Lectures

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M. Prest, Topological and geometric aspects of the Ziegler spectrum.
J. Rickard, Bousfield Localization for Representation Theorists.
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Krull, Gelfand-Kirillov and Filter dimension

Vladimir Bavula

In the first lecture a connection will be established between Krull, Gelfand-Kirillov and Filter dimension of simple affine algebras, and an analog of Bernstein's inequality will be proved for the nonzero finitely generated modules over simple affine algebras. These results will be applied to compute the Krull dimension of the ring of differential operators on a smooth affine variety. Detailed proofs will be given.

In the second lecture we give a review of (recent) results on the Krull dimension of skew Laurent extensions and generalized Weyl algebras with left Noetherian coefficients, based on papers of P. Gabriel, K.R. Goodearl, D. Jordan, T.J. Hodges, T.H. Lenagan, J.C. McConnell, J.J. Pettit, R. Rentschler, J.C. Robson, P. Smith, F. Van Oystaeyen, R.B. Warfield and V. Bavula.

These lectures can be considered as an addition to and continuation of the lectures given by Tom Lenagan.

1 Krull, Gelfand-Kirillov and Filter dimensions of simple affine algebras

Let K be a field, a module means a left one, $\otimes = \otimes_K$. Let A_n be the n'th **Weyl algebra**, i. e. the $K-$algebra with $2n$ generators $X_1, \ldots, X_n$, $\partial_1, \ldots, \partial_n$ and relations

$$\partial_i X_j - X_j \partial_i = \delta_{ij}, \text{ the Kronecker delta, } X_i X_j - X_j X_i = \partial_i \partial_j - \partial_j \partial_i = 0.$$

If char $K = 0$, then A_n is a simple Noetherian algebra.

Let K.dim, GK and fil.dim be the *(left) Krull* (in the sense of Rentschler and Gabriel, [RG]), *Gelfand-Kirillov* and *Filter* dimension respectively.

Theorem 1.1 (the Bernstein's inequality, [Be]). *Let A_n be the n'th Weyl algebra over a field of characteristic zero. Then* $\mathrm{GK}(M) \geq n$ *for every nonzero finitely generated A_n-module M.* ▪

For each function f from $\mathbf{N}_0$ to $\mathbf{R}'$, where $\mathbf{R}' = \{r \in \mathbf{R}, r \geq 1\}$ we can associate a number $\gamma(f) \in \mathbf{R}$, a *degree* of f,

$$\gamma(f) = \inf\{r \in \mathbf{R} : \ f(n) \leq n^r \text{ for sufficiently large} \, n >> 0\}.$$

Let A be an affine algebra with generators $a_1, \ldots, a_s$. Then A is equipped with a standard finite dimensional filtration F: $A = \cup_{i\geq 0} A_i$, where $A_0 = K$, $A_1 = K + \sum_{i=1}^{s} Ka_i$, $A_i := A_1^i$, $i \geq 2$.

If the algebra A is **simple**, define the **return function** $\nu_F : \mathbf{N}_0 \to \mathbf{N}_0 \cup \{\infty\}$ of A associated with F in the following way

$$\nu_F(i) = \inf\{j \in \mathbf{N}_0 \cup \{\infty\} : \ A_j a A_j \ni 1 \text{ for all } 0 \neq a \in A_i\},$$

where $A_j a A_j$ is the subspace of A generated by the products xay, for all $x, y \in A_j$.

Definition. ([Bav1]). The **filter dimension** of A is defined as $\text{fil.dim}\, A = \gamma(\nu_F)$.

Theorem 1.2 ([Bav1]). (An analog of the Bernstein's inequality for the simple affine algebras).

Let A be a simple affine algebra; then

$$\text{GK}(A) \leq \text{GK}(M)(\text{fil.dim}\, A + \max\{\text{fil.dim}\, A, 1\})$$

for every nonzero finitely generated $A-$module M. Moreover, if A is infinite dimensional, then

$$\text{fil.dim}\, A \geq 1/2. \blacksquare$$

We say that an algebra S is *(left) finitely partitive* ([Jos 1] or [MR], 8.3.17) if, given any finitely generated S-module M, there is an integer $n > 0$ such that for every chain

$$M = M_0 \supset M_1 \supset \ldots \supset M_m$$

with $\text{GK}(M_i/M_{i+1}) = \text{GK}(M)$, one has $m \leq n$.

Theorem 1.3 ([Bav2]). *Let A be a finitely partitive simple affine algebra with $GK(A) < \infty$. Then, for any nonzero finitely generated A-module M, the Krull dimension*

$$\text{K.dim}(M) \leq \text{GK}(M) - \text{GK}(A)/(\text{fil.dim}\, A + \max\{\text{fil.dim}\, A, 1\}).$$

In particular case,

$$\text{K.dim}(A) \leq \text{GK}(A)(1 - 1/(\text{fil.dim}\, A + \max\{\text{fil.dim}\, A, 1\})). \blacksquare$$

Let B be an integral regular domain, affine over a field of characteristic zero and $\mathcal{D}(B)$ be its ring of differential operators. It is well-known that $\mathcal{D}(B)$ is a simple affine Noetherian algebra ([MR], ch. 15).

Theorem 1.4 ([Bav2]). *The filter dimension*

$$\text{fil.dim}\, \mathcal{D}(B) = 1. \blacksquare$$

As an application we compute the Krull dimension of $\mathcal{D}(B)$.

Theorem 1.5 ([MR], ch. 15).

$$\text{K.dim}\mathcal{D}(B) = \text{GK}(\mathcal{D}(B))/2 = \text{K.dim}(B). \blacksquare$$

Theorem 1.6 ([MR], 15.4.3). *Let M be a nonzero finitely generated $\mathcal{D}(B)$-module. Then*

$$\text{GK}(M) \geq \text{GK}(\mathcal{D}(B))/2 = \text{K.dim}(B). \blacksquare$$

2 Krull dimension of skew Laurent extensions and generalized Weyl algebras

Let R be a ring, $\sigma \in \mathrm{Aut}(R)$ and $a \in Z(R)$. Denote by S (resp. T) a *skew Laurent extension* $S = R[X, X^{-1}; \sigma]$ (resp. a *generalized Weyl algebra* $T = R(\sigma, a)$). Note that if $a = 1$ we have $S = T$.

Theorem 2.1 *([Ho]). Let the commutative Noetherian ring R have finite Krull dimension; then*

$$\mathcal{K}(R[X, X^{-1}; \sigma]) = \sup\{\mathcal{K}(R), ht\,\mathbf{q} + 1 \mid \mathbf{q} \textit{ is a } \sigma - \textit{semistable prime of } R\}. \blacksquare$$

Here $ht\,\mathbf{p}$ is the *height* of an ideal $\mathbf{p}$ and $\mathbf{p}$ is σ-*semistable*, if $\sigma^n(\mathbf{p}) = \mathbf{p}$ for some integer n. If there is no such n, the ideal is called σ-*unstable.*

Theorem 2.2 *([BVO]). Let R be a commutative Noetherian ring with $\mathcal{K}(R) < \infty$ and $T = R(\sigma, a)$ be a generalized Weyl algebra. Then*

$$\mathcal{K}(T) = \sup\{\mathcal{K}(R),\ ht\,\mathbf{p} + 1,\ ht\,\mathbf{q} + 1 \mid \mathbf{p} \textit{ is a } \sigma - \textit{unstable prime ideal of } R \textit{ for which}$$

$$\textit{there exist infinitely many integers } i \textit{ with } a \in \sigma^i(\mathbf{p});$$

$$\mathbf{q} \textit{ is a } \sigma - \textit{semistable prime ideal of } R\}. \blacksquare$$

Let K be a field and $Usl(2)$ be the K-enveloping algebra of the Lie algebra

$$sl(2) =< X, Y, H \mid [H, X] = X,\ [H, Y] = -Y,\ [X, Y] = 2H\ >,$$

The Casimir element $C = YX + H(H + 1)$ is central in $Usl(2)$).

$$Usl(2) \simeq K[H, C](\sigma, a = C - H(H + 1)),\ X \to X,\ Y \to Y,\ H \to H,$$

σ given by $H \to H - 1$, $C \to C$. Applying Theorem 2.2 we obtain:

Corollary 2.3 *([Sm] for $K = \mathbf{C}$; see also [MR], 8.6.15) $\mathcal{K}(Usl(2))$ is either two when char $K = 0$, or three when char $K = p > 0$.* $\blacksquare$

Definition, [GL]. A critical R-module M is called S-**clean** if M is $S \otimes_R M$ is a critical S-module.

Definition, [GL]. Let A be a simple R-module and let M be an arbitrary R-module. Then the **height** $h(A : M)$ of A in M is defined to be the supremum of those non-negative integers n for which there exists a sequence $A = A_0, A_1, \ldots, A_n$ of S-clean R-modules such that A_i is isomorphic to a minor subfactor of A_{i+1}, for $i = 0, \ldots, n - 1$, while A_n is isomorphic to a subfactor (not necessarily minor) of M.

Theorem 2.4 *([GL]). Let R be a left noetherian ring with finite Krull dimension. Then*

$$\mathcal{K}(S) = \mathcal{K}(R)$$

unless there exists a simple R-module A such that

$$\mathcal{K}(S \otimes_R A) = 1 \text{ and } h(A : R) = \mathcal{K}(R),$$

in which case

$$\mathcal{K}(S) = \mathcal{K}(R) + 1. \blacksquare$$

Definition, ([BL]). Let A be a simple R-module and let M be an arbitrary R-module.

1. If A is a-torsionfree, then $h(A : M)$ is defined to be the supremum of those non-negative integers n for which there exists a sequence $A = A_0, A_1, \dots, A_n$ of T-clean R-modules such that A_i is isomorphic to a minor subfactor of A_{i+1}, for $i = 0, \dots, n-1$, while A_n is isomorphic to a subfactor (not necessarily minor) of M.

2. If A is a-torsion, then $h_+(A : M)$, $(h_-(A : M))$, is defined to be the supremum of those non-negative integers n for which there exists a sequence $A = A_0, A_1, \dots, A_n$ of R-modules such that each of the R-modules $A = A_0, A_1, \dots, A_i$, for some $i \geq 0$, is $(T, +)$-clean, $((T, -)$-clean), while each of the R-modules $A_{i+1}, \dots, A_n$ is T-clean; and A_j is isomorphic to a minor subfactor of A_{j+1}, for $j = 0, \dots n-1$, while A_n is isomorphic to a subfactor (not necessarily minor) of M.

Theorem 2.5 *([BL]). Let R be a left noetherian ring with finite Krull dimension. Then*

$$\mathcal{K}(T) = \mathcal{K}(R)$$

unless there exists either
(i) a simple a-torsionfree R-module A such that $h(A : R) = \mathcal{K}(R)$ and ${}^{\sigma^i}A \simeq A$ for some i; or
(ii) a simple $(T, +)$-clean R-module A such that $h_+(A : R) = \mathcal{K}(R)$ and the set $St_+(A)$ is infinite; or
(iii) a simple $(T, -)$-clean R-module A such that $h_-(A : R) = \mathcal{K}(R)$ and the set $St_-(A)$ is infinite;

in which case

$$\mathcal{K}(T) = \mathcal{K}(R) + 1,$$

where $St_\pm(A) = \{\pm i \geq 0 \,|\, \sigma^{\mp i}(a)A = 0\}$. $\blacksquare$

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Infinite dimensional modules for finite groups.

Three lectures by Dave Benson, Bielefeld, September 1998.

This series of lectures is about the recent developments in modular representation theory of finite groups, centered around the properties of certain infinite dimensional modules and their influence on the theory of finite dimensional modules.

Let G be a finite group and k a field of characteristic p dividing the order of G. Then kG is a finite dimensional symmetric algebra, and we are interested in the category $\mathsf{mod}(kG)$ of finite dimensional kG-modules and the category $\mathsf{Mod}(kG)$ of all kG-modules. The corresponding stable categories $\mathsf{stmod}(kG)$ and $\mathsf{StMod}(kG)$ are triangulated categories, and homomorphisms of degree n in this category correspond to elements of Tate Ext in degree n.

Some of the problems I wish to discuss are as follows:

1. How do you tell if a kG-module is projective?

2. What are the thick subcategories of $\mathsf{stmod}(kG)$?

3. Given a simple module S in the principal block of kG, is it true that there exists $n > 0$ such that $H^n(G, S) \neq 0$?

4. For what groups G is it true that for all nonprojective finite dimensional modules M in the principal block of kG, there exists $n > 0$ such that $H^n(G, M) \neq 0$?

5. Given a finite dimensional indecomposable periodic module, can we form a direct sum with an appropriate projective module in such a way that the result is induced from a suitable p-local subgroup? What about nonperiodic modules?

The study of these questions leads naturally to the study of infinitely generated modules, and in particular to some infinitely generated modules introduced by Rickard. Associated to any thick subcategory $\mathcal{C}$ of the stable module category $\mathsf{stmod}(kG)$ and any module M, we have a short exact sequence

$$0 \to \mathcal{E}_{\mathcal{C}}(M) \to M \oplus (\text{projective}) \to \mathcal{F}_{\mathcal{C}}(M) \to 0.$$

The functors $\mathcal{E}_{\mathcal{C}}$ and $\mathcal{F}_{\mathcal{C}}$ are idempotent in the sense that repeated application is equivalent to just one application. If $\mathcal{C}$ is closed under tensor products with arbitrary finitely generated modules then the functors

$\mathcal{E}_\mathcal{C}$ and $\mathcal{F}_\mathcal{C}$ are given by tensor product with modules $E_\mathcal{C}$ and $F_\mathcal{C}$. These modules are orthogonal idempotents in the sense that

$$E_\mathcal{C} \otimes E_\mathcal{C} \cong E_\mathcal{C} \oplus (\text{projective}) \tag{1}$$

$$F_\mathcal{C} \otimes F_\mathcal{C} \cong F_\mathcal{C} \oplus (\text{projective}) \tag{2}$$

$$E_\mathcal{C} \otimes F_\mathcal{C} \cong (\text{projective}). \tag{3}$$

My goal in these lectures is to introduce these idempotent modules and explain their relevance to the questions described above. Some time will be spent developing the general theory of infinitely generated modules, and the last lecture will be devoted to the theory of phantom maps and their relationship with purity. A homomorphism between infinitely generated modules $M \to N$ is said to be *phantom* if its restriction to every finitely generated submodule of M factors through a projective module. I shall describe some work of my student Gilles Gnacadja and some of our joint work on these phantom maps.

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MODULES WITH STRANGE DECOMPOSITION PROPERTIES

PAUL EKLOF

We will consider classes of abelian groups and modules where the Kaplansky test problems have negative answers. Kaplansky [9, pp. 12f] posed two test problems in order to "know when we have a satisfactory [structure] theorem. ... We suggest that a tangible criterion be employed: the success of the alleged structure theorem in solving an explicit problem." The two problems were:

(I) If A is isomorphic to a direct summand of B and conversely, are A and B isomorphic?

(II) If $A \oplus A$ and $B \oplus B$ are isomorphic, are A and B isomorphic?

Of particular interest is the method developed by Corner ([2], [3]) which, by realizing certain rings as endomorphism rings (sometimes modulo an ideal), provides negative answers to both test problems (for a given class) as special cases of an even more extreme pathology. More precisely, Corner's method — where applicable — yields, for any positive integer r, a module M_r (in the class) such that for any positive integers m and k, the direct sum of m copies of M_r is isomorphic to the direct sum of k copies of M_r if and only if m is congruent to k mod r. Then we obtain negative answers to both test problems by letting $A = M_2$ ($\cong M_2 \oplus M_2 \oplus M_2$) and $B = M_2 \oplus M_2$. We shall refer to the existence of such a module M_r (for each r) as the "Corner pathology".

For a number of classes of modules, the Corner pathology has been achieved with the help of "prediction principles", either the "diamond principles" that are a consequence of $V = L$, the Axiom of Constructibility, or some form of the "Black Box": a prediction principle, due to Shelah, which is provable in ordinary set theory, ZFC: see [5].

Shelah [11] has derived the Corner pathology for the class of all modules over any ring which is not pure semisimple. As in the following, this is a consequence of a very weak form of realization result.

A subring A of $\mathrm{End}(M)$ is said to be *algebraically closed* in $\mathrm{End}(M)$ when every finite set of ring equations with parameters from A (i.e., polynomials in several variables over A) which is satisfied in $\mathrm{End}(M)$ is also satisfied in A. For each r there is a ring A_r considered by Leavitt, Cohn and Corner (cf. [10],[1],[4]) which is universal for the property that any $R - A_r$-bimodule M is isomorphic to M^r. This ring has the property that if A_r is algebraically closed in $\mathrm{End}(M)$ then for any positive integers m and k, M^m is isomorphic to M^k only if m is congruent to k mod r.

In [7] we consider the class of $\aleph_1$-separable abelian groups (of cardinality $\aleph_1$). An abelian group M is called $\aleph_1$-separable if every countable subset is contained in a countable free direct summand of M. Although some structural properties of $\aleph_1$-separable groups are independent of ZFC (cf. [6, Chap. VIII]) we prove in ZFC the Corner pathology for this class. The proof involves a direct construction of the pathological group M using a tree-like ladder system and a "countable template" which comes from the Corner example [3] for countable torsion-free groups with the help of results from [8]. The Corner pathology holds because A_r is algebraically closed in $\mathrm{End}(M)$; no stronger result realizing A_r as an endomorphism ring is known in this case (without assuming more than ZFC).

The construction can be extended inductively to obtain $\aleph_n$-separable abelian groups of cardinality $\aleph_n$ with the Corner pathology for each $n > 1$.

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Failure of the Krull-Schmidt property for artinian modules and for serial modules

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We shall consider unital modules over associative rings with identity. A module is *uniserial* if its lattice of submodules is linearly ordered under inclusion. A module is *serial* if it is a direct sum of uniserial submodules.

Both the class of artinian modules and the class of serial modules of finite Goldie dimension are classes of modules with semilocal endomorphism rings. If a module M_R has a semilocal endomorphism ring $\mathrm{End}(M_R)$, then all direct sum decompositions of M_R have only finitely many nonzero direct summands, and there are only finitely many direct sum decompositions of M_R up to isomorphism. For classes for which the Krull-Schmidt property holds, e.g. the class of all modules M_R that are finite direct sums of modules with local endomorphism rings, there is only one direct sum decomposition of M_R up to isomorphism.

It is possible to prove that every module-finite algebra over a semilocal noetherian commutative ring may be realized as the endomorphism ring of a suitable artinian module over a suitable ring. It follows that the Krull-Schmidt property fails for the class of artinian modules.

For what concerns serial modules the situation is the following. If M and N are two modules, we say that M and N *belong to the same monogeny class*, and write $[M]_m = [N]_m$, if there exist a monomorphism $M \to N$ and a monomorphism $N \to M$. Similarly, we say that M and N *belong to the same epigeny class*, and write $[M]_e = [N]_e$, if there exist an epimorphism $M \to N$ and an epimorphism $N \to M$.

Theorem. [Weak Krull-Schmidt Theorem for serial modules of finite Goldie dimension] *Let $U_1, \ldots, U_n, V_1, \ldots, V_t$ be uniserial modules over an arbitrary ring. Then the direct sums $U_1 \oplus \ldots \oplus U_n$ and $V_1 \oplus \ldots \oplus V_t$ are isomorphic if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \ldots, n\}$ such that $[U_{\sigma(i)}]_m = [V_i]_m$ and $[U_{\tau(i)}]_e = [V_i]_e$ for every $i = 1, 2, \ldots, n$.*

This allows to show that the Krull-Schmidt property fails for the class of serial modules of finite Goldie dimension.

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Classifying modules over particular rings. – Can we find a border-line in terms of endomorphism rings?

Rüdiger Göbel

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From Paul Eklof [3] ('s abstract) we learn how decomposition properties of modules in the sense of Kaplansky-Corner are related with particular rings often - but not only - constructed in the 60ties and then realized as endomorphism rings. Eklof will also consider the connection of prediction principles in V = L with what we called "Shelah's Black Box in [1], see Shelah [14]. This gives us the advantage to concentrate on more specific topics, some of them dealt with very recently. Nevertheless I will illustrate the use of the Black Box giving one and only one educational example:

(1) Find a (large) cotorsion-free R-module M with

$$(*) \quad \mathrm{End}\,(M) = R.$$

We may be very special and choose $R = \mathbb{Z}$.

I will take this opportunity to make clear that Shelah's Black Box is really an easy "Kinder-garden counting argument" which everybody can understand - however it is a very clever one, which turns out to be useful in many places in algebra., see [5, 2] for instance. After the most desired application as in (1) are well-understood, it seems fascinating to see how we can put the module class under additional strains and still get realization theorem as in $(*)$. We will concentrate on two obviously opposite main directions investigated quite recently:

(2) How free can a module M as above be? — see [6].

(3) How close to pure-injectivity [16] can we choose M with $(*)$ — see [7, 8]

Keywords are almost free modules on one side and splitter or stones on the other side, cf. [11, 12, 13]. Other applications are related to questions concerning the Kronecker module, see e.g. [9, 10]

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ALGEBRAIC COMPACTNESS, DIRECT SUM DECOMPOSITIONS, AND REPRESENTATION TYPE

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LECTURE I. Algebraic compactness, product-preserving functors, and direct sum decompositions.
LECTURE II. Generic modules and representation type.

The first lecture will start by tracing the historical origins of algebraic compactness (= pure injectivity), a concept which already arose in various contexts during the 1950's.

The red thread through the core of the lecture will be provided by the following problem which goes back to Koethe, Cohen, and Kaplansky: Characterize the rings for which each (left) module is a direct sum of finitely generated components. This decomposition property obviously constitutes the strongest possible tie one can ask for between finitely and non-finitely generated modules. First headway towards a solution was made by Koethe, Cohen-Kaplansky, Chase, Griffith, Warfield, and Ringel- Tachikawa.

It was this problem that triggered new, important descriptions of algebraically compact and Σ-algebraically compact modules in terms of finite matrix subgroups (= subgroups of finite definition) which were simultaneously discovered by Gruson-Jensen and W. Zimmermann in the mid-70's. Finite matrix subgroups of a module will be introduced from a functorial point of view emphasizing one of their pivotal properties, namely compatibility with direct products. The most important therorems in this context will be stated, and proofs of a few crucial links will be sketched. On this occasion, it will become apparent how, in various guises, pure injectivity lies at the heart of many arguments in classical module theory.

Combined with an idea of Chase, the theory of algebraic compactness will then lead us to a series of decomposition theorems for direct products of modules, due to Gruson-Jensen, W. Zimmermann, and the speaker. These, in turn, will take us back to our original problem, allowing us to at least satisfactorily characterize the rings for which all left AND right modules have the stipulated decomposition properties. Aiming at a full solution of the problem - unfortunately, not available to date - we will further outline results due to Auslander, Simson, Prest, W. Zimmermann, Herzog, and the speaker.

The second lecture will start with a discussion of a duality between the lattices of finite matrix subgroups of left modules and their character modules relative to an injective cogenerator. This duality will then be invested into a sketch of an argument backing one of the culminating theorems of Lecture I. We will follow with a brief outline of other dualities that have arisen in the context of algebraic compactness and modules of finite endolength. (The topic of duality is to be addressed in greater depth by other speakers.)

This will guide us to the main subject of Lecture II, namely generic modules over Artin algebras in the sense of Crawley- Boevey. We will closely follow work of Crawley-Boevey in outlining how the generic modules 'govern' families of finitely generated indecomposable modules and in deducing their impact on the representation type of the underlying algebra. In particular, we will address the coincidence of classical tameness with generic tameness in the case of a finite dimensional algebra over an algebraically closed base field.

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FINITE VERSUS INFINITE DIMENSIONAL REPRESENTATIONS – A NEW DEFINITION OF TAMENESS

HENNING KRAUSE

The connection between finite and infinite dimensional representations of a finite dimensional algebra Λ will be the main theme of my talk. First I shall explain the concept of a definable subcategory of the category $\operatorname{Mod}\Lambda$ of all Λ-modules. A classical result says that a definable subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$ is determined by its intersection $\mathcal{X} \cap \operatorname{Ind}\Lambda$ with the set $\operatorname{Ind}\Lambda$ of isomorphism classes of indecomposable pure-injective Λ-modules. We present another description, using only the category $\operatorname{mod}\Lambda$ of finite dimensional Λ-modules. Here, we denote for any class $\mathcal{X}$ of Λ-modules by $[\mathcal{X}]$ the class of maps in $\operatorname{mod}\Lambda$ which factor through a module in $\mathcal{X}$. Furthermore, viewing $\operatorname{mod}\Lambda$ as a ring with several objects, we say that an ideal $\mathfrak{I}$ in $\operatorname{mod}\Lambda$ is fp-idempotent if the finitely presented functors $\operatorname{mod}\Lambda \to \mathrm{A}b$ vanishing on $\mathfrak{I}$ are closed under forming extensions.

Theorem. *The map $\mathcal{X} \mapsto [\mathcal{X}]$ defines a bijection between the set of definable subcategories of $\operatorname{Mod}\Lambda$ and the set of fp-idempotent ideals in $\operatorname{mod}\Lambda$.*

In the second part, I shall apply this result in order to give a new characterization of tameness. Recall that an algebra Λ has tame representation type if there exists for every $n \in \mathbb{N}$ a finite number of continuous 1-parameter families which parametrize all n-dimensional indecomposable Λ-modules. Let us denote by $\operatorname{ind}_n \Lambda$ the isomorphism classes of n-dimensional indecomposables. We need also the Ziegler topology on $\operatorname{Ind}\Lambda$; a subset $\mathbf{U}$ is Ziegler-closed if $\mathbf{U} = \mathcal{X} \cap \operatorname{Ind}\Lambda$ for some definable subcategory $\mathcal{X}$ of $\operatorname{Mod}\Lambda$.

Theorem. *The following conditions are equivalent:*

1. *the algebra Λ is of tame representation type;*
2. *for every $n \in \mathbb{N}$ the Ziegler closure of $\operatorname{ind}_n \Lambda$ contains only finitely many modules which do not belong to $\operatorname{ind}_n \Lambda$;*
3. *for every $n \in \mathbb{N}$ there are only finitely many fp-idempotent and nilpotent ideals in $\operatorname{mod}\Lambda$ which are contained in the ideal $[\operatorname{ind}_n \Lambda]$.*

Note that the characterization (3) involves only finite dimensional modules, although the original definition of tameness is based on infinite dimensional modules. Using this example, I shall demonstrate that the Ziegler spectrum $\operatorname{Ind}\Lambda$ and the fp-idempotent ideals in $\operatorname{mod}\Lambda$ provide an efficient link between infinite and finite dimensional representations.

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An abstract for three talks by Nick Kuhn
Generic Representation Theory: a survey of basic structure.

With $\mathbf{F}_q$ a finite field of characteristic p, let $\mathcal{F}(q)$ be the category whose objects are functors from finite dimensional $\mathbf{F}_q$–vector spaces to $\mathbf{F}_q$–vector spaces.

A connection between this abelian category and $GL_n(\mathbf{F}_q)$-modules is easy to see: if $F \in \mathcal{F}(q)$, then $F(\mathbf{F}_q^n)$ is a $GL_n(\mathbf{F}_q)$-module for all n. The category $\mathcal{F}(q)$ blends the categories of $GL_n(\mathbf{F}_q)$-modules together for all n in a very precise way. One is quickly led to lots of naturally occurring infinite length objects of interest.

The three lecturers will survey some of the basic structure of $\mathcal{F}(q)$ and related categories. As envisioned, the talks will go roughly as follows.

Lecture 1: Generic representations and the recollment tower.

The category $\mathcal{F}(q)$ can be regarded as the category of representations of a small additive category. Thus some basic structure is easy to see: e.g. the projectives and injectives. Two other useful bits of structure are a scalar decomposition and a duality transformation. To see the beginnings of deeper structure, we make two observations.

The first goes as follows. Let $\mathcal{GL}_n(q)$ and $\mathcal{M}_n(q)$ denote the categories of modules over the group ring $\mathbf{F}_q[GL_n(\mathbf{F}_q)]$ and the semigroup ring $\mathbf{F}_q[M_n(\mathbf{F}_q)]$. Then $\mathcal{F}(q)$ is equivalent to the inverse limit of the tower of categories

$$\ldots \to \mathcal{M}_{n+1}(q) \to \mathcal{M}_n(q) \to \mathcal{M}_{n-1}(q) \to \ldots$$

and each 'fiber sequence'

$$\mathcal{GL}_n(q) \to \mathcal{M}_n(q) \to \mathcal{M}_{n-1}(q)$$

is part of a 'recollement diagram'. This observation has many consequences. For example, it follows formally that simple objects in $\mathcal{F}(q)$ correspond to the simple $GL_n(\mathbf{F}_q)$-modules, for all n.

The second observation is that the simple-minded operation of direct sum in the category of finite dimensional $\mathbf{F}_q$-vector spaces induces all sorts of interesting structure on $\mathcal{F}(q)$. For example, this leads to the notion of a polynomial functor in the sense of Eilenberg-MacLane, which leads to more recollement diagrams involving the symmetric groups. Another consequence is a remarkable Kunneth formula for computing many Ext groups.

Lecture 2: Finite functors and the role of the symmetric powers.

Using the notion of a polynomial functor, it is easy to characterize the finite objects, and then the locally finite objects. An unresolved (and perhaps misnamed) **Artinian Conjecture** goes as follows: $\mathcal{F}(q)$ is a locally Noetherian category. (See Geoff Powell's talk for more about this.)

We then turn our attention to two (dual) families of finite objects in $\mathcal{F}(q)$: S_n and S^n, $n \geq 0$, defined by $S_n(V) = (V^{\otimes n})^{\Sigma_n}$, and $S^n(V) = (V^{\otimes n})_{\Sigma_n}$. Two fundamental theorems are the **Embedding Theorem** which says that every finite object embeds in a sum of S^n's, and the **Vanishing Theorem** which says that S^n, when localized by inverting the Frobenious map, becomes injective in the category of locally finite objects.

Via the Gabriel–Popescu Theorem, these theorems translate into statements about the representation category $\mathcal{U}(q) = Rep(S^*)$. When q is a prime, the category $\mathcal{U}(q)$ is revealed to be the category of unstable modules over the Steenrod algebra: a category that has been of interest to algebraic topologists for roughly half a century!

Lecture 3: Ext calculations, K-theory, and rational cohomology.

In this talk, I will survey work of Franjou-Lannes-Schwartz, Franjou-Friedlander-Skorichenko-Suslin, Betley, Pirashvili, myself, and others concerning calculations of Ext groups in $\mathcal{F}(q)$, and how these Ext groups are related to algebraic K–theory, Topological Hochshild Cohomology, finite group cohomology, and rational cohomology.

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Dimension Theory of Noetherian Rings

T H Lenagan

This is a series of three lectures introducing the basic ideas of dimension theory for noetherian rings and modules. The talks will be aimed at postgraduate and postdoctoral algebraists whose main interest is not in noetherian rings, but who may find that noetherian methods are sometimes needed in their own areas.

The series will begin by looking at desireable properties of dimension functions as generalizations of known ways of measuring the size of rings and modules in the artinian and commutative contexts, in order to extract the definition of a **dimension function**. Next, two of the most important dimensions, **Krull dimension** and **Gelfand-Kirillov dimension**, will be introduced and their basic properties developed. The series will finish with some applications of the methods introduced, chosen to illustrate typical uses of the methods.

The lectures of Vladimir Bavula, which follow this series, may well build on some of the notions developed here.

The books listed below will give an introductory idea to the type of material to be covered.

G.R. Krause and T.H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Pitman, 1985

C Nastasescu and F Van Oystaeyen, *Dimensions of ring theory*, Mathematics and its Applications, 36. D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1987.

J C McConnell and J C Robson, *Noncommutative noetherian rings*, Wiley, Chichester, 1987.

Cohen - Macaulay Modules and Approximations

Alex Martsinkovsky (Boston/USA)

In these lectures, we shall discuss some module-theoretics aspects of local algebra and their relations to neighboring disciplines. Traditionally, a great deal of research has been focused on Cohen - Macaulay local rings. It has been long recognized that, over such rings, Cohen - Macaulay modules play a special role. As two extremes, this class includes maximal Cohen - Macaulay (mCM) modules and modules of finite length. Especially amenable to homological methods is the case when the local ring is Gorenstein, i.e., of finite injective dimension over itself. This finiteness condition allows to realize mCM's as high enough syzygy modules, and thus introduces the first major theme to be discussed in the lectures: the relationships between a module and the "tail" of its projective resolution.

One such relationship is reflected by the concepts of (maximal) Cohen - Macaulay approximation and of hulls of finite injective dimensions, introduced by Auslander and Buchweitz in the mid-80's. In the first lecture, we shall discuss various ways of constructing mCM approximations and hulls. This includes the pitchfork construction of Auslander - Buchweitz and the gluing construction, appearing in joint work with Herzog. The latter is, in fact, an algebraic reformulation of a simple-minded homotopy-theoretic construct. Minimal MCM approximations are unique and thus give rise to various invariants of modules. One of them, the delta-invariant of Auslander seems to be of particular importance. The vanishing of that invariant has been the focus of research in the last decade, and we shall discuss some recent developments.

We then intend to show how mCM approximations can be used to study isolated singularities. This direction targets criteria for the quasihomogeneity of isolated singularities (over a field of characteristic zero). The distinct feature of this class of algebras is the presence of a certain module of finite length. This module is the transpose of the module of Kaehler differentials. In the hypersurface case, this module becomes a finite- dimensional algebra and is called the moduli algebra (or transgradient algebra) of the singularity. The famous criterion of K. Saito asserts that an isolated singularity is quasihomogeneous if and only if the defining equation belongs to its own jacobian ideal (this is the ideal generated by the partial derivatives). This condition is identical to one of the criteria for the vanishing of the delta-invariant of the moduli algebra (as a module over the singularity).

Another module-theoretic approach to the problem of quasi-homogeneity, at least in dimension two, is to compare the module of Zariski differentials with the mCM approximation of the maximal ideal. The latter is called the Auslander module of the singularity. In the case when the singularity is a rational double point, and therefore is of type A, D, or E, the indecomposable non-projectives are described by the simple roots of the corresponding Dynkin diagram. The unique projective can be thought of as the maximal root, and the Auslander module corresponds to the simple root(s) adjacent to the maximal root in the extended Dynkin diagram. A conjecture of the speaker asserts that a two-dimensional singularity is quasihomogeneous if and only if the module of Zariski differentials and the Auslander module are isomorphic. The "only if" part has been established, but the other direction is wide open. There are reasons to believe that this conjecture is the "next" step after the Zariski - Lipman conjecture.

We shall also discuss how the above approach generalizes to higher dimensions. Two modules of finite length come to the forefront: the residue field and the transpose of Kaehler differentials. Another conjecture of the speaker asserts that the singularity is quasihomogeneous if and only if the tails of the projective resolutions of the two modules are the same up to a shift. The gluing construction mentioned above allows to prove the "only if" part for complete intersections. A heuristic explanation of this phenomenon is nothing more complicated than the Euler identity for quasihomogeneous polynomials.

Returning to the delta-invariant, it is of interest to define it not just for Gorenstein rings but for arbitrary local rings, or even for more general algebraic structures. In the second lecture, we shall show how this can be done. The definition is based on a remarkable construction of Pierre Vogel that greatly generalizes and simplifies the definition of Tate cohomology. The new invariant, called the ksi-invariant, allows to generalize many results originally proved for Gorenstein rings. Sometimes the proofs are identical to the old ones, sometimes modifications are needed, and sometimes completely different techniques are required. We show how this new invariant can be used in local algebra to study the syzygy modules of the residue field. More precisely, we establish a new regularity criterion for local rings: a local ring is non-regular if and only if the ksi-invariant of each of the syzygy module of the residue field is zero. Besides providing non-trivial new information, this result links some seemingly unrelated results established earlier. The proof, including the definition of the ksi-invariant, will be done without any reference to high-powered machinery. It is accessible to any student who has had or is having a first course in homological algebra.

Artinian objects in the category of functors between $\mathbb{F}_2$-vector spaces

Geoffrey M.L. Powell

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The category $\mathcal{F}(k)$ associated to a field k is the category of functors $\mathcal{V}_k^f \to \mathcal{V}_k$, where $\mathcal{V}_k$ is the category of k-vector spaces and $\mathcal{V}_k^f$ is the full sub-category of finite-dimensional spaces. When the field k is infinite, this category is related to the study of polynomial representations of the general linear groups on k. The case of a finite field is less-well understood but is of interest since it is related to the category of unstable modules over the k-Steenrod algebra, a basic object of consideration in algebraic topology. The calculation of extension groups in $\mathcal{F}$ between polynomial functors is equivalent to the calculation of MacLane cohomology with coefficients; more recently, Suslin has established that this is in fact equal to the stable groups calculated over the general linear groups, an affirmative response to the conjecture that topological Hochschild homology is equal to stable K-theory. The talk will concentrate on the structure of objects in the category rather than homological considerations; throughout it will be assumed that $k = \mathbb{F}_2$, the prime field with two elements.

The evaluation functor $F \mapsto F(\mathbb{F}_2^n)$ defines a functor $\mathcal{F} \to \mathcal{M}_n$, where $\mathcal{M}_n$ denotes the category of left $\mathrm{End}(\mathbb{F}_2^n)$-modules. Restriction defines a functor $\mathcal{M}_n \to \mathcal{GL}_n$ to the category of $GL_n\mathbb{F}_2 = \mathrm{Aut}(\mathbb{F}_2^n)$-modules. This theory has been developed by Kuhn in the 'recollement' framework [K2]; in particular, this provides a construction of the simple functors in $\mathcal{F}$ from the simple $\mathrm{GL}_n\mathbb{F}_2$ modules, as n varies. There is a connection with the category of modules over the symmetric groups S_d, which arises from studying the polynomial filtration of the category $\mathcal{F}$. This may be described using the functor $\mathrm{Hom}_{\mathcal{F}}(T^d, -) : \mathcal{F} \to \mathrm{mod} - S_d$, where T^d is the d^{th} tensor power functor, $V \mapsto V^{\otimes d}$; the S_d action is induced by the evident (right) action on T^d. This provides a natural setting for the correspondence between the representation theory of the symmetric groups and the general linear groups. The basic material on the categories $\mathcal{F}$ will be covered in the talks by Kuhn and Schwartz.

A functor is said to be finite if it has a finite composition series and locally finite if it is the colimit of its finite sub-objects. The full sub-category of locally finite objects is denoted by $\mathcal{F}_\omega$; the injective functors in $\mathcal{F}$ belong to the category $\mathcal{F}_\omega$, which has enough injectives. The non-constant indecomposable injectives are of interest because they are *not* finite objects. The injective envelope of the functors Λ^1 is habitually written $\overline{I}$ and is a uniserial object, with socle series $\Lambda^1, \Lambda^2, \ldots, \Lambda^n, \ldots$. However, for all other non-constant indecomposable injectives, the structure is far richer and there is little hope of giving an explicit description of the full lattice of sub-objects. Nevertheless, Lionel Schwartz has proposed the following conjecture, in conjunction with Kuhn and Lannes:

The Artinian Conjecture The indecomposable injectives in $\mathcal{F}$ are Artinian objects in $\mathcal{F}$ (that is: every descending sequence of sub-objects stabilizes).

The talk will discuss this conjecture, along with related questions. A reduction is provided by introducing a *co-Weyl* object, $J_S \in \mathcal{F}$, associated to the simple functor S; this may be defined as the functor $R_n S(\mathbb{F}_2^n)$, where R_n is the right adjoint to the functor $\mathcal{F} \to \mathcal{M}_n$ and n is the least integer such that $S(\mathbb{F}_2^n) \neq 0$. A concrete description is that J_S is the largest sub-functor $J \subset I_S$ of the injective envelope of S for which $J(\mathbb{F}_2^n) = S(\mathbb{F}_2^n)$. The fundamental importance of these objects is shown by the following result, analogous to a property of highest weight categories.

Theorem 1 *[P2] Each indecomposable injective in $\mathcal{F}$ admits a finite filtration with each filtration quotient isomorphic to some co-Weyl object.*

It follows immediately that the Artinian conjecture reduces to the affirmation that each object J_S is Artinian.

The essential tool in the study of global properties of the lattice of sub-objects of a functor is the shift functor $\widetilde{\Delta} : \mathcal{F} \to \mathcal{F}$, defined by $\widetilde{\Delta}F(V) = F(V \oplus \mathbb{F}_2)$; this gives a generalization to the category $\mathcal{F}$of restriction arguments in representation theory. The functor $\widetilde{\Delta}$ is the right adjoint to the functor $-\otimes P$, where P is the projective (not locally finite) functor $V \mapsto \mathbb{F}_2[V]$; this admits a natural filtration by powers of the augmentation ideal and hence quotients $P \to q_n P$. The right adjoints to the functors $-\otimes q_n P$ are written as $[p_n \widetilde{\Delta}]$ and there are inclusions of endo-functors of $\mathcal{F}$, $[p_n \widetilde{\Delta}] \subset \widetilde{\Delta}$. The quotients $\widetilde{\nabla}_n = \widetilde{\Delta}/[p_{n-1}\widetilde{\Delta}]$ restrict to functors $\mathcal{F}_\omega \to \mathcal{F}_\omega$ which are not exact but preserve injections and surjections [P]. The following result establishes the most general result known concerning the structure of these functors:

Theorem 2 *[P2] Suppose that S, n, J_S are as above, then $\widetilde{\nabla}_{n+1} J_S = 0$ and there is an inclusion $J_S \subset \widetilde{\nabla}_n J_S$. Moreover, if $F \subset J_S$ is a proper sub-functor of J_S, then there exists a natural number t such that $(\widetilde{\nabla}_n)^k F = 0$.*

This result is used to establish the following particular case of the Artinian conjecture:

Theorem 3 *[PS] The injective object (not indecomposable) $\overline{I} \otimes \overline{I}$ is injective in $\mathcal{F}$.*

The proof of this theorem requires the study of the functors $\overline{I} \otimes \Lambda^n$, as n varies. More generally, one studies the functors $\overline{I} \otimes S$, where S is a simple functor. One says that a functor is *simple modulo finites* if every proper sub-object is a finite object. The class of simple modulo finite objects is a generalization of the class of uniserial objects. It is established easily that the functor $\overline{I} \otimes S$ is Artinian if and only if it admits a finite filtration of which each filtration quotient is simple modulo finites. The approach to the study of $\overline{I} \otimes S$ relies heavily on the usage of the functors ∇, mimicking the argument of [P1].

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Topological and geometric aspects of the Ziegler spectrum

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The (right) Ziegler spectrum of a ring R is a topological space which was introduced by Ziegler in his pivotal paper [18] on the model theory of modules. This space Zg_R may be defined purely in terms of the category Mod-R of (right) R-modules. The points of the space are the (isomorphism classes of) indecomposable pure-injective (=algebraically compact) R-modules. The topology may be defined in a number of equivalent ways: in terms of the model theory of modules (as Ziegler did); in terms of morphisms in the category of R-modules; in terms of the associated category $(R{-}\mathrm{mod}, \mathbf{Ab})$ of additive functors from the category of finitely presented left R-modules to that of abelian groups.

For the purposes of Ziegler's paper, the central properties of this space are that its closed sets correspond to complete theories of modules and that isolation of points of the space may be described by properties of certain types (the notion of "type" is one of the basic concepts of model theory).

Understanding the Ziegler spectrum of a ring may, with good justification, be regarded as the key to understanding most aspects of the model theory of modules over that ring.

If R is an artin algebra then every indecomposable module of finite length is a point of the Ziegler spectrum. So the ubiquity of wild algebras warns us that, in general, we cannot expect anything like an explicit description of the Ziegler spectrum. But we may hope that, at least for tame algebras, we may describe the Ziegler spectrum and, in the limited cases that have been thoroughly investigated so far, this hope has been borne out.

Typically, the Ziegler spectrum has many infinite-length points. Indeed, over an artin algebra R, the finite length indecomposables are precisely the isolated points of the space and hence, if R is not of finite representation type, then, since the space is compact (meaning just that every open cover has a finite subcover: Zg_R is seldom Hausdorff), there must be an infinite length point. In fact, over an artin algebra, the subset of Zg_R consisting of points of finite length is an open dense subset and so all infinite-length indecomposable pure-injectives are, in the topological sense, limits of finite-length indecomposable modules.

The connection between finite and infinite length modules also goes the other way. Crawley-Boevey describes an indecomposable module as generic if it is an infinite length point of Zg_R which is of finite length over its endomorphism ring. He showed, for finite-dimensional algebras over algebraically closed fields, that the existence of generic modules implies the existence of one-parameter families of finite-length points (see [4] for a precise statement). Since one can deduce the existence of generic modules from the existence of infinite-length points satisfying a rather weak finiteness condition, one sees that the infinite-length points of the spectrum are intimately related to families of finite-length points.

All this suggests that, beside the original model-theoretic motivation, there are strong representation-theoretic motivations for investigating the Ziegler spectrum.

The "weak finiteness condition" referred to above is the possession of what Garavaglia [7] called elementary Krull dimension, equivalently the possession of what is called rank in [18], equivalently m-dimension in [10] (again, equivalently, $N \in \mathrm{Zg}_R$ has these iff the functor $N \otimes -$ has Krull-Gabriel dimension in the category $(R\text{−mod}, \mathbf{Ab})$). This is closely related to (conjecturally equivalent to) the point having Cantor-Bendixson rank in the space Zg_R.

This topological, Cantor-Bendixson, rank, which is defined by inductively throwing away all the isolated points of a space, seems to be the right measure of the topological complexity of the space Zg_R. We will discuss the general significance of this rank and we will present some examples where it has been computed ([3],[12],[16], [17]), with the aim of illustrating how useful it is to know the Cantor-Bendixson layering of Zg_R.

In the second part of my talk I will discuss another topology on the underlying set of the Ziegler spectrum. This is obtained by taking, for a basis of open sets, the complements of the compact open sets of the Ziegler topology [11], [13]. It may reasonably be called the Gabriel-Zariski topology of R since if we follow Gabriel [6] in formulating the definition of the Zariski topology of a commutative noetherian ring in terms of the category of modules and then apply this definition to the category $(R\text{−mod}, \mathbf{Ab})$, we obtain this space (up to the identification of an indecomposable pure-injective module N with the corresponding indecomposable injective functor $N \otimes -$).

This space comes equipped with a sheaf of rings, the sheaf of rings of locally definable scalars [14], which generalises the structure sheaf of a commutative noetherian ring. The rings which occur in this sheaf may be defined

[2] via model theory (as rings of definable functions), module theory (as bi-endomorphism rings) or via functor categories (as endomorphism rings of localised functors). The centre of this sheaf is a sheaf of local rings.

We will present some examples of this ringed space and, in particular, we will compare the sheaf of locally definable scalars of a tame hereditary algebra with the sheaves discussed in [1], [8] and [5].

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BOUSFIELD LOCALIZATION FOR REPRESENTATION THEORISTS

JEREMY RICKARD

Bousfield localization is a technique that has been used extensively in algebraic topology, specifically in stable homotopy theory, over the last quarter century. Its name derives from the fundamental work of Bousfield [3], although this is based on earlier work of Brown [4] and Adams [1]. In abstract terms, Bousfield localization deals with the inclusion of a thick subcategory (i.e., a triangulated subcategory closed under direct summands) into a triangulated category and the existence of adjoint functors to such an inclusion. In the original topological setting, the triangulated category was typically the stable homotopy category, and the thick subcategory was defined by the vanishing of some homology theory, but the techniques involved work much more generally. In particular, the triangulated category can be one of interest to representation theorists, such as a stable module category or the derived category of a module category: since the techniques rely heavily on limiting procedures, however, one is forced to work with infinite dimensional modules.

In this pair of talks, I shall discuss Bousfield localization from a fairly abstract point of view, using examples from representation theory. In my first talk, I will discuss 'finite localization', which is the simplest case, and which has been sufficient for most of the existing applications in representation theory: some of these applications will be discussed in more detail in the talks of Dave Benson. For finite localization, one deals with a triangulated category and a thick subcategory that are both 'compactly generated': this means that every nonzero object has a nonzero map from some 'compact' (or 'small') object (i.e., an object X such that $\mathrm{Hom}\,(X, ?)$ commutes with infinite direct sums or, in other words, every map from X to an infinite direct sum factors through some finite subsum). For example, the compact objects of the derived category $D(\mathrm{Mod}(R))$ of the module category of a ring R are the objects isomorphic to bounded complexes of finitely generated projective R-modules, and since every nonzero object of $D(\mathrm{Mod}(R))$ has nonzero homology in some degree, it has a nonzero map from the compact object $R[n]$ for some n: hence $D(\mathrm{Mod}(R))$ is compactly generated.

In my second talk I shall deal with more general versions of Bousfield localization, where the condition of compact generation is weakened.

Much of what I know about Bousfield localization I learned from the papers of Neeman [2, 5, 6, 7, 8, 9], who has enthusiastically argued that these techniques are much underused outside algebraic topology and has shown that they can be used to simplify substantially the proofs of certain theorems of algebraic geometry and algebraic K-theory. My talks will owe a lot to this work of Neeman, though I will mostly restrict my attention to examples from representation theory.

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Birational classification of moduli spaces of representations and the endomorphism rings of bricks.

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One knows that the bricks for a quiver are parametrised by the Schur rays, that is, the 1-dimensional subspaces of the space of dimension vectors that contain a Schur root. One can show fairly easily that the endomorphism rings of the bricks are 'stably free division rings' and it is an interesting question to know whether they are actually free division rings. This question takes on greater importance from the realisation that this problem is equivalent to describing the function fields of the moduli spaces of representations of the quiver of all dimension vectors lying in the Schur ray. Thus I show that these endomorphism rings are in fact free division rings and that if α is a Schur root then a moduli space of representations of dimension vector α is birational to n h by h matrices up to simultaneous conjugacy where $h = \mathrm{gcd}_v(\alpha(v))$ and $n = 1 - \langle\beta, \beta\rangle$ where $\beta = \alpha/h$.

The degree filtration on polynomial functors and unstable modules

Lionel Schwartz (Paris/France)

One considers on one side the category of functors from the category of finite dimensional vector spaces over a finite prime field to the category of vector spaces. For simplicity of notations we will restrict to the caracteristic 2. The morphims are the natural transformations. This category contains as a full subcategory the category of polynomials functors in the sense of Eilenberg and Mac Lane.

On the other side one considers the category $\mathcal{U}$ of unstable modules over the Steenrod algebra (again for simplicity we will consider the case where the caracteristic is 2). The singular mod 2 cohomology of a space is such a module. One will introduce the notion of nilpotent module, and the quotient category $\mathcal{U}/\mathcal{N}/il$. We will explain briefly topological motivations for this study, in particular state Lannes' theorem on maps from the classifying space of an elementary abelien p-group.

Then, we compare the filtration induced by the Eilenberg-MacLane degree on the category of polynomial functors, on one side, wi th on the other side, the weight filtration on the category of reduced unstable modules over the Steenrod algebra. Or, what is better on the (abelian) quotient category $\mathcal{U}/\mathcal{N}il$. One will show that they are in strict correspondance.

A reduced unstable module is said to be of weight less than n if it is trivial in degrees d such that $\alpha(d) > n$.

In particular we will show how to associate to a reduced unstable module (or to an object in $\mathcal{U}/\mathcal{N}il$) a modular representation of the symmetric group $\mathcal{S}_n$ for any n. We will derive a brief description of simple objects in both settings, i.e. the one of polynomial functors and the one of unstable modules. The techniques involved there are quite close from group-representations theoretic ones.

As an application of what precedes we will discuss the classification of indecomposable injective objects in the category of polynomial functors, or in $\mathcal{U}/\mathcal{N}il$. Then, we will Krull-Schmidt decomposition of Carlsson's unstable modules. This is equivalent to the description of the Krull-Schmidt decomposition of the functors $S(i)$ which are direct limits under the square maps of the symmetric powers $S^{2^h i}$ (see N. Kuhn's talks).

Tame algebras and degenerations of modules

Grzegorz Zwara

Let A be a finite dimensional associative algebra over algebraically closed field k. This talk will be divided in two parts.

1) Tame algebras and generic modules.

Definition 1 *We say that A is of tame representation type if for all $d \geq 1$ there are a finite number of A-$k[X]$-bimodules $M_1, \ldots, M_n$ which are free of rank d as right $k[X]$-modules, and such that all but finitely many isomorphism classes of indecomposable A-modules of dimension d are of the form $M_i \otimes_{k[X]} k[X]/(X-\lambda)$ for some $1 \leq i \leq k$ and $\lambda \in k$.*

By Drozd's Tame and Wild Theorem it is known that the class of finite dimensional associative k-algebras may be didided in two disjoint classes. One class consists of tame algebras and the second class is formed by the wild algebras whose representation theory is as complicated as the study of finite dimensional vector spaces together with two non-commuting endomorphisms, for which the classification is a well-known unsolved problem. Hence, we can realistically hope to classify modules only for tame algebras.

Definition 2 *If G is an A-module, then we call its length as an $End_A(G)$-module the endolength of G. We say that that G is a generic module if it is indecomposable, of infinite length over A, but finite endolength.*

Theorem 3 ([2]) *The following are equivalent:*

1. *A has tame representation type;*
2. *For each $d \geq 1$ there are only finitely many isomorphism classes of generic left A-modules of endolength d.*

Moreover, generic modules are related to one-parameter families of indecomposable modules as follows.

Theorem 4 ([2]) *Assume that A is tame. Then for any generic module G, $End_A(G)/radEnd_A(G)$ is a rational function field in one variable over k and we can choose an A-R_G-bimodule M_G, such that the following hold.*

1. *R_G is a finitely generated localization of $k[X]$. As a right R_G-module, M_G is free of rank equal to the endolength of G. If K_G is the quotient field of R_G, then $M_G \otimes_{R_G} K_G \simeq G$.*
2. *The functor $M_G \otimes_{R_G} -$ from R_G-modules to A-modules preserves isomorphism classes, indecomposability and Auslander-Reiten sequences.*
3. *For each $d \geq 1$, all but finitely many isomorphism classes of d-dimensional A-modules arise as $M_G \otimes_{R_G} R_G/(r)$ for some G and some $0 \neq r \in R_G$.*

By condition (2) almost all indecomposable d-dimensional A-modules lie in homogeneous tubes (see [CB1]). For any such tube $\mathcal{T} = (\mathcal{T}_\rangle)_{\rangle \geq \infty}$ we define a direct limit T of modules T_i, $i \geq 1$. Then the following holds.

Theorem 5 ([3]) *Module T is Σ-pure-injective, indecomposable, but not generic. The set of generic modules which belong to the Ziegler closure of $\mathcal{T}$ is non-empty and finite.*

These modules are direct summands of infinite product of T. We shall present a more general construction of generic modules (in fact endofinite modules) given in the proof of the following theorem due to Ringel.

Theorem 6 ([5]) *Let X be an infinite dimensional A-module, let φ be a locally nilpotent endomorphism of X with finite dimensional kernel. Let I be some infinite set. Then the product X^I is the direct sum of a non-zero module P of finite endolength and of copies of X. The endolength of P is bounded by the minimum of the dimension of the vector spaces $Ker\varphi^t/Ker\varphi^{t-1}$, with $t \geq 1$.*

2) Degenerations of finite dimensional modules.

If $a_1 = 1, \ldots, a_\alpha$ is a basis of A over k, we have the structure constants a_{ijk} defined by $a_i a_j = \sum a_{ijk} a_k$. The affine variety $mod_A^d(k)$ of A-module structures on k^d consists of α-tuples $m = (m_1, \ldots, m_\alpha)$ of $d \times d$-matrices with coefficients in k such that m_1 is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$ holds for all indices i and j. Any such α-tuple m corresponds to a d-dimensional A-module M in the obvious way. The general linear group $Gl_d(k)$ acts on $mod_A^d(k)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional A-modules. By abuse of notation N is a degeneration of M if n belongs to the Zariski closure $\overline{Gl_d(k)(m)}$ of $Gl_d(k)(m)$ in $mod_A^d(k)$, and we denote this fact by $M \leq_{deg} N$.

Theorem 7 ([4]) *Let M, N, Z be finite dimensional A-modules such that there is an exact sequence of one of the forms*

$$0 \to N \to M \oplus Z \to Z \to 0 \qquad or \qquad 0 \to Z \to Z \oplus M \to N \to 0.$$

Then $M \leq_{deg} N$.

Let m, n be two points in $mod_A^d(k)$ such that $M \leq_{deg} N$. Then there is a non-singular curve $\mathcal{C}$, a point c_0 in $\mathcal{C}$ and a morphism $\mu : \mathcal{C} \to \overline{\mathcal{G}\updownarrow_{\lceil}(\|)\Updownarrow}$ such that $\mu(c_0) = n$ and $\mu(c) \in Gl_d(k)m$ for all $c \neq c_0$. Let R be the local ring of the point c_0 in $\mathcal{C}$ with the maximal ideal I. Then R is a discrete valuation k-algebra and μ corresponds to A-R-bimodule Y which is as R-module free of rank d. We shall show the following proposition:

Proposition 8 ([6]) *Assume that there is a monomorphism $\varphi : M \otimes_k R \to Y$ of A-R-bimodules, for some d-dimensional A-module M. Then, for a sufficiently large h, there is an A-module isomorphism*

$$Y/YI^{h+1} \simeq Y/YI^h \oplus M.$$

We apply the above two facts in the proof of the following characterization of degenerations:

Theorem 9 ([6]) *Let m, n be points in $mod_A^d(k)$, $d \geq 1$. Then the following conditions are equivalent:*

1. *$M \leq_{\deg} N$.*

2. *There is a short exact sequence* $0 \to N \to M \oplus Z \to Z \to 0$ *for some finite dimensional* A*-module* Z.

3. *There is a short exact sequence* $0 \to Z \to Z \oplus M \to N \to 0$ *for some finite dimensional* A*-module* Z.

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