Workshop on Operads

Osnabrück June $15^{\text{th}} - 19^{\text{th}}$

Sonderforschungsbereich 343

Bielefeld, F.R. Germany

In the third week of June several mathematicians were visiting the University of Osnabrück on invitation of the Sonderforschungsbereich 343. Since all of them were interested in operads and their applications in algebra and topology it was spontaneously decided to have a little workshop on the subject. Eight lectures were given, the rest of the time available was spent with extensive discussions. Other members of the Sonderforschungsbereich coming from Bielefeld attended the lectures on Thursday.

Rainer Vogt,
Fachbereich Mathematik/Informatik,
Universität Osnabrück,
49069 Osnabrück,
F.R. Germany

Program

Tuesday, June 16

14.15-15.15	M. Brinkmeier:	“The Milgram non-operad”
16.15-17.15	C. Berger:	“Milgram’s and Smith’s models

Wednesday, June 17

10.15-11.15	M. Markl:	“Homotopy algebras and resolutions of operads”
11.30-12.30	S. Klaus:	“On simplicial loops and H-spaces”

Thursday, June 18

13.30-14.30	J. McClure:	“The C_2-operad and topological Hochschild cohomology”
14.45-15.45	Z. Fiedorowicz:	“Constructions of E_n operads”
16.15-17.15	C. Berger:	“Cellular structures for E_n operads”
17.30-18.30	R. Vogt:	“The barconstruction for operads”

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Clemens Berger

Introduction

These notes are a detailed account of two lectures I gave during a workshop on operads in Osnabrück (16-19 June 1998). I would like to thank Rainer Vogt for organizing this really stimulating meeting which gave the participants the wonderful chance to exchange their ideas in a very lively atmosphere.

The purpose of my lectures is fourfold :

1. to show "on the nose" that the well known configuration space model for $\Omega^n S^n X$ is homotopy equivalent to Milgram's permutohedral model;
2. to indicate a "recipe" for constructing cellular decompositions of E_n-operads;
3. to give a simplicial splitting of $\Omega^n S^n X$ using Jeff Smith's filtration of the "symmetric monoidal" operad;
4. to outline some interaction between E_n-operads and *immersion theory.*

1 Configuration spaces and permutohedra.

Initially, the interest in iterated loop spaces arose from *homotopy theory*, more precisely from the fact that several important classifying spaces were known to be infinite loop spaces. In Peter May's theory of E_n-operads [20], the *recognition principle* for n-fold iterated loop spaces is based on the *approximation theorem* which (in its crude form) states that (for a connected, well pointed space X) the weak homotopy type of $\Omega^n S^n X$ can be realized as a coend $F(\mathbb{R}^n, -) \otimes_\Lambda \underline{X}$. The ingredients for this coend are the *configuration spaces*

$$F(\mathbb{R}^n, k) = \{(t_1, \ldots, t_k) \in (\mathbb{R}^n)^k \,|\, t_i \neq t_j \text{ for } i \neq j\}.$$

This May-Segal model for $\Omega^n S^n X$ is predated by Milgram's model which can also be written as a coend $J^{(n)} \otimes_\Lambda \underline{X}$, where this time the coend ingredients are constructed out of convex polytopes P_k, nowadays called *permutohedra*. The permutohedron P_k is by definition the *convex hull of the symmetric group* $\mathfrak{S}_k$ "generically" embedded in some affine euclidean space. There are different ways of doing so. Milgram considered the orbit $\mathfrak{S}_k.(1,2,\ldots,k)$ in $\mathbb{R}^k$, the symmetric group acting by permutation of the coordinates. Baues [4] considered the $k!$ edge-paths in $[0,1]^k$ joining the vertex $(0,\ldots,0)$ to the vertex $(1,\ldots,1)$, realizing this way the permutohedron as a *combinatorial path space*. Probability theorists consider $\mathfrak{S}_k$ as the set of permutation matrices embedded in the affine space of $k \times k$-matrices with real coefficients. The permutohedron is then identified with the space of the so called *probabilistic matrices* with column and line sum equal to 1.

The following lemma is crucial to Milgram's construction, cf. also Michael Brinkmeier's lecture [10] :

Lemma 1.1. *The face-poset of the permutohedron P_k is canonically isomorphic to the set of (right) cosets of subgroups $\mathfrak{S}_{i_1} \oplus \cdots \oplus \mathfrak{S}_{i_r}$ of $\mathfrak{S}_k$ with $k = i_1 + \cdots + i_r$, partially ordered by inclusion.*

The purpose of this section is to define equivariant cellular decompositions of both "building blocks" $F(\mathbb{R}^n, k)$ and $J_k^{(n)}$ in order to deduce that the May-Segal model and the Milgram model have the same homotopy type. Incidentally, this provides us with a nice compact model $J_k^{(2)}/\mathfrak{S}_k$ for the classifying space of the braid group on k strands $BB_k = F(\mathbb{R}^2, k)/\mathfrak{S}_k$ which is actually known in the literature as the *Salvetti-complex* of the symmetric group $\mathfrak{S}_k$ and which exists for every finite Coxeter group, cf. Charney-Davis [11].

The common structure of the families $F(\mathbb{R}^n, k), J_k^{(n)}, k \geq 1$, is that of a *preoperad* (or coefficient system, cf. [14]).

Definition 1.2. *A (topological) preoperad is a functor $\mathcal{O} : \Lambda^{op} \to \mathrm{Top}$, where Λ is the category of (non-empty) finite sets $\underline{k} = \{1, \ldots, k\}$ and injective maps. Each pointed space $(X, *)$ defines a functor $\underline{X} : \Lambda \to \mathrm{Top}_* : \underline{k} \mapsto X^k$ so that by the usual coend construction we end up with a bifunctor*

$$\begin{array}{rccl} -\otimes_\Lambda - : & \mathrm{Top}^{\Lambda^{op}} \times \mathrm{Top}_* & \to & \mathrm{Top} \\ & (\mathcal{O}, X) & \mapsto & \mathcal{O} \otimes_\Lambda \underline{X} = (\coprod_{k\geq 1} \mathcal{O}_k \times X^k)/\sim . \end{array}$$

where $(\phi^*(t), x) \sim (t, \phi_*(x))$ for all $t \in \mathcal{O}_l, x \in X^k, \phi : \underline{k} \to \underline{l}$.

In order to make sure that this bifunctor preserves weak equivalences in both arguments, we have two choices : either we look for the "correct" notion of weak equivalence for both arguments, or we restrict to objects for which the "naive" (= pointwise) notion of weak equivalence works. We choose the latter variant which is sufficient for our purpose. Indeed, if we restrict to preoperads $\mathcal{O}$ with *free* $\mathfrak{S}_k$-action on $\mathcal{O}_k$ for all k, and to *well* pointed spaces X (i.e. such that the basepoint inclusion is a cofibration), then the bifunctor $-\otimes_\Lambda -$ preserves (the naive notion of) weak equivalences in both arguments. The proof of this fact relies on the unique factorization property of the category Λ which allows to write $\mathcal{O}\otimes_\Lambda \underline{X}$ as an iterated pushout along cofibrations with values in $\mathcal{O}_k\times_{\mathfrak{S}_k} X^k$, cf. [14].

Notation 1.3. Each morphism of the category Λ factorizes uniquely as a *permutation* followed by an *increasing* map. For $\phi\in\Lambda(\underline{k},\underline{l})$, we write $\phi = \phi^{inc}\circ\phi^\natural$ with $\phi^\natural\in\Lambda(\underline{k},\underline{k})$ and $\phi^{inc}\in\Lambda^{inc}(\underline{k},\underline{l}) = \{\phi\in\Lambda(\underline{k},\underline{l})|\,\phi(i)<\phi(j)\text{ for } i<j\}$.

Remark 1.4. The increasing maps $\phi:\underline{k}\to\underline{l}$ are generated by elementary *degeneracy* operators $D_i:\underline{k}\to\underline{k+1}$ mapping j to j (resp. $j+1$) if $j<i$ (resp. $j\geq i$). A preoperad $\mathcal{O}$ is thus uniquely determined by a family of $\mathfrak{S}_k$-spaces $\mathcal{O}_k$ together with degeneracy maps $D_i^*:\mathcal{O}_{k+1}\to\mathcal{O}_k$ $(i=1,\dots,k+1)$ satisfying the appropriate relations. For an *operad* $\mathcal{O}$, these degeneracy maps are given by *evaluation* :

$$D_i^*:\mathcal{O}_{k+1}\to\mathcal{O}_k : t\mapsto t(1,\dots,1,\underset{i}{*},1,\dots,1)$$

where $1\in\mathcal{O}_1$ is the unit of the operad, and $*\in\mathcal{O}_0$ is the *unique* constant of the operad. Observe that in our definition of preoperads we discard constants.

Examples 1.5. (a) The configuration preoperads.
The collection of configuration spaces $F(\mathbb{R}^n,k)$ defines a topological preoperad $F(\mathbb{R}^n,-):\Lambda^{op}\to\mathrm{Top}$ by setting for $\phi\in\Lambda(\underline{k},\underline{l})$:

$$\begin{array}{rcl}\phi^*:F(\mathbb{R}^n,l) & \to & F(\mathbb{R}^n,k)\\ (t_1,\dots,t_l) & \mapsto & (t_{\phi(1)},\dots,t_{\phi(k)}).\end{array}$$

(b) The permutation preoperad.
The collection of symmetric groups $\mathfrak{S}_k=\Lambda(\underline{k},\underline{k})$ defines a set-valued preoperad $\mathfrak{S}:\Lambda^{op}\to\mathrm{Sets}$ by setting for $\phi\in\Lambda(\underline{k},\underline{l})$:

$$\begin{array}{rcl}\phi^*:\mathfrak{S}_l & \to & \mathfrak{S}_k\\ \sigma & \mapsto & (\sigma\circ\phi)^\natural.\end{array}$$

Equivalently, the permutation preoperad can also be defined as the composite functor $\pi_0 \circ F(\mathbb{R},-) : \Lambda^{op} \to \text{Sets}$, identifying $\mathfrak{S}_k$ with the path components of $F(\mathbb{R},k)$.

(c) The permutohedral preoperad. By convex extension of the permutation preoperad we get the permutohedral preoperad $P : \Lambda^{op} \to \text{Top}$, cf. [5].

Definition 1.6. *Milgram's preoperads $J^{(n)} : \Lambda^{op} \to \text{Top}$ are defined by*

$$J_k^{(n)} = (P_k)^{n-1} \times \mathfrak{S}_k / \sim$$

where the equivalence relation identifies certain boundary cells. Explicitly, for each point $(\tau^*(t_1), \dots, \tau^*(t_{n-1}); \sigma) \in (P_k)^{n-1} \times \mathfrak{S}_k$ such that t_s belongs to the convex hull of a proper subgroup $\mathfrak{S}_{i_1} \oplus \cdots \oplus \mathfrak{S}_{i_r} \subset \mathfrak{S}_k$ and such that τ^{-1} is a $(i_1, \dots, i_r)$-shuffle of $\mathfrak{S}_k$, we have the relation

$$\begin{gathered}(\tau^*(t_1), \dots, \tau^*(t_{n-1}); \sigma) \sim \\ (t_1, \dots, t_s, D^*_{i_1,\dots,i_r}(t_{s+1}), \dots, D^*_{i_1,\dots,i_r}(t_{n-1}); \tau\sigma)\end{gathered}$$

where $D^*_{i_1,\dots,i_r}$ denotes the projection of P_k onto the convex hull of $\mathfrak{S}_{i_1} \oplus \cdots \oplus \mathfrak{S}_{i_r}$. The Λ-structure is induced by

$$\phi^*(t_1, \dots, t_{n-1}; \sigma) = (t_1^{(\sigma\phi)^{inc}}, \dots, t_{n-1}^{(\sigma\phi)^{inc}}; (\sigma\phi)^\natural).$$

For $n = 1$, $J^{(1)} \otimes_\Lambda \underline{X}$ reduces to *James' model* for $\Omega S X$; indeed, $J_k^{(1)} = \mathfrak{S}_k$ and $\mathfrak{S} \otimes \underline{X}$ is nothing but the free monoid generated by the pointed space $(X, *)$.

For $n = 2$, the space $J_k^{(2)}$ is obtained by gluing together $k!$ copies of P_k according to the (right) $\mathfrak{S}_k$-action. This action is only "partially free" and the quotient space $J_k^{(2)}$ can be interpreted as the *free extension* of this partially free action. Indeed, the equivalence relation $(t, \sigma) \sim (t^\sigma, id_k)$ applies precisely when σ acts freely and orientation preserving on the minimal face containing t. The resulting $\mathfrak{S}_k$-action on $J_k^{(2)}$ is free.

Proposition 1.7. *The configuration preoperad $F(\mathbb{R}^n, -)$ and Milgram's preoperad $J^{(n)}$ are equivalent as preoperads. In particular, the $\mathfrak{S}_k$-spaces $F(\mathbb{R}^n, k)$ and $J_k^{(n)}$ have the same equivariant homotopy type and for a well pointed space $(X, *)$, the May-Segal model $F(\mathbb{R}^n, -) \otimes_\Lambda \underline{X}$ and the Milgram model $J^{(n)} \otimes_\Lambda \underline{X}$ are homotopy equivalent.*

Proof. – We shall outline a proof for the case $n = 2$ (cf. [5], [6]). The idea is to define equivariant cellular decompositions for $F(\mathbb{R}^2, k)$ and $J_k^{(2)}$ and to compare the associated cell posets, cf. definition 2.2. For the configuration space $F(\mathbb{R}^2, k)$ this is done by ordering its points antilexicographically, i.e. $(x_1, x_2) < (y_1, y_2)$ iff either $x_2 < y_2$ or $x_2 = y_2 \wedge x_1 < y_1$. In the first case we write $x \leq_1 y$, whereas in the second case we write $x \leq_0 y$. Each labeled linear graph

$$\sigma^{-1}(1) \xrightarrow{\alpha_{12}} \sigma^{-1}(2) \xrightarrow{\alpha_{23}} \cdots \xrightarrow{\alpha_{k-1,k}} \sigma^{-1}(k) \text{ with } \alpha_{i,i+1} \in \{0, 1\}$$

defines then the closed contractible subset

$$F_k^{(\sigma^*\alpha)} = \{(t_1, \ldots, t_k) \in F(\mathbb{R}^2, k) \,|\, t_{\sigma^{-1}(1)} \underset{\alpha_{12}}{\leq} t_{\sigma^{-1}(2)} \underset{\alpha_{23}}{\leq} \cdots \underset{\alpha_{k-1,k}}{\leq} t_{\sigma^{-1}(k)}\}$$

Let us write $\mathcal{K}(F)_k^{(2)}$ for the poset of the so defined "cells". – Similarly, to each labeled linear graph as above, we associate the closed contractible set

$$J_k^{(\sigma^*\alpha)} = Hull(\mathfrak{S}_{i_1} \oplus \mathfrak{S}_{i_2} \oplus \cdots \oplus \mathfrak{S}_{i_r}) \times \{\sigma\} \subset J_k^{(2)}$$

where the "partition" $(i_1, \ldots, i_r)$ of k is defined by the $1'$s in the sequence $(\alpha_{12}, \ldots, \alpha_{k-1,k})$; we write $\mathcal{K}(J)_k^{(2)}$ for the poset of these cells. One then shows that $\mathcal{K}(F)_k^{(2)}$ and $\mathcal{K}(J)_k^{(2)}$ are *antiisomorphic* posets and deduces the statement from the following diagram of Λ-equivalences, cf. [6] :

$$F(\mathbb{R}^2, k) \overset{\sim}{\leftarrow} \mathrm{hocolim} F_k^{(\sigma^*\alpha)} \overset{\sim}{\rightarrow} |\mathcal{K}(F)_k^{(2)}| \cong |\mathcal{K}(J)_k^{(2)}| \overset{\sim}{\leftarrow} \mathrm{hocolim} J_k^{(\sigma^*\alpha)} \overset{\sim}{\rightarrow} J_k^{(2)}$$

□

Remark 1.8. More generally, there are cellular decompositions of $F(\mathbb{R}^n, k)$ resp. $J_k^{(n)}$ with antiisomorphic cell-posets $\mathcal{K}(F)_k^{(n)}$ resp. $\mathcal{K}(J)_k^{(n)}$. The poset $\mathcal{K}(F)_k^{(n)}$ implicitly appears already in Fadell-Neuwirth's paper [16] and has also been studied by Getzler-Jones [17]. The poset $\mathcal{K}(J)_k^{(n)}$ plays a central role in Balteanu-Fiedorowicz-Schwänzl-Vogt's study of n-fold monoidal categories [2].

Remark 1.9. In [5], [6] I wrongly claimed to be able to endow the Milgram preoperad $J^{(n)} : \Lambda^{op} \to \mathrm{Top}$ with an operad structure. Although the *permutohedral* preoperad *is* an operad, this operad structure doesn't carry over to $J^{(n)}$, since its multiplication doesn't preserve the boundary. I tried to force the permutohedral multiplication to preserve the boundary by passing from convex to *cubical* extension, but I overlooked that cubical extension commutes with composition *only up to homotopy* so that I really got some kind of lax operad where all relations only hold up to (a uniquely specified) homotopy. Michael Brinkmeier shows in his lecture [10] that just assuming the ordinary (strict) relations between the "cubical" multiplication $J_2^{(2)} \times J_2^{(2)} \times J_1^{(2)} \to J_3^{(2)}$ (resp. $J_2^{(2)} \times J_1^{(2)} \times J_2^{(2)} \to J_3^{(2)}$) and the various degeneracies already implies that a hypothetical $J^{(!!2)}$-algebra would be a commutat ive monoid, which is incompatible with the approximation theorem.

Remark 1.10. Jim McClure and Jeff Smith construct an E_2-operad which acts on *topological Hochschild cohomology* and which is closely related to $J^{(2)}$, see Jim McClure's lecture [22]. This E_2-operad is a kind of semi-direct product of the little 1-cubes operad with the permutohedral operad. Its multiplication uses *prismatic* decompositions of the permutohedra P_k (labeled by "formulae") which can be described as follows : The image of $P_2 \times P_1 \times P_{k-1} \to P_k$ is a prism $\Delta^1 \times P_{k-1}$, thus by induction endowed with a prismatic decomposition; it turns out that the (closure of the) complement of the image also admits a prismatic decomposition labeled by the set of proper faces of P_{k-1}. The good thing about this prismatic decomposition of P_k is that all its cells are convex hulls of vertex-sets of P_k and thus allow a very neat combinatorial description.

2 The complete graph operad

While the configuration spaces $F(\mathbb{R}^n, k)$ suffice to approximate $\Omega^n S^n X$ for connected spaces X, they allow us neither to reconstruct the n different loop structures of $\Omega^n S^n X$ nor to recognize general n-fold iterated loop spaces. The appropriate gadget for these tasks is the *little n-cubes operad* of Boardman-Vogt which we shall denote by $\mathcal{C}^{(n)}$ [9]. As a preoperad, the little n-cubes are equivalent to $F(\mathbb{R}^n, -)$, but thanks to the canonical inclusion $\mathcal{C}_k^{(n)} \hookrightarrow Top_*(S^n, \bigvee_{i=1}^k S^n)$ they come equipped with a multiplication (indeed a *substitution product*)

$$\mathcal{C}_k^{(n)} \times \mathcal{C}_{i_1}^{(n)} \times \cdots \times \mathcal{C}_{i_k}^{(n)} \longrightarrow \mathcal{C}_{i_1+\cdots+i_k}^{(n)}$$

turning them into an *operad*. The endofunctor $\mathcal{C}^{(n)} \otimes_\Lambda - : \mathrm{Top}_* \to \mathrm{Top}_*$ is then a *monad* such that the category of n-fold iterated loop spaces fully embeds into the category of $\mathcal{C}^{(n)}$-spaces. Moreover, each connected $\mathcal{C}^{(n)}$-space is $\mathcal{C}^{(n)}$-equivalent to an n-fold loop space [20]. Operads equivalent to the little n-cubes operad are called *E_n-operads.*

The following question then naturally arises : Are there other interesting

E_n-operads or is there even an internal characterization of E_n-operads ? For the moment, we are far away from such a characterization if $n \neq 1, 2$ or ∞, however using "cellular techniques", Zig Fiedorowicz found a method of *constructing* new E_n-operads out of spaces having the equivariant homotopy type of an $(n-1)$-sphere, see his lecture [15].

In this section I shall describe a poset-valued E_n-operad which may serve to define an equivariant cellular decomposition of a topological E_n-operad $\mathcal{O}$. The leading idea is that the k-level space $\mathcal{O}_k$ should be determined by the family of $\binom{k}{2}$ degeneracies $\phi_{ij}^* : \mathcal{O}_k \to \mathcal{O}_2,\ i < j$, whereas the 2-level space $\mathcal{O}_2$ has to have the $\mathfrak{S}_2$-equivariant homotopy type of an $(n-1)$-sphere. Here and later on, $\phi_{ij} : \underline{2} \to \underline{k}$ denotes the map which sends $(1,2)$ to (i,j), and $\sigma_{ij} = \phi_{ij}^*(\sigma)$:

Definition 2.1. *Set* $\mathcal{K}_k^{(n)} = \{0, 1, \ldots, n-1\}^{\binom{k}{2}} \times \mathfrak{S}_k$ *and denote the elements of* $\mathcal{K}_k^{(n)}$ *by* (μ, σ) *where* $\mu = (\mu_{ij})_{1 \leq i < j \leq k}$*. Define a partial order by*

$$(\mu, \sigma) \leq (\nu, \tau) \overset{def}{\Leftrightarrow} \text{ either } \sigma_{ij} = \tau_{ij} \wedge \mu_{ij} \leq \nu_{ij} \text{ or } \sigma_{ij} \neq \tau_{ij} \wedge \mu_{ij} < \nu_{ij}$$

There is a nice combinatorial description of $\mathcal{K}_k^{(n)}$, due to Fiedorowicz [2] : Each element $(\mu, \sigma) \in \mathcal{K}_k^{(n)}$ corresponds to an *acyclic orientation* of the *complete graph on k vertices* together with a $\{0, 1, \ldots, n-1\}$-*coloring* of its edges. Explicitly,

$$(\mu, \sigma) \longleftrightarrow \begin{cases} \text{acyclic orientation} & \sigma^{-1}(1) \longrightarrow \sigma^{-1}(2) \longrightarrow \cdots \longrightarrow \sigma^{-1}(k) \\ \text{edge} - \text{coloring} & i \overset{\mu_{ij}}{\longleftrightarrow} j \end{cases}$$

In this setting, $(\mu, \sigma) \leq (\nu, \tau)$ iff for all $i < j$, we have $\mu_{ij} \leq \nu_{ij}$ with $\mu_{ij} < \nu_{ij}$ whenever the induced orientations on the edge i—j differ.

Observe that throughout these notes I use the convention that $\mathfrak{S}_k$ acts from the *right*, which is the natural choice with respect to the category Λ. The reader who prefers left actions has simply to replace every permutation by its inverse, but he has to be aware of the fact that this choice also affects the equivariance relations to be hold by an operad !

Now, let us describe the operad structure of $(\mathcal{K}_k^{(n)})_{k \geq 1}$: Since the edge-colorings of the complete graphs form themselves a preoperad whose degeneracies are given by restriction, we get a preoperad $\mathcal{K}^{(n)} : \Lambda^{op} \to$ Posets with $\phi^*(\mu, \sigma) = (\phi^*\mu, \phi^*\sigma)$. Like for the little n-cubes, the multiplication is defined as a *substitution product* :

$$\begin{array}{rcl}\mathcal{K}_k^{(n)} \times \mathcal{K}_{i_1}^{(n)} \times \cdots \times \mathcal{K}_{i_k}^{(n)} & \to & \mathcal{K}_{i_1+\cdots+i_k}^{(n)} \\ ((\mu,\sigma);(\mu_1,\sigma_1),\ldots,(\mu_k,\sigma_k)) & \mapsto & (\mu(\mu_1,\ldots,\mu_k),\sigma(\sigma_1,\ldots,\sigma_k)),\end{array}$$

where $(\mu(\mu_1,\ldots,\mu_k),\sigma(\sigma_1,\ldots,\sigma_k))$ corresponds to the complete (oriented and colored) graph on $i_1+\cdots+i_k$ vertices obtained from (μ,σ) by substituting to each of its k vertices the complete graphs defined by $(\mu_1,\sigma_1),\ldots,(\mu_k,\sigma_k)$ respectively. For sake of precision we give the explicit formulae :
$\sigma(\sigma_1,\ldots,\sigma_k)=\sigma(i_1,\ldots,i_k)\circ(\sigma_1\oplus\cdots\oplus\sigma_k)$, where $\sigma(i_1,\ldots,i_k)$ permutes the k blocks $\{1,\ldots,i_1\},\{i_1+1,\ldots,i_1+i_2\},\ldots,\{i_1+\cdots+i_{k-1}+1,\ldots,i_1+\cdots+i_k\}$ according to σ, and:

$$\mu(\mu_1,\ldots,\mu_k)_{rs}=\begin{cases}(\mu_j)_{r',s'} & \text{if } r,s \text{ belong to the } j\text{-th block,}\\ \mu_{jj'} & \text{if } r \text{ belongs to the } j\text{-th block and}\\ & \quad s \text{ belongs to the } j'\text{-th block, } j<j'.\end{cases}$$

Definition 2.2. *Let A be a partially ordered set and X a topological space. A collection $(c_\alpha)_{\alpha\in A}$ of closed contractible subspaces (the* "cells"*) of X will be called a* cellular A-decomposition *of X iff the following three conditions hold (where $\check{c}_\alpha=c_\alpha\setminus\bigcup_{\alpha'<\alpha}c_{\alpha'}$) :*

1. *$\alpha\le\beta\Rightarrow c_\alpha\subseteq c_\beta$ and for all $\alpha,\beta\in A:\check{c}_\alpha\cap\check{c}_\beta=\emptyset$;*
2. *the cell-inclusions are cofibrations ;*
3. *$X=\varinjlim_A c_\alpha$, i.e. X equals the union of its cells and carries the weak topology with respect to its cells.*

Cells c_α with non-empty cell-interiors $\check{c}_\alpha$ will be called *proper*. By the usual homotopy colimit argument, the nerve of the poset A has the same homotopy type as X : $X\xleftarrow{\sim}\mathrm{hocolim}_A c_\alpha\xrightarrow{\sim}|A|$. The same is true for the subposet of proper cells. Moreover, the cell-interiors of the proper cells partition X.

Definition 2.3. *An operad $\mathcal{O}$ is called a* cellular E_n-operad *iff*

1. *$\mathcal{O}_2$ admits an equivariant cellular $\mathcal{K}_2^{(n)}$-decomposition by cells denoted $\mathcal{O}_2^{(\alpha)},\alpha\in\mathcal{K}_2^{(n)}$;*
2. *the closed subsets $\mathcal{O}_k^{(\alpha)}=\{x\in\mathcal{O}_k\,|\,\phi_{ij}^*(x)\in\mathcal{O}_2^{(\phi_{ij}^*\alpha)}\},\alpha\in\mathcal{K}_k^{(n)}$, form a cellular $\mathcal{K}_k^{(n)}$-decomposition of $\mathcal{O}_k$;*

3. the operad multiplication sends each cell-product $\mathcal{O}_k^{(\alpha)} \times \mathcal{O}_{i_1}^{(\alpha_1)} \times \cdots \times \mathcal{O}_{i_k}^{(\alpha_k)}$ *into the cell* $\mathcal{O}_{i_1+\cdots+i_k}^{(\alpha(\alpha_1,\ldots,\alpha_k))}$ *prescribed by the complete graph operad.*

For the second condition of a cellular E_n-operad it is *sufficient* to show : a) the contractibility of the cells, b) the cofibration property of the cell-inclusions and c) the existence of "ordered" points, i.e. in each $\mathfrak{S}_k$-orbit of $\mathcal{O}_k$ there has to be a point x such that all degeneracies $\phi_{ij}^*(x)$ with $i < j$ belong to "upper hemispheres" $\mathcal{O}_2^{(\cdot,id_2)}$ (this last condition implies that the union of the $\mathcal{O}_k^{(\alpha)}$ covers $\mathcal{O}_k$).

Theorem 2.4. *The little n-cubes operad is a cellular E_n-operad and any two cellular E_n-operads are equivalent as operads.*

The first part of the theorem is due to Fiedorowicz and relies on the analysis of a given configuration of little n-cubes by means of separating hyperplanes perpendicular to the coordinate axis, cf. [6]. The second part (which implies consistency of our terminology) follows from the above homotopy colimit argument, more precisley : for each cellular E_n-operad $\mathcal{O}$ there are operad equivalences

$$\mathcal{O} \overset{\sim}{\leftarrow} \mathrm{hocolim}_{\alpha\in\mathcal{K}^{(n)}}\mathcal{O}^{(\alpha)} \overset{\sim}{\rightarrow} |\mathcal{K}^{(n)}|.$$

The existence of a cellular structure allows several constructions which would not be available otherwise. For instance, assume that $\mathcal{O}$ is a *cellular E_∞-operad.* The filtration of $\mathcal{K} = \varinjlim \mathcal{K}^{(n)}$ by the E_n-suboperads $\mathcal{K}^{(n)}$ induces an analogous filtration of $\mathcal{O}$ by cellular E_n-suboperads $\mathcal{O}^{(n)}$! As example, consider the simplicial E_∞-operad $\Gamma = E\mathfrak{S}$ where E stands for the universal bundle construction. This operad has a long history : Barratt-Eccles [3] based a general recognition principle for infinite loop spaces on it; Peter May [21] showed that Γ acts canonically on nerves of (strict) symmetric monoidal categories, but it was only Jeff Smith who discovered the existence of a filtration of Γ by suboperads $\Gamma^{(n)}$ [24]. Using the cellular structure of Γ, this filtration comes *for free* as well as the fact that $|\Gamma_k^{(n)}|$ has the $\mathfrak{S}_k$-equivariant homotopy type of $F(\mathbb{R}^n, k)$. The latter was only conjectured by Smith in view of his approximation theorem and later on proved by T. Kashiwabara [18]. For an explicit operad map $|\mathcal{K}^{(n)}| \xrightarrow{\sim} |\Gamma^{(n)}|$ inducing the equivalence, see [2].

3 A simplicial splitting of $\Omega^n S^n X$

The purpose of this section is to use the Smith filtration $\Gamma^{(n)}$ of the symmetric monoidal operad Γ in order to derive the Snaith-splitting of $\Omega^n S^n X$, cf. [25]. Adapting the method of Cohen-May-Taylor [14] to the simplicial setting I obtain combinatorial estimates for the number of suspensions necessary to split $\Omega^n S^n X$ globally at a given filtration level.

To fix notation, $\Gamma_k = E\mathfrak{S}_k$ is the universal bundle on $\mathfrak{S}_k$: a d-simplex of Γ_k is thus a $(d+1)$-tuple $(\sigma_0, \ldots, \sigma_d)$ of permutations in $\mathfrak{S}_k$ with the usual (homogenous) simplicial operators. The permutation operad $\mathfrak{S} : \Lambda^{op} \to \text{Sets}$ induces componentwise an operad structure on $\Gamma : \Lambda^{op} \to \text{SimpSets}$. Let us check that the E_∞-operad Γ is *cellular* : $\Gamma_2 = E\mathfrak{S}_2$ is the "Milnor-sphere" of infinite dimension with one non-degenerate simplex per hemisphere. There exists thus a natural cellular $\mathcal{K}_2$-decomposition of Γ_2. The c(an)onical contraction of Γ_k stays within the formally defined cells $\Gamma_k^{(\alpha)}, \alpha \in \mathcal{K}_k$. Moreover, cell-inclusions are cofibrations and "ordered" points exist. Finally, the multiplication is cellular since induced by the permutation operad.

Theorem 2.4 then implies that the union of the cells labeled by $\mathcal{K}^{(n)}$ defines a cellular E_n-suboperad $\Gamma^{(n)}$ of Γ, explicitly :

$$\Gamma_k^{(n)} = \{\omega \in \Gamma_k \,|\, \phi_{ij}^*(\omega) \in \Gamma_2^{(n)}\} = \{\omega \in \Gamma_k \,|\, \phi_{ij}^*(\omega) \in sk_{n-1}\Gamma_2\}.$$

This is precisely the suboperad defined by J. Smith [24]. Since the geometric realization preserves arbitrary colimits and finite limits, we get for each *connected* simplicial set $(X, *)$:

$$|\Gamma^{(n)} \otimes_\Lambda \underline{X}| \cong |\Gamma^{(n)}| \otimes_\Lambda \underline{|X|} \sim \Omega^n S^n |X|$$

thus the coend $\Gamma^{(n)} \otimes_\Lambda \underline{X}$ is a "simplicial model" for $\Omega^n S^n|X|$. Moreover, the inclusion $\Gamma^{(n)} \hookrightarrow \Gamma^{(n+1)}$ induces the stabilization map $\Omega^n S^n|X| \to \Omega^{n+1} S^{n+1}|X|$ as is the case for the "cellular" filtration of the little cubes by little n-cubes.

Remark 3.1. For $n = 1$, we get the James-construction $\Gamma^{(1)} \otimes_\Lambda \underline{X} = \mathfrak{S} \otimes_\Lambda \underline{X}$ whose group completion is isomorphic to the famous Kan-Milnor model $FX = GSX$ for $\Omega S|X|$ valid also for non-connected simplicial sets $(X, *)$. In general, as $\mathfrak{S} = \Gamma^{(1)}$ is a suboperad of $\Gamma^{(n)}$ the coend $\Gamma^{(n)} \otimes_\Lambda \underline{X}$ is a simplicial monoid which even turns out to be free, cf. [3]; its group completion is thus a free simplicial group model for $\Omega^n S^n|X|$ valid also for non-connected X, cf. [24].

Remark 3.2. Sets with a $\Gamma^{(1)}$-action are precisely monoids ; in [7], it is shown that the nerve of a *braided monoidal category* admits a $\Gamma^{(2)}$-action. Recall that in the terminology of Baez-Dolan [1] braided monoidal categories are 2-*tuply monoidal* 1-*categories*. I conjecture that in general the "nerve" of a "n-tuply monoidal $(n-1)$-category" admits a $\Gamma^{(n)}$-action. There is now at least one additional case, where this conjecture can be verified, the case $n = 3$, i.e. the nerve of a *sylleptic monoidal* 2-*category* should admit a $\Gamma^{(3)}$-action, cf. S. Crans [13].

For simplicity, we shall write $\Gamma^{(n)}(X)$ for the coend $\Gamma^{(n)} \otimes_\Lambda \underline{X}$. The Snaith-splitting can be summarized by the following diagram whose existence we are going to establish (where X is assumed to be *connected*):

$$
\begin{array}{ccccccccccc}
\Omega S X & \sim & \Gamma^{(1)}(X) & \overset{stably}{\longrightarrow} & D_1^{(1)}(X) & \vee & D_2^{(1)}(X) & \vee & D_3^{(1)}(X) & \vee & \cdots \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \vdots \\
\Omega^2 S^2 X & \sim & \Gamma^{(2)}(X) & \overset{stably}{\longrightarrow} & D_1^{(2)}(X) & \vee & D_2^{(2)}(X) & \vee & D_3^{(2)}(X) & \vee & \cdots \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \vdots \\
\Omega^3 S^3 X & \sim & \Gamma^{(3)}(X) & \overset{stably}{\longrightarrow} & D_1^{(3)}(X) & \vee & D_2^{(3)}(X) & \vee & D_3^{(3)}(X) & \vee & \cdots \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \vdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \vdots \\
\Omega^\infty S^\infty X & \sim & \Gamma(X) & \overset{stably}{\longrightarrow} & D_1(X) & \vee & D_2(X) & \vee & D_3(X) & \vee & \cdots
\end{array}
$$

$$D_m^{(n)}(X) = \Gamma_{\leq m}^{(n)}(X)/\Gamma_{\leq m-1}^{(n)}(X) \text{ where } \Gamma_{\leq m}^{(n)}(X) = (\coprod_{k=1}^{m} \Gamma_k^{(n)} \times X^k)/\sim$$

$$\text{In particular, we have : } D_1^{(n)}(X) = X \text{ and } D_m^{(1)}(X) = \underbrace{X \wedge \cdots \wedge X}_{m}$$

$$\text{In general : } D_m^{(n)}(X) = \Gamma_{m,+}^{(n)} \wedge_{\mathfrak{S}_m} \underbrace{X \wedge \cdots \wedge X}_{m}$$

Our purpose here is to show that *James' combinatorial maps*

$$j_m^{(1)} : \Gamma^{(1)}(X) \to \Gamma^{(1)}(D_m^{(1)}(X))$$

naturally extend to maps $j_m^{(n)} : \Gamma^{(n)}(X) \to \Gamma^{(t_{nm})}(D_m^{(n)}(X))$ which *by adjunction* induce "global" Snaith-splittings for each n and m. For $n = t_{nm} = \infty$

this was already observed by Barratt-Eccles [3], and Cohen-May-Taylor [14] used the same idea to define "global" Snaith-splittings in full generality. It is somewhat surprising that one gets purely combinatorial estimates for the number $t = t_{nm}$ of suspensions needed to split $\Omega^n S^n|X|$ at filtration level m :

$$S^t\Omega^n S^n(X) \xrightarrow{\sim} S^tX \vee S^tD_2^{(n)}(X) \vee S^tD_3^{(n)}(X) \vee \cdots \vee S^tD_m^{(n)}(X) \vee R$$

I obtain as estimate $\boxed{t_{nm} = (n-1)(2m-1)+1}$ which gives for $n = 1$

$$S\Omega S(X) \xrightarrow{\sim} SX \vee S(X \wedge X) \vee S(X \wedge X \wedge X) \vee \cdots \qquad \text{(Hilton-Milnor)}$$

and which seems to be quite sharp for $n = 2$, cf. [14]. Snaith's estimates [25] as well as Vogt's estimates [27] are lower for large n,m but they only allow a "non-global" splitting of $S^t\Gamma^{(n)}_{\leq m}(X)$.

Definition 3.3. *For each k and m define (cf. notation 1.3)*

$$\begin{array}{rcl} h_m^k : \mathfrak{S}_k & \to & \mathfrak{S}_{\Lambda^{inc}(\underline{m},\underline{k})} \stackrel{def}{=} \mathfrak{S}_{\binom{k}{m}} \\ \sigma & \mapsto & (\phi \mapsto (\sigma \circ \phi)^{inc}). \end{array}$$

Lemma 3.4. *If for each k, the set $\Lambda^{inc}(\underline{m},\underline{k})$ is antilexicographically ordered, then the following diagram commutes for all $\phi : \underline{k} \to \underline{k}'$ (cf. [14], 4.3 and [3], III.4):*

$$\begin{array}{ccc} \mathfrak{S}_{k'} & \xrightarrow{h_m^{k'}} & \mathfrak{S}_{\binom{k'}{m}} \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \binom{\phi}{m}^*} \\ \mathfrak{S}_k & \xrightarrow{h_m^k} & \mathfrak{S}_{\binom{k}{m}} \end{array}$$

Lemma 3.5. *The maps j_m^k below are compatible with the coend relations and induce the* stable *James maps $j_m : \Gamma(X) \to \Gamma(D_m(X))$ (satisfying that $j_m|_{\Gamma_{\leq m}(X)}$ is equal to the projection composed with the unit of $\Gamma(-)$) :*

$$\begin{array}{rcl} j_m^k : \Gamma_k \times X^k & \to & \Gamma_{\binom{k}{m}} \times D_m(X)^{\binom{k}{m}} \\ (\omega; x_1, \ldots, x_k) & \mapsto & \big(h_m^k(\omega); [\phi^*(\omega); x_{\phi(1)}, \ldots, x_{\phi(m)}]_{\phi \in \Lambda^{inc}(\underline{m},\underline{k})}\big). \end{array}$$

Proposition 3.6. *The stable James maps $j_m : \Gamma(X) \to \Gamma(D_m(X))$ restrict to* unstable *James maps $j_m^{(n)} : \Gamma^{(n)}(X) \to \Gamma^{(t_{nm})}(D_m(X))$ where t_{nm} can be chosen equal to $(n-1)(2m-1)+1$.*

Proof. – We have to show that $\omega \in \Gamma_k^{(n)}$ implies $h_m^k(\omega) \in \Gamma^{(t_{nm})}_{\Lambda^{inc}(\underline{m},\underline{k})}$ for all k.

Let $\omega = (\sigma_0, \dots, \sigma_d) \in \Gamma_k^{(n)}$, i.e. for each couple $i < j$, the simplex $\phi_{ij}^*(\omega)$ belongs to the $(n-1)$-skeleton of Γ_2, which means that $\phi_{ij}^*(\omega)$ contains at most $n-1$ changes or equivalently : the sequence

$$(\sigma_c(i) \lesseqgtr \sigma_c(j))_{c=0,\dots,d} \tag{1}$$

contains at most $n-1$ inversions of the order relation.

Similarly, let $\Omega = h_m^k(\omega) \in \Gamma^{(t)}_{\Lambda^{inc}(\underline{m},\underline{k})}$, i.e. for each couple $\{i_1, \dots, i_m\} < \{j_1, \dots, j_m\}$ in $\Lambda^{inc}(\underline{m}, \underline{k})$ the simplex $\phi^*_{\{i_1,\dots,i_m\},\{j_1,\dots,j_m\}}(\Omega)$ contains at most $t-1$ changes, or equivalently : the sequence

$$(\{\sigma_c(i_1), \dots, \sigma_c(i_m)\} \lesseqgtr \{\sigma_c(j_1), \dots, \sigma_c(j_m)\})_{c=0,\dots d} \tag{2}$$

contains at most $t-1$ inversions of the (antilexicographical) order relation.

Each inversion in (2) induces an inversion in (1) for at least one couple $(i,j) \in \{i_1, \dots, i_m\} \times \{j_1, \dots, j_m\}$. The "worst" case occurs when an inversion in (2) induces inversions in (1) for *as few couples as possible*. In this optic, the "worst" configuration is $i_1 < j_1 < i_2 < j_2 < \dots < i_m < j_m$ (thus $k \geq 2m$) with simplex $\omega = (\sigma_0, \dots, \sigma_{2m-1})$ such that $\sigma_0 = id_k$, σ_1 puts i_m on the top, σ_2 puts j_{m-1} on the top and i_m on the second place, σ_3 puts i_{m-1} on the top, j_{m-1} on the second place and i_m on the third place, and so on, until finally σ_{2m-1} reverses completely the order : $\sigma_{2m-1}(i_1, j_1, \dots, i_m, j_m) = (j_m, i_m, \dots, j_1, i_1)$.

For each couple $(i,j) \in \{i_1, \dots, i_m\} \times \{j_1, \dots, j_m\}$, the simplex $\phi_{ij}^*(\omega)$ contains then exactly one change, whereas $\phi^*_{\{i_1,\dots,i_m\},\{j_1,\dots,j_m\}}(\Omega)$ contains $2m-1$ changes. Repeating the above construction $n-1$ times in alternating directions defines a simplex $\omega \in \Gamma_k$ such that $\phi_{ij}^*(\omega)$ contains $n-1$ changes for each $(i,j) \in \{i_1, \dots, i_m\} \times \{j_1, \dots, j_m\}$, whereas $\phi^*_{\{i_1,\dots,i_m\},\{j_1,\dots,j_m\}}(\Omega)$ contains $(n-1)(2m-1)$ changes. Since this is presumably the "worst" case, it suffices to put $t_{nm} = (n-1)(2m-1)+1$. □

Let me finish this section by recalling how to associate to the unstable James maps $j_m^{(n)} : \Gamma^{(n)}(X) \to \Gamma^{(t_{nm})}(D_m^{(n)}(X))$ *higher Hopf invariants* for homotopy classes in $[S^n Z, S^n X]$ and state *conjecturally* the properties they should share (extending what is known for $n = 1$, see Boardman-Steer [8], and for general n, if X is a "Thom-space", see Koschorke-Sanderson [19]).

Conjecture 3.7. *For each $f : S^n Z \to S^n X$ there is a family of higher Hopf invariants $\gamma_m(f) : S^t Z \to S^t D_m^{(n)}(X)$ $(t = t_{nm})$ satisfying*

1. $\gamma_1(f) = f$;

2. $\gamma_m(S^n g) \sim 0$ *for* $m > 1$;

3. $\gamma_m(f+f') \overset{stably}{\sim} \gamma_m(f) + \gamma_{m-1}(f)\gamma_1(f') + \cdots + \gamma_1(f)\gamma_{m-1}(f') + \gamma_m(f')$;

4. $m!\,\gamma_m(f) \overset{stably}{\sim} \gamma_1(f)^m$.

The Hopf invariant is defined by the formula $\gamma_m(f) = \widehat{j_m^{(n)} \circ \check{f}}$ where $(\check{-})$ and $(\widehat{-})$ are given by the S^n-Ω^n-adjunction.

The product of Hopf invariants is given by a *cup product*, compare [8] :
For $\alpha : S^t Z \to S^t D_m^{(n)}(X)$ and $\beta : S^{t'} Z \to S^{t'} D_{m'}^{(n)}(X)$ we put

$$\alpha\beta : S^{t+t'} Z \xrightarrow{\nabla} S^t Z \wedge S^{t'} Z \xrightarrow{\alpha\wedge\beta} S^t D_m^{(n)}(X) \wedge S^{t'} D_{m'}^{(n)}(X) \xrightarrow{\times} S^{t+t'} D_{m+m'}^{(n)}(X)$$

where the last map is induced by the monoidal structure of $\Gamma^{(n)}(X)$.

Boardman-Steer's proof of properties 1-4 for the case $n = 1$ should go through in general, since it is combinatorial in nature. Their uniqueness result strongly suggests that properties 1-3 determine the higher Hopf invariants up to some suspensions. I think that the existence of subdivided power operations in the stable homotopy of $\Omega^n S^n X$ (X connected !) should at least contain some information about the Hurewicz homomorphism $\pi_*^{st}(\Omega^n S^n X) \to H_*(\Omega^n S^n X)$ and perhaps interact with Fred Cohen's [12] calculations of $H_*(\Omega^n S^n X; \mathbb{Z}/p\mathbb{Z})$ where he completely determines $H_*(D_m^{(n)}(X); \mathbb{Z}/p\mathbb{Z})$ in terms of the reduced homology of X and of suitably weighted *Dyer-Lashof operations*.

4 Decompressible immersions

In this final very "sketchy" section I will indicate how the configuration space model of $\Omega^n S^n X$ for a "Thom space" X appears quite naturally in immersion theory, giving rise to a "more concrete" construction of the higher Hopf invariants. The existence of such a relationship between E_n-operads and immersion theory has been brought to my attention by Jean Lannes, for which I am very grateful to him. In the literature, different approaches have successively been developed by R. Wells [28], P. Vogel [26] and U. Koschorke-B. Sanderson [19]. My presentation heavily relies on the last reference.

Definition 4.1. *An* immersion $j : M \looparrowright V$ *admits a ξ-structure with respect to a vector bundle $\xi : E \to B$ iff there exists a differentiable map $\phi : M \to B$ and a short exact sequence of vector bundles over M :*

$$0 \longrightarrow TM \longrightarrow j^*TV \longrightarrow \phi^*\xi \longrightarrow 0$$

The group of ξ-corbordism classes of closed ξ-embeddings in V will be denoted by $\Omega^\xi(V)$.

Theorem 4.2. *(Thom-Pontryagin) $\Omega^\xi(V) \cong [\overline{V}, \mathrm{Th}(\xi)]$, where $\overline{V}$ is the one-point compactification of V and $\mathrm{Th}(\xi) = D(\xi)/S(\xi)$ is the Thom-space of ξ.*

Definition 4.3. *The suspension of an immersion $j : M \looparrowright V$ is the immersion $j \times (0 \hookrightarrow \mathbb{R}) : M \looparrowright V \times \mathbb{R}$.*

Lemma 4.4. *The suspension of embeddings corresponds under 4.2 to ordinary suspension: $[\overline{V}, \mathrm{Th}(\xi)] \longrightarrow [\overline{V} \wedge \overline{\mathbb{R}}, \mathrm{Th}(\xi) \wedge \mathrm{Th}(\epsilon_1)]$ where we identify $\overline{V \times \mathbb{R}} \cong \overline{V} \wedge \overline{\mathbb{R}}$ and $\mathrm{Th}(\xi \oplus \epsilon_1) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\epsilon_1)$.*

Here and later on, I write ϵ_k for the trivial vector bundle $\mathbb{R}^k \to *$.
A *framed* cobordism (resp. immersion) is by definition an ϵ_k-cobordism (resp. ϵ_k-immersion) which in both cases means that the *normal bundle* is endowed with a trivialization. The group

$$\varinjlim{}_k \Omega^{\epsilon_k}(\mathbb{R}^{m+k}) \cong \varinjlim{}_k [S^{m+k}, S^k] = \{S^m, S^0\} = \pi_m^{st}$$

then defines simultaneously stable framed cobordism classes of m-dimensional framed manifolds and the m-th stem of the stable homotopy groups of spheres.

Immersions are in some sense *"desuspended embeddings"* since by a sufficient number of suspensions any immersion (of positive codimension) can be made "regularly homotopic" to an embedding.

Definition 4.5. *A* n-decompression *for an immersion $j : M \looparrowright V$ is a differentiable map $\lambda : M \to \mathbb{R}^n$ such that $(j, \lambda) : M \to V \times \mathbb{R}^n$ is an embedding. Immersions for which an n-decompression exists are called n-*decompressible.

Theorem 4.6. *(Koschorke-Sanderson) The group of $(\xi \oplus \epsilon_n)$-cobordism classes of n-decompressed ξ-immersions $(j, \lambda) : M \to V \times \mathbb{R}^n$ is canonically isomorphic to $[\overline{V}, \Omega^n S^n \mathrm{Th}(\xi)]$. Moreover, under $S^n\Omega^n$-adjunction, this group is isomorphic to the classical Thom-Pontryagin cobordism group (we assume* $\mathrm{rg}(\xi) > 0$) *:*

$$[\overline{V}, \Omega^n S^n \mathrm{Th}(\xi)] \cong [S^n\overline{V}, S^n \mathrm{Th}(\xi)] = [\overline{V \times \mathbb{R}^n}, \mathrm{Th}(\xi \oplus \epsilon_n)] = \Omega^{\xi \oplus \epsilon_n}(V \times \mathbb{R}^n)$$

Proof. The proof consists essentially in the construction of an "adjoint" representative of (j, λ) inside $[\overline{V}, \Omega^n S^n \mathrm{Th}(\xi)]$ using the configuration space model for the target space as well as the classical Thom-Pontryagin construction. The immersion $j : M \looparrowright V$ defines for a small neighborhood V_y of each $y \in M$ a neighborhood of $j(V_y)$ which by hypothesis can be given the structure of a $D(\xi)$-bundle over V_y. This defines a "fattening" $\overline{j} : M \times D(\xi) \looparrowright V$ of j. The "adjoint" representative is then given by

$$\begin{array}{rcl} \widehat{\tau}_{j,\lambda} : \overline{V} & \to & F(\mathbb{R}^n, -) \otimes_\Lambda \mathrm{Th}(\xi) \\ x & \mapsto & (\lambda(\overline{j}^{-1}(x)); \overline{j}^{-1}(x)) \end{array}$$

□

Corollary 4.7. *The "stable group" of ξ-cobordism classes of n-decompressible ξ-immersions is canonically isomorphic to*

$$\{\overline{V}, \Omega^n S^n \mathrm{Th}(\xi)\} \cong \bigoplus_{k \geq 1} \{\overline{V}, D_k^{(n)}(\mathrm{Th}(\xi))\}.$$

Remark 4.8. The split summand $D_k^{(n)}(\mathrm{Th}(\xi))$ can be identified with the Thom space of the vector bundle $F(\mathbb{R}^n, k) \times_{\mathfrak{S}_k} \xi^k$. For example, let $\xi = \epsilon_1$, $n = 2$, then $D_k^{(2)}(S^1) = D_k^{(2)}(\mathrm{Th}(\epsilon_1)) = \mathrm{Th}(F(\mathbb{R}^2, k) \times_{\mathfrak{S}_k} (\epsilon_1)^k)$ which is thus the Thom-space of the vector bundle $F(\mathbb{R}^2, k) \times_{\mathfrak{S}_k} \mathbb{R}^k \to BB_k$.

Remark 4.9. The projections inducing the above isomorphism are induced by composition with *James' combinatorial maps*

$$j_k^{(n)} : \Omega^n S^n \mathrm{Th}(\xi) \to \Omega^t S^t D_k^{(n)}(Th(\xi)).$$

The remarkable fact here is that there exists a direct construction for immersions performing this composition. Namely, assume that the immersion $j : M \looparrowright V$ is *generic*, i.e. for each k, the natural map $F(M, k) \to V^k$ is *transverse* to the diagonal ; then the subset

$$\tilde{M}(k) = \{(m_1, \ldots, m_k) \in F(M, k) \,|\, j(m_1) = \cdots = j(m_k)\}$$

is a *manifold* with free $\mathfrak{S}_k$-action and the quotient manifold $M(k) = \tilde{M}(k)/\mathfrak{S}_k$ is called the *k-tuple manifold* of $j : M \looparrowright V$.

It turns out (cf. [19]) that the immersion $j(k) : M(k) \looparrowright V$ represents precisely the projected class

$$[(j(k), ?)] \in [\overline{V}, \Omega^t S^t D_k^{(n)}(\mathrm{Th}(\xi))].$$

If the estimates of the last section are correct, then $j(k) : M(k) \looparrowright V$ is t-decompressible with $t = t_{nk} = (n-1)(2k-1)+1$, and this universally for every generic n-decompressible immersion $j : M \looparrowright V$!

Remark 4.10. Let us come back to the higher Hopf invariants. Given a class $[f] \in [S^n\overline{V}, S^n\mathrm{Th}(\xi)]$ represented by $(j,\lambda) : M \hookrightarrow V \times \mathbb{R}^n$ with generic immersion j. How can the Hopf invariant

$$[\gamma_k(f)] \in [S^t\overline{V}, S^t D_k^{(n)}(\mathrm{Th}(\xi))] = [S^t\overline{V}, S^t\mathrm{Th}(F(\mathbb{R}^n,k) \times_{\mathfrak{S}_k} \xi^k)]$$

be represented ? According to the preceding remark we have only to find a t-decompression $\lambda(k) : M(k) \to \mathbb{R}^t$ for $j(k) : M(k) \looparrowright V$ and the embedding $(j(k),\lambda(k)) : M(k) \hookrightarrow V \times \mathbb{R}^t$ will be a representative for $[\gamma_k(f)]$. Koschorke-Sanderson [19] prove that for the so defined higher Hopf invariants properties 1-3 of conjecture 3.7 hold, see also Boardman-Steer [8] where the same construction is carried out for $n = 1$.

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Author's address : Université de Nice-Sophia Antipolis, Lab. J.-A. Dieudonné, Parc Valrose, F-06108 Nice, Cedex 2, France.

E-mail : cberger@math.unice.fr

The Milgram non-operad

Michael Brinkmeier

Introduction

In [11] R.J. Milgram introduced geometric models J^n for the iterated loop-space operads $\Omega^n\Sigma^n$. Later J.M. Boardman, R.M. Vogt and P. May proved that n-fold loop spaces are closely related to E_n operads in general, and thus to the little cube operads (see [6], [7], [9]). Hence the question arises if the operad structure of $\Omega^n\Sigma^n$ translates to the geometric model J^n.

In [1] C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R.M. Vogt construct an operad $\mathcal{M}_n$, which codifies n-fold monoidal categories, a categorial analog of n-fold loop spaces. They observe that an equivalent *preoperad* is embedded in $\mathcal{M}_n$, whose free space is of the same homotopy type as J^nX . For $n = 2$ the spaces are even homeomorphic (see [1, 3.12 – 14]).

Due to the underlying polytopes, the permutohedra, the Milgram-construction J^n is of some importance for the examination of coherent homotopy-commutativity. Similar to the associahedra introduced by Stasheff in [13], Williams uses the permutohedra and the Milgram-construction in [14] to define his notion of C_n-spaces, which is used in several subsequent papers, and is occuring in papers of McGibbon and Hemmi (see for example [10] and [8]).

In fact there exists an operad structure with the permutohedra as underlying spaces (this was pointed out to me by Clemens Berger and Zig Fiedorowicz), simply by using the convex extension of the permutation operad. But since the permutohedra are contractible this is an E_∞ operad in the sense of Boardman and Vogt (i.e. the symmetric group action does not need to be free). Therefore an algebra of this operad is homotopy equivalent to an infinite loop space and hence its associated monad can not be of the same homotopy type as the Milgram construction J^n which is just an n-fold loop space.

In [4] and [5] C. Berger conjectured an E_n operad structure of the permutohedra, whose associated monad is the Milgram construction J^n.

I will show that Berger's construction does not work. The first observation is that the would-be operad bears a structure far too strong, namely that of strictly abelian monoids. This collapse of structure is then used to show that the suggested structure does not define an operad at all. In fact the proof shows that the multiplication defined by Berger does not respect the degeneration conditions.

I would like to thank Zig Fiedorowicz and especially Clemens Berger for some helpful and enlightening discussions, and James McClure for the discovery of some mistakes and errors in an earlier version of this paper.

1 The Permutohedra and Berger's construction

Let Λ be the category of finite sets $\mathbf{n} = \{1, \ldots, n\}$ and injective maps. Each injective map $\varphi : \mathbf{n} \to \mathbf{m}$ has an unique decomposition of the form $\varphi = \varphi^{inc} \circ \varphi^{\natural}$, such that $\varphi^{inc} : \mathbf{n} \to \mathbf{m}$ is increasing and $\varphi^{\natural} : \mathbf{n} \to \mathbf{n}$ is a permutation.

Definition 1.1. *The Permutation preoperad $\Sigma : \Lambda^{op} \to \mathbf{T}op$ is the functor with $\Sigma(\mathbf{n}) = \Sigma_n$ and $\varphi^* := \Sigma(\varphi) : \Sigma_m \to \Sigma_n$ given by $\sigma \mapsto (\sigma \circ \varphi)^{\natural}$ for $\varphi \in \Lambda(\mathbf{n}, \mathbf{m})$*

The multiplications of the Permutation Operad Σ are given by

$$\begin{aligned} \gamma^{\Sigma}_{n;i_1,\ldots,i_n} : \Sigma_n \times \Sigma_{i_1} \times \cdots \times \Sigma_{i_n} &\to \Sigma_{i_1+\cdots+i_n} \\ (\sigma; \tau_1, \ldots, \tau_n) &\mapsto \sigma(i_1, \ldots, i_n) \circ (\tau_1 \oplus \cdots \oplus \tau_n) \end{aligned}$$

where $\sigma(i_1, \ldots, i_n)$ permutes the blocks (see [5, 1.15.(a)]) and $- \oplus - : \Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ is the canonical product of permutations.

The product $\Sigma_{i_1} \oplus \cdots \oplus \Sigma_{i_n} \subset \Sigma_{i_1+\cdots+i_n}$ will be denoted with $\Sigma(i_1, \ldots, i_n)$.

For more details about (pre)operads in general the reader is referred to [4], [5] or [9].

In contrary to Clemens Berger I will use the left action of the symmetric group on $\mathbb{R}^n$, fixing some problems regarding the description of the skeletal structure of the permutohedra.

Definition 1.2. *For $n \geq 1$ the symmetric group Σ_n acts on $\mathbb{R}^n$ from the left by permuting the coordinates in the following way.*

$$\sigma(x_1, \ldots, x_n) = \big(x_{\sigma^{-1}(1)}, \ldots x_{\sigma^{-1}(n)}\big).$$

The n-th Permutohedron $P_n \subset \mathbb{R}^n$ is the convex hull of the orbit of $(1, 2, \ldots, n) \in \mathbb{R}^n$ under this operation.

The convex hull of the orbit of $\Sigma(i_1, \ldots, i_r) \subset \Sigma_n$ will be denoted with $P(i_1, \ldots, i_r) \subset P_n$.

The point $\sigma(1, \ldots, n) \in P_n$ will be denoted with σ.

Remark 1.3. The notiation of σ for the point $\sigma(1, \ldots, n) = (\sigma^{-1}(1), \ldots, \sigma^{-1}(n))$ seems somewhat confusing. But since we will extend the permutation operad to the permutohedra, we can calculate the vertices right from the permutations, without applying it to the $\mathbb{R}^n$.

The geometric and simplicial properties of these polygons were examined in [11], [14] and [2]. Here I will give only a rough sketch of the few details I will use.

P_1 consists of only one point, P_2 is homeomorphic to the unit interval in $\mathbb{R}$ and P_2 to the hexagon in $\mathbb{R}^2$. In general P_n is a $(n-1)$ dimensional polytope.

Obviously there exists a left Σ_n-action on P_n. The vertices are mapped to vertices, and for each $\sigma \in \Sigma_n$ the map $P_n \to P_n$ with $x \mapsto \sigma x$ is a homeomorphism. But unfortunaltely this action is not free, since the barycenter of each permutohedron is a fixed point.

An arbitrary point $x \in P_n$ will be denoted by a linear combination of permutations

$$x = \sum_{\sigma \in \Sigma_n} t_\sigma \sigma \qquad \text{with} \qquad \sum_{\sigma \in \Sigma_n} t_\sigma = 1.$$

If $x = \sum_{\sigma \in \Sigma_n} s_\sigma \sigma$ and $y = \sum_{\tau \in \Sigma_m} t_\tau \tau$ are points of P_n resp. P_m, the point $x \oplus y \in P_{n+m}$ is given by

$$x \oplus y = \sum_{\sigma \in \Sigma_n} \sum_{\tau \in \Sigma_m} s_\sigma t_\tau \sigma \oplus \tau.$$

In [11] Milgram defined maps $I_k : P_k \times P_{n-k} \to P_n$ given by $(x, y) \mapsto x \oplus y$ mapping the product of two permutohedra into certain faces of a higher dimensional permutohedron. More general the codimension $(r-1)$ faces of P_n are in one-to-one correspondence with the ordered partitions of $\{1, \ldots, n\}$ of type $(i_1, \ldots, i_r)$ with $i_1 + \cdots + i_r = n, i_k \geq 1$.

Remark 1.4. Each partition of type $(i_1, \ldots, i_r)$ can be interpreted as a permutation of $i_1 + \cdots + i_r$ elements. If the classes are given by

$$\{j_1, \ldots, j_{i_1}\}, \{j_{i_1+1}, \ldots, j_{i_1+i_2}\}, \ldots, \{j_{i_1+\cdots+i_{r-1}+1}, \ldots, j_{i_1+\cdots+i_r}\}$$

with $j_k < j_l$ for $i_m \leq k < l < i_{m+1}$, then the corresponding permutation is given by $k \mapsto j_k$ for $1 \leq k \leq i_1 + \cdots + i_r$.

The converse does not hold, since the same permutation can be associated to different partitions. For example the identity in Σ_3 corresponds to the partion $\{1\}, \{2,3\}$ and to $\{1,2\}, \{3\}$.

The vertices of the codimension $(r-1)$ face, corresponding to a partition of type $(i_1, \ldots, i_r)$, with associated permutation $\sigma \in \Sigma_n$, are given by the coset $\sigma\Sigma(i_1, \ldots, i_r)$. In addition there is a homeomorphism $I_\sigma : P_{i_1} \times \cdots \times P_{i_r} \to \sigma P(i_1, \ldots, i_r) \subset P_n$ with

$$(x_1, \ldots, x_r) \mapsto \sigma(x_1 \oplus \cdots \oplus x_r).$$

Using the *weak Bruhat order* on the symmetric groups, the 1-skeleton of these faces can be oriented.

Definition 1.5. *The inversion index $inv(\sigma)$ of a permutation $\sigma \in \Sigma_n$ is the number of ordered pairs $(i,j), 1 \leq i < j \leq n$, whose orders are inverted by σ, i.e. $\sigma(i) > \sigma(j)$.*

The Weak Bruhat Order of Σ_n is the partial order generated by $\sigma < \tau$, if τ is the composition of σ and a transposition of two subsequent numbers, that is $\tau = \sigma \circ (i, i+1)$, and $inv(\sigma) < inv(\tau)$.

Example 1.6. Σ_3 is given by the Poset

$$\begin{tikzcd} & (123) \arrow[dl] \arrow[dr] & \\ (213) \arrow[d] & & (132) \arrow[d] \\ (231) \arrow[dr] & & (312) \arrow[dl] \\ & (321) & \end{tikzcd}$$

Here and in the following the permutation

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$$

will be denoted with $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ (the commas will be left, if unnecessary).

Remark 1.7. If the vertices of the poset (i.e. the permutations) are interpreted as points in $\mathbb{R}^3$, it does not seem to describe the border of P_3 - the points $(2,3,1)$ and $(3,1,2)$ have to be switched. But since we use the left action of Σ_n on $\mathbb{R}^n$, the permutation (231) corresponds to the point $(3,1,2)$ and the permutation (312) to the point $(2,3,1)$. In fact the correspondence holds for all vertices of P_3 and the poset Σ_3.

If we would use the right action of the symmetric groups on $\mathbb{R}^n$ the weak Bruhat order would not describe the skeletal structure of P_n, since for example the permutation (312) would be equivalent with the point $(3,1,2)$.

Applying this partial order to the permutohedra the edges will be oriented. The face corresponding to a certain partition of type $(i_1,\ldots,i_r)$ and associated permutation σ has exactly one initial vertex, given by σ and a unique terminal vertex, given by the permutation that turns the classes of the partition "upside-down", i.e.

$$l \mapsto k_{l'} \quad \text{with} \quad l' = 2(i_1 + \ldots i_l) + i_{l+1} - l + 1 \quad \text{if} \quad i_l < l \leq i_{l+1}.$$

Definition 1.8. *An interval $[\sigma_1,\sigma_2]$ in Σ_n with the weak Bruhat order is called* admissible, *if it is the vertex set of some coset $\sigma\Sigma(i_1,\ldots,i_r)$. Hence σ_1 is the initial and σ_2 the terminal vertex of a face of P_n.*

The geodesic of an arbitrary interval $[\sigma_1,\sigma_2]$ of the weak Bruhat order in Σ_n, is the average of all oriented edge-paths between σ_1 and σ_2 in P_n. The barycenter of $[\sigma_1,\sigma_2]$ is the barycenter of its geodesic.

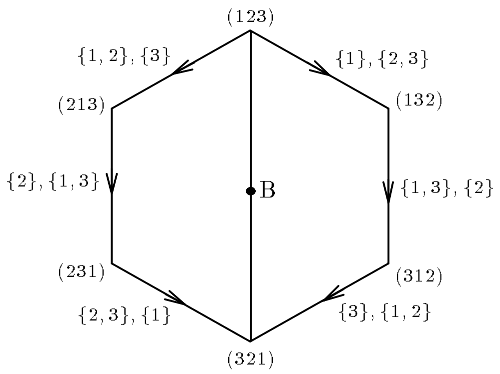

Abbildung 1: The geodesic and barycenter of P_3

The barycenter of an admissible interval coincides with the barycenter of the corresponding cell. For example the geodesic of $[(123),(213)] \subset \Sigma_3$ is the corresponding edge of P_3, and its barycenter the barycenter of the interval. The geodesic of $[(123),(321)] \subset \Sigma_3$ is the line between (123) and (321) and its barycenter is the point B. The geodesic of the non-admissible intervall $[(123),(231)] \subset \Sigma_3$ consists of the two edges $[(123),(213)]$ and $[(213),(231)]$ of P_3. Hence its barycenter is the point $(213) \in P_3$ (see Fig. 1).

To define an operad-multiplication on the permutohedra we have to define maps $P_r \times P_{i_1} \times \cdots \times P_{i_r} \to P_{i_1+\cdots+i_r}$ which satisfy the associativity conditions. But the intention to formulate an operad, whose associated monad is the Milgram construction gives certain additional restrictions to the choice of the multiplication.

The operad structure has to extend the permutation operad Σ. This can be done very easily by mapping the vertices of the product $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ to the corresponding vertices of $P_{i_1+\cdots+i_r}$, given by Σ. But the extension of this map can be done in two different ways.

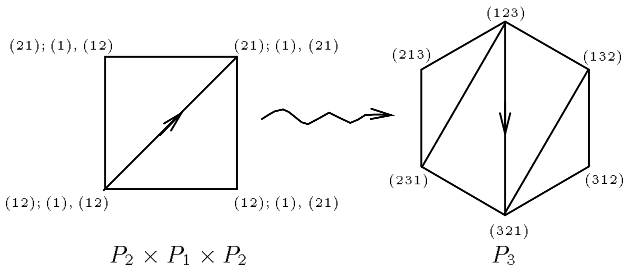

Abbildung 2: The convex multiplication

The first possibility is the *convex* extension of the permutation operad, by mapping $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ to the convex hull of the image vertices (see Fig. 2). This construction does in fact define an operad whose n-th space is P_n. It is E_∞ in the sense of Boardman and Vogt, i.e. its underlying spaces are contractible but the actions of the symmetric groups does not need to be free. Thus its algebras are homotopy equivalent to infinte loop spaces (cmp. [7] section VI.3). But since the Milgram construction is only a model for n-fold loop spaces the associated monad of the convex extension is of the wrong homotopy type.

Berger tried to get the correct operad by application of two changes to the convex extension of the permutation operad. First he deformed the multiplication such that it respects the relations on the borders of the permutohedra, given in Milgrams construction. But this does not affect the homotopy type of the spaces. In a second step he made the action of the symmetric group free, which would give the correct homotopy type.

In the Milgram construction the border of $P_r \times P_{i_1} \times \cdots \times P_{i_r}$ has to be mapped to the border of $P_{i_1+\cdots+i_r}$. Berger did this by using a *cubical* extension of the symmetric operad instead of the convex one. He defined the operad-multiplication $\gamma^P_{n;i_1,\ldots,i_n} : P_n \times P_{i_1} \times \cdots \times P_{i_n} \to P_{i_1+\cdots+P_{i_n}}$ as the affine extension of the permutation operad, such that the barycenter of any interval in $\Sigma_{i_1+\cdots+i_n}$ is mapped to the barycenter of its geodesics in $P_{i_1+\cdots+i_n}$ (see Fig. 3). Hence for $n = 2$ and $i_1 + i_2 = 3$ the squares A,B,C and D of $P_2 \times P_1 \times P_2$, resp. $P_2 \times P_2 \times P_1$ are mapped to the corresponding segments of P_3.

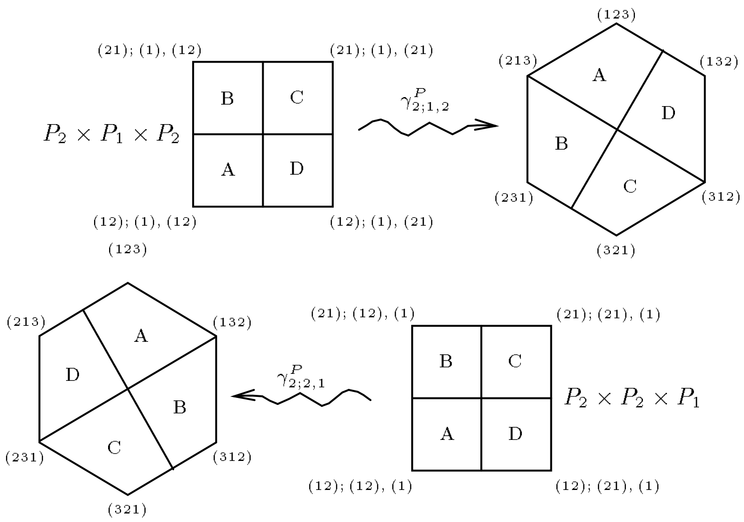

Abbildung 3: The cubical multiplication

The second step, in which the symmetric group action is made free, is done in the definition of the would-be operad. For each permutation in Σ_n a copy of P_n^k is added and an appropriate quotient of the space $P_n^k \times \Sigma_n$ is taken to be the $n-th$ space of the new E_k operad.

Here I will only describe the suggested construction for the operad $J^{(2)}$ corresponding to J_2, i.e. 2-fold loop spaces.

Definition 1.9. *(cmp. [5, 2.12]) Let $J_n^{(2)}$ be the quotient space of $P_n \times \Sigma_n$ under the relation*

$$(\tau x; \sigma) \sim (x; \tau\sigma)$$

for any partition $\tau \in \Sigma_n$ of type $(i_1, \ldots, i_r), i_1+\cdots+i_r = n$, $x \in P\,(i_1, \ldots, i_r)$ and $\sigma \in \Sigma_n$.

The action of $\varphi \in \Lambda(\mathbf{n}, \mathbf{n})$ is induced by

$$\begin{aligned} \varphi^* : P_m \times \Sigma_m &\to P_n \times \Sigma_n \\ (x; \sigma) &\mapsto \big(x(\sigma\varphi)^{inc}; (\sigma\varphi)^{\natural}\big). \end{aligned}$$

Remark 1.10. Since I use the left instead of the right action, the relation given by Berger has to be changed slightly.

Following [5, 2.14] the Λ structure and the maps $\gamma^P_{i_1,\ldots,i_r} \times \gamma^\Sigma_{i_1,\ldots,i_r}$ induce an E_2-operad structure on $J^{(2)}_n$.

2 The commutativity of Berger's constructi- on

In the following we assume that Berger's construction defines an operad. If $(X, *)$ is a $J^{(2)}$-algebra, there exist maps $F_n : P_n \times \Sigma_n \times X^n \to X$. These fulfil certain conditions, induced by the operad structure and the relations on $J^{(2)}_n$. Used here are

1. the associativity condition

$$\begin{aligned} &F_n\big(s; F_{i_1} = (t_1; x^1_1, \ldots, x^1_{i_1}), \ldots, F_{i_n}(t_n; x^n_1, \ldots, x^n_{i_n})\big) \\ &\quad = F_{i_1+\cdots+i_n}\big(\gamma_{n;i_1,\ldots,i_n}(s; t_1, \ldots, t_n); x^1_1, \ldots x^n_{i_n}\big), \end{aligned}$$

2. the degeneration relations on the associated monad, induced by the two maps $\mathbf{1} \to \mathbf{2}$ $F_2(s; x, *) = F_1(*; x) = x = F_2(s; *, x)$

3. and the Σ_n-equivariance: $F_n(s\sigma; x_1, \ldots, x_n) = F_n\big(s; x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\big)$.

Remark 2.1. Since only the identities of Σ_n are used, in the future this coordinate will be dropped.

There is a multiplication on X, given by $xy = F_2\big((12); x, y\big)$. The associativity of the permutation operad shows the associativity of this multiplication.

Since the degeneration relations hold $*$ is a unit. Therefore X is a associative monoid with strict unit.

Obviously the map $\phi : I \to P_2$ with $t \mapsto (12)(1-t) + (21)t$ is a homeomorphism. Thus there exists a homotopy $h_t : X \times X \to X$ with

$$h_t(x, y) = F_2\big((12)(t-1) + (21)t; x, y\big),$$

running from $h_0(x, y) = xy$ to $h_1(x, y) = F_2\big((21); x, y)\big) = F_2\big((12); y, x\big) = yx$.

Via F_3 and the degenerations one gets even stricter conditions for the commuting homotopy h_t. The maps $\gamma^P_{2;1,2}$ and $\gamma^P_{2;2,1}$ are homeomorphisms. Therefore there exist $s_i, t_i \in P_2, i = 1, 2$ for each $r \in P_3$ such that

$$\gamma^P_{2:2,1}\big(s_2; t_2, (1)\big) = r = \gamma^P_{2;1,2}\big(s_1; (1), t_1\big).$$

Hence for $x, y \in X$ one gets

$$F_2(s_2; x, y) = F_3(r; *, x, y) = F_2(t_1; x, y)$$

In the first case the homotopy h_t is mapped to the edges $[(123), (132)]$ and $[(231), (321)]$ (cmp. Fig. 2). In the second case h_t is mapped to the geodesics of the intervals $[(123), (312)]$ and $[(213), (321)]$, such that the center point of the homotopy $h_{\frac{1}{2}}(x, y)$ is mapped to (132) and (231). Thus the homotopy needs to be equal to xy on its first half and equal to yx on the second half (In the first case the edge $[(213), (231)]$ is mapped to xy and the edge $[(132), (312)]$ to yx). Thus $(X, *)$ must be an abelian monoid.

Remark 2.2. Since the permutation coordinate wasn't used, the freeness of the symmetric group action is not involved in the failure of the suggested construction. In fact the cubical extension of the multiplication does not fulfil the needed degeneration properties.

Now let X be a 2-connected CW-complex with non-degenrate base point $*$ (i.e. the inclusion $* \hookrightarrow X$ is a closed cofibration). Then the two-fold Moore loop space $Y := \Omega^2_M X$ of X is a connected CW-complex. The canonical evaluation map $e : \Sigma^2 \Omega^2_M X \to X$ induces a homomorphism of monoids $\Omega^2_M e : \Omega^2_M \Sigma^2 Y \to Y$.

If $J^{(2)}$ is an E_2 operad, whose associated monad is the Milgram construction, there exists a homomorphism of monoids $\psi : J^{(2)}Y \to \Omega^2_M \Sigma^2 Y$ (the map is given in [11, 5.2.]). Therefore we get a homomorphism $\varphi : J^{(2)}Y \to Y$. From

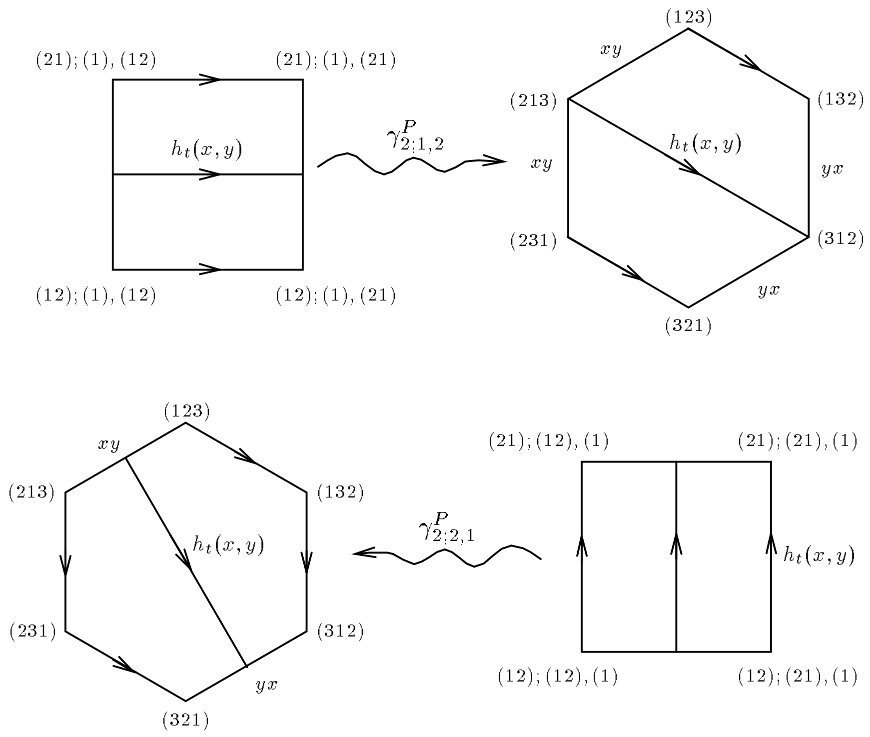

Abbildung 4: The commuting homotopy

the construction of ψ in [11] one can see that the diagram

$$\begin{tikzcd}
J^{(2)}Y \arrow[r, "\psi"] & \Omega^2_M\Sigma^2 Y \arrow[r, "\Omega^2_M e"] & Y \\
& Y \arrow[ul, hook] \arrow[u, hook] \arrow[ur, equal] &
\end{tikzcd}$$

with the inclusion $i : Y \hookrightarrow \Omega^2_M\Sigma^2 Y$ given by

$$(i(y))(s,t) = y \wedge s \wedge t,$$

commutes. Therefore The homomorphism $\varphi = \Omega^2_M\psi$ is an extension of the identity and hence surjective.

Since $J^{(2)}Y$ is the free $J^{(2)}$-algebra of the space Y, it is an abelian monoid. The surjectivity of φ now shows that $Y = \Omega^2_M X$ is strictly commutative, too. But obviously this is wrong. Therefore $J^{(2)}$ can not be an operad whose associated monad is the Milgram construction.

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Constructions of E_n Operads

Z. Fiedorowicz

Throughout this talk I will use the following conventions and notations. I will primarily consider operads in the category of compactly generated Hausdorff topological spaces having the homotopy type of CW-complexes. When I refer to simplicial operads or operads in the category of posets, it will be understood that they can be converted to topological operads by taking geometric realization or geometric realization of the nerve, respectively. I will only consider operads $\mathcal{A}$ satisfying $\mathcal{A}(0) = *$. This allows us to define degeneracy maps

$$d_i : \mathcal{A}(n) \longrightarrow \mathcal{A}(n-1) \quad i = 1, 2, \ldots, n$$

by composing with $* \in \mathcal{A}(0)$. We will refer to collections of spaces $\mathcal{A} = \{\mathcal{A}(k)\}_{k \geq 0}$ equipped with symmetric group actions and such degeneracy maps, satisfying the evident relations, but without any further operad composition maps, as *preoperads*. (We refer the reader to Berger's talk in this volume [2] (or [3]) for a precise definition.)

Definition. An operad map $\mathcal{A} \to \mathcal{B}$ is said to be an *equivalence* if for each $k \geq 0$, $\mathcal{A}(k) \to \mathcal{B}(k)$ is a Σ_k-equivariant homotopy equivalence. We say that an operad $\mathcal{A}$ is E_n $(n = 1, 2, 3, \ldots, \infty)$ if there is a chain of operad equivalences (in either or both directions) connecting $\mathcal{A}$ to the Boardman-Vogt little n-cubes operad $\mathcal{C}_n$ (cf. [4]).

Remark. It can be shown that an operad $\mathcal{A}$ is E_n iff there is an operad equivalence $W\mathcal{C}_n \to \mathcal{A}$, where $W\mathcal{C}_n$ is a certain enlarged cofibrant model of the little n-cubes operad. (See Vogt's talk in this volume [12] (or [4]) for details.) However this is of little use in recognizing E_n operads, since the only practical way to construct such an equivalence $W\mathcal{C}_n \to \mathcal{A}$ is to be given a chain of operad equivalences connecting $\mathcal{A}$ to $\mathcal{C}_n$.

The following results provide useful criteria for recognizing E_n operads.

Theorem (Recognition principles for E_n operads.)

1. $\mathcal{A}$ is an E_∞ operad iff

(i) Each space $\mathcal{A}(k)$ is contractible.

(ii) Σ_k acts freely on $\mathcal{A}(k)$ for each $k \geq 0$.

2. $\mathcal{A}$ is an E_1 (a.k.a. A_∞) operad iff

   (i) Each path component of each space $\mathcal{A}(k)$ is contractible.

   (ii) Σ_k acts freely and transitively on $\pi_0\mathcal{A}(k)$ for each $k \geq 0$.

3. $\mathcal{A}$ is an E_2 operad iff each space $\mathcal{A}(k)$ is connected and the collection of universal covering spaces $\{\widetilde{\mathcal{A}}(k)\}_{k\geq 0}$ forms a B_∞ operad. That is

   (i) Each space $\widetilde{\mathcal{A}}(k)$ is contractible.

   (ii) The braid group B_k acts freely on $\widetilde{\mathcal{A}}(k)$ for each $k \geq 0$.

Proof Sketch. The first two results are classical, due to Boardman-Vogt [4]. The fact that E_∞, resp. E_1, operads satisfy these criteria is immediate from the fact that $\mathcal{C}_\infty$, resp. $\mathcal{C}_1$, satisfy these criteria, and that these criteria are preserved by operad equivalences. For the reverse implications, one notes that the given criteria insure that the projections provide a chain of operad equivalences

$$\mathcal{A} \xleftarrow{\simeq} \mathcal{A} \times \mathcal{C}_\infty \xrightarrow{\simeq} \mathcal{C}_\infty$$

in the E_∞ case, and a similar chain of operad equivalences

$$\mathcal{A} \xleftarrow{\simeq} \mathcal{A} \times_{\pi_0(\mathcal{A})} \mathcal{C}_1 \xrightarrow{\simeq} \mathcal{C}_1$$

in the E_1 case.

The recognition principle for E_2 operads is due to the author (cf. [8]). It first requires noting that the definition of operad can be reformulated using actions by braid groups in place of actions by symmetric groups. One then notes that the operad structure of an E_2 operad $\mathcal{A}$ can be lifted to a (braided) operad structure on the universal covers $\{\widetilde{\mathcal{A}}\}_{k\geq 0}$. The lifting requires a consistent choice of basepoints in $\mathcal{A}$. Such a choice is provided by mapping an E_1 operad into $\mathcal{A}$ as follows:

$$W\mathcal{C}_1 \longrightarrow W\mathcal{C}_2 \xrightarrow{\simeq} \mathcal{A},$$

(cf. remark above). This gives the requisite B_∞ structure on $\widetilde{\mathcal{A}}$. Conversely given a braided operad structure on $\widetilde{\mathcal{A}}$, one can recover a regular operad structure on $\mathcal{A}$, by taking the orbit spaces of the actions by the pure braid groups PB_k. The rest of the proof now proceeds as in the E_∞ case:

$$\mathcal{A} \xleftarrow{\simeq} \left(\widetilde{\mathcal{A}} \times \widetilde{\mathcal{C}_2}\right) \Big/ PB_* \xrightarrow{\simeq} \mathcal{C}_2.$$

Unfortunately at this time, there is nothing like a recognition principle for E_n operads for $3 \leq n < \infty$. Indeed until the last five years or so, the only known E_n operads in such cases were minor variants of the little n-cubes operads, eg. little n-disks operads.

This situation changed with the discovery of a nice family of simplicial E_n operads and operad equivalences (for all n):

$$\begin{tikzcd}
\text{Getzler-Jones poset operad (cf. [9])} \arrow[d, "\simeq"] \\
n\text{-fold monoidal (poset) operad } \mathcal{M}_n \text{ (cf. [1])} \arrow[d, hook, "\simeq"] \\
\text{complete graphs (poset) operad } \mathcal{K}^{(n)} \text{ (cf. [3])} \arrow[d, "\simeq"] \\
\text{Smith (simplicial) operad (cf. [11])}
\end{tikzcd}$$

I will not discuss these examples and instead refer the reader to Berger's talk in this volume [2] for more details. However I would like to briefly discuss the method developed by Berger to prove that these examples are E_n operads.

This method is the notion of a cellular decomposition of a topological operad over a poset operad. Let me first discuss the simpler notion of a cellular decomposition of a topological space X over a finite poset $\mathcal{P}$. By this we mean a decomposition $X = \cup_{P \in \mathcal{P}} C_P$, where each cell C_P is a closed subset of X. We further suppose that $P < Q \Longrightarrow C_P \subseteq C_Q$ and define $\mathrm{bd}(C_P) = \cup_{P' < P} C_{P'}$, $\mathrm{int}(C_P) = C_P - \mathrm{bd}(C_P)$. If we assume in addition that

1. Each C_P is contractible.
2. The inclusions $\mathrm{bd}(C_P) \subseteq C_P$ are cofibrations.
3. $\mathrm{int}(C_P) \cap \mathrm{int}(C_Q) = \emptyset$ if $P \neq Q$.

Then we have a sequence of equivalences and homeomorphisms:

$$|\mathcal{P}| \xleftarrow{\simeq} \mathrm{hocolim}_{P\in\mathcal{P}} C_P \xrightarrow{\simeq} \mathrm{colim}_{P\in\mathcal{P}} C_P \xrightarrow{\cong} X,$$

where $|\mathcal{P}|$ denotes the geometric realization of the nerve of $\mathcal{P}$. Briefly, we need the hypothesis (1) to insure that the first arrow is an equivalence, hypothesis (2) to insure that the second arrow is an equivalence, and hypothesis (3) to insure that the third arrow is a homeomorphism. It is useful to note that if X has a cellular decomposition over a poset $\mathcal{P}$, then the product X^m has a cellular decomposition indexed over the product poset $\mathcal{P}^m$.

Please note that the word "cellular" in the above definition is being used in a very loose sense. It is **not** to be supposed that either the cells or their interiors look anything like disks. We merely assume that the cells satisfy the above three conditions.

To upgrade the notion of cellular decomposition to the context of operads, we assume that the posets $\{\mathcal{P}(k)\}_{k\geq 0}$ indexing the cellular decompositions of the topological operad $\mathcal{A}$, themselves form an operad in the category of posets. In addition we assume the evident compatibility conditions between the operad structures of $\mathcal{P}$ and $\mathcal{A}$:

1. $1_{\mathcal{A}} \in C_{1_{\mathcal{P}}}$, where $1_{\mathcal{A}} \in \mathcal{A}(1)$, $1_{\mathcal{P}} \in \mathcal{P}(1)$ are the units of the operads $\mathcal{A}$, $\mathcal{P}$ respectively.
2. $(C_P)\sigma = C_{P\sigma}$, for any $\sigma \in \Sigma_k$, $P \in \mathcal{P}(k)$.
3. If $\mu_{\mathcal{A}}$, $\mu_{\mathcal{P}}$ denote the operad composition maps in $\mathcal{A}$, $\mathcal{P}$ respectively, then
$$\mu_{\mathcal{A}}\left(C_P \times C_{Q_1} \times C_{Q_2} \times \ldots C_{Q_k}\right) \subseteq C_{\mu_{\mathcal{P}}(P;Q_1,Q_2,\ldots,Q_k)}$$

Under these hypotheses the sequence:

$$\{|\mathcal{P}(k)|\}_{k\geq 0} \xleftarrow{\simeq} \{\mathrm{hocolim}_{P\in\mathcal{P}} C_P(k)\}_{k\geq 0} \xrightarrow{\simeq} \{\mathrm{colim}_{P\in\mathcal{P}(k)} C_P\}_{k\geq 0} \xrightarrow{\cong} \{\mathcal{A}(k)\}_{k\geq 0}$$

is a chain of operad equivalences. Thus under these circumstances the topological operad $\mathcal{A}$ is E_n iff the poset operad $\mathcal{P}$ is E_n. Currently this is the only practical method for recognizing E_n operads for $3 \leq n < \infty$.

Remark. The Getzler-Jones poset operad has not been explicitly described in the literature to our knowledge. However Getzler-Jones describe a cellular decomposition of their topological operad given by compactifications of configuration spaces, and it is not difficult to see that their decomposition is indexed by a poset operad which maps into the n-fold monoidal operad, and it can be shown that this map of poset operads is an equivalence. This also furnishes a proof that the Getzler-Jones topological operad is E_n. (There may be unpublished more direct proofs of this fact, but we are unaware of the details.)

I recently noticed that this method can be combined with the following right adjoint construction to produce many new examples of E_n operads.

Theorem 1. (i) The forgetful functor

$$U: \text{ Preoperads} \longrightarrow \mathbb{Z}/2\text{ -spaces}$$

$$\mathcal{A} \longmapsto \mathcal{A}(2)$$

has a right adjoint

$$R: \ \mathbb{Z}/2\text{ -spaces} \longrightarrow \text{Preoperads}$$

(ii) For any $\mathbb{Z}/2$-space X, RX has a functorial operad structure.
(iii) If $\mathcal{A}$ is an operad such that $\mathcal{A}(1) = \{1_{\mathcal{A}}\}$, then

$$\mathcal{A} \longrightarrow RU\mathcal{A} = R(\mathcal{A}(2))$$

is a map of operads.

Remark. Without the hypothesis that $\mathcal{A}(1) = \{1_{\mathcal{A}}\}$

$$\mathcal{A} \longrightarrow RU\mathcal{A} = R(\mathcal{A}(2))$$

is only a map of preoperads, ie. it is only compatible with the symmetric group actions and composition with constants, **not** with general compositions. We shall illustrate this phenomenon below with the little n-cubes operad. I should add that I initially overlooked the necessity of this condition and noticed it only while preparing this talk.

Proof Sketch. Given a $\mathbb{Z}/2$-space X, we need to construct a sequence of spaces $\{RX(k)\}_{k\geq 0}$ with an appropriate structure. As a topological space

$RX(k)$ is just the product space $X^{\binom{k}{2}}$, where $\binom{k}{2}$ is the binomial coefficient $k(k-1)/2$. However to describe the symmetric group actions and the operad compositions, it is convenient to describe $RX(k)$ as follows.

Consider the complete graph on the set of vertices $\{1, 2, \ldots, k\}$. By an orientation of this graph we mean the assignment of a direction to each edge of the graph. By a labelling of this graph, we mean the assignment of a point in X to each edge of the graph. (Different edges may labelled by different points.) We further impose the equivalence relation that changing the direction of an edge is equivalent to changing its label from x to $\overline{x}$, where $x \mapsto \overline{x}$ denotes the $\mathbb{Z}/2$ action on X. We define $RX(k)$ as the space of all such orientations and labellings of the complete graph on $\{1, 2, \ldots, k\}$. Formally we can describe the elements of $RX(k)$ as functions $f(i, j)$, where $i \neq j \in \{1, 2, \ldots, k\}$, taking values in X, and satisfying the condition that $f(j, i) = \overline{f(i, j)}$. $RX(k)$ can be identified with the product $X^{\binom{k}{2}}$ by noting that using the equivalence relation, we can choose unique representatives in $RX(k)$ with the edges oriented in the direction of the natural order on $\{1, 2, \ldots, k\}$, and with an arbitrary labelling of the edges.

The symmetric group action on $RX(k)$ is induced by the natural action of Σ_k on the vertex set $\{1, 2, \ldots, k\}$. The operad composition

$$RX(k) \times RX(i_1) \times RX(i_2) \times \cdots \times RX(i_k) \longrightarrow RX(i_1 + i_2 + \cdots + i_k)$$

is also easily described with the complete graph formalism. We need to describe an orientation and labelling of the complete graph on the set $\{1, 2, \ldots, i_1 + i_2 + \cdots + i_k\}$ using the orientation and labelling information encoded in the product space on the left. It is almost selfevident how to do this: for edges connecting vertices in a given block

$$\{(i_1 + i_2 + \cdots + i_{j-1}) + 1, (i_1 + i_2 + \cdots + i_{j-1}) + 2, \ldots, i_1 + i_2 + \cdots + i_{j-1} + i_j\}$$

use the orientation and labelling information from the space $RX(i_j)$. For edges connecting vertices in different blocks use the orientation and labelling information encoded in the space $RX(k)$. The fact that our construction specifies an operad is then an exercise in understanding the notation.

For a preoperad $\mathcal{A}$ the unit of the adjunction

$$\mathcal{A}(k) \longrightarrow (RU\mathcal{A})(k) = (\mathcal{A}(2))^{\binom{k}{2}}$$

is specified by the obvious iterated degeneracies. It is obvious that RU specifies an idempotent monad on the category of preoperads (since $URU = U$). Hence it follows that R is right adjoint to U.

The proof of part (iii) is deferred for now. It will be proved below using a more structured right adjoint construction R_2 which coincides with R under the hypothesis that $\mathcal{A}(1) = \{1_{\mathcal{A}}\}$.

Example. The little 2-cubes operad $\mathcal{C}_2$ does not satisfy the hypothesis of part (iii) of the theorem: $\mathcal{C}_2(1) \neq \{1_{\mathcal{C}_2}\}$. We would like to show that

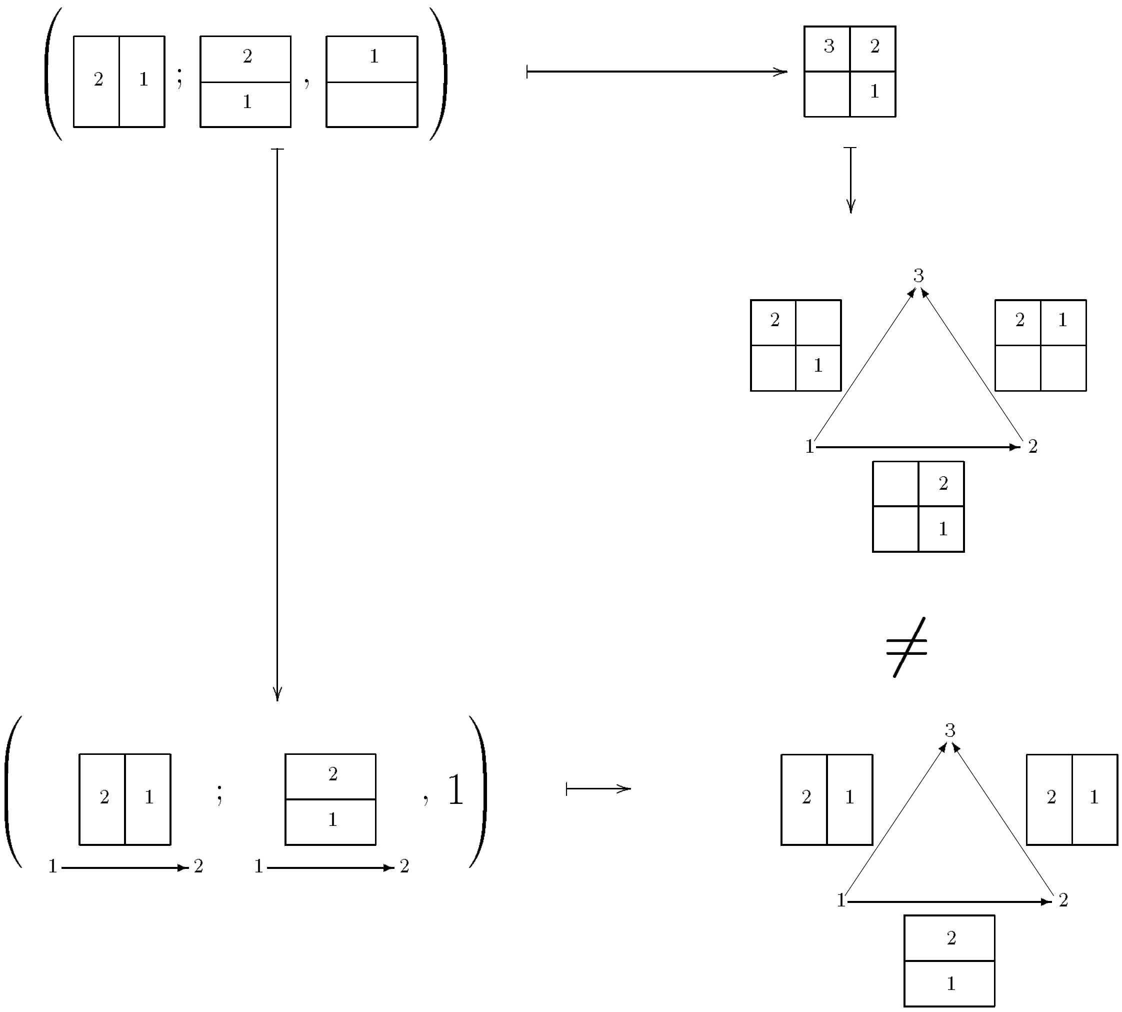
$$\mathcal{C}_2 \longrightarrow RU\mathcal{C}_2$$

is not an operad map by showing that the diagram below does **not** commute

$$\begin{tikzcd}
\mathcal{C}_2(2) \times \mathcal{C}_2(2) \times \mathcal{C}_2(1) \arrow[r] \arrow[d] & \mathcal{C}_2(3) \arrow[d] \\
(RU\mathcal{C}_2)(2) \times (RU\mathcal{C}_2)(2) \times (RU\mathcal{C}_2)(1) \arrow[r] & (RU\mathcal{C}_2)(3)
\end{tikzcd}$$

where the vertical maps are given by the unit $\mathcal{C}_2 \to RU\mathcal{C}_2$ and the horizontal maps are compositions in the operads $\mathcal{C}_2$, $RU\mathcal{C}_2$ respectively.

To see this we take a typical element in the top left corner and chase it around both sides of the diagram:



Thus we see that the image of $\mathcal{C}_2$ in $RU\mathcal{C}_2$ is not closed under operad composition in $RU\mathcal{C}_2$.

Remark. Theorem 1 has a poset version: the forgetful functor

$$U : \text{Poset preoperads} \longrightarrow \mathbb{Z}/2\text{ -posets}$$

has a right adjoint

$$R : \mathbb{Z}/2\text{ -posets} \longrightarrow \text{Poset preoperads}$$

satisfying the same conditions as above. The construction of the functor R is identical to the case of spaces. If a $\mathbb{Z}/2$-space X has a cellular decomposition over $\mathbb{Z}/2$-poset $\mathcal{P}$, with the $\mathbb{Z}/2$ actions being compatible, then the topological operad RX has a cellular decomposition over the poset operad $R\mathcal{P}$. Then using Berger's argument we have a chain of operad equivalences connecting $|R\mathcal{P}|$ to RX.

For any $\mathbb{Z}/2$-space X the operad RX is not E_n for at least two reasons:

1. The spaces $RX(k)$ have the wrong homotopy type. For by definition $RX(k) = X^{\binom{k}{2}}$, whereas the k-th space of an E_n operad can not have the homotopy type of a product of multiple copies of a space unless $n = \infty$, $k = 0$ or $k = 1$.

2. For $k > 2$ the action of Σ_k on $RX(k)$ isn't free. For example for any $x \in X$ the following point in $RX(3)$

$$\begin{tikzcd} & 3 \arrow[dl, "x"'] & & & & 3 & \\ 1 \arrow[rr, "x"] & & 2 \arrow[ul, "x"'] & = & 1 \arrow[ur, "\overline{x}"] \arrow[rr, "x"] & & 2 \arrow[ul, "x"'] \end{tikzcd}$$

is fixed by the cyclic permutation $(1\ 2\ 3)$.

However under suitable conditions, RX has many E_n suboperads. We begin with a definition.

Definition. Let X be a space which is homotopy equivalent to S^{n-1} with a free $\mathbb{Z}/2$ action. If $n = 1$ we say that X has a *hemispherical cellular decomposition* if the $\mathbb{Z}/2$ action permutes the two path components. If $n > 1$ we define the notion of a hemispherical cellular decomposition for X by recursively requiring:

(i) X has a decomposition $X = D \cup \overline{D}$, where $\overline{D}$ is the image of D under the $\mathbb{Z}/2$ action.

(ii) D is contractible by a contraction $H : D \times I \to D$ having the property that $H(x,t) \notin \overline{D}$ for $t > 0$.

(iii) $D \cap \overline{D}$ is homotopy equivalent to S^{n-2} and has a hemispherical cellular decomposition.

We can think of $D \subset X$ as being a kind of fundamental domain for the $\mathbb{Z}/2$ action.

It is an immediate consequence that if X has a hemispherical cellular decomposition, then X has a cellular decomposition over the $\mathbb{Z}/2$ poset

$$\mathcal{K}^{(n)}(2) = \{1, 2, \ldots, n\} \times \mathbb{Z}/2$$

with partial order

$$(i, \sigma) < (j, \tau) \quad \text{if } i < j$$

and with $\mathbb{Z}/2$ action via the second factor. As suggested by the notation, $\mathcal{K}^{(n)}(2)$ is the 2-space of Berger's complete graphs operad $\mathcal{K}^{(n)}$. Now by the remark above, the topological operad RX has a cellular decomposition over the poset operad $R\left(\mathcal{K}^{(n)}(2)\right) = RU\mathcal{K}^{(n)}$. Now for every poset suboperad $\mathcal{P} \subset RU\mathcal{K}^{(n)}$ we define

$$\left(R_{\mathcal{P}}X\right)(k) = \cup_{P \in \mathcal{P}(k)} \mathrm{int} C_P \subset RX(k)$$

and we denote $R_{\mathcal{P}}X = \{(R_{\mathcal{P}}(k)X)\}_{k \geq 0}$.

Our main result is

Theorem 2. Let X be a space homotopy equivalent to S^{n-1}, with a free $\mathbb{Z}/2$ action and a hemispherical cellular decomposition. Let $\mathcal{P}$ be an E_n poset suboperad of $RU\mathcal{K}^{(n)}$ (eg. $\mathcal{P} = \mathcal{K}^{(n)}$ or $\mathcal{P} = \mathcal{M}_n$). Then $R_{\mathcal{P}}X$ is an E_n suboperad of RX.

Proof Sketch. It is almost clear that $R_{\mathcal{P}}X$ has a cellular decomposition over the poset operad $\mathcal{P}$. The only nonobvious thing to check is that the cells of the decomposition are contractible. For the cells of $R_{\mathcal{P}}X$ are obtained from the cells of RX by removing boundary cells indexed by elements outside the poset $\mathcal{P}$. Their contractibility is insured by condition (ii) in the definition above. (One also has to appeal to results of [6] to check that the required cofibration conditions for a cellular decomposition continue to hold in $R_{\mathcal{P}}X$.) Now we can apply Berger's argument to obtain a chain of operad equivalences connecting $R_{\mathcal{P}}X$ to the E_n operad $|\mathcal{P}|$.

Remark. The poset operad $RU\mathcal{K}^{(n)}$ has many other E_n suboperads besides the n-fold monoidal operad $\mathcal{M}_n$ and the complete graphs operad $\mathcal{K}^{(n)}$. It would be interesting to determine the maximal E_n suboperads of $RU\mathcal{K}^{(n)}$.

Examples. Theorem 2 allows us to construct many new examples of E_n operads.

1. If we take $X = S^{n-1}$ with the antipodal $\mathbb{Z}/2$ action, then $R_{\mathcal{P}}X$ can be identified with an open suboperad of the topological operad $|RU\mathcal{K}^{(n)}|$ containing the closed suboperad $|\mathcal{P}|$ as a strong deformation retract. This follows from the fact that $|\mathcal{K}^{(n)}(2)| \cong S^{n-1}$.

2. Consider the preoperad $F(\mathbb{R}^n, \text{-})$ of configuration spaces in $\mathbb{R}^n$. Then $UF(\mathbb{R}^n, \text{-}) = F(\mathbb{R}^n, 2)$ has a hemispherical cellular decomposition (see Berger [3]), and $R_{\mathcal{P}}F(\mathbb{R}^n, 2)$ is an E_n operad containing $F(\mathbb{R}^n, \text{-})$ as a subpreoperad.

3. Consider the little n-cubes operad $\mathcal{C}_n$. Again by Berger [3], $U\mathcal{C}_n = \mathcal{C}_n(2)$ has a hemispherical cellular decomposition. Then $R_{\mathcal{P}}\mathcal{C}_n$ is an E_n operad containing $\mathcal{C}_n$ as a subpreoperad, but **not** as a suboperad (cf. example above).

4. Let X, Y be spaces with free $\mathbb{Z}/2$ actions, having the homotopy type of S^{m-1}, S^{n-1} respectively, and with hemspherical cellular decompositions. Let $\mathcal{P}_m$, $\mathcal{P}_n$, $\mathcal{P}_{m+n}$ denote either the triple $\mathcal{M}_m$, $\mathcal{M}_n$, $\mathcal{M}_{m+n}$ or $\mathcal{K}^{(m)}$, $\mathcal{K}^{(n)}$, $\mathcal{K}^{(m+n)}$. Then the join $X * Y$ has the homotopy type of S^{m+n-1}, with a free $\mathbb{Z}/2$ action and with a hemispherical cellular decomposition. Hence $R_{\mathcal{P}_{m+n}}(X * Y)$ is an E_{m+n} operad containing the E_m operad $R_{\mathcal{P}_m}X$ and the E_n operad $R_{\mathcal{P}_n}Y$ as suboperads. Moreover $R_{\mathcal{P}_{m+n}}(X * Y)$ is a kind of "homotopy tensor product" of the operads $R_{\mathcal{P}_m}X$ and $R_{\mathcal{P}_n}Y$: if $R_{\mathcal{P}_{m+n}}(X * Y)$ acts on a space Z then any element of $R_{\mathcal{P}_m}X(k)$ determines a map $Z^k \to Z$, which is a homotopy homomorphism with respect to the $R_{\mathcal{P}_n}Y$ actions on Z^k and Z, and similarly with the roles of $R_{\mathcal{P}_m}X$ and $R_{\mathcal{P}_n}Y$ reversed (cf. [4] and [5] and the discussion of tensor products of operads below.)

5. Let Y_1, Y_2, ... , Y_n be arbitrary contractible spaces. Then by iteration of the previous example, the iterated join

$$X = (Y_1 \times \mathbb{Z}/2) * (Y_2 \times \mathbb{Z}/2) * \cdots * (Y_n \times \mathbb{Z}/2)$$

has the homotopy type of S^{n-1} with free $\mathbb{Z}/2$ action and a hemispherical cellular decomposition. Then $R_{\mathcal{P}_n}X$ is an E_n operad containing the E_1 operads $R_{\mathcal{P}_1}Y_i$ as suboperads, and $R_{\mathcal{P}_n}X$ can be regarded as a homotopy tensor product of these suboperads.

I now return to the proof of part (iii) of Theorem 1. As I mentioned previously, this is accomplished via a more structured right adjoint construction.

Definition. An *ℓ-truncated operad* is a collection of topological spaces $\{\mathcal{A}(i)\}_{0\leq i\leq \ell}$ (with $\mathcal{A}(0) = *$) with a Σ_i action on $\mathcal{A}(i)$ and with compositions

$$\mathcal{A}(k) \times \mathcal{A}(i_1) \times \mathcal{A}(i_2) \times \ldots \mathcal{A}(i_k) \longrightarrow \mathcal{A}(i_1 + i_2 + \cdots + i_k)$$

defined whenever the spaces on both sides are defined, with these structures satisfying all the relations required of an ordinary operad. The forgetful functor

$$\text{Operads} \longrightarrow \ell\text{-truncated operads}$$

will be denoted T_ℓ.

Example. A 1-truncated operad is just a monoid $\mathcal{A}(1)$ (since $\mathcal{A}(0) = *$).

Example. A 2-truncated operad consists of the following data:

1. A monoid $\mathcal{A}(1)$.
2. A $\mathbb{Z}/2$-space $\mathcal{A}(2)$.
3. A left action
$$\mathcal{A}(1) \times \mathcal{A}(2) \longrightarrow \mathcal{A}(2)$$
equivariant with respect to the $\mathbb{Z}/2$ action on both sides.
4. A right action
$$\mathcal{A}(2) \times (\mathcal{A}(1) \times \mathcal{A}(1)) \longrightarrow \mathcal{A}(2)$$
commuting with the left $\mathcal{A}(1)$ action and equivariant with respect to the $\mathbb{Z}/2$ action on both sides. ($\mathbb{Z}/2$ acts on the left hand side via its action on $\mathcal{A}(2)$ and by permuting the factors of $\mathcal{A}(1) \times \mathcal{A}(1)$.)
5. A pair of maps
$$(d_1, d_2) : \mathcal{A}(2) \longrightarrow \mathcal{A}(1) \times \mathcal{A}(1)$$
which are equivariant with respect to the left action of $\mathcal{A}(1)$ (diagonally on the right hand side), the right action of $\mathcal{A}(1) \times \mathcal{A}(1)$, and the $\mathbb{Z}/2$ action.

Theorem 3. (i) The forgetful functor

$$T_\ell : \text{Operads} \longrightarrow \ell\text{-truncated operads}$$

has a right adjoint

$$R_\ell : \ell\text{-truncated operads} \longrightarrow \text{Operads}$$

(ii) If $\mathcal{B}$ is an operad with $\mathcal{B}(1) = \{1_{\mathcal{B}}\}$, then $R_2 T_2 \mathcal{B} = RU\mathcal{B}$ (where U and R are as in Theorem 1).

Proof Sketch. We describe the space $R_\ell\mathcal{A}(k)$ as the space of all orientations and labellings of the $(\ell-1)$-skeleton of the $(k-1)$-simplex with vertices $\{1, 2, \ldots, k\}$. By an orientation of an $(i-1)$-subsimplex $(i \leq \ell)$ we mean a choice of total order of its vertices. By a labelling of such a subsimplex we mean assigning a point in $\mathcal{A}(i)$ corresponding to such a simplex. We impose an equivalence relation on the orientations and labellings by specifying that changing the total order of the vertices of an i-simplex by means of a permutation in Σ_i is equivalent to acting on its label by the same permutation. We further impose the consistency condition on the orientations and labellings of the subsimplices. The orientation and labelling on a face of a subsimplex must be equivalent to the one obtained by restricting the total order to the vertices and acting via the appropriate degeneracy. (The total order on the vertices of the $(i-1)$-subsimplex determines a bijection with the set $\{1, 2, \ldots, i\}$ and thus determines the appropriate degeneracy.) More formally we can specify the points of $R_\ell\mathcal{A}(k)$ as functions defined on the set of pairs (S, λ), where S is a nonvoid subset of $\{1, 2, \ldots, k\}$ of cardinality $\leq \ell$ and λ is a total order on S, taking values in the disjoint union $\mathcal{A}(1) \amalg \mathcal{A}(2) \amalg \cdots \amalg \mathcal{A}(\ell)$, and satisfying the conditions described above.

The action of Σ_k on $R_\ell\mathcal{A}(k)$ is via its action on the vertices and orientations. The operad composition

$$R_\ell\mathcal{A}(k) \times R_\ell\mathcal{A}(i_1) \times R_\ell\mathcal{A}(i_2) \times \cdots \times R_\ell\mathcal{A}(i_k) \longrightarrow R_\ell\mathcal{A}(i_1 + i_2 + \cdots + i_k)$$

can be described as follows. Consider an $(i-1)$ simplex with vertices in $\{1, 2, \ldots, i_1 + i_2 + \cdots + i_k\}$. Group them into blocks corresponding to the factors on the left hand side. Then order the vertices within the j-th block using the order in $R_\ell\mathcal{A}(i_j)$. Order vertices in different blocks using the order specified in $R_\ell\mathcal{A}(k)$. If there are p different blocks of vertices and the blocks contain q_1, q_2, $\ldots$, q_p vertices respectively, pick the labels specified in the product space on the left hand side and multiply them together using the ℓ-truncated operad composition

$$\mathcal{A}(p) \times \mathcal{A}(q_1) \times \mathcal{A}(q_2) \times \cdots \times \mathcal{A}(q_p) \longrightarrow \mathcal{A}(q_1 + q_2 + \cdots + q_p)$$

The unit of the adjunction $\mathcal{B} \to R_\ell T_\ell \mathcal{B}$ is given by appropriate degeneracy maps. Part (ii) is obvious (which also proves Theorem 1(iii)).

Remark. The construction R_1 is due to Igusa [10]. He calls this construction the "atomic operad generated by a monoid". We have

$$R_1 M(k) = M^k$$

(where $M = \mathcal{A}(1)$ is the given monoid, a.k.a. 1-truncated operad) with the operad composition

$$M^k \times M^{i_1} \times M^{i_2} \times \cdots \times M^{i_k} \longrightarrow M^{i_1+i_2+\cdots+i_k}$$

specified by the formula

$$\begin{aligned} &(m_1, m_2, \ldots, m_k; x_{11}, \ldots, x_{1i_1}, x_{21}, \ldots, x_{2i_2}, \ldots, x_{k1}, \ldots, x_{ki_k}) \\ &\quad \mapsto \quad (m_1 x_{11}, \ldots, m_1 x_{1i_1}, m_2 x_{21}, \ldots, m_2 x_{2i_2}, \ldots, m_k x_{k1}, \ldots, m_k x_{ki_k}) \end{aligned}$$

(It would be more accurate to call $R_1 M$ the operad "cogenerated by M", since this is a right adjoint construction and the forgetful functors T_ℓ also have left adjoints.)

Remark. The little n-cubes operads $\mathcal{C}_n$ are 2-cogenerated, ie.

$$\mathcal{C}_n \xrightarrow{\cong} R_2 T_2 \mathcal{C}_n.$$

To see this note that a k-fold configuration of little n-cubes in $\mathcal{C}_n(k)$ is determined by specifying each little n-cube in the configuration, ie. by the vertex labels in $R_2 T_2 \mathcal{C}_n(k)$. The labelling of the edges in $R_2 T_2 \mathcal{C}_n(k)$ specifies that each pair of little n-cubes in the configuration specified by the vertex labels determines an element of $\mathcal{C}_n(2)$. This is equivalent to specifying that the interiors of any two little n-cubes in the configuration specified by the vertex labels are disjoint. Thus the configuration specified by the vertex labels in $R_2 T_2 \mathcal{C}_n(k)$ determines a unique element of $\mathcal{C}_n(k)$. A similar argument show that the linear isometries E_∞ operad $\mathcal{L}$ (cf. [4]) is 2-cogenerated. Recall that the elements of $\mathcal{L}(k)$ are linear isometries $(\mathbb{R}^\infty)^k \to \mathbb{R}^\infty$, or equivalently a

k-tuple of linear isometries $\mathbb{R}^\infty \to \mathbb{R}^\infty$ whose images are mutually orthogonal. The vertex labels of an element of $R_2T_2\mathcal{L}(k)$ specify a k-tuple of linear isometries $\mathbb{R}^\infty \to \mathbb{R}^\infty$, whereas the edge labels specify that their images are mutually orthogonal. It follows that we have a homeomorphism of operads

$$\mathcal{L} \xrightarrow{\cong} R_2T_2\mathcal{L}.$$

I will now turn to an additional method of constructing E_n operads, via generalized tensor products.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be operads acting on a topological space X. We say that the actions of $\mathcal{A}$ and $\mathcal{B}$ are *interchangeable* if for any k and any element $\alpha \in \mathcal{B}(k)$ the induced map $\alpha : X^k \longrightarrow X$ is a homomorphism of $\mathcal{B}$-spaces, where $\mathcal{B}$ acts on X^k coordinatewise via its action on X. Equivalently for any $\beta \in \mathcal{B}(\ell)$ we have

$$\alpha \cdot (\beta \times \beta \times \cdots \times \beta)(x_{\substack{1\leq i\leq k\\ 1\leq j\leq \ell}}) = \beta \cdot (\alpha \times \alpha \times \cdots \times \alpha)\tau(x_{\substack{1\leq i\leq k\\ 1\leq j\leq \ell}}),$$

where $\tau : X^{k\ell} \to X^{k\ell}$ is the permutation which reorders the coordinates of $X^{k\ell}$ from lexicographic to reverse lexicographic order.

Definition. Let $\mathcal{A} \to \mathcal{C}$ and $\mathcal{B} \to \mathcal{C}$ be operad maps. We say that these operad maps are *interchangeable* if the following diagrams commute for all k and ℓ:

$$(*) \qquad \begin{array}{ccccc}
\mathcal{A}(k) \times \mathcal{B}(\ell) & \xrightarrow{id\times\Delta} & \mathcal{A}(k) \times \mathcal{B}(\ell)^k & \longrightarrow & \mathcal{C}(k) \times \mathcal{C}(\ell)^k \\
\downarrow{\scriptstyle \Delta\times id} & & & & \downarrow{\scriptstyle \mu} \\
\mathcal{A}(k)^\ell \times \mathcal{B}(\ell) & & & & \mathcal{C}(k\ell) \\
\downarrow{\scriptstyle \cong} & & & & \uparrow{\scriptstyle \tau} \\
\mathcal{B}(\ell) \times \mathcal{A}(k)^\ell & \longrightarrow & \mathcal{C}(\ell) \times \mathcal{C}(k)^\ell & \xrightarrow[\mu]{} & \mathcal{C}(k\ell)
\end{array}$$

Here μ denotes composition in the operad $\mathcal{C}$ and τ denotes the same permutation as in the preceding definition.

It is clear that if $\mathcal{C}$ acts on a space X, then the induced actions by $\mathcal{A}$ and $\mathcal{B}$ on X are interchangeable. It was shown in [4] (cf. also [5]) that for any two operads $\mathcal{A}$ and $\mathcal{B}$ there is a pair of operad maps $\mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$, $\mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$ which are universal for interchangeable pairs of operad maps, ie. any other interchangeable pair $\mathcal{A} \to \mathcal{C}$, $\mathcal{B} \to \mathcal{C}$ factors through a unique operad map $\mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$. The operad $\mathcal{A} \otimes \mathcal{B}$ is called the *tensor product* of $\mathcal{A}$ and $\mathcal{B}$.

In general analyzing the homotopy type of the tensor product of operads is an intractable problem. However Dunn [7] showed that the n-fold iterated tensor product of little 1-cubes operads is an E_n operad, more precisely the suboperad of decomposable little n-cubes in $\mathcal{C}_n$. We shall see below another example of a tensor product of operads which can be shown to be E_n. It is much easier to construct examples of E_n operads which contain pairs of interchangeable suboperads. We shall refer to such examples as generalized tensor products. More precisely if $\mathcal{A}$ is an E_m operad, $\mathcal{B}$ is an E_n operad, and $\mathcal{C}$ is an E_{m+n} operad containing $\mathcal{A}$ and $\mathcal{B}$ as interchangeable suboperads, we call $\mathcal{C}$ a *generalized tensor product* of $\mathcal{A}$ and $\mathcal{B}$.

Definition. A *partial acyclic orientation* of the complete graph on the set of vertices $\{1, 2, 3, \dots, k\}$ is an assignment of direction to some of the edges of the graph such that no directed cycles occur. A *partial coloring* of the complete graph on k vertices is an assignment of colors to some of the edges of the graph from the countable set of colors $\{1, 2, 3, \dots\}$. The poset $\widehat{\mathcal{K}}(k)$ has as elements pairs (μ, σ), where μ is a partial coloring and σ is a partial acyclic orientation of the complete graph on k vertices, with the condition that uncolored edges are also unoriented and vice-versa. The order relation on $\widehat{\mathcal{K}}(k)$ is determined as follows: we say that $(\mu_1, \sigma_1) \leq (\mu_2, \sigma_2)$ if every uncolored unoriented edge in (μ_1, σ_1) is also uncolored unoriented in (μ_2, σ_2), and for any colored oriented edge $a \xrightarrow{i} b$ in (μ_1, σ_1) the corresponding edge in (μ_2, σ_2) is either uncolored unoriented or has either orientation and coloring $a \xrightarrow{j} b$ with $j \geq i$ or $b \xrightarrow{j} a$ with $j > i$. The n-th filtration $\widehat{\mathcal{K}}^{(n)}(k)$ is the subposet of $\mathcal{K}(k)$ where the colorings are restricted to take values in the subset $\{1, 2, 3, \dots, n\}$.

The action of the symmetric group Σ_k on $\widehat{\mathcal{K}}(k)$ is via permutation of the vertices. The composition

$$\widehat{\mathcal{K}}(k) \times \widehat{\mathcal{K}}(m_1) \times \widehat{\mathcal{K}}(m_2) \times \cdots \times \widehat{\mathcal{K}}(m_k) \longrightarrow \widehat{\mathcal{K}}_(m_1 + m_2 + \cdots + m_k)$$

assigns to a tuple of partial orientations and colorings in $\widehat{\mathcal{K}}(k) \times \widehat{\mathcal{K}}(m_1) \times \widehat{\mathcal{K}}(m_2) \times \cdots \times \widehat{\mathcal{K}}(m_k)$ the partial orientation and coloring obtained by subdividing the set of $m_1 + m_2 + \cdots + m_k$ vertices into k blocks containing m_1, m_2, ... , m_k vertices respectively. The edges connecting vertices within the

i-th block are oriented and colored (or unoriented and uncolored) according to the given element in $\widehat{\mathcal{K}}(m_i)$. The edges connecting vertices between blocks i and j are all oriented and colored (or not) according to the corresponding edge in the given element of $\widehat{\mathcal{K}}(k)$. It is easy to check that this specifies the structure of a filtered operad on $\widehat{\mathcal{K}}$ containing the complete graphs operad $\mathcal{K}$ as a filtered suboperad. Note however that $\widehat{\mathcal{K}}^{(n)}$ is not E_n: $\widehat{\mathcal{K}}^{(n)}(k)$ is equivariantly contractible to the Σ_k fixed point specified by the complete graph on $\{1, 2, 3, \ldots, k\}$ with all its edges unoriented and uncolored. We shall refer to $\widehat{\mathcal{K}}$ and its filtrations as the *augmented complete graphs operad*.

Definition. An E_n operad $\mathcal{A}$ is said to have an *augmented cellular decomposition* over the complete graphs operad if

1. $\mathcal{A}$ has a cellular decomposition over $\mathcal{K}^{(n)}$.
2. $R_1\mathcal{A}(1)$ has a cellular decomposition over $\widehat{\mathcal{K}}^{(n)}$, where R_1 is the right adjoint construction from monoids to operads of Theorem 3.
3. The adjunction map $\mathcal{A} \to R_1\mathcal{A}(1)$ is an imbedding onto the union of cells indexed by $\mathcal{K}^{(n)}$.

The basic example of an E_n operad with an augmented cellular decomposition over the complete graphs operad is the little n-cubes operad $\mathcal{C}_n$. We specify the cellular decomposition of $(R_1\mathcal{C}_n(1))(k)$ over $\widehat{\mathcal{K}}^{(n)}(k)$ as follows. Given a partial orientation and coloring of the complete graph on $\{1, 2, 3, \ldots, k\}$ we define the corresponding cell to be the subspace of $\mathcal{C}_n(1)^k$ consisting of k-tuples of subcubes of the unit n-cube satisfying

1. For every oriented and colored edge $a \xrightarrow{i} b$, the interior of the a-th subcube must be separated from the interior of the b-subcube by a hyperplane perpendicular to the j-th coordinate axis for some $j \leq i$. If $j = i$, then the a-th subcube is required to lie on the negative side of the hyperplane, and the b-th subcube on the positive side.
2. If the edge joining a to b is unoriented and uncolored, no condition is imposed on the relative positions of the a-th and b-th subcubes of the k-tuple, eg. their interiors are allowed to intersect, even coincide.

It is not difficult to check that this is a cellular decomposition satisfying the requirements of the definition, using the arguments of [3].

Remark. It may be worthwhile to note that requirements for an operad to have an augmented cellular decomposition over the complete graphs operad are completely antithetical to the requirement $\mathcal{A}(1) = \{1_{\mathcal{A}}\}$ for the construction of Theorem 1 to have good properties. Moreover while there are many examples of finite simplicial E_n operads satisfying $\mathcal{A}(1) = \{1_{\mathcal{A}}\}$, this is precluded for operads with an augmented cellular decomposition. Indeed the mere condition that

$$(d_1, d_2) : \mathcal{A}(2) \longrightarrow (R_1\mathcal{A}(1)) = \mathcal{A}(1)^2$$

be injective precludes $\mathcal{A}(k)$ from being a finite simplicial complex for any $k > 0$.

To see this, pick any vertex $c \in \mathcal{A}(2)$ and let $a = d_1(c) \in \mathcal{A}(1)$. Then, assuming that $\mathcal{A}(1)$ is a finite complex, the positive powers a^m (with respect to the monoid structure on $\mathcal{A}(1)$) can't be all distinct. Suppose $a^m = a^{m+r}$. Consider the element $c' = a^m c\,(a^{r-1}d_2(c), 1)$, where we use the left action of $\mathcal{A}(1)$ on $\mathcal{A}(2)$ together with the right action of $\mathcal{A}(1) \times \mathcal{A}(1)$. Then

$$(d_1, d_2)(c') = \big(a^{m+k}d_2(c), a^m d_2(c)\big) = \big(a^m d_2(c), a^{m+k}d_2(c)\big) = (d_1, d_2)(c'\,\tau),$$

where τ is the transposition in Σ_2. It now follows that $c' = c'\,\tau$, contradicting freeness of the Σ_2 action. Thus $\mathcal{A}(1)$ can't be a finite complex. It can be shown that for any $k \geq 2$, $\mathcal{A}(k)$ can't be a finite complex, by assuming the contrary, taking a vertex $c \in \mathcal{A}(k)$, letting $a = d_1 d_1 \dots d_1(c) \in \mathcal{A}(1)$, noting that $a^m c$ can't all be distinct, and arguing as above.

Theorem 4. (i) Let $\mathcal{A}$ be an E_m operad with an augmented cellular decomposition over the complete graphs operad. Let X be a space homotopy equivalent to S^{n-1}, with a free $\mathbb{Z}/2$ action and a hemispherical cellular decomposition. Then there is a generalized tensor product $\mathcal{A}\hat{\otimes}R_{\mathcal{K}(n)}X$, an E_{m+n} operad containing $\mathcal{A}$ and $R_{\mathcal{K}(n)}X$ as interchangeable suboperads.
(ii) If $X = S^0$, then

$$\mathcal{A}\hat{\otimes}R_{\mathcal{K}(n)}S^0 = \mathcal{A} \otimes R_{\mathcal{K}(n)}S^0 = \mathcal{A} \otimes \mathcal{M},$$

the actual tensor product of $\mathcal{A}$ with $\mathcal{M}$, the A_∞ operad which acts on (strict) monoids.

Proof Sketch. First consider the product operad $R_1\mathcal{A}(1) \times R_{\mathcal{K}(n)}X$. The elements of the k-th space of this operad can be described as the complete graph on the set $\{1, 2, \ldots, k\}$, with the vertices being labelled by elements of $\mathcal{A}(1)$ and the edges labelled by elements of X.

Now let us impose the quotient relation that if the endpoints of an edge are labelled by a pair of elements in $\mathcal{A}(1) \times \mathcal{A}(1)$ which is in the image of $\mathcal{A}(2)$, then we are allowed to replace the X label on that edge by any other element of X. It is straightforward to check that the composition operation in $R_1\mathcal{A}(1) \times R_{\mathcal{K}(n)}X$ passes to the resulting collection of quotient spaces. Thus we obtain an operad we denote $\mathcal{A}\hat{\otimes}R_{\mathcal{K}(n)}X$.

There is an evident inclusion of operads $R_{\mathcal{K}(n)}X \subset \mathcal{A}\hat{\otimes}R_{\mathcal{K}(n)}X$, given by labelling all vertices in the complete graph by $1_{\mathcal{A}}$. There is also an inclusion of operads $\mathcal{A} \subset \mathcal{A}\hat{\otimes}R_{\mathcal{K}(n)}X$ given by labelling all edges of the complete graph by arbitrary elements of X (which is well-defined by the quotient relation we have imposed).

These suboperads are easily seen to be interchangeable: if we take $\mathcal{B} = R_{\mathcal{K}(n)}X$ in diagram (*) and chase an element $(\alpha, \beta) \in \mathcal{A}(k) \times (R_{\mathcal{K}(n)}X)(\ell)$, around the two sides of the diagram, we obtain the same labelling of the complete graph on $k\ell$ vertices either way. For if we identify $\{1, 2, \ldots, k\ell\}$ with the product $\{1, 2, \ldots, k\} \times \{1, 2, \ldots, \ell\}$ via lexicographic ordering on the latter, then the labelling of the complete graph obtained by going along the top and right of the diagram assigns to vertex (i, j) the label $\alpha_i = \alpha(* \times * \times \ldots 1_{\mathcal{A}} \times \cdots \times *)$, where $1_{\mathcal{A}}$ is in the i-th place. The label assigned to the edge joining vertices (i_1, j_1) and (i_2, j_2) is $\beta_{j_1 j_2}$ provided that $i_1 = i_2$ and $j_1 \neq j_2$, otherwise it is unspecified (which is allowed by the quotient relation imposed above). If we go around the other side of the diagram, we obtain the same labelling on the vertices. The labelling assigned to the edge joining vertices (i_1, j_1) and (i_2, j_2) is $\beta_{j_1 j_2}$ provided that $j_1 \neq j_2$. However the quotient relation allows us to discard those labels when $i_1 \neq i_2$, so we obtain the same labelling as before.

We show that $\mathcal{A}\hat{\otimes}R_{\mathcal{K}(n)}X$ is an E_{m+n} operad by displaying a cellular decomposition over $\mathcal{K}^{(m+n)}$. Given an element $\lambda = (\mu, sigma) \in \mathcal{K}^{(m+n)}(k)$, the corresponding cell in $\left(\mathcal{A}\hat{\otimes}R_{\mathcal{K}(n)}X\right)(k)$ can be described as follows. Consider the cell in $\mathcal{A}(1)^k$ specified by $(\mu_1, \sigma_1) \in \widehat{\mathcal{K}}^{(m)}(k)$ specified by deleting directions and colorings of those edges of the complete graph whose colors are $> m$. This specifies the allowable labellings of vertices of the complete graph of the elements in $\left(\mathcal{A}\hat{\otimes}R_{\mathcal{K}(n)}X\right)(k)$. Now for each edge whose color is $j > m$, let the allowable labellings of that edge be elements of X specified by the given orientation of that edge and color $j-m$. Then this cell in $\left(\mathcal{A}\hat{\otimes}R_{\mathcal{K}^{(n)}}X\right)(k)$ is homeomorphic to the pushout of the following diagram

$$\begin{tikzcd}
\bigcup_{\lambda_1'<\lambda_1} C_{\lambda_1'} \times D_{\lambda-\lambda_1} \arrow[r, hook] \arrow[d] & C_{\lambda_1} \times D_{\lambda-\lambda_1} \\
\bigcup_{\lambda_1'<\lambda_1} C_{\lambda_1'} \times D_{\lambda-\lambda_1'} &
\end{tikzcd}$$

where $\lambda_1(\mu_1, \sigma_1) \in \widehat{\mathcal{K}}^{(m)}(k)$, and C_{λ_1} is the corresponding cell in $\mathcal{A}(1)^k$. $D_{\lambda-\lambda_1} \subset X^r$, where r denotes the number of edges which are not assigned a coloring and orientation by λ_1, and which therefore specify elements of X as indicated above. Similarly for $\lambda_1' < \lambda_1 \in \widehat{\mathcal{K}}^{(m)}(k)$. This pushout is contractible since $C_{\lambda_1} \times D_{\lambda-\lambda_1}$ is contractible, the horizontal arrow is a cofibration, and the vertical arrow is an equivalence.

Part (ii) follows by comparing with the explicit description of $\mathcal{A} \otimes \mathcal{M}$ given in [5].

Remark. In general $\mathcal{A} \otimes R_{\mathcal{K}^{(n)}}X$ is obtained from $R_1\mathcal{A}(1) \times R_{\mathcal{K}^{(n)}}X$ by imposing a finer equivalence than that used to define $\mathcal{A}\hat{\otimes}R_{\mathcal{K}^{(n)}}X$. Although one could define "cells" in the resulting operad similarly to the above, it is difficult to check under what circumstances they are contractible. One would certainly need to impose further conditions on X.

Theorem 5. Let $\mathcal{A}$ be an E_m operad and let $\mathcal{B}$ be an E_n operad, both with augmented cellular decompositions over the complete graphs operad. Then there is a generalized tensor product $\mathcal{A}\hat{\otimes}\mathcal{B}$, an E_{m+n} operad containing $\mathcal{A}$ and $\mathcal{B}$ as interchangeable suboperads.

Proof Sketch. First consider the product operad $R_1\mathcal{A}(1) \times R_1\mathcal{B}$. The elements of the k-th space of this operad can be described as the complete graph on the set $\{1, 2, \ldots, k\}$, with the vertices being labelled by elements of $\mathcal{A}(1) \times \mathcal{B}(1)$. We define $\mathcal{A}\hat{\otimes}\mathcal{B}$ to be the suboperad consisting of those elements such that for any pair of vertex labels $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ at least one of (α_1, α_2) or (β_1, β_2) is in $\mathcal{A}(2) \subset \mathcal{A}(1) \times \mathcal{A}(1)$ or $\mathcal{B}(2) \subset \mathcal{B}(1) \times \mathcal{B}(1)$.

Then $\mathcal{A} \subset R_1\mathcal{A} \times R_1\{1_{\mathcal{B}}\}$ and $\mathcal{B} \subset R_1\{1_{\mathcal{A}}\} \times R_1\mathcal{B}$ are both suboperads of $\mathcal{A}\hat{\otimes}\mathcal{B}$, and are easily seen to be interchangeable.

Now for each $(\mu,\sigma) \in \mathcal{K}^{(m+n)}(k)$ define $(\mu_1,\sigma_1) \in \widehat{\mathcal{K}}^{(m)}(k)$ to be the partial coloring and orientation obtained from (μ,σ) by removing the coloring and orientation of any edge colored by a color $> m$. Similarly define $(\mu_2,\sigma_2) \in \widehat{\mathcal{K}}^{(n)}(k)$ to be the partial coloring and orientation obtained from (μ,σ) by removing the coloring and orientation of any edge colored by a color $\leq m$ and changing the color of all the remaining edges according to the rule $\imath \mapsto i-m$. Now let the cell corresponding to (μ,σ) be $C_{(\mu_1,\sigma_1)} \times D_{(\mu_2,\sigma_2)} \subset \mathcal{A}(1)^k \times \mathcal{B}(1)^k$. It is easy to check that this specifes a cellular decomposition of $\mathcal{A}\hat{\otimes}\mathcal{B}$ over $\mathcal{K}^{(m+n)}$.

Remarks. (i) If in the above construction we take $\mathcal{A}$ and $\mathcal{B}$ to be the little m-cubes, resp. little n-cubes operads, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is the little $(m+n)$-cubes operad.
(ii) It is easy to see that $\mathcal{A}\hat{\otimes}\mathcal{B}$ actually is an E_{m+n} operad with an augmented cellular decomposition over the complete graphs operad. Thus the construction can be iterated.

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On Simplicial Loops and H-Spaces

Stephan Klaus

1 Minimal simplicial sets

First, we recall some facts on simplicial sets (see [M], for example). There are the category $\mathcal{T}$ of topological spaces and the category $\mathcal{S}$ of simplicial sets, and the total singular set functor $S : \mathcal{T} \to \mathcal{S}$ and the geometric realization functor $| \ | : \mathcal{S} \to \mathcal{T}$. According to Quillen, these give equivalences of the associated homotopy categories. Here, we have to be careful in defining homotopy for simplicial maps between simplicial sets X and Y: The naive notion of a homotopy as a map $X \times I \to Y$ only gives an equivalence relation on the set $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ of simplicial maps if Y satisfies the *Kan extension condition* (i.e., Y is fibrant). In this case, we have $[X, Y] = [|X|, |Y|]$.

Now, a Kan set Y is called *minimal*, if homotopic simplices coincide. There are two main results on minimal Kan sets:

- Every Kan set Y contains a minimal Kan subset M which is a strong deformation retract.
- A homotopy equivalence between minimal Kan sets is an isomorphism

Thus, the homotopy types of CW-complexes are in bijective correspondence to the isomorphism types of minimal Kan sets. We speak of the *minimal model* of a space.

We are interested in the following problem: How we can see if a minimal Kan set M belongs to an H-space, or A_n-space, or E_n-space? Which additional structures on M reflect these properties?

2 Minimal models of H-spaces

We prove a generalization of a Theorem of James [J]:

Proposition: *A connected space X is an H-space iff its minimal model M is a simplicial loop.*

Recall from [B] that an *algebraic loop* L is a set together with a map $+ : L \times L \to L$ and an element $0 \in L$ such that

- 0 is a two-sided zero for $+$.
- The equation $x + y = z$ has a unique solution after fixing any two elements.

Thus, a loop is a group without associativity. We prefer to write the composition additively despite the lack of associativity and commutativity. We denote the unique solutions of the equation above by $y = [-x + z]$ and $x = [z - y]$. A homomorphism of loops $f : L \to L'$ respects addition. It follows, that f also respects the zeros and unique solutions. We have a category $\mathcal{L}$ of loops and thus also the notion of a simplicial object in $\mathcal{L}$.

Proof of the proposition: Deform X such that the basepoint 0 acts as a strict unit. Let M be the minimal model of X which is a simplicial H-set $+ : X \times X \to X$ such that 0 acts as a two-sided unit. ($+$ is defined using the functor S and the retraction $S(X) \to M$.) The shearing map $M \times M \to M \times M$, $(x, y) \mapsto (x, x+y)$, induces an isomorphism on the homotopy groups, hence is a homotopy equivalence. As $M \times M$ is minimal, it is an isomorphism, so $x + y = z$ has a unique solution y for any x, z. Similar for x.

Corollary [J]: *Let X be a connected H-space and Y a CW-complex. Then $[Y, X]$ is an algebraic loop.*

Corollary [KKM]: *Let E^* be a generalized cohomology theory. Then E^* can be represented as the homology of a loop-valued cochain functor C^*.*

Remark: There is a 'no go' Theorem [BCF], which says that this is not possible with an abelian group-valued cochain functor.

Sketch of proof of the second corollary: Represent E^* by an Ω-spectrum and consider the minimal models M^n of the representing spaces E^n. By $E^n \simeq \Omega E^{n+1}$, we see that we can identify M^n with the 'small loop space' LM^{n+1}. Define the cochains by $C^n(X) := \mathcal{C}(\mathcal{S}(\mathcal{X}), \mathcal{PM}^{\backslash+\infty})$ where PM^{n+1} denotes the 'small path space'. The differential is defined by $PM^n \to M^n = LM^{n+1} \to PM^{n+1}$.

3 Simplicial loops

We recall from [B] some basics on loops: A subset U of a loop L is called a subloop if $0 \in U$ and U is closed under $x+y$, $[-x+y]$, $[x-y]$. A subloop N of L is called normal if $x+N = N+x$, $x+(y+N) = (x+y)+N$ and $(N+x)+y = N+(x+y)$. Then one can define the quotient loop $L/N := \{x+N\}$ by $(x+N)+(y+N) := (x+y)+N$. The center $Z(L)$ of a loop L is the set of elements a such that $a+x = x+a$, $a+(x+y) = (a+x)+y$, $(x+y)+a = x+(y+a)$ for all x, y. This is a normal abelian subgroup of L. A central extension is an epimorphism $p: L \to Q$ of loops such that $A := ker(p) \subset Z(L)$. Equivalently, L is isomorphic to $A \times Q$ with loop structure

$$(a,q)+(a',q') := (a+a'+c(q,q'), q+q')$$

for some map $c: Q \times Q \to A$. c is called the cochain of the extension.

Now, for simplicial groups, Moore has proved the following results:

- An epimorphism of simplicial groups is a Kan fibration. In particular, any simplicial group is Kan.
- The homology of the Moore complex N_* of a simplicial group (which is a chain complex of nonabelian groups) gives its homotopy groups.
- A simplicial group is minimal iff its Moore differential is trivial.

These results are not true for simplicial monoids. But we have shown:

Theorem: *All remains true in the case of simplicial loops. In particular, one can define the Moore complex, which is a chain complex of loops and gives the homotopy groups by taking its homology.*

Now, we consider the natural simplicial Postnikov decomposition of a simplicial loop. We recall that the n-th Postnikov stage $X^{(n)}$ of a Kan set X is defined by identifying simplices which have equal faces of dimension $\leq n$.

Theorem: *Let L be a connected minimal simplicial loop. Then $L^{(n)} \to L^{(n-1)}$ is a central loop extension.*

Sketch of proof: The kernel of the above epimorphism is the minimal Eilenberg-MacLane 'space' $K := K(\pi, n)$ with $\pi := \pi_n(L)$. This shows that K is an abelian normal subloop. In order to prove that it is central, we consider the simplicial map

$$f: K \times L^{(n)} \to K, \quad f(a,x) := [(x+a)-x].$$

Using the adjoint map $L^{(n)} \to K^K$ and the known description of the endomorphism complex of an EM-space, we get $f(a, x) = a$, hence $x + a = a + x$. Similar for the equations $(x+y)+a = x+(y+a)$ and $a+(x+y) = (a+x)+y$.

4 2-antilocal H-spaces

Using difficult methods of equivariant homotopy theory, Iriye and Kono proved that the localization of a connected H-space at an odd prime admits a homotopy commutative H-space structure [IK]. We show the following generalization:

Theorem: *Let X be a connected H-space which is 2-antilocal (i.e., multiplication by 2 is an isomorphism in all homotopy groups). Then its minimal model M has a strict commutative simplicial loop structure.*

Sketch of proof: Consider the following procedure of constructing a new simplicial loop structure $+'$ on M from a given one $+$: The doubling map $M \to M$, $x \mapsto x + x$, induces an isomorphism in homotopy and so is an isomorphism of simplicial sets. Denote the inverse map by $\frac{1}{2} : M \to M$ and form

$$x +' y := \frac{1}{2}((x + y) + (y + x)).$$

It is straightforward to check that this is again a simplicial loop structure on M. Caution: if $+$ is not commutative, $+'$ in general is also not commutative. But if $+$ is commutative in a range of dimensions, we have $+' = +$ in the same range. Moreover, we show that if $+$ is commutative on $M^{(n)}$, then $+'$ is commutative on $M^{(n+1)}$. This follows easily from the structure of central extensions. Thus, the sequence $+$, $+'$, $+''$, ... converges to a strictly commutative simplicial loop structure on M.

Corollary: *Let E^* be a generalized cohomology theory such that 2 is a unit in the coefficients. Then E^* can be represented as the homology of a commutative loop-valued cochain functor C^*.*

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Stephan Klaus
Universität Mainz, FB 17 Mathematik
D-55099 Mainz, Germany
klaus@topologie.mathematik.uni-mainz.de

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Homotopy Algebras via Resolutions of Operads

Based on a talk given on June 16, 1998,
at University of Osnabrück, Germany

Martin Markl

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All algebraic objects in this note will be considered over a fixed field $\mathbf{k}$ of characteristic zero. If not stated otherwise, all operads live in the category of differential graded vector spaces over $\mathbf{k}$. For standard terminology concerning operads, algebras over operads, etc., see either the original paper by J.P. May [May72], or an overview [Lod].

The aim of this note is mainly to advocate our approach to homotopy algebras based on the minimal model of an operad. Our intention is to expand this note to a paper on homotopy properties of the category of homotopy algebras; some possible results in this direction are indicated in Section 3.

I would like to express my thanks to Rainer Vogt for his kind invitation to Osnabrück. I also owe my thanks to Jim Stasheff for careful reading the manuscript and many useful comments.

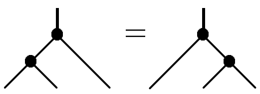
1 Motivations

Example 1.1. An associative algebra consists of a vector space A together with a multiplication $\mu : A \otimes A \to A$, which is supposed to be associative:

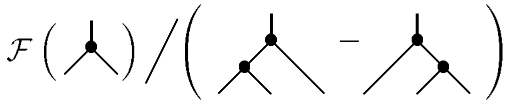
$$\mu(\mu(a,b),c) = \mu(a,\mu(b,c));$$

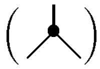
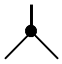
we neglect the role of a unit.

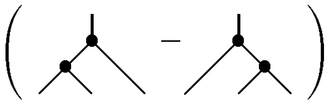
We can 'visualize' μ as an 'operation' with two inputs and one output, $\mu =$ . The associativity can be then depicted as



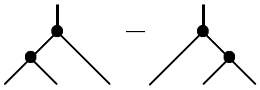
This means that the operad Ass describing associative algebras in the category of differential graded vector spaces can be presented as the quotient



where $\mathcal{F}$  is the free non-Σ operad on the operation  representing the product μ and



is the operadic ideal generated by the element



corresponding to the associativity axiom. As differential graded vector spaces, the pieces $Ass(n)$, $n \geq 1$, of the operad Ass are concentrated in degree zero and have trivial differentials.

Homotopy versions of associative algebras are so-called A(∞)-algebras (also called strongly homotopy associative algebras), introduced by J. Stasheff [Sta63]. An A(∞)-algebra $A = (A, \partial, \mu_2, \mu_3, \dots)$ consists of a differential graded vector space $A = (A, \partial)$ and multilinear operations $\mu_n : A^{\otimes n} \to A$ of degree $n-2$ that satisfy the following infinite set of axioms:

$$\begin{aligned} \mu_2(\mu_2(a,b)c) - \mu_2(a, \mu_2(b,c)) &= [\mu_3, \partial](a,b,c), \qquad (3) \\ \mu_3(\mu_2(a,b),c,d) - \mu_3(a,\mu_2(b,c),d) + & \\ +\mu_3(a,b,\mu_2(c,d)) - \mu_2(\mu_3(a,b,c),d) - & \\ -(-1)^{|a|}\cdot\mu_2(a,\mu_3(b,c,d)) &= [\mu_4, \partial](a,b,c,d), \\ &\;\vdots \\ \sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{s=0}^{n-j} (-1)^{\epsilon}\cdot\mu_i(a_1,\dots,a_s,\mu_j(a_{s+1},\dots,a_{s+j}),a_{s+j+1},\dots,a_n) &= \\ &= [\mu_n,\partial](a_1,\dots,a_n), \end{aligned}$$

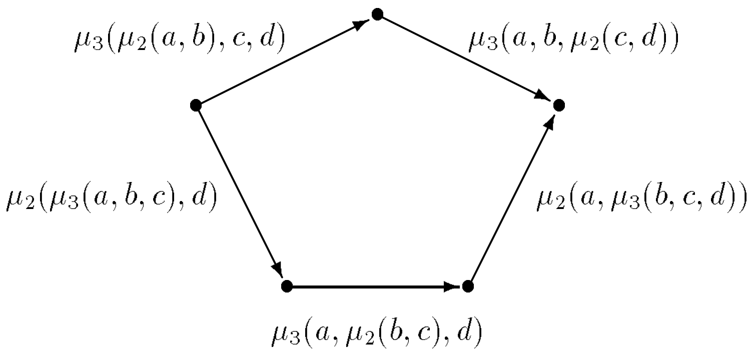
where $[\mu_n, \partial](a_1, \dots, a_n)$ denotes the induced differential in the endomorphism complex $\mathrm{Hom}_{\mathbf{k}}(A^{\otimes n}, A)$,

$$\begin{aligned} &[\mu_n, \partial](a_1,\dots,a_n) := \\ &\quad := \sum_{1\leq s\leq n} (-1)^{|a_1|+\cdots+|a_{s-1}|}\cdot\mu_n(a_1,\dots,\partial a_s,\dots,a_n) - (-1)^n\cdot\partial\mu_n(a_1,\dots,a_n). \end{aligned}$$

The sign is given by

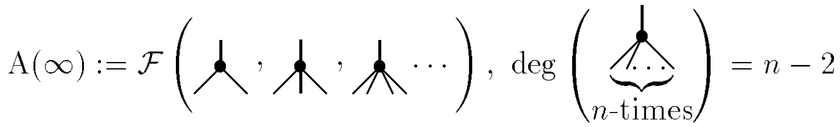
$$\epsilon := j + s(j+1) + j(|a_1| + \cdots + |a_{s-1}|).$$

The first axiom of (3) expresses the homotopy associativity of the product μ_2. The second axiom is a linearized version of the Mac Lane pentagon condition, with the terms in the left hand side corresponding to the edges of the pentagon:

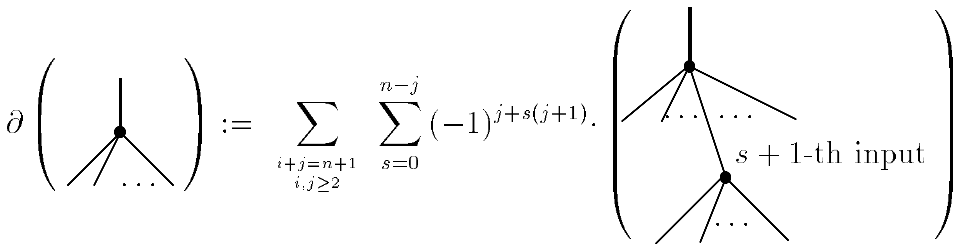


This miraculous phenomenon is explained by the fact that the operad $\mathrm{A}(\infty)$ is isomorphic to the operad $\{CC_*(K_n)\}_{n\geq 1}$ of cellular chains of a certain cellular operad $\{K_n\}_{n\geq 1}$, introduced by J. Stasheff in his pioneering work [Sta63]. The spaces K_n are now called associahedra; the polyhedron K_4 that parameterizes 4-ary operations is the pentagon mentioned above.

A more straightforward description of $\mathrm{A}(\infty)$ can be obtained from the axioms as follows. The differential graded operad $\mathrm{A}(\infty)$ is free as an operad,



with the differential given on generators by

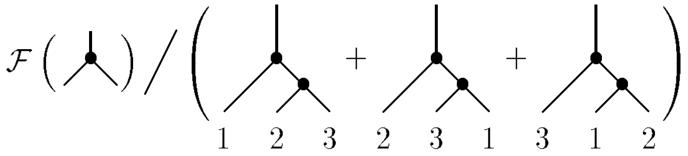


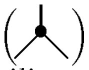
and extended by the derivation property. A(∞)-algebras occur in Nature as chain algebras of loop spaces, a result which is today classical, see again [Sta63].

Example 1.2. A Lie algebra is a vector space L with an antisymmetric product $[-,-] : L \otimes L \to L$ (the 'bracket'), satisfying the Jacoby identity:

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0.$$

Thus the operad *Lie* for Lie algebras is the quotient



Here $\mathcal{F}$  now denotes the free *symmetric* (Σ-) operad on one antisymmetric bilinear operation; the labels of the inputs encode the action of the symmetric group in a standard manner.

Homotopy versions of Lie algebras are strongly homotopy Lie algebras (also called sh Lie algebras or L(∞)-algebras). They were introduced and systematically studied in [LS93], though they had existed in the literature, in various disguises, even before; see also [LM95].

An L(∞) algebra is a differential graded vector space $L = (L, \partial)$, together with a set $\{l_n\}_{n \geq 2}$ of graded antisymmetric operations $l_n : L^{\otimes n} \to L$ of degree $n-2$ such that the following infinite set of axioms is satisfied for any $n \geq 2$:

$$\sum_{\substack{i+j=n+1\\ i,j\geq 2}} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(a_{\sigma(1)}, \ldots, a_{\sigma(i)}), a_{\sigma(i+1)}, \ldots, a_{\sigma(n)}) =$$

$$= (-1)^n \cdot [l_n, \partial](a_1, \ldots, a_n)$$

The summation is taken over all $(i, n-i)$-unshuffles

$$\sigma \in \Sigma_n,\ \sigma(1) < \cdots < \sigma(i),\ \sigma(i+1) < \cdots < \sigma(n),$$

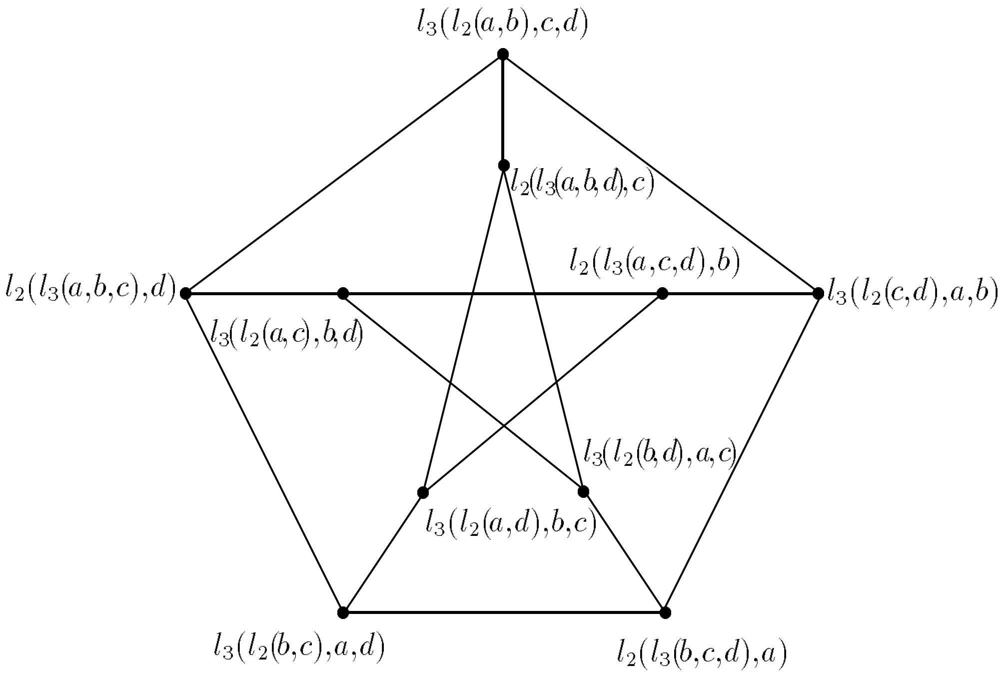
with $n-1 \geq i \geq 1$, and $\chi(\sigma)$ is a certain sign which we will not specify here, see [LM95]. We write the first two axioms explicitly (though, to save paper, without signs). The first axiom says that the 'bracket' l_2 satisfies the Jacobi identity up to a homotopy:

$$l_2(l_2(a,b),c) + l_2(l_2(b,c),a) + l_2(l_2(c,a),b) = [l_3, \partial](a,b,c).$$

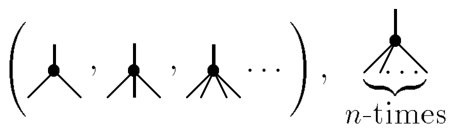
The second condition reads

$$
\begin{aligned}
&l_2(l_3(a,b,c),d) + l_2(l_3(a,c,d),b) + l_2(l_3(a,b,d),c)+\\
&\quad +l_2(l_3(b,c,d),a) + l_3(l_2(a,b),c,d) + l_3(l_2(a,c),b,d) + l_3(l_2(a,d),b,c) +\\
&\quad +l_3(l_2(b,c),a,d) + l_3(l_2(c,d),a,b) + l_3(l_2(b,d),a,c) = [\partial, l_4](a,b,c,d).
\end{aligned}
$$

To indicate an interpretation of the last equality, recall that there is a Lie-analog of K_4, which we introduced in [MS97]. We denoted it by L_4 and called the Lie-hedron. It is not a polyhedron, but just a graph, and the 10 terms in the left hand side of the above equation correspond to the vertices (not edges!) of L_4, which is the Peterson graph:



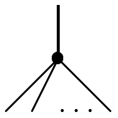
As in Example 1.1, the operad $\mathrm{L}(\infty)$ for strongly homotopy Lie algebras is a differential graded operad, which is free as an operad,

$\mathrm{L}(\infty) := \mathcal{F}$ 

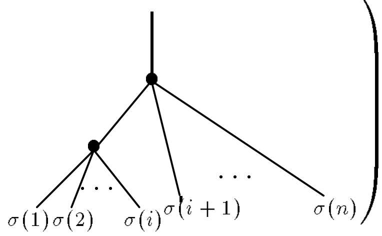
antisymmetric of degree $n-2$,

where $\mathcal{F}(-)$ is the free *symmetric* (Σ-) operad. The action of the differential is given by

$$(-1)^n \cdot \partial \left(\right) :=$$



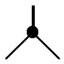
$$:= \sum_{\substack{i+j=n+1 \\ i,j \geq 2}} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} \cdot \left(\right)$$



There is a stunning example of a strongly homotopy Lie algebra taken from Nature – the convolution product of functionals on the space of closed strings, see [SZ89, Sta92]. Another place where strongly homotopy Lie algebras naturally occur is the space of horizontal forms on the infinite jet bundle, where they appear as lifts of brackets on the space of Lagrangian functionals, see [BFLS97, MS98].

Homotopy versions of commutative associative algebras were introduced by T. Kadeishvili in [Kad85]. They are called $\mathrm{C}(\infty)$, commutative, or balanced $\mathrm{A}(\infty)$-algebras (compare also [Mar92]).

Let us define a map $\alpha_{Ass} : \mathrm{A}(\infty) \to Ass$ by



$$\alpha_{Ass}(\quad) = \text{ the multiplication } \mu \in Ass(2),$$

while α_{Ass} is trivial on the remaining generators of $\mathrm{A}(\infty)$. We recommend as an exercise to verify that $\alpha_{Ass} \circ \partial = 0$, which means that α_{Ass} is a map of differential operads, if we interpret Ass as a differential operad with trivial differential. It can be shown that α_{Ass} induces an isomorphism of homology, $H_*(\mathrm{A}(\infty), \partial) \cong Ass$. There is also a map $\alpha_{Lie} : \mathrm{L}(\infty) \to Lie$ having similar properties.

2 Concepts

Let us try to sum up common features of the above examples. In both cases, homotopy algebras were algebras over a differential graded operad which was free as an operad. Moreover, we had natural maps $\alpha_{Ass} : (\mathrm{A}(\infty), \partial) \to (Ass, 0)$ and $\alpha_{Lie} : (\mathrm{L}(\infty), \partial) \to (Lie, 0)$ inducing isomorphisms in homology.

Recall that, in [Mar96], we formulated the following definition.

Definition 2.1. *Let $\mathcal{P}$ be an operad in the category of differential graded vector spaces. A minimal model of $\mathcal{P}$ is a differential graded operad $\mathcal{M}_{\mathcal{P}} = (\mathcal{F}(E), \partial)$, where $\mathcal{F}(E)$ is the free operad on a collection E, together with a map $\alpha_{\mathcal{P}} : \mathcal{M}_{\mathcal{P}} \to \mathcal{P}$ that is a homology isomorphism. The minimality means that we assume that $\partial(E)$ consists of decomposable elements of the free operad $\mathcal{F}(E)$.*

We proved, for each differential graded operad $\mathcal{P}$, the existence and a kind of uniqueness of the minimal model. When $\mathcal{P}$ has trivial differential (which was the case of all our examples), the minimal model is in fact bigraded, but we will not use this property. Having in mind the above examples, we proposed, in [Mar96], the following definition.

Definition 2.2. *A homotopy $\mathcal{P}$-algebra is a differential graded vector space $A = (A, \partial)$ with an action of the minimal model $\mathcal{M}_{\mathcal{P}}$ of $\mathcal{P}$, in other words, a differential graded algebra over the operad $\mathcal{M}_{\mathcal{P}}$.*

For so called Koszul quadratic operads, the minimal model coincides with the cobar construction on the quadratic dual of the operad, and our definition agrees with the one given by V. Ginzburg and A. Kapranov in [GK94].

Our approach keeps the symmetric group action (i.e., on the level of algebras, the commutativity, anticommutativity, etc.) strict, so we cannot handle algebraic variants of homotopy-everything operads such as those considered in [Smi82, Smi94], nor various other homotopy algebras where the symmetry is relaxed up to a homotopy – like homotopy Gerstenhaber algebras in the sense of M. Gerstenhaber and A.A. Voronov [GV95].

In the following section we try to indicate what must be done to justify our approach.

3 Perspectives

There exists a natural concept of a map $\mathbf{f} : A \to B$ of homotopy $\mathcal{P}$-algebras. It is a system of degree zero multilinear maps $\mathbf{f} = \{f_n : A^{\otimes n} \to B\}_{n \geq 1}$ satisfying a certain set of axioms. One of the axioms postulates that $f_1 : (A, \partial) \to (B, \partial)$ is a morphism of the underlying differential graded vector spaces. Let $h\mathcal{P}$ denote the category of homotopy $\mathcal{P}$-algebras in the sense of Definition 2.2 and their maps.

Definition 3.1. *Let A and B be two homotopy $\mathcal{P}$-algebras and $g : (A, \partial) \to (B, \partial)$ a map of underlying differential graded vector spaces. A homotopy $\mathcal{P}$-structure on g is an $h\mathcal{P}$-map $\mathbf{f} : A \to B$ such that $f_1 = g$.*

In order to justify the notion of homotopy $\mathcal{P}$-algebras and their maps, we need to show that homotopy $\mathcal{P}$-structures are 'stable under a homotopy.' The precise meaning of this was explained, for topological spaces, in Chapter 1 of [BV73]. Corresponding statements translated to our algebraic language are the following.

1. For each homotopy $\mathcal{P}$-algebra A, differential graded vector space $B = (B, \partial)$ and a map $g : (A, \partial) \to (B, \partial)$ that is a homology isomorphism, there exist a homotopy $\mathcal{P}$-structure on (B, ∂) and a homotopy $\mathcal{P}$-structure $\mathbf{f} : A \to B$ on g.

2. Suppose A and B are two homotopy $\mathcal{P}$-algebras and $\mathbf{f} : A \to B$ a homotopy $\mathcal{P}$-algebra map. Suppose that $g : (A, \partial) \to (B, \partial)$ is a differential map that induces the same map of homology as f_1. Then there exists a homotopy $\mathcal{P}$-structure on g.

3. Suppose that $\mathbf{f} : A \to B$ is a homotopy $\mathcal{P}$-algebra map such that f_1 is a homology isomorphism. Suppose that $g : (B, \partial) \to (A, \partial)$ is a homology inverse of f_1. Then there exists a homotopy $\mathcal{P}$-structure on g.

We recommend, as an easy exercise, to prove that conditions 1 and 2 imply

1′. For each homotopy $\mathcal{P}$-algebra B, differential graded vector space $A = (A, \partial)$ and a map $g : (A, \partial) \to (B, \partial)$ that is a homology isomorphism, there exist a homotopy $\mathcal{P}$-structure on (A, ∂) and a homotopy $\mathcal{P}$-structure $\mathbf{f} : A \to B$ on g.

As far as we know, nobody has considered the above properties in full generality, though there are several partial results in this direction. Let us quote at least the following theorem due to T. Kadeishvili [Kad80, p. 232], which is a special case of 1′ for the category of $\mathrm{A}(\infty)$-algebras, with $(A, \partial) = (H(C), \partial = 0)$ and $B = (C, \partial, \mu_2, 0, \dots)$.

Theorem 3.2. *Let (C, ∂) be a chain algebra. Then there exists an $\mathrm{A}(\infty)$-structure $\{X_k; k \geq 2\}$ on the graded space $H(C)$ having $X_2(a, b) = a \cdot b$ (the multiplication induced by μ_2), together with an $\mathrm{A}(\infty)$-homomorphism*

$$\mathbf{f} : (H(C), 0, X_2, X_3, \dots) \to (C, \partial, \mu_2, 0, \dots)$$

such that $f_1 : H(C) \to C$ is a homology isomorphism.

A similar statement for balanced A(∞)-algebras was proved by the author in [Mar92]. Another direction of results that indicates a certain homotopy stability of homotopy algebraic structures is represented by various versions of the Perturbation Lemma, see [GS86, HK91].

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Mathematical Institute of the Academy, Žitná 25, 115 67 Praha 1, Czech Republic,
email: `markl@math.cas.cz`

LITTLE 2-CUBES AND HOCHSCHILD COHOMOLOGY
(preliminary announcement)

Jim McClure and Jeff Smith
Purdue University
West Lafayette, IN 47907–1395

Motivation: Gerstenhaber proved (Annals 1962) that the natural cup product on the Hochschild cohomology of an associative ring is commutative. He also proved that there is a Lie bracket which shifts degrees by one. Since this is the same structure that one would find in the homology of a 2-fold loop space, Deligne asked (1993) whether the chains of $\mathcal{C}_2$ (the little 2-cubes operad) act on the (algebraic) Hochschild cohomology complex.

Our results give a positive answer to this question and also prove something more: we construct a topological operad which is equivalent as operads to $\mathcal{C}_2$, we show that the chains of our operad act on the Hochschild cohomology complex of an associative ring, and we also show that our operad itself acts on the topological Hochschild cohomology of an associative ring spectrum.

Cup and circle operations in Hochschild cohomology.

Let R be an associative ring (but everything generalizes to associative ring spectra).

A Hochschild p-cochain is a map of abelian groups

$$\phi : R^{\otimes p} \to R$$

$$(d^i\phi)(r_1, \dots, r_{p+1}) = \begin{cases} r_1\phi(r_2, \dots, r_{p+1}) & \text{if } i = 0 \\ \phi(\dots, r_i r_{i+1}, \dots) & \text{if } 0 < i < p+1 \\ \phi(r_1, \dots, r_p) r_{p+1} & \text{if } i = p+1 \end{cases}$$

If ϕ is a p-cochain and ψ is a q-cochain then $\phi \smile \psi$ is the $(p+q)$-cochain defined by

$$(\phi \smile \psi)(r_1 \otimes \cdots \otimes r_{p+q}) =$$
$$\phi(r_1 \otimes \cdots \otimes r_p) \cdot \psi(r_{p+1} \otimes \cdots \otimes r_{p+q})$$

There is another kind of operation in the Hochschild complex: for each i with $1 \le i \le p$ define

$$\phi \circ_i \psi$$

to be the $(p+q-1)$-cochain with

$$(\phi \circ_i \psi)(r_1 \otimes \ldots \otimes r_{p+q-1}) =$$
$$\phi(r_1 \otimes \ldots \otimes r_{i-1} \otimes \psi(r_i \otimes \ldots \otimes r_{i+q-1}) \otimes \ldots \otimes r_{p+q-1})$$

Theorem 1. There is an operad equivalent to $\mathcal{C}_2$ which acts on $\mathrm{Tot}'(X^\bullet)$ whenever $X^\bullet$ has $\smile$ and $\circ_i$ operations which satisfy the same relations as those in the Hochschild complex.

Here X can be a cosimplicial space or a cosimplicial spectrum, and Tot' is a modification of Tot which will be defined below.

Question: Is there an analogous result for $\mathcal{C}_n$ with $n > 2$?

Cup pairings and A_∞ structures on Tot

(This is a warmup for the proof of Theorem 1).

Question: Let $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ be cosimplicial spaces. How can you recognize on the cosimplicial level when there is a pairing in Tot:

$$\mathrm{Tot}(X^\bullet) \times \mathrm{Tot}(Y^\bullet) \to \mathrm{Tot}(Z^\bullet)$$

Obvious answer: A cosimplicial map

$$X^\bullet \times Y^\bullet \to Z^\bullet$$

induces a pairing in Tot (recall that a cosimplicial map is a family of maps

$$X^n \times Y^n \to Z^n$$

which preserve coface and codegeneracy maps).

However: the cobar construction on a space A does not have a cosimplicial pairing, although there is a pairing on Tot.

Definition A cup-pairing is a family of maps

$$\smile : X^p \times Y^q \to Z^{p+q}$$

satisfying

(i) $d^i(x \smile y) = \begin{cases} d^i x \smile y & \text{if } i \leq p \\ x \smile d^{i-p} y & \text{if } i > p \end{cases}$

(ii) $(d^{p+1}x) \smile y = x \smile d^0 y$

(iii) $s^i(x \smile y) = \begin{cases} s^i x \smile y & \text{if } i \leq p-1 \\ x \smile s^{i-p} y & \text{if } i \geq p \end{cases}$

Theorem 2. (i) A cup-pairing

$$X^\bullet \times Y^\bullet \to Z^\bullet$$

induces a pairing in Tot.

(ii) If $X^\bullet$ has a strictly associative cup pairing then $\mathrm{Tot}(X^\bullet)$ has an A_∞ structure.

Remarks: (i) The definition of cup-pairing is modeled on the relation between the coface and codegeneracy maps and the cup product in the Hochschild complex.

(ii)The cobar construction on A has an obvious cup-pairing.

(iii) A cosimplicial map $\mu : X^\bullet \times Y^\bullet \to Z^\bullet$ induces a cup pairing by

$$x \smile y = \mu(d^{p+q} d^{p+q-1} \cdots d^{p+1} x, d^0 d^0 \cdots d^0 y)$$

Proof of Theorem 2.

Part (i). Recall that a point in $\mathrm{Tot}(X^\bullet)$ is a sequence

$$f_0 \in X^0, f_1 : \Delta^1 \to X^1, f_2 : \Delta^2 \to X^2, \ldots$$

which is *consistent*, i.e.,

$$d^i \circ f_n = f_{n+1} \circ d^i$$

and

$$s^i \circ f_n = f_{n-1} \circ s^i$$

Thus what is required is to take consistent sequences $f_n : \Delta^n \to X^n$ and $g_n : \Delta^n \to Y^n$ and construct a consistent sequence $h_n : \Delta^n \to Z^n$. This is done by means of *Grayson subdivision.*

Grayson Subdivision of a simplex (Grayson, Batanin)

This is a homeomorphism

$$\left((\coprod_{p+q=n} \Delta^p \times \Delta^q)/\sim \right) \xrightarrow{\approx} \Delta^n$$

where $\sim$ consists of certain identifications along the boundaries. The map on $\Delta^p \times \Delta^q$ takes

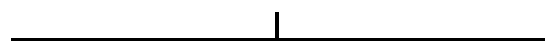
$$(s_0, \ldots, s_p), (t_0, \ldots, t_q)$$

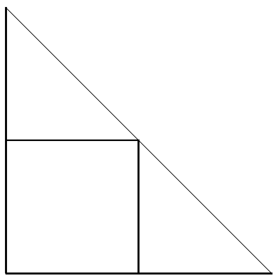
to

$$\frac{1}{2}(s_0, \ldots, s_{p-1}, s_p + t_0, t_1, \ldots, t_q)$$

This is the two-fold Grayson subdivision; there is a similar k-fold Grayson subdivision.

Here are the two-fold Grayson subdivisions of the 1-simplex and the 2-simplex:

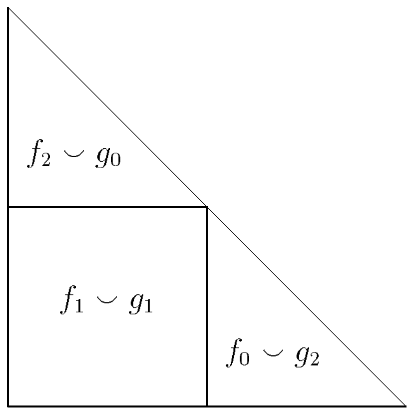




Proof of Part (i) (concluded).

If we are given two consistent sequences $f_n : \Delta^n \to X^n$ and $g_n : \Delta^n \to Y^n$ then for each n we can construct a map $h_n : \Delta^n \to Z^n$ by applying Grayson subdivision to the domain of h_n and letting the restriction of h_n to the piece $\Delta^p \times \Delta^q$ be $f_p \smile g_q$.

Here is what this looks like for $n = 1$ and $n = 2$:



Proof of Part (ii). It is possible to write down an explicit action of the little 1-cubes operad on $\mathrm{Tot}(X^\bullet)$, but it is sufficient, and more pleasant, to show that $\mathrm{Tot}(X^\bullet)$ is a deformation retract of a strictly associative monoid. In order to do this we define a construction Tot' which is related to Tot in the same way that the Moore loop space is related to the loop space. First we define Δ^n_r, for each $r \geq 0$, to be the set

$$\{ (s_0, \ldots, s_n) \,|\, s_i \geq 0 \text{ for all } i \text{ and } s_0 + \cdots + s_n = r \}$$

Note that $\Delta^\bullet_r$ is a cosimplicial space for each r. Then we define $\mathrm{Tot}'(X^\bullet)$ to consist of pairs $(r, \{f_n\})$, where r is ≥ 0 and $\{f_n\}$ is a consistent sequence of maps $\Delta^n_r \to X^n$; we topologize $\mathrm{Tot}'(X^\bullet)$ in a way analogous to the way the Moore loop space is topologized.

The proof that $\mathrm{Tot}'(X^\bullet)$ is a deformation retract of $\mathrm{Tot}(X^\bullet)$ is the same as the corresponding proof for the Moore loop space. The proof that $\mathrm{Tot}'(X^\bullet)$ has a strictly associative multiplication when $X^\bullet$ has an associative cup product is analogous to the proof of part (i) of Theorem 2, except that we use Grayson subdivision with *parameters*; this is a homeomorphism

$$\left(\Big(\coprod_{p+q=n} \Delta^p_{r_1} \times \Delta^q_{r_2} \Big) / \sim \right) \xrightarrow{\approx} \Delta^n_{r_1+r_2}$$

which takes $(s_0, \ldots, s_p), (t_0, \ldots, t_q)$ to $(s_0, \ldots, s_{p-1}, s_p + t_0, t_1, \ldots, t_q)$

Description of the operad $\mathcal{C}$.

The structure of the operad $\mathcal{C}$ is based on that of the Hochschild cohomology complex.

First we define the space $\mathcal{P}(k)$, for each $k \geq 0$, to be the set of k-tuples $(r_1, \ldots, r_k)$ of positive real numbers.

The k-th space $\mathcal{C}(k)$ of the operad $\mathcal{C}$ is the product of the (contractible) space $\mathcal{P}(k)$ with another space $\mathcal{F}(k)$. (The idea here is that the spaces $\mathcal{F}(k)$ by themselves have a structure like that of an operad but the composition is only A_∞-associative. By combining the $\mathcal{F}(k)$ with the $\mathcal{P}(k)$ we will get a true operad).

Cells of $\mathcal{F}(k)$: The cells of $\mathcal{F}(k)$ are in one-to-one correspondence with the formulas constructed from the numbers $1, 2, \ldots k$ (with no repetitions) by formal cup products and formal compositions. (The idea is that a formula is like a Hochschild cochain, the formal cup products are like cup products in the Hochschild complex, and the formal compositions are like $\circ_i$ products in the Hochschild complex.)

To each entry in the formula we associate a simplex whose dimension is the number of inputs for that entry; to the formula as a whole we associate the product of these simplices. This product of simplices is a cell of $\mathcal{F}(k)$.

Example: $\mathcal{F}(3)$

The top dimensional cells are those with no cup products:

- 1(2(3)) and its permutations correspond to $\Delta^1 \times \Delta^1$'s
- 1(2,3) and its permutations correspond to Δ^2's

The lower dimensional cells are:

- $1(2) \smile 3$ and its permutations corresponding to Δ^1's
- $1 \smile 2(3)$ and its permutations corresponding to Δ^1's
- $1(2 \smile 3)$ and its permutations corresponding to Δ^1's
- $1 \smile 2 \smile 3$ and its permutations corresponding to Δ^0's.

Another example: 3(1,2(4,5),6) is a cell of $\mathcal{F}(6)$ of type $\Delta^3 \times \Delta^2$

Write $\mathcal{F}_f$ from now on for the cell corresponding to a formula f.

Identifications Along The Boundary:

These are modelled on the Hochschild coface maps.

Example: the cell corresponding to 1(2) is an interval whose two endpoints are identified with the cells corresponding to $1 \smile 2$ and $2 \smile 1$

Compare this with the formulas

$$(d^0\phi)(r_0) = r_0 \cdot \phi \qquad (d^1\phi)(r_0) = \phi \cdot r_0$$

Example: the cell corresponding to $1(2,3)$ is a triangle whose three faces are identified with $2 \smile 1(3)$, $1(2 \smile 3)$ and $1(2) \smile 3$.

Compare this with the formulas

$$(d^0\phi)(r_0, r_1) = r_0\phi(r_1) \quad (d^1\phi)(r_0, r_1) = \phi(r_0 r_1) \quad (d^2\phi)(r_0, r_1) = \phi(r_0)r_1$$

Example: the cell corresponding to $1(2(3))$ is a square whose four faces are identified with $2(3) \smile 1$, $1 \smile 2(3)$, $1(2 \smile 3)$, and $1(3 \smile 2)$

Note that the space $\mathcal{F}(2)$ is homeomorphic to a circle.

The Operad Structure Maps

It is well known that to specify the structure maps of an operad $\mathcal{O}$ we just have to explain how to "plug a point of the k-th space into the j-th position of a point in the n-th space," that is, for each k and n and each $j \leq n$ we have to give a map

$$\gamma_j : \mathcal{O}(n) \times \mathcal{O}(k) \to \mathcal{O}(n+k-1)$$

First we describe an operad structure on the collection of spaces $\mathcal{P}(k)$: when we plug a point $(s_1, \ldots, s_k)$ of $\mathcal{P}(k)$ into the j-th position of a point $(r_1, \ldots, r_n)$ of $\mathcal{P}(n)$ the result is defined to be

$$(r_1, \ldots, r_{j-1}, r_j s_1, \ldots, r_j s_k, r_{j+1}, \ldots, r_n).$$

Now recall that $\mathcal{C}(k) = \mathcal{P}(k) \times \mathcal{F}(k)$, so we need to define

$$\gamma_j : (\mathcal{P}(n) \times \mathcal{F}(n)) \times (\mathcal{P}(k) \times \mathcal{F}(k)) \to (\mathcal{P}(n+k-1) \times \mathcal{F}(n+k-1))$$

The projection of γ_j on the $\mathcal{P}(n+k-1)$ factor is defined to be the corresponding structure map of the operad $\mathcal{P}$. The projection of γ_j on the $\mathcal{F}(n+k-1)$ factor will be a map

$$\mathcal{P}(k) \times \mathcal{F}(n) \times \mathcal{F}(k) \to \mathcal{F}(n+k-1)$$

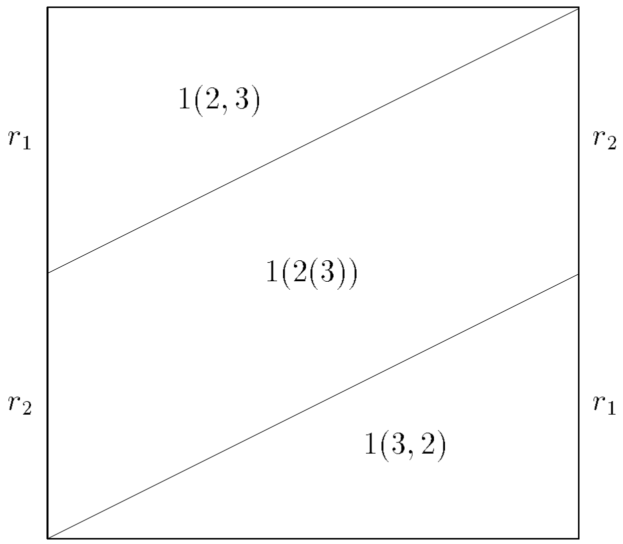
which we define next. (This means that the operad structure of $\mathcal{C}$ is like a "semidirect product" of $\mathcal{P}$ with $\mathcal{F}$, except that $\mathcal{F}$ is not an operad.)

In fact it suffices to say what happens to a point of $\mathcal{P}(k)$ and a product of cells $\mathcal{F}_f \times \mathcal{F}_g$, where f is a formula made from $1, \ldots, n$ and g is a formula made from $1, \ldots, k$.

Example: Here is how to define

$$\gamma_1 : \mathcal{F}_{1(2)} \times \mathcal{F}_{1(2)} \to \mathcal{F}(3)$$

when one is given a point (r_1, r_2) in $\mathcal{P}(2)$.



How to define the map

$$\mathcal{P}(k) \times \mathcal{F}_f \times \mathcal{F}_g \to \mathcal{F}(n+k-1)$$

in general

The idea is that the j-th entry of f corresponds to a simplex Δ, and we subdivide $\Delta \times \mathcal{F}_g$ (in a way that depends on the point of $\mathcal{P}(k)$) so that the resulting pieces of $\mathcal{F}_f \times \mathcal{F}_g$ are cells of $\mathcal{F}(n+k-1)$

Fattening Formulas: To fatten a formula means to add extra inputs (denoted by dashes) to the entries of the formula. The new formula is called a p-fattening if there are p dashes.

Example: the 1-fattenings of $1(2)$ are $1(-,2)$, $1(2,-)$, and $1(2(-))$.

The cells of the subdivision of $\Delta^p \times F_g$ will correspond to the p-fattenings of g (where a fattened formula is interpreted as a product of simplices as usual).

"Fiberwise Grayson Subdivision" Given a cell $\mathcal{F}_g$ of $\mathcal{F}(k)$ associated to a formula g, a simplex Δ^p and a point of $\mathcal{P}(k)$, there is a homeomorphism

$$(\coprod_h \mathcal{F}_h)/\sim \xrightarrow{\approx} \Delta^p \times \mathcal{F}_g$$

where h runs through the p-fattenings of g and $\sim$ consists of certain identifications along the boundaries. (We will not give the precise definition of this homeomorphism in this note).

If π denotes the projection of this homeomorphism on the second factor then the fibers of π are Grayson-subdivided copies of Δ^p.

The action on Tot′

The action of $\mathcal{C}$ on $\mathrm{Tot}'(X^\bullet)$ is defined in a way similar to the definition of the operad structure.

For example, to define the map

$$\mathcal{C}(2) \times \mathrm{Tot}'(X^\bullet) \times \mathrm{Tot}'(X^\bullet) \to \mathrm{Tot}'(X^\bullet)$$

it suffices to define, for each point (r_1, r_2) of $\mathcal{P}(2)$ and each pair (s_1, s_2) of nonnegative real numbers, a sequence of compatible maps

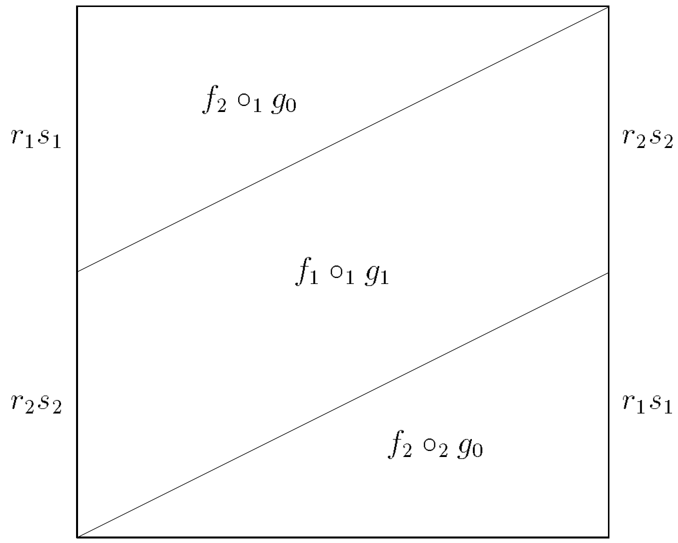
$$\mathcal{F}(2) \times \mathrm{Hom}(\Delta^\bullet_{s_1}, X^\bullet) \times \mathrm{Hom}(\Delta^\bullet_{s_2}, X^\bullet) \to \mathrm{Map}(\Delta^n_{r_1 s_1 + r_2 s_2}, X^n)$$

as n varies, and by adjointness it suffices to give compatible maps

$$\Delta^n_{r_1 s_1 + r_2 s_2} \times \mathcal{F}_{1(2)} \times \mathrm{Hom}(\Delta^\bullet_{s_1}, X^\bullet) \times \mathrm{Hom}(\Delta^\bullet_{s_2}, X^\bullet) \to X^n$$

and similarly for the other cell $\mathcal{F}_{2(1)}$.

Here is the case $n = 1$.



Cofibrant Operads and Universal E_∞ Operads

R.M. Vogt

1 Introduction

Contrary to general opinion the concept of an operad was already introduced in 1968 under the name "category of operators in standard form" in joint work of J.M. Boardman and myself [5], [15], at least in an obvious implicit way. It was strongly inspired by MacLane's notions of a PROP and a PACT [10] and Lawvere's work on functional semantics [9]. To be precise, a *category of operators* B is a permutative topologically enriched category with object $0, 1, 2, \ldots$ and continuous structure functor $\oplus$ such that $m \oplus n = m + n$ and all structure isomorphisms are homeomorphisms. It is called *in standard form* if it is determined by its morphisms spaces $\mathrm{B}(r, 1)$, i.e. if

1.1.

$$\begin{array}{rcl} \coprod\limits_{r_1+\ldots+r_n=m} \mathrm{B}(r_1,1)\times\ldots\times\mathrm{B}(r_n,1)\times_{\Sigma_{r_1}\times\ldots\times\Sigma_{r_n}}\Sigma_m & \to & \mathrm{B}(m,n) \\ (f_1,\ldots,f_n,\sigma) & \mapsto & (f_1\oplus\ldots\oplus f_n)\circ\sigma \end{array}$$

is a homeomorphism.

Any category of operators B defines an operad $Op\mathrm{B}$ by $(Op\mathrm{B})(n) = \mathrm{B}(n, 1)$, and any operad C defines a category of operators in standard form $C\mathrm{C}$ by $C\mathrm{C}(n, 1) = \mathrm{C}(n)$ and $C\mathrm{C}(m, n)$ obtained from formula (1.1). Clearly $Op(C\mathrm{C}) = \mathrm{C}$, and there is a continuous functor $C(Op\mathrm{B}) \to \mathrm{B}$ which is a homeomorphism if B is in standard form.

For this reason we worked with categories of operators in standard form or rather their operads. In fact, almost all constructions and proofs in [5], [6], and [15] make use of the underlying operads and not of the ambient categories. We introduced the little cubes operads and the linear isometry operad and illustrated their importance for infinite loop space theory.

P. May in his work on iterated loop spaces [12] combined our operad approach with ideas of Beck from [4], such as the use of the functorial twosided barconstruction or an n-fold delooping in one step. He discarded the ambient category from a category of operators in standard form and introduced the name "operad".

In his version of the theory of iterated loop spaces May neglected the concept of homotopy invariance which we consider as one of the basic features of the subject. To tackle homotopy invariance we introduced the bar construction $W\mathrm{B}$ for operads B. This construction has been considered a bit mysterious in the past and the twosided bar construction was a means to avoid it. Recently it has experienced a revival through the works of Ginzburg-Kapranov [8], Getzler-Jones [7], Batanin [2], and others . In the present paper we readdress this construction and show that the augmentation $\varepsilon : W\mathrm{B} \to \mathrm{B}$ can be considered as a cofibrant resolution of the operad B with respect to a suitable cofibration category structure on $\mathcal{O}pr$, the category of operads.

2 The bar construction

The bar construction, also called W-construction, is quite formal and, for example, makes sense over the categories of spaces, simplicial abelian groups, chain complexes, and suitable module spectra, but for the sake of an easy presentation we restrict ourselves to operads in the category $\mathcal{T}op$ of compactly generated spaces in the sense of [16, 5(ii)].

We should add a warning: when working over chain complexes, the corresponding constructions of some of the functors constructed below are not additive. Hence it is not obvious that our results also hold in this case. We intend to address this important case in a separate paper.

Consider the following diagram of topologically enriched categories and continuous forgetful functors.

2.1.

$$\begin{tikzcd} & & \mathbb{S}-\mathcal{T}op \arrow[dr, "U_4"] & \\ \mathcal{O}pr \arrow[r, "U_1"] & Eq-\mathcal{T}op \arrow[ur, "U_2"] \arrow[dr, "U_3"'] & & \mathbb{N}-\mathcal{T}op \\ & & \mathbb{N}-\mathcal{T}op^* \arrow[ur, "U_5"'] & \end{tikzcd}$$

The objects of $\mathbb{N}-\mathcal{T}op$ are collections $X = \{X_n; n \in \mathbb{N}\}$ of topological spaces, and the morphisms $f : X \to Y$ are collections of maps $f_n : X_n \to Y_n$. The category $\mathbb{N}-\mathcal{T}op^*$ is obtained from $\mathbb{N}-\mathcal{T}op$ by requiring that X_1 is based and $f_1 : X_1 \to Y_1$ preserves base points. $\mathbb{S}-\mathcal{T}op$ is obtained form $\mathbb{N}-\mathcal{T}op$ and $Eq-\mathcal{T}op$ from $\mathbb{N}-\mathcal{T}op^*$ by requiring that the symmetric group Σ_n acts on X_n for all n and that the maps f_n are equivariant. $\mathcal{O}pr$ is the category of operads. All these categories are topologically enriched in the obvious way.

2.2. *The operad of grown trees*

A *tree* θ is a finite contractible directed graph except that the edges need not have vertices on both ends. Each vertex v has a finite set $In(v)$ of incoming edges and exactly one outgoing edge. $In(v) = \emptyset$ is allowed. Hence each tree θ has a finite set $In(\theta)$ of inputs, i.e. incoming edges with no start vertices, and exactly one output, i.e. edge with no end vertex. We allow the *trivial tree* with no vertex

(directed from top to bottom).

For $X \in \mathbb{N} - \mathcal{T}op$ we define the *operad* TX *of grown trees on* X as follows. An element of $TX(n)$ is a triple (θ, f, g) consisting of a tree θ with $|In\theta| = n$, a function f assigning to each vertex v of θ an element $x \in X_{|In\,v|}$, and a bijection $g : In(v) \to \underline{n} = \{1, 2, \ldots, n\}$. Here $|M|$ denotes the cardinality of the set M. We give $TX(n)$ the obvious topology inherited from X.

We usually suppress f and g from the notation and think of an element of $TX(n)$ as a tree with vertices v labelled by $x \in X_{|In\,v|}$ and inputs labelled by $1, \ldots, n$ according to g. Composition in TX

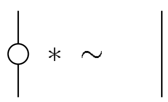
$$\begin{array}{rcl} TX(n) \times TX(r_1) \times \ldots \times TX(r_n) & \to & TX(r_1 + \ldots + r_n) \\ (\theta; \psi_1, \ldots, \psi_n) & \mapsto & \varphi \end{array}$$

is defined as follows: First relabel the input of ψ_i with label $k \in \underline{r_i}$ by $r_i + \ldots + r_{i-1} + k$, then stick ψ_i with all its (new) labels onto the input of θ with label i.

There is a Σ_n-operation on $TX(n)$ given by $(\theta, f, g) \cdot \sigma = (\theta, f, \sigma^{-1} \circ g)$. It is easy to check that these data make TX an operad.

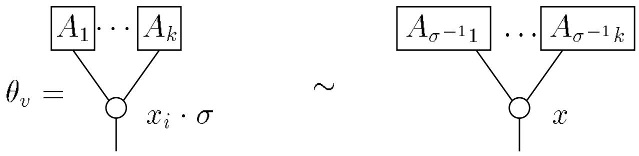
Relations 2.3. *If $X \in \mathbb{N} - \mathcal{T}op^*$, $\mathbb{S} - \mathcal{T}op$, or $Eq - \mathcal{T}op$ we can impose relations on TX:*

(1) For $X \in \mathbb{N} - \mathcal{T}op^*$ or $Eq - \mathcal{T}op$ with base point $* \in X_1$ the following relation makes sense for subtrees



(2) For $X \in \mathbb{S} - \mathcal{T}op$ or $Eq - \mathcal{T}op$ we consider the following relation: Let v be a vertex of a grown tree $\theta \in TX(n)$ and θ_v the subtree (including all

labels) consisting of v and all directed paths ending in v. If v has label $x \cdot \sigma$, $\sigma \in \Sigma_k$, then for subtrees



The proof of the following result is straight forward. For details see [6, p.31ff].

Theorem 2.4. *The following functors are left adjoint to the corresponding forgetful functors*

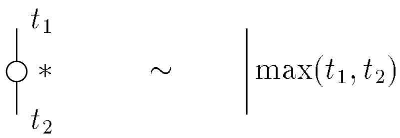
$$\begin{array}{llllll} \mathbb{N}-\mathcal{T}op & \longrightarrow & \mathcal{O}pr, & X & \longmapsto & TX \\ \mathbb{N}-\mathcal{T}op^* & \longrightarrow & \mathcal{O}pr, & X & \longmapsto & TX/relation(2.3.1) \\ \mathbb{S}-\mathcal{T}op & \longrightarrow & \mathcal{O}pr, & X & \longmapsto & TX/relation(2.3.2) \\ Eq-\mathcal{T}op & \longrightarrow & \mathcal{O}pr, & X & \longmapsto & TX/relations(2.3.1),(2.3.2) \end{array}$$

□

2.5. *The operad of trees*

The *operad $\widetilde{T}X$ of trees* is a modified version of TX. An element of $\widetilde{T}X(n)$ is a quadruple (θ, f, g, h) consisting of a tree (θ, f, g) and a *length function* $h : Edges\ \theta \to [0,1]$ such that each input and the output of θ have lengths 1. As before we suppress f, g, h from the notation. We give $\widetilde{T}X$ the obvious topology defined by the edge lengths and the vertex labels. Composition and the actions of the symmetric groups are defined as in TX; the new edges obtained via composition by sticking trees on inputs get lengths 1. These data define an operad. An element in $\widetilde{T}X$ is a non-trivial composite iff an internal edges has length 1.

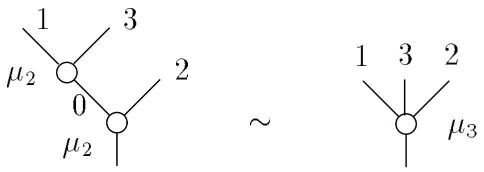
Relations 2.6. *(1) Relation (2.3.1) has to be modified: for $X \in \mathbb{N}-\mathcal{T}op^*$ or $Eq-\mathcal{T}op$ we consider the relation*



(t_1 and t_2 are the lengths of the edges).

(2) If $\mathrm{X} \in \mathcal{O}pr$, we consider the following relation: An edge of length 0 may be shrunk away by composing its vertices using the composition in X.

Example 2.7. Let M be the operad of monoids and $\mu_n \in \mathrm{M}(n)$ the n-fold multiplication. In $\widetilde{T}\mathrm{M}(3)$ we can consider the relation



Definition 2.8. *The bar construction for operads also called W-construction is the continuous functor*

$$W : \mathcal{O}pr \longrightarrow \mathcal{O}pr, \quad \mathrm{B} \longmapsto \widetilde{T}\mathrm{B}/\textit{(relations (2.3.2),(2.6.1),(2.6.2))}$$

Definition 2.9. *Let $\mathcal{C}$ be any of the categories of Diagram 2.1 and U : $\mathcal{O}pr \to \mathcal{C}$ the forgetful functor. A map f : $\mathrm{A} \to \mathrm{B}$ of operads is called a U-equivalence if $U(f)$ is a homotopy equivalence in $\mathcal{C}$ with respect to the homotopy relation induced by the canonical cylinder functor in $\mathcal{C}$.*

The following result and its relative version was the main tool in the development of homotopy invariant structures in [5] and [6]. For a proof see [6, Prop.3.6, Thm.3.17].

Theorem 2.10. *(1) The augmentation $\varepsilon : W\mathrm{B} \to \mathrm{B}$ obtained by shrinking all edges is a U_1-equivalence.*

(2) Given a diagram of operads with A *well-pointed (i.e. $\{1\} \subset \mathrm{A}(1)$ is a*

$$\begin{tikzcd} & \mathrm{B} \arrow[d, "g"] \\ W\mathrm{A} \arrow[r, "f"'] & \mathrm{C} \end{tikzcd}$$

closed cofibration) and g a U_1-equivalence then there exists a lift $\overline{f} : W\mathrm{A} \to \mathrm{B}$ such that $g \circ \overline{f} \simeq f$ in $\mathcal{O}pr$. Any two such lifts are homotopic in $\mathcal{O}pr$.

3 The category $\mathcal{O}pr$

It is well-known that $\mathcal{O}pr$ is complete and cocomplete (e.g. see [7], [13], or apply [1]). But we know more: $\mathcal{O}pr$ is topologically complete and cocomplete, i.e. the natural bijections

$$\begin{aligned} \mathcal{O}pr(\operatorname{colim} D, \mathrm{B}) &\cong \lim \mathcal{O}pr(D, \mathrm{B}) \\ \mathcal{O}pr(\mathrm{B}, \operatorname{colim} D) &\cong \lim \mathcal{O}pr(\mathrm{B}, D) \end{aligned}$$

are homeomorphisms for any small diagram D in $\mathcal{O}pr$. Moreover, $\mathcal{O}pr$ is *tensored* and *cotensored* over $\mathcal{T}op$: there are continuous functors

$$\begin{array}{lclcl} \mathcal{O}pr \times \mathcal{T}op & \to & \mathcal{O}pr, & (\mathrm{B}, K) & \mapsto & \mathrm{B} \otimes K \\ \mathcal{O}pr \times \mathcal{T}op^{op} & \to & \mathcal{O}pr, & (\mathrm{B}, K) & \mapsto & \mathrm{B}^K \end{array}$$

and natural homeomorphisms

3.1.

$$\mathcal{O}pr(\mathrm{B} \otimes K, \mathrm{C}) \cong \mathcal{T}op(K, \mathcal{O}pr(\mathrm{B}, \mathrm{C})) \cong \mathcal{O}pr(\mathrm{B}, \mathrm{C}^K)$$

The same is true for the other categories of Diagram 2.1. In particular, we have canonical cylinder functors $- \otimes I$ and path space functors $(-)^I$ in each of these categories. Using these functors we can define cofibrations and (Hurewicz) fibrations as morphisms having the homotopy extension and homotopy lifting property respectively.

Let $\mathcal{C}$ be any of the categories of Diagram 2.1 and $U : \mathcal{O}pr \longrightarrow \mathcal{C}$ be the forgetful functor.

Definition 3.2. *$f : \mathrm{A} \to \mathrm{B}$ in $\mathcal{O}pr$ is called*

(1) a U-fibration if $U(f)$ is a fibration,

(2) a U-cofibration if it has the left lifting property for all U-fibration which are U-equivalences.

The next results provide a better understanding of the W-construction.

Proposition 3.3. *$\mathcal{O}pr$ with U-equivalences as weak equivalences and U-cofibrations as cofibrations is a cofibration category with all objects fibrant.*

Remark 3.4. Cofibration categories were extensively studied by Baues in [3]. We use the weaker version of Majewski [11] who only requires the Axioms (C1), (C2)(b), (C3), and (C4) of [3, p.6].

Proposition 3.3 is a special case of the main result of [14].

The lifting lemma [3, p.90] also holds for Majewski's version [11, (3.9)].

3.5. ***Lifting Lemma:*** *Let $\mathcal{C}$ be a cofibration category and*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

a commutative diagram in $\mathcal{C}$ with p a weak equivalence between fibrant objects and i a cofibration. Then there exists a morphism $h : B \to X$ uniquely up to homotopy $\mathrm{rel} A$, such that $h \circ i = f$ and $p \circ h \simeq g\, \mathrm{rel} A$.

The Lifting Theorem 2.10 and its relative versions follow form (3.5) and the following result, which can be deduced from the analysis of the W-construction [6, p.76ff].

Theorem 3.6. *Take $U = U_1$ in Prop. 3.3. Then $\varepsilon : W\mathrm{B} \to \mathrm{B}$ is a U_1-cofibrant resolution of B for any well-pointed operad B.*

These results allow the construction of universal E_∞-operads. Recall, that the notion of an E_∞-operad in [12] is a modification of the terminology of [5] and [6] where E_∞-operads (called E-categories there) are not required to be Σ-free.

Definition 3.7. *An operad B is called an E_∞-operad if the unique morphism $\mathrm{B} \to \mathcal{C}om$ into the operad of commutative monoids is an $(U_5 \circ U_3 \circ U_1)$-equivalence, i.e. if each space $\mathrm{B}(n)$ is contractible.*

For $U = U_5 \circ U_3 \circ U_1$ any U-cofibrant resolution $\mathrm{Q} \to \mathcal{C}om$ of $\mathcal{C}om$ is universal by (3.5).

Proposition 3.8. *Let B be an E_∞-operad and Q a U-cofibrant resolution of $\mathcal{C}om$ then there is a functor of operads $\mathrm{Q} \to \mathrm{B}$ uniquely up to homotopy in $\mathcal{O}pr$ which makes*

$$\begin{tikzcd} \mathrm{Q} \arrow[rr] \arrow[dr] & & \mathrm{B} \arrow[dl] \\ & \mathcal{C}om & \end{tikzcd}$$

commute. Any two such Q are homotopy equivalent in $\mathcal{O}pr$.

Since cofibrant resolutions exists in every cofibration category we have universal E_∞-operads. In our case there exist functorial cofibrant resolutions (see [14]). To obtain other examples we can construct cofibrant resolutions in steps. We first observe that Proposition 3.3 holds for any composition of forgetful functors $U : \mathcal{C} \to \mathcal{D}$ in Diagram 2.1 giving $\mathcal{C}$ a cofibration category structure as in Prop. 3.3. Starting with $Id : \mathbb{N} - \mathcal{T}op \to \mathbb{N} - \mathcal{T}op$ we can equip all categories $\mathcal{C}$ of (2.1) with cofibration structures with respect to the forgetful functor $\mathcal{C} \longrightarrow \mathbb{N} - \mathcal{T}op$.

Proposition 3.9. *With respect to the forgetful functor $U : \mathcal{C} \to \mathbb{N} - \mathcal{T}op$ an object $X \in \mathcal{C}$ is U-cofibrant if*

(1) *X_1 is well-pointed if $\mathcal{C} = \mathbb{N} - \mathcal{T}op^*$*

(2) *$X_n \to X_n/\Sigma_n$ is a numerable principal Σ_n-bundle for all n if $\mathcal{C} = \mathbb{S} - \mathcal{T}op$*

(3) *Condition (1) and (2) hold if $\mathcal{C} = Eq - \mathcal{T}op$.*

The proof is straight forward.

Starting with any E_∞-operad B it is now easy to construct a $(U_5 \circ U_3 \circ U_1)$-cofibrant resolution of B: first whisker $\mathrm{B}(1)$ to obtain an operad B' such that $U_5 \circ U_3(\mathrm{B}')$ is cofibrant (cp. [5, p.1120]), replace it by the operad $\overline{\mathrm{B}}$ with

$$\overline{\mathrm{B}}(n) = B(\mathrm{B}'(n), \Sigma_n, \Sigma_n)$$

where the right hand side is the two-sided bar construction. Then $U_1(\overline{\mathrm{B}})$ is cofibrant in $Eq - \mathcal{T}op$, and $W\overline{\mathrm{B}}$ is a universal E_∞-operad. More generally we have

Proposition 3.10. *Let B be an operad such that $U_1(\mathrm{B})$ is $(U_5 \circ U_3)$-cofibrant, then $W\mathrm{B}$ is $(U_5 \circ U_3 \circ U_1)$-cofibrant. If, in addition, each $\mathrm{B}(n)$ is contractible, then $W\mathrm{B}$ is a universal E_∞-operad.*

Corollary 3.11. *Let Q_∞ be the infinite little cubes operad and L the linear isometry operad of [5], then $W\mathrm{Q}_\infty$ and $W\mathrm{L}$ are universal E_∞-operads.*

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