-1}{2} > 2^{k-2}$, respectively $2^{k-3}+2^{k-5}+...+2^2+2^0+\frac{t^*-1}{2} > 2^{k-2}$, hence the last value is $t-2(l+1)) > 2^{k-1}+2^{k-3}+2^{k-5}+...+2^3+2^1+1$, respectively $t-2(l+1)) > 2^{k-1} + 2^{k-3} + 2^{k-5} + ... + 2^2 + 2^0 + 2$. Again, the $b_\Gamma(l+1)$ are up to $2^{k-2} - 1$ congruent $0 \pmod 2$ by our induction hypothesis.

(P_1) and (P_2):
$b_\Gamma(2^{k-1}+t^*-1)\cdot b_\Gamma(2^{k-2}) \equiv b_\Gamma(t^*-1)\cdot b_\Gamma(2^{k-1}) \pmod 2$, depending on $b_\Gamma(t^*-1)$ which is either $0 \pmod 2$ or $1 \pmod 2$, hence in the whole sum the two products vanish modulo 2.

(Σ_2) and (Σ_4):
$\sum_{l=2^{k-2}}^{2^{k-2}+\frac{t^*-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \equiv \sum_{l=2^{k-1}}^{2^{k-1}+\frac{t^*-3}{2}=\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \pmod 2 \equiv$ $b_\Gamma(t^*+1) - 3 \cdot b_\Gamma(t^*) \pmod 2$. This follows by the congruence argument and, since both sums have the same value, their sum vanishes modulo 2, too.

(Σ_3):
$\sum_{l=2^{k-2}+\frac{t^*-1}{2}}^{2^{k-1}-2} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \equiv 0 \pmod 2$
As in the first sum we analyzed, the products are congruent $0 \pmod 2$ by the switch of the ranges for which $b_\Gamma(l+1) \equiv 0 \pmod 2$, respectively $b_\Gamma(t-2(l+1) \equiv 0 \pmod 2$.

The congruence argument (Observation 1) holds for $r \geq 1$, so it is left to investigate the case $t+1 = 2^k$ for $k \geq 1$. We consider the case $2 \nmid k$, the case $2 \mid k$ follows the same arguments and is an analogue calculation with the same results.

$$
\begin{aligned}
b_\Gamma(t+1) &= 3 \cdot b_\Gamma(t) + \sum_{l=0}^{\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\
&= 3 \cdot b_\Gamma(t) \\
&+ \sum_{l=0}^{2^{k-3}+2^{k-5}+\ldots+2^2+2^0-2} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\
&+ b_\Gamma(2^{k-1}+2^{k-3}+2^{k-5}+\ldots+2^2+2^0) \cdot b_\Gamma(2^{k-3}+2^{k-5}+\ldots+2^2+2^0) \\
&+ \sum_{l=2^{k-3}+2^{k-5}+\ldots+2^2+2^0}^{2^{k-2}-2} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\
&+ \sum_{l=2^{k-2}-1}^{2^{k-2}+2^{k-4}+\ldots+2^3+2^1-1} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\
&+ \sum_{l=2^{k-2}+2^{k-4}+\ldots+2^3+2^1}^{2^{k-1}-2} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\
&\equiv 1 \pmod 2
\end{aligned}
\tag{2.35}
$$

$3 \cdot b_\Gamma(t) \equiv 0 \pmod 2$ since $t = 2^k - 1$ and not of form $(**)$, hence by our induction hypothesis $b_\Gamma(t) \equiv 0 \pmod 2$.

$$\sum_{l=0}^{2^{k-3}+2^{k-5}+...+2^2+2^0-2} b_\Gamma(t-2(l+1))\cdot b_\Gamma(l+1) \equiv 0 \pmod 2$$ because we have $2^{k-1}+2^{k-3}+2^{k-5}+...+2^2+2^0+2 \leq t-2(l+1) \leq 2^k-1$ and by the induction hypothesis $b_\Gamma(t-2(l+1)) \equiv 0 \pmod 2$.

$b_\Gamma(2^{k-1}+2^{k-3}+2^{k-5}+...+2^2+2^0)\cdot b_\Gamma(2^{k-3}+2^{k-5}+...+2^2+2^0) \equiv 1 \pmod 2$. Both values $2^{k-1}+2^{k-3}+2^{k-5}+...+2^2+2^0$ and $2^{k-3}+2^{k-5}+...+2^2+2^0$ are of form $(**)$, hence the product is congruent $1 \pmod 2$.

$$\sum_{l=2^{k-3}+2^{k-5}+...+2^2+2^0}^{2^{k-2}-2} b_\Gamma(t-2(l+1))\cdot b_\Gamma(l+1) \equiv 0 \pmod 2.$$ We have $2^{k-3}+2^{k-5}+...+2^2+2^0 < l+1 \leq 2^{k-2}-2$ and by the induction hypothesis $b_\Gamma(l+1) \equiv 0 \pmod 2$.

$$\sum_{l=2^{k-2}-1}^{2^{k-2}+2^{k-4}+...+2^3+2^1-1} b_\Gamma(t-2(l+1))\cdot b_\Gamma(l+1) \equiv 0 \pmod 2.$$ $2^{k-2}+2^{k-4}+...+2^1+2^0 \leq t-2(l+1) \leq 2^{k-1}-1$ and so $b_\Gamma(t-2(l+1) \equiv 0 \pmod 2$ follows by our induction hypothesis.

$$\sum_{l=2^{k-2}+2^{k-4}+...+2^3+2^1}^{2^{k-1}-2} b_\Gamma(t-2(l+1))\cdot b_\Gamma(l+1) \equiv 0 \pmod 2$$ since $2^{k-2}+2^{k-4}+...+2^1+1 \leq l+1 \leq 2^{k-1}-1$ and so $b_\Gamma(l+1) \equiv 0 \pmod 2$.

Case a_2) $2 \mid t+1$ and $t+1$ not of form $(**)$
In this case we have the same recursion as in case a_1). We apply the same congruence argument (Observation 1). In fact it holds up to:

Observation 2:
$b_\Gamma(2^k+r) \equiv b_\Gamma(r)$ for all $r = 1, ..., 2^{k-1}$.

For the case $2^k < t+1 < 2^k+2^{k-1}$ we use the sum splitting as in formula 2.32 and get by the same arguments used in that case $b_\Gamma(t+1) \equiv 0 \pmod 2$.

Considering $2^k + 2^{k-1} < t+1 \leq 2^{k+1}-1$, we have $t+1-(2^k+2^{k-1}) \leq 2^j$ for an $j \leq k-1$, hence

$$\begin{aligned} b_\Gamma(t+1) &= 3 \cdot b_\Gamma(t) + \sum_{l=0}^{\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\ &= 3 \cdot b_\Gamma(t) + \sum_{l=0}^{2^{j-1}-1} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\ &+ \sum_{2^{j-1}}^{\frac{t-3}{2}} b_\Gamma(t-2(l+1)) \cdot b_\Gamma(l+1) \\ &\equiv 0 \pmod 2 \end{aligned} \tag{2.36}$$

where because after the first sum, which is congruent $0 \pmod 2$, by our induction hypothesis all $b_\Gamma(t-2(l+1)) \equiv 0 \pmod 2$, we reach the range for our congruence argument (Observation 2) and the sum splitting 2.32 gives that the value is congruent $0 \pmod 2$.

Case b_1) $2 \nmid t+1$ and $t+1$ is of form $(**)$
Here we have for the recursion formula

$$\begin{aligned} b_\Gamma(t+1) = ((t(4t+6)+3) \cdot b_\Gamma(t) \\ &+ 3 \cdot (t+1) \cdot \sum_{\substack{t_1,t_2 \geq 1 \\ t_1+t_2=t}} b_\Gamma(t_1) b_\Gamma(t_2) \\ &+ \sum_{\substack{t_1,t_2,t_3 \geq 1 \\ t_1+t_2+t_3=t}} b_\Gamma(t_1) b_\Gamma(t_2) b_\Gamma(t_3) \\ &\equiv 3 \cdot b_\Gamma(t) + \sum_{\substack{t_1,t_2 \geq 1 \\ t_1+t_2=t}} b_\Gamma(t_1) b_\Gamma(t_2) + \sum_{\substack{t_1,t_2,t_3 \geq 1 \\ t_1+t_2+t_3=t}} b_\Gamma(t_1) b_\Gamma(t_2) b_\Gamma(t_3) \pmod 2. \end{aligned} \tag{2.37}$$

Further,

$$\begin{aligned} \sum_{\mu=1}^{t-1} b_\Gamma(\mu) \cdot b_\Gamma(t-\mu) &= 2 \cdot \sum_{\mu=1}^{\frac{t-1}{2}-1} b_\Gamma(\mu) \cdot b_\Gamma(t-\mu) \quad + (b_\Gamma(\frac{t}{2}))^2 \\ &\equiv (b_\Gamma(\frac{t}{2}))^2 \pmod 2. \end{aligned} \tag{2.38}$$

With $2 \nmid (t+1)$ and $(t+1)$ of form $(**)$ we have $t+1 = 2^k + 2^{k-2} + ... + 2^0$ but then t is also of form $(**)$, hence $3 \cdot b_\Gamma(t) \equiv 1 \pmod 2$ and $\sum_{\mu=1}^{t-1} b_\Gamma(\mu) \cdot b_\Gamma(t-\mu) \equiv 1 \pmod 2$ since $\frac{t}{2}$ is of form $(**)$, too.

It remains to calculate

$$b_\Gamma(t+1) = \sum_{\substack{t_1,t_2,t_3 \geq 1 \\ t_1+t_2+t_3=t}} b_\Gamma(t_1) b_\Gamma(t_2) b_\Gamma(t_3) \equiv 1 \pmod 2. \tag{2.39}$$

The investigation of this sum follows the same ideas as used in the previous cases and we suppress the calculations here.

Case $\boldsymbol{b_2}$) $2 \nmid t+1$ and $t+1$ is not of form $(**)$
Like in case b_1) we have to analyze the same recursion and get with $t+1$ not of form $(**)$ either t is of form $(**)$ or t is not of form $(**)$. In both cases $\frac{t}{2}$ is then of the same form as t, hence we have that the sum is congruent 0 modulo 2. It is left to analyze

$$b_\Gamma(t+1) = \sum_{\substack{t_1,t_2,t_3 \geq 1 \\ t_1+t_2+t_3=t}} b_\Gamma(t_1) b_\Gamma(t_2) b_\Gamma(t_3) \equiv 0 \pmod 2, \tag{2.40}$$

which follows the same ideas and calculations similar to those used in the previous cases. $\square$

Again, for virtually free groups of rank $\mu(\Gamma) = 3$ we observe two divisibility patterns for primes $p \leq 13$ apart from the more complex one stated above. Either $b_\Gamma(\lambda) \equiv c \pmod p$ for all $\lambda \geq 1$ or $b_\Gamma(\lambda)$ has a period modulo p. As in the previous section, we will state the divisibility conditions for $b_\Gamma(\lambda)$ according to the type $\tau(\Gamma)$.

Theorem 2.18 *Let Γ be of the type $\tau(\Gamma) = (2k; 0, ..., 0, 2_k, 0_{2k})$ and recursion formula 2.18. Then we get for primes $p \leq 7$ and $k \equiv 1 \pmod p$ the following divisibility properties for $b_\Gamma(\lambda)$:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv 0 \pmod 2 \quad \forall \lambda \geq 1$
3	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 3 & \text{if } 2 \mid \lambda \\ 2 \pmod 3 & \text{if } 2 \nmid \lambda \end{cases}$
5	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 5 & \text{if } \lambda \equiv 0 \pmod 4 \\ 2 \pmod 5 & \text{if } \lambda \equiv 1 \pmod 4 \\ 3 \pmod 5 & \text{if } \lambda \equiv 3 \pmod 4 \\ 4 \pmod 5 & \text{if } \lambda \equiv 2 \pmod 4 \end{cases}$
7	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 7 & \text{if } \lambda \equiv 0, 9, 15, 10 \pmod{24} \\ 2 \pmod 7 & \text{if } \lambda \equiv 1, 7, 12, 16 \pmod{24} \\ 3 \pmod 7 & \text{if } \lambda \equiv 3, 10, 21, 22 \pmod{24} \\ 4 \pmod 7 & \text{if } \lambda \equiv 4, 8, 17, 23 \pmod{24} \\ 5 \pmod 7 & \text{if } \lambda \equiv 5, 6, 11, 18 \pmod{24} \\ 6 \pmod 7 & \text{if } \lambda \equiv 2, 13, 14, 19 \pmod{24} \end{cases}$

Proof:
The recursion formula 2.18 states

$$\begin{aligned} b_\Gamma(1) &= m_\Gamma k^2 \\ b_\Gamma(\lambda+1) &= \frac{1}{4} m_\Gamma^2 \cdot (\lambda(4\lambda+8)+5) \cdot b_\Gamma(\lambda) \\ &\quad + \frac{1}{2} m_\Gamma \cdot (3\lambda+4) \cdot \sum_{\substack{\lambda_1,\lambda_2 \geq 1 \\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) \\ &\quad + \sum_{\substack{\lambda_1,\lambda_2,\lambda_3 \geq 1 \\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1) b_\Gamma(\lambda_2) b_\Gamma(\lambda_3), \qquad \lambda \geq 1, \end{aligned}$$

and with $m_\Gamma = 2\,k,\ k \equiv 1 \pmod p$ we rewrite

$$\begin{aligned} b_\Gamma(\lambda+1) &= (\lambda(4\lambda+8)+5)\cdot b_\Gamma(\lambda) \\ &\quad + (3\lambda+4)\cdot \sum_{\substack{\lambda_1,\lambda_2\geq 1\\ \lambda_1+\lambda_2=\lambda}} b_\Gamma(\lambda_1)b_\Gamma(\lambda_2) \\ &\quad + \sum_{\substack{\lambda_1,\lambda_2,\lambda_3\geq 1\\ \lambda_1+\lambda_2+\lambda_3=\lambda}} b_\Gamma(\lambda_1)b_\Gamma(\lambda_2)b_\Gamma(\lambda_3). \end{aligned}$$

Case $p = 2$
Induction on λ. For $\lambda = 1$ we have $b_\Gamma(1) = m_\Gamma\, k^2 = 2\,k^3 \equiv 0 \pmod 2$. Suppose the assertion is true for all $\lambda \leq t$ and we consider $t+1$.

$$\begin{aligned} b_\Gamma(t+1) &= (t(4t+8)+5)\cdot b_\Gamma(t) \\ &\quad + (3t+4)\cdot \sum_{\substack{t_1,t_2\geq 1\\ t_1+t_2=t}} b_\Gamma(t_1)b_\Gamma(t_2) \\ &\quad + \sum_{\substack{t_1,t_2,t_3\geq 1\\ t_1+t_2+t_3=t}} b_\Gamma(t_1)b_\Gamma(t_2)b_\Gamma(t_3) \\ &\equiv 0 \pmod 2, \end{aligned}$$

since for all $t_i,\ i \in \{1,...,3\},\quad t_i \leq t$ and by the induction assumption $b_\Gamma(t_i) \equiv 0 \pmod 2$. Hence, $b_\Gamma(t+1) \equiv 0 \pmod 2$ follows.

Case $p = 3$
Induction on λ. For $\lambda = 1$ we calculate $b_\Gamma(1) = m_\Gamma\, k^2 = 2\,k^3 \equiv 2 \pmod 3$. We suppose the statement is true for all $\lambda \leq t$ and consider $t+1$.

$$\begin{aligned} b_\Gamma(t+1) &= (t(4t+8)+5)\cdot b_\Gamma(t) \\ &\quad + (3t+4)\cdot \sum_{\substack{t_1,t_2\geq 1\\ t_1+t_2=t}} b_\Gamma(t_1)b_\Gamma(t_2) \\ &\quad + \sum_{\substack{t_1,t_2,t_3\geq 1\\ t_1+t_2+t_3=t}} b_\Gamma(t_1)b_\Gamma(t_2)b_\Gamma(t_3) \end{aligned}$$

and $t+1$ is of the form $t+1 = 2k - s$ with $s \in \{0,1\},\ k \geq 1$. We distinguish two cases:

a) $\quad t+1 = 2k, \quad s = 0$

b) $\quad t+1 = 2k-1, \quad s = 1$

We investigate the two sums separately. Calculating $\sum\limits_{\substack{t_1,t_2\geq 1\\ t_1+t_2=t}} b_\Gamma(t_1)b_\Gamma(t_2)$ gives, since $t_i < t$ with the induction hypothesis iterated sequences of $(2,1)$ for $t_1 = 1,...,t-1$, respectively

$(1,2)$ for $t_2 = t-1, ..., 1$. Like in Theorem 2.8 we use the blocks B_0 and the iterated block B_1.

B_0			
$(t-1)...$	$b_\Gamma(t-1)...$	$b_\Gamma(\mu),\ \mu = 1, ..., 2-s$	
2k	1	2	
2k-1	2	1	2
$\sum b_\Gamma(\mu)b_\Gamma(t-\mu) \pmod 3$		1	1
		(B_{01})	(B_{02})

B_1			
$(t-\mu)$	$b_\Gamma(t-\mu)...$	$b_\Gamma(\mu),\ \mu = 2-s+1, ..., t-1$	
2k	1	2	1
2k-1	2	1	2
$\sum b_\Gamma(\mu)b_\Gamma(t-\mu) \pmod 3$		1	2
		(B_{11})	(B_{12})

Analyzing the second sum $\sum\limits_{\substack{t_1,t_2,t_3\geq 1\\ t_1+t_2+t_3=t}} b_\Gamma(t_1)b_\Gamma(t_2)b_\Gamma(t_3)$ we fix w.l.o.g. $t_1 \in \{1, ..., t-2\}$ and consider $\sum\limits_{\substack{t_1,t_2\geq 1\\ t_1+t_2=t}} b_\Gamma(t_1)b_\Gamma(t_2)$ where we use the blocks B_0 and the iterated block B_1.

Case a)
$s = 0,\ t+1 = 2k,\ t = 2k-1,\ t-1 = 2(k-1), t-2 = 2(k-1)-1,\ 2 \nmid t-2$
Setting $t_1 = t-2-i, \quad i \in \{0, ..., t-3\}$ we get
Case $a_0)$ $i = 0$

$$b_\Gamma(t-2)b_\Gamma(1)^2 \equiv 2 \cdot 2^2 \pmod 3 \equiv 2 \pmod 3 \tag{2.41}$$

Case a_1) $\quad 2 \mid i \Leftrightarrow 2 \nmid i+2-1$
Here, we have $i \in \{2, ..., t-3\},\ 2 \mid i$ and so

$$\sum_{2|i}^{t-3} \left(b_\Gamma(t-2-i) \cdot \sum_{t_2=1}^{i+1} b_\Gamma(t_2) b_\Gamma(i+2-t_2) \right)$$

(induction hypothesis, blockargument)

$$\equiv \sum_{2|i}^{t-3} b_\Gamma(t-2-i) \cdot (B_{02} + \frac{i}{2} B_{12}) \pmod 3$$

$(b_\Gamma(t-2-i) \equiv 2\ (3),\ 2 \nmid t-2-i)$

$$\equiv 2 \cdot \sum_{2|i}^{t-3} (B_{02} + \frac{i}{2} B_{12}) \pmod 3$$

$$\equiv 2 \cdot \frac{t-3}{2} \left(B_{02} + B_{12}\, \frac{t-1}{4} \right) \pmod 3$$

Case a_2) $\quad 2 \nmid i \Leftrightarrow 2 \mid i+2-1$
Here we have $i \in \{1, ..., t-4\},\ 2 \nmid i$ and so

$$\sum_{2\nmid i}^{t-4} \left(b_\Gamma(t-2-i) \cdot \sum_{t_2=1}^{i+1} b_\Gamma(t_2) b_\Gamma(i+2-t_2) \right)$$

(induction hypothesis, blockargument)

$$\equiv \sum_{2\nmid i}^{t-4} b_\Gamma(t-2-i) \cdot \frac{i+1}{2} B_{11} \pmod 3$$

$(b_\Gamma(t-2-i) \equiv 1\ (3),\ 2 \mid t-2-i,\ B_{11} \equiv 1(3))$

$$\equiv \sum_{2\nmid i}^{t-4} \frac{i+1}{2} \pmod 3$$

$$\equiv \frac{(t-3)(t-1)}{8} \pmod 3$$

Case b)
$s = 1,\ t+1 = 2k-1,\ t = 2(k-1),\ t-1 = 2(k-1)-1, t-2 = 2(k-2),\ 2 \mid t-2$
Setting $t_1 = t-2-i, \quad i \in \{0, ..., t-3\}$ we get
Case b_0) $i = 0$

$$b_\Gamma(t-2) b_\Gamma(1)^2 \equiv 1 \cdot 2^2 \equiv 1 \pmod 3 \tag{2.42}$$

Again, we consider two cases:
Case b_1) $\quad 2 \mid i \Leftrightarrow 2 \nmid i+2-1$

Here, we have $i \in \{2, ..., t-4\}$ $2 \mid i$ and so

$$\sum_{2|i}^{t-4} \left(b_\Gamma(t-2-i) \cdot \sum_{t_2=1}^{i+1} b_\Gamma(t_2) b_\Gamma(i+2-t_2) \right)$$

(induction hypothesis, blockargument)

$$\equiv \sum_{2|i}^{t-4} b_\Gamma(t-2-i) \cdot (B_{02} + \frac{i}{2} B_{12}) \pmod 3$$

$(b_\Gamma(t-2-i) \equiv 1\ (3),\ 2 \mid t-2-i)$

$$\equiv \sum_{2|i}^{t-4} B_{02} + \frac{i}{2} B_{12} \pmod 3$$

$$\equiv \frac{t-4}{2} \left(B_{02} + B_{12} \, \frac{t-2}{4} \right) \pmod 3$$

Case b_2) $\quad 2 \nmid i \Leftrightarrow 2 \mid i+2-1$

Here, we have $i \in \{1, ..., t-3\}$ $2 \nmid i$ and so

$$\sum_{2\nmid i}^{t-3} \left(b_\Gamma(t-2-i) \cdot \sum_{t_2=1}^{i+1} b_\Gamma(t_2) b_\Gamma(i+2-t_2) \right)$$

(induction hypothesis, blockargument)

$$\equiv \sum_{2\nmid i}^{t-3} b_\Gamma(t-2-i) \cdot \frac{i+1}{2} \, B_{11} \pmod 3$$

$(b_\Gamma(t-2-i) \equiv 2\ (3),\ 2 \nmid t-2-i)$

$$\equiv 2 \cdot \sum_{2\nmid i}^{t-3} \frac{i+1}{2} \, B_{11} \pmod 3$$

$$\equiv 2 \cdot B_{11} \frac{t(t-2)}{8} \pmod 3$$

Now we can consider the recursion for $s \in \{0, 1\},\ k \geq 1$, adding the previously calculated sums.

Case $s = 0:\quad t+1 = 2k,\ t = 2k-1,\ t-1 = 2(k-1), b_\Gamma(2k-1) \equiv 2 \pmod 3$

$$\begin{aligned}
b_\Gamma(t+1) = {} & (t(4t+8)+5)\cdot b_\Gamma(t) \\
& + (3t+4)\cdot \sum_{\substack{t_1,t_2\geq 1\\ t_1+t_2=t}} b_\Gamma(t_1)b_\Gamma(t_2) \\
& + \sum_{\substack{t_1,t_2,t_3\geq 1\\ t_1+t_2+t_3=t}} b_\Gamma(t_1)b_\Gamma(t_2)b_\Gamma(t_3) \\
& \equiv (t(4t+8)+5)\cdot 2 \\
& + (3t+4)\cdot((k-1)\cdot 1) \\
& + 2 + 2\cdot\left(\frac{t-3}{2} + 2\cdot\frac{(t-3)(t-1)}{8}\right) + \frac{(t-3)(t-1)}{8} \pmod 3 \\
& \equiv (16(k-2)(k-1) + 24((k-2)+(k-1)) + 41)\cdot 2 \\
& + 6(k-2)(k-1) + 13(k-1) + (k-2)(2+(k-1)) + 2 \pmod 3 \\
& \equiv 1 \pmod 3
\end{aligned}$$

Case $s = 1:\quad t+1 = 2k-1,\ t = 2(k-1),\ t-1 = 2(k-1)-1,\ b_\Gamma(2(k-1)) \equiv 1 \pmod 3$

$$\begin{aligned}
b_\Gamma(t+1) = {} & (t(4t+8)+5)\cdot b_\Gamma(t) \\
& + (3t+4)\cdot \sum_{\substack{t_1,t_2\geq 1\\ t_1+t_2=t}} b_\Gamma(t_1)b_\Gamma(t_2) \\
& + \sum_{\substack{t_1,t_2,t_3\geq 1\\ t_1+t_2+t_3=t}} b_\Gamma(t_1)b_\Gamma(t_2)b_\Gamma(t_3) \\
& \equiv (t(4t+8)+5)\cdot 1 \\
& + (3t+4)\cdot(1+(k-2)\cdot 2) \\
& + 1 + \frac{t-4}{2} + 2\cdot\frac{(t-4)(t-2)}{8} + 2\cdot\frac{t(t-2)}{8} \pmod 3 \\
& \equiv \big(16(k-2)^2 + 48(k-2) + 37\big)\cdot 1 \\
& + (6(k-2)+10)\cdot(1+(k-2)\cdot 2) \\
& + 1 + (k-3) + 2\cdot(k-2)^2 \pmod 3 \\
& \equiv 2 \pmod 3
\end{aligned}$$

The proofs of the cases $p = 5,\ p = 7$ follow exactly the same arguments as in the case $p = 3$ but we suppress the calculations. □

The next theorems are stated without proofs. The proofs follow exactly the same ideas mentioned in the theorem above and effort a lot of calculation which we like to avoid here.

Theorem 2.19 *Let Γ be of the type $\tau(\Gamma) = (2k; 0, ..., 0, 3_k, -1_{2k})$ and recursion formula 2.19. Then we get for primes $p \leq 7$ and $k \equiv 1 \pmod p$ the following divisibility properties for $b_\Gamma(\lambda)$:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 2 & \text{if } \lambda \text{ of form } (**) \\ 0 \pmod 2 & \text{if } \lambda \text{ else} \qquad \text{(Theorem 2.17)} \end{cases}$
3	$b_\Gamma(\lambda) \equiv 1 \pmod 3 \quad \forall \lambda \geq 1$
5	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 5 & \text{if } \lambda \equiv 0, 1, 5 \pmod{12} \\ 2 \pmod 5 & \text{if } \lambda \equiv 3, 4, 8 \pmod{12} \\ 3 \pmod 5 & \text{if } \lambda \equiv 2, 9, 10 \pmod{12} \\ 4 \pmod 5 & \text{if } \lambda \equiv 6, 7, 11 \pmod{12} \end{cases}$
7	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 7 & \text{if } \lambda \equiv 0, 1, 7, 20 \pmod{24} \\ 2 \pmod 7 & \text{if } \lambda \equiv 12, 16, 17, 23 \pmod{24} \\ 3 \pmod 7 & \text{if } \lambda \equiv 5, 10, 11, 22 \pmod{24} \\ 4 \pmod 7 & \text{if } \lambda \equiv 4, 8, 9, 15 \pmod{24} \\ 5 \pmod 7 & \text{if } \lambda \equiv 6, 13, 18, 19 \pmod{24} \\ 6 \pmod 7 & \text{if } \lambda \equiv 2, 3, 14, 21 \pmod{24} \end{cases}$

Theorem 2.20 *Let Γ be of the type $\tau(\Gamma) = (2k; 0, ..., 0, 1_k, 1_{2k})$ and recursion formula 2.20. Then $b_\Gamma(\lambda) \equiv 0 \pmod 2 \quad \forall \lambda \geq 1$.*

Theorem 2.21 *Let Γ be of the form $\tau(\Gamma) = (4k; 0, ..., 0, 1_k, 0, ..., 0_{4k})$ and recursion formula 2.22. Then we get for primes $p \leq 7$, $p = 13$ and $k \equiv 1 \pmod{p}$ the following divisibility properties for $b_\Gamma(\lambda)$:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv 0 \pmod 2 \quad \forall \lambda \geq 1$
3	$b_\Gamma(\lambda) \equiv 0 \pmod 3 \quad \forall \lambda \geq 1$
5	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 5 & \text{if } \lambda \equiv 0 \pmod 4 \\ 2 \pmod 5 & \text{if } \lambda \equiv 1 \pmod 4 \\ 3 \pmod 5 & \text{if } \lambda \equiv 3 \pmod 4 \\ 4 \pmod 5 & \text{if } \lambda \equiv 2 \pmod 4 \end{cases}$
7	$b_\Gamma(\lambda) \equiv \begin{cases} 3 \pmod 7 & \text{if } \lambda \equiv 0 \pmod 3 \\ 5 \pmod 7 & \text{if } \lambda \equiv 1 \pmod 3 \\ 6 \pmod 7 & \text{if } \lambda \equiv 2 \pmod 3 \end{cases}$
13	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod{13} & \text{if } \lambda \equiv 4, 8 \pmod{12} \\ 1 \pmod{13} & \text{if } \lambda \equiv 0 \pmod{12} \\ 4 \pmod{13} & \text{if } \lambda \equiv 5 \pmod{12} \\ 7 \pmod{13} & \text{if } \lambda \equiv 10, 11 \pmod{12} \\ 8 \pmod{13} & \text{if } \lambda \equiv 7 \pmod{12} \\ 9 \pmod{13} & \text{if } \lambda \equiv 6 \pmod{12} \\ 11 \pmod{13} & \text{if } \lambda \equiv 2, 9 \pmod{12} \\ 12 \pmod{13} & \text{if } \lambda \equiv 1, 3 \pmod{12} \end{cases}$

Theorem 2.22 *Let Γ be of the type $\tau(\Gamma) = (6k; 0, ..., 0, 1_k, 0, ..., 0_{6k})$ and recursion formula 2.25. Then we get for primes $p \leq 7$ and $k \equiv 1 \pmod p$ the following divisibility properties for $b_\Gamma(\lambda)$:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv 0 \pmod 2 \quad \forall \lambda \geq 1$
3	$b_\Gamma(\lambda) \equiv 0 \pmod 3 \quad \forall \lambda \geq 1$
5	$b_\Gamma(\lambda) \equiv 0 \pmod 5 \quad \forall \lambda \geq 1$
7	$b_\Gamma(\lambda) \equiv \begin{cases} 1 \pmod 7 & \text{if } \lambda \equiv 0 \pmod 3 \\ 2 \pmod 7 & \text{if } \lambda \equiv 1 \pmod 3 \\ 4 \pmod 7 & \text{if } \lambda \equiv 2 \pmod 3 \end{cases}$

Theorem 2.23 *Let Γ be of the type $\tau(\Gamma) = (4k; 0, ..., 0, 1_k, 1_{2k}, -1_{4k})$ and recursion formula 2.26. Then we get for primes $p \leq 7$ and $k \equiv 1 \pmod p$ the following divisibility properties for $b_\Gamma(\lambda)$:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv 0 \pmod 2 \quad \forall \lambda \geq 1$
3	$b_\Gamma(\lambda) \equiv 0 \pmod 3 \quad \forall \lambda \geq 1$
5	$b_\Gamma(\lambda) \equiv 1 \pmod 5 \quad \forall \lambda \geq 1$
7	$b_\Gamma(\lambda) \equiv \begin{cases} 3 \pmod 7 & \text{if } \lambda \equiv 0 \pmod 3 \\ 5 \pmod 7 & \text{if } \lambda \equiv 2 \pmod 3 \\ 6 \pmod 7 & \text{if } \lambda \equiv 1 \pmod 3 \end{cases}$

Theorem 2.24 *Let* Γ *be of the type* $\tau(\Gamma) = (6k; 0, ..., 0, 1_k, 0, ..., 0, 1_{3k}, -1_{6k})$ *and recursion formula 2.27. Then we get for primes* $p \leq 7$ *and* $k \equiv 1 \pmod p$ *the following divisibility properties for* $b_\Gamma(\lambda)$*:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv 1 \pmod 2 \quad \forall \lambda \geq 1$
3	$b_\Gamma(\lambda) \equiv 0 \pmod 3 \quad \forall \lambda \geq 1$
5	$b_\Gamma(\lambda) \equiv 0 \pmod 5 \quad \forall \lambda \geq 1$
7	$b_\Gamma(\lambda) \equiv 1 \pmod 7 \quad \forall \lambda \geq 1.$

Theorem 2.25 *Let* Γ *be of the type* $\tau(\Gamma) = (6k; 0, ..., 0, 1_{2k}, 1_{3k}, -1_{6k})$ *and recursion formula 2.28. Then we get for primes* $p \leq 11$ *and* $k \equiv 1 \pmod p$ *the following divisibility properties for* $b_\Gamma(\lambda)$*:*

Prime p	$b_\Gamma(\lambda),\ \lambda \geq 1$
2	$b_\Gamma(\lambda) \equiv 0 \pmod 2 \quad \forall \lambda \geq 1$
3	$b_\Gamma(\lambda) \equiv 0 \pmod 3 \quad \forall \lambda \geq 1$
5	$b_\Gamma(\lambda) \equiv 4 \pmod 5 \quad \forall \lambda \geq 1$
7	$b_\Gamma(\lambda) \equiv \begin{cases} 2 \pmod 7 & \text{if } \lambda \equiv 0 \pmod 3 \\ 3 \pmod 7 & \text{if } \lambda \equiv 1 \pmod 3 \\ 5 \pmod 7 & \text{if } \lambda \equiv 2 \pmod 3 \end{cases}$
11	$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod{11} & \text{if } \lambda \equiv 5, 6, 10 \pmod{15} \\ 1 \pmod{11} & \text{if } \lambda \equiv 2, 3, 7 \pmod{15} \\ 2 \pmod{11} & \text{if } \lambda \equiv 1, 11, 12 \pmod{15} \\ 4 \pmod{11} & \text{if } \lambda \equiv 0, 4, 14 \pmod{15} \\ 7 \pmod{11} & \text{if } \lambda \equiv 8, 9, 13 \pmod{15} \end{cases}$

2.4 Further results

In the previous sections we observed three different types of divisibility patterns for $b_\Gamma(\lambda)$ and primes $p \leq 13$. Some groups exhibit an interesting and complex pattern, but most of the investigated groups satisfy either $b_\Gamma(\lambda) \equiv c \pmod p$ for all $\lambda \geq 1$ or $b_\Gamma(\lambda)$ is periodic modulo p. Here, we will give, for $\mu(\Gamma) \leq 3$, some necessary and sufficient conditions involving m_Γ and $b_\Gamma(1)$, for $b_\Gamma(\lambda)$ to be constant modulo any prime p. Especially for the type $\tau(\Gamma) = (n; 0, ..., 0, r-1)$ and $\zeta_{m_\Gamma} \geq 0$ we state divisibility patterns for $p \in \{2, 3\}$ and for every rank r, depending on the parity of the rank. First, we focus on conditions for $b_\Gamma(\lambda)$ is constant modulo p and our approach to determine such conditions is a direct calculation using the explicit recursion formulas.

Theorem 2.26 *Let Γ be a virtually free group of rank $\mu(\Gamma) = 2$ and of one of the following forms:*

a) Γ is a free product $\Gamma = G_1 \underset{S}{*} G_2$ of finite groups G_i with amalgamated subgroup S and indices $\{[G_1 : S], [G_2 : S]\} = \{2, 3\}$

b) Γ is a free product $\Gamma = G_1 \underset{S}{*} G_2$ of finite groups G_i with amalgamated subgroup S and indices $[G_1 : S] = 3 = [G_2 : S]$

c) Γ is of the form $\Gamma = (G_1 \underset{G_{12}}{*} G_2) \underset{G_{23}}{*} G_3$ with finite groups G_i and indices $[G_1 : G_{12}] = 2 = [G_2 : G_{12}]$ and $[G_2 : G_{23}] = 2 = [G_3 : G_{23}]$.

With $|S| \equiv 1 \pmod p$, respectively $|G_{12}| \equiv 1 \pmod p$ we get $b_\Gamma(\lambda) \equiv b_\Gamma(1) \not\equiv 0 \pmod p$ for all $\lambda \geq 1$ iff $2 \cdot m_\Gamma \equiv 1 \pmod p$ and $m_\Gamma + b_\Gamma(1) \equiv 0 \pmod p$. $b_\Gamma(\lambda) \equiv 0 \pmod p$ for all $\lambda \geq 1$ iff $b_\Gamma(1) \equiv 0 \pmod p$.

Proof:
For all three types of groups the recursion formula for $b_\Gamma(\lambda)$ is given by

$$b_\Gamma(\lambda+1) = (\lambda+1) \cdot m_\Gamma \cdot b_\Gamma(\lambda) + \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) b_\Gamma(\lambda-\mu), \qquad \lambda \geq 1,$$

and with initial values $b_\Gamma(1) = (m_\Gamma - |S|)|S|$ for the first two types of groups, and $b_\Gamma(1) = \left(\frac{m_\Gamma}{2}\right)^2$ for the last one.

$\Longrightarrow$
We use induction on λ.
Let $b_\Gamma(1) \equiv c \pmod p$ and $b_\Gamma(\lambda) \equiv c \pmod p$ for all $\lambda \leq t$. We consider $t+1$:

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1)\, m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv (t+1) \cdot m_\Gamma \cdot c + (t-1) \cdot c^2 \pmod p \\
&\equiv (t-1) \cdot (c \cdot (m_\Gamma + c)) + 2 \cdot m_\Gamma \cdot c \pmod p
\end{aligned}$$

Since $2 \cdot m_\Gamma \cdot c \pmod p$ is a constant $\pmod p$ for all t we need $(t-1)\cdot(c\cdot(m_\Gamma+c))$ also to be a constant for all t. Because of the changing values for t and the constants m_Γ, c, $(t-1)\cdot(c\cdot(m_\Gamma+c)) \equiv 0 \pmod p$ is the only possible value for a constant $\pmod p$. Either

a) $c \pmod p \equiv 0 \pmod p$ then $b_\Gamma(t+1) \equiv 0 \pmod p$ $\forall t \geq 1$ or

b) $m_\Gamma + c \pmod p \equiv 0 \pmod p$ then $b_\Gamma(t+1) \equiv 2 \cdot m_\Gamma \cdot c \pmod p$ and with $2 \cdot m_\Gamma \equiv 1 \pmod p$, $b_\Gamma(t+1) \equiv c \pmod p$ follows.

$\Longleftarrow$
Let $2m_\Gamma \equiv 1 \pmod 2$ and $m_\Gamma + b_\Gamma(1) \equiv 0 \pmod p$. Again, we calculate the recursion above: We have $b_\Gamma(1) = (m_\Gamma - |S|)|S| \equiv m_\Gamma - 1 \pmod p$. For $b_\Gamma(2) = 2 \cdot m_\Gamma \cdot b_\Gamma(1) \equiv b_\Gamma(1)$ with $2 \cdot m_\Gamma \equiv 1 \pmod p$ and further we calculate $b_\Gamma(3) = 3 \cdot m_\Gamma \cdot b_\Gamma(2) + b_\Gamma(1)^2 \equiv 3 \cdot m_\Gamma \cdot b_\Gamma(1) + b_\Gamma(1)^2 \equiv b_\Gamma(1) \cdot (2 \cdot m_\Gamma + m_\Gamma + b_\Gamma(1)) \equiv b_\Gamma(1) \pmod p$ with $2m_\Gamma \equiv 1 \pmod 2$ and $m_\Gamma + b_\Gamma(1) \equiv 0 \pmod p$. Suppose $b_\Gamma(\lambda) \equiv b_\Gamma(1) \pmod p$ for all $\lambda \leq t$ and consider $t+1$.

$$\begin{aligned}
b_\Gamma(t+1) &= (t+1) \cdot m_\Gamma \cdot b_\Gamma(t) + \sum_{\mu=1}^{t-1} b_\Gamma(\mu) b_\Gamma(t-\mu) \\
&\equiv (t+1) \cdot m_\Gamma \cdot b_\Gamma(1) + (t-1) \cdot b_\Gamma(t)^2 \pmod p \\
&\equiv b_\Gamma(1)\,(2m_\Gamma + (t-1) \cdot (b_\Gamma(1) + m_\Gamma)) \pmod p \\
&\equiv b_\Gamma(1) \pmod p
\end{aligned}$$

since $2m_\Gamma \equiv 1 \pmod p$ and $b_\Gamma(1) + m_\Gamma \equiv 0 \pmod p$. In case $b_\Gamma(1) \equiv 0 \pmod p$ then $b_\Gamma(t+1) \equiv 0 \pmod p$ follows. $\square$

Theorem 2.27 *Let Γ be a virtually free group of rank $\mu(\Gamma) = 2$ and Γ contains a finite normal group H with factor group $\Gamma/H \cong F_2$, free of rank 2.Then with $|S| \equiv 1 \pmod p$ we have $b_\Gamma(\lambda) \equiv b_\Gamma(1) \not\equiv 0 \pmod p$ for all $\lambda \geq 1$ iff $3 \cdot m_\Gamma \equiv 1 \pmod p$ and $m_\Gamma + b_\Gamma(1) \equiv 0 \pmod p$. $b_\Gamma(\lambda) \equiv 0 \pmod p$ for all $\lambda \geq 1$ iff $b_\Gamma(1) \equiv 0 \pmod p$.*

Proof:
The proof is analogue to the one of Theorem 2.26. $\square$

Theorem 2.28 *Let Γ be a virtually free group of rank $\mu(\Gamma) = 2$ and of one of the following forms:*

a) *Γ is an HNN-extension $\Gamma = G_1 \underset{S,\sigma}{*}$ with finite base group G_1 and index $[G_1 : S] = 2$.*

b) *Γ is a free product $\Gamma = G_1 \underset{S}{*} G_2$ where G_1 contains a finite normal subset $H_1 \supset S$ with infinite-cyclic factor group and G_2 is finite with $[H_1 : S] = 2 = [G_2 : S]$.*

Then with $|S| \equiv 1 \pmod p$ $b_\Gamma(\lambda) \equiv b_\Gamma(1) \not\equiv 0 \pmod p$ for all $\lambda \geq 1$ iff $5|S| \equiv 1 \pmod p$ and $m_\Gamma + b_\Gamma(1) \equiv 0 \pmod p$. $b_\Gamma(\lambda) \equiv 0 \pmod p$ for all $\lambda \geq 1$ iff $b_\Gamma(1) \equiv 0 \pmod p$.

Proof:
Again, the proof is analogue to the one of Theorem 2.26. □

We focus now on finitely generated virtually free groups of rank $\mu(\Gamma) = 3$ and the conditions of $b_\Gamma(\lambda)$ to be constant modulo p. One can easily see, that the recursions are not integral in the cases 2.17 and 2.21. Besides those cases we can state conditions for all the remaining cases.

Theorem 2.29 *Let Γ be a virtually free group of rank $\mu(\Gamma) = 3$ and of one of the types described in Proposition 2.16, except the cases i) and v). Assuming $k \equiv 1 \pmod p$, we have $b_\Gamma(\lambda) \equiv b_\Gamma(1) \not\equiv 0 \pmod p$ for all $\lambda \geq 1$ iff all of the following three conditions are fulfilled:*

a) $m_\Gamma + b_\Gamma(1) \equiv 0 \pmod p$

b) $2\, m_\Gamma^2 + b_\Gamma(1)\,(3m_\Gamma + b_\Gamma(1)) \equiv 0 \pmod p$

c) *For the specific recursions we need*

i) *(2.18)* $17\, k^2 \equiv 1 \pmod p$
ii) *(2.19)* $13\, k^2 \equiv 1 \pmod p$
iii) *(2.20)* $22\, k^2 \equiv 1 \pmod p$
iv) *(2.22)* $67k^2 \equiv 1 \pmod p$
v) *(2.23)* $149k^2 \equiv 1 \pmod p$
vi) *(2.24)* $51k^2 \equiv 1 \pmod p$
vii) *(2.25)* $113\, k^2 \equiv 1 \pmod p$
viii) *(2.26)* $116\, k^2 \equiv 1 \pmod p$

$b_\Gamma(\lambda) \equiv 0 \pmod p$ for all $\lambda \geq 1$ if and only if $b_\Gamma(1) \equiv 0 \pmod p$.

Proof:
The proof follows the same arguments as the one of Theorem 2.26 and we suppress the calculations. We just present the equations for $b_\Gamma(\lambda)$ in order to read off the conditions stated above.

The recursion formulas write as:

i) (2.18)

$$\begin{aligned} b_\Gamma(t+1) \equiv \frac{1}{2}\ c\ (t-1)(t-2)\ [2\ m_\Gamma^2 + c\ (3m_\Gamma + c)] \\ + m_\Gamma\ (t-1)\ c\ (5(m_\Gamma + c)) + \frac{1}{4}\ 17\ m_\Gamma^2\ c \pmod p \end{aligned}$$

ii) (2.19)

$$\begin{aligned} b_\Gamma(t+1) \equiv \frac{1}{2}\ c\ (t-1)(t-2)\ [2\ m_\Gamma^2 + c\ (3m_\Gamma + c)] \\ + m_\Gamma\ (t-1)\ c\ \frac{1}{2}(9(m_\Gamma + c)) + \frac{1}{4}\ 13\ m_\Gamma^2\ c \pmod p \end{aligned}$$

iii) (2.20)

$$\begin{aligned} b_\Gamma(t+1) \equiv \frac{1}{2}\ c\ (t-1)(t-2)\ [2\ m_\Gamma^2 + c\ (3m_\Gamma + c)] \\ + m_\Gamma\ (t-1)\ c\ \frac{1}{2}(11(m_\Gamma + c)) + \frac{1}{2}\ 11\ m_\Gamma^2\ c \pmod p \end{aligned}$$

iv) (2.22)

$$\begin{aligned} b_\Gamma(t+1) \equiv \frac{1}{2}\ c\ (t-1)(t-2)\ [2\ m_\Gamma^2 + c\ (3m_\Gamma + c)] \\ + m_\Gamma\ (t-1)\ c\ (5(m_\Gamma + c)) + \frac{1}{16}\ 67\ m_\Gamma^2\ c \pmod p \end{aligned}$$

v) (2.23)

$$\begin{aligned} b_\Gamma(t+1) \equiv \frac{1}{2}\ c\ (t-1)(t-2)\ [2\ m_\Gamma^2 + c\ (3m_\Gamma + c)] \\ + m_\Gamma\ (t-1)\ c\ (5(m_\Gamma + c)) + \frac{1}{36}\ 149\ m_\Gamma^2\ c \pmod p \end{aligned}$$

vi) (2.24)

$$\begin{aligned} b_\Gamma(t+1) \equiv \frac{1}{2}\ c\ (t-1)(t-2)\ [2\ m_\Gamma^2 + c\ (3m_\Gamma + c)] \\ + m_\Gamma\ (t-1)\ c\ \frac{9}{2}\ (m_\Gamma + c) + \frac{1}{16}\ 51\ m_\Gamma^2\ c \pmod p \end{aligned}$$

vii) (2.25)

$$b_\Gamma(t+1) \equiv \frac{1}{2}\, c\,(t-1)(t-2)\,[2\,m_\Gamma^2 + c\,(3m_\Gamma + c)]$$
$$+\, m_\Gamma\,(t-1)\,c\,\frac{9}{2}\,(m_\Gamma + c) + \frac{1}{36}\,113\,m_\Gamma^2\,c \pmod p$$

viii) (2.26)

$$b_\Gamma(t+1) \equiv \frac{1}{2}\, c\,(t-1)(t-2)\,[2\,m_\Gamma^2 + c\,(3m_\Gamma + c)]$$
$$+\, m_\Gamma\,(t-1)\,c\,\frac{9}{2}\,(m_\Gamma + c) + \frac{1}{9}\,29\,m_\Gamma^2\,c \pmod p$$

Case a) $c \pmod p \equiv 0 \pmod p$ then $b_\Gamma(t+1) \equiv 0 \pmod p$.

Case b) $m_\Gamma + c \pmod p \equiv 0 \pmod p$ and $m_\Gamma^2 + \frac{1}{2}\,c \cdot (3m_\Gamma + c) \equiv 0 \pmod p$ and the appropriate condition of *c)* gives $b_\Gamma(t+1) \equiv c \pmod p \equiv b_\Gamma(1) \pmod p$.

$\square$

We found for the groups of type $\tau(\Gamma) = (m_\Gamma; 0, ..., 0, r-1)$ conditions for divisibility properties for the parity and for $p = 3$ for every rank $r \geq 1$.

Theorem 2.30 *Let Γ be a finitely generated virtually free group of rank $\mu(\Gamma) = r \geq 1$ and of type $\tau(\Gamma) = (m_\Gamma; 0, ..., 0, r-1)$. If $m_\Gamma \equiv 1 \pmod 2$ then $b_\Gamma(\lambda) \equiv 1 \pmod 2$ for all $\lambda \geq 1$.*

Proof:
By Proposition 1.30 and formula 1.16 we have

$$b_\Gamma(1) = A_0 = m_\Gamma \cdot \prod_{k \mid m_\Gamma} \prod_{\substack{1 \leq l \leq m_\Gamma \\ (m_\Gamma, l) = k}} l^{\zeta_k}.$$

For the type $\tau(\Gamma) = (m_\Gamma; 0, ..., 0, r-1)$ this rewrites with $m_\Gamma \equiv 1 \pmod 2$ to

$$A_0 = m_\Gamma \cdot m_\Gamma^{r-1} \equiv 1 \pmod 2.$$

Using the recursion formula 1.10 and induction on λ where the and $a_\Gamma(\lambda)$ are calculated by ([Mü89, (34)])

$$a_\Gamma(\lambda) = a_\Gamma(\lambda - 1) \cdot \frac{\mathcal{P}_\Gamma(\lambda)}{m_\Gamma \cdot \lambda}, \tag{2.43}$$

where the polynomial $\mathcal{P}_\Gamma(\lambda)$ is given by

$$\mathcal{P}_\Gamma(\lambda) := (m_\Gamma \cdot \lambda)^{1+\zeta_{m_\Gamma}} \prod_{k<m_\Gamma} \prod_{\substack{1\leq l\leq m_\Gamma \\ (m_\Gamma, l)=k}} ((\lambda-1)\cdot m_\Gamma + l)^{\zeta_k}.$$

Hence,

$$a_\Gamma(\lambda) = a_\Gamma(\lambda-1) \cdot \prod_{k|m_\Gamma} \prod_{\substack{1\leq l\leq m_\Gamma \\ (m_\Gamma, l)=k}} ((\lambda-1)\cdot m_\Gamma + l)^{\zeta_k},$$

and

$$a_\Gamma(\lambda) = a_\Gamma(\lambda-1)\cdot(\lambda\cdot m_\Gamma)^{r-1}, \quad \lambda \geq 1.$$

Case $r = 1$:

$$b_\Gamma(1) = A_0 = m_\Gamma \equiv 1 \pmod 2$$

Further, by the transformation formula 1.10

$$b_\Gamma(1) = m_\Gamma \cdot a_\Gamma(1) \equiv 1 \pmod 2$$

gives $a_\Gamma(1) \equiv 1 \pmod 2$. So we have $a_\Gamma(\lambda) = a_\Gamma(\lambda-1)$ and with $a_\Gamma(1) \equiv 1 \pmod 2$ we get $a_\Gamma(\lambda) \equiv 1 \pmod 2$ for all $\lambda \geq 1$.
Calculating the $b_\Gamma(\lambda)$ we obtain

$$\begin{aligned} b_\Gamma(\lambda) &= \lambda - \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) \\ &\equiv \lambda - (\lambda-1)\cdot 1 \pmod 2 \\ &\equiv 1 \pmod 2. \end{aligned}$$

Case $r > 1$:
Then

$$a_\Gamma(\lambda) = a_\Gamma(\lambda-1)\cdot(\lambda\cdot m_\Gamma)^{r-1}$$

and $a_\Gamma(\lambda) \equiv \lambda!^{r-1} \pmod 2$. We calculate $a_\Gamma(\lambda) \equiv 0 \pmod 2$ for $\lambda \geq 2$ and $a_\Gamma(1) \equiv 1 \pmod 2$. With that values and $b_\Gamma(1) = m_\Gamma \cdot a_\Gamma(1) \equiv 1 \pmod 2$ we calculate for $\lambda \geq 2$

$$\begin{aligned} b_\Gamma(\lambda) &= m_\Gamma \cdot \lambda \cdot a_\Gamma(\lambda) - \sum_{\mu=1}^{\lambda-1} a_\Gamma(\lambda-\mu) b_\Gamma(\mu) \\ &\equiv a_\Gamma(1)\cdot b_\Gamma(\lambda-1) \pmod 2 \\ &\equiv b_\Gamma(\lambda-1) \pmod 2. \end{aligned} \tag{2.44}$$

With $b_\Gamma(1) \equiv 1 \pmod 2$ it follows that $b_\Gamma(\lambda) \equiv 1 \pmod 2$ for all $\lambda \geq 1$. $\square$

Investigating this type for $p = 3$ we detect a divisibility behavior depending on the parity of the rank r. With $m_\Gamma \equiv 1 \pmod 3$ either $b_\Gamma(\lambda)$ is periodic with length 8 or constant modulo p.

Theorem 2.31 *Let Γ be a virtually free group of rank $\mu(\Gamma) = r \geq 1$ and of type $\tau(\Gamma) = (m_\Gamma; 0, ..., 0, r-1)$ with $m_\Gamma \equiv 1 \pmod 3$. If $2 \nmid r$ then $b_\Gamma(\lambda) \equiv 1 \pmod 3$ and if $2 \mid r$ then*

$$b_\Gamma(\lambda) \equiv \begin{cases} 0 \pmod 3 & \text{if } \lambda \equiv 2, 6 \pmod 8 \\ 1 \pmod 3 & \text{if } \lambda \equiv 0, 1, 3 \pmod 8 \\ 2 \pmod 3 & \text{if } \lambda \equiv 4, 5, 7 \pmod 8 \end{cases} \tag{2.45}$$

Proof:
By Proposition 1.30 and formula 1.16 we have for the type $\tau(\Gamma) = (m_\Gamma; 0, ..., 0, r-1)$

$$\begin{aligned} b_\Gamma(1) = A_0 &= m_\Gamma \cdot \prod_{k \mid m_\Gamma} \prod_{\substack{1 \leq l \leq m_\Gamma \\ (m_\Gamma, l) = k}} l^{\zeta_k} \\ &= m_\Gamma \cdot m_\Gamma^{r-1} \\ &\equiv 1 \pmod 3. \end{aligned}$$

This gives by 2.43

$$a_\Gamma(\lambda) = a_\Gamma(\lambda - 1) \cdot (\lambda \cdot m_\Gamma)^{r-1}, \quad \lambda \geq 1.$$

Case $r = 1$:

$$b_\Gamma(1) = A_0 = m_\Gamma \equiv 1 \pmod 3$$

Further, by the transformation formula 1.10

$$b_\Gamma(1) = m_\Gamma \cdot a_\Gamma(1) \equiv 1 \pmod 3$$

gives $a_\Gamma(1) \equiv 1 \pmod 3$. So we have $a_\Gamma(\lambda) = a_\Gamma(\lambda - 1)$ and with $a_\Gamma(1) \equiv 1 \pmod 3$ are $a_\Gamma(\lambda) \equiv 1 \pmod 3$ for all $\lambda \geq 1$. Calculating the $b_\Gamma(\lambda)$

$$\begin{aligned} b_\Gamma(\lambda) &= \lambda - \sum_{\mu=1}^{\lambda-1} b_\Gamma(\mu) \\ &\equiv \lambda - (\lambda - 1) \cdot 1 \pmod 3 \\ &\equiv 1 \pmod 3. \end{aligned}$$

Case $r > 1$:
Then

$$a_\Gamma(\lambda) = a_\Gamma(\lambda - 1) \cdot (\lambda \cdot m_\Gamma)^{r-1}$$

and $a_\Gamma(\lambda) \equiv \lambda!^{r-1} \pmod 3$. We calculate $a_\Gamma(\lambda) \equiv 0 \pmod 3$ for $\lambda \geq 3$ and $a_\Gamma(1) \equiv 1 \pmod 3$. Further for $a_\Gamma(2)$ we obtain

$$a_\Gamma(2) \equiv \begin{cases} 1 & \pmod 3 \text{ if } 2 \mid r-1 \\ 2 & \pmod 3 \text{ if } 2 \nmid r-1 \end{cases}$$

For our induction again $b_\Gamma(1) = A_0 \equiv 1 \pmod 3$ and we calculate $b_\Gamma(2)$ as

$$\begin{aligned} b_\Gamma(2) &= m_\Gamma \cdot 2 \cdot a_\Gamma(2) - a_\Gamma(1) \cdot b_\Gamma(1) \\ &\equiv \begin{cases} 1 & \pmod 3 \text{ if } 2 \mid r-1 \\ 0 & \pmod 3 \text{ if } 2 \nmid r-1 \end{cases} \end{aligned} \tag{2.46}$$

And for $\lambda \geq 3$ we obtain

$$\begin{aligned} b_\Gamma(\lambda) &= m_\Gamma \cdot \lambda \cdot a_\Gamma(\lambda) - \sum_{\mu=1}^{\lambda-1} a_\Gamma(\lambda-\mu) b_\Gamma(\mu) \\ &\equiv -(a_\Gamma(2) \cdot b_\Gamma(\lambda-2) + a_\Gamma(1) b_\Gamma(\lambda-1)) \pmod 3 \\ &\equiv -\begin{cases} b_\Gamma(\lambda-2) + b_\Gamma(\lambda-1) & \pmod 3 \text{ if } 2 \mid r-1 \\ 2 \cdot b_\Gamma(\lambda-2) + b_\Gamma(\lambda-1) & \pmod 3 \text{ if } 2 \nmid r-1 \end{cases} \end{aligned} \tag{2.47}$$

Case a) $b_\Gamma(\lambda) \equiv -(b_\Gamma(\lambda-2) + b_\Gamma(\lambda-1)) \pmod 3$, $\quad 2 \mid r-1$. Then by induction on λ and initial values $b_\Gamma(1) = b_\Gamma(2) \equiv 1 \pmod 3$ and the assumption $b_\Gamma(\lambda) \equiv 1 \pmod 3$ for all $\lambda \leq t-1$, we have $b_\Gamma(t) \equiv -(b_\Gamma(t-2) + b_\Gamma(t-1)) \pmod 3 \equiv 1 \pmod 3$.

Case b) $b_\Gamma(\lambda) \equiv -(2 \cdot b_\Gamma(\lambda-2) + b_\Gamma(\lambda-1)) \pmod 3$, $\quad 2 \nmid r-1$. Here we calculate the first 10 values for λ:

λ	$b_\Gamma(\lambda)$
1	1 (3)
2	0 (3)
3	1 (3)
4	2 (3)
5	2 (3)
6	0 (3)
7	2 (3)
8	1 (3)
9	1 (3)
10	0 (3)

Since $b_\Gamma(\lambda)$ depends only on the two previous values, the period of length 8 modulo 3 is given. $\square$

3 Divisibility properties of the function $s_\Gamma(n)$

We recall from Chapter 1 that $s_\Gamma(n)$ counts all subgroups of given index in Γ. Further, the problem of counting the subgroups of a finitely generated virtually free group Γ is connected to the enumeration of Γ-actions $|Hom(\Gamma, S_n)|$ on finite sets via the formula 1.6.

In this chapter we consider the special case when Γ is a free product $\Gamma = G_1 * G_2$ of two finite groups. Then

$$|Hom(\Gamma, S_n)| = |Hom(G_1, S_n)| \cdot |Hom(G_2, S_n)|,$$

where $|Hom(G_i, S_n)|$ denotes the number of homomorphisms of the finite group G_i into S_n. The exponential generating function for finite groups is given by formula 1.8.

Furthermore, we can give for finite groups G a recursive description:

Proposition 3.1 *Let (Γ, Σ, S) be an admissible triple where $S = \mathbb{N}$, $\Sigma = \varnothing$ and G is a finite group with $|G| = m$. The number $|Hom(G, S_{n+1})|$ of G-actions on an $[n+1]$-set S_{n+1} can be calculated as:*

$$|Hom(G, S_{n+1})| = \sum_{d|m} \binom{n}{n-d+1} (d-1)!\; |Hom(G, S_{n-d+1})|\; s_G(d), \tag{3.1}$$

where $n \geq 0$ and $|Hom(G, S_0)| = 1$.

Proof:
We consider the special case 1.8 of Theorem 1.18

$$\sum_{n=0}^{\infty} \frac{|Hom(G, S_n)|}{n!}\, z^n = \exp\left(\sum_{d|m} \frac{s_G(d)}{d}\, z^d\right), \qquad m = |G|. \tag{3.2}$$

Calculating the first derivative of $\exp\left(\sum_{d|m} \frac{s_G(d)}{d}\, z^d\right)$ we obtain

$$\begin{aligned}
\left\{\exp\left(\sum_{d|m} \frac{s_G(d)}{d}\, z^d\right)\right\}' &= \exp\left(\sum_{d|m} \frac{s_G(d)}{d}\, z^d\right) \cdot \left(\sum_{d|m} s_G(d)\, z^{d-1}\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{|Hom(G, S_n)|}{n!}\, z^n\right)' \\
&= \sum_{n=0}^{\infty} \frac{|Hom(G, S_n)|}{(n-1)!}\, z^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{|Hom(G, S_{n+1})|}{n!}\, z^n
\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} \frac{|Hom(G, S_{n+1})|}{n!} z^n = \exp\left(\sum_{d|m} \frac{s_G(d)}{d} z^d\right) \cdot \left(\sum_{d|m} s_G(d) z^{d-1}\right)$$
$$= \sum_{n=0}^{\infty} \frac{|Hom(G, S_n)|}{n!} z^n \cdot \left(\sum_{d|m} s_G(d) z^{d-1}\right).$$

Comparing coefficients we get

$$\frac{|Hom(G, S_{n+1})|}{n!} = \sum_{d|m} s_G(d) \frac{|Hom(G, S_{n-d+1})|}{(n-d+1)!},$$

or, equivalently,

$$|Hom(G, S_{n+1})| = \sum_{d|m} \binom{n}{n-d+1} (d-1)! \, |Hom(G, S_{n-d+1})| \, s_G(d).$$

□

The principle for obtaining a recurrence for $s_\Gamma(n)$ is analogue to the case of $b_\Gamma(\lambda)$ with one significant difference. Here, we do not have a general differential equation with explicit coefficients for the generating function of $|Hom(\Gamma, S_n)|$. We can only use the transformation formula 1.9 and the generating functions for $s_\Gamma(n)$ and $|Hom(\Gamma, S_n)|$ in order to obtain a differential equation first for the generating function of $|Hom(\Gamma, S_n)|$ and then via the transformation formula 1.9 and coefficient comparing the recurrence for $s_\Gamma(n)$. From now on we consider cyclic groups C_p for the finite groups G_i, where p is an odd prime. The number of homomorphisms of a free product of cyclic groups $\Gamma = C_{p_1} * C_{p_2}$ into S_n is the product of the number of homomorphisms of the cyclic groups C_{p_i} into S_n. For each cyclic group C_{p_i} we have by Proposition 3.1 a linear recursion with polynomial coefficients for $|Hom(C_{p_i}, S_n)|$. For $|Hom(\Gamma, S_n)|$ we also need a linear recurrence with polynomial coefficients as product sequence from the two recurrences for $|Hom(C_{p_i}, S_n)|$, $i = 1, 2$, which is the pointwise product, called the Hadamard product. With this recurrence for $|Hom(\Gamma, S_n)|$ we consider Hall's formula 1.9 with $h_\Gamma(n) := \frac{|Hom(\Gamma, S_n)|}{n!}$

$$s_\Gamma(n) = n \cdot h_\Gamma(n) - \sum_{k=1}^{n-1} h_\Gamma(n-k) \, s_\Gamma(k)$$

where $s_\Gamma(1) = 1$.

Let $G(z)$ denote the ordinary generating function for $s_\Gamma(n)$:

$$G(z) = \sum_{n=0}^{\infty} s_\Gamma(n+1)\ z^n$$

and the generating function for the number of homomorphisms of Γ into S_n is the exponential generating function

$$F(z) = \sum_{n=0}^{\infty} \frac{|Hom(\Gamma, S_n)|}{n!}, \quad |Hom(\Gamma, S_0)| = 1.$$

Then rewriting 1.9 we get the identity

$$G(z) = \frac{F'(z)}{F(z)} \tag{3.3}$$

which transforms the differential equation for $F(z)$ into one for $G(z)$ and by comparing coefficients we get then a recurrence for $s_\Gamma(n)$.

3.1 The Hadamard product of P-recursive functions

First, we present some results dealing with P-recursive functions, their Hadamard product and the existence of a differentiably finite power series with P-recursive linear coefficients, like we are going to use for our considerations.

Definition 3.2 *A formal power series* $\sum f(n)\, z^n$ *is said to be differentiably finite (D-finite) if it satisfies a linear differential equation with polynomial coefficients.*

We focus on such polynomial coefficients which are recursively defined.

Definition 3.3 *A function* $f : \mathbb{N} \to \mathbb{C}$ *is called polynomial recursive (P-recursive) if there exist finitely many polynomials* $P_0(n), ..., P_k(n)$ *with* $P_k(n) \neq 0$*, such that for all* $n \in \mathbb{N}$

$$P_k(n)\, f(n+k) + P_{k-1}(n)\, f(n+k-1) + ... + P_0(n)\, f(n) = 0.$$

Theorem 3.4 ([Sta80, Thm. 1.5]) *The formal power series* $y = \sum_{n \geq 0} f(n)\, z^n \in \mathbb{C}[[z]]$ *is D-finite if and only if* $f(n)$ *is P-recursive.*

Definition 3.5 *If* $x = \sum f(n)\, z^n$ *and* $y = \sum g(n)\, z^n$ *then the Hadamard product* $x * y$ *is the power series*

$$x * y = \sum f(n)\, g(n)\, z^n.$$

Theorem 3.6 ([Sta80, Thm. 2.10]) *Let* x *and* y *be D-finite power series. Then* $x * y$ *is D-finite. Equivalently, if* $f(n)$ *and* $g(n)$ *are P-recursive, then* $f(n)\, g(n)$ *is P-recursive.*

The question of how to find the Hadamard product of two linear recurrences is in principle answered by an algorithm *rec* $*$ *rec* implemented in the *gfun-package* for *Maple* [SZ92]:

Step 0	Initializing: Two linear recurrences $L_1(f_n, ..., f_{n+p}) = 0$ and $L_2(g_n, ..., g_{n+q}) = 0$
Step 1	Define $h_n = f_n\, g_n$
Step 2	Shift repetitively this equation ($n \longmapsto n+1$), reducing f_{n+p} and g_{n+q} according to L_1 and L_2.
Step 3	Once $p\,q+1$ equations have been built, Gaussian elimination yields a linear dependency between $h_n, h_{n+1}, ..., h_{n+pq}$.

Figure 4: Algorithm rec $*$ rec of the gfun-package under Maple

Unfortunately, this algorithm does not have a good performance even for small cases, where the recursion depth is low. With this algorithm we were not able to treat the Hadamard product for the simplest of our cases of linear recurrences. By a thorough analysis of the recurrences and occurring substitutions we were able to provide more efficient procedures for the special cases considered in this chapter.

Hadamard Product $\boldsymbol{C_p * C_p}$

Initializing

With $f_n = f_{n-1} + (n-1) \cdot ... \cdot (n-p+1) f_{n-p}$ we set
A: $\quad h_n = h_{n-1} + a_{n-1}(1)\ f_{n-1} f_{n-p} + a_{n-p}(1)\ h_{n-p}$

The $a_n(k),\ b_n(k)$ denote polynomial coefficients modified in the k-th iteration

Step 0

A:
$$\begin{aligned} a_n(k)\ h_n &= a_{n-1}(k)\ h_{n-1} + ... + a_{n-kp}(k)\ h_{n-kp} \\ &\quad + a_{n-k}(k)\ f_{n-k} f_{n-p} \end{aligned}$$

If $k \geq 2$ then
B: $b_{n-k+1-p}(k)\ f_{n-k+1} f_{n-p} = b_{n-1}(k)\ h_{n-1} + ... + b_{n-kp}(k)\ h_{n-kp}$

Step 1 Substitution of f_{n-k} in $f_{n-k} f_{n-p}$ in equation A

C:
$$\begin{aligned} a_n(k)\ h_n &= a_{n-1}(k)\ h_{n-1} + ... + a_{n-kp}(k)\ h_{n-kp} \\ &\quad + a_{n-k}(k)\ \{f_{n-k-1} f_{n-p} + a_{n-k-p}(k+1)\ f_{n-k-p} f_{n-p}\} \end{aligned}$$

D:
$$\begin{aligned} a_{n-k}(k)\ f_{n-k-1} f_{n-p} &= a_n(k)\ h_n - a_{n-1}(k)\ h_{n-1} - ... - a_{n-kp}(k)\ h_{n-kp} \\ &\quad - a_{n-k}(k)\ a_{n-k-p}(k+1)\ f_{n-k-p} f_{n-p} \end{aligned}$$

Step 2 Substitution of f_{n-p} in $f_{n-k-p} f_{n-p}$ in equation C

E:
$$\begin{aligned} a_n(k)\ h_n &= a_{n-1}(k)\ h_{n-1} + ... + a_{n-kp}(k)\ h_{n-kp} \\ &\quad + a_{n-k}(k)\ f_{n-k-1} f_{n-p} + a_{n-k}(k)\ a_{n-k-p}(k+1) \cdot \\ &\quad \cdot \{f_{n-p-1} f_{n-k-p} + a_{n-2p}(k+1) f_{n-k-p} f_{n-2p}\} \end{aligned}$$

Step 3 Equation A with $n := n - p$

F:
$$a_{n-p}(k+1)\ h_{n-p} = a_{n-p-1}(k+1)\ h_{n-p-1} + ... + a_{n-(k+1)p}(k+1) \cdot \cdot\ h_{n-(k+1)p} + a'_{n-k-p}(k+1)\ f_{n-k-p} f_{n-2p}$$

Equation B with $n := n - 1$

G:
$$b_{n-k-p}(k+1)\ f_{n-k-p} f_{n-p-1} = b_{n-p-1}(k+1)\ h_{n-p-1} + ... + b_{n-kp-1}(k+1)\ h_{n-kp-1}$$

Linear combination of equation E with F and G
$E := a'_{n-k-p}(k+1)\ E - a_{n-k}(k)\ a_{n-k-p}(k+1)\ a_{n-2p}(k+1)\ F$

$E := b_{n-k-p}(k+1)\ E - a_{n-k}(k)\ a_{n-k-p}(k+1)\ a'_{n-k-p}(k+1)\ G$
gives

H:
$$\begin{aligned} a_n(k+1)\ h_n &= a_{n-1}(k+1)\ h_{n-1} + ... + a_{n-kp}(k+1)\ h_{n-kp} \\ &+ a_{n-kp-1}(k+1)\ h_{n-kp-1} + a_{n-(k+1)p}(k+1)\ h_{n-(k+1)p} \\ &+ a_{n-k}(k+1)\ f_{n-k-1} f_{n-p} \end{aligned}$$

If $k + 1 = p$
$$h_{n-p} = f_{n-(k+1)} f_{n-p}$$
Equation H is the desired recursion
Stop

Else insertion of equation D in H:

I:
$$\begin{aligned} b_{n-k-p}(k+1)\ f_{n-k-p} f_{n-p} &= b_{n-1}(k+1)\ h_{n-1} + ... \\ &+ b_{n-kp-1}(k+1)\ h_{n-kp-1} \\ &+ b_{n-(k+1)p}(k+1)\ h_{n-(k+1)p} \end{aligned}$$

A := H
B := I
$\rightarrow$ Step 1

Theorem 3.7 *Given an arithmetic function $f(n)$ and a recursion of the form $f_n = f_{n-1} + (n-1) \cdot ... \cdot (n-p+1) f_{n-p}$, the algorithm Hadamard product $C_p * C_p$ produces a recurrence with polynomial coefficients for the Hadamard product $h_n = f_n \cdot f_n$. The complexity of the algorithm measured in the number of equations is $O(4p)$.*

Proof:
Induction on k.
For $k = 1$ we have equation A: $h_n = h_{n-1} + a_{n-1}(1)\ f_{n-1} f_{n-p} + a_{n-p}(1)\ h_{n-p}$. In order to obtain equation B we substitute f_{n-1} in $f_{n-1} f_{n-p}$ (Step 1). This gives equation C.

$$C: \quad h_n = h_{n-1} + a_{n-1}(1) \{ f_{n-2} f_{n-p} + a_{n-1-p}(2)\ f_{n-1-p} f_{n-p} \} + a_{n-p}(1)\ h_{n-p}.$$

Further,

$$D: \quad a_{n-1}(1) f_{n-2} f_{n-p} = h_n - h_{n-1} - a_{n-p}(1)\ h_{n-p}\ a_{n-1}(1)\ a_{n-1-p}(2)\ f_{n-1-p} f_{n-p}.$$

Step 2, substitution of f_{n-p} in $f_{n-1-p} f_{n-p}$ of equation C:

$$\begin{aligned} E: \quad h_n = {} & h_{n-1} + a_{n-1}(1)\ f_{n-2} f_{n-p} + a_{n-p}(1)\ h_{n-p} \\ & + a_{n-1}(1)\ a_{n-1-p}(2)\ h_{n-p-1} \\ & + a_{n-1}(1)\ a_{n-1-p}(2)\ a_{n-p}(2)\ f_{n-1-p} f_{n-2p}. \end{aligned}$$

Applying equation A with $n := n - p$ (Step 3) gives

$$F: \quad h_{n-p} = h_{n-p-1} + a_{n-2p}(2)\ f_{n-1-p} f_{n-2p} + a_{n-2p}(2)\ h_{n-2p}.$$

Linear combination with equation E results in equation H:

$$\begin{aligned} H: \quad a_{n-2p}(2)\ h_n = {} & a_{n-2p}(2)\ h_{n-1} + (a_{n-2p}(2) + a_{n-1}(1)\ a_{n-1-p}(2)\ a_{n-p}(2))\, h_{n-p} \\ & + a_{n-1}(1)\ a_{n-1-p}(2)\ (a_{n-2p}(2) - a_{n-p}(2))\, h_{n-p-1} \\ & - a_{n-1}(1)\ a_{n-1-p}(2)\ a_{n-p}(2)\ h_{n-2p} \\ & + a_{n-2p}(2)\ a_{n-1}(1)\ f_{n-2} f_{n-p}. \end{aligned}$$

Finally, the insertion of equation D in H gives equation I:

$$I: \quad a_{n-2p}(2)\ f_{n-1-p} f_{n-p} = a_{n-p}(2)\ h_{n-p} + (a_{n-2p}(2) - a_{n-p}(2))\, h_{n-p-1} - a_{n-p}(2)\ h_{n-2p}.$$

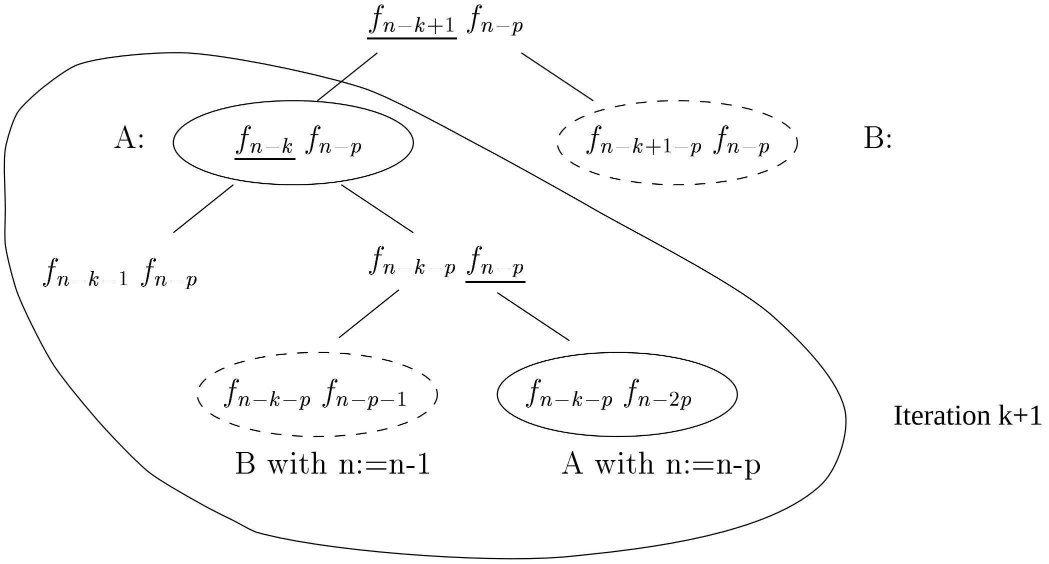
Setting $A := H$ and $B := I$ is the end of the first iteration.

Induction step $k \to k+1$:
We start with our initial equations A and B.

$$\begin{aligned} A: \quad a_n(k)\ h_n = {} & a_{n-1}(k)\ h_{n-1} + ... + a_{n-kp}(k)\ h_{n-kp} \\ & + a_{n-k}(k)\ f_{n-k} f_{n-p} \end{aligned}$$

$$B: \quad b_{n-k+1-p}(k)\ f_{n-k+1} f_{n-p} = b_{n-1}(k)\ h_{n-1} + ... + b_{n-kp}(k)\ h_{n-kp}$$

We do not perform the algorithm step by step again, but show the substitutions in the (k+1)-th's iteration via a binary tree, neglecting the coefficients a_{n-i}. The substituted f_{n-i} is underlined.



First, we substitute f_{n-k} in $f_{n-k}f_{n-p}$ as $f_{n-k} = f_{n-k-1} + a_{n-k-p}\ f_{n-k-p}f_{n-p}$ which gives an equation C for h_n with the products $f_{n-k-1}f_{n-p}$ and $f_{n-k-p}f_{n-p}$ (Step 1), and expressing $f_{n-k-1}f_{n-p}$ (equation D) using equation C. Once more, we substitute now f_{n-p} in $f_{n-k-p}f_{n-p}$ (Step 2) and obtain equation E with products $f_{n-k-p}f_{n-p-1}$ and $f_{n-k-p}f_{n-2p}$. We can replace both terms by using the equations A and B:

$$f_{n-k-p}f_{n-p-1} \simeq \text{equation B with n:=n-1}$$
$$f_{n-k-p}f_{n-2p} \simeq \text{equation A with n:=n-p.}$$

Linear combinations of those 3 equations (Step 3) results in an equation H only with $f_{n-k-1}f_{n-p}$ as a "mixed" (not h_{n-i}) term which replaces equation A. Expressing equation H by use of equation D gives also an equation for $f_{n-k-p}f_{n-p}$ which replaces equation B in the next iteration.

Iterating this procedure $(p-1)$ times gives an equation H:

$$\begin{aligned} h_n &= a_{n-1}(p-1)\ h_{n-1} + \ldots + a_{n-(p-1)k}(p-1)\ h_{n-(p-1)p} \\ &\quad + a_{n-p^2}(p-1)\ f_{n-p}f_{n-p} \end{aligned}$$

and with $f_{n-p}f_{n-p} = h_{n-p}$ the algorithm terminates.

To compare this algorithm with the one for general products of P-recursive functions used in the gfun-package, we count the equations we have built in every iteration. In

Step 1 and Step 3 we perform the substitutions and in Step 4 we have equations A and B with replaced values for n. This gives $4p$ equations instead of p^2+1 in the general algorithm. We do not consider the 3 linear combinations in Step 4 and Step 5 since the general algorithm performs after building the p^2+1 equations and substitutions Gaussian elimination. $\square$

We present now an analogous procedure to solve the problem of determining the Hadamard product of two recursions of the form

$$\begin{aligned} f_n &= f_{n-1} + (n-1) \cdot f_{n-2} \\ g_n &= g_{n-1} + (n-1) \cdot \ldots \cdot (n-p+1) \cdot g_{n-p}. \end{aligned} \tag{3.4}$$

Hadamard Product $\boldsymbol{C_2 * C_p}$

Initializing

$$
\begin{aligned}
h_n &= f_n g_n \\
&= h_{n-1} + (n-1)\cdot \ldots \cdot (n-p+1)\ f_{n-1}g_{n-p} \\
&+ (n-1)^2(n-2)\cdot \ldots \cdot (n-p+1)\ f_{n-2}g_{n-p} \\
&+ (n-1)\ f_{n-2}g_{n-1}
\end{aligned}
$$

Step 1 Substitution of f_{n-1} in $f_{n-1}g_{n-p}$ and handling $f_{n-2}g_{n-p}$ separated.

$$
\begin{aligned}
f_{n-1}g_{n-p} &= f_{n-2}g_{n-p} + (n-2)\ f_{n-3}g_{n-p} \\
&= f_{n-3}g_{n-p} + (n-3)\ f_{n-4}g_{n-p} + (n-2)\ \{f_{n-4}g_{n-p} \\
&+(n-4)\ f_{n-5}g_{n-p}\} \\
&\vdots \\
&= c_1^1(n-1)\ h_{n-p} + c_2^1(n-1)\ f_{n-p-1}g_{n-p} \\
&+ c_1^2(n-1)\ h_{n-p} + c_2^2(n-1)\ f_{n-p-1}g_{n-p} \\
&= C_1(n-1)\ h_{n-p} + C_2(n-1)\ f_{n-p-1}g_{n-p}
\end{aligned}
$$

$c_i^j(n-1)$ and $C_i(n-1)$ denote polynomial coefficients

Step 2

$$
\begin{aligned}
h_n &= h_{n-1} \\
&+ (n-1)\cdot \ldots \cdot (n-p+1)\left\{(C_1(n-1)\ + (n-1)\ c_1^1(n-1))\ h_{n-p}\right. \qquad (*) \\
&\left.+(C_2(n-1)\ + (n-1)\ c_2^1(n-1))\ f_{n-p-1}g_{n-p}\right\} \\
&+ (n-1)\ f_{n-2}g_{n-1}
\end{aligned}
$$

Step 3 General substitution of g_n in $f_{n-1}g_n$

$$
f_{n-1}g_n = f_{n-1}g_{n-1} + (n-1)\cdot \ldots \cdot (n-p+1)\ f_{n-1}g_{n-p}
$$

Applying the substitution of $f_{n-1}g_{n-p}$ (Step 1)

$$
\begin{aligned}
f_{n-1}g_n &= h_{n-1} + (n-1)\cdot \ldots \cdot (n-p+1)\cdot \{C_1(n-1)\ h_{n-p} \qquad (**) \\
&+ C_2(n-1)\ f_{n-p-1}g_{n-p}\}
\end{aligned}
$$

Step 4 Substitution of $f_{n-p-1}g_{n-p}$ and $f_{n-2}g_{n-1}$ in $(*)$ with form $(**)$

$$
\begin{aligned}
f_{n-p-1}g_{n-p} &= h_{n-p-1} + (n-p-1)\cdot \ldots \cdot \\
&\cdot (n-2p+1)\cdot \{C_1(n-p-1)\ h_{n-2p}+ \\
&C_2(n-p-1)\ f_{n-2p-1}g_{n-2p}\}
\end{aligned}
$$

$$f_{n-2}g_{n-1} = h_{n-2} + (n-2)\cdot ... \cdot (n-p)\cdot \{C_1(n-2)\; h_{n-p-1} + C_2(n-2)\; f_{n-p-2}g_{n-p-1}\}$$

Plugging those formulas into the equation for h_n

(4.1)
$$h_n = h_{n-1} + (n-1)\cdot ... \cdot (n-p+1)\,(C_1(n-1) + (n-1)\cdot c_1^1(n-1)\big)\; h_{n-p} + (n-1)\cdot ... \cdot (n-p+1)\,\big(C_2(n-1) + (n-1)\; c_2^1(n-1)\big)\cdot \cdot \{h_{n-p-1} + (n-p-1)\cdot ... \cdot (n-2p+1)\cdot \{C_1(n-p-1)\; h_{n-2p} + C_2(n-p-1)\; f_{n-2p-1}g_{n-2p}\}\} + (n-1)\,\{h_{n-2} + (n-2)\cdot ... \cdot (n-p)\cdot \{C_1(n-2)\; h_{n-p-1} + C_2(n-2)\; f_{n-p-2}g_{n-p-1}\}\}$$

Step 5 $(*)$ with $n := n-p$

(5.1)
$$h_{n-p} = h_{n-p-1} + (n-p-1)\cdot ... \cdot (n-2p+1)\cdot \cdot \big\{\big(C_1(n-p) + (n-p-1)\; c_1^1(n-p)\big)\; h_{n-2p} + (C_2(n-p) + (n-p-1)\; c_2^1(n-p)\big)\;\cdot f_{n-2p-1}g_{n-2p}\big\} + (n-p-1)\; f_{n-p-2}g_{n-p-1}$$

Linear combination of (4.1) and (5.1) eliminates one "mixed" product, e.g. $f_{n-p-2}g_{n-p-1}$

Step 6 Elimination of the remaining "mixed" product $f_{n-2p-1}g_{n-2p}$

(6.1) $h_n = h_{n-1} + ... + C\cdot f_{n-2p-1}g_{n-2p}$
C is a polynomially constant

$(**)$ with $n := n-2p$

(6.2)
$$f_{n-2p-1}g_{n-2p} = h_{n-2p-1} + (n-2p-1)\cdot ... \cdot (n-3p+1)\cdot \cdot \{C_1(n-2p-1)\; h_{n-3p} + C_2(n-2p-1)\; f_{n-3p-1}g_{n-3p}\}$$

(6.3) Equation (6.1) with $n := n-p$

Replacing $f_{n-2p-1}g_{n-2p}$ in (6.1) with (6.2)

Linear combination of equation (6.1) and (6.3)

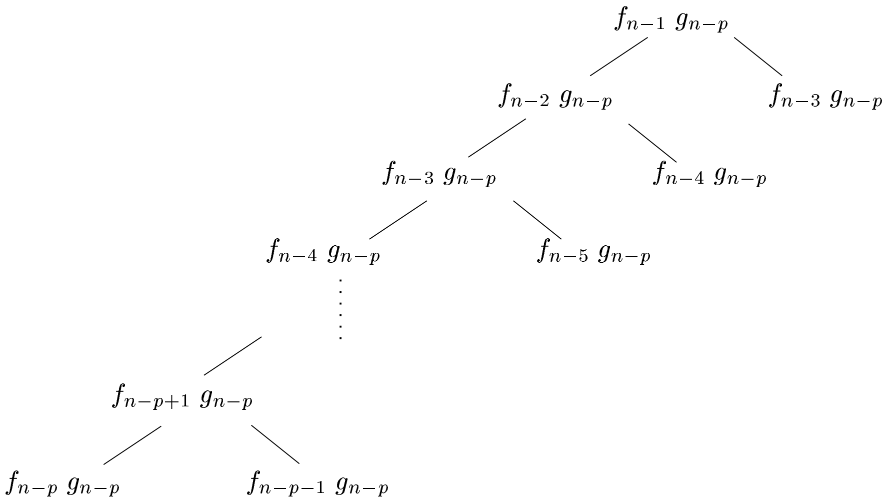
Theorem 3.8 *Given two arithmetic functions $f(n)$, $g(n)$ and recursions of the form*

$$f_n = f_{n-1} + (n-1) \cdot f_{n-2}$$

$$g_n = g_{n-1} + (n-1) \cdot \ldots \cdot (n-p+1) \cdot g_{n-p}.$$

The algorithm Hadamard product $C_2 * C_p$ produces a recurrence with polynomial coefficients for the Hadamard product $h_n = f_n \cdot g_n$. The algorithm requires p equations.

Proof:
In Step 1 we break down $f_{n-1}g_{n-p}$ (respectively $f_{n-2}g_{n-p}$ as part of the substitution of $f_{n-1}g_{n-p}$) using the iterated substitution of $f_{n-i}, i = 1, ..., p-1$. With the recursion of $f_n = f_{n-1} + (n-1)f_{n-2}$ we get in every substitution always $f_{n-k}g_{n-p} = f_{n-k-1}g_{n-p} + (n-k-1)f_{n-k-2}g_{n-p}$. The iterated substitution is best shown in a binary tree (neglecting the coefficients).



We need to consider only the $f_{n-k-1}g_{n-p}$ in every iteration, since $f_{n-k-2}g_{n-p}$ is a subtree of $f_{n-k-1}g_{n-p}$, the same substitutions are involved, but the tree with $f_{n-k-2}g_{n-p}$ will terminate earlier and finishes by the definition of the f_n also with the products $f_{n-p}g_{n-p}$ and $f_{n-p-1}g_{n-p}$. The idea of handling the subtree with root $f_{n-k-2}g_{n-p}$ distinct from the remaining substitutions is that we have to add $f_{n-k-2}g_{n-p}$ as the second mixed term in the recursion for h_n and we need the substituted $f_{n-1}g_{n-p}$ for $f_{n-2}g_{n-1}$ again.

After $(p-1)$ substitutions we can express $f_{n-1}g_{n-p}$ and $f_{n-2}g_{n-p}$ only as multiples of h_{n-p} and $f_{n-p-1}g_{n-p}$ by the definition of f_n. Hence, h_n rewrites to (Step 2)

$$\begin{aligned} h_n = {} & h_{n-1} \\ & + (n-1) \cdot \ldots \cdot (n-p+1) \left\{ (C_1(n-1) \ + (n-1)\ c_1^1(n-1))\ h_{n-p} \right. \\ & \left. +(C_2(n-1) \ + (n-1)\ c_2^1(n-1))\ f_{n-p-1} g_{n-p} \right\} \\ & + (n-1)\ f_{n-2} g_{n-1}. \end{aligned} \tag{3.5}$$

Here, both products $f_{n-p-1}g_{n-p}$ and $f_{n-2}g_{n-1}$ are of the general form $f_{n-1}g_n$, and we can substitute (Step 3)

$$f_{n-1}g_n = f_{n-1}g_{n-1} + (n-1) \cdot \ldots \cdot (n-p+1)\ f_{n-1}g_{n-p}, \tag{3.6}$$

where we already calculated $f_{n-1}g_{n-p}$ in Step 1.

Expressing $f_{n-p-1}g_{n-p}$ and $f_{n-2}g_{n-1}$ with 3.6 and substituting in h_n, we get equation (4.1). Considering 3.5 with $n := n-p$ we get an equation for h_{n-p} (5.1) also with $f_{n-2p-1}g_{n-2p}$ as mixed products. Linear combination of those two equations (4.1) and (5.1) eliminates the product $f_{n-2p-1}g_{n-2p}$. For h_n remains only the product $f_{n-p-2}g_{n-p-1}$ which we can eliminate by linear combination of 3.6 with $n := n-2p$, resulting in formula (6.2) and equation (6.1) with $n := n-p$. The algorithm terminates with a recursion for $h_n = f_n g_n$.

Hadamard product $C_2 * C_p$ produces in Step 1 $(p-1)$ equations plus one equation for the general product $f_{n-1}g_n$. This gives p equations. We do not count the six equations in the remaining steps since the general algorithm uses $2p+1$ equations plus Gaussian elimination. $\square$

3.2 The groups $\Gamma = C_p * C_p$

In this section we focus on the free product $\Gamma = C_p * C_p$ of two cyclic groups of the same order $|C_p| = p$, where p is a prime number. For such groups the Euler characteristic $\chi(\Gamma)$ and the free rank $\mu(\Gamma)$ are $\chi(\Gamma) = -\frac{p-2}{p}$ and $\mu(\Gamma) = p - 1$. In case Γ is the free product of two cyclic groups of order 2, $\Gamma = C_2 * C_2$, then the number of subgroups of index n in Γ is given as [Sto78]

$$s_\Gamma(2n) = s_\Gamma(2n+1) = 2\,n + 1, \quad n \geq 1, s_\Gamma(1) = 1.$$

Obviously, for the parity of $s_\Gamma(n)$ we have $s_\Gamma(n) \equiv 1 \pmod 2$ for all $n \geq 1$.

Considering other primes p, a result of Grady and Newman states that $s_\Gamma(n)$ can be expressed as a linear recurrence modulo $p > 3$.

Theorem 3.9 ([GN92, Thm. 2]) *Let Γ be a free product $\Gamma = C_p * C_p$, $p > 3$. Then Hall's formula reduces to the $p-1$ term linear recurrence:*

$$s_\Gamma(n) \equiv -\sum_{i=1}^{p-1} h_\Gamma(p-i)\; s_\Gamma(n-p+i) \pmod p, \qquad n \geq p.$$

The case $p = 3$ is not covered by the above theorem. We study the divisibility properties for $\Gamma = C_3 * C_3$. For the parity of Γ we obtain that $s_\Gamma(n) \equiv 1 \pmod 2$ if and only if n is of the form $n = 2^k - 3$ for $k \geq 2$. We are also able to describe the divisibility pattern for $s_\Gamma(n)$ modulo 3, which is periodic with length 8. Furthermore, we present a result for the groups $\Gamma = C_5 * C_5$. Here, $s_\Gamma(n) \equiv 1 \pmod 2$ if and only if n is of the form $n = \frac{2\cdot 4^k - 5}{3}$ with $k \geq 1$.

3.2.1 The group $\Gamma = C_3 * C_3$

Proposition 3.10 *The number of homomorphisms $H_n = |Hom(\Gamma, S_n)|$ where $\Gamma = C_3 * C_3$ is given by the following recurrence:*

$$\begin{aligned} H_n = {} & H_{n-1} \\ & + (n-1)(n-2)\; H_{n-2} \\ & + (n-1)(n-2)\left(2\,(n-2)^2 + 1\right)\; H_{n-3} \\ & + (n-1)(n-2)^2(n-3)\; H_{n-4} \\ & - (n-1)(n-2)(n-3)^2(n-4)^2\; H_{n-5} \\ & - (n-1)(n-2)^2(n-3)(n-4)^2(n-5)^2\; H_{n-6} \end{aligned} \tag{3.7}$$

with initial value $H_0 = 1$ and $H_i = 0$ for $i < 0$.

Proof:
With Proposition 3.1 we obtain for the number $f_n = |Hom(C_3, S_n)|$ of C_3-actions on S_n

$$f_n = f_{n-1} + (n-1)(n-2)\ f_{n-3}, \quad f_0 = 1$$

and apply our algorithm Hadamard product $C_p * C_p$. $\square$

Proposition 3.11 *Let $\Gamma = C_3 * C_3$. The number of subgroups $s_\Gamma(n)$ of given index n in Γ can be recursively calculated as*

$$\begin{aligned} s_\Gamma(n) = {} & s_\Gamma(n-2) + 2\ (n-1)\ s_\Gamma(n-3) + s_\Gamma(n-4) \\ & - (n-2)\ s_\Gamma(n-5) - (22 + (n-7)(n+2))\ s_\Gamma(n-6) \\ & + 2 \cdot \sum_{\substack{n_1+n_2=n-3 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2) - \sum_{\substack{n_1+n_2=n-5 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2) \\ & - 10 \cdot \sum_{\substack{n_1+n_2=n-6 \\ n_1 \geq 1, n_2 \geq 2}} s_\Gamma(n_1)\ s_\Gamma(n_2) - 2 \cdot \sum_{\substack{n_1+n_2=n-6 \\ n_1 \geq 1, n_2 \geq 2}} (n_2 - 1)\ s_\Gamma(n_1)\ s_\Gamma(n_2) \\ & - \sum_{\substack{n_1+n_2+n_3=n-5 \\ n_1, n_2 \geq 1, n_3 \geq 2}} (n_3 - 1)\ s_\Gamma(n_1)\ s_\Gamma(n_2)\ s_\Gamma(n_3) - \sum_{\substack{n_1+n_2+n_3=n-6 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2)\ s_\Gamma(n_3) \end{aligned} \tag{3.8}$$

with $n \geq 7$ and initial values $s_\Gamma(1) = 1,\ s_\Gamma(2) = 0\ s_\Gamma(3) = 4,\ s_\Gamma(5) = 5$ and $s_\Gamma(6) = 36$.

Proof:
Multiplying 3.7 by $z^n/(n!)$ and running over $n \geq 1$ we find for the generating function

$$F(z) = \sum_{n=0}^{\infty} \frac{H_n}{n!}\ z^n$$

$$\begin{aligned} F(z) = 1 + \sum_{n=1}^{\infty} [\ & H_{n-1} + (n-1)(n-2)\ H_{n-2} + (n-1)(n-2)\left(2\ (n-2)^2 + 1\right)\ H_{n-3} \\ & + (n-1)(n-2)^2(n-3)\ H_{n-4} - (n-1)(n-2)(n-3)^2(n-4)^2\ H_{n-5} \\ & - (n-1)(n-2)^2(n-3)(n-4)^2(n-5)^2\ H_{n-6}\ ]\ \cdot\ z^n/(n!) \end{aligned}$$

Investigating the individual summands we find:

$$
\begin{aligned}
F_1(z) &= \sum_{n=1}^{\infty} \frac{H_{n-1}}{n!}\ z^n \\
F_1'(z) &= \sum_{n=1}^{\infty} \frac{H_{n-1}}{(n-1)!}\ z^{n-1} = F(z) \\
F_2(z) &= \sum_{n=2}^{\infty} \frac{(n-1)(n-2)\ H_{n-2}}{n!}\ z^n \\
F_2'(z) &= \sum_{n=3}^{\infty} \frac{H_{n-2}}{(n-3)!}\ z^{n-1} = z^2\ F'(z) \\
F_3(z) &= \sum_{n=3}^{\infty} \frac{(n-1)(n-2)\left(2(n-2)^2+1\right)\ H_{n-3}}{n!}\ z^n \\
F_3'(z) &= \sum_{n=3}^{\infty} \frac{\left(2(n-2)^2+1\right)\ H_{n-3}}{(n-3)!}\ z^{n-1} \\
&= 2 \cdot \sum_{n=5}^{\infty} \frac{H_{n-3}}{(n-5)!}\ z^{n-1} + 6 \cdot \sum_{n=4}^{\infty} \frac{H_{n-3}}{(n-4)!}\ z^{n-1} + 3 \cdot \sum_{n=3}^{\infty} \frac{H_{n-3}}{(n-3)!}\ z^{n-1} \\
&= 2\ z^4\ F^{(2)}(z) + 6\ z^3\ F'(z) + 3\ z^2\ F(z) \\
F_4(z) &= \sum_{n=4}^{\infty} \frac{(n-1)(n-2)^2(n-3)\ H_{n-4}}{n!}\ z^n \\
F_4'(z) &= \sum_{n=4}^{\infty} \frac{(n-2)\ H_{n-4}}{(n-4)!}\ z^{n-1} \\
&= \sum_{n=5}^{\infty} \frac{H_{n-4}}{(n-5)!}\ z^{n-1} + 2 \cdot \sum_{n=4}^{\infty} \frac{H_{n-4}}{(n-4)!}\ z^{n-1} \\
&= z^4\ F'(z) + 2\ z^3\ F(z)
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
F_5(z) &= \sum_{n=5}^{\infty} \frac{(n-1)(n-2)(n-3)^2(n-4)^2\ H_{n-5}}{n!}\ z^n \\
F_5'(z) &= \sum_{n=5}^{\infty} \frac{(n-3)(n-4)^2\ H_{n-5}}{(n-5)!}\ z^{n-1} \\
&= \sum_{n=7}^{\infty} \frac{H_{n-5}}{(n-7)!}\ z^{n-1} + 4 \cdot \sum_{n=6}^{\infty} \frac{H_{n-5}}{(n-6)!}\ z^{n-1} + 2 \cdot \sum_{n=5}^{\infty} \frac{H_{n-5}}{(n-5)!}\ z^{n-1} \\
&= z^6\ F^{(2)}(z) + 4\ z^5\ F'(z) + 2\ z^4\ F(z) \\
F_6(z) &= \sum_{n=6}^{\infty} \frac{(n-1)(n-2)^2(n-3)(n-4)^2(n-5)^2\ H_{n-6}}{n!}\ z^n \\
F_6'(z) &= \sum_{n=6}^{\infty} \frac{(n-2)(n-4)(n-5)\ H_{n-6}}{(n-6)!}\ z^{n-1} \\
&= \sum_{n=9}^{\infty} \frac{H_{n-6}}{(n-9)!}\ z^{n-1} + 10 \cdot \sum_{n=8}^{\infty} \frac{H_{n-6}}{(n-8)!}\ z^{n-1} + 22 \cdot \sum_{n=7}^{\infty} \frac{H_{n-6}}{(n-7)!}\ z^{n-1} \\
&+ 8 \cdot \sum_{n=6}^{\infty} \frac{H_{n-6}}{(n-6)!}\ z^{n-1} \\
&= z^8\ F^{(3)}(z) + 10\ z^7\ F^{(2)}(z) + 22 \cdot z^6\ F'(z) + 8\ z^5\ F(z)
\end{aligned} \tag{3.10}
$$

Hence, we obtain the differential equation

$$
\begin{aligned}
F'(z) = {} & (1 + 3\ z^2 + 2\ z^3 - 2\ z^4 - 8\ z^5)\ F(z) \\
& + (z^2 + 6\ z^3 + z^4 - 4\ z^5 - 22\ z^6)\ F'(z) \\
& + (2\ z^4 - z^6 - 10\ z^7)\ F''(z) \\
& - z^8\ F'''(z).
\end{aligned} \tag{3.11}
$$

Using the identity $G(z) = \frac{F'(z)}{F(z)}$ we express the derivatives $F^{(i)}(z)$ as:

$$
\begin{aligned}
F'(z) &= G(z)\ F(z) \\
F''(z) &= G'(z)\ F(z) + G(z)^2\ F(z) \\
F'''(z) &= G^{(2)}(z)\ F(z) + 2\ G'(z)\ G(z)\ F(z) + G'(z)\ G(z)^2\ F(z) + G(z)^3\ F(z)
\end{aligned}
$$

If we now divide our differential equation 3.11 by $F(z)$ we obtain a differential equation in terms of the function $G(z)$:

$$
\begin{aligned}
G(z) = {} & (1 + 3\ z^2 + 2\ z^3 - 2\ z^4 - 8\ z^5) \\
& + (z^2 + 6\ z^3 + z^4 - 4\ z^5 - 22\ z^6)\ G(z) \\
& + (2\ z^4 - z^6 - 10\ z^7)\ (G'(z) + G(z)^2) \\
& - z^8\ (\ G''(z) + 2\ G'(z)\ G(z) + G'(z)\ G(z)^2 + G(z)^3\ )
\end{aligned} \tag{3.12}
$$

It is left to determine the different Cauchy products:

$$
\begin{aligned}
G(z)^2 &= \sum_{n=0}^{\infty} \Bigg(\sum_{n_1+n_2=n+2} s_\Gamma(n_1) s_\Gamma(n2) \Bigg) \; z^n \\
G(z)^3 &= \sum_{n=0}^{\infty} \Bigg(\sum_{n_1+n_2+n_3=n+3} s_\Gamma(n_1) s_\Gamma(n_2) \Bigg) \; z^n \\
G(z)\; G'(z) &= \sum_{n=0}^{\infty} \Bigg(\sum_{\substack{n_1+n_2=n+3 \\ n_2 \geq 2}} s_\Gamma(n_1) s_\Gamma(n_2) \Bigg) \; z^n \\
G(z)^2\; G'(z) &= \sum_{n=0}^{\infty} \Bigg(\sum_{\substack{n_1+n_2+n_3=n+4 \\ n_3 \geq 2}} s_\Gamma(n_1) s_\Gamma(n_2) s_\Gamma(n_3) \Bigg) \; z^n
\end{aligned}
\tag{3.13}
$$

Comparing coefficients now yields the following recursion for $s_\Gamma(n)$:

$$
\begin{aligned}
s_\Gamma(n) = {} & s_\Gamma(n-2) + 2\;(n-1)\; s_\Gamma(n-3) + s_\Gamma(n-4) \\
& - (n-2)\; s_\Gamma(n-5) - (22 + (n-7)(n+2))\; s_\Gamma(n-6) \\
& + 2 \cdot \sum_{\substack{n_1+n_2=n-3 \\ n_i \geq 1}} s_\Gamma(n_1)\; s_\Gamma(n_2) - \sum_{\substack{n_1+n_2=n-5 \\ n_i \geq 1}} s_\Gamma(n_1)\; s_\Gamma(n_2) \\
& - 10 \cdot \sum_{\substack{n_1+n_2=n-6 \\ n_1 \geq 1, n_2 \geq 2}} s_\Gamma(n_1)\; s_\Gamma(n_2) - 2 \cdot \sum_{\substack{n_1+n_2=n-6 \\ n_1 \geq 1, n_2 \geq 2}} (n_2 - 1)\; s_\Gamma(n_1)\; s_\Gamma(n_2) \\
& - \sum_{\substack{n_1+n_2+n_3=n-5 \\ n_1, n_2 \geq 1, n_3 \geq 2}} (n_3 - 1)\; s_\Gamma(n_1)\; s_\Gamma(n_2)\; s_\Gamma(n_3) - \sum_{\substack{n_1+n_2+n_3=n-6 \\ n_i \geq 1}} s_\Gamma(n_1)\; s_\Gamma(n_2)\; s_\Gamma(n_3).
\end{aligned}
\tag{3.14}
$$

with initial values $s_\Gamma(1) = 1,\; s_\Gamma(2) = 0\; s_\Gamma(3) = 4,\; s_\Gamma(5) = 5$, and $s_\Gamma(6) = 36$. □

Theorem 3.12 *Let $\Gamma = C_3 * C_3$. Then for the number of subgroups $s_\Gamma(n)$ of given index n in Γ we have $s_\Gamma(n) \equiv 1 \pmod 2$ if and only if n has the form $n = 2^k - 3$ for $k \geq 2$.*

Proof:
Working modulo 2 the equation 3.14 reduces to

$$\begin{aligned} s_\Gamma(n) \equiv\, & s_\Gamma(n-2) + s_\Gamma(n-4) - (n-2)\, s_\Gamma(n-5) \\ & - \sum_{\substack{n_1+n_2=n-5 \\ n_i \geq 1}} s_\Gamma(n_1)\, s_\Gamma(n_2) \\ & - \sum_{\substack{n_1+n_2+n_3=n-5 \\ n_1,n_2 \geq 1, n_3 \geq 2}} (n_3 - 1)\, s_\Gamma(n_1)\, s_\Gamma(n_2)\, s_\Gamma(n_3) \\ & \sum_{\substack{n_1+n_2+n_3=n-6 \\ n_i \geq 1}} s_\Gamma(n_1)\, s_\Gamma(n_2)\, s_\Gamma(n_3) \pmod 2 \end{aligned} \tag{3.15}$$

In order to prove the assertion of the theorem by induction on n, we first investigate the different occurring sums in 3.14 separately in order to reduce the calculation effort.

Analyzing the first sum

$$\sum_{\substack{n_1+n_2=n-5 \\ n_i \geq 1}} s_\Gamma(n_1)\, s_\Gamma(n_2) = \sum_{i=1}^{n-6} s_\Gamma(i) s_\Gamma(n-5-i).$$

Case i) $\quad 2 \mid n-6 \quad \Leftrightarrow 2 \mid n$
By the symmetry of the $s_\Gamma(i), s_\Gamma(n-5-i)$ we have

$$\sum_{i=1}^{n-6} s_\Gamma(i) s_\Gamma(n-5-i) = 2 \cdot \sum_{i=1}^{\frac{n-6}{2}} s_\Gamma(i) s_\Gamma(n-5-i) \equiv 0 \pmod 2. \tag{3.16}$$

Case ii) $\quad 2 \nmid n-6 \quad \Leftrightarrow 2 \nmid n$
Here, one product remains

$$\begin{aligned} \sum_{i=1}^{n-6} s_\Gamma(i) s_\Gamma(n-5-i) &= 2 \cdot \sum_{i=1}^{\frac{n-5}{2}-1} s_\Gamma(i) s_\Gamma(n-5-i) + s_\Gamma(\frac{n-5}{2})^2 \\ &\equiv s_\Gamma(\frac{n-5}{2})^2 \pmod 2. \end{aligned} \tag{3.17}$$

For the sum

$$\sum_{\substack{n_1+n_2+n_3=n-5 \\ n_1,n_2 \geq 1, n_3 \geq 2}} (n_3 - 1)\, s_\Gamma(n_1)\, s_\Gamma(n_2)\, s_\Gamma(n_3)$$

we fix w.l.o.g. $n_3 \in \{2, ..., n-7\}$ and consider

$$S = \sum_{i=2}^{n-6} \left((n-6-i) \cdot s_\Gamma(n-5-i) \sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l) \right).$$

Investigating the factor $(n-6-i)$ we obtain

Case i) $\quad 2 \mid n-6-i \Leftrightarrow (2 \mid n \text{ and } 2 \mid i) \text{ or } (2 \nmid n \text{ and } 2 \nmid i)$

Then $(n-6-i) \cdot s_\Gamma(n-5-i) \cdot \sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l) \equiv 0 \pmod 2$.

Case ii) $\quad 2 \nmid n-6-i$

Then

a) $2 \nmid n$ and $2 \mid i$

b) $2 \mid n$ and $2 \nmid i$

Here, we have to calculate the inner sum $\sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l)$.

a) $\quad 2 \nmid n$ and $2 \mid i$. Then

$$\begin{aligned} \sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l) &= 2 \cdot \sum_{l=1}^{\frac{i}{2}-1} s_\Gamma(l) s_\Gamma(i-l) + s_\Gamma(\frac{i}{2})^2 \\ &\equiv s_\Gamma(\frac{i}{2})^2 \pmod 2 \end{aligned} \tag{3.18}$$

b) $\quad 2 \mid n$ and $2 \nmid i$. Then

$$\begin{aligned} \sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l) &= 2 \cdot \sum_{l=1}^{\frac{i-1}{2}} s_\Gamma(l) s_\Gamma(i-l) \\ &\equiv 0 \pmod 2 \end{aligned} \tag{3.19}$$

So, it remains to calculate the sum only for odd values of n and even values for i:

$$S := \sum_{\substack{i \equiv 0(2) \\ 2 \nmid n}}^{n-6} (n-6-i) \cdot s_\Gamma(n-5-i) s_\Gamma(\frac{i}{2})^2 \tag{3.20}$$

Finally, analyzing the last sum

$$\sum_{\substack{n_1+n_2+n_3=n-6 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2)\ s_\Gamma(n_3)$$

we fix w.l.o.g. $n_3 \in \{2, ..., n-8\}$ and rewrite the sum to

$$\sum_{i=2}^{n-7} s_\Gamma(n-6-i) \cdot \sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l).$$

Again, calculating $\sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l)$ we distinguish

a) $2 \mid i \Leftrightarrow 2 \nmid i-1$. Then

$$\sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l) = 2 \cdot \sum_{l=1}^{\frac{i}{2}-1} s_\Gamma(l) s_\Gamma(i-l) + s_\Gamma(\frac{i}{2})^2 \equiv s_\Gamma(\frac{i}{2})^2 \pmod 2$$

b) $2 \nmid i \Leftrightarrow 2 \mid i-1$. Then

$$\sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l) = 2 \cdot \sum_{l=1}^{\frac{i}{2}-1} s_\Gamma(l) s_\Gamma(i-l) \equiv 0 \pmod 2$$

Hence,

$$\sum_{i=2}^{n-7} s_\Gamma(n-6-i) \cdot \sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l) \equiv \sum_{i \equiv 0(2)}^{n-7} s_\Gamma(n-6-i) s_\Gamma(\frac{i}{2})^2 \pmod 2 \tag{3.21}$$

Summarizing the 3 sums in the recurrence for $s_\Gamma(n)$ it is left to calculate
For the case $2 \nmid n$:

$$\sum_{\substack{i \equiv 0(2) \\ i \geq 2}}^{n-6} (n-6-i) \cdot s_\Gamma(n-5-i) s_\Gamma(\frac{i}{2})^2 + \sum_{\substack{i \equiv 0(2) \\ i \geq 2}}^{n-7} s_\Gamma(n-6-i) s_\Gamma(\frac{i}{2})^2 \pmod 2$$

and

$$\sum_{\substack{i \equiv 0(2) \\ i \geq 2}}^{n-6} s_\Gamma(n-5-i) s_\Gamma(\frac{i}{2})^2 + \sum_{\substack{i \equiv 0(2) \\ i \geq 2}}^{n-7} s_\Gamma(n-6-i) s_\Gamma(\frac{i}{2})^2 \pmod 2 \tag{3.22}$$

where $n-6-i \equiv 1 \pmod 2$.

In the case $2 \mid n$ it is left to calculate:

$$\sum_{\substack{i \equiv 0(2) \\ i \geq 2}}^{n-7} s_\Gamma(n-6-i) s_\Gamma(\frac{i}{2})^2 \pmod 2$$

After those investigations we start now with our induction on n. For the first values we get: $s_\Gamma(1) \equiv 1 \pmod 2$, $s_\Gamma(2) \equiv 0 \pmod 2$, $s_\Gamma(3) \equiv 0 \pmod 2$, $s_\Gamma(4) \equiv 0 \pmod 2$, $s_\Gamma(5) \equiv 1 \pmod 2$. We assume the statement is true for all $n \leq t-1$ and consider the recurrence for $s_\Gamma(t)$. We distinguish $2 \mid t$ and $2 \nmid t$.

Case A: $2 \mid t$
Applying our investigations of the sums we need to calculate in this case

$$
\begin{aligned}
s_\Gamma(t) &\equiv s_\Gamma(t-2) + s_\Gamma(t-4) - s_\Gamma(t-2)\, s_\Gamma(t-5) \\
&\quad - \sum_{\substack{t_1+t_2=t-5 \\ t_i \geq 1}} s_\Gamma(t_1)\, s_\Gamma(t_2) \\
&\quad - \sum_{\substack{t_1+t_2+t_3=t-5 \\ t_1,t_2 \geq 1, t_3 \geq 2}} (t_3-1)\, s_\Gamma(t_1)\, s_\Gamma(t_2)\, s_\Gamma(t_3) \\
&\sum_{\substack{t_1+t_2+t_3=t-6 \\ t_i \geq 1}} s_\Gamma(t_1)\, s_\Gamma(t_2)\, s_\Gamma(t_3) \\
&\equiv s_\Gamma(t-2) + s_\Gamma(t-4) \\
&\quad - \sum_{\substack{i \equiv 0(2) \\ i \geq 2}}^{t-7} s_\Gamma(t-6-i) s_\Gamma(\frac{i}{2})^2 \pmod 2
\end{aligned}
\tag{3.23}
$$

by 3.16, 3.19 and 3.21. Since $2 \mid n-6-i$ for $2 \mid i$, $n-6-i$ is not of the form 2^k-3 for an $k \geq 2$ and by our induction assertion is $s_\Gamma(n-6-i) \equiv 0 \pmod 2$. With $2 \mid t-2$ and $2 \mid t-4$ both are not of form 2^k-3 and so again by induction assertion $s_\Gamma(t-2) = s_\Gamma(t-4) \equiv 0 \pmod 2$. Hence, we have $s_\Gamma(t) \equiv 0 \pmod 2$ for $2 \mid t$.

Case B: $2 \nmid t$
Here, we have by our arguments above, 3.18 and 3.22, to consider

$$\begin{aligned} s_\Gamma(t) \equiv\ & s_\Gamma(t-2) + s_\Gamma(t-4) - s_\Gamma(t-2)\, s_\Gamma(t-5) \\ & - s_\Gamma(\frac{t-5}{2})^2 \\ & - \sum_{\substack{i\equiv 0(2)\\ i\geq 2}}^{t-6} s_\Gamma(t-5-i) s_\Gamma(\frac{i}{2})^2 \\ & - \sum_{\substack{i\equiv 0(2)\\ i\geq 2}}^{t-7} s_\Gamma(n-6-i) s_\Gamma(\frac{i}{2})^2 \pmod 2. \end{aligned} \tag{3.24}$$

By our induction hypothesis and with $2 \mid i$ we get $t-5-i \equiv 0 \pmod 2$, hence not of form 2^k-3 and so is $s_\Gamma(t-5-i) \equiv 0 \pmod 2$. Since $2^{k+1}-3-(2^k-3) = 2^k$ and $k \geq 3$, $t-2$ and $t-4$ are not of form 2^k-3, hence $s_\Gamma(t-2) \equiv s_\Gamma(t-4) \equiv 0 \pmod 2$ and since $2 \mid t-5$ is $s_\Gamma(t-5) \equiv 0 \pmod 2$.
Further, we have

$$\sum_{\substack{i\equiv 0(2)\\ i\geq 2}}^{t-6} s_\Gamma(t-5-i) s_\Gamma(\frac{i}{2})^2 \equiv 0 \pmod 2$$

because with $2 \nmid t$ we get $(t-5-i) \equiv 0 \pmod 2$ and so by case A $s_\Gamma(t-5-i) \equiv 0 \pmod 2$ for all $i \equiv 0(2)$.

It remains to analyze

$$\begin{aligned} s_\Gamma(t) \equiv & - \sum_{\substack{i\equiv 0(2)\\ i\geq 2}}^{t-7} s_\Gamma(n-6-i) s_\Gamma(\frac{i}{2})^2 \\ & - s_\Gamma(\frac{t-5}{2})^2 \pmod 2. \end{aligned} \tag{3.25}$$

Subcase B_1: t is of the form 2^k-3, $k \geq 3$
For the calculation of the sum

$$\sum_{\substack{i\equiv 0(2)\\ i\geq 2}}^{t-7} s_\Gamma(t-6-i) s_\Gamma(\frac{i}{2})^2 = \sum_{\substack{j\equiv 0(2)\\ i\geq 2}}^{\frac{t-7}{2}} s_\Gamma(t-6-2j) s_\Gamma(j)^2$$

we need to determine the values for $t-6-2j$ and j such that both are of the form 2^k-3 for $k \geq 3$. Only in that case we get $s_\Gamma(t-6-2j)s_\Gamma(j) \equiv 1 \pmod 2$, in all other cases at least one factor is congruent 0 modulo 2. So let $t = 2^K-3$ and $s_\Gamma(t-6-2j)s_\Gamma(j) \equiv 1 \pmod 2$. Then $j = 2^L-3$ and $t-6-2j = 2^M-3$ and $2^K-3-6-2^{L+1}+6 = 2^M-3$, respectively $2^K = 2^M + 2^{L+1}$. The equation is true only for the values $M = K - 1$ and $L = K - 2$. So $j = 2^{K-2} - 3$ is the only possible value which is in the range for $j \in \{1, ..., 2^{K-1} - 5\}$. Calculating $t - 6 - 2j = t - 6 - 2(2^{K-2} - 3) = 2^{K-1} - 3$ confirms the value. Hence, we obtain

$$\sum_{\substack{j \equiv 0(2) \\ i \geq 2}}^{\frac{t-7}{2}} s_\Gamma(t - 6 - 2j) s_\Gamma(j) \equiv 1 \pmod 2.$$

With $t = 2^K - 3$, $K \geq 3$ we have $\frac{t-5}{2} = 2^{k-1} - 4$ an even number, hence by case A $s_\Gamma(\frac{t-5}{2}) \equiv 0 \pmod 2$ and $s_\Gamma(t) \equiv 1 \pmod 2$ for $t = 2^K - 3$.

Subcase B_2: $2 \nmid t$ and t not of form $2^K - 3$
We need to know the possible values for t and j such that t is not of form $2^k - 3$ and $s_\Gamma(t - 6 - 2j) s_\Gamma(j) \equiv 1 \pmod 2$. That is only the case for $t = 2^M + 2^L - 3$ with $M \neq L, M \geq 3, L \neq 0$ and $j = 2^{L-1} - 3$ or $j = 2^{M-1} - 3$. Then we have $t - 6 - 2j = 2^M + 2^L - 3 - 6 - 2(2^{L-1} - 3) = 2^M - 3$ or $t - 6 - 2j = 2^M + 2^L - 3 - 6 - 2(2^{M-1} - 3) = 2^L - 3$. That those values are sufficient is seen by setting $j = 2^L - 3$ and $t - 6 - 2j = 2^M - 3$. Then t calculates to $t = 2^M + 3 + 2j = 2^M + 3 + 2(2^L - 3) = 2^M + 2^{L+1} - 3$. If $t = 2^M + 2^L - 3$ we reach those values for j, since $\frac{t-7}{2} = 2^{M-1} + 2^{L-1} - 5$ and $M \geq 3$. So we get the case of 2 values for $j \in \{2^{L-1} - 3, 2^{M-1} - 3\}$ and so $s_\Gamma(t - 6 - 2j) s_\Gamma(j) \equiv 1 \pmod 2$, hence the whole sum is congruent 0 modulo 2. With $t = 2^M + 2^L - 3$ we have $\frac{t-5}{2} = 2^{M-1} + 2^{L-1} - 4$ and, by case A, $s_\Gamma(\frac{t-5}{2}) \equiv 0 \pmod 2$.

Subcase B_3: $2 \nmid t$ and $\frac{t-5}{2} = 2^M - 3$
Then by our induction assertion $s_\Gamma(\frac{t-5}{2}) \equiv 1 \pmod 2$. We have $t - 5 = 2^{M+1} - 6$ and $t = 2^{M+1} - 1$ and t is not of form $2^k - 3$ or of form B_2, hence $s_\Gamma(t) \equiv 0 \pmod 2$. But then $s_\Gamma(t-2) = s_\Gamma(2^{M+1} - 3) \equiv 1 \pmod 2$ by our induction hypothesis and so $s_\Gamma(t) \equiv 0 \pmod 2$ for $2 \nmid t$ where t is not of form $2^k - 3$. □

Further, we find a divisibility pattern for the case $p = 3$ for this type of groups.

Theorem 3.13 *For $\Gamma = C_3 * C_3$ we have*

$$s_\Gamma(n) \equiv \begin{cases} 0 \pmod 3, & \text{if } n \equiv 2,6 \pmod 8 \\ 1 \pmod 3, & \text{if } n \equiv 0,1,3 \pmod 8 \\ 2 \pmod 3, & \text{if } n \equiv 4,5,7 \pmod 8. \end{cases}$$

Proof:
We prove the theorem by induction on n. For the first eight values we calculate $s_\Gamma(n)$ as given in Proposition 3.11:

$$\begin{aligned} s_\Gamma(1) &\equiv 1 \pmod 3 \\ s_\Gamma(2) &\equiv 0 \pmod 3 \\ s_\Gamma(3) &\equiv 1 \pmod 3 \\ s_\Gamma(4) &\equiv 2 \pmod 3 \\ s_\Gamma(5) &\equiv 2 \pmod 3 \\ s_\Gamma(6) &\equiv 0 \pmod 3 \\ s_\Gamma(7) &\equiv 2 \pmod 3 \\ s_\Gamma(8) &\equiv 1 \pmod 3 \end{aligned}$$

We assume that the assertion is true for all $t \leq n-1$ and we consider n. In our analysis of the different sums in the recurrence for $s_\Gamma(n)$ the most complex one is the sum

$$\begin{aligned} &\sum_{\substack{n_1+n_2+n_3=n-5 \\ n_1,n_2\geq 1, n_3 \geq 2}} (n_3-1)\ s_\Gamma(n_1)\ s_\Gamma(n_2)\ s_\Gamma(n_3) \\ &= \sum_{i=2}^{n-7} \left((n-6-i)\ s_\Gamma(n-5-i) \sum_{l=1}^{i-1} s_\Gamma(l)\ s_\Gamma(i-l) \right), \end{aligned} \tag{3.26}$$

and we show in detail the calculation procedure for this sum. The same arguments and similar calculations apply for the investigations of the other sums.

We have to evaluate the sum for every case for $n \in \{8k, 8k-1, ..., 8k-7\},\ k \geq 1$. Because of the factor $(n-6-i)$ we get for $(n-6-i) \pmod 3$ a period of length three together with the period of length eight for the $s_\Gamma(n-5-i)$. The least common multiple of both is 24, hence we need to consider the products $(n-6-i)\ s_\Gamma(n-5-i)$, starting with $i = n-7$, respectively $s_\Gamma(n-5-i) = s_\Gamma(2) = 0$ and $(n-6-i) = 1$, which is best visualized in form of a table:

(n-6-i)	(n-6-i) (mod 3)	$s_\Gamma(n-5-i)$ (mod 3)	(n-6-i) $\cdot s_\Gamma(n-5-i)$ (mod 3)
1	1	0	0
2	2	1	2
3	0	2	0
4	1	2	2
5	2	0	0
6	0	2	0
7	1	1	1
8	2	1	2
9	0	0	0
10	1	1	1
11	2	2	1
12	0	2	0
13	1	0	0
14	2	2	1
15	0	1	0
16	1	1	1
17	2	0	0
18	0	1	0
19	1	2	2
20	2	2	1
21	0	0	0
22	1	2	2
23	2	1	2
24	0	1	0

Figure 5: Table 1

The next step is to calculate the values for the interior sum of formula (3.26)

$$\sum_{l=1}^{i-1} s_\Gamma(l)\ s_\Gamma(i-l).$$

Using the same idea of the "shifted sequences" as in most proofs of theorems in Chapter 2, we calculate the product of the two sequences both periodic of length 8; for $s_\Gamma(1), ..., s_\Gamma(i-1)$ and for $s_\Gamma(i-1), ..., s_\Gamma(1)$ where we know the values for the s_Γ by our induction assertion. We have two blocks, one initial block B_0 and the iterated B_1.

B_0									
$(i-1)$...	$s_\Gamma(i-1)$...	$s_\Gamma(l),\ l=1,...,8-s,\ s=0,...,7$							
8k	1	1							
8k-1	2	0	1						
8k-2	0	1	0	1					
8k-3	2	2	1	0	1				
8k-4	2	2	2	1	0	1			
8k-5	1	0	2	2	1	0	1		
8k-6	0	2	0	2	2	1	0	1	
8k-7	1	1	2	0	2	2	1	0	1
$\sum s_\Gamma(l)s_\Gamma(i-l) \pmod 3$		1	0	1	2	1	2	0	1
		B_{00}	B_{01}	B_{02}	B_{03}	B_{04}	B_{05}	B_{06}	B_{07}

B_1									
$(i-l)$	$s_\Gamma(i-l)$	$s_\Gamma(l)$							
8(k-1)	1	1	1	2	0	2	2	1	0
8(k-1)-1	2	0	1	1	2	0	2	2	1
8(k-1)-2	0	1	0	1	1	2	0	2	2
8(k-1)-3	2	2	1	0	1	1	2	0	2
8(k-1)-4	2	2	2	1	0	1	1	2	0
8(k-1)-5	1	0	2	2	1	0	1	1	2
8(k-1)-6	0	2	0	2	2	1	0	1	1
8(k-1)-7	1	1	2	0	2	2	1	0	1
$\sum s_\Gamma(l)s_\Gamma(i-l) \pmod 3$		1	1	2	0	2	2	1	0
		B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}

Figure 6: Table 2

With this two blocks we read off the values for each sum, depending from the start value of $s_\Gamma(i-1)$. For example, let $i-1 = 8k-2$, then we get for the initial block $B_{02} \equiv 1 \pmod 3$ and for the iterative block $B_{12} \equiv 2 \pmod 3$, which occurs $(k-1)$ times. So, in total we have $\sum\limits_{l=1}^{i-1=8k-2} s_\Gamma(i)s_\Gamma(i-l) \equiv 1 + 2\cdot(k-1) \pmod 3$. Obviously, the sum values depend on the congruence class of k and the sequence of the sum values which is periodic of length 24. We calculate the three sequences for the sum using Table 2 for the three values for k modulo 3. Again, best presented in form of a table.

(i-1)	$\sum_{l=1}^{i-1} s_\Gamma(l)s_\Gamma(i-l)$	$k \equiv 0(3)$	$k \equiv 1(3)$	$k \equiv 2(3)$
8(k-1)	(k-1) (3)	2	0	1
8(k-1)-1	0 + (k-2)(3)	1	2	0
8(k-1)-2	$1 + 2 \cdot (k-2)(3)$	0	2	1
8(k-1)-3	$2 + 0 \cdot (k-2)(3)$	2	2	2
8(k-1)-4	$1 + 2 \cdot (k-2)(3)$	0	2	1
8(k-1)-5	$2 + 2 \cdot (k-2)(3)$	1	0	2
8(k-1)-6	$0 + (k-2)(3)$	1	2	0
8(k-1)-7	$1 + 0 \cdot (k-2)(3)$	1	1	1
8(k-2)	(k-2) (3)	1	2	0
8(k-2)-1	0 + (k-3)(3)	0	1	2
8(k-2)-2	$1 + 2 \cdot (k-3)(3)$	1	0	2
8(k-2)-3	$2 + 0 \cdot (k-3)(3)$	2	2	2
8(k-2)-4	$1 + 2 \cdot (k-3)(3)$	1	0	2
8(k-2)-5	$2 + 2 \cdot (k-3)(3)$	2	1	0
8(k-2)-6	$0 + (k-3)(3)$	0	1	2
8(k-2)-7	$1 + 0 \cdot (k-3)(3)$	1	1	1
8(k-3)	(k-3) (3)	0	1	2
8(k-3)-1	0 + (k-4)(3)	2	0	1
8(k-3)-2	$1 + 2 \cdot (k-4)(3)$	2	1	0
8(k-3)-3	$2 + 0 \cdot (k-4)(3)$	2	2	2
8(k-3)-4	$1 + 2 \cdot (k-4)(3)$	2	1	0
8(k-3)-5	$2 + 2 \cdot (k-4)(3)$	0	2	1
8(k-3)-6	$0 + (k-4)(3)$	2	0	1
8(k-3)-7	$1 + 0 \cdot (k-4)(3)$	1	1	1

Figure 7: Table 3

Now we reduced the interior sum, which starts for $i = n-7$ with the product $s_\Gamma(1)s_\Gamma(n-8)$, to a periodic sequence of values modulo 3 with length 24. It remains to calculate the sum in total from the combinations of the sequence for $(n-6-i)s_\Gamma(n-5-i)$, which can be obtained from Table 1, and the values of the interior sum from Table 3. Both sequences are periodic with length 24, hence, following the same procedure of the "shifted sequences" we need to calculate an initial block, depending on $k \pmod 3$ and on $n \in \{8k, ..., 8k-7\}$ plus an iterative block which has length 24. From now on we consider the case $k \equiv 0 \pmod 3$. The argumentation and calculation is analogue for the cases $k \equiv 1, 2 \pmod 3$.

$B_0, \quad k \equiv 0 \pmod 3$									
$(i-1)...$	$\sum_{l=1}^{i-1} s_\Gamma(l)s_\Gamma(i-l)$	$(n-6-i)\cdot s_\Gamma(n-5-i),\ i=n-7,...$							
8(k-1)	2	0							
8(k-1)-1	1	2	0						
8(k-1)-2	0	0	2	0					
8(k-1)-3	2	2	0	2	0				
8(k-1)-4	0	0	2	0	2	0			
8(k-1)-5	1	0	0	2	0	2	0		
8(k-1)-6	1	1	0	0	2	0	2	0	
8(k-1)-7	1	2	1	0	0	2	0	2	0
8(k-2)	1	0	2	1	0	0	2	0	2
8(k-2)-1	0	1	0	2	1	0	0	2	0
8(k-2)-2	1	1	1	0	2	1	0	0	2
8(k-2)-3	2	0	1	1	0	2	1	0	0
8(k-2)-4	1	0	0	1	1	0	2	1	0
8(k-2)-5	2	1	0	0	1	1	0	2	1
8(k-2)-6	0	0	1	0	0	1	1	0	2
8(k-2)-7	1	1	0	1	0	0	1	1	0
$\sum(n-6-i)s_\Gamma(n-5-i)\cdot \sum s_\Gamma(l)s_\Gamma(i-l) \pmod 3$		1	0	2	1	2	0	2	0
		B_{00}	B_{01}	B_{02}	B_{03}	B_{04}	B_{05}	B_{06}	B_{07}

$B_1, \quad k \equiv 0 \pmod 3$									
$(i-l)$	$\sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l)$	$(n-6-i) \cdot s_\Gamma(n-5-i)$							
8(k-3)	0	0	1	0	1	0	0	1	1
8(k-3)-1	2	0	0	1	0	1	0	0	1
8(k-3)-2	2	2	0	0	1	0	1	0	0
8(k-3)-3	2	1	2	0	0	1	0	1	0
8(k-3)-4	2	0	1	2	0	0	1	0	1
8(k-3)-5	0	2	0	1	2	0	0	1	0
8(k-3)-6	2	2	2	0	1	2	0	0	1
8(k-3)-7	1	0	2	2	0	1	2	0	0
8(k-4)	2	0	0	2	2	0	1	2	0
8(k-4)-1	1	2	0	0	2	2	0	1	2
8(k-4)-2	0	0	2	0	0	2	2	0	1
8(k-4)-3	2	2	0	2	0	0	2	2	0
8(k-4)-4	0	0	2	0	2	0	0	2	2
8(k-4)-5	1	0	0	2	0	2	0	0	2
8(k-4)-6	1	1	0	0	2	0	2	0	0
8(k-4)-7	1	2	1	0	0	2	0	2	0
8(k-5)	1	0	2	1	0	0	2	0	2
8(k-5)-1	0	1	0	2	1	0	0	2	0
8(k-5)-2	1	1	1	0	2	1	0	0	2
8(k-5)-3	2	0	1	1	0	2	1	0	0
8(k-5)-4	1	0	0	1	1	0	2	1	0
8(k-5)-5	2	1	0	0	1	1	0	2	1
8(k-5)-6	0	0	1	0	0	1	1	0	2
8(k-5)-7	1	1	0	1	0	0	1	1	0
$\sum (n-6-i) s_\Gamma(n-5-i) \cdot \sum s_\Gamma(l) s_\Gamma(i-l) \pmod 3$		2	0	2	2	1	0	1	1
		B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}

Figure 8: Table 4

Finally, we can read from Table 4 the values for the sum (formula 3.16) consisting of values for the block B_0 and $\frac{k-3}{3}$, $k-3 \equiv 0 \pmod 3$ times block B_1. For example, let $i-1 = 8k-2$, then we get for the initial block $B_{02} \equiv 2 \pmod 3$ and for the iterative $B_{12} \equiv 2 \pmod 3$, which occurs $\frac{(k-3)}{3}$ times, where $k-3 \equiv 0 \pmod 3$, since $k \equiv 0 \pmod 3$. So, in total we have for this case

$$\sum_{i=2}^{n-7} \left((n-6-i)\ s_\Gamma(n-5-i) \sum_{l=1}^{i-1} s_\Gamma(l)\ s_\Gamma(i-l) \right) \equiv 2 + 2 \cdot \frac{(k-3)}{3} \pmod 3.$$

Using the same idea of analyzing the sums by the “shifted sequences” of the s_Γ we get for the remaining sums in (3.14) the following values modulo 3 where $k \equiv 0 \pmod 3$:

Blocks	$\sum_{\substack{n_1+n_2=n-6 \\ n_2\geq 2}} (n_2-1)s_\Gamma(n_1)s_\Gamma(n_2) = \sum_{i=2}^{n-7} s_\Gamma(i)(i-1)s_\Gamma(n-6-i)$	$\sum_{n_1+n_2+n_3=n-6} s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3) = \sum_{i=2}^{n-7}\left(s_\Gamma(n-6-i)\sum_{l=1}^{i-1} s_\Gamma(l)s_\Gamma(i-l)\right)$
B_{00}	2 (3)	1 (3)
B_{01}	0 (3)	0 (3)
B_{02}	2 (3)	1 (3)
B_{03}	0 (3)	2 (3)
B_{04}	0 (3)	0 (3)
B_{05}	0 (3)	0 (3)
B_{06}	2 (3)	2 (3)
B_{07}	0 (3)	0 (3)
B_{10}	0 (3)	$1 \cdot \frac{(k-3)}{3}$ (3)
B_{11}	0 (3)	$0 \cdot \frac{(k-3)}{3}$ (3)
B_{12}	0 (3)	$1 \cdot \frac{(k-3)}{3}$ (3)
B_{13}	0 (3)	$1 \cdot \frac{(k-3)}{3}$ (3)
B_{14}	0 (3)	$2 \cdot \frac{(k-3)}{3}$ (3)
B_{15}	0 (3)	$0 \cdot \frac{(k-3)}{3}$ (3)
B_{16}	0 (3)	$2 \cdot \frac{(k-3)}{3}$ (3)
B_{17}	0 (3)	$2 \cdot \frac{(k-3)}{3}$ (3)

Figure 9: Table 5

For the values of the remaining sums we can use Table 2 calculating the blocks B_0 and B_1 with $k \equiv 0 \pmod 3$. After those investigations we are able to calculate now $s_\Gamma(n)$ where $k \equiv 0 \pmod 3$ and $n \in \{8k, 8k-1, ..., 8k-7\}$.

Let $n = 8k, \quad k \equiv 0 \pmod 3$. Then we obtain for the recurrence of $s_\Gamma(8k)$ according to Proposition 3.11:

$s_\Gamma(8k-2) \equiv 0 \pmod 3$, $2 \cdot (8k-1) \cdot s_\Gamma(8k-3) \equiv 2 \pmod 3$, $s_\Gamma(8k-4) \equiv 2 \pmod 3$, $(8k-2) \cdot s_\Gamma(8k-5) \equiv 1 \pmod 3$, $(22+(8k-7)(8k+2)) \cdot s_\Gamma(8k-6) \equiv 0$

(mod 3), by our induction hypothesis. Further,

$$
\begin{aligned}
2 \cdot \sum_{n_1+n_2=n-3} s_\Gamma(n_1)\, s_\Gamma(n_2) &= 2 \cdot \sum_{i=1}^{n-4=8k-4} s_\Gamma(i) s_\Gamma(n-3-i) \\
&\equiv 1 \pmod 3 \qquad (\mathit{Table}\ 2) \\
\sum_{n_1+n_2=n-5} s_\Gamma(n_1)\, s_\Gamma(n_2) &= \sum_{i=1}^{n-6=8k-6} s_\Gamma(i) s_\Gamma(n-5-i) \\
&\equiv 2 \pmod 3 \qquad (\mathit{Table}\ 2) \\
\sum_{n_1+n_2=n-6} s_\Gamma(n_1)\, s_\Gamma(n_2) &= \sum_{i=1}^{n-7=8k-7} s_\Gamma(i) s_\Gamma(n-6-i) \\
&\equiv 1 \pmod 3 \qquad (\mathit{Table}\ 2) \\
2 \cdot \sum_{\substack{n_1+n_2=n-6 \\ n_1 \geq 1, n_2 \geq 2}} (n_2-1)\, s_\Gamma(n_1)\, s_\Gamma(n_2) &= 2 \cdot \sum_{i=2}^{n-7=8k-7} s_\Gamma(i)(i-1) s_\Gamma(n-6-i) \\
&\equiv 1 \pmod 3 \qquad (\mathit{Table}\ 5) \\
\sum_{\substack{n_1+n_2+n_3=n-5 \\ n_1,n_2 \geq 1, n_3 \geq 2}} (n_3-1)\, s_\Gamma(n_1)\, s_\Gamma(n_2)\, s_\Gamma(n_3) & \\
= \sum_{i=2}^{n-7=8k-7} &\left((n-6-i) s_\Gamma(n-5-i) \sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l) \right) \\
&\equiv 1 + 2 \cdot \frac{(k-3)}{3} \pmod 3 \qquad (\mathit{Table}\ 4) \\
\sum_{\substack{n_1+n_2+n_3=n-6 \\ n_i \geq 1}} s_\Gamma(n_1)\, s_\Gamma(n_2)\, s_\Gamma(n_3) &= \sum_{i=2}^{n-7=8k-7} \left(s_\Gamma(n-6-i) \sum_{l=1}^{i-1} s_\Gamma(l) s_\Gamma(i-l) \right) \\
&\equiv 1 + 1 \cdot \frac{(k-3)}{3} \pmod 3 \qquad (\mathit{Table}\ 5)
\end{aligned}
\tag{3.27}
$$

In total, respecting the signs for each term we get:

$$
s_\Gamma(8k) \equiv 0 + 2 + 2 - 1 + 0 + 1 - 2 - 1 - 1 - 1 - 1 - \frac{(k-3)}{3}(1+2) \equiv 1 \pmod 3.
$$

With similar calculations we get the desired values for the cases $n = 8k-1, ..., 8k-7$. For the cases $k \equiv 1 \pmod 3$ and $k \equiv 2 \pmod 3$ we apply the same principle of the "shifted sequences" but we like to avoid the presentation of analogue calculations. □

3.2.2 The group $\Gamma = C_5 * C_5$

Here, we focus on divisibility properties of $s_\Gamma(n)$ for groups of the form $\Gamma = C_5 * C_5$. In order to get a recurrence for $s_\Gamma(n)$ we first need a recurrence for the Hadamard product of $|Hom(\Gamma, S_n)| = |Hom(C_5, S_n)| \cdot |Hom(C_5, S_n)|$.

Proposition 3.14 *The number of homomorphisms $H_n = |Hom(\Gamma, S_n)|$ where $\Gamma = C_5 * C_5$ is given by the following recurrence:*

$$\begin{aligned}
H_n = H_{n-1} & \\
&+ (n-1)(n-2)(n-3)(n-4)\ H_{n-3} \\
&+ (n-1)(n-2)(n-3)(n-4)\ [\ (n-3)(n-4)\ (2(n-2)(n-3) + (n-5)(n-6)) + 1\ ]\ H_{n-5} \\
&+ (n-1)(n-2)^2(n-3)^2(n-4)^2(n-5)\ H_{n-6} \\
&+ (n-1)(n-2)(n-3)^2(n-4)^2(n-5)(n-6)\ H_{n-7} \\
&- 2 \cdot (n-1)(n-2)(n-3)(n-4)^2(n-5)^2(n-6)^3(n-7)^2\ H_{n-8} \\
&- 2 \cdot (n-1)(n-2)(n-3)(n-4)(n-5)^2(n-6)^3(n-7)^2(n-8)\ H_{n-9} \\
&- 3 \cdot (n-1)(n-2)(n-3)^2(n-4)^2(n-5)(n-6)^3(n-7)^2(n-8)^2(n-9)^2\ H_{n-10} \qquad (3.28) \\
&- 2 \cdot (n-1)(n-2)(n-3)^2(n-4)^2(n-5)(n-6)(n-7)^2(n-8)^3(n-9)^2(n-10)\ H_{n-11} \\
&+ (n-1)(n-2)(n-3)(n-4)^2(n-5)^2(n-6)^2(n-7)^2(n-8)(n-9)^2(n-10)^2 \cdot \\
&\cdot (n-11)^2(n-12)^2\ H_{n-13} \\
&+ (n-1)(n-2)(n-3)^2(n-4)^2(n-5)(n-6)^2(n-7)^2(n-8)^2(n-9)^2(n-10) \cdot \\
&\cdot (n-11)^2(n-12)^2(n-13)^2(n-14)^2\ H_{n-15}
\end{aligned}$$

with initial values $H_1 = 1,\ H_2 = 1,\ H_3 = 1,\ H_4 = 1,\ H_5 = 625,\ H_6 = 21025,$ $H_7 = 255025,\ H_8 = 1809025,\ H_9 = 9150625,\ H_{10} = 6181890625,\ H_{11} = 655168830625,$ $H_{12} = 23126721450625,\ H_{13} = 432125353140625,\ H_{14} = 5284799286390625,$ *and* $H_{15} = 38478103178681406 25.$

Proof:
With Proposition 3.1 we obtain for the number $f_n = |Hom(C_5, S_n)|$ of C_5-actions on S_n

$$f_n = f_{n-1} + (n-1)(n-2)(n-3)(n-4)\ f_{n-5}, \quad f_0 = 1$$

and apply our algorithm Hadamard product $C_p * C_p$. □

With this recurrence for $|Hom(\Gamma, S_n)|$ in hand we can calculate a recurrence for $s_\Gamma(n)$.

Proposition 3.15 *Let $\Gamma = C_5 * C_5$. Modulo 2 we have*

$$
\begin{aligned}
s_\Gamma(n) \equiv\ & (n-4)\ s_\Gamma(n-3) + (n-8)\ s_\Gamma(n-7) \\
& + \sum_{\substack{n_1+n_2=n-3 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2) \\
& + \sum_{\substack{n_1+n_2=n-7 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2) \\
& + \sum_{\substack{n_1+n_2=n-5 \\ n_1 \geq 1, n_2 \geq 2}} (n_1-1)(n_2-1)\ s_\Gamma(n_1)\ s_\Gamma(n_2) \\
& + \sum_{\substack{n_1+\ldots+n_4=n-4 \\ n_1,n_2,n_3 \geq 1, n_4 \geq 2}} s_\Gamma(n_1)\ s_\Gamma(n_2)\ s_\Gamma(n_3)\ (n_4-1)\ s_\Gamma(n_4) \\
& + \sum_{\substack{n_1+\ldots+n_3=n-6 \\ n_i \geq 1}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3) \\
& + \sum_{\substack{n_1+\ldots+n_4=n-5 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4) \\
& - \sum_{\substack{n_1+\ldots+n_4=n-10 \\ n_1 \geq 1,\ n_i \geq 2}} s_\Gamma(n_1)\ (n_2-1)(n_3-1)(n_4-1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4) \\
& - \sum_{\substack{n_1+\ldots+n_6=n-9 \\ n_5,n_6 \geq 2}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ (n_5-1)(n_6-1) s_\Gamma(n_5)\ s_\Gamma(n_6) \\
& - \sum_{\substack{n_1+\ldots+n_6=n-10 \\ n_6 \geq 2}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ s_\Gamma(n_5)\ (n_6-1)\ s_\Gamma(n_6) \\
& - \sum_{\substack{n_1+\ldots+n_7=n-10 \\ n_i \geq 1}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ s_\Gamma(n_5)\ s_\Gamma(n_6)\ s_\Gamma(n_7) \\
& + \sum_{\substack{n_1+\ldots+n_4=n-13 \\ n_i \geq 2}} (n_1-1)(n_2-1)(n_3-1)(n_4-1)\ s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4) \\
& + \sum_{\substack{n_1+\ldots+n_5=n-13 \\ n_1,n_2 \geq 1,\ n_3,n_4,n_5 \geq 2}} s_\Gamma(n_1) s_\Gamma(n_2)\ (n_3-1)(n_4-1)(n_5-1)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ s_\Gamma(n_5) \\
& + \sum_{\substack{n_1+\ldots+n_6=n-12 \\ n_1,n_2,n_3 \geq 1, n_4,n_5,n_6 \geq 2}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ (n_4-1)(n_5-1)(n_6-1)\ s_\Gamma(n_4)\ s_\Gamma(n_5)\ s_\Gamma(n_6) \\
& + \sum_{\substack{n_1+\ldots+n_6=n-13 \\ n_1,\ldots,n_4 \geq 1,\ n_5,n_6 \geq 2}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ (n_5-1)(n_6-1)\ s_\Gamma(n_5)\ s_\Gamma(n_6) \\
& + \sum_{\substack{n_1+\ldots+n_7=n-12 \\ n_1,\ldots,n_5 \geq 1,\ n_6,n_7 \geq 2}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ s_\Gamma(n_5)\ (n_6-1)(n_7-1)\ s_\Gamma(n_6)\ s_\Gamma(n_7)
\end{aligned}
\tag{3.29}
$$

$$+ \sum_{\substack{n_1+\ldots+n_8=n-13 \\ n_i\geq 1}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ s_\Gamma(n_5)\ s_\Gamma(n_6)\ s_\Gamma(n_7)\ s_\Gamma(n_8)$$

$$+ \sum_{\substack{n_1+\ldots+n_5=n-15 \\ n_i\geq 2}} (n_1-1)(n_2-1)(n_3-1)(n_4-1)(n_5-1)\ s_\Gamma(n_1)\cdot\ldots\cdot s_\Gamma(n_5)$$

$$+ \sum_{\substack{n_1+\ldots+n_7=n-14 \\ n_1,n_2,n_3\geq 1,\ n_4,\ldots,n_7\geq 2}} (n_4-1)(n_5-1)(n_6-1)(n_7-1)\ s_\Gamma(n_1)\cdot\ldots s_\Gamma(n_7)$$

$$+ \sum_{\substack{n_1+\ldots+n_8=n-15 \\ n_1,\ldots,n_6\geq 1,\ n_7,n_8\geq 2}} (n_7-1)(n_8-1)\ s_\Gamma(n_1)\cdot\ldots\cdot s_\Gamma(n_8)$$

$$+ \sum_{\substack{n_1+\ldots+n_9=n-15 \\ n_1,\ldots,n_8\geq 1,\ n_9\geq 2}} (n_9-1)\ s_\Gamma(n_1)\cdot\ldots\cdot s_\Gamma(n_9)$$

$$+ \sum_{\substack{n_1+\ldots+n_9=n-14 \\ n_1,\ldots,n_7\geq 1,\ n_8,n_9\geq 2}} (n_8-1)(n_9-1)\ s_\Gamma(n_1)\cdot\ldots\cdot s_\Gamma(n_9)$$

$$+ \sum_{\substack{n_1+\ldots+n_{10}=n-15 \\ n_i\geq 1}} s_\Gamma(n_1)\cdot\ldots\cdot s_\Gamma(n_{10}) \pmod 2$$

with initial values:

$s_\Gamma(1) = 1$	$s_\Gamma(2) = 0$	$s_\Gamma(3)=0$
$s_\Gamma(4)=0$	$s_\Gamma(5)= 26$	$s_\Gamma(6) = 144$
$s_\Gamma(7)=168$	$s_\Gamma(8)=64$	$s_\Gamma(9)=9$
$s_\Gamma(10)=16800$	$s_\Gamma(11)=160512$	$s_\Gamma(12)=387648$
$s_\Gamma(13)=369460$	$s_\Gamma(14)=171024$	$s_\Gamma(15)=43503120$
$s_\Gamma(16)=571582336$	$s_\Gamma(17)=2075904340$	$s_\Gamma(18)=3153767616$
$s_\Gamma(19)=2481628608$	$s_\Gamma(20)= 289187863040$	$s_\Gamma(21)=4725118414170$
$s_\Gamma(22)=22759304428608$	$s_\Gamma(23)=47243815037640$	$s_\Gamma(24)=52209437187136.$

Proof:
The proof follows the same principle as the one of Proposition 3.11. We suppress the massive calculations involved, but present the differential equation for the generating function $F(z)$ and the higher derivatives occurring in this differential equation. First, after expressing each H_{n-i} in the Hadamard product of Proposition 3.14 for $F(z)$ like

in 3.9 we obtain the following differential equation:

$$
\begin{aligned}
F'(z) = & (\underline{1} + \underline{25\ z^4} + 24\ z^5 + 12\ z^6 - 96\ z^7 - 144\ z^8 - 24192\ z^9 - 8064\ z^{10} \\
& + 72576\ z^{12} + 9580032\ z^{14})\ F(z) \\
& + (120\ z^5 + 36\ z^6 + 8\ z^7 - 624\ z^8 - 336\ z^9 - 94752\ z^{10} - 26496\ z^{11} \\
& + 532224\ z^{13} + 84768768\ z^{15})\ F'(z) \\
& + (\ \underline{z^4} + 120\ z^6 + 12\ z^7 + \ \underline{z^8} - 768\ z^9 - 192\ z^{10} - 138528\ z^{11} - 23472\ z^{12} \\
& + 857088\ z^{14} + 169900416\ z^{16})\ F^{(2)}(z) \\
& + (36\ z^7 + \ \underline{z^8} - 304\ z^{10} - 36\ z^{11} - 79704\ z^{12} - 8136\ z^{13} + 527904\ z^{15} \\
& + 135390528\ z^{17})\ F^{(3)}(z) \\
& + (\underline{3\ z^8} - 44\ z^{11} - 2\ z^{12} - 20544\ z^{13} - 1248\ z^{14} + 153024\ z^{16} \\
& + 53419968\ z^{18})\ F^{(4)}(z) \\
& + (-2\ z^{12} - 2532\ z^{14} - 84\ z^{15} + 22800\ z^{17} + 11626272\ z^{19})\ F^{(5)}(z) \\
& + (-144\ z^{15} - 2\ z^{16} + 1776\ z^{18} + 1471680\ z^{20})\ F^{(6)}(z) \\
& + (\underline{-3\ z^{16}} + 68\ z^{19} + 109992\ z^{21})\ F^{(7)}(z) \\
& + (\ \underline{z^{20}} + 4740\ z^{22})\ F^{(8)}(z) \\
& + 108\ z^{23}\ F^{(9)}(z) \\
& + \ \underline{z^{24}}\ F^{(10)}(z)
\end{aligned} \tag{3.30}
$$

Inspecting $F'(z)$ we see, that most of the coefficients are even, hence do not influence the parity of $F'(z)$, respectively the parity of $s_\Gamma(n)$. The terms which are of effect on $s_\Gamma(n) \pmod 2$, the odd coefficients, are underlined. In order to obtain the recursive description of $s_\Gamma(n)$ by coefficient comparing, we need to transform the differential equation 3.30 to one for $G(z)$ by use of the identity $G(z) = \frac{F'(z)}{F(z)}$. Additionally to the higher derivatives already used in Proposition 3.11 we need to calculate $F^{(4)}, ..., F^{(10)}$ and present $F^{(2)}(z)$ again.

$$F^{(2)}(z) = \underline{G'(z)\ F(z)} + \underline{G(z)^2\ F(z)}$$

$$\begin{aligned}F^{(4)}(z) = &G^{(3)}(z)F(z)\ + 4\ G^{(2)}(z)\ G(z)F(z)\ + 3\ G'(z)^2\ F(z)\ + 5\ G(z)^2\ G'(z)F(z)\\ &+\ G(z)^3\ G'(z)F(z)\ +\ G(z)^4F(z)\end{aligned}$$

$$\begin{aligned}F^{(5)}(z) = &G^{(4)}(z)\ F(z)\ + 5\ G^{(3)}(z)\ G(z)\ F(z)\ + 10\ G^{(2)}(z)\ G'(z)\ F(z)\\ &+ 10\ G^{(2)}(z)\ G(z)^2\ F(z)\ + 11\ G'(z)^3\ F(z)\ + 4\ G'(z)^2\ G(z)^2\ F(z)\\ &+ 9\ G'(z)\ G(z)^3\ F(z)\ +\ G'(z)\ G(z)^4\ F(z)\ +\ G(z)^5\end{aligned}$$

$$\begin{aligned}F^{(6)}(z) = &G^{(5)}(z)\ F(z)\ + 6\ G^{(4)}(z)\ G(z)\ F(z)\ + 15\ G^{(3)}(z)\ G'(z)\ F(z)\\ &+ 15\ G^{(3)}(z)\ G(z)^2\ F(z)\ + 10\ G^{(2)}(z)\ G(z)^2\ F(z)\ + 50\ G^{(2)}(z)\ G'(z)\ G(z)\ F(z)\\ &+ 10\ G^{(2)}(z)\ G'(z)\ G(z)^2\ F(z)\ + 20\ G^{(2)}(z)\ G(z)^3\ F(z)\ + 15\ G'(z)^3\ F(z)\\ &+ 36\ G'(z)^2\ G(z)^2\ F(z)\ + 9\ G'(z)^2\ G(z)^3\ F(z)\ + 14\ G'(z)\ G(z)^4\ F(z)\\ &+\ G'(z)\ G(z)^5\ F(z)\ +\ G(z)^6\end{aligned}$$

$$\begin{aligned}F^{(7)}(z) = &G^{(6)}(z)\ F(z)\ + 7\ G^{(5)}(z)\ G(z)\ F(z)\ + 21\ G^{(4)}(z)\ G(z)^2\ F(z)\\ &+ 35\ G^{(3)}(z)\ G^{(2)}(z)\ F(z)\ + 85\ G^{(3)}(z)\ G'(z)\ G(z)\ F(z)\\ &+ 20\ G^{(3)}(z)\ G'(z)\ G(z)^2\ F(z)\ + 35\ G^{(3)}(z)\ G(z)^3\ F(z)\\ &+ 60\ G^{(2)}(z)^2\ G(z)\ F(z)\ + 105\ G^{(2)}(z)\ G'(z)^2\ F(z)\\ &+ 135\ G^{(2)}(z)\ G'(z)\ G(z)^2\ F(z)\ + 25\ G^{(2)}(z)\ G'(z)\ G(z)^3\ F(z)\\ &+ 35\ G^{(2)}(z)\ G(z)^4\ F(z)\ + \underline{81\ G'(z)^3\ G(z)\ F(z)}\\ &+ 24\ G'(z)^3\ G(z)^2\ F(z)\ + 90\ G'(z)^2\ G(z)^3\ F(z)\\ &+ \underline{15\ G'(z)^2\ G(z)^4\ F(z)}\ + 20\ G'(z)\ G(z)^5\ F(z)\\ &+ 10\ G^{(2)}(z)\ G(z)^3\ F(z)\ + 50\ G^{(2)}(z)\ G'(z)\ G(z)^2\ F(z)\\ &+\ \underline{G'(z)\ G(z)^6\ F(z)}\ +\ \underline{G(z)^7}\end{aligned}$$

$$
\begin{aligned}
F^{(8)}(z) = {} & G^{(7)}(z)\ F(z)\ + 8\ G^{(6)}(z)\ G'(z)\ F(z)\ + 28\ G^{(5)}(z)\ G(z)^2\ F(z) \\
& + 56\ G^{(4)}(z)\ G^{(2)}(z)\ F(z)\ + 133\ G^{(4)}(z)\ G'(z)\ G(z)\ F(z) \\
& + 35\ G^{(4)}(z)\ G'(z)\ G(z)^2\ F(z)\ + 56\ G^{(4)}(z)\ G(z)^3\ F(z) \\
& + 35\ G^{(3)}(z)^2\ F(z)\ + 280\ G^{(3)}(z)\ G^{(2)}(z)\ G(z)\ F(z) \\
& + 210\ G^{(3)}(z)\ G'(z)^2\ F(z)\ + 365\ G^{(3)}(z)\ G(z)^2\ G'(z)\ F(z) \\
& + 55\ G^{(3)}(z)\ G'(z)\ G(z)^3\ F(z)\ + 70\ G^{(3)}(z)\ G(z)^4\ F(z) \\
& + 210\ G^{(2)}(z)^2\ G'(z)\ F(z)\ + 270\ G^{(2)}(z)^2\ G(z)^2\ F(z) \\
& + 686\ G^{(2)}(z)\ G'(z)^2\ G(z)\ F(z)\ + 154\ G^{(2)}(z)\ G'(z)^2\ G(z)^2\ F(z) \\
& + 464\ G^{(2)}(z)\ G'(z)\ G(z)^3\ F(z)\ + 46\ G^{(2)}(z)\ G'(z)\ G(z)^4\ F(z) \\
& + 56\ G^{(2)}(z)\ G(z)^5\ F(z)\ + \underline{105\ G'(z)^4\ F(z)} \\
& + \underline{333\ G'(z)^3\ G(z)^2\ F(z)}\ + \underline{87\ G'(z)^3\ G(z)^3\ F(z)} \\
& + 188\ G'(z)^2\ G(z)^4\ F(z)\ + 22\ G'(z)^2\ G(z)^5\ F(z) \\
& + \underline{27\ G'(z)\ G(z)^6\ F(z)}\ + 10\ G^{(2)}(z)\ G(z)^4\ F(z) \\
& + 50\ G^{(2)}(z)\ G'(z)\ G(z)^3\ F(z)\ + \ \underline{G'(z)\ G(z)^7\ F(z)}\ +\ \underline{G(z)^8}
\end{aligned}
$$

$$
\begin{aligned}
F^{(9)}(z) = {} & G^{(8)}(z)\ F(z)\ + 9\ G^{(7)}(z)\ G(z)\ F(z)\ + 36\ G^{(6)}(z)\ G'(z)\ F(z) \\
& + 36\ G^{(6)}(z)\ G(z)^2\ F(z)\ + 84\ G^{(5)}(z)\ G^{(2)}(z)\ F(z) \\
& + 196\ G^{(6)}(z)\ G'(z)\ G(z)\ F(z)\ + 56\ G^{(5)}(z)\ G'(z)\ G(z)^2\ F(z) \\
& + 84\ G^{(5)}(z)\ G(z)^3\ F(z)\ + 129\ G^{(4)}(z)\ G^{(3)}(z)\ F(z) \\
& + 504\ G^{(4)}(z)\ G^{(2)}(z)\ G(z)\ F(z)\ + 378\ G^{(4)}(z)\ G'(z)^2\ F(z) \\
& + 651\ G^{(4)}(z)\ G'(z)\ G(z)^2\ F(z)\ + 105\ G^{(4)}(z)\ G'(z)\ G(z)^3\ F(z) \\
& + 126\ G^{(4)}(z)\ G(z)^4\ F(z)\ + 315\ G^{(3)}(z)^2\ G(z)\ F(z) \\
& + 1260\ G^{(3)}(z)\ G^{(2)}(z)\ G'(z)\ F(z)\ + 1260\ G^{(3)}(z)\ G^{(2)}(z)\ G(z)^2\ F(z) \\
& + 1506\ G^{(3)}(z)\ G'(z)^2\ G(z)\ F(z)\ + 384\ G^{(3)}(z)\ G'(z)^2\ G(z)^2\ F(z) \\
& + 1149\ G^{(3)}(z)\ G'(z)\ G(z)^3\ F(z)\ + 111\ G^{(3)}(z)\ G'(z)\ G(z)^4\ F(z) \\
& + 126\ G^{(3)}(z)\ G(z)^5\ F(z)\ + 28\ G^{(2)}(z)^2\ G(z)^2\ F(z) \\
& + 2090\ G^{(2)}(z)^2\ G'(z)\ G(z)\ F(z)\ + 280\ G^{(2)}(z)^2\ G'(z)\ G(z)^2\ F(z) \\
& + 830\ G^{(2)}(z)^2\ G(z)^3\ F(z)\ + 1260\ G^{(2)}(z)\ G'(z)^3\ F(z) \\
& + 2774\ G^{(2)}(z)\ G'(z)^2\ G(z)^2\ F(z)\ + 606\ G^{(2)}(z)\ G'(z)^2\ G(z)^3\ F(z) \\
& + 1136\ G^{(2)}(z)\ G'(z)\ G(z)^4\ F(z)\ + 74\ G^{(2)}(z)\ G'(z)\ G(z)^5\ F(z) \\
& + 84\ G^{(2)}(z)\ G(z)^6\ F(z)\ + \underline{753\ G'(z)^4\ G(z)\ F(z)} \\
& + 192\ G'(z)^4\ G(z)^2\ F(z)\ + \underline{1053\ G'(z)^3\ \ G(z)^3\ F(z)} \\
& + \underline{207\ G'(z)^3\ G(z)^4\ F(z)}\ + 348\ G'(z)^2\ G(z)^5\ F(z) \\
& + 80\ G^{(2)}(z)\ G'(z)\ G(z)^3\ F(z)\ + 400\ G^{(2)}(z)\ G'(z)^2\ G(z)^2\ F(z) \\
& + 30\ G'(z)^2\ G(z)^6\ F(z)\ + 8\ G(z)^7\ F(z)\ + \underline{27\ G'(z)\ G(z)^7\ F(z)} \\
& + 10\ G^{(2)}(z)\ G(z)^5\ F(z)\ + 50\ G^{(2)}(z)\ G'(z)\ G(z)^4\ F(z) \\
& +\ \underline{G'(z)\ G(z)^8\ F(z)}\ +\ \underline{G(z)^9}
\end{aligned}
$$

$$
\begin{aligned}
F^{(10)}(z) = {} & G^{(9)}(z)\ F(z)\ +10\ G^{(8)}(z)\ G(z)\ F(z)\ +45\ G^{(7)}(z)\ G'(z)\ F(z) \\
& +45\ G^{(7)}(z)\ G(z)^2\ F(z)\ +120\ G^{(6)}(z)\ G^{(2)}(z)\ F(z) \\
& +276\ G^{(6)}(z)\ G'(z)\ G(z)\ F(z)\ +84\ G^{(6)}(z)\ G'(z)\ G(z)^2\ F(z) \\
& +120\ G^{(6)}(z)\ G(z)^3\ F(z)\ +210\ G^{(5)}(z)\ G^{(3)}(z)\ F(z) \\
& +840\ G^{(5)}(z)\ G^{(2)}(z)\ G(z)\ F(z)\ +630\ G^{(5)}(z)\ G'(z)^2\ F(z) \\
& +1078\ G^{(5)}(z)\ G'(z)\ G(z)^2\ F(z)\ +182\ G^{(5)}(z)\ G'(z)\ G(z)^3\ F(z) \\
& +210\ G^{(5)}(z)\ G(z)^4\ F(z)\ +126\ G^{(4)}(z)\ G^{(4)}(z)\ F(z) \\
& +1263\ G^{(4)}(z)\ G^{(3)}(z)\ G(z)\ F(z)\ +2520\ G^{(4)}(z)\ G^{(2)}(z)\ G'(z)\ F(z) \\
& +2520\ G^{(4)}(z)\ G^{(2)}(z)\ G(z)^2\ F(z)\ +2961\ G^{(4)}(z)\ G'(z)^2\ G(z)\ F(z) \\
& +819\ G^{(4)}(z)\ G'(z)^2\ G(z)^2\ F(z)\ +2289\ G^{(4)}(z)\ G'(z)\ G(z)^3\ F(z) \\
& +231\ G^{(4)}(z)\ G'(z)\ G(z)^4\ F(z)\ +252\ G^{(4)}(z)\ G(z)^5\ F(z) \\
& +1575\ G^{(3)}(z)^2\ G(z)^2\ F(z)\ +1575\ G^{(3)}(z)^2\ G'(z)\ F(z) \\
& +840\ G^{(3)}(z)\ G^{(2)}(z)\ G(z)^2\ F(z)\ +11040\ G^{(3)}(z)\ G^{(2)}(z)\ G'(z)\ G(z)\ F(z) \\
& +1560\ G^{(3)}(z)\ G^{(2)}(z)\ G'(z)\ G(z)^2\ F(z) \\
& +4200\ G^{(3)}(z)\ G^{(2)}(z)\ G(z)^3\ F(z)\ +3150\ G^{(3)}(z)\ G'(z)^3\ F(z) \\
& +7815\ G^{(3)}(z)\ G'(z)^2\ G(z)^2\ F(z)\ +1635\ G^{(3)}(z)\ G'(z)^2\ G(z)^3\ F(z) \\
& +2955\ G^{(3)}(z)\ G'(z)\ G(z)^4\ F(z)\ +195\ G^{(3)}(z)\ G'(z)\ G(z)^5\ F(z) \\
& +210\ G^{(3)}(z)\ G(z)^6\ F(z)\ +1260\ G^{(3)}(z)\ G^{(2)}(z)^2\ F(z) \\
& +2160\ G^{(2)}(z)^3\ \ G(z)\ F(z)\ +5670\ G^{(2)}(z)^2\ G'(z)^2\ F(z) \\
& +9380\ G^{(2)}(z)^2\ G'(z)\ G(z)^2\ F(z)\ +1180\ G^{(2)}(z)^2\ G'(z)\ G(z)^3\ F(z) \\
& +2090\ G^{(2)}(z)^2\ G(z)^4\ F(z)\ +10350\ G^{(2)}(z)\ G'(z)^3\ G(z)\ F(z) \\
& +2250\ G^{(2)}(z)\ G'(z)^3\ G(z)^2\ F(z)\ +10190\ G^{(2)}(z)\ G'(z)^2\ G(z)^3\ F(z) \\
& +1560\ G^{(2)}(z)\ G'(z)^2\ G(z)^4\ F(z)\ +2360\ G^{(2)}(z)\ G'(z)\ G(z)^5\ F(z) \\
& +360\ G^{(2)}(z)^2\ G(z)^3\ F(z)\ +1800\ G^{(2)}(z)^2\ G'(z)\ G(z)^2\ F(z) \\
& +36\ G^{(2)}(z)\ G'(z)\ G(z)^6\ F(z)\ +120\ G^{(2)}(z)\ G(z)^7\ F(z) \\
& +\underline{945\ G'(z)^5\ F(z)}\ +3750\ G'(z)^4\ G(z)^2\ F(z) \\
& +\underline{975\ G'(z)^4\ G(z)^3\ F(z)}\ +\underline{2745\ G'(z)^3\ G(z)^4\ F(z)} \\
& +\underline{405\ G'(z)^3\ G(z)^5\ F(z)}\ +\underline{591\ G'(z)^2\ G(z)^6\ F(z)} \\
& +170\ G^{(2)}(z)\ G'(z)\ G(z)^4\ F(z)\ +850\ G^{(2)}(z)\ G'(z)^2\ G(z)^3\ F(z) \\
& +\underline{39\ G'(z)^2\ G(z)^7\ F(z)}\ +8\ G(z)^8\ F(z)\ +36\ G'(z)\ G(z)^8\ F(z) \\
& +10\ G^{(2)}(z)\ G(z)^6\ F(z)\ +50\ G^{(2)}(z)\ G'(z)\ G(z)^5\ F(z) \\
& +\ \underline{G'(z)\ G(z)^9\ F(z)}\ +\ \underline{G(z)^{10}\ F(z)}\ +28\ G^{(2)}(z)\ G^{(2)}(z)\ G(z)^2
\end{aligned}
$$

Regarding the derivatives, we note, that most of the terms have even coefficients, hence do not matter for $s_\Gamma(n)$ modulo 2. Many terms include derivatives $G^{(i)}(z),\ i \geq 2$, where

$$G^{(i)}(z) = \sum_{n=0}^{\infty}(n+1)\cdot \ldots \cdot (n+i)s_\Gamma(n+i+1)\ z^n,$$

hence at least one factor is even and so the whole sum. In the Cauchy product containing such an $G^{(i)}(z),\ i \geq 2$, we get also those factors belonging to $G^{(i)}(z)$ in every summand, hence, these sums are congruent 0 modulo 2. The terms which remain to be considered, because of odd coefficients or missing derivatives $G^{(i)}(z),\ i \geq 2$ are underlined. We only consider the remaining terms and proceed - like in Proposition 3.11 - with the calculation of the different Cauchy products. Then we are able to compare coefficients and get the recursion 3.29 for $s_\Gamma(n) \pmod 2$. $\square$

Now we obtained the desired recurrence for $s_\Gamma(n)$ and start with the investigation of the divisibility properties of this function.

Theorem 3.16 *Let $\Gamma = C_5 * C_5$. Then $s_\Gamma(n)$ is odd if and only if n is of the form*

$$n = \frac{2\cdot 4^k - 5}{3} \qquad (*)$$

with $k \geq 1$.

Proof:
We analyze the recurrence for $s_\Gamma(n)$ given by Proposition 3.15 and prove the assertion via induction on n. For $k = 1$ we have $s_\Gamma(1) = 1 \equiv 1 \pmod 2$ and for $i = 2, \ldots, 8$ we calculate $s_\Gamma(i) \equiv 0 \pmod 2$. With $k = 2$ we obtain $s_\Gamma(9) = 9 \equiv 1 \pmod 2$. It is easy to see that $n = \frac{2\cdot 4^k-5}{3}$ is always odd, since $2\cdot 4^k - 5 = 3\cdot n$ and $2\cdot 4^k - 5 \equiv 1 \pmod 2$, $k \geq 1$, hence $n \equiv 1 \pmod 2$, too.

Suppose that the assertion is true for all $t \leq n-1$ and consider n. First, we investigate the sums which contain a factor $(n_i - 1)$ originating from the derivative $G'(z)$. Those sums are always congruent $0 \pmod 2$. By our induction hypothesis $s_\Gamma(n_j) \equiv 1 \pmod 2$ if and only if $n_j = \frac{2\cdot 4^k-5}{3}$ and we have that all $n_j < n$. In a sum

$$\sum_{\substack{n_1+\ldots+n_k=n-l \\ n_j \geq 2}} (n_j - 1)\ s_\Gamma(n_1)\cdot \ldots \cdot s_\Gamma(n_k)$$

we get $s_\Gamma(n_1)\cdot \ldots \cdot s_\Gamma(n_k) \equiv 1 \pmod 2$ if and only if n_i is of form $(*)$ for all $i = 1, \ldots, k$. With $n_j \geq 2$ and all n_i are odd we have $(n_j - 1) \equiv 0 \pmod 2$, hence $(n_j - 1)\ s_\Gamma(n_1)\cdot \ldots \cdot s_\Gamma(n_k) \equiv 0 \pmod 2$. In case not all n_i are of the form $(*)$, then at least one n_i is even or odd but not of the form $(*)$. Then by our induction hypothesis $s_\Gamma(n_i) \equiv 0 \pmod 2$ and so also $(n_j - 1)\ s_\Gamma(n_1)\cdot \ldots \cdot s_\Gamma(n_k) \equiv 0 \pmod 2$.

We can reduce the recurrence for $s_\Gamma(n)$ modulo 2 to

$$\begin{aligned} s_\Gamma(n) \equiv\ & (n-4)\ s_\Gamma(n-3) + (n-8)\ s_\Gamma(n-7) \\ & + \sum_{\substack{n_1+n_2=n-3\\ n_i\geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2) \\ & + \sum_{\substack{n_1+n_2=n-7\\ n_i\geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2) \\ & + \sum_{\substack{n_1+\ldots+n_3=n-6\\ n_i\geq 1}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3) \\ & + \sum_{\substack{n_1+\ldots+n_4=n-5\\ n_i\geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4) \\ & - \sum_{\substack{n_1+\ldots+n_7=n-10\\ n_i\geq 1}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ s_\Gamma(n_5)\ s_\Gamma(n_6)\ s_\Gamma(n_7) \\ & + \sum_{\substack{n_1+\ldots+n_8=n-13\\ n_i\geq 1}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ s_\Gamma(n_5)\ s_\Gamma(n_6)\ s_\Gamma(n_7)\ s_\Gamma(n_8) \\ & + \sum_{\substack{n_1+\ldots+n_{10}=n-15\\ n_i\geq 1}} s_\Gamma(n_1)\cdot \ldots \cdot s_\Gamma(n_{10}) \pmod 2 \end{aligned} \tag{3.31}$$

We consider two cases for $n \geq 25$:

A) $n \equiv 0 \pmod 2$

B) $n \equiv 1 \pmod 2$

Case A: $n \equiv 0 \pmod 2$
Then we get $(n-4)\ s_\Gamma(n-3) \equiv 0 \pmod 2$ since $(n-4) \equiv 0 \pmod 2$ and $(n-8)\ s_\Gamma(n-7) \equiv 0 \pmod 2$ since $(n-8) \equiv 0 \pmod 2$. We need to consider for the sums the compositions $n_1 + \ldots + n_k = n - l,\ l \geq 1$, respectively the different possible partitions $n_1, \ldots, n_k$. We observe two possible cases for the partitions with $n \equiv 0 \pmod 2$:

S_1) The number of the n_i is even and $n-l$ is odd.

S_2) The number of the n_i is odd and $n-l$ is even.

For the analysis of each summand $s_\Gamma(n_1) \cdot \ldots \cdot s_\Gamma(n_k)$ we fix w.l.o.g. one n_i, let us say n_1 and consider

i) $2 \mid n_1$

Then by our induction hypothesis $s_\Gamma(n_1) \equiv 0 \pmod 2$, hence, $s_\Gamma(n_1) \cdot ... \cdot s_\Gamma(n_k) \equiv 0 \pmod 2$.

ii) $2 \nmid n_1$

Here we investigate the two cases:

S_1) $2 \mid k$ and $2 \nmid n - l$

With $2 \nmid n_1$ and $2 \nmid n - l$ we have $n_2 + ... + n_k \equiv 0 \pmod 2$ with an odd number of summands. Hence, at least one n_i, say n_2, is even, and by our induction hypothesis we get $s_\Gamma(n_2) \equiv 0 \pmod 2$, further,$s_\Gamma(n_1)\cdot...\cdot s_\Gamma(n_k) \equiv 0 \pmod 2$.

S_2) $2 \nmid k$ and $2 \mid n - l$

From $2 \nmid n_1$ and $2 \mid n - l$ follows $n_2 + ... + n_k \equiv 1 \pmod 2$, with an even number of summands. Not all n_i can be odd, at least one, say n_2, is congruent 0 modulo 2. Hence, by induction hypothesis $s_\Gamma(n_2) \equiv 0 \pmod 2$ and also $s_\Gamma(n_1) \cdot ... \cdot s_\Gamma(n_k) \equiv 0 \pmod 2$.

For $n \equiv 0 \pmod 2$ we obtain that all sums and the first terms in the recurrence for $s_\Gamma(n)$ are congruent 0 hence, $s_\Gamma(n)$ vanishes modulo 2.

Case B: $n \equiv 1 \pmod 2$

We have to show that $s_\Gamma(n) \equiv 1 \pmod 2$ if and only if $n = \frac{2 \cdot 4^k - 5}{3}$ and $s_\Gamma(n) \equiv 0 \pmod 2$ for n odd but not of the form $n = \frac{2 \cdot 4^k - 5}{3}$. With n odd $(n-3)$ and $(n-7)$ are even we get by the considerations above (Case A) $s_\Gamma(n-3) \equiv s_\Gamma(n-7) \equiv 0 \pmod 2$. It is left to analyze the different sums for the cases where

$$\sum_{\substack{n_1+...+n_k=n-l \\ n_i \geq 1}} s_\Gamma(n_1) \cdot ... \cdot s_\Gamma(n_k) \quad \equiv 1 \pmod 2,$$

which is possible if and only if there is an odd number of partitions $n_1, ..., n_k$ of $n - l$ with all n_i of the form $(*)$. We use a shorthand notation of partitions for the summands in each sum. For example, consider $n_1 + n_2 = n - 3$. We calculate the partitions of 2, denoted as 1^2 and 2^1, which means that for 1^2 we have different values for n_1 and n_2, and 2^1 encodes that we have one pair of equal values, $n_1 = n_2$. We get a representation of n as $n = \frac{1}{3}(2 \cdot (4_1^k + 4_2^k) - 1)$ for the case 1^2 with two different values for $n_1 = \frac{1}{3}(2 \cdot 4^{k_1} - 5)$, and $n_2 = \frac{1}{3}(2 \cdot 4^{k_2} - 5)$, $k_1, k_2 \geq 1$, both of form $(*)$ with $k_1 \neq k_2$. The number of permutations of such partitions is given by the multinomial coefficient. For the case 1^2 we calculate $\frac{2!}{1!\,1!} = 2$, hence an even number and so of no influence on the parity of $\sum_{n_1+n_2=n-3} s_\Gamma(n_1)\, s_\Gamma(n_2)$. The second multinomial coefficient calculates to $\frac{2!}{2!} = 1$ and we have to consider this representation for n of the form $(*)$ with two n_1, n_2 of the form $(*)$, as well. The following table shows all possible partitions for each sum with the different n_i of form $(*)$ and the equation for n together with the number of permutations.

Partition	**Permutations**	**Representation of n**
	First sum $m = 2$	
1^2 2^1	2 1	- $n = \frac{1}{3}(4^{k+1} - 1)$
	Second sum $m = 2$	
1^2 2^1	2 1	- $n = \frac{1}{3}(4^{k+1} + 11)$
	Third sum $m = 3$	
$1^1\ 2^1$ 3^1	3 1	$n = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 3)$ $n = \frac{1}{3}(6 \cdot 4^k + 3) = 2 \cdot 4^k + 1$
	Fourth sum $m = 4$	
1^4 $1^2\ 2^1$ 2^2 $1^1\ 3^1$ 4^1	24 12 6 4 1	- - - - $n = \frac{1}{3}(2 \cdot 4^{k+1} - 5)$
	Fifth sum $m = 7$ (Remaining cases)	
$1^1\ 2^1\ 4^1$ $3^1\ 4^1$ $1^1\ 6^1$ 7^1	105 35 7 1	$n = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5)$ $n = \frac{1}{3}(6 \cdot 4^{k_1} + 2 \cdot 4^{k_2+1} - 5)$ $n = \frac{1}{3}(2 \cdot 4^{k_1} + 3 \cdot 4^{k_2+1} - 5)$ $n = \frac{1}{3}(14 \cdot 4^k - 5)$
	Sixth sum $m = 8$ (Remaining cases)	
8^1	1	$n = \frac{1}{3}(4^{k+2} - 1)$
	Seventh sum $m = 10$ (Remaining cases)	
$2^1\ 8^1$ 10^1	45 1	$n = \frac{1}{3}(4^{k_1+1} + 4^{k_2+2} - 5)$ $n = \frac{1}{3}(5 \cdot 4^{k+1} - 5)$

We obtain from the fourth sum listed in the table that if $n = \frac{1}{3}(2 \cdot 4^{k+1} - 5)$, hence of the form $(*)$, then $\sum_{\substack{n_1+\ldots+n_4=n-5 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4) \equiv 1 \pmod 2$. It remains to show that all other remaining sums are with this n of form $(*)$ congruent 0 modulo 2. Thatfore, we prove if $n = \frac{1}{3}(2 \cdot 4^{k+1} - 5)$ for an $k \geq 1$ then it has no other representation of n listed in the table. We set $n = \frac{1}{3}(4^{k'+1} - 1) = \frac{1}{3}(2 \cdot 4^{k+1} - 5)$ which is equivalent to

$4^{k'+1} + 4 = 2 \cdot 4^{k+1}$ and rewrite $4 \cdot (4^{k'} + 1) = 4 \cdot (2 \cdot 4^k)$. Then

$$\begin{aligned} 4^{k'} + 1 &\equiv \begin{cases} 1(4) & \text{if } k' \geq 1 \\ 2(4) & \text{if } k' = 0 \end{cases} \\ 2 \cdot 4^k &\equiv \begin{cases} 0(4) & \text{if } k \geq 1 \\ 2(4) & \text{if } k = 0 \end{cases} \end{aligned} \tag{3.32}$$

We get equality if and only if $k' = k = 0$ and calculate with those values $n = 1$ and we know already that $s_\Gamma(1) \equiv 1 \pmod 2$. Proceeding with analogue calculations for the remaining cases in our table we get if n is of the form $(*)$ no other presentation for n also occurring in the table is possible. Hence, if n is of the form $(*)$ then $s_\Gamma(n) \equiv 1 \pmod 2$.

It is left to prove if $n \equiv 1 \pmod 2$ and not of the form $(*)$ then $s_\Gamma(n) \equiv 0 \pmod 2$, which is the case when

$$\begin{aligned} s_\Gamma(n) \equiv &\sum_{\substack{n_1+n_2=n-3 \\ n_i \geq 1}} s_\Gamma(n_1)\, s_\Gamma(n_2) \\ &+ \sum_{\substack{n_1+n_2=n-7 \\ n_i \geq 1}} s_\Gamma(n_1)\, s_\Gamma(n_2) \\ &+ \sum_{\substack{n_1+\ldots+n_3=n-6 \\ n_i \geq 1}} s_\Gamma(n_1) s_\Gamma(n_2)\, s_\Gamma(n_3) \\ &- \sum_{\substack{n_1+\ldots+n_7=n-10 \\ n_i \geq 1}} s_\Gamma(n_1) s_\Gamma(n_2)\, s_\Gamma(n_3)\, s_\Gamma(n_4)\, s_\Gamma(n_5)\, s_\Gamma(n_6)\, s_\Gamma(n_7) \\ &+ \sum_{\substack{n_1+\ldots+n_8=n-13 \\ n_i \geq 1}} s_\Gamma(n_1) s_\Gamma(n_2)\, s_\Gamma(n_3)\, s_\Gamma(n_4)\, s_\Gamma(n_5)\, s_\Gamma(n_6)\, s_\Gamma(n_7)\, s_\Gamma(n_8) \\ &+ \sum_{\substack{n_1+\ldots+n_{10}=n-15 \\ n_i \geq 1}} s_\Gamma(n_1) \cdot \ldots \cdot s_\Gamma(n_{10}) \\ &\equiv 0 \pmod 2 \end{aligned} \tag{3.33}$$

For our investigations of the sums we can neglect the fourth sum, since it is congruent 1 modulo 2 if and only if n is of the form $(*)$.

First, we have to check if n is odd and of one of the forms listed in the table, then there might be other representations for such an n also listed in the table. We consider first the different representations of n belonging to the same sum:

I. Third sum:

$$n = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 3) = \frac{1}{3}(6 \cdot 4^{l} + 3) \tag{3.34}$$

with $k_1 \neq k_2$, then $4^{k_1} + 2 \cdot 4^{k_2} = 3 \cdot 4^{l}$. We consider

a) $k_1 = k_2 = l \quad \lightning$

b) $k_1 < k_2 < l$
$4^{k_1}\underbrace{(1 + 2 \cdot 4^{k_2-k_1})}_{\equiv 1(4)} = 4^{k_1}\underbrace{(3 \cdot 4^{l-k_1})}_{\equiv 0(4)} \quad \lightning$

c) $k_2 < k_1 < l$
$4^{k_2}\underbrace{(4^{k_1-k_2} + 2)}_{\equiv 2(4)} = 4^{k_2}\underbrace{(3 \cdot 4^{l-k_2})}_{\equiv 0(4)} \quad \lightning$

Hence, both representations of n are not possible.

II. Fifth sum:
Here we have to consider 60 possible combinations of representations of n. We skip the calculations which are quite similar to the one above and present the cases which are possible for several representations of n:

(3.35)

a) $n = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5) = \frac{1}{3}(6 \cdot 4^{l_1} + 2 \cdot 4^{l_2+1} - 5)$
with $k_1 = l_2 + 1,\ k_2 = l_1,\ k_3 = l_1 - 1$, and $k_1 < k_3 < k_2$.

b) $n = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5) = \frac{1}{3}(2 \cdot 4^{l_1} + 3 \cdot 4^{l_2+1} - 5)$
with $k_1 = l_2 + 1,\ k_2 = l_2$, and $k_3 = l_1 - 1$.

c) $n = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5) = \frac{1}{3}(2 \cdot 4^{l_1} + 12 \cdot 4^{l_2} - 5)$
with $k_1 = l_2 + 1,\ k_2 = l_2,\ k_3 = l_1 - 1$, and $k_2 < k_1 < k_3$ and $l_2 < l_1$.

d) $n = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5) = \frac{1}{3}(14 \cdot 4^{l} - 5)$
with $k_1 = l + 1,\ k_2 = l,\ k_3 = l - 1$, and $k_3 < k_2 < k_1$.

III. Seventh sum:

$$(3.36)$$

a) $n = \frac{1}{3}(4^{k_1+1} + 4^{k_2+2} - 5) = \frac{1}{3}(5 \cdot 4^{l+1} - 5)$
with $k_1 = l+1,\ k_2 = l-1$.

With that information we consider now the possible cases for the six remaining sums that every pair of sums is congruent 0 or 1 modulo 2, respectively n has more than one representation. We denote the six sums by 6-tuples with values 0 or 1, stating the value of the corresponding sum modulo 2. Considering each possible combination of the different expressions for the n_i of each pair of sums, a lot of case analysis has to be done to exclude most of the cases. We present for the first two sums the argumentation and suppress the other calculations. By the definition of possible cases for other representations of n of the form $(*)$ we have always for the exponents $k_i \geq 1$ and $k_i \neq k_j$ for all pairs i, j occurring in such a representation.

1.1.0.0.0.0 $\lightning$ Since

$$\sum_{\substack{n_1+n_2=n-3 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2) \equiv \sum_{\substack{n_1+n_2=n-7 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2) \equiv 1 \pmod 2$$

$$\Longrightarrow \quad n = \tfrac{1}{3}(4^{k+1} - 1) = \tfrac{1}{3}(4^{k'+1} + 11)$$

$$\begin{aligned} 4^{k+1} - 1 &= 4^{k'+1} + 11 \\ 4^{k+1} - 4^{k'+1} &= 3 \cdot 4 \\ 4^{k} - 4^{k'} &= 3 \end{aligned} \tag{3.37}$$

$\Longrightarrow \quad k' = 0$ and $k = 1$
$\Longrightarrow \quad n = 5$ and $s_\Gamma(5) \equiv 0 \pmod 2$

1.0.1.0.0.0 $\lightning$ Since

$$\sum_{\substack{n_1+n_2=n-3 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2) \equiv \sum_{\substack{n_1+\ldots+n_3=n-6 \\ n_i \geq 1}} s_\Gamma(n_1) s_\Gamma(n_2)\ s_\Gamma(n_3) \equiv 1 \pmod 2$$

$\Longrightarrow$

i) $n = \frac{1}{3}(4^{k+1} - 1) = 2 \cdot 4^{k'} + 1$

$$\begin{aligned} 4^{k+1} &= 6 \cdot 4^{k'} + 4 \\ &= 2 \cdot 4^{k'} + 4^{k'+1} + 4 \end{aligned}$$

$k' < k$ then $\quad 4^{k'}(4^{k-k'+1} - 6) = 4$
If $k' = 0$ then $4^{k+1} - 6 = 4 \quad \lightning$ or
if $k' \geq 1$ then $4 \cdot (4^k - 6) > 4 \quad \lightning$.

ii) $n = \frac{1}{3}(4^{k+1} - 1) = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 3)$
Then we have $2 \cdot 4^{k_1} + 4^{k_2+1} + 4 = 4^{k+1} \Longrightarrow k_2 < k$. We consider

$$4 = 4^{k+1} - 4^{k_2+1} - 2 \cdot 4^{k_1}$$

If $k_1 \leq k_2$ then $4 = 4^{k_1}(4^{k+1-k_1} - 4^{k_2+1-k_1} - 2)$. If $k_1 = 0$ then $4 \nmid (4^{k+1-k_1} - 4^{k_2+1-k_1} - 2)$ $\lightning$. If $k_1 \geq 1$ then $4 < 4^{k+1-k_1} - 4^{k_2+1-k_1} - 2$ $\lightning$. Finally, if $k_1 > k_2$ then $4 = 4^{k_2+1}(4^{k-k_2} - 1 - 2 \cdot 4^{k_1-k_2-1})$ $\lightning$.

1.0.0.1.0.0 $\lightning$

1.0.0.0.1.0 $n = \frac{1}{3}(4^{k+1} - 1) = \frac{1}{3}(4^{l+2} - 1)$
with $l = k - 1$.

1.0.0.0.0.1 $\lightning$

0.1.1.0.0.0 $\lightning$

0.1.0.1.0.0 $n = \frac{1}{3}(4^{k+1} + 11) = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5)$
with $k_1 = k,\ k_2 = 1,\ k_3 = k - 1$

0.1.0.0.1.0 $\lightning$

0.1.0.0.0.1 $\lightning$

0.0.1.1.0.0

i) $n = 2 \cdot 4^k + 1 = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5)$
with $k_1 = 1,\ k_2 = k,\ k_3 = k - 1$

ii) $n = \frac{1}{3}(2 \cdot 4^{l_1} + 4^{l_2+1} + 3) = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5)$
with

  i) $k_1 = 1,\ k_2 = l_2,\ k_3 = l_1 - 1$
  ii) $k_1 = l_2,\ k_2 = 1,\ k_3 = l_2 - 1,\ l_1 = 1$

iii) $n = \frac{1}{3}(2 \cdot 4^{l_1} + 4^{l_2+1} + 3) = \frac{1}{3}(6 \cdot 4^{k_1} + 2 \cdot 4^{k_2+1} - 5)$
with $k_1 = 1,\ k_2 = l_1 - 1,\ l_2 = 1$

iv) $n = \frac{1}{3}(2 \cdot 4^{l_1} + 4^{l_2+1} + 3) = \frac{1}{3}(2 \cdot 4^{k_1+1} + 10 \cdot 4^{k_2} - 5)$
with $k_1 = l_2,\ k_2 = 1,\ l_1 = 2$

v) $n = \frac{1}{3}(2 \cdot 4^{l_1} + 4^{l_2+1} + 3) = \frac{1}{3}(2 \cdot 4^{k_1} + 12 \cdot 4^{k_2} - 5)$
with $k_1 = 1,\ k_2 = l_2,\ l_1 = l_2 + 1$

0.0.1.0.1.0 $\lightning$

0.0.1.0.0.1 $n = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 3) = \frac{1}{3}(4^{l_1+1} + 4^{l_2+2} - 5)$
with $k_1 = 1,\ k_2 = l_2 + 1,\ l_1 = 1$

0.0.0.1.1.0 $\lightning$

0.0.0.1.0.1

i) $n = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5) = \frac{1}{3}(4^{l_1+1} + 4^{l_2+2} - 5)$
with $k_1 = l_1\ k_2 = l_2 + 1,\ k_3 = l_1 - 1$

ii) $n = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5) = \frac{1}{3}(5 \cdot 4^{l+1} - 5)$
with $k_1 = l,\ k_2 = l+1,\ k_3 = l-1$

iii) $n = \frac{1}{3}(2 \cdot 4^{k_1} + 3 \cdot 4^{k_2+1} - 5) = \frac{1}{3}(5 \cdot 4^{l+1} - 5)$
with $k_1 = l+1,\ k_2 = l$

0.0.0.0.1.1 $\lightning$

Now we consider every sum congruent 1 modulo 2 and the combinations with all 5 other sums with regard to the different representations for n.

First sum:

$$\sum_{\substack{n_1+n_2=n-3 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2)$$

With 1.1.0.0.0.0 $\lightning$, 1.0.1.0.0.0 $\lightning$, 1.0.0.1.0.0 $\lightning$, and 1.0.0.0.0.1 $\lightning$ only the case

$$n = \frac{1}{3}(4^{k+1} - 1) = \frac{1}{3}(4^{l+2} - 1)$$

is left with $l = k-1$. For n not of the form $(*)$, n has a representation in the first and fifth sum with n_i of the form $(*)$, but in both sums an odd number of permutations of such representations. So, both sums are congruent 1 modulo 2, hence $s_\Gamma(n) \equiv 0 \pmod 2$.

Second sum:

$$\sum_{\substack{n_1+n_2=n-7 \\ n_i \geq 1}} s_\Gamma(n_1)\ s_\Gamma(n_2)$$

With 1.1.0.0.0.0 $\lightning$, 0.1.1.0.0.0 $\lightning$, 0.1.0.0.1.0 $\lightning$, and 0.1.0.0.0.1 $\lightning$ it is left

$$n = \frac{1}{3}(4^{k+1} + 11) = \frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5)$$

with $k_1 = k,\ k_2 = 1,\ k_3 = k-1$. With $k_2 < k_3 < k_1$ we find no other representation of $\frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5)$ in 3.35. Again, for n not of the form $(*)$, n has a representation in the second and fourth sum with n_i of the form $(*)$, but in both sums an odd number of permutations of such representations. So, both sums are congruent 1 modulo 2, hence $s_\Gamma(n) \equiv 0 \pmod 2$.

Third sum:

$$\sum_{\substack{n_1+\ldots+n_3=n-6\\ n_i\geq 1}} s_\Gamma(n_1)s_\Gamma(n_2)\ s_\Gamma(n_3)$$

Here, we detected 1.0.1.0.0.0 $\lightning$, 0.1.1.0.0.0 $\lightning$ and 0.0.1.0.1.0 $\lightning$. The two cases 0.0.1.0.0.1 and 0.0.1.1.0.0 remain. Since both cases $n = 2\cdot 4^k+1$ and $n=\frac{1}{3}(2\cdot 4^{l_1}+4^{l_2+1}+3)$ are not possible by (3.34), we have because of 0.0.1.1.0.0 $n = 2\cdot 4^k+1 = \frac{1}{3}(2\cdot 4^{k_1}+4^{k_2+1}+2\cdot 4^{k_3+1}-5)$ with $k_1=1,\ k_2=k,\ k_3=k-1$, and $1=k_1<k_3<k_2$, and this order of the k_i is case 3.35 a). But with $k_1=1$ we get $l_2=0$ $\lightning$ because all exponents are greater or equal 1. Hence, only this additional representation for $n=2\cdot 4^k+1$ is possible and so the third sum is equivalent 1 modulo 2 and also the fourth, hence $s_\Gamma(n)\equiv 0 \pmod 2$.

For $n = \frac{1}{3}(2\cdot 4^{l_1}+4^{l_2+1}+3) = \frac{1}{3}(2\cdot 4^{k_1}+4^{k_2+1}+2\cdot 4^{k_3+1}-5)$ we distinguish by 0.0.1.1.0.0. ii) two cases for l_1. Let $l_1=1$ (case ii)ii) , then $1=k_2<k_3<k_1$ and the case ii)i) is not possible, since then we would get $k_3=l_1-1=0$ $\lightning$. With $l_1=1$ also none of the cases iii), iv), and v) are possible. With 0.0.1.0.0.1 we get $n=\frac{1}{3}(2\cdot 4^{l_1}+4^{l_2+1}+3)=\frac{1}{3}(4^{l_1'+1}+4^{l_2'+2}-5)$ with $l_1'=1,\ l_2'=l_2-1$ and with these specific values we also obtain $n=\frac{1}{3}(2\cdot 4^{l_1}+4^{l_2+1}+3)=\frac{1}{3}(2\cdot 4^{1}+4^{l_2+1}+3)=\frac{1}{3}(4^{l_2+1}+11)$ which is a representation for n in the second sum. In total, we calculate for this case 4 sums congruent 1 modulo 2 that gives $s_\Gamma(n)\equiv 0 \pmod 2$.

Now we have to consider $n=\frac{1}{3}(2\cdot 4^{l_1}+4^{l_2+1}+3)=\frac{1}{3}(2\cdot 4^{k_1}+4^{k_2+1}+2\cdot 4^{k_3+1}-5)$ with $k_1=1,\ k_2=l_2,\ k_3=l_1-1$, where k_1 is the least value and by 3.35 only case a) can occur. Then we get $l_1=k_2-1$ and $l_2=k_1-1=0$ $\lightning$. So, only this additional value for n is possible, hence two sums are congruent 1 modulo 2 and so $s_\Gamma(n)\equiv 0 \pmod 2$ follows. For the remaining cases iii), iv), and v) we get by 3.35 that they exclude each other hence, only one combination is possible for each case and since 0.0.1.0.0.1 applies only for $l_1=1$ we get that two sums are congruent 1 modulo 2 and so $s_\Gamma(n)\equiv 0 \pmod 2$.

Fourth sum:

$$\sum_{\substack{n_1+\ldots+n_7=n-10\\ n_i\geq 1}} s_\Gamma(n_1)s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ s_\Gamma(n_5)\ s_\Gamma(n_6)\ s_\Gamma(n_7)$$

With 1.0.0.1.0.0 $\lightning$, 0.0.0.1.1.0 $\lightning$ three cases remain: 0.1.0.1.0.0, 0.0.1.1.0.0, 0.0.0.1.0.1. For $\frac{1}{3}(2\cdot 4^{k_1}+4^{k_2+1}+2\cdot 4^{k_3+1}-5)=\frac{1}{3}(2\cdot 4^{l_1}+4^{l_2+1}+3)=\frac{1}{3}(2\cdot 4^{l_1'}+4^{l_2'+2}-5)$ is by the argumentation for the third sum only possible if $l_1=1, l_2=l_2'+1, l_1'=1$, but then also the case $n=\frac{1}{3}(2\cdot 4^{k+1}+11)$ occurs and with $l_1'<l_2'$ by 3.36 $n=\frac{1}{3}(5\cdot 4^{l+1}-5)$ does not occur. So, four sums are congruent 1 modulo 2 and then $s_\Gamma(n)\equiv 0 \pmod 2$.

If one of the cases listed in 3.35 occur, then we get by 3.35 that each such case has a representation with $\frac{1}{3}(2\cdot 4^{k_1}+4^{k_2+1}+2\cdot 4^{k_3+1}-5)$ and no other is possible. So, two representations within the fourth sum in each case with an odd number of permutations yield $s_\Gamma(n)\equiv 0 \pmod 2$.

The case where $\frac{1}{3}(2 \cdot 4^{k_1} + 4^{k_2+1} + 2 \cdot 4^{k_3+1} - 5) = \frac{1}{3}(2 \cdot 4^{l_1} + 4^{l_2+1} + 3)$ states that $k_1 = 1$ and we do not find with this condition another case in 3.35. Only this case occurs and hence two sums are congruent $0 \pmod 2$.

Fifth sum:

$$\sum_{\substack{n_1+...+n_8=n-13 \\ n_i \geq 1}} s_\Gamma(n_1)s_\Gamma(n_2)\ s_\Gamma(n_3)\ s_\Gamma(n_4)\ s_\Gamma(n_5)\ s_\Gamma(n_6)\ s_\Gamma(n_7)\ s_\Gamma(n_8)$$

In this case we read from our sum analysis that 0.1.0.0.1.0 $\lightning$, 0.0.1.0.1.0 $\lightning$, 0.0.0.1.1.0 $\lightning$, 0.0.0.0.1.1 $\lightning$, and 1.0.0.0.1.0 remains. We have $n = \frac{1}{3}(4^{k+1} - 1) = \frac{1}{3}(4^{l+2} - 1)$ with $l = k - 1$ and so two sums are congruent 1 modulo 2, hence $s_\Gamma(n) \equiv 0 \pmod 2$.

Sixth sum:

$$\sum_{\substack{n_1+...+n_{10}=n-15 \\ n_i \geq 1}} s_\Gamma(n_1) \cdot ... \cdot s_\Gamma(n_{10})$$

Here we have 1.0.0.0.0.1 $\lightning$, 0.1.0.0.0.1 $\lightning$, 0.0.0.1.0.1 $\lightning$, 0.0.0.0.1.1 $\lightning$. It is left to analyze the two cases 0.0.1.0.0.1 and 0.0.0.1.0.1 . We consider first $n = \frac{1}{3}(4^{l_1+1} + 4^{l_2+2} - 5)$. If this case occurs then we find by 0.0.0.1.0.1 i) another representation in the fourth sum, hence two sums are congruent 1 modulo 2 and then $s_\Gamma(n) \equiv 0 \pmod 2$. In the special case $l_1 = 1$ we obtain by 0.0.1.0.0.1 another representation in the third sum. Because $l_1 = 1$ we would get for our condition in the sixth sum $k_3 = 0$, this case is not possible, hence only two sums are congruent 1 modulo 2. By 3.36 we have the case that both representations for n in the sixth sum are possible. But then all three subcases in 0.0.0.1.0.1 occur, hence we get four different representations for n with odd permutations for each representation, hence $s_\Gamma(n) \equiv 0 \pmod 2$. In the case $n = \frac{1}{3}(5 \cdot 4^{l+1} - 5)$ we get the subcases 0.0.0.1.0.1 ii) and iii) plus by 3.36 another representation for n. In total, four different representations for n yield $s_\Gamma(n) \equiv 0 \pmod 2$. $\square$

3.3 The groups $\Gamma = C_2 * C_p$

In this last section we investigate the Hecke groups $\Gamma = C_2 * C_p$ with regard to divisibility properties of the function $s_\Gamma(n)$. For such groups the Euler characteristic is $\chi(\Gamma) = -\frac{p-2}{2p}$ and for the free rank we have $\mu(\Gamma) = p - 1$. Considering this type of groups, there exists only a divisibility result for the modular group $\Gamma = PSL(2, \mathbb{Z})$: Stothers [Sto78] derives for the parity of the subgroups of $\Gamma = C_2 * C_3$ that the $s_\Gamma(n)$ are odd if and only if n is of the form $2^k - 3$ or $2(2^k - 3)$, $k \geq 2$, proved via coset diagrams. Later Godsil, Imrich and Razen [GIR78] give another proof using similar methods to the ones we apply for our divisibility investigations of $s_\Gamma(n)$. In this section we are concerned with the group $\Gamma = C_2 * C_5$ and divisibility properties of $s_\Gamma(n)$. First, we obtain a recurrence for $H_n = |Hom(\Gamma, S_n)|$ and get this time such a recurrence via the algorithm Hadamard product $C_2 * C_p$.

Proposition 3.17 *The number of homomorphisms $H_n = |Hom(\Gamma, S_n)|$ where $\Gamma = C_2 * C_5$ is given by the following recurrence:*

$$
\begin{aligned}
(3\,n-22)\cdot H_n = {} & (3\,n-22)H_{n-1} \\
& + (3\,n-22)(n-1)\;H_{n-2} \\
& + (n-1)\cdot\ldots\cdot(n-4)\;[\;(n-3)(3\,n-22)(5\,n-12) \\
& \qquad + (n-5)(n-7)(6\,n-14)\;]\;H_{n-5} \\
& + (n-1)\cdot\ldots\cdot(n-5)\;[(3\,n-22)(n-2)^2-(n-7)(6\,n-14)\;]\;H_{n-6} \\
& - (n-1)\cdot\ldots\cdot(n-7)(6\,n-14)\;H_{n-7} \\
& + (n-1)\cdot\ldots\cdot(n-9)(n-8)\;[(3\,n-22)\;((n-2)(n+1)(n-5) \\
& \qquad -(n-3)(3\,n-17)) - (n-7)6\,n-14)(5\,n-37)\;]\;H_{n-10} \\
& - (n-1)\cdot\ldots\cdot(n-10)(n-7)(6\,n-14)\;[\;2\cdot(n-7)(n-10)+2\;]\;H_{n-11} \\
& + (n-1)\cdot\ldots\cdot(n-14)(n-13)(n-7)(6\,n-14) \\
& \qquad [\;(n-10)\;((3\,n-22)-(n-4)(n-7)) + 2\cdot(n-8)(3\,n-22)\;]\;H_{n-15}
\end{aligned}
\tag{3.38}
$$

with initial values $H_1 = 1, H_2 = 2, H_3 = 4, H_4 = 10, H_5 = 650, H_6 = 11020,$ $H_7 = 117160, H_8 = 1027580, H_9 = 7925500, H_{10} = 746623000, H_{11} = 28893234800,$ $H_{12} = 673994471800, H_{13} = 11817847963000, H_{14} = 173779828130000,$ *and* $H_{15} = 20301480343948000.$

Proof:
With Proposition 3.1 we obtain for the number $f_n = |Hom(C_2, S_n)|$ of C_2-actions on S_n

$$f_n = f_{n-1} + (n-1)\;f_{n-2}, \quad f_0 = 1$$

and for $g_n = |Hom(C_5, S_n)|$

$$g_n = g_{n-1} + (n-1)(n-2)(n-3)(n-4)\;g_{n-5}, \quad g_0 = 1.$$

With those recurrences our algorithm Hadamard product $C_2 * C_p$ generates the recurrence for $H_n = |Hom(\Gamma, S_n)|$. $\square$

We obtain the following recurrences for $s_\Gamma(n)$ modulo 2 and 3:

Proposition 3.18 *Let $\Gamma = C_2 * C_5$. Modulo 2 we have for $s_\Gamma(n)$, $n \geq 21$*

$$
\begin{aligned}
(22-3n)\cdot s_\Gamma(n) \equiv &-3\ s_\Gamma(n-1) - 3\ s_\Gamma(n-2) - 101\ (n-6)\ s_\Gamma(n-5) \\
&- 3\ (n-11)(n-12)(n-13)(n-14)\ s_\Gamma(n-10) \\
&+ 3\cdot \sum_{n_1+n_2=n} s_\Gamma(n_1)s_\Gamma(n_2) \\
&- 101\cdot \sum_{n_1+n_2=n-5} s_\Gamma(n_1)s_\Gamma(n_2) \\
&- 21\cdot \sum_{\substack{n_1+n_2+n_3=n-4\\ n_3\geq 2}} (n_3-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3) \\
&- 21\cdot \sum_{n_1+n_2+n_3=n-5} s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3) \\
&- 15\cdot \sum_{\substack{n_1+n_2=n-10\\ n_2\geq 4}} (n_2-1)(n_2-2)(n_2-3)\ s_\Gamma(n_1)s_\Gamma(n_2) \\
&- 33\cdot \sum_{\substack{n_1+n_2+n_3=n-10\\ n_i\geq 2}} (n_1-1)(n_2-1)(n_3-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3) \\
&- 27\cdot \sum_{\substack{n_1+\ldots+n_4=n-10\\ n_4\geq 2}} (n_4-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3)s_\Gamma(n_4) \\
&- 3\cdot \sum_{\substack{n_1+\ldots+n_5=n-10\\ n_5\geq 2}} (n_5-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3)s_\Gamma(n_4)s_\Gamma(n_5) \\
&- 3\cdot \sum_{n_1+\ldots+n_5=n-10} s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3)s_\Gamma(n_4)s_\Gamma(n_5) \pmod 2
\end{aligned}
\tag{3.39}
$$

where $n_i \geq 1$ if not stated otherwise.

Modulo 3 we get for $s_\Gamma(n)$, $n \geq 21$

$$\begin{aligned}
(22-3n)\cdot s_\Gamma(n) \equiv\ & 56\ s_\Gamma(n-5) - 116\ s_\Gamma(n-6) + 34\ s_\Gamma(n-7) + 4300\ s_\Gamma(n-11) \\
& - 101\ (n-6)\ s_\Gamma(n-5) + 8\ (n-7)\ s_\Gamma(n-6) - 1130\ (n-11)\ s_\Gamma(n-10) \\
& - 458\ (n-13)(n-12)\ s_\Gamma(n-11) \\
& - 68\ (n-13)(n-12)(n-11)\ s_\Gamma(n-10) \\
& - 38\ (n-19)(n-18)(n-17)(n-16)\ s_\Gamma(n-15) \\
& - 101\cdot \sum_{n_1+n_2=n-5} s_\Gamma(n_1)s_\Gamma(n_2) \\
& + 8\cdot \sum_{n_1+n_2=n-6} s_\Gamma(n_1)s_\Gamma(n_2) \\
& - 1130\cdot \sum_{n_1+n_2=n-10} s_\Gamma(n_1)s_\Gamma(n_2) \\
& - 916\cdot \sum_{\substack{n_1+n_2=n-10\\ n_2\geq 2}} (n_2-1)\ s_\Gamma(n_1)s_\Gamma(n_2) \\
& + 568\cdot \sum_{\substack{n_1+n_2=n-11\\ n_2\geq 2}} (n_2-1)\ s_\Gamma(n_1)s_\Gamma(n_2) \\
& - 458\cdot \sum_{\substack{n_1+n_2+n_3=n-9\\ n_3\geq 2}} (n_3-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3) \\
& + 284\cdot \sum_{\substack{n_1+n_2+n_3=n-10\\ n_3\geq 2}} (n_3-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3) \qquad (3.40) \\
& - 458\cdot \sum_{n_1+n_2+n_3=n-10} s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3) \\
& + 284\cdot \sum_{n_1+n_2+n_3=n-11} s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3) \\
& + 272\cdot \sum_{\substack{n_1+n_2=n-10\\ n_2\geq 3}} (n_2-1)(n_2-2)\ s_\Gamma(n_1)s_\Gamma(n_2) \\
& - 340\cdot \sum_{\substack{n_1+n_2+n_3=n-10\\ n_3\geq 2}} (n_3-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3) \\
& - 68\cdot \sum_{\substack{n_1+\ldots+n_4=n-9\\ n_4\geq 2}} (n_4-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3)s_\Gamma(n_4) \\
& - 68\cdot \sum_{n_1+\ldots+n_4=n-10} (n_4-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3)s_\Gamma(n_4) \\
& - 190\cdot \sum_{\substack{n_1+n_2=n-15\\ n_2\geq 4}} (n_2-1)(n_2-2)(n_2-3)\ s_\Gamma(n_1)s_\Gamma(n_2)
\end{aligned}$$

$$
\begin{aligned}
&- 380 \cdot \sum_{\substack{n_1+n_2=n-15\\ n_1\geq 2, n_2\geq 3}} (n_1-1)(n_2-1)(n_2-2)\ s_\Gamma(n_1)s_\Gamma(n_2)\\
&- 380 \cdot \sum_{\substack{n_1+n_2+n_3=n-15\\ n_3\geq 3}} (n_3-1)(n_3-2)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3)\\
&- 418 \cdot \sum_{\substack{n_1+n_2+n_3=n-15\\ n_i\geq 2}} (n_1-1)(n_2-1)(n_3-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3)\\
&- 152 \cdot \sum_{\substack{n_1+\ldots+n_4=n-15\\ n_3, n_4\geq 2}} (n_3-1)(n_4-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3)s_\Gamma(n_4)\\
&- 38 \cdot \sum_{\substack{n_1+\ldots+n_5=n-15\\ n_5\geq 2}} (n_5-1)\ s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3)s_\Gamma(n_4)s_\Gamma(n_5)\\
&- 38 \cdot \sum_{n_1+\ldots+n_5=n-15} s_\Gamma(n_1)s_\Gamma(n_2)s_\Gamma(n_3)s_\Gamma(n_4)s_\Gamma(n_5) \pmod 3
\end{aligned}
$$

with initial values:

$s_\Gamma(1) = 1$	$s_\Gamma(2) = 1$
$s_\Gamma(3) = 0$	$s_\Gamma(4) = 0$
$s_\Gamma(5) = 26$	$s_\Gamma(6) = 60$
$s_\Gamma(7) = 56$	$s_\Gamma(8) = 32$
$s_\Gamma(9) = 9$	$s_\Gamma(10) = 1766$
$s_\Gamma(11) = 5192$	$s_\Gamma(12) = 7152$
$s_\Gamma(13) = 5928$	$s_\Gamma(14) = 3388$
$s_\Gamma(15) = 194200$	$s_\Gamma(16) = 695680$
$s_\Gamma(17) = 1189932$	$s_\Gamma(18) = 1288377$
$s_\Gamma(19) = 979032$	$s_\Gamma(20) = 30538720$

Proof:

The proof follows the same principle as the proof of Theorem 3.11. We suppress the calculations here but present the differential equation for $G(z)$. The derivatives involved are already presented in Proposition 3.15. Applying then the same procedure, i.e., calculating the different Cauchy products and comparing the coefficients we get the recurrences for $s_\Gamma(n)$ modulo 2 and modulo 3 as stated.

$$
\begin{aligned}
&(19 + 3\,z + 3\,z^2 - 56\,z^5 + 116\,z^6 - 34\,z^7 + 1164\,z^{10} - 4300\,z^{11} + 1656192\,z^{15}) \cdot G(\,z) \\
&= (16\,z + 182\,z^4 - 150\,z^5 - 540\,z^9 + 2080\,z^{10} - 1036032\,z^{14}) \\
&\quad + (3\,z - 101\,z^6 + 8\,z^7 + 6\,z^8 - 1130\,z^{11} + 1956\,z^{12} - 731712\,z^{16}) \cdot (G'(z) + G(z)^2\,) \\
&\quad + (-21\,z^7 - 6\,z^8 - 458\,z^{12} + 284\,z^{13} - 122832\,z^{17}) \cdot \\
&\quad (G^{(2)}(z) + 2\,G'(z)\,G(z) + G'(z)\,G(z)^2 + G(z)^3\,) \\
&\quad - (68\,z^{13} - 12\,z^{14} + 6702\,z^{18}) \cdot \\
&\quad (G^{(3)}(z) + 4\,G^{(2)}(z)\,G(z) + 3\,G'(z)^2 + 5\,G(z)^2\,G'(z) + G(z)^3\,G'(z) + G(z)^4\,) \\
&\quad - (3\,z^{14} + 38\,z^{19}) \cdot \\
&\quad (G^{(4)}(z) + 5\,G^{(3)}(z)\,G(z) + 10\,G^{(2)}(z)\,G'(z) + 10\,G^{(2)}(z)\,G(z)^2 \\
&\qquad + 11\,G'(z)^3 + 4\,G'(z)^2\,G(z)^2 + 9\,G'(z)\,G(z)^3 + G'(z)\,G(z)^4 + G(z)^5\,) \\
&\qquad + 6\,z^{20} \cdot (G^{(5)}(z) + 6\,G^{(4)}(z)\,G(z) + 15\,G^{(3)}(z)\,G'(z) + 15\,G^{(3)}(z)\,G(z)^2 \\
&\qquad + 10\,G^{(2)}(z)\,G(z)^2 + 50\,G^{(2)}(z)\,G'(z)\,G(z) + 10\,G^{(2)}(z)\,G'(z)\,G(z)^2 \\
&\qquad + 20\,G^{(2)}(z)\,G(z)^3 + 15\,G'(z)^3 + 36\,G'(z)^2\,G(z)^2 + 9\,G'(z)^2\,G(z)^3 \\
&\qquad + 14\,G'(z)\,G(z)^4 + G'(z)\,G(z)^5 + G(z)^6\,)
\end{aligned}
\tag{3.41}
$$

□

As one can easily see, we obtain a coefficient $(22 - 3n)$ together with $s_\Gamma(n)$ in the recurrence. Hence, in the analysis of $s_\Gamma(n) \pmod 2$ we are not able to treat the case $n \equiv 0 \pmod 2$. So, unfortunately we can not complete the proof of our conjecture on the parity of $s_\Gamma(n)$ which is based on massive computer calculations:

Conjecture 3.19 *Let $\Gamma = C_2 * C_5$. Then $s_\Gamma(n) \equiv 1 \pmod 2$ if and only if $n = \frac{2^{2k+1}-5}{3}$ or $n = \frac{2\,(2^{2k+1}-5)}{3}$ where $k \geq 1$.*

Considering $s_\Gamma(n)$ modulo 3 we have for the coefficient $(22 - 3n) \equiv 1 \pmod 3$, hence, we are able to analyze the recurrence for $s_\Gamma(n)$. But in this case we do not find a recognizable divisibility pattern for $s_\Gamma(n) \pmod 3$. We exhibit the values for $s_\Gamma(n) \pmod 3$, $n \leq 300$, in an appendix. At least we want to present the recurrence for $s_\Gamma(n)$ modulo 3. In case someone is able to describe the divisibility modulo 3 for $s_\Gamma(n)$ it might be of some use for the proof.

Appendix

In this appendix we present the results for the divisibility properties of the function $s_\Gamma(n)$ for the groups considered in Chapter 3. These results are obtained by massive computer calculations.

We have for the group $\Gamma = C_3 * C_3$ and indices $n \leq 2050$ that $s_\Gamma(n) \equiv 1 \pmod 2$ for the following values of n:

Index n where $s_\Gamma(n) \equiv 1 \pmod 2$								
1	5	13	29	61	125	509	1021	2045

For all other values $n \leq 2050$ we got $s_\Gamma(n) \equiv 0 \pmod 2$.

Investigating the divisibility properties of $s_\Gamma(n)$ for $\Gamma = C_5 * C_5$ our computer runs delivered for the parity of $s_\Gamma(n)$ for $n \leq 2730$ that $s_\Gamma(n)$ is odd if n belongs to the values listed below. In all other cases we obtained $s_\Gamma(n) \equiv 0 \pmod 2$.

Index n where $s_\Gamma(n) \equiv 1 \pmod 2$					
1	9	41	169	681	2729

For the parity of subgroups of $\Gamma = C_2 * C_p$ we calculated $s_\Gamma(n)$ up to $n = 2000$ and found for the following values $s_\Gamma(n) \equiv 1 \pmod 2$:

Index n where $s_\Gamma(n) \equiv 1 \pmod 2$									
1	2	9	18	41	82	169	338	681	1362

We obtain for all other values $n \leq 2000$ that $s_\Gamma(n) \equiv 0 \pmod 2$.

Investigating the function $s_\Gamma(n)$ modulo 3 for the two groups $\Gamma = C_3 * C_3$ and $\Gamma = C_2 * C_p$ we obtained the following divisibility patterns:

1	1	41	2	81	0	121	0	161	0	201	0	241	0	281	0
2	1	42	0	82	2	122	1	162	0	202	1	242	1	282	0
3	0	43	0	83	2	123	0	163	0	203	1	243	0	283	1
4	0	44	0	84	0	124	1	164	1	204	1	244	0	284	2
5	2	45	1	85	0	125	1	165	1	205	2	245	1	285	2
6	0	46	1	86	1	126	0	166	2	206	1	246	1	286	0
7	2	47	1	87	2	127	1	167	2	207	2	247	2	287	1
8	2	48	1	88	2	128	1	168	0	208	2	248	2	288	2
9	0	49	1	89	1	129	0	169	1	209	0	249	0	289	2
10	2	50	0	90	1	130	0	170	0	210	0	250	1	290	1
11	2	51	0	91	1	131	2	171	0	211	1	251	0	291	0
12	0	52	2	92	0	132	0	172	0	212	2	252	0	292	2
13	0	53	2	93	0	133	0	173	0	213	0	253	0	293	0
14	1	54	0	94	2	134	1	174	2	214	2	254	0	294	1
15	1	55	2	95	1	135	1	175	2	215	1	255	2	295	1
16	1	56	2	96	2	136	1	176	2	216	1	256	2	296	2
17	0	57	0	97	0	137	1	177	0	217	2	257	2	297	0
18	0	58	0	98	1	138	0	178	0	218	2	258	0	298	1
19	0	59	1	99	0	139	0	179	0	219	0	259	0	299	2
20	1	60	2	100	2	140	2	180	2	220	0	260	0	300	1
21	0	61	2	101	1	141	0	181	2	221	1	261	2		
22	1	62	1	102	0	142	2	182	2	222	1	262	2		
23	1	63	0	103	2	143	2	183	1	223	0	263	2		
24	1	64	0	104	0	144	1	184	1	224	1	264	1		
25	2	65	2	105	0	145	0	185	2	225	0	265	1		
26	1	66	0	106	2	146	0	186	0	226	2	266	2		
27	0	67	2	107	2	147	0	187	1	227	0	267	0		
28	0	68	2	108	0	148	0	188	1	228	0	268	1		
29	1	69	2	109	0	149	0	189	0	229	1	269	1		
30	1	70	1	110	0	150	1	190	1	230	2	270	1		
31	2	71	2	111	0	151	0	191	1	231	2	271	2		
32	2	72	1	112	0	152	2	192	0	232	0	272	2		
33	0	73	1	113	0	153	0	193	0	233	1	273	0		
34	1	74	0	114	0	154	2	194	2	234	0	274	0		
35	0	75	0	115	0	155	0	195	1	235	0	275	1		
36	0	76	2	116	0	156	0	196	1	236	2	276	1		
37	0	77	1	117	2	157	1	197	2	237	0	277	0		
38	0	78	0	118	2	158	2	198	0	238	2	278	1		
39	2	79	1	119	2	159	1	199	0	239	2	279	0		
40	2	80	2	120	0	160	2	200	1	240	1	280	2		

Figure 10: $s_\Gamma(n) \pmod 3$, $n \leq 300$, for $\Gamma = C_2 * C_5$

1	1	41	1	81	1	121	1	161	1
2	0	42	0	82	0	122	0	162	0
3	1	43	1	83	1	123	1	163	1
4	2	44	2	84	2	124	2	164	2
5	2	45	2	85	2	125	2	165	2
6	0	46	0	86	0	126	0	166	0
7	2	47	2	87	2	127	2	167	2
8	1	48	1	88	1	128	1	168	1
9	1	49	1	89	1	129	1	169	1
10	0	50	0	90	0	130	0	170	0
11	1	51	1	91	1	131	1	171	1
12	2	52	2	92	2	132	2	172	2
13	2	53	2	93	2	133	2	173	2
14	0	54	0	94	0	134	0	174	0
15	2	55	2	95	2	135	2	175	2
16	1	56	1	96	1	136	1	176	1
17	1	57	1	97	1	137	1	177	1
18	0	58	0	98	0	138	0	178	0
19	1	59	1	99	1	139	1	179	1
20	2	60	2	100	2	140	2	180	2
21	2	61	2	101	2	141	2	181	2
22	0	62	0	102	0	142	0	182	0
23	2	63	2	103	2	143	2	183	2
24	1	64	1	104	1	144	1	184	1
25	1	65	1	105	1	145	1	185	1
26	0	66	0	106	0	146	0	186	0
27	1	67	1	107	1	147	1	187	1
28	2	68	2	108	2	148	2	188	2
29	2	69	2	109	2	149	2	189	2
30	0	70	0	110	0	150	0	190	0
31	2	71	2	111	2	151	2	191	2
32	1	72	1	112	1	152	1	192	1
33	1	73	1	113	1	153	1	193	1
34	0	74	0	114	0	154	0	194	0
35	1	75	1	115	1	155	1	195	1
36	2	76	2	116	2	156	2	196	2
37	2	77	2	117	2	157	2	197	2
38	0	78	0	118	0	158	0	198	0
39	2	79	2	119	2	159	2	199	2
40	1	80	1	120	1	160	1	200	1

Figure 11: $s_\Gamma(n) \pmod 3$, $n \leq 200$, for $\Gamma = C_3 * C_3$

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