Generalized Verma Modules

Volodymyr Mazorchuk

Contents

1 Introduction

The foundations of the theory of highest weight modules over simple complex finite-dimensional Lie algebras due to the original paper by Verma ([113]), where the family of universal highest weight modules (Verma modules) was introduced and studied. Although in [113] only basic properties of Verma modules were studied, it was clear, that Verma modules should have some deep connection with the general theory of Lie algebras, Weyl groups, flag manifolds and so on.

The major famous results about Verma modules were obtained in a series of celebrated papers ([9, 10, 11]) by Bernstein - Gelfand - Gelfand (BGG). In a few words, these three results describe the following: The first one, usually called the BGG Theorem, gives, in terms of the Weyl group action on the weight space, a criterion for the existence of a Verma submodule in a Verma module. The second one presents a combinatorially constructed resolution of a simple finite-dimensional module by Verma modules. This resolution is usually called the BGG resolution. The last result shows, that in a natural category, which is now known as category $\mathcal{O}$, Verma modules play the "intermediate" role between simple and projective objects.

It is quite difficult to overestimate the influence of these results on modern representation theory of Lie algebras. The bibliography of papers on this and related subjects is enormous (the reader can consult, for example, resent monographs [96] or [66], although it is impossible to find the complete bibliography anywhere outside Math. Reviews). The results by BGG not only led to the development of new branches of algebra (like quasi-hereditary algebras), but also found applications in functional analysis, combinatorics, modern physics and so on.

During the 30 years since the paper of Verma, many famous and deep results about Verma modules were obtained. It is worth mentioning, for example, the Kazhdan-Lusztig Theorem ([70, 5, 18]) or Soergel's description of the category $\mathcal{O}$ ([107]). Some results have been generalized to certain infinite-dimensional Lie algebras, e.g. (affine) Kac-Moody Lie algebras, Virasoro algebra, quantum algebras, Yangians and so forth. So far the theory of Verma modules is not completed and there are many interesting unsolved questions and problems.

There have been several attempts to generalize the theory of Verma modules, and one of the most natural of them is to generalize Verma modules themselves. This can be done in different ways (compare, for example, [56, 82, 103]). Generalized Verma modules (GVM) were studied from different points of view and many properties of classical Verma modules were established for or generalized to GVMs. The aim of this manuscript is to present a systematic study of GVM obtained by so-called parabolic induction for a parabolic subalgebra of a simple Lie algebra and to give a historical overview of the development of the subject.

In particular, we present an analogue of the BGG Theorem (in partial cases, covering explicitly the case of induction from a weight $sl(2,\mathbb{C})$-module), an analogue of the BGG resolution for simply laced algebras, an analogue of the Kazhdan-Lusztig Theorem and an analogue of category $\mathcal{O}$, including Soergel's type description for its blocks. In order to avoid technical calculations (which are quite difficult in some cases), we present mostly the ideological foundations of the proofs. For complete details the reader can consult the corresponding original paper.

The results, described here, were obtained by several authors during the last 14 years (see [22, 37, 38, 46, 47, 48, 49, 44, 45, 89, 90, 92, 71, 72, 73, 74, 75]).

This manuscript was written during the authors stay as an Alexander von Humboldt fellow at Universität Bielefeld. The finansial support of Humboldt Foundation and the hospitality of Bielefeld University are gratefully acknowledged. Finally, I want to thank all my colleagues for their help in my research work, leading to the appearance of this manuscript. I am especially indebt to Prof. M.Klucznik for his generous help in preparation of this manuscript.

2 Verma modules

In this Chapter we give a definition of classical Verma modules over simple complex finite-dimensional Lie algebras and list their main properties. We follow closely [28, 66, 96].

2.1 Definition

Let $\mathbb{C}$ denote the complex field, $\mathbb{Z}$ the ring of integers, $\mathbb{Z}_+$ the set of all non-negative integers and $\mathbb{N}$ the set of all positive integers. For a Lie algebra $\mathfrak{A}$ let $U(\mathfrak{A})$ denote the universal enveloping algebra of $\mathfrak{A}$ and $Z(\mathfrak{A})$ the center of $U(\mathfrak{A})$.

Let $\mathfrak{G}$ be a simple complex finite-dimensional Lie algebra with a fixed Cartan subalgebra $\mathfrak{H}$. Let $\Delta \subset \mathfrak{H}^*$ denote the corresponding root system. Fixing a basis π of Δ we can write Δ as a disjoint union of positive Δ_+ and negative Δ_- roots with respect to π. For $\alpha \in \Delta$ let $\mathfrak{G}_\alpha$ denote the corresponding root subspace of $\mathfrak{G}$. Denote by $\mathfrak{N}_\pm$ the subalgebras of $\mathfrak{G}$ generated by all $\alpha \in \Delta_\pm$. Then we have a triangular decomposition $\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+$. Fix a Weyl-Chevalley basis X_α, $\alpha \in \Delta$, H_α, $\alpha \in \pi$ of $\mathfrak{G}$ and let ρ denote half the sum of all positive roots. Denote by W the Weyl group of Δ, which acts on $\mathfrak{H}^*$ in a natural way. Define a dot-action of W on $\mathfrak{H}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$. For $\alpha \in \Delta$ let s_α denote the corresponding reflection. Let $(\cdot\,,\cdot)$ be the standard W-invariant form on $\mathfrak{H}^*$. The corresponding dual form on $\mathfrak{H}$ will be also denoted by $(\cdot\,,\cdot)$.

Fix $\lambda \in \mathfrak{H}^*$ and consider a one-dimensional $\mathfrak{B} = \mathfrak{H} \oplus \mathfrak{N}_+$-module $\mathbb{C} = \mathbb{C}_\lambda$, such that $(a + h)(z) = (\lambda - \rho)(h)z$ for all $a \in \mathfrak{N}_+$, $h \in \mathfrak{H}$ and $z \in \mathbb{C}$. The module

$$M(\lambda) = U(\mathfrak{G}) \bigotimes_{U(\mathfrak{B})} \mathbb{C}_\lambda$$

is called the *Verma module*, associated with $\mathfrak{G}$, $\mathfrak{H}$, π and λ (see [28, Chapter 7]).

We note, that there are two standard ways to parameterize the module $M(\lambda)$. They differ by the replacement of $\lambda - \rho$ with λ in the above definition of $\mathbb{C}_\lambda$. Both possibilities occur in the literature; we have chosen to follow the convention of Dixmier in his standard reference [28].

2.2 Basic properties

Originally, Verma modules appeared as the universal highest weight modules. The first branch of basic properties of $M(\lambda)$ connected precisely with this notion.

Let M be a $\mathfrak{G}$-module and $\mu \in \mathfrak{H}^*$. denote by M_μ the set of all elements of M on which any $h \in \mathfrak{H}$ acts via the scalar $\mu(h)$. M_μ is called the *weight space* corresponding to the *weight* μ. A $\mathfrak{G}$-module M is called *weight* if it is a direct sum of its weight spaces. The set of all non-zero weights of a weight $\mathfrak{G}$-module M is called the *support* of M and is usually denoted by $\mathrm{supp}(M)$.

Let M be a weight $\mathfrak{G}$-module. A non-zero element $v \in V_\mu$ is called a *highest weight element* provided $\mathfrak{N}_+ v = 0$. A weight $\mathfrak{G}$-module M is called a *highest weight module* if M

is generated by a highest weight vector, whose weight is called the *highest weight* of M. We also recall the standard partial order $\leqslant$ (which depends on π) on $\mathfrak{H}^*$ defined as follows: $\mu_1 \leqslant \mu_2$ if and only if $\mu_2 - \mu_1$ can be written as a linear combination of positive roots with non-negative integer coefficients. Define the *Kostant partition function* $P : \mathfrak{H}^* \to \mathbb{Z}_+$ as the number of different decompositions of $\mu \in \mathfrak{H}^*$ in a linear combination of positive roots with non-negative integer coefficients. Clearly $P(\mu) > 0$ if and only if $0 \leqslant \mu$.

Theorem 2.2.1. *1. $M(\lambda)$ is a highest weight module with the highest weight $\lambda - \rho$.*

2. *$M(\lambda)$ is a $U(\mathfrak{N}_-)$-free module of rank 1. In particular, $M(\lambda)$ is isomorphic to $U(\mathfrak{N}_-)$ as a vector space (graded by weights).*

3. $\mathrm{supp}(M(\lambda))$ *coinsides with the set of all $\mu \in \mathfrak{H}^*$ such that $\mu \leqslant \lambda - \rho$. Moreover,* $\dim(M(\lambda)_\mu) = P(\lambda - \rho - \mu)$ *for any $\mu \in \mathfrak{H}^*$.*

4. *Any highest weight $\mathfrak{G}$-module with the highest weight $\lambda - \rho$ is a quotient of $M(\lambda)$.*

5. *$M(\lambda)$ has a unique simple quotient $L(\lambda)$.*

6. *$M(\lambda)$ has a central character χ_λ.*

7. *Any endomorhpism of $M(\lambda)$ is scalar.*

The complete proof can be found, for example in [28, Chapter 7]. We note, that under our definition the parameter λ of $M(\lambda)$ does not coincide with the highest weight of $M(\lambda)$, which is $\lambda - \rho$. Of course one can parameterize Verma modules by their highest weights. This leads to the second way of describing them (as was mentioned above). Further we will see that our way is a bit more convenient for some properties of Verma modules. We also note the the fifth statement of the Theorem 2.2.1 is usually called the universal property of Verma modules. From this statement it follows, for example, that any simple finite-dimensional $\mathfrak{G}$-module is a quotient of an appropriate Verma module (and hence coincides with the corresponding $L(\lambda)$). The set of λ such that $L(\lambda)$ is finite-dimensional is usually denoted P^{++} and called the set of *regular dominant* parameters.

The next collection of properties of Verma modules is closely related to the structure of $M(\lambda)$.

Theorem 2.2.2. *1. $\chi_\lambda = \chi_\mu$ if and only if $\lambda \in W(\mu)$.*

2. *For any $\chi \in Z(\mathfrak{G})^*$ there exists $\lambda \in \mathfrak{H}^*$ such that $\chi = \chi_\lambda$.*

3. *$M(\lambda)$ has a composition series.*

4. $\dim \mathrm{Hom}_{\mathfrak{G}}(M(\lambda), M(\mu)) \leqslant 1$ *and any non-zero homomorphism from this space is injective.*

Again, the proof can be found in [28, Chapter 7]. The first two statements of this Theorem are easy corollaries of the famous Harish-Chandra isomorphism Theorem, which states, that under the Harish-Chandra homomorphism φ, $Z(\mathfrak{G})$ maps isomorphically onto the algebra $S(\mathfrak{H})^{(W,\cdot)}$ of polynomials in $\mathfrak{H}$ invariant under the dot-action of the Weyl group. Composing φ with the shift of indetermines in $\mathfrak{H}$, associated with ρ, we obtain a new map, whose image coincide with the algebra $S(\mathfrak{H})^W$ of usual invariants under the standard action of W (see [28, Section 7.4]).

2.3 BGG Theorem

The following Theorem is the celebrated BGG criterion for the existence of a non-trivial homomorphism between two Verma modules ([10]). The sufficiency of it was proved by Verma ([113]), who also conjectured the necessity part.

Theorem 2.3.1. *For $\lambda, \mu \in \mathfrak{H}^*$ the following are equivalent*

1. *$M(\mu) \subset M(\lambda)$.*
2. *$L(\mu)$ is a subquotient of $M(\lambda)$.*
3. *There exist a sequence $\alpha_1, \alpha_2, \dots, \alpha_k$ of positive roots such that*

$$\mu \leqslant s_{\alpha_1}(\mu) \leqslant s_{\alpha_2}(s_{\alpha_1}(\mu)) \leqslant s_{\alpha_k}(\dots(s_{\alpha_2}(s_{\alpha_1}(\mu)))\dots) = \lambda.$$

The proof can be found in [28, 96, 10]. From this theorem one can easily deduce the following criterion for $M(\lambda)$ to be simple.

Corollary 2.3.1. *$M(\lambda)$ is simple if and only if $s_\alpha(\lambda) \not\leqslant \lambda$ for any $\alpha \in \Delta_+$.*

Originally, the proof of Theorem 2.3.1 crucially used the first statement from Theorem 2.2.2. Later ([69]), an elementary proof was found. It uses the notion of the Shapovalov form on $M(\lambda)$, defined in [106]. Denote by σ the Chevalley involution on $\mathfrak{G}$ and by v_λ the canonical generator of the module $\hat{M}(\lambda) = M(\lambda+\rho)$ (in what follows we will keep this notation). According to Theorem 2.2.1, any element of $\hat{M}(\lambda)$ can be written as uv_λ, $u \in U(\mathfrak{N}_-)$. Define

$$F(u_1v_\lambda, u_2v_\lambda) = \varphi(\mathrm{proj}_{U_0}(\sigma(u_1)u_2))(\lambda),$$

where U_0 denotes the centralizer of $\mathfrak{H}$ in $U(\mathfrak{G})$. F is called the *Shapovalov form* on $\hat{M}(\lambda)$.

Theorem 2.3.2.
1. *F is a symmetric contravariant bilinear form on $\hat{M}(\lambda)$.*
2. *The weight subspaces of $\hat{M}(\lambda)$ are orthogonal with respect to F.*
3. *The radical of F coincides with the maximal submodule of $\hat{M}(\lambda)$.*
4. *$\hat{M}(\lambda)$ is simple if and only if F is non-degenerate.*

5.

$$\det F|_{\hat{M}(\lambda)_\mu} = \prod_{\alpha\in\Delta_+}\prod_{n=1}^{\infty}(\lambda(H_\alpha)+\rho(H_\alpha)-n(\alpha,\alpha)/2)^{P(\lambda-\mu-n\alpha)}.$$

Theorem 2.3.1 is an easy corollary of Theorem 2.3.2. But the proof of Theorem 2.3.2 in [106] was based on Theorem 2.3.1. The proof in [69] is quite elementary and is independent of Theorem 2.3.1 (and even on Harish-Chandra Theorem). Hence the last gives us an elementary (but not easy) proof of Theorem 2.3.1.

2.4 BGG resolution

As was mentioned, $L(\lambda)$ is finite-dimensional if and only if $\lambda \in P^{++}$, moreover, each simple finite-dimensional module is of the form $L(\lambda)$ for an appropriate $\lambda \in P^{++}$. Let $l : W \to \mathbb{Z}_+$ denote the standard length function with respect to π. Fix $\lambda \in P^{++}$. For $i = 0, 1, \dots, |\Delta_+|$ denote by C_i the direct sum of all $M(w(\lambda))$, with $l(w) = i$ (we will also denote by W_i the set of such w). According to Theorem 2.2.2, any homomorphism $d_i : C_i \to C_{i-1}$ can be defined via a complex matrix $(d(i)_{a,b})_{a\in W_{i-1}}^{b\in W_i}$. According to [9] or [103], one can choose $d(i)_{a,b}$ simultaneously such that $d(i)_{a,b} \neq 0$ if and only if $a < b$ with respect to the Bruhat order on W and $d_{i-1} \circ d_i = 0$ for all i.

Theorem 2.4.1. *The sequence*

$$0 \to C_{|\Delta_+|} \xrightarrow{d_{|\Delta_+|}} C_{|\Delta_+|-1} \xrightarrow{d_{|\Delta_+|-1}} \dots C_1 \xrightarrow{d_1} M(\lambda) \xrightarrow{p} L(\lambda) \to 0,$$

where p is the canonical projection, is exact.

The exact sequence from Theorem 2.4.1 is known as the *BGG resolution* of $L(\lambda)$. The proof of Theorem 2.4.1 can be found in [9]. Some interesting facts about it also can be found in [82, 103, 104, 51]. We have to note, that sometimes this BGG resolution is called the strong BGG resolution in order to distinguish it from another (quite similar) resolution, constructed in [9] by using the cohomology of $\mathfrak{G}$. The equivalence of both resolutions was proved in [103].

2.5 Category $\mathcal{O}$

One of the most important structures arising in the representation theory of $\mathfrak{G}$ is the so-called category $\mathcal{O}$, defined and investigated in [11]. In particular, it led to the appearence of such new objects in modern algebra, as quasi-hereditary algebras, highest weight categories, BGG catregories and so on (see [20, 60]).

Let $\mathcal{O}$ denote a full subcategory of the category of all $\mathfrak{G}$-modules, which consists of those modules M, which are

- finitely generated;

- $\mathfrak{H}$ diagonalizable;
- $\mathfrak{N}_+$ finite (i.e. $U(\mathfrak{N}_+)v$ is finite-dimensional for any $v \in M$).

Theorem 2.5.1. *1. $\mathcal{O}$ is closed under taking submodules, quotients, finite direct sums and under tensoring with finite-dimensional modules.*

2. All Verma modules belong to $\mathcal{O}$.

3. $L(\lambda)$, $\lambda \in \mathfrak{H}^*$ exhaust the set of simple modules in $\mathcal{O}$.

4. Any module in $\mathcal{O}$ has a composition series.

5. $\mathcal{O} = \oplus_{\chi \in Z(\mathfrak{G})^*} \mathcal{O}(\chi)$, where $\mathcal{O}(\chi)$ is a full subcategory consisting of all modules, which are annihilated by a big enough power of $z - \chi(z)$ for all $z \in Z(\mathfrak{G})$.

The proof can be found in [11] or in [96]. It is quite easy. The first non-trivial property of $\mathcal{O}$ states that it can be decomposed into a direct sum of full subcategories, each of which is equivalent to the category of (finite-dimensional) modules over a finite-dimensional algebra.

Theorem 2.5.2. *1. $\mathcal{O}$ has enough projective modules (i.e. any module in $\mathcal{O}$ is a quotient of a projective module in $\mathcal{O}$).*

2. There is a bijection between simple modules and indecomposable projective modules in $\mathcal{O}$. We will denote by $P(\lambda)$ the projective cover of $L(\lambda)$, $\lambda \in \mathfrak{H}^*$.

3. Each $\mathcal{O}(\chi)$ is equivalent to the category of (finite-dimensional) modules over a finite-dimensional algebra.

Again, the proof can be found in [11] or in [96]. We should mention that all finite-dimensional algebras, which appear in Theorem 2.5.2 are quasi-hereditary (in the sence of [20]). Moreover, Theorem 2.5.2 was one of the reasons, which led to the abstract definition of quasi-hereditary algebras in [20]. The next Theorem is the celebrated BGG reciprocity principle for $\mathcal{O}$.

Theorem 2.5.3. *1. Any projective module in $\mathcal{O}$ admits a Verma flag (i.e. a filtration, whose subquotients are Verma modules).*

2. For any $\lambda, \mu \in \mathfrak{H}^*$ holds $[P(\lambda) : M(\mu)] = (M(\mu) : L(\lambda))$, where $[P(\lambda) : M(\mu)]$ denotes the number of occurrences of $M(\mu)$ in a Verma flag of $P(\lambda)$. In particular, the last is a well-defined number (i.e. does not depend on a Verma flag).

The proof can be found in [11] or in [96]. This Theorem was generalized to arbitrary quasi-hereditary algebras having a duality ([60]). In particular, it shows that Verma modules play in $\mathcal{O}$ a role of intermediate modules between simples and projectives.

The next natural question for the category $\mathcal{O}$ is to give a precise (combinatorial) description of the algebras which appear in Theorem 2.5.2. This was solved (much later) by Soergel in [107]. In order to state Soergel's results, we need a bit more notation.

Denote by C the quotient of the polynomial algebra $S(\mathfrak{H})$ over the ideal generated by all non-constant homogeneous polynomials in $S(\mathfrak{H})$, invariant under the action of W (we emphasize that here we mean the ordinary action). The algebra C is usually called the *coinvariant algebra*. Let $\lambda_0 \in P^{++}$ be such that the module $L(\lambda_0)$ is one-dimensional (trivial module). Then the corresponding block $\mathcal{O}(\chi_{\lambda_0})$ is called *principal*. Let w_0 denote the unique longest element in W ($l(w_0) = |\Delta_+|$). The projective module $P(w_0(\lambda_0))$ is usually called the *big projective module*. It is easy to see that $[P(w_0(\lambda_0)) : M(w(\lambda_0))] = 1$ for any $w \in W$ and $[P(w(\lambda_0)) : M(w_0(\lambda_0))] = 0$ for any $w \neq w_0$.

Theorem 2.5.4. *1. The endomorphism algebra of $P(w_0(\lambda_0))$ is isomorphic to C.*

2. *The finite-dimensional algebra, associated with $\mathcal{O}(\chi_{\lambda_0})$ coinsides with the endomorphism algebra of $P(w_0(\lambda_0))$, viewed as a module over its endomorphism algebra (i.e. viewed as a C-module).*

3. *If $\mu_1, \mu_2 \in P^{++}$, then $\mathcal{O}(\chi_{\mu_1})$ and $\mathcal{O}(\chi_{\mu_1})$ are equivalent.*

4. *Any $\mathcal{O}(\chi)$ is equivalent to some $\mathcal{O}(\chi_\lambda)$ with an integral λ with respect to the semisimple part of the Levi factor of a parabolic subalgebra of $\mathfrak{G}$.*

The proof can be found in [107, 8]. The second statement of this Theorem is know as the *double centralizer property*. The third statement is standard and was stated (without proof) in [11]. From the last statement it follows, in particular, that for a fixed algebra $\mathfrak{G}$ the set of non-isomorphic finite-dimensional algebras arizing from the corresponding $\mathcal{O}$ is finite.

The Chevalley antiinvolution σ on $\mathfrak{G}$ leads to a natural duality on $\mathcal{O}$ (by a *duality* we mean a contravariant exact functor, which preserves simple objects). Denote this functor by $\mathcal{F}$. Let O_1 (resp. O_2) denote the full subcategory of $\mathcal{O}$ which consists of all modules admitting a Verma flag (resp. dual Verma flag, i.e. a flag, whose subquotients are isomorphic to $\mathcal{F}(M(\lambda))$, $\lambda \in \mathfrak{H}^*$). Then, according to the general result of Ringel for quasi-hereditary algebras ([101]), the indecomposable modules in the intersection O of O_1 and O_2 are naturally parametrized by $\lambda \in \mathfrak{H}^*$ (i.e. there is a natural bijection between simple modules in $\mathcal{O}$ and indecomposable modules in O), moreover O coincides with the full subcategory consisting of all direct sums of these indecomposable modules. We will denote by $T(\lambda)$ the indecomposable module in O which corresponds to $\lambda \in \mathfrak{H}^*$ (this means that any Verma flag of $T(\lambda)$ starts with $M(\lambda)$). $T(\lambda)$ is usually called a *tilting module*. If we restrict our consideration to $\mathcal{O}(\chi_\lambda)$, $\lambda \in \mathfrak{H}^*$ and denote by W_λ the subgroup of W stabilizing λ, then the direct sum $T = \oplus_{w\in W/W_\lambda} T(w(\lambda))$ is called the *characteristic tilting module* of $\mathcal{O}(\chi_\lambda)$. In this case $O(\chi_\lambda)$ coinsides with $\mathrm{add}\, T$ (see [77]). Having this characteristic tilting module T one can consider its endomorphism algebra, which is known (for our example, or in general, for quasi-hereditary algebras) to be a quasi-hereditary algebra ([101, 102]). This algebra is called the *Ringel dual* for the algebra, corresponding to $\mathcal{O}(\chi_\lambda)$.

Theorem 2.5.5. *1. The principal block $\mathcal{O}(\chi_{\lambda_0})$ of $\mathcal{O}$ is its own Ringel dual (i.e. the corresponding algebra is isomorphic to its Ringel dual).*

2. For $w_1, w_2 \in W$ holds $[T(w_1(\lambda_0)) : M(w_2(\lambda_0))] = (M(w_2 w_0(\lambda_0)) : L(w_1 w_0(\lambda_0)))$.

This Theorem is a recent result by Soergel ([110]), where the reader can find a proof of both statements. Some more interesting results about category $\mathcal{O}$ and related structures can be found in [12, 14, 15, 7, 58, 23, 24, 35, 59, 61, 62, 63, 96, 105, 108, 109].

3 Introducing Generalized Verma Modules

In this Chapter we introduce the main object of our interest: Generalized Verma modules. We establish their basic properties and introduce one important tool: the generalized Harish-Chandra homomorphism.

3.1 Definition

Let $\mathcal{P} \supset \mathfrak{B}$ be a parabolic subalgebra of $\mathfrak{G}$. Denote by $\mathfrak{N}$ the nilpotent radical of $\mathcal{P}$ and by $\mathfrak{A}'$ the Levi factor of $\mathcal{P}$. Then $\mathfrak{N} \subset \mathfrak{N}_+$, $\mathfrak{A}'$ is a reductive Lie algebra, which can be decomposed $\mathfrak{A}' = \mathfrak{A} \oplus \mathfrak{H}_{\mathfrak{A}}$, where $\mathfrak{A}$ is semi-simple and $\mathfrak{H}_{\mathfrak{A}} \subset \mathfrak{H}$ is abelian and central in $\mathfrak{A}'$. Let V be an $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable $\mathfrak{A}'$-module. Set $\mathfrak{N}V = 0$. In this way we turn V into a $\mathcal{P}$-module. Now we can use the induction from $\mathcal{P}$ to $\mathfrak{G}$ and obtain the module

$$M_{\mathcal{P}}(V) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} V,$$

associated with $\mathfrak{G}$, $\mathcal{P}$ and V. This induction, which is associated with a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{G}$ is usually called a *parabolic induction*.

If the module V is simple we will call $M_{\mathcal{P}}(V)$ a *Generalized Verma module* (GVM). If the parabolic algebra $\mathcal{P}$ is fixed or clear from the context we will usually omit it in the notation and will simply write $M(V)$.

We will also need some more notation. Let $\Delta(\mathfrak{A}) \subset \Delta$ be the root system of $\mathfrak{A}$. Then $\pi(\mathfrak{A}) = \pi \cap \Delta(\mathfrak{A})$ is a basis in $\Delta(\mathfrak{A})$ and $\Delta(\mathfrak{A})_{\pm} = \Delta_{\pm} \cap \Delta(\mathfrak{A})$ are the corresponding sets of positive (negative) roots with respect to $\pi(\mathfrak{A})$. Denote by $W(\mathfrak{A})$ the Weyl group of $\Delta(\mathfrak{A})$, which is a subgroup of W. Let $\mathfrak{N}(\mathfrak{A})$ denote the subalgebra of $\mathfrak{N}_-$ generated by X_α, $\alpha \in (\Delta_- \setminus \Delta(\mathfrak{A}))$. Clearly, $\mathfrak{N}(\mathfrak{A})$ is the image of $\mathfrak{N}$ under σ.

Denote by $\mathfrak{H}(\mathfrak{A})$ the intersection $\mathfrak{H} \cap \mathfrak{A}$. Since $\mathfrak{H}_{\mathfrak{A}}$ is central in $\mathfrak{A}'$, we have that $\mathfrak{H}_{\mathfrak{A}}$ coincides with the orthogonal complement to $\mathfrak{H}(\mathfrak{A})$ in $\mathfrak{H}$ with respect to $(\cdot\,,\cdot)$. Hence we can identify $\mathfrak{H}(\mathfrak{A})^*$ with the subspace of $\mathfrak{H}^*$, generated by all $\alpha \in \Delta(\mathfrak{A})$ and $\mathfrak{H}_{\mathfrak{A}}^*$ with the orthogonal complement to $\mathfrak{H}(\mathfrak{A})^*$ in $\mathfrak{H}^*$. Let $p_{\mathfrak{A}} : \mathfrak{H}^* \to \mathfrak{H}^*$ denote the projection on $\mathfrak{H}_{\mathfrak{A}}^*$ with respect to $\mathfrak{H}(\mathfrak{A})^*$. $\mathfrak{H}_{\mathfrak{A}}^*$ inherits a natural partial order from the order $\leqslant$ on $\mathfrak{H}^*$. We will denote this order by the same symbol and set $\mu_1 \leqslant \mu_2$ for $\mu_1, \mu_2 \in \mathfrak{H}_{\mathfrak{A}}^*$ if there exists $\mu_3 \in \mathfrak{H}^*$ such that $p_{\mathfrak{A}}(\mu_3) = \mu_2$ and $\mu_1 \leqslant \mu_3$ (as elements in $\mathfrak{H}^*$).

3.2 Basic properties

For this Section we fix a GVM $M(V) = M_{\mathcal{P}}(V)$ and keep the notation of Section 3.1. Since V is a simple $\mathfrak{A}'$-module and $\mathfrak{H}_{\mathfrak{A}}$ is abelian and central, $\mathfrak{H}_{\mathfrak{A}}$ acts on V by scalars, i.e. via some $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. So for the rest of this section we fix V, $M(V)$ and λ above. In order to give a formal description of the action of $\mathfrak{H}_{\mathfrak{A}}$ we include it in a more general context.

Let A be a Lie algebra and B be an abelian subalgebra of A. An A-module M will be called *weight* module with respect to B (or, simply, *B-weight module*), if

$$M = \oplus_{\lambda \in B^*} M_\lambda,$$

where $M_\lambda = \{v \in M \mid bv = \lambda(b)v \text{ for all } b \in B\}$. For a weight A-module M we will denote by $\mathrm{supp}_B(M)$ the set of all $\lambda \in B^*$ such that $M_\lambda \neq 0$. We will also call elements $\lambda \in B^*$ the *B-weights*.

Under this notation, the weight $\mathfrak{G}$-modules, defined in Section 2.2 are just $\mathfrak{H}$-weight $\mathfrak{G}$-modules. Further we will also call $\mathfrak{H}$-weight modules just weight module, in all other cases the corresponding abelian subalgebra of $\mathfrak{G}$ will be mentioned.

Let M be an $\mathfrak{H}_\mathfrak{A}$-weight $\mathfrak{G}$-module. A non-zero element $v \in M_\lambda$, $\lambda \in \mathfrak{H}_\mathfrak{A}^*$ will be called a *$\mathfrak{A}$-highest weight element* (or a *$\mathfrak{A}$-primitive element*, or just a *semiprimitive element*, if $\mathfrak{A}$ is clear from the context) provided $\mathfrak{N}v = 0$. An $\mathfrak{H}_\mathfrak{A}$-weight $\mathfrak{G}$-module M will be called a *$\mathfrak{A}$-highest weight module* if it is generated by a $\mathfrak{A}$-highest weight vector v such that $U(\mathfrak{A})v$ is a simple $\mathfrak{A}$-module. The $\mathfrak{A}$-weight of v will be called the *$\mathfrak{A}$-highest weight* of M.

Proposition 3.2.1. *1. $M(V)$ is a $\mathfrak{A}$-highest weight module with the $\mathfrak{A}$-highest weight λ. Moreover $M(V)_\lambda \simeq V$ as an $\mathfrak{A}$-module.*

2. *$M(V)$ is a $U(\mathfrak{N}(\mathfrak{A}))$-free module and any basis of V is a $U(\mathfrak{N}(\mathfrak{A}))$-basis of $M(V)$.*

3. *$\mathrm{supp}_{\mathfrak{H}_\mathfrak{A}}(M(V))$ coincides with the set of all $\mu \in \mathfrak{H}_\mathfrak{A}^*$ such that $\mu \leqslant \lambda$.*

4. *Any $\mathfrak{A}$-highest weight module M such that λ is the $\mathfrak{A}$-highest weight of M and $M_\lambda \simeq V$ (as an $\mathfrak{A}$-module) is a quotient of $M(V)$.*

5. *$M(V)$ has a unique simple quotient $L(V) = L_\mathcal{P}(V)$.*

6. *$\mathrm{End}_\mathfrak{G}(M(V)) \simeq \mathrm{End}_\mathfrak{A}(V)$. In particular, any $\mathfrak{G}$-endomorphism of $M(V)$ is scalar.*

Proof. The second statement follows from PBW Theorem. The fourth statement follows from the universal property of the tensor product. The last statement follows from [28, Proposition 2.6.5]. The rest is trivial. □

The forth statement of Proposition 3.2.1 is nothing more than the universal property of a GVM. One sees that Proposition 3.2.1 is almost analogous to Theorem 2.2.1, although one statement is missing. We will discuss it in the next Section.

The last basic property of a GVM, which will be explained in this Section is connected with its $\mathfrak{H}_\mathfrak{A}$-weight structure. Consider $U(\mathfrak{G})$ as an $\mathfrak{A}$-module under the adjoint action. Then this module is $\mathfrak{H}_\mathfrak{A}$-weight. Given an $\mathfrak{H}_\mathfrak{A}$ weight λ of $U(\mathfrak{G})$, we will denote by $U(\mathfrak{G})^\lambda$ the vector space, generated by all monomials in $U(\mathfrak{G})_\lambda$, which do not contain any factor from $\mathfrak{A}$. Clearly, $U(\mathfrak{G})^\lambda$ is finite-dimensional and is an $\mathfrak{A}$-submodule of $U(\mathfrak{G})$.

Lemma 3.2.1. *Let λ be the $\mathfrak{A}$-highest weight of $M_\mathcal{P}(V)$ and $\mu \in \mathfrak{H}_\mathfrak{A}^*$. Then $M_\mathcal{P}(V)_\mu$ is isomorphic to $U(\mathfrak{G})^{\mu-\lambda} \otimes V$ as a $\mathfrak{A}$-module.*

Proof. Since $M_\mathcal{P}(V) = U(\mathfrak{G}) \otimes_{U(\mathcal{P})} V$, we have $M_\mathcal{P}(V)_\mu = U(\mathfrak{G})^{\mu-\lambda} \otimes V$ as a vector space. If we consider $U(\mathfrak{G})^{\mu-\lambda}$ as a $\mathfrak{A}$-module under adjoint action, the equality above becomes an isomorphism of $\mathfrak{A}$-modules. □

3.3 Generalized Harish-Chandra homomorphism

One of the most important tools in studing of GVMs is the so-called generalized Harish-Chandra homomorphism, defined and studied in [32]. We will follow [32] in this section so the reader can consult [32] for any technical details which will be missed.

Again, we fix a GVM $M(V)$ as in the previous section. The reason to introduce a generalization of the Harish-Chandra homomorphism is to study of the action of $Z(\mathfrak{G})$ on $M(V)$. This will be explained in this section.

Define $L(\mathfrak{A}) = U_0 \cap U(\mathfrak{G})\mathfrak{N}$. Then it is clear, that $L(\mathfrak{A})$ is a two-sided ideal in U_0, $L(\mathfrak{A}) = U_0 \cap \mathfrak{N}(\mathfrak{A})U(\mathfrak{G})$ and $U_0 = L(\mathfrak{A}) \oplus (U(\mathfrak{A})_0 \otimes U(\mathfrak{H}_{\mathfrak{A}}))$. We define the *generalized Harish-Chandra homomorphism* (or *$\mathfrak{A}$-Harish-Chandra homomorphism*) $\varphi_{\mathfrak{A}}$ as the projection of U_0 on $U(\mathfrak{A})_0 \otimes U(\mathfrak{H}_{\mathfrak{A}})$ with respect to $L(\mathfrak{A})$. We note, that $\varphi_{\mathfrak{A}}$ is completely determined by a subset $\pi(\mathfrak{A})$ of π and if $\pi(\mathfrak{A}) = \varnothing$, i.e. $\mathfrak{A} = 0$, then $\pi(\mathfrak{A})$ coincides with the classical Harish-Chandra homomorphism φ (see [28, Chapter 7]).

The next lemma explains an importance of $\mathfrak{A}$-Harish-Chandra homomorphism in the study of $\mathfrak{A}$-highest weight modules, in particular, in the study of GVMs.

Lemma 3.3.1. *Let M be a $\mathfrak{A}$-highest weight module with the $\mathfrak{A}$-highest weight $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Then for any $u \in U_0$ and for any $v \in M_\lambda$ holds $uv = \varphi_{\mathfrak{A}}(u)v$.*

Proof. Follows directly form the definitions of $\mathfrak{A}$-highest weight and $\mathfrak{A}$-Harish-Chandra homomorphism. ☐

Since $Z(\mathfrak{G}) \subset U_0$, we can compute the action of any $z \in Z(\mathfrak{G})$ on a GVM $M(V)$ directly by an application of Lemma 3.3.1. Hence, it is necessary to study the image of $Z(\mathfrak{G})$ under $\varphi_{\mathfrak{A}}$.

Let $\rho_{\mathfrak{A}}$ denote half the sum of all positive roots in $\Delta(\mathfrak{A})$ and let $\rho^{\mathfrak{A}} = \rho - \rho_{\mathfrak{A}}$. Denote by γ the automorphism of $S(\mathfrak{H})$ defined by $\gamma(f)(\lambda) = f(\lambda - \rho)$ (we say that γ is the *shift by* ρ). Analogously, let $\gamma_{\mathfrak{A}}$ be the automorphism of $S(\mathfrak{H}(\mathfrak{A}))$ defined by $\gamma_{\mathfrak{A}}(f)(\lambda) = f(\lambda - \rho_{\mathfrak{A}})$. Let $\gamma^{\mathfrak{A}} = \gamma|_{S(\mathfrak{H}_{\mathfrak{A}})}$.

Denote by i the restriction of the composition $(1 \otimes \gamma^{\mathfrak{A}}) \circ \varphi_{\mathfrak{A}} : U_0 \to U(\mathfrak{A})_0 \otimes U(\mathfrak{H}_{\mathfrak{A}})$ to $Z(\mathfrak{G})$ and by $i_{\mathfrak{A}}$ the restriction of $(\gamma_{\mathfrak{A}} \circ \varphi) \otimes 1 : U(\mathfrak{A})_0 \otimes S(\mathfrak{H}_{\mathfrak{A}}) \to S(\mathfrak{H}(\mathfrak{A})) \otimes S(\mathfrak{H}_{\mathfrak{A}})$ to $Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$. Let $m : S(\mathfrak{H}(\mathfrak{A})) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \to S(\mathfrak{H})$ be a canonical isomorphism. For a semisimple Lie algebra A with a fixed Cartan subalgebra we will denote by a_A the canonical inclusion of $Z(A)$ into $U(A)_0$.

Theorem 3.3.1. *The following diagrams commute:*

1.

$$\begin{array}{ccc}
U_0 & \xrightarrow{(1\otimes\gamma^{\mathfrak{A}})\circ\varphi_{\mathfrak{A}}} & U(\mathfrak{A})_0 \otimes S(\mathfrak{H}_{\mathfrak{A}}) \\
{\scriptstyle \gamma\circ\varphi}\downarrow & & \downarrow{\scriptstyle (\gamma_{\mathfrak{A}}\circ\varphi)\otimes 1} \\
S(\mathfrak{H}) & \xleftarrow{m} & S(\mathfrak{H}(\mathfrak{A})) \otimes S(\mathfrak{H}_{\mathfrak{A}})
\end{array}$$

2.

$$\begin{tikzcd}
Z(\mathfrak{G}) \arrow[r, "i"] \arrow[d, "a_{\mathfrak{G}}"'] & Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[r, "m \circ i_{\mathfrak{A}}"] \arrow[d, "a_{\mathfrak{A}} \otimes 1"] & S(\mathfrak{H}) \arrow[d, "\mathrm{id}"'] \\
U_0 \arrow[r, "(1 \otimes \gamma^{\mathfrak{A}}) \circ \varphi_{\mathfrak{A}}"] & U(\mathfrak{A})_0 \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[r, "m \circ ((\gamma_{\mathfrak{A}} \circ \varphi) \otimes 1)"] & S(\mathfrak{H})
\end{tikzcd}$$

3.

$$\begin{tikzcd}
Z(\mathfrak{G}) \arrow[r, "i"] \arrow[d, "\gamma \circ \varphi"'] & Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[d, "(\gamma_{\mathfrak{A}} \circ \varphi) \otimes 1"] \\
S(\mathfrak{H})^W \arrow[r] & S(\mathfrak{H})^{W(\mathfrak{A})}
\end{tikzcd}$$

Proof. Direct verification, which uses Harish-Chanda isomorphism Theorem ([28, Theorem 7.4.5]). $\square$

Corollary 3.3.1. $\varphi_{\mathfrak{A}}(Z(\mathfrak{G})) \subset Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$.

Hence, the action of $Z(\mathfrak{G})$ on $M(V)$ can be computed from the action of $Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$ on V. Since V is a simple $\mathfrak{A}'$-module, $\mathfrak{H}_{\mathfrak{A}}$ and hence $S(\mathfrak{H}_{\mathfrak{A}})$ acts on V by scalars, which correspond to $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Since V is a simple $\mathfrak{A}$-module, we can apply Quillen's Lemma ([28, Proposition 2.6.8]) and obtain that $Z(\mathfrak{A})$ acts on V by scalars. Hence $Z(\mathfrak{G})$ acts on $M(V)$ by scalars. This means that we have already proved the following result.

Corollary 3.3.2. *M(V) admits a central character.*

Having the map $i : Z(\mathfrak{G}) \to Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$, we can define a natural dual map $i^* : (Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}))^* \to Z(\mathfrak{G})^*$. One more important consequence of Theorem 3.3.1 and the classical Galois theory is the following statement.

Corollary 3.3.3. *For each $\chi \in Z(\mathfrak{G})^*$ the set $(i^*)^{-1}(\chi)$ is finite. Moreover $|(i^*)^{-1}(\chi)|$ is less or equal to the index of $W(\mathfrak{A})$ in W.*

4 Motivations and Tools

In this Chapter we present some motivation for why one should be interested in studying GVMs. In particular, GVMs appear in different generalizations of the $\mathcal{O}$ category, e.g. in the category $\mathcal{O}_S$ introduced by Rocha-Caridi ([103]) or in the category $\mathcal{O}^\alpha$ introduced by Coleman and Futorny ([22]). GVMs also play an important role in the classification of all simple weight $\mathfrak{G}$-modules.

In the last two sections of this chapter we recall a Theorem of Kostant ([79, Theorem 5.1]) and a localization of $U(\mathfrak{G})$ introduced by Mathieu ([86]). They will be essential tools in our study.

4.1 Classification of simple weight modules

The main problem in the representation theory is to classify the representations of some algebraic object. Usually, this problem is very difficult. For example, in some cases it is equivalent to the problem of classification of all representations for all objects of the same kind. Such a situation is called *wild* (of course this is not a formal definition, see, for example, [29]). Thus usually one restricts the general problem to classification of some reasonable classes of representations. One of the most natural classes of representations of a simple complex finite-dimensional Lie algebra is the class of all weight modules. For example, all finite-dimensional modules ($L(\lambda)$, $\lambda \in P^{++}$) are weight modules. Hence, the problem to clasify all weight (in particular, all simple weight) $\mathfrak{G}$-modules is natural and very important. In this section we describe, what is known here and how this problem relates to the study of GVMs.

First of all about the history of the question. A classification of all simple $\mathfrak{G}$-modules is known only for $\mathfrak{G} = sl(2,\mathbb{C})$ ([13] or [4] for a more general result). A classification of simple weight $\mathfrak{G}$-modules with finite-dimensional weight spaces was recently completed by Mathieu ([86]), some partial results towards it were obtained in [39, 17] (see also references therein). For weight modules with infinite-dimensional weight spaces the problem is still open, however there are some partial results, see [39, 30].

Now, about the problem itself. One of the most important results for the solution of this problem is the Theorem, proved independently in some special cases by Fernando ([36]) and Futorny ([40]) and then in a complete generality in [25] (using a computer). Recently an easy proof was found in [27]. To state the theorem, we have to introduce the notion of a dense module.

Recall that $\mathfrak{G}$ and $\mathfrak{H}$ are fixed. A weight $\mathfrak{G}$-module M is called *dense* if $\mathrm{supp}(M)$ coincides with the set $\lambda + \mathbb{Z}\Delta$ for some $\lambda \in \mathfrak{H}^*$. It is clear, that if M is indecomposable, then $\mathrm{supp}(M)$ is a subset of some $\lambda + \mathbb{Z}\Delta$. Hence, for an indecomposable M, dense means, that M has the maximal possible support. Now the mentioned result (which we will call the Fernando-Futorny Theorem) is the following.

Theorem 4.1.1. *Let M be a simple weight $\mathfrak{G}$-module. Then M is either dense or there exists a parabolic subalgebra $\mathcal{P} \supset \mathfrak{B}$ in Γ with the Levi decomposition $\mathcal{P} = \mathfrak{A}' \oplus \mathfrak{N}$ and a simple dense module V over $\mathfrak{A}'$, such that $M \simeq L_{\mathcal{P}}(V)$.*

We note, that the module V, given in Theorem 4.1.1 is automatically $\mathfrak{H}(\mathfrak{A})$-weight. This result is quite non-trivial, especially if one looks at [25] and the computer list presented there. The proof in [27] is tricky and uses both ideas from [36] and [40].

In principal, Theorem 4.1.1 decomposes our classification problem into two parts. The first one: to classify all dense modules. Exactly this was done in [86] for weight modules with finite-dimensional weight spaces. The second problem is to study modules $L_{\mathcal{P}}(V)$, or more general, modules $M_{\mathcal{P}}(V)$, which are GVMs, associated with $\mathfrak{H}(\mathfrak{A})$-weight modules. We also note, that some information about $L_{\mathcal{P}}(V)$, in the case when V has finite-dimensional $\mathfrak{A}$-weight spaces was also obtained in [86]. In fact, the character of $L_{\mathcal{P}}(V)$ was computed, but not directly, only by reducing the problem to the character of some $L(\lambda)$.

In principal, the claim that Theorem 4.1.1 reduces the classification of simple weight modules do the classification of simple dense modules is a bit unfair, since given $\mathcal{P}$ and a simple $\mathfrak{A}$-weight module V the construction of $L_{\mathcal{P}}(V)$ may be a very difficult problem. This is not easy even for Verma modules, since it involves at least a description of a maximal submodule in $M(\lambda)$, which is far from being trivial in general.

Anyway, Theorem 4.1.1 presents a good motive to study GVMs and the corresponding simple quotients.

4.2 Category $\mathcal{O}_S$

The Verma modules appear as important intermediate objects in the category $\mathcal{O}$. Analogously, GVMs appear in different generalizations of $\mathcal{O}$. One of them, which we will describe in this Section was introduced and studied by Rocha-Caridi in [103]. We fix notation from Section 3.1. Original notation $\mathcal{O}_S$ relates to a subset S of π, which defines the semisimple part of the Levi factor of $\mathcal{P}$. In our situation $S = \pi(\mathfrak{A})$, but we retain the classical notation for the whole category.

Define $\mathcal{O}_S$ to be the full subcategory of the category of $\mathfrak{G}$-modules consisting of the modules M such that

- M is finitely generated;
- M is a direct sum of simple finite-dimensional $\mathfrak{A}'$-modules, when viewed as an $\mathfrak{A}'$-module;
- M is $\mathfrak{N}$-finite, i.e. $U(\mathfrak{N})v$ is finite-dimensional for any $v \in M$.

We note, that the second condition is equivalent to saying, that M is $\mathfrak{H}$-diagonalizable and is a direct sum of finite-dimensional $\mathfrak{A}$-modules, when viewed as an $\mathfrak{A}$-module.

It happened that $\mathcal{O}_S$ has properties, similar to those of $\mathcal{O}$.

Theorem 4.2.1. *1. $\mathcal{O}_S$ is a full subcategory of $\mathcal{O}$.*

2. $\mathcal{O}_S$ is closed under taking submodules, quotients, finite direct sums and under tensoring with finite-dimensional modules.

3. *All GVMs $M_{\mathcal{P}}(V)$, where V is finite-dimensional, belong to $\mathcal{O}_S$.*

4. *$L_{\mathcal{P}}(V)$, V is simple finite-dimensional, exhaust the set of simple modules in $\mathcal{O}_S$.*

5. *Any module in $\mathcal{O}_S$ has a composition series.*

6. *$\mathcal{O}_S = \oplus_{\chi\in Z(\mathfrak{G})^*}\mathcal{O}_S(\chi)$, where $\mathcal{O}_S(\chi)$ is a full subcategory consisting of all modules, which are annihilated by a big enough power of $z-\chi(z)$ for all $z\in Z(\mathfrak{G})$.*

As in $\mathcal{O}$, category $\mathcal{O}_S$ has a block decomposition, with each block being a module category over a finite-dimensional algebra.

Theorem 4.2.2. *1. $\mathcal{O}_S$ has enough projective modules (i.e. any module in $\mathcal{O}_S$ is a quotient of a projective module in $\mathcal{O}_S$).*

2. *There is a bijection between simple modules and indecomposable projective modules in $\mathcal{O}_S$. We will denote by $P(L)$ the projective cover of a simple module $L\in\mathcal{O}_S$.*

3. *Each $\mathcal{O}_S(\chi)$ is equivalent to the category of (finite-dimensional) modules over a finite-dimensional algebra.*

And finally, there is an analogue of BGG reciprocity.

Theorem 4.2.3. *1. Any projective module in $\mathcal{O}_S$ admits a generalized Verma flag (i.e. a filtration, whose subquotients are $M_{\mathcal{P}}(V)$, V finite-dimensional).*

2. *For any finite-dimensional $\mathfrak{A}'$-modules V_1 and V_2 holds $[P(L_{\mathcal{P}}(V_1)) : M_{\mathcal{P}}(V_2)] = (M_{\mathcal{P}}(V_2) : L_{\mathcal{P}}(V_1))$, where $[P(L_{\mathcal{P}}(V_1)) : M_{\mathcal{P}}(V_2)]$ denotes the number of occurrences of $M_{\mathcal{P}}(V_2)$ in a generalized Verma flag of $P(L_{\mathcal{P}}(V_1))$. In particular, the last is a well-defined number (i.e. does not depend on a generalized Verma flag).*

The proofs of all theorems can be found in [103], for a more general context see also [105, 96].

The finite-dimensional algebras, arising in $\mathcal{O}_S$ are related the the algebras, arizing from $\mathcal{O}$. In fact, they are Koszul dual to algebras corresponding to the singular blocks of $\mathcal{O}$ ([7]). From Theorem 4.2.2 and Theorem 4.2.3 it follows easily, that the blocks of $\mathcal{O}_S$ correspond to quasi-hereditary algebras, so one can construct, for example, tilting modules. But so far this theory is not completed.

So, we see, that GVMs naturally occur and play an important role in the category $\mathcal{O}_S$.

4.3 Category $\mathcal{O}^\alpha$

GVMs appear also in another generalization of $\mathcal{O}$, proposed by Coleman and Futorny in [22]. Assume that $\mathfrak{A}$ is isomorphic to $sl(2,\mathbb{C})$. Then $\pi(\mathfrak{A})$ contains the unique element, say α. This is why we will denote the category, associated with such $\mathcal{P}$ by $\mathcal{O}^\alpha$. To proceed we need one more notation, introduced in [22]. A weight $\mathfrak{G}$-module M with finite-dimensional weight spaces will be called α-stratified provided X_α and $X_{-\alpha}$ act injectively (and hence bijectively) on M. We also fix the quadratic Casimir operator C of $U(\mathfrak{G})$ ([68, p.22]).

We define $\mathcal{O}^\alpha$ the be the full subcategory of the category of $\mathfrak{G}$-modules, consisting of all modules M such that

- M is finitely generated;
- M is $\mathfrak{H}$-diagonalizable;
- M is α-stratified;
- M is $\mathfrak{N}$-finite, i.e. $U(\mathfrak{N})v$ is finite-dimensional for all $v \in M$;
- M is C-diagonalizable.

The fifth condition seems to be quite disputable. The reason to add it in the definition above was to avoid the self-extensions of GVMs inside $\mathcal{O}^\alpha$. But it happens that exacly this condition causes difficulty in naturally generalizing some of the properties of category $\mathcal{O}$ to $\mathcal{O}^\alpha$. We improve the situation later in Chapter 10. Nevertheless, now we list the basic properties of $\mathcal{O}^\alpha$.

Theorem 4.3.1. *1. $\mathcal{O}^\alpha$ is closed under taking submodules, quotients and finite direct sums.*

2. All GVMs $M_{\mathcal{P}}(V)$, where V is a simple α-stratified $\mathfrak{A}$-module, belong to $\mathcal{O}^\alpha$.

3. $L_{\mathcal{P}}(V)$, where V is a simple α-stratified $\mathfrak{A}$-module, exhaust the set of all simple modules in $\mathcal{O}^\alpha$.

4. Any module in $\mathcal{O}^\alpha$ has a composition series.

5. $\mathcal{O}^\alpha = \oplus_{\chi\in Z(\mathfrak{G})^*}\mathcal{O}^\alpha(\chi)$, where $\mathcal{O}^\alpha(\chi)$ is a full subcategory consisting of all modules, which are annihilated by a big enough power of $z-\chi(z)$ for all $z \in Z(\mathfrak{G})$.

The proof can be found in [22, Section 4].

First, we note, that unlike $\mathcal{O}$ and $\mathcal{O}_S$, the category $\mathcal{O}^\alpha$ is not closed under tensoring with finite-dimensional modules. The reason is that such a tensor product does not preserve the fifth condition (but it preserves all other conditions).

Secondly, all simple α-stratified $\mathfrak{A}$-modules can be easily classified. This is a part of classical $sl(2,\mathbb{C})$ theory. We will need this later, so now we present a construction of all such modules.

Fix $a, b \in \mathbb{C}$ and denote by $\mathfrak{c} = (H_\alpha+1)^2 + 4X_{-\alpha}X_\alpha$ the quadratic Casimir oparator of $U(\mathfrak{A})$. Denote by $V(a,b)$ the $\mathfrak{A}$-module with the basis v_i, $i \in \mathbb{Z}$ and the action of generators of $\mathfrak{A}$ defined as follows:

$$H_\alpha v_i = (a+2i)v_i, \qquad X_{-\alpha}v_i = v_{i-1}, \qquad X_\alpha v_i = \frac{1}{4}(b-(a+2i+1)^2)v_{i+1}.$$

We note that b is a unique eigenvalue of $\mathfrak{c}$ on $V(a,b)$.

Proposition 4.3.1. *1. $V(a,b)$ is an indecomposable dense weight module of length $\leqslant 3$ with one dimensional weight spaces.*

2. $V(a,b)$ is simple if and only if $V(a,b)$ is α-stratified if and only if $b \neq (a+2j+1)^2$ for all $j \in \mathbb{Z}$.

3. The set of simple $V(a,b)$ exhaust the set of simple α-stratified $\mathfrak{A}$-modules.

4. $V(a,b) \simeq V(a',b')$ if and only if $b = b'$ and $a = a' + 2k$ for some $k \in \mathbb{Z}$.

Proof. The first and the second statements follow directly form the definition of $V(a,b)$. To prove the rest we recall one known fact ([32, Lemma 3]), that for any simple $\mathfrak{H}(\mathfrak{A})$-weight $\mathfrak{A}$-module M any weight space M_λ, $\lambda \in \mathfrak{H}(\mathfrak{A})^*$ is a simple $U(\mathfrak{A})_0$-module and any simple $\mathfrak{H}(\mathfrak{A})$-weight $U(\mathfrak{A})_0$-module N has a unique extension to a simple $\mathfrak{H}(\mathfrak{A})$-weight $\mathfrak{A}$-module. The algebra $U(\mathfrak{A})_0$ is generated by H_α and $\mathfrak{c}$ and thus is commutative. Hence, any simple $\mathfrak{H}(\mathfrak{A})$-weight $U(\mathfrak{A})_0$-module is one-dimensional and thus, coincides with some $V(a,b)_\lambda$ for some $a, b \in \mathbb{C}$ and $\lambda \in \mathfrak{H}(\mathfrak{A})^*$. Now the third statement follows from the first and the second ones. The rest is obvious. □

Using Proposition 4.3.1, one can characterize $V(a,b)$ as a unique $\mathfrak{A}$-module, having the following properties:

- a is an eigenvalue of H_α on $V(a,b)$;
- b is the eigenvalue of $\mathfrak{c}$ on $V(a,b)$;
- $V(a,b)$ is dense;
- $X_{-\alpha}$ acts bijectively (or injectively) on $V(a,b)$;
- all weight spaces of $V(a,b)$ are one-dimensional.

We will need this characterization later.

We return to the category $\mathcal{O}^\alpha$. We have already established its basic properties. The second step is the result about finite-dimensional algebras arising from $\mathcal{O}^\alpha$.

Theorem 4.3.2. *1. $\mathcal{O}^\alpha$ has enough projective modules (i.e. any module in $\mathcal{O}^\alpha$ is a quotient of a projective module in $\mathcal{O}^\alpha$).*

2. There is a bijection between simple modules and indecomposable projective modules in $\mathcal{O}^\alpha$.

3. Each $\mathcal{O}^\alpha(\chi)$ can be decomposed into a direct sum of full subcategories, each of which is equivalent to the category of (finite-dimensional) modules over a finite-dimensional algebra.

The proof of this fact is essentialy [22, Theorem 4.7]. Again, one can see some differences between $\mathcal{O}$ and $\mathcal{O}_S$. The direct summand $\mathcal{O}^\alpha(\chi)$ of $\mathcal{O}^\alpha$ has infinitely many simple objects, hence one should decompose it further in order to reduce the situation to finite-dimensional algebras. The fifth condition from the definition of $\mathcal{O}^\alpha$ causes the fact that the finite-dimensional algebras arising from $\mathcal{O}^\alpha$ are not quasi-hereditary in general (at least in a natural presentation). The problem is that the projective modules in $\mathcal{O}^\alpha$ do not always admit a generalized Verma flag (i.e. a filtration, with quotients of the form $M_{\mathcal{P}}(\mathrm{V(a,b)})$). We will improve this situation in Chapter 10.

Category $\mathcal{O}^\alpha$ is another generalization of $\mathcal{O}$, in which some GVMs appear as natural objects, and whose simple objects are exhaust by the unique simple quotients of GVMs.

We hope, that we have presented enough motivation to claim that the study of GVMs is an interesting and important problem in representation theory of Lie algebras.

4.4 Kostant's Theorem

We have already mentioned, that a lot of information about $\mathfrak{G}$-modules, in particular, about Verma modules or GVMs, can be obtained, studying the action of $Z(\mathfrak{G})$ on these modules. For example, this was a motivation for introducing the generalized Harish-Chandra homomorphism in Section 3.3 (and in [32]). Further, one of the most powerful technical tools for studying Verma modules is the Jantzen translation functor, which is a composition of the tensor product with a finite-dimensional module followed by a projection to $\mathcal{O}(\chi)$. This motivates the exploration of how a central character behaves under tensoring with a finite-dimensional module. This is the content of a famous result of Kostant ([79, Theorem 5.1]). This Theorem will be an important statement for us and we present a complete formulation of it, and will call it Kostant's Theorem in what follows.

We recall, that, according to Quillen's Lemma, any simple $\mathfrak{G}$-module M has a central character. This means that for some $\chi = \chi_M \in Z(\mathfrak{G})^*$ holds: $zv = \chi(z)v$ for all $v \in M$, $z \in Z(\mathfrak{G})$. We also recall, that according to Theorem 2.2.2, for any $\chi \in Z(\mathfrak{G})^*$ there exists $\lambda \in \mathfrak{H}^*$ such that $\chi = \chi_\lambda$, i.e. χ is the central character of $M(\lambda)$.

Theorem 4.4.1. *Let M be a simple $\mathfrak{G}$-module having a central character $\chi_M = \chi_\lambda$ for some $\lambda \in \mathfrak{H}^*$ and F be a finite-dimensional $\mathfrak{G}$-module. Then for any $z \in Z(\mathfrak{G})$ the element*

$$\prod_{\mu \in \mathrm{supp}(F)} (z - \chi_{\lambda+\mu}(z))$$

annihilates $M \otimes F$.

We remark, that under our notations $\mathrm{supp}(F)$ is an ordinary set and not a multi-set. Hence any weight of F is taken only once in the product above. A proof can be found in [79]. We emphasize one useful corollary of this Theorem.

Corollary 4.4.1. *Keep the notation of Theorem 4.4.1 and assume that for any $\mu_1, \mu_2 \in \mathrm{supp}(F)$, $\mu_1 \neq \mu_2$ holds $\chi_{\lambda+\mu_1} \neq \chi_{\lambda+\mu_2}$. Then $M \otimes F$ is $Z(\mathfrak{G})$-diagonalizable.*

In particular, the last means that for almost all (with respect to, say, Lebesgue measure) $\lambda \in \mathfrak{H}^*$ the module $M(\lambda) \otimes F$ is $Z(\mathfrak{G})$-diagonalizable for all finite-dimensional F.

Theorem 4.4.1 can be easily applied for example, to the modules $V(a, b)$ introduced in Section 4.3. Fix $a, b \in \mathbb{C}$ and a finite-dimensional module F of dimension n. Since F is n-dimensional, its highest weight is $n-1$ and the eigenvalue of $\mathfrak{c}$ on F equals n^2. The eigenvalue of $\mathfrak{c}$ on $V(a,b)$ equals b, which coincides with the eigenvalue of $\mathfrak{c}$ on the Verma module $M(\sqrt{b}-1)$ (since $\sqrt{b}$ is defined up to a sign, there can exist, in general, two Verma modules having the same eigenvalue of $\mathfrak{c}$). According to Theorem 4.4.1, the element

$$\prod_{i=0}^{n-1}(\mathfrak{c} - (\sqrt{b} - n + 2i + 1)^2)$$

annihilates $F \otimes V(a,b)$. Hence, the only subquotient that can appear in $F \otimes V(a,b)$ are subquotients of $V(a, (\sqrt{b} - n + 2i + 1)^2)$, $i = 0, 1, \dots, n-1$. Moreover, if $V(a,b)$ is simple and b is not the square of an integer, the module $F \otimes V(a,b)$ is completely reducible. One more thing, if $V(a,b)$ is simple, then any indecomposable submodule of $F \otimes V(a,b)$ is of the length 1 or 2, and any such submodule of the length 2 is a self-extension of some $V(a,k^2)$, $k \in \mathbb{Z}$.

Comparing the dimensions of the weight spaces one can show (we will do it later on in Section 5.7), that $F \otimes V(a,b)$ has precisely $\dim(F)$ subquotients (counted with multiplicities), which has the form $V(a,b')$. This is a nice property, which has no analogue for a finite-dimensional substitution of $V(a,b)$. Really, the length of $F \otimes E$ for F, E finite-dimensional can be smaller than $\dim(F)$.

We also note that for a general simple M it is not known if the module $F \otimes M$ (F, finite-dimensional) has finite length. This was proved by Kostant ([79]) for simple Harish-Chandra modules (i.e. modules coming from group representations), but in general the question is open. We do not know any counterexample, but we also can not prove this, so we will have to be careful later, especially in Chapter 10.

4.5 Mathieu's localization

Another important tool for us will be a special localization of the universal enveloping algebra together with a family of automorphisms, introduced by Mathieu in [86]. As mentioned above, Mathieu studied weight modules with finite-dimensional weight spaces, so his construction works perfectly only in that case. This means that we will not be able to apply it everywhere. Nevertheless, we will use it for the study of α-stratified modules (it can also be applied in a more general situation, which we will not discuss here). We also note, that we will not present Mathieu's construction in complete generality. Since we are going to use it only for α-stratified modules, we will define everything to cover only this case. For the general case we refer the reader to [86].

So, assume that $\mathfrak{A}$ is isomorphic to $sl(2,\mathbb{C})$ and $\pi(\mathfrak{A}) = \{\alpha\}$. The main object of our interests here will be the element $X_{-\alpha}$. Since $\mathfrak{G}$ is a simple finite-dimensional complex Lie algebra, $X_{-\alpha}$ is locally ad-nilpotent on $\mathfrak{G}$ and thus on $U(\mathfrak{G})$. Hence, the multiplicative set S of powers of $X_{-\alpha}$ in $U(\mathfrak{G})$ satisfies the Ore conditions for localizability (see also [86, Lemma 4.2]). Denote by U_S the corresponding localization of U. The following proposition presents a nice family of automorphisms of U_S. This is a partial case of [86, Lemma 4.3] and we have just copied the proof from there.

Proposition 4.5.1. *There exists a unique family of automorphisms $\theta_x : U_S \to U_S$, $x \in \mathbb{C}$ such that*

- *$\theta_x(u) = X_{-\alpha}^x u X_{-\alpha}^{-x}$ for all integer x;*
- *the map $x \mapsto \theta_x(u)$ is polynomial in x for any $u \in U_S$.*

Proof. Clearly, $X_{-\alpha}$ is locally ad-nilpotent on U_S. Thus for a fixed $u \in U_S$ there is a positive integer N such that $\mathrm{ad}\,(X_{-\alpha})^{N+1}(u) = 0$. For $x \geqslant 0$ set

$$\theta_x(u) = \sum_{i=0}^{N} (-1)^i \binom{x}{i} \mathrm{ad}\,(X_{-\alpha})^i(u) X_{-\alpha}^{-i}.$$

As the binomial coefficient can be extended to a polynomial on $\mathbb{C}$, we can polynomially extend θ_x for all $x \in \mathbb{C}$. Uniqueness follows from the uniqueness of the polynomial extension. □

Proposition 4.5.1 allows one to twist special $U(\mathfrak{G})$-modules by θ_x. Indeed, let M be a $\mathfrak{G}$-module on which $X_{-\alpha}$ acts bijectively (for example, M can be α-stratified). Then M extends to a U_S-module $\hat{M}$ in a natural way and we can apply θ_x, $x \in \mathbb{C}$ to obtain the twisted U_S-module $\theta_X(\hat{M})$. Then we can restrict the result on $U(\mathfrak{G})$ and obtain a new $U(\mathfrak{G})$ module, which we will denote by $\theta_x(M)$. We note, that if we start from an α-stratified module M the result $\theta_x(M)$ is not necessary α-stratified. This means that it is possible that the action of X_α on $\theta_x(M)$ is not injective (the action of $X_{-\alpha}$ on $\theta_x(M)$ is bijective by construction). In a more general context one can start from an arbitrary module M and use an induction from $U(\mathfrak{G})$ to U_S, but if $X_{-\alpha}$ acts not injectively on M, some element of M will be annihilated during the induction process. We emphasize the following important property of θ_x.

Lemma 4.5.1. *Let M be a weight $\mathfrak{G}$-module on which $X_{-\alpha}$ acts bijectively and $\lambda \in \mathfrak{H}^*$. Then for any $h \in \mathfrak{H}$, $v \in M_\lambda$ and $x \in \mathbb{C}$ holds $\theta_x(h)v = (\lambda + x\alpha)(h)v$. In particular, $\mathrm{supp}(\theta_x(M)) = \mathrm{supp}(M) + x\alpha$.*

Proof. Suppose that $x \in \mathbb{Z}$. Then $\theta_x(h)v = X_{-\alpha}^x h X_{-\alpha}^{-x} v = (h + x\alpha(h))v = (\lambda + x\alpha)(h)v$. Since θ_x is polynomial in x, $\theta_x(h)v = (\lambda + x\alpha)(h)v$ for any $x \in \mathbb{C}$. The statement about the support is now trivial. □

Lemma 4.5.2. *$\theta_x(z) = z$ for any $x \in \mathbb{C}$ and any $z \in Z(\mathfrak{G})$.*

Proof. Again the statement is trivial for $x \in \mathbb{Z}$. For arbitrary x everything follows now from polynomiality of θ_x. □

As an example, one can take $\mathfrak{G} = \mathfrak{A}$ and $M = V(a,b)$, the module constructed in the Section 4.3. Then $X_{-\alpha}$ acts bijectively on $V(a,b)$ (but $V(a,b)$ is not necessary α-stratified) and one can consider $\theta_x(V(a,b))$. Clearly from Proposition 4.3.1 it follows that $\theta_x(V(a,b)) \simeq V(a',b')$ for some $a', b' \in \mathbb{C}$. By Lemma 4.5.2, $b' = b$. By Lemma 4.5.1, $a' = a + 2x$. Hence $\theta_x(V(a,b)) \simeq V(a+2x, b)$. Further we will use θ_x to α-stratified module, and will see that all the results can be derived from the last calculation with θ_x.

5 An analogue of the BGG Theorem, I

We begin the study of GVMs with an attempt to generalize the BGG Theorem about the existence of non-trivial homomorphisms between Verma modules (Theorem 2.3.1). In this Chapter we cover the case $\mathfrak{A} \simeq sl(2, \mathbb{C})$, which was worked out in [87, 89, 46, 71]. We note, that the results of this chapter will be generalized and reproved in Chapter 8 in a much easier way. Nevertheless we will present the scheme of the original proof here, because the same scheme can be applied for another situation (Chapter 7), which will not be covered in Chapter 8.

As a first step, we present one general sufficient condition for simplicity of $M_{\mathcal{P}}(V)$ ([74]).

5.1 Naive sufficient condition for simplicity

In this section with each GVM we associate a Verma module. We will prove (Theorem 5.1.1), that under some conditions, the simplicity of this Verma module implies the simplicity of the original GVM. Finally, we conjecture that this is the case for any GVM. This is the easiest result about the structure of a GVM and it is based on a study of properties of the generalized Harish-Chandra homomorphism, combined with Kostant's Theorem. We retain the notation from Chapter 3.

Consider a GVM, $M_{\mathcal{P}}(V)$. Let χ_V be the central character of V, where V is considered as a $\mathfrak{A}$-module. Assume that $\mathfrak{H}_{\mathfrak{A}}$ acts on V via $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$ (or, equivalently, λ is the $\mathfrak{A}$-highest weight of $M_{\mathcal{P}}(V)$). Suppose that $\chi_V = \chi_\mu$, $\mu \in \mathfrak{H}(\mathfrak{A})^*$ and μ belongs to the closure of the antidominant Weyl chamber. Now we can consider the module $M_{\mathcal{P}}(M(\mu))$, which is, in fact, the Verma module $M(\nu)$, $\nu \in \mathfrak{H}^*$, such that $p_{\mathfrak{A}}(\nu) = \lambda + \rho^{\mathfrak{A}}$ and $\nu - p_{\mathfrak{A}}(\nu) = \mu$. Set $M(\nu) = \mathfrak{f}(M_{\mathcal{P}}(V))$. We remark, that $M(\mu)$ is a simple Verma module over $\mathfrak{A}$, since μ is chosen to be in the closure of the antidomimant Weyl chamber.

Theorem 5.1.1. *Suppose that* $\mathrm{supp}(M(\nu)) \cap W(\nu) - \rho = \{\nu - \rho\}$. *Then* $M_{\mathcal{P}}(V)$ *is simple.*

We remark, that $\mathrm{supp}(M(\nu)) \cap W(\nu) - \rho = \{\nu - \rho\}$ automatically implies that $M(\nu)$ is simple, according to the BGG Theorem (Theorem 2.3.1).

Proof. In this proof all weight subspaces will be taken with respect to the abelian algebra $\mathfrak{H}_{\mathfrak{A}}$.

Consider $M(\nu)$ as a GVM $M_{\mathcal{P}}(M(\mu))$. Suppose that $M_{\mathcal{P}}(V)$ is not simple and has a non-trivial submodule N. Clearly, N contains a non-zero $\mathfrak{A}$-primitive element $v \in M_{\mathcal{P}}(V)_\xi$, for some $\xi \in \mathfrak{H}_{\mathfrak{A}}^*$ such that $\lambda \neq \xi$. By Lemma 3.2.1, there exists a finite-dimensional $\mathfrak{A}$-module F, such that $M_{\mathcal{P}}(V)_\xi$ is isomorphic to $F \otimes V$, as an $\mathfrak{A}$-module. Applying Kostant's Theorem, can assume that $Z(\mathfrak{A})$ acts on v via some character, say χ.

Consider a $\mathfrak{A}$ module $M_{\mathcal{P}}(M(\mu))_\xi$. Clearly, it is isomorphic to $F \otimes M(\mu)$ as a $\mathfrak{A}$-module. Moreover, since $M(\mu)$ is a Verma module, then $M(\mu) \otimes F$ has a Verma flag ([28, Lemma 7.6.14]). Comparing the filtration in [28, Lemma 7.6.14] with Kostant's Theorem, we conclude that necessarily one of the Verma sub-quotients of $M_{\mathcal{P}}(M(\mu))_\xi$, say $M(\eta)$, has the central character χ.

Consider the GVM $M_{\mathcal{P}}(M(\eta))$ with the $\mathfrak{A}$-highest weight ξ (clearly, $M_{\mathcal{P}}(M(\eta))$ is a Verma module over $\mathfrak{G}$), and let w be its canonical generator. Since w is a $\mathfrak{A}$-primitive element, we can calculate the action of $Z(\mathfrak{G})$ on w in terms of χ and ξ using the generalized Harish-Chandra homomorphism (see Lemma 3.3.1). Moreover, the same can be done for v. This implies, that the central characters of $M_{\mathcal{P}}(M(\eta))$ and $M_{\mathcal{P}}(V)$ and thus the central characters of $M_{\mathcal{P}}(M(\eta))$ and $M(\nu)$ coincide. Let $M_{\mathcal{P}}(M(\eta)) \simeq M(\nu')$ for some $\nu' \in \mathfrak{H}^*$. From Theorem 2.2.2 we immediately obtain, that $\nu' \in W(\nu)$, which contradicts our conditions. This completes the proof. $\square$

Conjecture 5.1.1. *If $\mathfrak{f}(M_{\mathcal{P}}(V))$ is simple then $M_{\mathcal{P}}(V)$ is simple.*

We also believe that this conjecture is true if one replaces $\mathfrak{G}$ with a symmetrizable Kac-Moody Lie algebra and $\mathcal{P}$ with a standard parabolic subalgebra of $\mathfrak{G}$, such that $\mathfrak{A}$ is finite-dimensional.

5.2 The category $\mathcal{K}^\alpha$

In this section we establish some basic properties of weight GVMs in the case $\mathfrak{A} \simeq sl(2,\mathbb{C})$. As in Section 4.3, we assume $\pi(\mathfrak{A}) = \{\alpha\}$. If V is a simple weight $\mathfrak{A}$-module without highest or lowest weights (otherwise we will have $M_{\mathcal{P}}(V)$ is a Verma module), then $V \simeq V(a,b)$ for some $a, b \in \mathbb{C}$, $b \neq (a+2i+1)^2$ for all $i \in \mathbb{Z}$, according to Proposition 4.3.1. In this case both X_α and $X_{-\alpha}$ act bijectively on $V(a,b)$. By Lemma 3.2.1, any $M_{\mathcal{P}}(V(a,b))_\mu$, $\mu \in \mathfrak{H}_{\mathfrak{A}}^*$ is isomorphic to $F \otimes V(a,b)$ for some finite-dimensional $\mathfrak{A}$-module F. Let $V(a',b')$ be a simple subquotient of $F \otimes V(a,b)$. If we recall our example from Section 4.4, we will see, that $b' \neq (a'+2i+1)^2$ for all $i \in \mathbb{Z}$. Hence $M_{\mathcal{P}}(V(a,b))$ is α-stratified.

It will be more convenient to study modules $M_{\mathcal{P}}(V(a,b))$ for all $a, b \in \mathbb{C}$. From the previous paragraph it follows, that $M_{\mathcal{P}}(V(a,b))$ is α-stratified if and only if $V(a,b)$ is α-stratified. To be able to work with any $a, b \in \mathbb{C}$ we introduce a new category of $\mathfrak{G}$-modules. Let $\mathcal{K}^\alpha$ denote the full subcategory of the category of all $\mathfrak{G}$-modules, which consists of all weight modules with finite-dimensional weight spaces, on which $X_{-\alpha}$ acts bijectively. Clearly, all $M_{\mathcal{P}}(V(a,b))$ are objects of $\mathcal{K}^\alpha$. In this section, we will study the properties of $M_{\mathcal{P}}(V(a,b))$ inside the category $\mathcal{K}^\alpha$.

Lemma 5.2.1. *K^α is closed under taking kernels and cokernels of morphisms and under taking finite direct sums.*

Proof. Let $M_1, M_2 \in K^\alpha$ and $f : M_1 \to M_2$ be a homormorphism. First we prove that $X_{-\alpha}$ acts bijectively on the kernel of f. Indeed, $X_{-\alpha}$ acts injectively on it, since it is a submodule of M_1. Let $x \in M_1$, $f(x) = 0$ and $y \in M_1$ be such that $X_{-\alpha}y = x$. Then $X_{-\alpha}f(y) = f(x) = 0$ and hence $f(y) = 0$, since $X_{-\alpha}$ acts injectively on M_2. This means that $X_{-\alpha}$ acts bijectively on the kernel of f.

Since M_1 has finite-dimensional weight spaces, we obtain, that $X_{-\alpha}$ also acts bijectively on $M_1/\ker(f)$. Thus, $X_{-\alpha}$ acts bijectively on the image of f. Finally, since M_2 has finite-dimensional weight spaces, we obtain that $X_{-\alpha}$ acts bijectively on the cokernel of f. $\square$

It is convenient to reparameterize our GVMs. The module $M_{\mathcal{P}}(V(a,b))$ is uniquely determined by the $\mathfrak{H}$-weight $\lambda-\rho$ of one of its primitive generators v and by the mentioned $b=p^2$ for some $p\in\mathbb{C}$, which is the eigenvalue of $\mathfrak{c}$ on v. We set $M(\lambda,p)=M_{\mathcal{P}}(V(a,b))$. Under this notation we always have $M(\lambda,p)\simeq M(\lambda,-p)\simeq M(\lambda+k\alpha,p)$ for all $k\in\mathbb{Z}$. Clearly, $M(\lambda,p)$ is α-stratified if and only if $\pm p\neq\lambda(H_\alpha)+2i$ for all $i\in\mathbb{Z}$. We can always choose λ such that $(\lambda-\rho)(H_\alpha)=a$. Sometimes it will be convenient to fix such λ.

Lemma 5.2.2. *Each $M(\lambda,p)$ has a unique simple quotient object $L(\lambda,p)$ in $\mathcal{K}^\alpha$. Moreover, $L(\lambda,p)$ is a simple $\mathfrak{G}$-module if and only if $M(\lambda,p)$ is α-stratified.*

Proof. The proof of the first statement is analogous to the proof of the corresponding statement for Verma modules. The second statement is obvious. □

Let v_i be the basis of $V(a,b)$, defined in Section 4.3. Then, by Proposition 3.2.1 (the second part of it can be easily extended on the case of a non-simple V), $1\otimes v_i$, $i\in\mathbb{Z}$ is a $U(\mathfrak{N}(\mathfrak{A}))$-basis of $M(\lambda,p)$.

Lemma 5.2.3. *For any $w\in M(\lambda,p)$ there exists $i\in\mathbb{Z}$ and $u\in U(\mathfrak{N}_-)$ such that $w=uv_i$.*

Proof. For some $k\in\mathbb{N}$, $a_j\in\mathbb{C}$ and $u_j\in U(\mathfrak{N}(\mathfrak{A}))$, $-k\leqslant j\leqslant k$, we have

$$w=\sum_{j=-k}^{k}(a_ju_jv_j)=\sum_{j=-k}^{k}(a_ju_jX_{-\alpha}^{k-j}v_k)=\left(\sum_{j=-k}^{k}a_ju_jX_{-\alpha}^{k-j}\right)v_k.$$

This completes the proof. □

Proposition 5.2.1. *1. Any two non-trivial subobjects (in K^α) of $M(\lambda,p)$ have a non-trivial intersection. In particular, $M(\lambda,p)$ contains a unique minimal subobject.*

2. $\dim\mathrm{Hom}(M(\lambda,p),M(\mu,q))\leqslant 1$ and any non-zero homomorphism from this space is a monomorphism.

Proof. Follows from Lemma 5.2.3 and the fact that $U(\mathfrak{N}_-)$ has no zero divisors by standard arguments (see, for example [28, Proposition 7.6.3,Theorem 7.6.6]). □

5.3 A_2 case

In this Section we explain the results from [38]. This was the first attempt to study the structure of GVMs in the simplest case, when $\mathfrak{G}$ is isomorphic to $sl(3,\mathbb{C})$. In this case we can assume that $\mathfrak{A}\simeq sl(2,\mathbb{C})$, $\pi(\mathfrak{A})=\{\alpha\}$, $\pi=\{\alpha,\beta\}$. We retain notation from Section 5.1. Our main result here is the following statement.

Theorem 5.3.1. *Denote $n^\pm=n^\pm(\lambda,p)=(\lambda(H_\alpha+2H_\beta)\pm p)/2$.*

1. $M(\mu,q)\subset M(\lambda,p)$ if and only if $\mu=\lambda-n\beta+k\alpha$, $n\in\mathbb{Z}_+$, $k\in\mathbb{Z}$, and one of the following conditions holds:

(i) $n = 0$ and $q = \pm p$;

(ii) $n \in \{n^{\pm}(\lambda, p)\}$ and $q^2 = (p \mp n^{\pm}(\lambda, p))^2$.

Moreover, if $M(\lambda, p)$ contains a unique non-trivial Verma submodule $V(\mu, q)$, then $M(\lambda, p) \supset M(\mu, q) \supset 0$ is a composition series of $M(\lambda, p)$ (in $\mathcal{K}^{\alpha}$).

2. *If $n_1 = n^-(\lambda, p) \in \mathbb{N}$ and $n_2 = n^+(\lambda, p) \in \mathbb{N}$, then*

$$M(\lambda, p) \supset M(\lambda - n_1\beta, p + n_1) \supset M(\lambda - n_2\beta, p - n_2) \supset 0,$$

moreover, this is a composition series of $M(\lambda, p)$ (in $\mathcal{K}^{\alpha}$).

3. *The module $M(\lambda, p)$ is a simple object in $\mathcal{K}^{\alpha}$ if and only if $n^{\pm}(\lambda, p) \notin \mathbb{N}$.*

We note, that this statement gives us more information than we have expected. In fact, the statement additionally presents a composition series for $M(\lambda, p)$. This is a phenomena for $sl(3, \mathbb{C})$ case and we will not recover it in this Chapter for other cases. One can also note, that the proof of this additional information is ideologically the most difficult part of the whole proof.

Proof. The original proof from [38] is based on direct technical calculation. We will omit these technical detailes, which can be found in [38].

Clearly, to prove the theorem, we have to determine the set of $\mathfrak{A}$-primitive elements in $M(\lambda, p)$ (this will be a direct generalization of the BGG Theorem). Suppose that $v_0 \in M(\lambda, p)_{\lambda-\rho}$ (i.e. $(\lambda - \rho)(H_{\alpha}) = a$). Let $u \in M(\lambda, p)_{\lambda-\rho+k\alpha-n\beta}$, $k \in \mathbb{Z}$, $n \in \mathbb{Z}_+$ be a non-zero element. Then

$$u = \sum_{j=0}^{n} a_j X_{-\alpha-\beta}^{j} X_{-\beta}^{n-j} \otimes v_{j+k}. \tag{1}$$

The element u is $\mathfrak{A}$-primitive if and only if $X_{\beta}u = X_{\alpha+\beta}u = 0$. Using (1), we can write these conditions as a system of linear equations on a_j. Solving this sistem, we obtain that a non-trivial solution exists if and only if $n = 0$ or $p^2 = (-2n + \lambda(H_{\alpha} + 2H_{\beta}))^2$. This, together with Proposition 5.2.1, in fact, proves all statements about the embeddings of GVMs.

To complete the proof we have to show that the natural embeddings of GVMs form a composition series. We will prove this for the second statement. The corresponding result for the first statement can be obtained by the same arguments. We have $M(\lambda - n_1\beta, p + n_1) \subset M(\lambda, p)$, $M(\lambda - n_2\beta, p - n_2) \subset M(\lambda, p)$ and $M(\lambda - n_2\beta, p - n_2) \subset M(\lambda - n_1\beta, p + n_1)$ by virtue of the first part of the Theorem. Now one can easily calculate, that $\dim(M(\lambda, p)_{\lambda-\rho-n\beta}) = n+1$. Let $L_1 = M(\lambda, p)/M(\lambda - n_1\beta, p + n_1)$. From Proposition 5.2.1 it follows that $\dim(L_1)_{\lambda-\rho-n\beta} = n+1$, if $n < n_1$ and $\dim(L_1)_{\lambda-\rho-n\beta} = n_1$, if $n \geqslant n_1$. Denote by $\mathfrak{G}_1$ the $sl(2, \mathbb{C})$ subalgebra of $\mathfrak{G}$, generated by $X_{\pm\beta}$. Since α and β are not orthogonal, from the standard $sl(2, \mathbb{C})$ calculation it follows that there exists $i \in \mathbb{Z}$ such that the module $U(\mathfrak{G}_1)v_i$ is simple. Hence, $X_{-\beta}$ acts injectively on $L_{\lambda-\rho+i\alpha-j\beta}$ for any quotient L

of $M(\lambda, p)$ and any $j \in \mathbb{Z}_+$. This implies, in particular, that L_1 is simple. By the same arguments the module $L_2 = M(\lambda - n_1\beta, p + n_1)/M(\lambda - n_2\beta, p - n_2)$ is also simple as good as $M(\lambda - n_2\beta, p - n_2)$ itself. This completes the proof. $\square$

On the set of isomorphism classes of all GVMs we can defene a natural partial order with respect to inclusion. According to Theorem 5.3.1, the obtained partially ordered set decomposes into a non-ordered union of chains of lengths 1,2 or 3. This shows that the situation with GVMs is simpler than the analogous situation of Verma modules. For example, for the principal block of $\mathcal{O}$ the corresponding poset of Verma modules is, according to the BGG Theorem, the poset of Bruhat order on the corresponding Weyl group. In particular, the order on this poset is not linear.

5.4 B_2 case

In this Section we present the results, analogous to Theorem 5.3.1 in the case, when $\mathfrak{G}$ is a Lie algebra of type B_2. These results were obtained in [46, Section 4], which is, in fact, the most difficult technical part of [46]. The proof of these results is absolutely analogous to the proof of Theorem 5.3.1, but involves more complicated calculations. This is why we will omit the proof here and refer the reader to [46]. We retain the notations from Section 5.1.

Let again $\pi(\mathfrak{A}) = \{\alpha\}$ and $\pi = \{\alpha, \beta\}$. Unlike the case of $sl(3, \mathbb{C})$, the Dynkin diagram of type B_2 has no non-trivial automorphisms and hence we have to consider two different cases:

- α is a short root.
- α is a long root.

We will see that these cases, as well as the corresponding criteria, are quite different.

Consider the first case (α is a short root). Then $\Delta_+ = \{\alpha, \beta, \beta+\alpha, \beta+2\alpha\}$. For $\lambda \in \mathfrak{H}^*$ and $p \in \mathbb{C}$ denote $n^{\pm}(\lambda, p) = (\lambda(H_\alpha + 2H_\beta) \pm p)/2$ and $n(\lambda, p) = n^+(\lambda, p) + n^-(\lambda, p) = \lambda(H_\alpha + 2H_\beta)$.

Theorem 5.4.1. *1. $M(\mu, q) \subset M(\lambda, p)$ if and only if $\mu = \lambda + k\alpha - n\beta$, $k \in \mathbb{Z}$, $n \in \mathbb{Z}_+$ and one of the following conditions holds:*

(i) $n = 0$ and $q = \pm p$;

(ii) $n \in \{n^{\pm}(\lambda, p), n(\lambda, p)\}$ and $q^2 = p^2 + 4n\lambda(H_\alpha + 2H_\beta) - 4n^2$.

2. The module $M(\lambda, p)$ is a simple object in $\mathcal{K}^\alpha$ if and only if $\{n^{\pm}(\lambda, p), n(\lambda, p)\} \cap \mathbb{N} = \varnothing$.

From Theorem 5.4.1 follows immediately, that the poset of GVMs with respect to inclusions is a non-ordered union of chains of lengths 1,2 or 4, which is analogous to the case of $sl(3, \mathbb{C})$. As we have already mentioned, the proof of Theorem 5.4.1 can be found in [46] and is analogous to the first part of the proof of Theorem 5.3.1. We also note, that unlike Theorem 5.3.1, Theorem 5.4.1 does not give us a composition series of $M(\lambda, p)$.

Now consider the second case (α is a long root). Then $\Delta_+ = \{\alpha, \beta, \beta+\alpha, 2\beta+\alpha\}$. For $\lambda \in \mathfrak{H}^*$ and $p \in \mathbb{C}$ denote $n^{\pm}(\lambda, p) = (\lambda(H_\alpha + H_\beta) \pm p)$ and $n(\lambda, p) = n^+(\lambda, p) + n^-(\lambda, p) = 2\lambda(H_\alpha + H_\beta)$.

Theorem 5.4.2. *1. $M(\mu, q) \subset M(\lambda, p)$ if and only if $\mu = \lambda + k\alpha - n\beta$, $k \in \mathbb{Z}$, $n \in \mathbb{Z}_+$ and one of the following conditions holds:*

(i) $n = 0$ and $q = \pm p$;

(ii) $n \in \{n^{\pm}(\lambda, p)\}$ and $q^2 = (p \mp n^{\pm}(\lambda, p))^2$;

(iii) $n = n(\lambda, p)$, $n/2 \in \mathbb{N}$ and $q = \pm p$;

(iv) $n = n(\lambda, p)$, $n/2 \notin \mathbb{N}$, $n^{\pm}(\lambda, p) \in \mathbb{N}$ and $q = \pm p$.

2. The module $M(\lambda, p)$ is a simple object in $\mathcal{K}^\alpha$ if and only if $n^{\pm}(\lambda, p) \notin \mathbb{N}$ and $n(\lambda, p)/2 \notin \mathbb{N}$.

From Theorem 5.4.2 it follows that the inclusions of GVMs for this case can be more complicated that in previous cases and not only linear posets can occure. Thus, if we assume that $n_1 = n^-(\lambda, p) \in \mathbb{N}$ and $n_2 = n^+(\lambda, p) \in \mathbb{N}$, $n_2 > n_1$ then if $p \in \mathbb{N}$ we have

$$M(\lambda - (n_1 + n_2)\beta, p) \subset M(\lambda - n_2\beta, p - n_2) \subset M(\lambda - n_1\beta, p + n_1) \subset M(\lambda, p),$$

and if $p \notin \mathbb{N}$ we have

$$\begin{gathered} M(\lambda - (n_1 + n_2)\beta, p) \subset M(\lambda - n_2\beta, p - n_2) \subset M(\lambda, p), \\ M(\lambda - (n_1 + n_2)\beta, p) \subset M(\lambda - n_1\beta, p + n_1) \subset M(\lambda, p), \\ M(\lambda - n_1\beta, p + n_1) \not\subset M(\lambda - n_2\beta, p - n_2), \\ M(\lambda - n_2\beta, p - n_2) \not\subset M(\lambda - n_1\beta, p + n_1). \end{gathered}$$

5.5 A new action of the Weyl group and a generalization of the Harish-Chandra Theorem

We have already understood, that the space of parameters for modules $M(\lambda, p)$ is the linear space of all pairs (λ, p), $\lambda \in \mathfrak{H}^*$, $p \in \mathbb{C}$. So far we have not seen why we substituted b by p, $p^2 = b$. This will be clarified in this Section. Denote by $\Omega = \mathfrak{H}^* \oplus \mathbb{C}$ the set of parameters of our GVMs. The aim of this Section is to introduce an action of the Weyl group on Ω, which will play an important role in our generalization of the BGG Theorem. For Section 5.5 and Section 5.6 we assume that $\mathfrak{G}$ is a simple finite-dimensional complex Lie algebra and $\mathfrak{G}$ is not of type G_2. The reason is that the action of the Weyl group on Ω will be defined using rank two calculations, obtained in Theorems 5.3.1, 5.4.1 and 5.4.2. We did not cover the G_2 case there, and so will not be able to consider it here. We will handle the G_2 case in Section 5.7 by different methods, because of the technical difficulties.

Consider the following partition of π: $\pi = \pi_1 \cup \pi_2 \cup \pi_3 \cup \pi_4$, where $\pi_1 = \{\gamma \in \pi \mid \alpha + \gamma \in \Delta, |\alpha| = |\gamma|\}$, $\pi_2 = \{\gamma \in \pi \mid \alpha + \gamma \in \Delta, |\alpha| < |\gamma|\}$, $\pi_3 = \{\gamma \in \pi \mid \alpha + \gamma \in \Delta, |\alpha| > |\gamma|\}$, $\pi_4 = \{\gamma \in \pi \mid \alpha + \gamma \notin \Delta\}$. For $(\lambda, p) \in \Omega$ and $\beta \in \pi \setminus \pi_4$ denote

$$n^{\pm}_{\beta}(\lambda, p) = \begin{cases} \frac{1}{2}(\lambda(H_\alpha + 2H_\beta) \pm p), & \beta \in \pi_1 \cup \pi_2 \\ \lambda(H_\alpha + H_\beta) \pm p, & \beta \in \pi_3 \end{cases}$$

and define three pairs $(\lambda_\beta, p^i_\beta) \in \Omega$, $i = 1, 2, 3$, where $\lambda_\beta = \lambda - n^-_\beta(\lambda, p)\beta$, $p^1_\beta = n^+_\beta(\lambda, p)$, $p^2_\beta = p + 2n^-_\beta(\lambda, p)$, $p^3_\beta = p + n^-_\beta(\lambda, p)$.

For each $\beta \in \pi$ consider $l_\beta \in GL(\Omega)$ such that

$$l_\beta(\lambda, p) = \begin{cases} (\lambda, -p), & \beta = \alpha \\ (s_\beta \lambda, p), & \beta \in \pi_4 \setminus \{\alpha\} \\ (\lambda_\beta, p^i_\beta), & \beta \in \pi_i, \qquad i = 1, 2, 3. \end{cases} \tag{2}$$

If one looks at (2) carefully, it is easy to see, that for rank two algeras of type A_2 or B_2, formulae (2) coincide with those obtained in Theorems 5.3.1, 5.4.1 and 5.4.2. Further we will see that, in fact, (2) defines an action of W on Ω.

For $r \in \mathbb{C}$ denote by Ω_r the afiine hyperplane in Ω, which contains all (λ, p) such that $\lambda = r\alpha + \sum_{\beta \in \pi \setminus \{\alpha\}} a_\beta \beta$, $a_\beta \in \mathbb{C}$. Clearly, $\Omega_r = r\alpha + \Omega_0$, Ω_0 is a subspace of Ω and all Ω_r are invariant under l_β, $\beta \in \pi$.

Lemma 5.5.1. *Suppose that* $(\lambda, p) \in \Omega_{r_1}$, $(\mu, q) \in \Omega_{r_2}$ *and* $r_1 - r_2 \notin \mathbb{Z}$. *Then* $\mathrm{Hom}(M(\lambda, p), M(\mu, q)) = 0$.

Proof. Under the conditions of the Lemma, the weight lattices of $M(\lambda, p)$ and $M(\mu, q)$ are different, which completes the proof. □

Clearly, Lemma 5.5.1 can be generalized to any Ext between $M(\lambda, p)$ and $M(\mu, q)$ in the category of weight $\mathfrak{G}$-modules.

Let Δ° be a root system dual to Δ, $\eta' : \Delta \to \Delta^\circ$ be a canonical bijection and $\pi^\circ = \eta'(\pi)$. Construct a map $\eta^\circ_r : \Delta^\circ \to \Omega_r$ as follows. For $\beta \in \pi$ let

$$\eta^\circ_r(\beta^\circ) = \begin{cases} (r\alpha, |\alpha^\circ|^2), & \beta = \alpha \\ ((|\beta^\circ|^2/2)\beta + r\alpha, -\frac{1}{2}|\beta^\circ|^2), & \alpha + \beta \in \Delta, |\alpha| \geqslant |\beta| \\ (\frac{1}{2}\beta + r\alpha, -1), & \alpha + \beta \in \Delta, |\alpha| < |\beta| \\ ((|\beta^\circ|^2/2)\beta + r\alpha, 0), & \alpha + \beta \notin \Delta, \alpha \neq \beta. \end{cases} \tag{3}$$

The formulae (3) define a map from π° to Ω_r, which can be extended to the whole of Δ° by linearity. Define $\eta_r = \eta^\circ_r \circ \eta' : \Delta \to \Omega_r$. For a fixed r we set $\eta = \eta_r$, $\pi_r = \eta(\pi)$, $\Delta_{\alpha,r} = \eta(\Delta)$, $\Delta^+_{\alpha,r} = \eta(\Delta^+)$. Clearly, $\pi_{\alpha,r}$ is a basis of Ω_r.

Define a bilinear form $(\cdot, \cdot)_r : \Omega_r \times \Omega \to \mathbb{C}$ as follows. For $\beta \in \pi_{\alpha,r}$ and $(\lambda, p) \in \Omega$ let

$$(\beta, (\lambda, p))_r = \begin{cases} p, & \eta^{-1}(\beta) = \alpha \\ \lambda(H_{\eta^{-1}(\beta)}), & \eta^{-1}(\beta) \in \pi_4 \setminus \{\alpha\} \\ n^-_{\eta^{-1}(\beta)}(\lambda, p), & \eta^{-1}(\beta) \in \pi \setminus \pi_4. \end{cases} \tag{4}$$

One can extend (4) to the whole space by linearity. It is straightforward, that $(\cdot,\cdot)_r$ is non-degenerate on Ω_r and using this form one verifies that $\Delta_{\alpha,r}$ is a root system in $\Omega_r, (\cdot,\cdot)_r$ of the same type as Δ° and $\pi_{\alpha,r}$ is a basis of $\Delta_{\alpha,r}$ (see [46, Proposition 5.2,Lemma5.3]).

Lemma 5.5.2. *The rule $s_\beta(\lambda,p) = l_\beta(\lambda,p)$ defines an action of W on Ω.*

Proof. Follows from the discussion above and the fact that the Weyl groups of Δ and Δ° are isomorphic. □

Now we are ready to prove the first important result towards the BGG Theorem: a generalization of the Harish-Chandra Theorem. Consider a map $j : Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \to \mathbb{C}[t] \otimes S(\mathfrak{H}_{\mathfrak{A}})$, such that $j(c) = t$, and the algebra $\Lambda = \mathbb{C}[\sqrt{t}] \otimes S(\mathfrak{H}_{\mathfrak{A}})$, which acts on Ω_r by polynomial functions. Denote by $\Lambda^W \subset \Lambda$ the polynomial functions, invariant under W. Then $\Lambda^W \subset \mathbb{C}[t] \otimes S(\mathfrak{H}_{\mathfrak{A}})$, since $s_\alpha(\lambda,p) = (\lambda,-p)$.

We know already, that $M(\lambda,p)$ has a central character. We will denote this central character by $\chi_{\lambda,p}$.

Theorem 5.5.1. *$j \circ i : Z(\mathfrak{G}) \to \Lambda^W$ is an isomorphism that does not depend on the choice of π containing α.*

There is a small confusion in the formulation, since we do not yet know if $j \circ i(Z(\mathfrak{G}))$ is contained in Λ^W. We will prove this at the first time.

Proof. We will prove this theorem by reducing it to the classical Harish-Chandra Theorem. As the first part, we prove, that $j \circ i(Z(\mathfrak{G})) \subset \Lambda^W$.

First of all, we remark that for (λ,p), and (μ,q) in Ω_r such that $(\lambda,p) \in W(\mu,q)$ holds $\chi_{\lambda,p} = \chi_{\mu,q}$. Really, it is enough to show this in the case $(\lambda,p) = s_\beta(\mu,q)$ for $\beta \in \pi$. Using the generalized Harish-Chandra homomorphism we see, that the eigenvalue of any $z \in Z(\mathfrak{G})$ on $M(\lambda,p)$ is a polynomial function in (λ,p). Using the classical theory of Verma modules ([28, Chapter 7]) and Theorems 5.3.1, 5.4.1 and 5.4.2 one can find sufficiently many (λ,p) such that $\chi_{(\lambda,p)} = \chi_{s_\beta(\lambda,p)}$ and the statement follows from the polynomiality of the action of $Z(\mathfrak{G})$.

Let $0 \neq z \in Z(\mathfrak{G})$, $w \in W$ and $(\lambda,p) \in \Omega_r$. Then $w(j\circ i)(z)(\lambda,p) = (j\circ i)(z)(w^{-1}(\lambda,p))$. From Lemma 3.3.1 we have that the last expression equals $\chi_{w^{-1}(\lambda,p)}(z) = \chi_{(\lambda,p)}(z) = (j\circ i)(z)(\lambda,p)$. Hence $w(j\circ i)(z) = (j\circ i)(z)$ and we obtain that $j \circ i(Z(\mathfrak{G})) \subset \Lambda^W$.

Now consider the commutative diagram

$$\begin{array}{ccccc}
Z(\mathfrak{G}) & \xrightarrow{j\circ i} & \Lambda^W & \xrightarrow{(\varphi\otimes 1)\circ j^{-1}} & S(\mathfrak{H})^W \\
 & & \downarrow & & \downarrow \\
 & & \mathbb{C}[t] \otimes S(\mathfrak{H}_{\mathfrak{A}}) & \xrightarrow{(\varphi\otimes 1)\circ j^{-1}} & S(\mathfrak{H})^{W(\mathfrak{A})}.
\end{array}$$

Since $((\varphi\otimes 1)\circ j^{-1}) \circ j \circ i$ is an isomorphism, we have that $\ker(j\circ i) = 0$ and $((\varphi\otimes 1)\circ j^{-1})(\Lambda^W) = S(\mathfrak{H})^W$. On the other hand, $(\varphi\otimes 1)\circ j^{-1}$ is an isomorphism on the lower row. Thus $\ker((\varphi\otimes 1)\circ j^{-1}) = 0$, which implies that $j\circ i$ is an epimorphism and hence an isomorphism. It does not depend on the choice of π containing α by the Harish-Chandra Theorem. □

Corollary 5.5.1. *Suppose that $(\lambda, p), (\mu, q) \in \Omega_r$. Then $\chi_{\lambda,p} = \chi_{\mu,q}$ if and only if $(\lambda, p) \in W(\mu, q)$.*

Proof. We have already known the "if" part, so we have to prove the "only if" part. But two different orbits of W on Ω_r can be distinguished by the values of a polynomial function. Summing over W we can assume that this function is W-invariant. Applying Theorem 5.5.1, we obtain that there is a central element (which corresponds to the constructed function), which has different eigenvalues on the corresponding GVMs. $\square$

5.6 BGG Theorem and a simplicity criterion

In the previous Section we defined an action of W on the space Ω of parameters of GVMs and found out that this action is compatible with the central characters of GVMs. This is good evidence to expect that it is possible to generalize the BGG Theorem and give a criterion for the existence of a non-trivial homomorphism between two GVMs in terms of the defined action of W. This will be done in this Section.

Now we have to define a special partial order on Ω_r, which is analogous to the standard partial order on $\mathfrak{H}^*$. For (λ, p), (μ, q) in Ω_r set $(\mu, q) \prec (\lambda, p)$ if there exists $\beta \in \Delta^+_{\alpha,r}$ (here $\Delta^+_{\alpha,r}$ is taken with respect to $\pi_{\alpha,r}$) such that $s_\beta(\lambda, p) = (\mu, q)$ and for $\beta \neq \eta(\alpha)$ holds $(\beta, (\lambda, p))_r \in \mathbb{N}$. Consider the transitive closure of $\prec$, which we will also denote by $\prec$. The main result of this Section is the following Theorem, wich generalizes the BGG Theorem to the case of GVMs of the form $M(\lambda, p)$.

Theorem 5.6.1. *Let (λ, p) and (μ, q) be elements from Ω_r. Then the following statements are equivalent:*

1. $M(\mu, q) \subset M(\lambda, p)$.
2. *$L(\mu, q)$ is a subquotient of $M(\lambda, p)$.*
3. $(\mu, q) \prec (\lambda, p)$.

The detailed proof of this Theorem can be found in [46, Section 7]. In fact, it follows the general line of the original proof of BGG from [10] (see also [28, Theorem 7.6.23]). In order, to omit the standard technical details here we will explain only main ideological steps of this proof.

Proof. In the first part of the proof one has to show that the third condition implies the first one. For this it is enough to show, that if $\beta \in \Delta^+_{\alpha,r}$, $\beta \neq \eta(\alpha)$ such that $(\beta, (\lambda, p)) = n \in \mathbb{N}$, then $M(s_\beta(\lambda, p)) \subset M(\lambda, p)$. If one fixes n, then it is easy to see that the set of those (λ, p), such that $M(s_\beta(\lambda, p)) \subset M(\lambda, p)$ is Zariski closed in Ω_r ([46, Proposition 7.1]). Now one has only to find sufficiently many (λ, p) such that the inclusion $M(s_\beta(\lambda, p)) \subset M(\lambda, p)$ is obvious. This can be easily done if one considers (λ, p), which correspond to non α-stratified GVMs. To the highest weight submodules of such GVMs one can apply the classical BGG Theorem, which guarantees the desired inclusions ([46, Lemma 7.3]).

Note that the first condition easily implies the second one. Thus, in the second part of the proof one has to show that the second condition implies the third one. For this one can follow the original proof of BGG, the main tool of which is tensoring with finite-dimensional modules. This proof can be directly rewritten in our case, if we understand, how GVMs $M(\lambda, p)$ behave when we tensor them with a finite-dimensional module.

The key statement here is the following observation ([89, Lemma 4.5]) for $sl(2,\mathbb{C})$: Let F be an n-dimensional simple $sl(2,\mathbb{C})$-module. Then the module $F \otimes V(a, p^2)$ has length n in the category $\mathcal{K}^\alpha$. Moreover, the list of simple subquotients of $F \otimes V(a, p^2)$ is the following: $V(a,(p-n+1)^2)$, $V(a,(p-n+3)^2)$, $\dots$, $V(a,(p+n-1)^2)$. We note that it is possible, that some subquotients of the above list are isomorphic. This means that they occur in a composition series with the corresponding multiplicity. The proof of this fact is based on the idea, that one can substitute $V(a,b)$ with a (big enough) simple finite-dimensional module, namely, with a simple finite-dimensional module F', such that $\dim(F') > 3n$. Then the statement follows directly from the Littlewood-Richardson rule. This implies the statement for $V(a',b')$, which has F' as a subquotient. Now one has just to note, that the action of generators of $sl(2,\mathbb{C})$ is defined on the basis of $V(a,b)$ via some polynomials and the set of possible F' is infinite.

As soon as we know the result above, we can construct a filtration of $F \otimes M(\lambda', p')$, where F is a finite-dimensional $\mathfrak{G}$-module, such that each subquotient of this filtration is isomorphic to some $M(\mu', q')$ (a Verma flag of $F \otimes M(\lambda', p')$). Moreover, we can precisely determine all GVM subquotients of this filtration (see also Chapter 10, which is independent of the present Section), and see, that the picture for their parameters in Ω is quite analogous to the situation with parameters of Verma modules in $\mathfrak{H}^*$. Having this knowledge, we just rewrite the proof of [28, Theorem 7.5.23] (this was done, with all necessary notation and explanations in [46, Section 7]). $\square$

From Theorem 5.6.1 one can easily derive a simplicity criterion for $M(\lambda, p)$.

Corollary 5.6.1. *The module $M(\lambda, p)$ is a simple object in $\mathcal{K}^\alpha$ if and only if $(\beta,(\lambda,p))_r \notin \mathbb{N}$ for all $\beta \in \Delta^+_{\alpha,r} \setminus \{\eta(\alpha)\}$.*

5.7 G_2 case

Theorem 5.6.1 generalizes the BGG Theorem to the case of arbitrary simple complex finite-dimensional Lie algebras not of type G_2. The reason is that the definition of our action of W on Ω is based on the computational results for A_2 and B_2 algebras, obtained in Theorems 5.3.1, 5.4.1 and 5.4.2. We are not brave enough to try to do analogous calculations for the G_2 case, so we will try to guess the necessary action of W and then prove an analogue of the BGG Theorem, following the proof of Theorem 5.6.1. The hint, why it seems to be possible, is that we really did not use Theorems 5.3.1, 5.4.1 and 5.4.2 in the proof of Theorem 5.6.1 directly. In the first part of the proof we succeed to obtain enough information from highest weight submodules of non α-stratified modules $M(\lambda, p)$. The content of this Section is essentially the content of [72]. We also refer the reader to [72] for all technical details.

Retain the notation from Section 5.5. Let $\mathfrak{G}$ be a Lie algebra of type G_2, $\pi = \{\alpha, \beta\}$, $\pi(\mathfrak{A}) = \{\alpha\}$ (α can be a long root as well as a short one). Denote by $(\cdot,\cdot)^\circ$ the invariant bilinear form, which corresponds to Δ°. For $(\lambda, p) \in \Omega$ set

$$N_{(\lambda,p)} := \frac{\lambda(\alpha(H_\alpha)H_\beta - \alpha(H_\beta)H_\alpha) + \alpha(H_\beta)p}{\alpha(H_\alpha)}. \tag{5}$$

For all $\gamma^\circ \in \pi^\circ$ we define a map $\eta_r^\circ : \pi^\circ \to \Omega_r$ as follows

$$\eta_r^\circ(\gamma^\circ) := \begin{cases} (r\alpha, (\alpha^\circ, \alpha^\circ)^\circ), & \gamma^\circ = \alpha^\circ \\ \left(\frac{\alpha(H_\alpha)(\gamma^\circ,\gamma^\circ)^\circ - \alpha(H_\beta)(\alpha^\circ,\gamma^\circ)^\circ}{\alpha(H_\alpha)\beta(H_\beta) - \alpha(H_\beta)\beta(H_\alpha)}\beta + r\alpha, (\alpha^\circ, \gamma^\circ)^\circ\right), & \gamma^\circ \neq \alpha^\circ. \end{cases} \tag{6}$$

The map η_r° can be continued by linearity on Δ°. Set $\eta = \eta_r^\circ \circ \eta'$ and let $\Delta_{\alpha,r} = \eta(\Delta)$ with the fixed base $\eta(\pi)$. For all $(\lambda, p) \in \Omega, \gamma \in \pi$ we let

$$(\eta(\gamma), (\lambda, p))_r := \begin{cases} p & , \gamma = \alpha \\ N_{(\lambda,p)} & , \gamma \neq \alpha. \end{cases} \tag{7}$$

One can continue $(\cdot,\cdot)_r$ on Ω_r by linearity on the first argument and obtain a bilinear form $(\cdot,\cdot)_r : \Omega_r \times \Omega \to \mathbb{C}$ at the same way as it was done in Section 5.5.

We note that the formulae above are valid for any algebra of rank two, not only for type G_2. In the cases of algebras of type A_2 and B_2 they will coincide with the corresponding formulae from Section 5.5. It is easy to see that $(\cdot,\cdot)_r$ is non-degenerate on Ω_r and $\Delta_{\alpha,r}$ is a root system in Ω_r, $(\cdot,\cdot)_r$ of the same type, as Δ°. Now one can define an action of W on Ω via $s_{\eta(\gamma)}$, $\gamma \in \pi$. Retaining the notation from Section 5.6 and following all the proofs we obtain the following.

Theorem 5.7.1. *The statements of Theorem 5.5.1, Corollary 5.5.1, Theorem 5.6.1 and Corollary 5.6.1 extend to the case when $\mathfrak{G}$ is of type G_2.*

Finally, we remark that the structure of the poset of GVMs (with respect to inclusions), in this case, is the most difficult among all cases of algebras of rank two. In the case when α is a short root, the corresponding poset is a non-ordered union of linearly ordered sets and of quadruples occuring in B_2 case. In the case when α is a long root, to the posets descibed above one should add the poset of the Bruhat order on on a Weyl group of type A_2.

6 Gelfand-Zetlin modules

In the previous Chapter we generalized the BGG Theorem on GVMs, induced from a dense $sl(2,\mathbb{C})$-module. We crucially used the classification of such modules (classification of modules $V(a,b)$) and the fact that the modules $V(a,b)$ are, from some points of view, similar to finite-dimensional modules. In our case, this means that all the modules $V(a,b)$ have a natural basis, in which the action of the generators of $\mathfrak{A}$ is written using the same polynomial formulae, which occur in finite-dimensional modules. We really used this similarity in the proof of Theorem 5.6.1, where we determine all subquotients in a composition series of the tensor product of $V(a,b)$ with a simple finite-dimensional module. One of the most direct ways to generalize this machinery on bigger algebras, is to find a reasonable family of modules, similar to finite-dimensional modules in the above sense. Fortunately, this can be done, using the celebrated Gelfand-Zetlin basis for finite-dimensional $\mathfrak{A}$-modules ([54, 55], see also [3, 114]). The classical result by Gelfand and Zetlin presents a nice basis for finite-dimensional modules over simple Lie algebras of type A_n, B_n and D_n. Jimbo extended the same construction on the case of quantum algebra $U_q(gl_n)$ ([65]).

In this Chapter we present a construction of a family of modules "similar to finite-dimensional modules" for all mentioned algebras. We have to note, that there are also some recent results for the C_n case and for certain (non-standard) quantum algebras (see, for example, [94, 95, 52]), but we are not going to discuss them. These modules were first constructed by Drozd, Futorny and Ovsienko for the A_n case. The best explanation of their construction can be found in [30], although this is not the first paper on this subject. One can also consult [31, 33, 34]. This construction was extended to orthogonal Lie algebras in [91] and to $U_q(gl_n)$ in [93].

In this Chapter we will consider three cases (A_n, orthogonal algebras, $U_q(gl_n)$) separately. For each case we recall the classical rersult concerning finite-dimensional modules and present a construction of the new family of modules, which are called *generic Gelfand-Zetlin modules*. Regretfully, the only complete proof for the classical situation, available in the litarature ([114, Chapter 10]) covers only the A_n case. Some analogous calculations can also be found in [98, 99]. In the last Section we give a unified proof for the construction of generic modules in all cases.

6.1 $gl(n,\mathbb{C})$ $(sl(n,\mathbb{C}))$ case

In this Section we recall the results from [54]. For our convenience we will work with a reductive Lie algebra $\mathfrak{A}'$, isomorphic to $gl(n,\mathbb{C})$. We fix a standard basis in $\mathfrak{A}'$, consisting of matrix units $e_{i,j}$, $i,j=1,2,\dots,n$. By a tableau we will mean a doubly indexed vector $[\,l\,]=[\,l_{i,j}\,]$ from $\mathbb{C}^{n(n+1)/2}$, where $i=1,2,\dots,n$ and $j=1,2,\dots,i$. Denote by $[\,\delta^{k,j}\,]$ the Kronecker tableau, i.e. one with $\delta^{k,j}_{k,j}=1$ and $\delta^{k,j}_{a,b}=0$ if $k\neq a$ or $j\neq b$.

Theorem 6.1.1. *Let $V(m)$ be a simple $\mathfrak{A}'$-module with the highest weight $m=(m_{n,1}, m_{n,2},\dots, m_{n,n})$, $m_{n,i}\geqslant m_{n,i+1}$, $m_{n,i}\in\mathbb{N}$. Then $V(m)$ possesses a basis consisting of all tableau $[\,s\,]$ such that $s_{n,j}=m_{n,j}$, $j=1,2,\dots,n$ and $s_{i+1,j}\geqslant s_{i,j}\geqslant s_{i+1,j+1}$, $i=1,2,\dots,n$,*

$j=1,2,\dots,i$ *and the action of generators of* $\mathfrak{A}'$ *is given by the following formulae:*

$$e_{k,k}[\,s\,]=\left(\sum_{i=1}^{k}s_{k,i}-\sum_{i=1}^{k-1}s_{k-1,i}+k\right)[\,s\,],\quad k=1,2,\dots,n,$$

$$e_{k,k+1}[\,s\,]=\sum_{j=1}^{k}a^{+}_{k,j}([\,s\,])([\,s\,]+[\,\delta^{k,j}\,]),\quad e_{k+1,k}[\,s\,]=\sum_{j=1}^{k}a^{-}_{k,j}([\,s\,])([\,s\,]-[\,\delta^{k,j}\,]),$$

where $[\,\delta^{k,j}\,]$ *is the Kronecker tableau and for* $l_{r,t}=s_{r,t}-t$ *we have*

$$a^{\pm}_{k,j}([\,s\,])=\mp\frac{\prod_{i}(l_{k\pm1,i}-l_{k,j})}{\prod_{i\neq j}(l_{k,i}-l_{k,j})}.$$

The proof of this Theorem can be found in [114, Chapter 10]. We note, that not all simple $\mathfrak{A}$-modules occur as $V(m)$ in the above formulation. To cover all simple finite-dimensional $\mathfrak{A}$-modules one has to consider the modules $V(m,h)$, $h\in\mathbb{C}$ in which in all basis elements $[\,s\,]$ the entries $s_{i,j}$ are replaced by $s_{i,j}+h$. In this notation $V(m)=V(m,0)$. Nevertheless, if we consider the restrictions of all $V(m)$ from the above formulation on $sl(n,\mathbb{C})$ we will obtain all simple finite-dimensional $sl(n,\mathbb{C})$-modules (but this correspondence is not bijective).

Now we can see an analogue with $sl(2,\mathbb{C})$-case. Theorem 6.1.1 presents a basis of the finite-dimensional $\mathfrak{A}'$-module $V(m)$, and the action of generators of $\mathfrak{A}'$ on this basis is written by some polynomial (rational) formulae. The idea to construct a new family of $\mathfrak{A}'$-modules, is to take an analogous basis, in which all formulae are well-defined. More precisely, fix a tableau $[\,l\,]$ such that $l_{i,j}-l_{i,k}\notin\mathbb{Z}$ for all possible i and $j\neq k$. Consider the set $B([\,l\,])$, consisting of all tableaux $[\,l\,]$, such that

- $t_{n,j}=l_{n,j}$ for all j;
- $t_{i,j}-l_{i,j}\in\mathbb{Z}$ for all $1\leqslant i\leqslant n-1$ and all j.

Let $V([\,l\,])$ be a vector space with a basis $B([\,l\,])$. For $[\,t\,]\in B([\,l\,])$ set

$$e_{k,k}[\,t\,]=\left(\sum_{i=1}^{k}l_{k,i}-\sum_{i=1}^{k-1}l_{k-1,i}\right)[\,t\,],\quad k=1,2,\dots,n,$$

$$e_{k,k+1}[\,t\,]=\sum_{j=1}^{k}a^{+}_{kj}([\,t\,])([\,t\,]+[\,\delta^{k,j}\,]),\quad e_{k+1,k}[\,t\,]=\sum_{j=1}^{k}a^{-}_{kj}([\,t\,])([\,t\,]-[\,\delta^{k,j}\,]),$$

where

$$a^{\pm}_{kj}([\,t\,])=\mp\frac{\prod_{i}(t_{k\pm1,i}-t_{k,j})}{\prod_{i\neq j}(t_{k,i}-t_{k,j})}.$$

We will call the formulae above the *Gelfand-Zetlin (GZ) formulae.*

Theorem 6.1.2. *1. The GZ-formulae define on $V([\,l\,])$ a structure of a $\mathfrak{A}'$-module of finite length.*

2. $V([\,l\,])$ is simple if and only if $l_{i,j} - l_{i-1,k} \notin \mathbb{Z}$ for all i, j, k.

The proof will be given in Sectrion 6.4. Some more explanation and properties of $V([\,l\,])$ will be presented in Section 6.5. The modules $V([\,l\,])$ will be called *generic* Gelfand-Zetlin modules.

6.2 $\mathfrak{O}(n, \mathbb{C})$ case

In this Section we recall the Gelfan-Zetlin construction of simple finite dimensional modules over the Lie algebra $\mathfrak{A} = \mathfrak{O}(n, \mathbb{C})$ (see [55] or [3, Section 10.1.B]). The results presented here were obtained in [91]. Let $X_{i+1,i} = e_{i+1,i} - e_{i,i+1}$, $1 \leqslant i \leqslant n-1$ denote the standard generators of $\mathfrak{A}$. Assume that $n = 2k$ or $n = 2k+1$. In this Section, by a tableau we will mean the vector $[\,l\,]$ with complex entries, considered as a double indexed family $[\,l_{i,j}\,]$, where $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant s$ for $i = 2s-1$ or $i = 2s$. As usually, $[\,\delta^{i,j}\,]$ will denote the Kronecker tableau.

Fix a vector $m = (m_1, m_2, \dots, m_k)$ with integer or half-integer entries satisfying the following conditions:

1. For $n = 2k$: $m_1 \geqslant m_2 \geqslant \dots \geqslant m_{k-1} \geqslant |m_k|$.

2. For $n = 2k+1$: $m_1 \geqslant m_2 \geqslant \dots \geqslant m_{k-1} \geqslant m_k \geqslant 0$.

Consider the set $B(m)$ consisting of all tableaux with all integer or all half integer entries satisfying the following conditions:

$$\left.\begin{array}{ll} l_{n-1,i} = m_i, & \\ l_{2p+1,i} \geqslant l_{2p,i} \geqslant l_{2p+1,i+1}, & i = 1, 2, \dots, p-1, \\ l_{2p+1,p} \geqslant l_{2p,p} \geqslant |l_{2p+1,p+1}|, & \\ l_{2p,i} \geqslant l_{2p-1,i} \geqslant l_{2p,i+1}, & i = 1, 2, \dots, p-1, \\ l_{2p,p} \geqslant l_{2p-1,p} \geqslant -l_{2p,p} & \end{array}\right\}.$$

Consider the vectorspace $V(m)$ having $B(m)$ as basis (clearly $V(m)$ is finite-dimensional) and define an action of the generators $X_{i+1,i}$ on the basis elements as follows:

$$X_{2p+1,2p}[\,l\,] = \sum_{j=1}^{p} A(l_{2p-1,j})([\,l\,] + [\,\delta^{2p-1,j}\,]) - \sum_{j=1}^{p} A(l_{2p-1,j} - 1)([\,l\,] - [\,\delta^{2p-1,j}\,]),$$
$$X_{2p+2,2p+1}[\,l\,] = \sum_{j=1}^{p} B(l_{2p,j})([\,l\,] + [\,\delta^{2p,j}\,]) - \sum_{j=1}^{p} B(l_{2p,j} - 1)([\,l\,] - [\,\delta^{2p,j}\,]) + iC_{2p}[\,l\,],$$

where the functions A, B and C are defined in the following way: first we substitute $l_{2p-1,j}$ by $s_{2p-1,j} = l_{2p-1,j} + p - j$ and $l_{2p,j}$ by $s_{2p,j} = l_{2p,j} + p - j + 1$ for all possible p, then we define

$$A(l_{2p-1,j}) = \frac{1}{2}\left(\prod_{r=1}^{p-1}(s_{2p-2,r} - s_{2p-1,j} - 1)(s_{2p-2,r} + s_{2p-1\,j})\right)^{1/2} \times$$
$$\times \left(\prod_{r=1}^{p}(s_{2p,r} - s_{2p-1,j} - 1)(s_{2p,r} + s_{2p-1,j})\right)^{1/2} \times$$
$$\times \left(\prod_{r\neq j}(s^2_{2p-1,r} - s^2_{2p-1,j})(s^2_{2p-1,r} - (s_{2p-1\,j} + 1)^2)\right)^{-1/2},$$

$$B(l_{2p,j}) = \left(\prod_{r=1}^{p}(s^2_{2p-1,r} - s^2_{2p,j})\prod_{r=1}^{p+1}(s^2_{2p+1,r} - s^2_{2p,j})\right)^{1/2} \times$$
$$\times \left(s^2_{2p,j}(4s^2_{2p,j} - 1)\prod_{r\neq j}(s^2_{2p,r} - s^2_{2p,j})((s_{2p,r} - 1)^2 - s^2_{2p,j})\right)^{-1/2},$$

$$C_{2p} = \prod_{r=1}^{p} s_{2p-1,r}\prod_{r=1}^{p+1} s_{2p+1,r}\left(\prod_{r=1}^{p} s_{2p,r}(s_{2p,r} - 1)\right)^{-1}.$$

Theorem 6.2.1. *The formulae above define on $V(m)$ the structure of a simple $\mathfrak{A}$-module. Moreover, any simple finite-dimensional $\mathfrak{A}$-module is isomorphic to $V(m)$ for some m as above.*

We will call the formulae above the *Gelfand-Zetlin (GZ) formulae* (for the orthogonal case). Regretfully, there is no complete proof of this theorem (which is again a long straightforward calculation) available in the literature. The reader can find some related calculations in [98, 99]. Using Theorem 6.2.1, we define a new family of $\mathfrak{A}$-modules (which we will call *generic* Gelfan-Zetlin modules) as follows:

Fix a tableau $[\,l\,]$ with complex entries $l_{i,j}$, $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant k$ if $i = 2k-1$ or $i = 2k$ satisfying the following defining conditions:

- all $l_{i,j}$ are not integers and half-integers for $i < n-1$;
- $l_{i,j} \pm l_{i,k}$ is not an integer for all $1 \leqslant i \leqslant n-2$ and all $j \neq k$.

Consider a set $B([\,l\,])$ consisting of all tableaux $[\,t\,]$ such that

- $t_{n-1,j} = l_{n-1,j}$ for all j;
- $t_{i,j} - l_{i,j}$ is an integer for all $1 \leqslant i \leqslant n-2$ and all j.

Let $V([\,l\,])$ be a vector space with basis $B([\,l\,])$. For $[\,t\,] \in B([\,l\,])$ set

$$X_{2p+1,2p}[\,t\,] = \sum_{j=1}^{p} A(t_{2p-1,j})([\,t\,] + [\,\delta^{2p-1,j}\,]) - \sum_{j=1}^{p} A(t_{2p-1,j} - 1)([\,t\,] - [\,\delta^{2p-1,j}\,]),$$
$$X_{2p+2,2p+1}[\,t\,] = \sum_{j=1}^{p} B(t_{2p,j})([\,t\,] + [\,\delta^{2p,j}\,]) - \sum_{j=1}^{p} B(t_{2p,j} - 1)([\,t\,] - [\,\delta^{2p,j}\,]) + iC_{2p}[\,t\,],$$

where the functions A, B and C are taken from the GZ formulae for the orthogonal case. This action can be easily extended to $V([\,l\,])$ by linearity.

Theorem 6.2.2. *1. The formulae above define on $V([\,l\,])$ the structure of a completely reducible $\mathfrak{A}$-module of finite length.*

2. $V([\,l\,])$ is simple if and only if $l_{i,j} - l_{i-1,k} \notin \mathbb{Z}$ for all i, j, k.

The proof is postponed till Section 6.4.

6.3 $U_q(gl_n)$ case

In this Section we apply the same construction to the quantum analogue $U_q(gl_n)$ of the algebra $gl(n, \mathbb{C})$. The original generalization of the Gelfan-Zetlin construction to this case was obtained in [65], but we will follow [76], because [76] contains a more convenient version of the Gelfand-Zetlin formulae. The reader may also consult [111, 112], where the same results were obtained by different methods. The results of this Section were obtained in [93]. In this Section by a tableau we will mean a tableau as in Section 6.1. We begin with the definition of $U_q(gl_n)$ we will use.

Let q be a non-zero complex non-root of unity. For any complex x we set $[x]_q = (q^x - q^{-x})/(q - q^{-1}) = (e^{xh} - e^{-xh})/(e^h - e^{-h})$, where $q = \exp(h)$. We define $U_q(gl_n)$ as a unital associative complex algebra generated by E_i, F_i, $i = 1, 2, \ldots, n-1$, K_j, K_j^{-1}, $j = 1, 2, \ldots, n$ subject to the relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$
$$K_iE_jK_i^{-1} = q^{\delta_{ij}/2}q^{-\delta_{i,j+1}/2}E_j,$$
$$K_iF_jK_i^{-1} = q^{-\delta_{ij}/2}q^{\delta_{i,j+1}/2}F_j,$$
$$[E_i, F_j] = \delta_{ij}\frac{K_i^2K_{i+1}^{-2} - K_i^{-2}K_{i+1}^2}{q - q^{-1}},$$
$$[E_i, E_j] = [F_i, F_j] = 0, \quad |i - j| \geqslant 2,$$
$$E_i^2E_{i\pm1} - (q + q^{-1})E_iE_{i\pm1}E_i + E_{i\pm1}E_i^2 = 0,$$
$$F_i^2F_{i\pm1} - (q + q^{-1})F_iF_{i\pm1}F_i + F_{i\pm1}F_i^2 = 0$$

(see, for example [76, 111, 112]).

Theorem 6.3.1. *Let $V(m)$ be a simple $U_q(gl_n)$-module with a highest weight $m = (m_{n,1}, m_{n,2}, \dots, m_{n,n})$, $m_{n,i} \geqslant m_{n,i+1}$, $m_{n,j} \in \mathbb{N}$. Then $V(m)$ possesses a basis consisting of all tableau $[\,s\,] = (s_{i,j})_{i=1,2,\dots,n}^{j=1,2,\dots,i}$ such that $s_{n,j} = m_{n,j}$, $j = 1, 2, \dots, n$ and $s_{i+1,j} \geqslant s_{i,j} \geqslant s_{i+1,j+1}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, i$ and the action of generators of $U_q(gl_n)$ are given by the following formulae:*

$$K_k[\,s\,] = q^{a_k/2}[\,s\,], \quad a_k = \sum_{i=1}^{k} s_{k,i} - \sum_{i=1}^{k-1} s_{k-1,i}, \quad k = 1, 2, \dots, n,$$

$$E_k[\,s\,] = \sum_{j=1}^{k} a^{+}_{k,j}([\,s\,])([\,s\,] + [\,\delta^{k,j}\,]), \quad F_k[\,s\,] = \sum_{j=1}^{k} a^{-}_{k,j}([\,s\,])([\,s\,] - [\,\delta^{k,j}\,]),$$

where for $l_{r,t} = s_{r,t} - t$ we have

$$a^{\pm}_{k,j}([\,s\,]) = \mp \frac{\prod\limits_{i} [l_{k\pm 1,i} - l_{k,j}]_q}{\prod\limits_{i \neq j} [l_{k,i} - l_{k,j}]_q}.$$

The proof of this theorem can be found in [111]. Analogously to Section 6.1 we construct a family of *generic* Gelfand-Zetlin modules over $U_q(gl_n)$ as follows.

Let $1(q)$ be the set of all complex x such that $q^x = 1$. Fix a tableau $[\,l\,]$ with complex entries $l_{i\,j}$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant i$ satisfying the following condition:

- $2(l_{i,j} - l_{i,k}) \notin 1(q) + 2\mathbb{Z}$ for all $1 \leqslant i \leqslant n-1$ and all $j \neq k$.

Consider a set $B([\,l\,])$ consisting of all tableaux $[\,t\,]$ such that

- $t_{n,j} = l_{n,j}$ for all j;
- $t_{i,j} - l_{i,j}$ is an integer for all $1 \leqslant i \leqslant n-1$ and all j.

Let $V([\,l\,])$ be a vector space with a basis $B([\,l\,])$. For $[\,t\,] \in B([\,l\,])$ set

$$K_k[\,t\,] = q^{a_k/2}[\,t\,], \quad a_k = \sum_{i=1}^{k} l_{k,i} - \sum_{i=1}^{k-1} l_{k-1,i} + k, \quad k = 1, 2, \dots, n,$$

$$E_k[\,t\,] = \sum_{j=1}^{k} a^{+}_{kj}([\,t\,])([\,t\,] + [\,\delta^{k,j}\,]), \quad F_k[\,t\,] = \sum_{j=1}^{k} a^{-}_{kj}([\,t\,])([\,t\,] - [\,\delta^{k,j}\,]),$$

where

$$a^{\pm}_{kj}([\,t\,]) = \mp \frac{\prod\limits_{i} [t_{k\pm 1,i} - t_{k,j}]_q}{\prod\limits_{i \neq j} [t_{k,i} - t_{k,j}]_q}.$$

We will call the formulae above the *Gelfand-Zetlin (GZ) formulae.*

Theorem 6.3.2. *1. The GZ formulae define on $V([\,l\,])$ the structure of a $U_q(gl_n)$-module of finite length.*

2. The module $V([\,l\,])$ is simple if and only if $l_{i,j} - l_{i-1,k} \notin \mathbb{Z}$ for all i, j, k.

The proof will be given in the next Section.

6.4 Proof of Theorem 6.1.2, Theorem 6.2.2 and Theorem 6.3.2

Here we present a unified proof of Theorem 6.1.2, Theorem 6.2.2 and Theorem 6.3.2. For the unification we set $U(\mathfrak{A}) = U_q(gl_n)$ in Theorem 6.3.2.

First we show that the GZ formulae really define on $V([\,l\,])$ the structure of a $U(\mathfrak{A})$-module. Consider a relation $u = 0$ in $U(\mathfrak{A})$. Clearly, it is enough to show that u (as an element of the tensor algebra) acts trivially on $V([\,l\,])$, thus we have only to prove that $u[\,t\,] = 0$ for any $[\,t\,] \in B([\,l\,])$. Using the GZ formulae, we can write $u[\,t\,] = \sum_{[\,s\,]\in I(u,[\,t\,])} f([\,s\,])[\,s\,]$, where the set $I(u,[\,t\,]) - [\,t\,]$ depends only on u and for any fixed u each $f([\,t\,])$ is a rational function in $t_{i,j}$ (resp. $q^{t_{i,j}}$ in the quantum case). Thus, it is enough to show that each $f([\,s\,])$ is identically zero. Hence, we have to show that some polynomials in $t_{i,j}$ (resp. $q^{t_{i,j}}$) are zero. Let p be such a polynomial, k be its degree and s be the degree of u. Clearly, there exist a tableau $[\,\tilde{t}\,]$ such that all $\tilde{t}_{i,j}$ are positive integers and for any integers $-k-s \leqslant \tilde{s}_{i,j} \leqslant k+s$ the tableau $[\,\tilde{t}_{i,j} + \tilde{s}_{i,j}\,]$ occurs as a basis element in a finite-dimensional $U(\mathfrak{A})$-module. Taking into account in the quantum case, that $q^a \neq q^b$, if $a \neq b$ are positive integers, we conclude that p is identically zero, since GZ formulae really define the structure of an $\mathfrak{A}$-module on the finite-dimensional module as above. This complete the proof of the first step.

Now we are going to show that there exists a natural commutative subalgebra Γ in $U(\mathfrak{A})$, which distinguishes the basis elements $[\,t\,]$ from $B([\,l\,])$, i.e. for any different $[\,t\,]$ and $[\,s\,]$ from $B([\,l\,])$ there exists $z \in \Gamma$ such that $z[\,t\,] = a[\,t\,]$, $z[\,s\,] = b[\,s\,]$ and $a \neq b$. Consider an increasing chain of algebras $U(gl(1,\mathbb{C})) \subset U(gl(2,\mathbb{C})) \subset \cdots \subset U(gl(n,\mathbb{C}))$ (resp. $U(\mathfrak{O}(1,\mathbb{C})) \subset U(\mathfrak{O}(2,\mathbb{C})) \subset \cdots \subset U(g\mathfrak{O}(n,\mathbb{C}))$, resp. $U_q(gl_1) \subset U_q(gl_2) \subset \cdots \subset U_q(gl_n)$) with natural inclusions with respect to the first basis elements. Denote by Γ the subalgebra of $U(\mathfrak{A})$ generated by all $Z(\hat{\mathfrak{A}})$, for $U(\hat{\mathfrak{A}})$ occuring in the above filtration. Set $U(\mathfrak{A}_k) = U(gl(k,\mathbb{C}))$ (resp. $U(\mathfrak{O}(k,\mathbb{C}))$, resp. $U_q(gl_k)$) from the above filtration. Then any generator of $U(\mathfrak{A}_k)$, acting on $[\,t\,]$ changes only $t_{i,j}$ with $j < k$. From this it follows that the action of any $z \in Z(\mathfrak{A}_k)$ on $[\,t\,]$ is a rational function in $t_{k,j}$ (resp. $q^{t_{k,j}}$), $j = 1, 2, \ldots, k$. Reducing our consideration to $U(\mathfrak{A}_k)$ we obtain, that it is enough to prove that the central characters (if they exist) of $V([\,l(1)\,])$ and $V([\,l(2)\,])$ with $l(1)_{n,j} - l(2)_{n,j} \in \mathbb{Z}$ for all j and $l(a)_{n,j} - l(a)_{n,i} \notin \mathbb{Z}$ for all a and $j \neq i$ do not coincide. Consider a finite-dimensional $U(\mathfrak{A})$-module $V(m)$, as in Theorem 6.1.1 (resp. Theorem 6.2.1, resp. Theorem 6.3.1). The upper row of any tableau $[\,s\,]$ which indexes some GZ-basis element of $V(m)$ determines the highest weight of $V(m)$. Hence, any $z \in Z(\mathfrak{A})$ acts on $V(m)$ via a scalar, which is a polynomial in $m_{n,j}$, $j = 1, 2, \ldots, n$, that can be computed via the Harish-Chandra homomorphism. Denote this scalar by $f_z(m_{n,j})$. Since we have infinitely many finite-dimensional modules with a Gelfand-Zetlin basis, we obtain that $z[\,t\,] = f_z(t_{n,j})[\,t\,]$ for any

$[\,t\,] \in B([\,l\,])$. In particular, $V([\,l\,])$ has a central character. Using a trivial calculation with the Weyl group, we see that $(l(1)_{n,j})$ and $(l(2)_{n,j})$ do not belong to the same orbit, since $l(1)_{n,j} - l(2)_{n,j} \in \mathbb{Z}$ for all j and $l(a)_{n,j} - l(a)_{n,i} \notin \mathbb{Z}$ for all a and $j \neq i$. Now the desired statement follows immediately from the Harish-Chandra Theorem (Theorem 2.2.2, resp. [64, Claim 6.26,Section 8.30]).

From the previous paragraph it follows that any submodule of $V([\,l\,])$ is determined by the set of basis elements, $[\,t\,] \in B([\,l\,])$, which it contains. Consider a graph with vertices $[\,t\,] \in B([\,l\,])$, in which $[\,t\,]$ and $[\,t\,] + [\,\delta^{i,j}\,]$ are connected by an edge if there are generators X, Y of $U(\mathfrak{A})$, such that $X[\,t\,]$ has a non-zero coefficient in $[\,t\,] + [\,\delta^{i,j}\,]$ and $Y([\,t\,] + [\,\delta^{i,j}\,])$ has a non-zero coefficient in $[\,t\,]$, $i < n$, $j \leqslant i$. From the GZ formulae it follows easily, that this graph contains only finite many connected components. Hence $V([\,l\,])$ has a finite length. All other statements follow immediately from the GZ formulae.

6.5 General theory of Gelfand-Zetlin modules

In Section 6.4 we introduced an important technical tool in the study of $V([\,t\,])$ – the subalgebra Γ of $U(\mathfrak{A})$. Using it we can abstractly define a general class of $U(\mathfrak{A})$-modules, which contains all the modules $V([\,l\,])$. We say that a $U(\mathfrak{A})$-module M is a *Gelfand-Zetlin module*, if it decomposes into a direct sum of non-isomorphic finite-dimensional Γ-modules. Since Γ is commutative, an equivalent condition is that M is a Γ-root module (i.e. a direct sum of Γ-root subspaces) with finite-dimensional root subspaces (a Γ-root subspace M^{χ} of M, which corresponds to $\chi \in \Gamma^*$ is the set of all $v \in M$ such that $(z - \chi(z))^N v = 0$ for all $z \in \Gamma$ and for big enough $N \in \mathbb{N}$). From Section 6.4 it follows easily, that each $V([\,l\,])$ is a Gelfand-Zetlin module. The algebra Γ is usually called the Gelfand-Zetlin subalgebra of $U(\mathfrak{A})$.

In Section 6.4 we saw, that there is a natural (not bijective) parametrization of $\chi \in \Gamma^*$ by tableaux. Really, if $[\,l\,]$ is a tableau, we will say that it parametrizes a GZ weight $\chi \in \Gamma^*$ if for any $1 \leqslant k \leqslant n$ the highest weioght $U(\mathfrak{A}_k)$-module with the highest weight $(l_{k,j})_{j=1,2,\ldots,k}$ has the central character $\chi|_{Z(\mathfrak{A}_k)}$. It follows from the Harish-Chandra Theorem, that two tableaux parametrize the same chracter of Γ if and only if they belong to the same orbit of the natural action of the product $\hat{W}$ of all Weyl groups of $U(\mathfrak{A}_k)$ on the set of all tableaux.

The category of Gelfand-Zetlin modules is quite interesting. It was studied in different cases in [30, 31, 33, 34, 90, 91, 92, 93]. Some recent results were also obtained in [100]. In particular, the situation of the Gelfand-Zetlin subalgebra Γ in $U(\mathfrak{A})$ is a partial case of an abstract Harish-Chandra situation, defined and studied in [30]. Thus, Γ is a Harish-Chandra subalgebra of $U(\mathfrak{A})$ in the sense of [30] (see [30, 91, 93]). This means (for a commutative algebra Γ), that $\Gamma u \Gamma$ is a finitely generated bimodule for any $u \in U(\mathfrak{A})$. An abstract machinery, developed in [30], allows one to obtain many interesting properties of the category of Gelfand-Zetlin modules. For example, it follows easily, that any simple $V([\,l\,])$ has only trivial extensions with any other (non-isomorphic) simple Gelfand-Zetlin module. We will not need these properties, so we will not discuss them. The reader can consult [30] directly.

In what follows we will use constructed generic Gelfand-Zetlin modules $V([\,l\,])$ only over $gl(n, \mathbb{C})$ $(sl(n, \mathbb{C}))$ algebra. We have already mentioned, that the main advantage of modules $V([\,l\,])$, is their similarity to finite-dimensional modules. Using this, we will obtain an analogue of the BGG Theorem for modules $M_{\mathcal{P}}(V([\,l\,]))$ by the same methods as used in Chapter 5. For any simple $\mathfrak{A}$, bigger than $sl(2, \mathbb{C})$, this will not be recovered using the general machinery of Shapovalov form in Chapter 8.

7 An analogue of the BGG Theorem, II

In this Chapter we give a generalization of the BGG Theorem for GVMs $M_{\mathcal{P}}(V)$, induced from a simple generic GZ module $V \simeq V([\,l\,])$. The results, presented here were obtained in [90, 92]. The content in [92] is a generalization of the BGG Theorem, but in [90] the main content is different and there only a slight generalization of the results from [92] mentioned. This is why [92] is much more detailed than [90], so we refer the reader to [92] for all technical details.

7.1 GVMs induced from $V([\,l\,])$

In this Chapter we fix the Lie algebra $\mathfrak{G} = gl(n, \mathbb{C})$ (or $\mathfrak{G} = sl(n, \mathbb{C})$ just by restiction) with the standard Cartan subalgebra and the standard root basis. Then the algebra $\mathfrak{A}$ is a direct sum of some $gl(n_i, \mathbb{C})$ $(sl(k_i, \mathbb{C}))$, $i = 1, 2, \ldots, k$. In a natural way we extend the notion of a generic GZ module to the algebra $\mathfrak{A}$. Thus, any generic GZ module V over $\mathfrak{A}$ is defined by a sequence of tableaux $[\,l\,] = ([\,l(i)\,])_{i=1,2,\ldots,k}$. Let $V([\,l\,])$ be a simple $\mathfrak{A}'$-module, whose restricion on $\mathfrak{A}$ is a simple generic GZ $\mathfrak{A}$-module. Consider the corresponding GVM $M_{\mathcal{P}}(V([\,l\,]))$. It follows directly from the definition of GZ modules and from the generalized Harish-Chandra homomorphism, that $M_{\mathcal{P}}(V([\,l\,]))$ is a GZ module over $\mathfrak{G}$. We can parameterize $M_{\mathcal{P}}(V([\,l\,]))$ in a regular way by $[\,l\,]$ and by the $\mathfrak{H}_{\mathfrak{A}}$-highest weight $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$ of $M_{\mathcal{P}}(V([\,l\,]))$. Since $M_{\mathcal{P}}(V([\,l\,]))$ is a GZ module, this parameterization can be changed by a bit more convenient one. If we fix a Γ-weight vector $v \in M_{\mathcal{P}}(V([\,l\,]))_{\lambda}$, whose GZ weight is parameterized by some tableau $[\,t\,]$, then $[\,t\,]$ uniquely defines $V([\,l\,])$ and λ. Hence $[\,t\,]$ uniquely defines $M_{\mathcal{P}}(V([\,t\,]))$. For us it will be much more convenient to parameterize GVMs by such tableaux $[\,t\,]$, so we need to know, which $[\,t\,]$ can occur as parameters of GVMs. In the situation described above we set $M([\,t\,]) = M_{\mathcal{P}}(V([\,l\,]))$. Correspondingly, we set $L([\,t\,])$ to be the unique simple quotient of $M([\,t\,])$. To formulate such a condition, we assume that π is ordered in a natural way, $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$. From Section 6.5 we know that the tableaux parameterize the Γ weights up to the action of $\hat{W}$. This means that all our results should be also formulated up to this action. For a subset $A \in \mathbb{N}$ we will denote by $I(A)$ its convex hull in $\mathbb{N}$.

Lemma 7.1.1. *Up to the action of $\hat{W}$, the tableau $[\,t\,]$ parametrizes the GVM induced from a generic GZ module over $\mathfrak{A}$ if and only if the following conditions are satisfied:*

- $t_{i,j} - t_{i,k} \notin \mathbb{Z}$, $j \neq k$ *if* $\alpha_s \in \pi(\mathfrak{A})$ *for any* $s \in I(\{i, j, k\})$;
- $t_{i+1,j} = t_{i,j}$ *if* $\alpha_s \notin \pi(\mathfrak{A})$ *for some* $s \in I(\{i, j\})$.

Proof. Follows from the facts, that $l_{i,j} - l_{i,k} \notin \mathbb{Z}$, $i < n$, $j \neq k$ for a generic $V([\,l\,])$ and $l_{i+1,j} = l_{i,j}$ for all i, j if $[\,l\,]$ paramitrizes the highest weight of a highest weight module. $\square$

Let Ω denote the set of all tableaux, satisfying the conditions of Lemma 7.1.1. From the disscusion above it follows, that we can consider our GVMs as $M([\,t\,])$, $[\,t\,] \in \Omega$. Fix $[\,t\,] \in \Omega$, such that $M([\,t\,])$ is the GVM induced from a simple generic GZ module. Consider the vectorspace T, consisting of all tableaux, which satisfy the following conditions:

- $t_{i,j} = 0$ if $\alpha_s \in \pi(\mathfrak{A})$ for all $s \in I(\{i,j\})$;
- $t_{i+1,j} = t_{i,j}$ if $\alpha_s \notin \pi(\mathfrak{A})$ for some $s \in I(\{i,j\})$.

Denote by $\Omega_{[t]}$ the affine hyperplane $[t] + T$. In the next Section we will define an action of W on $\Omega_{[t]}$ and in Section 7.3 we will give a generalization of the BGG Theorem in terms of this action.

7.2 An action of the Weyl group and the Harish-Chandra Theorem

There is a natural action of the Weyl group W on the set of all tableaux: it just permutes the elements of the upper row of a tableau. From the definition of $\Omega_{[t]}$ is is clear, that we can not just restrict the natural action on $\Omega_{[t]}$, because $\Omega_{[t]}$ is not invariant under it. Nevertheless, it is easy to extend the natural action of W on the upper row of $[s] \in \Omega_{[t]}$ to an action of W on $\Omega_{[t]}$. Suppose that W acts naturaly on the upper row of any $[s] \in \Omega_{[t]}$. For $[s] \in \Omega_{[t]}$ and $w \in W$ set $w[s] = [m]$, where $m_{i,j} = s_{i,j}$ if $s_{i,j} \neq s_{n,j}$ and $m_{i,j} = s_{n,w(j)}$ if $s_{i,j} = s_{n,j}$. Clearly, this defines an action of W on $\Omega_{[t]}$.

Lemma 7.2.1. *For $[s] \in \Omega_{[t]}$ and $w \in W(\mathfrak{A})$ we have $M([s]) \simeq M(w[s])$.*

Proof. Follows from the fact that the action of $W(\mathfrak{A})$ does not change $[s]$ up to the action of $\hat{W}$. □

For $[s] \in \Omega_{[t]}$ let $\chi_{[s]}$ denote the central character of the module $M([s])$.

Lemma 7.2.2. *For $[s]$, $[m]$ from $\Omega_{[t]}$ holds $\chi_{[s]} = \chi_{[m]}$ if and only if $[s] \in W[m]$.*

Proof. $\chi_{[s]}$ (resp. $\chi_{[m]}$) coincides with $\chi_{(s_{n,j})_{j=1,2,\dots,n}}$ (resp. $\chi_{(n_{n,j})_{j=1,2,\dots,n}}$). Since the action of W on the upper row of $[s]$ (resp. $[m]$) coincides with the action of W on $\mathfrak{H}^*$, the statement follows from Theorem 2.2.2. □

This result can now be easily extended to a more general statement, which is a generalization of the Harish-Chandra isomorphism Theorem. Denote by $\mathcal{A}$ the vectorspace of all tableau $[t]$, satisfying the following condition: $t_{i+1,j} = t_{i,j}$ if $\alpha_s \notin \pi(\mathfrak{A})$ for some $s \in I(\{i,j\})$. Then the action of W on each $\Omega_{[t]}$ can be easily extended to an action of W on $\mathcal{A}$. Since any element of $\mathcal{A}$ parameterize some element of Γ^*, we have a natural map from Γ to $S(\mathcal{A}^*)$. Let p denote the composition of this map with the natural inclusion of $Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}})$ in Γ.

Consider the following commutative diagram:

$$\begin{tikzcd} & S(\mathcal{A}^*) \arrow[r, "j"] & S(\mathfrak{H}) \\ Z(\mathfrak{G}) \arrow[r, "(1\otimes\gamma^{\mathfrak{A}})\circ\varphi_{\mathfrak{A}}"] & Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[u, "p"] \arrow[r, "(\gamma_{\mathfrak{A}}\circ\varphi)\otimes 1"] & S(\mathfrak{H}(\mathfrak{A}))^{W(\mathfrak{A})} \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[u] \end{tikzcd}$$

Here j is the natural projection with respect to the upper row of the tableau. We can extend our commutative diagram by the row of the invariant algebras:

$$
\begin{tikzcd}
& S(\mathcal{A}^*) \arrow[r, "j"] & S(\mathfrak{H}) \\
Z(\mathfrak{G}) \arrow[r, "(1\otimes\gamma^{\mathfrak{A}})\circ\varphi_{\mathfrak{A}}"] & Z(\mathfrak{A}) \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[u, "p"] \arrow[r, "(\gamma_{\mathfrak{A}}\circ\varphi)\otimes 1"] & S(\mathfrak{H}(\mathfrak{A}))^{W(\mathfrak{A})} \otimes S(\mathfrak{H}_{\mathfrak{A}}) \arrow[u] \\
& S(\mathcal{A}^*)^W \arrow[u, "q"] \arrow[r, "j_{\mathfrak{A}}"] & S(\mathfrak{H})^W \arrow[u]
\end{tikzcd}
$$

where q is the canonical inclusion and $J_{\mathfrak{A}}$ is an isomorphism. Now we can consider a composition $\psi_{\mathfrak{A}} = q^{-1} \circ (1 \otimes \gamma^{\mathfrak{A}}) \circ \varphi_{\mathfrak{A}}$. From Lemma 7.2.2 and the commutativity of the diagram above we obtain.

Theorem 7.2.1. $Z(\mathfrak{G}) \overset{\psi_S}{\simeq} S(\mathcal{A}^*)^W$.

Proof. Follows from the discussion above. □

Note, that one can also prove Theorem 7.2.1 independently from Lemma 7.2.2 by methods, analogous to that used in the proof of Theorem 5.5.1. Theorem 7.2.1 is a generalization of the Harish-Chandra isomorphism Theorem.

7.3 The BGG Theorem and a simplicity criterion

Now we are ready to state and prove an analogue of the BGG Theorem for modules $M([\,s\,])$, $[\,s\,] \in \Omega_{[\,t\,]}$ in terms of the new action of W on $\Omega_{[\,t\,]}$. To formulate the Theorem we have to introduce a partial order $\prec$ on $\Omega_{[\,t\,]}$. For any transposition (ij), $i < j$ in W we will write $(ij)[\,s\,] \prec [\,s\,]$ provided $s_{n,i} - s_{n,j} \in \mathbb{Z}_+$. We also denote by $\prec$ the transitive closure of $\prec$.

Theorem 7.3.1. *For $[\,s\,], [\,m\,] \in \Omega_{[\,t\,]}$ the following conditions are equivalent:*

1. $M([\,s\,]) \subset M([\,m\,])$.
2. $L([\,s\,])$ *is a subquotient of* $M([\,m\,])$.
3. $[\,s\,] \prec [\,m\,]$

The complete proof of this theorem can be found in [92]. Here we present only a scheme of the proof. The original proof from [92] is, in fact, analogous to that of Theorem 5.6.1, or, more generally, to the original proof ([28, Theorem 7.6.23]) of the BGG Theorem. From Theorem 7.3.1 we immediately obtain the following criterion of simplicity for $M([\,s\,])$.

Corollary 7.3.1. *For $[\,s\,], [\,m\,] \in \Omega_{[\,t\,]}$ the following conditions are equivalent:*

1. $M([\,s\,])$ *is simple.*

2. *For any $\alpha \in \Delta_+ \setminus \Delta(\mathfrak{A})$ holds $s_\alpha[\,s\,] \not\succ [\,s\,]$.*

3. *For any $1 \leqslant i < j \leqslant n$ such that $\{\alpha_i, \alpha_{i+1}, \dots, \alpha_{j-1}\} \not\subset \pi(\mathfrak{A})$ holds $s_{n,i} - s_{n,j} \notin \mathbb{N}$.*

As we already mentioned in Section 5.6, in order to be able to follow the original proof of BGG, we have to know the subquotients of the tensor product $M([\,s\,]) \otimes F$ with a finite dimensional $\mathfrak{G}$-module F. Of course, for this it is sufficient to decompose $V([\,l\,]) \otimes F$, where $V([\,l\,])$ is a simple generic GZ module over $\mathfrak{A}$ and F is a simple finite-dimensional module over $\mathfrak{A}$. Clearly, we can assume that $\mathfrak{A} \simeq gl(n, \mathbb{C})$, $(sl(n, \mathbb{C}))$. Denote by $[\,\hat{p}\,]$ the natural inclusion of $\mathfrak{H}(\mathfrak{A})^*$ in the space of all tableaux, which sends $\lambda \in \mathfrak{H}(\mathfrak{A})^*$ to the tableau $[\,\hat{p}(\lambda)\,]$, such that $\hat{p}(\lambda)_{n,j} = \lambda_j$ and $\hat{p}(\lambda)_{i,j} = 0$ otherwise. Denote also by $\hat{p}^*$ the canonical inverse map, which sends any tableau to the weight defined by its upper row.

Lemma 7.3.1. *Let $V([\,l\,])$ be a simple generic GZ module over $\mathfrak{A}$ and F be a simple finite-dimensional $\mathfrak{A}$-module. Then any subquotient of $V([\,l\,]) \otimes F$ is of the form $V([\,t\,] + [\,\hat{p}(\lambda)\,])$ for $\lambda \in \mathrm{supp}(F)$. Moreover, the multiplicity of $V([\,t\,] + [\,\hat{p}(\lambda)\,])$ as a subquotient of $V([\,l\,]) \otimes F$ equals $\sum \dim(F_\mu)$, where the sum is taken over all $\mu \in \mathrm{supp}(F)$ such that $\hat{p}^*([\,t\,] + [\,\hat{p}(\mu)\,]) \in W(\hat{p}^*([\,t\,] + [\,\hat{p}(\mu)\,]))$.*

Proof. First we note that, according to [54], F decomposes into a direct sum of non-isomorphic one-dimensional Γ-modules. As we have seen in Section 6.4, the same is true for $V([\,l\,])$. Further, from the construction of $V([\,l\,])$ and Section 6.5 it follows that two modules $V([\,l(1)\,])$ and $V([\,l(2)\,])$, such that $l(1)_{i,j} = l(2)_{i,j}$ for all $1 \leqslant i \leqslant n-1$ and all j are isomorphic if and only if their central characters coincide.

Now the statement, that any subquotient of $V([\,l\,]) \otimes F$ is of the form $V([\,t\,] + [\,\hat{p}(\lambda)\,])$ for some $\lambda \in \mathrm{supp}(F)$, follows directly from Kostant's Theorem.

Denote by $\widetilde{\mathrm{supp}}(F)$ the multi-support of F, i.e. the support in which all weights are counted with their multiplicities. Denote by Γ' a natural subalgebra of Γ, which is the Gelfand-Zetlin subalgebra of $U(gl(n-1))$ $(U(sl(n-1)))$. By the construction, the module $V([\,l\,])$ is dense with respect to Γ' (i.e. its Γ'-support coincides with a weight lattice) and all Γ'-weight subspaces are one-dimensional. Since tensoring with a finite-dimensional module preserves the weight lattice, we conclude that $V([\,l\,]) \otimes F$ is a dense module and all non-trivial Γ'-weight subspaces of it are of dimension $\dim F$. Hence, applying [30, Corollary 33], we obtain that the length of $V([\,l\,]) \otimes F$ equals $\dim(F)$.

Now we want to substitute $V([\,l\,])$ by a finite-dimensional module E and apply a similarity of $V([\,l\,])$ with E, mentioned in Chapter 6. Suppose that E lies far enough from the walls, i.e. the length of $E \otimes F$ equals $\dim(F)$. We will call a Γ'-weight subspace of E *generic* provided the dimension of this weight subspace in $E \otimes F$ equals $\dim(F)$. Clearly, any E lying far from the walls has a generic Γ'-weight subspace. Fix E lying far from the walls and a generic Γ'-weight ν (i.e. E_ν is a generic Γ'-weight subspace). Fix $z \in Z(\mathfrak{A})$. Choose a basis in $(E \otimes F)_\nu$ and write the characteristic polynomial $f_z(X)$ of z in this basis. Let λ be a highest weight of E. From the Littlewood-Richardson rule we obtain

$$f_z(X) = \pm \left(\prod_{\lambda' \in \widetilde{\mathrm{supp}}(F)} (X - \chi_{\lambda+\lambda'}(z)). \right) \tag{8}$$

Note that from the GZ formulae, which define the action of the generators of $\mathfrak{G}$ on finite-dimensional modules, it follows that the coefficients of $f_z(X)$ are just the rational functions in the entries of the tableau, parametrizing ν. Since we can find sufficiently many modules E lying far from the walls and sufficiently many generic Γ'-weight subspaces in E we conclude for any generic Γ'-weight ν in any simple module M, defined using Gelfand-Zetlin formulae, the polynomial $f_z(X)$ has also the form (8).

To complete the proof we only have to recall that the module $V([\,l\,])$ was constructed using the GZ formulae and, as it was mentioned above, any Γ'-weight subspace of $V([\,l\,])$ is generic. $\square$

From Lemma 7.3.1, we see that generic GZ modules behave even more regular with respect to tensoring with finite-dimensional modules, than finite-dimensional modules themselves. One notes that Lemma 7.3.1 is a precise analogue of the corrsponding statement for modules $V(a,b)$ from Section 5.6. In fact, it also includes the mentioned statement in the case of simple $V(a,b)$.

In order to feel free while working with $M([\,s\,])$, $[\,s\,]\in\Omega_{[\,t\,]}$, one needs also some classical results about GVMs. More precisely, one needs an analogue of Proposition 5.2.1.

Proposition 7.3.1. *For* $[\,s\,],[\,m\,]\in\Omega_{[\,t\,]}$ *holds* $\dim\mathrm{Hom}(M([\,s\,]),M([\,m\,]))\leqslant 1$ *and any non-zero homomorphism from this space is injective.*

Proof. Analogous to the proof of the analogous statement for Verma modules, but uses some technical innovations, like relative Gelfand-Kirillov dimension. These are purely technical calculations so we omit them. The reader can consult [92, Section 6]. $\square$

Now the strategy to prove Theorem 7.3.1 is quite transparent.

Proof of Theorem 7.3.1. Again the first statement clearly implies the second one. So, first we prove that $[\,s\,]\prec[\,m\,]$ implies $M([\,s\,])\subset M([\,m\,])$. For this it is enough to consider $\alpha\in\Delta_+\setminus\Delta(\mathfrak{A})$ and $[\,s\,]=s_\alpha[\,m\,]\prec[\,m\,]$. First assume that $\pi=\pi(\mathfrak{A})\cup\{\alpha_{n-1}\}$ and that s_α transposes $m_{n,i}$ with $m_{n,n}$. If we assume that $m_{n,i}-m_{n,i'}\notin\mathbb{Z}$ for all $i'\neq i,n$, the module $M([\,m\,])$ can be constructed via GZ formulae and the statement follows by direct calculation. Now for the general $\alpha\in\Delta_+\setminus\Delta(\mathfrak{A})$ the proof repeats the proof of the analogous statement in Theorem 5.6.1.

Now we have to prove that the second statement implies the third one. Having Lemma 7.3.1 and Proposition 7.3.1, this is a direct translation of the original proof of BGG ([28, Theorem 7.6.23]). $\square$

7.4 Tableaux realization of Verma modules and GVMs

In the last two Chapters we saw, that it is quite convenient to work with a GZ $\mathfrak{G}$-module M, constructed directly from the GZ formulae. For this the set of tableaux, parametrizing the GZ weights of M, should be chosen such that the GZ formulae are well-defined on all these tableaux, i.e. all denominators of the GZ formulae are non-zero. This leads us to a useful notion of a $\mathfrak{G}$-module having a tableaux realization. In this section we define a family of $\mathfrak{G}$-modules having a tableaux realization and construct a tableaux realization for some Verma modules and some GVMs. The content of this section is taken from [90], where the notion of a module having a tableaux realization was introduced.

We fix $\mathfrak{G} = gl(n,\mathbb{C})$ (or $sl(n,\mathbb{C})$), but we note that analogous constructions can be done in other cases (see, for example, [91]). A tableau $[\,l\,]$ will be called good, provided $l_{i,j} - l_{i,k} \notin \mathbb{Z}$ for all $i < n$ and $j \neq k$. A GZ module M over $\mathfrak{G}$ is said to have a tableaux realization, if it decomposes into a direct sum of one-dimensional Γ-modules, parameterized by good tableaux. From Theorem 6.1.1 it follows that if an indecomposable $\mathfrak{G}$-module M has a tableaux parametrization, then there exists a set $B(M)$ of tableaux, which can be taken as a basis for M, and action of $\mathfrak{G}$ on this basis is defined by the GZ formulae. The simplest examples of modules having a tableaux realization are finite-dimensional $\mathfrak{G}$-modules or generic GZ modules $V([\,l\,])$. In this Section we also construct a tableaux realization for some Verma modules and some GVMs. Tha main advantage of such modules is that they are given by precise formulae, so they are quite easy for calculations.

Let $a = (a_1, \dots, a_n) \in \mathfrak{H}^*$ be such that $a_k - a_j \notin \mathbb{Z}$ and $a_j > a_k$ for all $1 \leqslant j < k \leqslant n$. Consider the tableau $[\,l\,] = [\,l\,](a)$ defined as follows: $l_{i,j} = a_j$ for all $1 \leqslant j \leqslant i \leqslant n$. Let $B([\,l\,])$ denote the set of all tableaux $[\,t\,]$ satisfying the following conditions:

1. $l_{n,j} = t_{n,j}$, $j = 1, \dots, n$;
2. $l_{i,j} - t_{i,j} \in \mathbb{Z}_+$ for all $1 \leqslant j \leqslant i < n$;
3. $t_{i,j} \geqslant t_{i-1,j}$ for all $1 < i \leqslant n$, $1 \leqslant j \leqslant i$.

It follows directly from the GZ formulae, that $[\,t\,] \pm [\,\delta^{i,j}\,] \in B([\,l\,])$ for $[\,t\,] \in B([\,l\,])$, if $a^{\pm}_{ij}([\,t\,]) \neq 0$. Let $M = M([\,l\,])$ denote the $\mathfrak{G}$-module with a basis $B([\,l\,])$ and the action of generators of $\mathfrak{G}$, defined by the GZ formulae (here we have an abuse of notation, but we will see that $M([\,l\,])$ is a Verma module with the highest weight $a - \rho$, so everything is correct). It follows directly from Section 6.4, that $M([\,l\,])$ is really a $\mathfrak{G}$-module.

Theorem 7.4.1. *1. $M([\,l\,]) \simeq M(a)$.*

2. $M([\,l\,])$ is simple.

3. $M([\,l\,]) \otimes F$ is completely reducible for any finite-dimensional $\mathfrak{G}$-module F.

4. $\mathrm{Ext}^1(M([\,l\,](a)), M([\,l\,](a'))) = \mathrm{Ext}^1(M([\,l\,](a')), M([\,l\,](a))) = 0$ for any a, a' as above.

Proof. It follows directly from the GZ formulae, that the GZ weight space of $M([\,l\,])$, corresponding to $[\,l\,]$ is a highest weight and generates $M([\,l\,])$. Thus $M([\,l\,])$ is a quotient of $M(a)$. Comparing the dimensions of the weight spaces we obtain that they coincide, hence $M([\,l\,]) \simeq M(a)$. Direct application of the GZ formulae also shows that $[\,l\,]$ is the only highest weight of $M([\,l\,])$. Hence $M([\,l\,])$ is simple. The third statement follows from Corollary 4.4.1. The last one follows from the third one and Theorem 2.5.3. $\square$

Now, as in Section 7.2, assume that $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and fix $\pi(\mathfrak{A}) \subset \pi$. Consider a tableau $[\,t\,]$, satisfying the following conditions:

- $t_{i,j} - t_{i,k} \notin \mathbb{Z}$ for any $i < n$ and any j, k;
- $t_{i+1,j} = t_{i,j}$ if $a_s \notin \pi(\mathfrak{A})$ for some $s \in I(\{i,j\})$.

Let $B([\,t\,])$ denote the set of all tableaux $[\,s\,]$, satisfying the following conditions:

1. $s_{n,j} = t_{n,j}$, $j = 1, \dots, n$;
2. $s_{i,j} - t_{i,j} \in \mathbb{Z}_+$ for all $1 \leqslant j \leqslant i < n$ such that $a_s \notin \pi(\mathfrak{A})$ for some $s \in I(\{i,j\})$;
3. $s_{i,j} \geqslant s_{i-1,j}$ for all $1 < i \leqslant n$, $1 \leqslant j \leqslant i$ such that $a_s \notin \pi(\mathfrak{A})$ for some $s \in I(\{i,j\})$;
4. $s_{i,j} - t_{i,j} \in \mathbb{Z}$ if $a_s \in \pi(\mathfrak{A})$ for any $s \in I(\{i,j\})$.

It follows directly from the GZ formulae, that $[\,s\,] \pm [\,\delta^{i,j}\,] \in B([\,t\,])$ for $[\,s\,] \in B([\,t\,])$, if $a_{ij}^{\pm}([\,s\,]) \neq 0$. Let $M = M([\,t\,])$ denote the $\mathfrak{G}$-module with a basis $B([\,t\,])$ and the action of generators of $\mathfrak{G}$, defined by the GZ formulae (here we have an abuse of notation, but we will see that $M([\,t\,])$ is a GVM, so again everything is correct). It follows directly from Section 6.4 that $M([\,t\,])$ is really a $\mathfrak{G}$-module.

Theorem 7.4.2. *$M([\,l\,])$ is a simple GVM, induced from a simple generic GZ module over $\mathfrak{A}$. Moreover, $M([\,l\,]) \otimes F$ is completely reducible for any finite-dimensional $\mathfrak{G}$-module F.*

Proof. Folows directly from Section 7.2 and Corollary 4.4.1. □

Theorem 7.4.2 differs from the description in Section 7.2, since here we present a precise construction of $M([\,t\,])$, which was not done in Section 7.2.

8 Generalized Shapovalov form

In this Chapter we generalize a modern technique of studying of Verma modules using the Shapovalov form ([69]) on GVMs. This machinery was worked out in [73], so we refer the reader to this paper for all missing technical details. In particular, we define a generalization of the Shapovalov form on a GVM $M(\lambda, p)$, see Chapter 5. We calculate the determinant of this form. Using the determinant formula we generalize the BGG Theorem about the embeddings of GVMs. In particular, these results covers all known generalizations of the BGG Theorem for α-stratified modules, obtained in Chapter 5 and in [43] in the case of an affine algebra of type $A_1^{(1)}$. Since the proposed machinery works in a more general situation than just simple finite-dimensional algebras, we begin with the setup.

8.1 Definition of the form

Let $\mathfrak{G}$ be a complex contragradient Lie algebra (or Chevalley algebra) associated with a complex $(n \times n)$-matrix $A = (a_{ij})$ (see [69]). We fix the standard triangular decomposition $(\mathfrak{G}_+, \mathfrak{H}, \Delta_+, \sigma)$ of $\mathfrak{G}$, where $\mathfrak{H}$ is Cartan subalgebra, Δ_+ is the set of roots of $\mathfrak{G}_+$ and σ is an antiinvolution on $\mathfrak{G}$ (see [96, 69] for details). Let Δ be the set of roots of the algebra $\mathfrak{G}$ i.e. $\Delta = \Delta_+ \cup -\Delta_+$ ([96]). For the rest of this Chapter we fix a basis π of Δ_+ and an element $\alpha \in \pi$ satisfying the following conditions: the subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ generated by $\mathfrak{G}_{\pm\alpha}$ should be isomorphic to $sl(2, \mathbb{C})$ and $\mathfrak{G}$ should be an integrable (i.e. direct sum of finite-dimensional modules) $\mathfrak{A}$-module under the adjoint action. This generalizes the situation, described in Chapter 5, where α was a basis of the root system $\Delta(\mathfrak{A})$ and $\mathfrak{A}$ was the semisimple part of the Levi factor of $\mathcal{P}$. We can just let $\mathcal{P}$ be generated by $\mathfrak{G}_+$, $\mathfrak{H}$ and $\mathfrak{G}_{-\alpha}$. So, we keep all the notation from Chapter 3. We fix the dual elements $H_\beta \in \mathfrak{H}$ for all $\beta \in \Delta_+$.

Under the above choice of α the simple reflection s_α on $\mathfrak{H}^*$ is correctly defined and satisfies all the standard properties of a simple reflection. Let P denote the standard Kostant partition function with respect to π and $\hat{P}$ denote the standard Kostant partition function with respect to $s_\alpha(\pi)$. By a *quasiroot* we will mean any element $q\alpha \in \mathfrak{H}^*$, where $\alpha \in \Delta_+$ and q is a positive rational number.

For a contragradient Lie algebra with a symmetrizable Cartan matrix let $(\cdot,\cdot)$ denote the bilinear form on $\mathfrak{G}$ ([68, 96]). The corresponding bilinear form on $\mathfrak{H}^*$ will also be denoted by $(\cdot,\cdot)$. For a restricted weight $\mathfrak{G}$-module V we introduce the action of the Kac-Casimir operator Ω ([69]) on V as follows: for $v \in V_\mu$, $\mu \in \mathfrak{H}^*$ let

$$\Omega v = (\mu + 2\rho, \mu)v + 2 \sum_{\beta \in \mathbb{Q}_+} \sum_i e_{-\beta}^{(i)} e_\beta^{(i)} v,$$

where ρ is an element in $\mathfrak{H}^*$ such that $(\rho, \gamma) = 1$ for all $\gamma \in \pi$, $e_\beta^{(i)}$ form a basis of $\mathfrak{G}_\beta$ and $e_{-\beta}^{(i)}$ form the dual basis of $\mathfrak{G}_{-\beta}$. One can easily check that the form $(\cdot,\cdot)$ on $\mathfrak{H}^*$ is invariant under s_α.

Set $\mathfrak{H}(\alpha) = U(\mathfrak{H}) \otimes \mathbb{C}[c]$. Consider the following decomposition of $U(\mathfrak{G})$ ([43, page 88]):

$$U(\mathfrak{G}) = (\mathfrak{N}(\mathfrak{A})U(\mathfrak{G}) + U(\mathfrak{G})\mathfrak{N}) \oplus \mathfrak{H}(\alpha)\mathbb{C}[X_\alpha]X_\alpha \oplus \mathfrak{H}(\alpha)\mathbb{C}[X_{-\alpha}]X_{-\alpha} \oplus \mathfrak{H}(\alpha).$$

Let p be the projection of $U(\mathfrak{G})$ on $\mathfrak{H}(\alpha)$ with respect to the above decomposition. We define an α-Shapovalov form (or a generalized Shapovalov form) F_α on $U(\mathfrak{G})$ as a symmetric bilinear form with values in $\mathfrak{H}(\alpha)$ as follows (see also [43, 69, 96, 106]):

$$F_\alpha(x, y) = p(\sigma(x)y), \quad x, y \in U(\mathfrak{G}).$$

It is straightforward that the graded components $U(\mathfrak{G})_\xi$, $\xi \in \mathbb{Z}\Delta$, are orthogonal with respect to F_α. Moreover, F_α is contravariant, i.e. $F_\alpha(zx, y) = F_\alpha(x, \sigma(z)y)$ for all $x, y, z \in U(\mathfrak{G})$.

Consider a vector subspace

$$\mathcal{M} = U(\mathfrak{N}(\mathfrak{A}) \oplus \mathfrak{G}_\alpha) + U(\mathfrak{N}(\mathfrak{A}) \oplus \mathfrak{G}_{-\alpha})$$

in $U(\mathfrak{G})$. For $\xi \in \mathbb{Z}\Delta$ we set $\mathcal{M}_\xi = \mathcal{M} \cap U(\mathfrak{G})_\xi$. Clearly, each $\mathcal{M}_\xi$ is finite-dimensional. To calculate the dimension of $\mathcal{M}_\xi$ we have to introduce the notion of the Kostant α-function P_α (see also [92]).

For $\gamma = \sum_{\beta\in\pi} a_\beta \beta \in \Delta$ set $\psi_\alpha(\gamma) = \sum_{\beta\in(\pi\setminus\{\alpha\})} a_\beta \beta$. Define the Kostant α-function P_α : $\mathfrak{H}^* \to \mathbb{N} \cup \{0\}$ as follows: for $\lambda \in \mathfrak{H}^*$ set $P_\alpha(\lambda)$ to be the maximum number of the decompositions

$$\lambda + n\alpha = \sum_{\beta\in\Delta_+\setminus\{\alpha\}} n_\beta \psi_\alpha(\beta)$$

with non-negative integer coefficients, where n runs through all integers. It follows easily from the definition of P_α that $\dim \mathcal{M}_{-\xi} = P_\alpha(\xi)$.

For $\eta \in \mathbb{Z}\Delta$ we denote by F_α^η the restriction of F_α to $\mathcal{M}_{-\eta}$.

Fix $a, b \in \mathbb{C}$ and denote by $\hat{V}(a, b)$ the $\mathfrak{A}$-module uniquely defined by the following properties: $\hat{V}(a, b)$ has the same subquotients as $V(a, b)$ and $\hat{V}(a, b)$ is generated by $\hat{V}(a, b)_a$ as a $\mathfrak{A}$-module. Clearly, $\hat{V}(a, b) \simeq V(a, b)$ if the module $V(a, b)$ is α-stratified (and hence simple by Proposition 4.3.1). We will denote by $M(\lambda, b)$, $\lambda(H_\alpha) = a$, the corresponding GVM $M_\mathcal{P}(\hat{V}(a, b))$. We note that $M_\mathcal{P}(\hat{V}(a, b)) \simeq M_\mathcal{P}(V(a, b))$ if and only if $V(a, b)$ is α-stratified, but these two modules always have the same subquotients. Let $0 \neq v_{(\lambda,b)} \in M(\lambda, b)_\lambda$ be a canonical generator of $M(\lambda, b)$. It is well-known (see for example [22]) that $\mathcal{M}v_{(\lambda,b)} = M(\lambda, b)$. We can naturally identify $\mathfrak{H}(\alpha)$ with the ring of polynomials on the $\mathbb{C}$-space $\{(\lambda, b) \mid \lambda \in \mathfrak{H}^*, b \in \mathbb{C}\}$ by setting $c^* = (0, 1)$. Thus we can define the value $F_\alpha^\eta((\lambda, b))$ of F_α^η at the point (λ, b).

Now we can define a bilinear $\mathbb{C}$-valued form, $\hat{F}_\alpha$, on $M(\lambda, b)$ by setting

$$\hat{F}_\alpha(u_1 v_{(\lambda,b)}, u_2 v_{(\lambda,b)}) = F_\alpha(u_1, u_2)((\lambda, b)), \quad u_1, u_2 \in \mathcal{M}.$$

The following Lemma presents the standard properties of $\hat{F}_\alpha$:

Lemma 8.1.1. *1. The kernel of $\hat{F}_\alpha$ coincides with the unique maximal submodule of the module $M(\lambda, b)$.*

2. $\hat{F}_\alpha$ is non-degenerate on $M(\lambda, b)$ if and only if $M(\lambda, b)$ is simple.

3. All weight subspaces of $M(\lambda, b)$ are orthogonal with respect to $\hat{F}_\alpha$.

Proof. Proof is analogous to that for the classical Shapovalov form (see for example [96]). □

8.2 Determinant formula

The main advantage of the generalized Shapovalov form is that it is possible to calculate the determinant of its restriction to a weight space of $M(\lambda, v)$. This is given by the following *determinant formula.*

Theorem 8.2.1. *Let $\mathfrak{G}$ be a contragradient Lie algebra with a symmetrisable Cartan matrix. Then for any $\eta \in \mathfrak{H}^*$*

$$\begin{aligned}
\det F_\alpha^\eta = \prod_{k=1}^{\infty} (X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))^{P(\eta - k\alpha)} \times \\
\times \prod_{k=1}^{\infty} (X_{-a}X_\alpha + (1-k)(H_\alpha + \rho(H_\alpha) - (1-k)))^{\hat{P}(\eta - k\alpha)} \times \\
\times \prod_{\substack{\beta \in \Delta_+ \setminus \{\alpha\},\\ s_\alpha(\beta) = \beta}} \prod_{k=1}^{\infty} \left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2} \right)^{P_\alpha(\eta - k\beta)} \times \\
\times \prod_{\substack{\{\beta, s_\alpha(\beta) |\\ \beta \in \Delta_+ \setminus \{\alpha\},\\ s_\alpha(\beta) \neq \beta\}}} \prod_{k=1}^{\infty} \left(\left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2} \right) \cdot \right. \\
\left. \cdot \left(H_{s_\alpha(\beta)} + \rho(H_{s_\alpha(\beta)}) - k\frac{(\beta,\beta)}{2} \right) + \alpha(H_\beta)\alpha(H_{s_\alpha(\beta)})X_{-\alpha}X_\alpha \right)^{P_\alpha(\eta - k\beta)} .
\end{aligned}$$

up to a non-zero constant factor, where all the roots β are taken with their multiplicities.

We note that the product in the last factor of the above formula runs through all non-orderd pairs $\{\beta, s_\alpha(\beta)\}$ such that $\beta \neq s_\alpha(\beta)$.

The formula from Theorem 8.2.1 looks quite unattractive, however we will later see (during the proof), that the division into four factors as above is quite natural. In this Section we present a proof of this formula, a more detailed version of which can be found in [73]. We begin with two necessary lemmas.

Lemma 8.2.1. *Up to a non-zero constant factor,* $\det F^\eta_\alpha$ *is a product of factors having one of the following forms:*

1. $(X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$;
2. $(X_{-a}X_\alpha + (1-k)(H_\alpha + \rho(H_\alpha) - (1-k)))$;
3. $\left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2}\right)$, *where* β *is a quasiroot such that* $s_\alpha(\beta) = \beta$.
4. $\left(\left(H_\beta + \rho(H_\beta) - k\frac{(\beta,\beta)}{2}\right) \cdot \left(H_{s_\alpha(\beta)} + \rho(H_{s_\alpha(\beta)}) - k\frac{(\beta,\beta)}{2}\right) + \alpha(H_\beta)\alpha(H_{s_\alpha(\beta)})X_{-\alpha}X_\alpha\right)$, *where* β *is a quasiroot such that* $s_\alpha(\beta) \neq \beta$.

Proof. Consider a GVM $M(\lambda, b)$ generated by a non-zero element $v_{(\lambda,b)} \in M(\lambda,b)_\lambda$. Clearly, $M(\lambda, b)$ is restricted, hence the action of Ω is well-defined on it. Applying Ω to $v_{(\lambda,b)}$ one obtains $\Omega v_{(\lambda,b)} = ((\lambda+2\rho,\lambda) + (b - ((\lambda,\alpha)+1)^2)/2)v_{(\lambda,b)}$ and thus Ω acts as $((\lambda+2\rho,\lambda) + (b-(\lambda+\rho,\alpha)^2)/2)id$ on $M(\lambda,b)$.

There are two general possibilities for $M(\lambda,b)$ to be reducible. The first one: $M(\lambda,b)$ is reducible if $\hat{V}(a,b)$ is. The second one: $M(\lambda,b)$ is reducible if there exists an α-highest weight vector in some $M(\lambda,b)_\mu$ with $\mu - \lambda \notin \mathbb{Z}\alpha$. First assume that $\hat{V}(a,b)$ is reducible. This is possible if and only if for some $m \in \mathbb{N}$, $X^m_\alpha X^m_{-\alpha} v_{(\lambda,b)} = 0$ or $X^m_{-\alpha}X^m_\alpha v_{(\lambda,b)} = 0$ holds. By direct calculations with $U(\mathfrak{G}_\alpha)$ we obtain

$$\prod_{k=1}^{m} (X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))\, v_{(\lambda,b)} = 0$$

or

$$\prod_{k=1}^{m} (X_{-a}X_\alpha + (1-k)(H_\alpha + \rho(H_\alpha) - (1-k)))\, v_{(\lambda,b)} = 0.$$

Further, suppose that there exists an α-highest weight vector w in $M(\lambda,b)_\mu$ for some $\mu \in \mathfrak{H}^*$ such that $\mu - \lambda \notin \mathbb{Z}\alpha$. Then the eigenvalues of Ω on $v_{(\lambda,b)}$ and w coincide and we obtain

$$(\lambda+2\rho,\lambda) + (b - ((\lambda,\alpha)+1)^2)/2 = (\mu+2\rho,\mu) + (b' - ((\mu,\alpha)+1)^2)/2 \tag{9}$$

for some $b' \in \mathbb{C}$. Clearly, the difference $b'-b$ polynomialy depends on $\sqrt{b}$ after fixing λ^α and $\mu - \lambda$ (see [46]). Thus the formula above can be applied to the case $(a+1+2n)^2 = b$, $n \in \mathbb{Z}$. For such $N(a,b)$ we get that $M(\lambda,b)$ is an extension of two Verma modules (with respect to different bases in Δ). Now, using the fact that the action of Ω on a Verma module can be calculated at the highest weight vector, we obtain that $b' = b + 2\sqrt{b}(\mu-\lambda,\alpha) + (\mu-\lambda,\alpha)^2$ (here $\sqrt{b}$ is the complex square root function which has two different values as soon as $b \neq 0$).

If $(\mu-\lambda,\alpha) = 0$ the equality (9) reduces to $(\lambda+2\rho,\lambda) = (\mu+2\rho,\mu)$ and we can use the same arguments as in proof of [69, Lemma 3.2] obtaining the factors $(H_\beta - \rho(H_\beta) - (\beta,\beta)/2)$ (here β is not necessary a quasiroot).

If $(\mu-\lambda,\alpha)\neq 0$ we can take two equalities of the form (9) corresponding to different values b_1 and b_2 of $\sqrt{b}$, transfer everything in the left-hand side and multiply them. We obtain the following (here $\beta=\lambda-\mu$):

$$(2(\lambda+\rho,\beta)-(\beta,\beta)-(\lambda+\rho,\alpha)(\beta,\alpha))^2-b(\beta,\alpha)^2=0.$$

The last equality can be rewritten in the form

$$(2(\lambda+\rho,\beta)-(\beta,\beta)-(\lambda+\rho,\alpha)(\beta,\alpha))^2-(\beta,\alpha)^2(\lambda+\rho,\alpha)^2-(\beta,\alpha)^2(b-(\lambda+\rho,\alpha)^2)=0.$$

We note that

$$(\lambda+\rho,\beta)-(\lambda+\rho,\alpha)(\beta,\alpha)=(\lambda+\rho,\beta-(\beta,\alpha)\alpha)=(\lambda+\rho,\beta-\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha)=(\lambda+\rho,s_\alpha(\beta)).$$

From this it follows that

$$\begin{aligned}(2(\lambda+\rho,\beta)-(\beta,\beta))\,(2(\lambda+\rho,s_\alpha(\beta))-(s_\alpha(\beta),s_\alpha(\beta)))&+\\ +4(\frac{1}{4}(b-(\lambda+\rho,\alpha)^2))(\alpha,\beta)(\alpha,s_\alpha(\beta))&=0.\end{aligned}$$

Taking into account that $\frac{1}{4}(b-(\lambda+\rho,\alpha)^2)$ is an eigenvalue of the operator $X_{-\alpha}X_\alpha$, we obtain the factor of the form

$$\big((H_\beta+\rho(H_\beta)-(\beta,\beta)/2)\,\big(H_{s_\alpha(\beta)}+\rho(H_{s_\alpha(\beta)})-(\beta,\beta)/2\big)+\alpha(H_\beta)\alpha(H_{s_\alpha(\beta)})X_{-\alpha}X_\alpha\big)$$

with the same arguments as in [69, Lemma 3.2].

Now we only need to show that all β appearing above are quasiroots. Suppose not. Thus we will have some factor of the determinant of F_α corresponding to a non-quasiroot β. Calculating F_α on a Verma submodule for some reducible $N(a,b)$ we obtain a contradiction with [69, Theorem 1]. Lemma is proved. □

To proceed we define a new *α-gradation* on $U(\mathfrak{G})$ by setting the degree of $X_{\pm\alpha}$ and the degree of H_α to be 0 and all the degrees of other base elements in $\mathfrak{G}$ to be 1.

Lemma 8.2.2. *Up to a factor of grade zero the leading term of* $\det F_\alpha^\eta$ *with respect to the α-gradation is equal to*

$$\prod_{\beta\in\Delta_+\setminus\{\alpha\}}\prod_{k=1}^{\infty}H_\beta^{P_\alpha(\eta-k\beta)}.$$

Proof. From the classical Shapovalov determinant formula ([69]) it follows that the above formula is correct for $\det F_\alpha^{\eta-l\alpha}$, where $l\in\mathbb{N}$ is big enough. To complete the proof it is sufficient to show that the leading term of $\det F_\alpha^\eta$ in the α-gradation does not depend on the shift on α.

Choose some PBW monomial base $v_1,\ \dots\ ,\ v_t$ in $\mathcal{M}_{-\eta}$ and suppose that as soon as some v_i contains $X_{-\alpha}$ this monomial should start with this $X_{-\alpha}$. Consider the elements

$X_\alpha v_1, \ldots, X_\alpha v_t$ and let W be a linear span of these elements. For $1 \leqslant i \leqslant t$ set $\hat{v}_i = X_\alpha v_i$ if v_i does not contain $X_{-\alpha}$ and $\hat{v}_i = w_i$ if $v_i = X_{-\alpha} w_i$. Clearly, elements $\hat{v}_1, \ldots, \hat{v}_t$ form a basis of $\mathcal{M}_{-\eta+\alpha}$. Moreover, it follows from the definition of $\hat{v}_i$ that up to a factor of zero degree the leading term of $\det F_\alpha^{\eta-\alpha}$ coincides with the leading term of the determinant of the form F_α restricted to W (we will denote it by $F_\alpha(W)$). Since the base change from $v_1, \ldots, v_t$ to $X_\alpha v_1, \ldots, X_\alpha v_t$ is defined by the elements of zero grade it follows that $\det F_\alpha^\eta$ differs from $\det F_\alpha(W)$ by a factor of grade zero. This implies that the leading term of $\det F_\alpha^\eta$ in the α-gradation does not depend on the shift on α. $\square$

The proof of the classical determinant formula for Shapovalov form ([69]) is based on the so-called Jantzen filtration of a Verma module. In order to prove Theorem 8.2.1 we have to generalize this notion for GVMs. Choose $z \in \mathfrak{H}^*$ such that $(z, \beta) \neq 0$ for all $\beta \in \mathbb{Z}\Delta_+ \setminus 0$. Let t be an indeterminate. By standard techniques, we can extend $M(\lambda, b)$ to the module $\widetilde{M}(\widetilde{\lambda, b})$ over the algebra $\widetilde{U(\mathfrak{G})} = U(\mathfrak{G}) \otimes \mathbb{C}[t]$, where $\widetilde{(\lambda, b)} = (\lambda, b) + t(z, 1) \in \widetilde{\mathfrak{H}(\alpha)^*} = \mathfrak{H}(\alpha)^* \otimes \mathbb{C}[t]$. Further we can trivially extend σ on $\widetilde{U(\mathfrak{G})}$ and construct a bilinear form $\widetilde{F_\alpha}$. Using $\widetilde{F_\alpha}$ one can define a bilinear $\mathbb{C}[t]$-valued form $\widetilde{F_\alpha^t}$ on $\widetilde{M}(\widetilde{\lambda, b})$. Setting $\widetilde{M^i}$ to be equal to the set of all elements v in $\widetilde{M}(\widetilde{\lambda, b})$ such that $\widetilde{F_\alpha^t}(v, w)$ is divisible by t^i for all $w \in \widetilde{M}(\widetilde{\lambda, b})$ we define a Jantzen filtration

$$\widetilde{M}(\widetilde{\lambda, b}) = \widetilde{M^0} \supset \widetilde{M^1} \supset \ldots$$

on $\widetilde{M}(\widetilde{\lambda, b})$. The canonical epimorphism $\varphi : \widetilde{M}(\widetilde{\lambda, b}) \to M(\lambda, b)$ $(t \to 0)$ induces a filtration

$$M(\lambda, b) = M^0 \supset M^1 \supset \ldots$$

of $M(\lambda, b)$ which we will also call a *Jantzen filtration.*

Proof of Theorem 8.2.1. We have only to calculate the degrees in $\det F_\alpha^\eta$ of the factors described in Lemma 8.2.1. For a quasiroot β, which is not proportional to α, the proof of this fact is exactly the same as in [69, Proof of Theorem 1] because of Lemma 8.2.2 and the remark that the functions $P_\alpha(x - y)$ $y \in \alpha^\perp$ are linearly independent (here $\alpha^\perp$ is taken with respect to $(\cdot, \cdot)$).

Thus we have only to calculate the degrees of the factors of the form

- $(X_{-a} X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$;
- $(X_{-a} X_\alpha + (1 - k)(H_\alpha + \rho(H_\alpha) - (1 - k)))$.

We will do it for the first kind of factors. One can apply analogous arguments for the second case. Consider a factor $(X_{-a} X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$ for some fixed $k \in \mathbb{N}$. Let $N(a, b)$ be such that it has the unique submodule starting at the highest weight $a - k\alpha$. We note that in this case $a \notin \mathbb{Z}$. One can easily choose $\lambda \in \mathfrak{H}^*$ $(\lambda(H_\alpha) = a)$ such that GVM $M(\lambda, b)$ has the unique non-trivial submodule N. Clearly, in the described case N is isomorphic to the Verma module $M(\lambda - k\alpha)$. From the definition of Jantzen filtration we have $M^0 = M(\lambda, b)$ and $M^1 = N$. Our goal is to prove that $M^2 = 0$. Since N is irreducible it follows that either $M^2 = N$ or $M^2 = 0$. Consider $\widetilde{U(\mathfrak{G})}$-modules $\widetilde{M(\lambda, b)}$ and $\widetilde{N}$ and let w be a canonical generator of $\widetilde{N}$. Use the definition of $\widetilde{F_\alpha}$ to calculate $\widetilde{F^t_\alpha}(w, w)$. By the direct application of $sl(2)$-theory we obtain that

$$\widetilde{F^t_\alpha}(w, w) = \prod_{i=1}^{k} f_k(t),$$

where $f_k(t) \in \mathbb{C}[t]$ such that $f'_k(0) \neq 0$ satisfy the following condition: the difference between constant terms in f_{k+1} and f_k is equal to $a - 2k$. Since a is not integer it follows that the product in the formula above is divisible by at most t. But it is divisible by t since $\widetilde{N}$ is a submodule. Thus the canonical generator of N belongs to M^1 and does not belong to M^2. Hence $M^2 = 0$. Now we can claim that from the construction of the Jantzen filtration, it follows immediately that $\det F^\eta_\alpha$ is divisible exactly by the $P(\eta - k\alpha)$-th power of $(X_{-a}X_\alpha + k(H_\alpha + \rho(H_\alpha) - k))$ (see [69, Proof of Theorem 1] and [96, Section 6.6]). This completes our proof. □

8.3 Generalization of the BGG Theorem

As in the classical case, the determinant formula for F_α enables one to prove a generalization of the BGG Theorem (see [69, Theorem 2] and [96, Section 6.7]). In this Section we will formulate an analogous result for our GVMs induced from $\mathfrak{A}$.

For $\lambda, \mu \in \mathfrak{H}^*$ and $b_1, b_2 \in \mathbb{C}$ we set $(\lambda, b_1) \rightharpoonup (\mu, b_2)$ in one of the following cases:

1. $b_1 = b_2$ and $\lambda = \mu - k\alpha$ for some $k \in \mathbb{Z}$;

2. $b_1 = b_2 \pm 2\sqrt{b_2}(k\beta, \alpha) + (k\beta, \alpha)^2$ for $k \in \mathbb{N}$ and $\beta \in \Delta_+ \setminus \{\alpha\}$ such that $\lambda = \mu - k\beta$ and
$$2(\lambda + \rho)(H_\beta) - k(\beta, \beta) - (\lambda + \rho)(H_\alpha)(\beta, \alpha) = \pm\sqrt{b_2}(\beta, \alpha).$$
(here an analytic branch of $\sqrt{z}$ function is fixed).

Denote by $\prec$ the transitive closure of the relation $\rightharpoonup$ on $\mathfrak{H}^* \times \mathbb{C}$.

For each pair $\beta \neq s_\alpha(\beta)$ of roots in Δ_+ we fix some bijective map

$$\mathrm{sign} : \{\beta, s_\alpha(\beta)\} \to \{\pm 1\}.$$

We also set $\mathrm{sign}(\beta) = 0$ if $(\alpha, \beta) = 0$ and fix some analytic branch of $\sqrt{z}$ function. For $\beta \in \Delta_+ \setminus \{\alpha\}$, $k \in \mathbb{N}$ and $b \in \mathbb{C}$ set $f_{\beta,k}(b) = b + 2\,\mathrm{sign}(\beta)\sqrt{b_2}(k\beta, \alpha) + (k\beta, \alpha)^2$.

First of all one can formulate the following criterion of simplicity for the module $M(\lambda, b)$, which follows immediately from Theorem 8.2.1 and Lemma 8.1.1.

Theorem 8.3.1. *$M(\lambda, b)$ is simple if and only if the two following conditions are satisfied:*

1. *$((\lambda + \rho, \alpha) + 2k)^2 \neq b$ for all $k \in \mathbb{Z}$.*

2. $((2(\lambda+\rho,\beta)-k(\beta,\beta))\,(2(\lambda+\rho,s_\alpha(\beta))-k(s_\alpha(\beta),s_\alpha(\beta)))+(\alpha,\beta)(\alpha,s_\alpha(\beta))\cdot$ $\cdot\,(b-(\lambda+\rho,\alpha)^2))\neq 0$ *for all* $\beta\in\Delta_+\setminus\{\alpha\}$ *and for all* $k\in\mathbb{N}$.

We remark that the first condition of the Theorem 8.3.1 is equivalent to the condition that the module $N(a,b)$ (see definition of $M(\lambda,b)$) and thus the module $M(\lambda,b)$ is α-stratified. Hence for α-stratified modules one needs to check only the second condition. The following theorem is a generalization of the BGG Theorem and [69, Theorem 2].

Theorem 8.3.2. *The following statements are equivalent:*

1. $L(\lambda,b_1)$ *is a subquotient of* $M(\mu,b_2)$.
2. $M(\lambda,b_1)\subset M(\mu,b_2)$.
3. $(\lambda,b_1)\prec(\mu,b_2)$.

The proof is standard and analogous to the proof of the corresponding result for Verma modules (see, for example, [96, 69]). One can also consult [73] for some technical details.

9 The BGG resolution

In this Chapter we begin the study of simple (as objects in $\mathcal{K}^\alpha$) quotients $L(\lambda, p)$ of the GVMs $M(\lambda, p)$ considered in Chapter 5. In fact, we construct an analogue of the BGG resolution for such modules and obtain an analogue of the Weyl character formula. We also present a BGG resolution for some simple quotients of GVMs, induced from generic GZ modules. The results from this Chapter were obtained in [47, 48, 88, 90], where the reader can find missing technical details. We retain the notation from Chapter 5.

9.1 General case – semidominant parameters

For this Section we assume that $\mathfrak{G}$ is an arbitrary simple finite-dimensional complex Lie algebra. We begin with a cohomological construction of an exact sequence of GVMs. Denote by W^+ the subgroup of W, generated by all s_β, $\beta \in \pi \setminus \pi(\mathfrak{A})$ (this means $\beta \in \pi$, $\beta \neq \alpha$). Recall the set Ω of parameters of GVMs, defined in Section 5.5, and the action of W on Ω constructed in the same Section. We will call an element $(\lambda, p) \in \Omega$ *minimal*, if $(\lambda, p) - s_\beta(p) = (\beta, p_\beta)$ for all $\beta \in \pi \setminus \pi(\mathfrak{A})$. For some time we fix a minimal element $(\lambda, p) \in \Omega$.

Let $\hat{\Delta}$ be the set of all positive roots, generated by $\pi \setminus \{\alpha\}$. Denote by $B = B(\alpha)$ the subalgebra of $\mathfrak{G}$ generated by all root subspaces $\mathfrak{G}_{-\beta}$, $\beta \in \hat{\Delta}$. Consider B as a module over the subalgebra $A = \mathfrak{H} \oplus \mathfrak{N}$ under the action $h \cdot a = [h, a] + \lambda(h)a$ for any $h \in \mathfrak{H}$ and $a \in B$, and

$$b \cdot a = \begin{cases} [b, a], & [b, a] \in B; \\ 0, & [b, a] \notin B. \end{cases}$$

for all $b \in \mathfrak{N}$ and $a \in B$. Clearly, this action can be naturally extended to an action on the exterior powers $\bigwedge^{k} B$, for all $k \in \mathbb{N}$.

Let ε be the unique eigenvalue on $M(\lambda, p)$ of the quadratic Casimir operator

$$C = h_0 + \sum_{\alpha \in \Delta_+} X_{-\alpha} X_\alpha,$$

where h_0 is a certain fixed element in $S(\mathfrak{H})$. Note that this eigenvalue is determined uniquely by (λ, p) via the generalized Harish-Chandra homomorphism.

Define $U_\varepsilon = U(\mathfrak{G})/(C - \varepsilon)$ and consider the following $\mathfrak{G}$-modules:

$$D_k = U_\varepsilon \bigotimes_{U(A)} \bigwedge^{k} B,$$

where $k \in \mathbb{Z}_+$.

Following [9], for $k \in \mathbb{N}$, define the homomorphisms $d_k : D_k \to D_{k-1}$ as follows:

$$d_k(X \otimes X_1 \wedge X_2 \wedge \cdots \wedge X_k) =$$
$$\sum_{i=1}^{k}(-1)^{i+1} X X_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k +$$
$$\sum_{1 \leqslant i < j \leqslant k} (-1)^{i-j} X \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k.$$

Since $d_k \circ d_{k+1} = 0$, we immediately obtain that the sequence

$$0 \leftarrow D_0/\operatorname{Im} d_1 \xleftarrow{\eta} D_0 \xleftarrow{d_1} D_1 \xleftarrow{d_2} D_2 \xleftarrow{d_3} \ldots,$$

where η is a natural projection, is a complex. We will denote this complex by $\mathfrak{V}_\alpha(\lambda, \varepsilon)$.

Theorem 9.1.1. *The complex $\mathfrak{V}_\alpha(\lambda, \varepsilon)$ is exact.*

Proof. The algebra U_ε inherits the natural gradation on $U(\mathfrak{G})$ by the degree of the monomials. This leads to a gradation on D_k. For $l \geqslant k$, let $D_k^{(l)}$ be a subspace spanned by the elements $x \otimes y$ where x is an element in U_ε of degree less than or equal to $l - k$ and $y \in \bigwedge^k B$. It is clear that $d_k(D_k^{(l)}) \subset D_{k-1}^{(l)}$ and thus d_k induces a homomorphism $d_k^{(l)} : D_k^{(l)}/D_k^{(l-1)} \to D_{k-1}^{(l)}/D_{k-1}^{(l-1)}$. Denote $\hat{D}_k^{(l)} = D_k^{(l)}/D_k^{(l-1)}$. Also set $M^{(l)} = \hat{D}_0^{(l)}/\operatorname{Im} d_1^{(l)}$ and let $\eta^{(l)}$ be the corresponding induced homomorphism.

It is sufficient to show that, for every l, the complex

$$0 \leftarrow M^{(l)} \xleftarrow{\eta^{(l)}} \hat{D}_0{}^{(l)} \xleftarrow{d_1^{(l)}} \hat{D}_1{}^{(l)} \xleftarrow{d_2^{(l)}} \hat{D}_2{}^{(l)} \xleftarrow{d_3^{(l)}} \ldots \tag{10}$$

is exact.

By the PBW Theorem, for every $k \in \mathbb{Z}_+$, one can write:

$$D_k = \left(U(\mathfrak{N}_-) \bigotimes \bigwedge^k B\right) \bigoplus \left(\sum_{m \geqslant 1} X_\alpha^m U(\mathfrak{N}(\mathfrak{A})) \bigotimes \bigwedge^k B\right)$$

and hence

$$\hat{D}_k^{(l)} \simeq \left(U(\mathfrak{N}_-)^{(l-k)} \bigotimes \bigwedge^k B\right) \bigoplus \left(\sum_{m=1}^{l-k} X_\alpha^m U(\mathfrak{N}(\mathfrak{A}))^{(l-k-m)} \bigotimes \bigwedge^k B\right).$$

We will denote by $s_\alpha \mathfrak{N}_-$ the subalgebra generated by $\mathfrak{N}(\mathfrak{A})$ and X_α. Let $\mathfrak{N}_-^B$ (resp. $s_\alpha \mathfrak{N}_-^B$) be the subalgebra generated by $X_{-\beta}$, $\beta \in \Delta_+$, $\beta \notin \hat{\Delta}$ (resp. $\beta \in s_\alpha \Delta_+$, $\beta \notin s_\alpha(\hat{\Delta})$) and let $S_j(B)$ be the set of all homogeneous elements of degree j in the symmetric algebra of B. Then

$$\hat{D}_k^{(l)} \simeq \left(\sum_{j=0}^{l-k} U(\mathfrak{N}_-^B)^{(l-j-k)} S_j(B) \bigotimes \bigwedge^k B\right) \bigoplus \left(\sum_{j=0}^{l-k} U(s_\alpha \mathfrak{N}_-^B)^{(l-j-k)} S_j(B) \bigotimes \bigwedge^k B\right).$$

For any homogeneous element $u \in U(\mathfrak{N}_-^B)$ (resp. $u \in U(s_\alpha \mathfrak{N}_-^B)$) of degree $l-j-k$ we have that $d_k^{(l)}(uS_j(B) \otimes \bigwedge^k B) \subset uS_{j+1}(B) \otimes \bigwedge^{k-1} B$. Therefore $d_k^{(l)}$ induces a complex which is in fact the Koszul complex, and hence is exact. Using the PBW theorem we conclude that the complex (10) decomposes into a direct sum of exact complexes and therefore is exact. The theorem is proved. $\square$

From Theorem 9.1.1 one immediately obtains the following formula for the formal character of $D_0/\operatorname{Im} d_1$, which we will need later on in Section 9.4:

Corollary 9.1.1.

$$\operatorname{ch}(D_0/\operatorname{Im} d_1) = \sum_{i \geqslant 0} (-1)^i \operatorname{ch} D_i.$$

Denote by $P(\alpha)^{++}$ the set of all $(\lambda, p) \in \Omega$, such that $w(\lambda, p) \prec (\lambda, p)$ for all $w \in W$. Let w_0 be the longest element in W and w_0^α be the longest element in W^+. Denote by $P(\alpha)^+$ the set of all $w_0^\alpha w_0(\lambda, p)$, $(\lambda, p) \in P(\alpha)^{++}$. We will call the elements from $P(\alpha)^{++}$ (resp. $P(\alpha)^+$) dominant (resp. semidominant). In this section we will construct the BGG resolution of the module $L(\lambda, p)$, $(\lambda, p) \in P(\alpha)^+$. Let W_k^+ denote the set of all elements of W of length $k \in \mathbb{Z}_+$.

Theorem 9.1.2. *Let $(\lambda, p) \in P(\alpha)^+$. Denote by C_k the direct sum of all $M(w(\lambda, p))$, $w \in W_k^+$. Then there exists an exact sequence*

$$0 \leftarrow L(\lambda, p) \xleftarrow{\eta} C_0 \xleftarrow{\delta_1} C_1 \xleftarrow{\delta_2} \dots \xleftarrow{\delta_m} C_m \leftarrow 0,$$

where η is a natural projection and δ_i are the standard homomorphisms, defined with respect to the Bruhat order on W^+, see [9, Lemma10.4].

Proof. It follows directly from Theorem 5.6.1 that all the homomorphisms are well-defined and that this sequence is a complex. First suppose that the statement is true for all minimal (λ, p). Then using the translation functors (i.e tensoring the corresponding sequences with finite-dimensional $\mathfrak{G}$-modules and taking the summand which corresponds to a central character) we produce the resolutions for all $(\lambda, p) \in P(\alpha)^+$. So, it is enough to prove the statement for minimal (λ, p). Fix such (λ, p) and consider the corresponding exact complex $\mathfrak{V}_\alpha(\lambda, \varepsilon)$. Assume that we know that $D_0/\operatorname{Im}(d_1) \simeq L(\lambda, p)$. Then, following [9] or [103], one constructs a sequence of homomorphisms $\nu^i : B_i \to C_i$ which makes the following diagram commutative:

$$\begin{array}{ccccccccc}
\dots \longrightarrow & B_2(\lambda, p) & \xrightarrow{d_2} & B_1(\lambda, p) & \xrightarrow{d_1} & B_0(\lambda, p) & \xrightarrow{\eta} & L(\lambda, p) & \longrightarrow 0 \\
& \downarrow{\scriptstyle \nu^2} & & \downarrow{\scriptstyle \nu^1} & & \downarrow{\scriptstyle \nu^0} & & \downarrow{\scriptstyle 1} & \\
\dots \longrightarrow & C_2(\lambda, p) & \xrightarrow{\delta_2} & C_1(\lambda, p) & \xrightarrow{\delta_1} & C_0(\lambda, p) & \xrightarrow{\eta} & L(\lambda, p) & \longrightarrow 0.
\end{array}$$

To complete the proof we have only to show that $D_0/\operatorname{Im}(d_1) \simeq L(\lambda, p)$, or in other words, that

$$M = M(\lambda, p)/\sum_{\beta\in(\pi\setminus\{\alpha\})} M(s_\beta(\lambda, p))$$

is a simple object in $\mathcal{K}^\alpha$. From Theorem 5.6.1, it follows that the only simple subquotients that can occur in M are $L(w(\lambda, p))$ for $w \in W^+$. Let $w \in W^+$, $w \neq 1$. Then there exists $\beta \in (\pi \setminus \{\alpha\})$ such that the length of $s_\beta w$ is less than the length of w. Hence, the growth of the dimensions of the weight spaces of $L(w(\lambda, p))_{\mu-k\beta}$ is strictly bigger the the growth of the dimensions of $M_{\mu-k\beta}$. This means that $L(w(\lambda, p))$ is not a subquotient of M. Hence, the only subquotient of M is $L(\lambda, p)$, which has multiplicity not bigger than 1 and we obtain $M \simeq L(\lambda, p)$. This completes the proof. $\square$

It is natural to call the sequence, constructed in Theorem 9.1.2, the BGG resolution of $L(\lambda, p)$.

9.2 Simply laced case – dominant parameters

It is clear, that almost all the arguments used in Section 9.1, except those about the simplicity of $D_0/\operatorname{Im}(d_1)$, can be extended to the case $(\lambda, p) \in P(\alpha)^{++}$. In fact, we can state that, from Section 9.1, the following statement follows easily.

Proposition 9.2.1. *Let $(\lambda, p) \in P(\alpha)^{++}$ be a minimal element. Then there exists an exact sequence*

$$0 \leftarrow D_0/\operatorname{Im}(d_1) \xleftarrow{\eta} C_0 \xleftarrow{\delta_1} C_1 \xleftarrow{\delta_2} \dots \xleftarrow{\delta_m} C_m \leftarrow 0,$$

where η and δ_i are as in Theorem 9.1.2.

To prove, that $D_0/\operatorname{Im}(d_1) \simeq L(\lambda, p)$ for a minimal $(\lambda, p) \in P(\alpha)^{++}$, we have to assume that $\mathfrak{G}$ has a simply-laced Dynkin diagram (note we can not prove the corresponding result for a non-simply laced case; we do not know if it is true there).

Lemma 9.2.1. *Assume, that $\mathfrak{G}$ has a simply-laced Dynkin diagram and $(\lambda, p) \in P(\alpha)^{++}$ is a minimal element. Then $D_0/\operatorname{Im}(d_1) \simeq L(\lambda, p)$.*

Proof. The proof is essentially the content of [47, Section 5], so we skip some technical details which can be found there. Let $K(\mathfrak{G})$ denote the subalgebra of $\mathfrak{G}$, generated by all $X_{-\beta}$, $\beta \in (\Delta_+ \setminus \hat{\Delta})$.

First we consider arbitrary $M(\mu, q)$ with a weight generator v. Call an element $u \in M(\mu, q)$ quasi-primitive, if there exists a submodule $M \subset M(\mu, q)$, $M \neq M(\mu, q)$, such that $\mathfrak{N}u = 0$ in the quotient $M(\mu, q)/M$. It is easy to see, that as soon as $K(\mathfrak{G})v$ intersects any proper submodule M of $M(\mu, q)$ then $K(\mathfrak{G})v$ contains a quasi-primitive element of F. The main observation, we are going to prove is the claim.

Lemma 9.2.2. *The only quasi-primitive elements of $K(\mathfrak{G})v$ are $\mathbb{C}X_{-\alpha}^k v$, $k \in \mathbb{Z}_+$.*

Proof. A direct calculation shows that for any $\tau \in \mathfrak{H}^*$ the existence of a non-zero α-primitive element in $K(\mathfrak{G})v$ of weight $\mu - \tau$ is equivalent to a system of linear equations on $\mu(H_\beta)$, $\beta \in \pi$, and does not depend on q. But this contradicts Theorem 5.6.1. Thus the only α-primitive elements in $K(\mathfrak{G})v$ are $\mathbb{C}X_{-\alpha}^k v$, $k \in \mathbb{Z}_+$.

Now suppose that $v' \in (K(\mathfrak{G})v)_\nu$ is quasi-primitive and $(K(\mathfrak{G})v)_\xi$ has no quasi-primitive elements if $p_{\mathfrak{A}}(\nu) \leqslant p_{\mathfrak{A}}(\xi)$. Consider a basis T in $\Delta_+ \setminus \{\alpha\}$ containing $\pi \setminus \{\alpha\}$. Then $X_\gamma v' = 0$ for all $\gamma \in \pi \setminus \{\alpha\}$. If $\gamma \in T \setminus \pi$, we have $(\gamma, \alpha) \neq 0$ since $\mathfrak{G}$ has a simply laced Dynkin diagram. Let $Q \simeq sl(2, \mathbb{C})$ be a subalgebra generated by $X_{\pm\gamma}$ and F be a Q-module generated by v'. Suppose that $X_\gamma v' \neq 0$. Since v' is quasi-primitive it implies that $v' \notin F'$, where F' is a Q-module generated by $X_\gamma v'$. Then F'_ν contains a non-zero element v'' such that $X_\gamma v'' = 0$ and hence F' has a finite-dimensional quotient. Since $M(\mu, q)$ is α-stratified then $v_k = X_{-\alpha}^k v'$ is quasi-primitive for all $k > 0$. Note that $X_\gamma v_k = 0$ for all k. Indeed, if $X_\gamma v_k \neq 0$ for some $k > 0$ then we can apply to v_k the same arguments as above and conclude that a Q-module generated by $X_\gamma v_k$ also has a finite-dimensional quotient of the same dimension. But $(\alpha, \gamma) \neq 0$ and hence these finite-dimensional modules have different highest weights which is a contradiction from the $sl(2, \mathbb{C})$-theory. Therefore, $X_\gamma v_k = 0$ for all $k > 0$. Using the fact that the root system Δ is finite we find $m \geqslant 0$ such that $X_\beta v_m = 0$ for all $\beta \in T$. Hence, v_m is α-primitive and thus belongs to $\mathbb{C}X_{-\alpha}^k v$ for some $k \in \mathbb{Z}_+$. We conclude that v' is α-primitive and belongs to $\mathbb{C}X_{-\alpha}^k v$ for some $k \in \mathbb{Z}_+$. □

Now it is clear, that for any quotient V of $M(\mu, q)$ holds $\dim V_\nu \geqslant \dim(K(\mathfrak{G})v)_\nu$. Consider the module $M(\lambda, p)$. Form Corollary 9.1.1 we deduce that $\dim(D_0/\operatorname{Im}(d_1))_\nu = \dim(K(\mathfrak{G})v)_\nu$ for all ν such that $(K(\mathfrak{G})v)_\nu$ is non-zero. This implies that $D_0/\operatorname{Im}(d_1)$ is a simple module, and hence $D_0/\operatorname{Im}(d_1) \simeq L(\lambda, p)$. □

From Lemma 9.2.1 and Proposition 9.2.1 we immediately obtain the BGG resolution of $L(\lambda, p)$, $(\lambda, p) \in P(\alpha)^{++}$ for $\mathfrak{G}$ having a simply-laced Dynkin diagram.

Corollary 9.2.1. *Assume that $\mathfrak{G}$ has a simply-laced Dynkin diagram and $(\lambda, p) \in P(\alpha)^{++}$. Then there exists an exact sequence*

$$0 \leftarrow L(\lambda, p) \xleftarrow{\eta} C_0 \xleftarrow{\delta_1} C_1 \xleftarrow{\delta_2} \dots \xleftarrow{\delta_m} C_m \leftarrow 0,$$

where η and δ_i are as in Theorem 9.1.2.

Proof. Analogous to that of Theorem 9.1.2. □

9.3 Corank one case of GZ modules

There is also a transparent way to construct an analogue for the BGG resolution in some cases considered in Chapter 7. For this Section we retain the notation from Chqpter 7. Our results are quite easy and far from being general, so we omit all technical details and refer the reader to [90].

Let $\mathfrak{G}$ be the $gl(\mathfrak{N}, \mathbb{C})$ (or $sl(n, \mathbb{C})$) algebra with $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ and $\pi \setminus \pi(\mathfrak{A}) = \{\alpha_1\}$. Let $[\,l\,]$ be a tableau, such that

- $l_{i,1} = l_{n,1}$ for all i;
- $l_{n,i} - l_{n,i+1} \in \mathbb{Z}$ for all i;
- $l_{k,i} - l_{k,j} \notin \mathbb{Z}$ for all $k < n$ and all i, j;
- $l_{k,i} - lk + 1, j \notin \mathbb{Z}$ for all $k < n$ and all $i, j > 1$.

We will denote by P the set of all tableaux satisfying the conditions above. Let $M([\,l\,])$ be the GVM constructed in the Section 7.4. Recall that, in Section 7.2, we defined an action of W on P. The BGG resolution of $M([\,l\,])$ for some special $[\,l\,]$ has the following form:

Theorem 9.3.1. *Assume that $l_{n,i} - l_{n,i+1} \in \mathbb{N}$ for all i. Then the following natural sequence is exact:*

$$0 \to M(s_{\alpha_1}([\,l\,])) \to M([\,l\,]) \to L([\,l\,]) \to 0.$$

Proof. In fact we have to prove that $M = M([\,l\,])/M(s_{\alpha_1}([\,l\,]))$ is a simple module. From the construction of both $M([\,l\,])$ and $M(s_{\alpha_1}([\,l\,]))$ from Section 7.4, it follows that M has no non-trivial α-primitive elements, and hence is simple. □

We have called this situation a corank one case, since the difference in ranks between $\mathfrak{G}$ and $\mathfrak{A}$ equals one. We also note that the corresponding result for another corank one situation, when $\pi \setminus \pi(\mathfrak{A}) = \{\alpha_{n-1}\}$ is still a conjecture for $n > 3$. It is also natural to call the sequence from Theorem 9.3.1 the BGG resolution of $L([\,l\,])$. We also note, that, using the notation from Section 9.1, W^+ is generated by s_{α_1} in our case, so this sequence has the same form, as ones constructed in Theorem 9.1.2 and Corollary 9.2.1.

9.4 Weyl character formula

Using Theorem 9.1.2 and Corollary 9.2.1 it is easy to write down the formal character of $L(\lambda, p)$, $(\lambda, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$. Let ρ' denote the halfsum of all roots in $\hat{\Delta}$ and $K = \Delta_+ \setminus \hat{\Delta}$.

Theorem 9.4.1. *Let $(\lambda, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$. Then there exists an element $a(\lambda, p) \in \mathfrak{H}^*$ such that*

$$\mathrm{ch}(L(\lambda,p)) = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right)\left(\prod_{\beta\in K\setminus\{\alpha\}} (1-e^{-\beta})^{-1}\right) \times$$
$$\times \left(\sum_{w\in W^+} (-1)^{l(w)} e^{w(\lambda+a(\lambda,p)+\rho')-a(\lambda,p)}\right)\left(\sum_{w\in W^+} (-1)^{l(w)} e^{w(\rho')}\right)^{-1}$$

Proof. We note that the character of $L(\lambda, p)$ is invariant under a shift by α. This gives us the first factor. Now using Theorem 9.1.2 and Corollary 9.2.1, one reduces the rest of the calculation to the Weyl character formula for the semisimple Lie algebra which corresponds to $\hat{\Delta}$. The reader can consult [47, Section 7] for more technical details. □

10 Categories $\mathcal{O}(\mathcal{P}, \mathcal{L})$

In this Chapter we present some recent results concerning the structure of certain categories connected with GVMs. These results were obtained in [44, 45, 49], where the reader can find all necessary technical details. We also use some classical results from [11, 103, 105, 22, 50]. In Sections 10.1 – 10.4 we will work in a general situation, so we fix all the notations from Chapter 3. The situation will be specialized to the case $\mathfrak{A} \simeq sl(2, \mathbb{C})$, beginning in Section 10.6, where we give all necessary additional information. We will not need the Weyl group W in this Chapter (except Section 10.5), so we will often use the letter W to indicate a $\mathfrak{G}$ (or $\mathfrak{A}$) module.

10.1 Admissible categories

Let $\mathcal{L}$ be a full abelian subcategory of the category of all finitely generated $\mathfrak{A}'$-modules. The category $\mathcal{L}$ will be called *admissible* if the following conditions are satisfied:

1. Any $M \in \mathcal{L}$ is finitely generated.

2. Any $M \in \mathcal{L}$ is weight with respect to the center of $\mathfrak{A}'$.

3. For any $M \in \mathcal{L}$ and any finite-dimensional simple $\mathfrak{A}'$-module F the module $F \otimes M$ decomposes into a direct sum of indecomposable modules from $\mathcal{L}$.

In this Chapter we will always assume that $\mathcal{L}$ is an admissible category and will often consider the objects in $\mathcal{L}$ only as $\mathfrak{A}$-modules.

Let $\mathcal{L}$ be an admissible category of $\mathfrak{A}'$-modules. Denote by $\mathcal{O}(\mathcal{P}, \mathcal{L})$ the full subcategory of the category of $\mathfrak{G}$-modules consisting of modules which are

1. finitely generated;

2. $\mathfrak{N}$-finite;

3. a direct sum of modules from $\mathcal{L}$, when viewed as $\mathfrak{A}'$-modules.

Proposition 10.1.1. *1. The category $\mathcal{O}(\mathcal{P}, \mathcal{L})$ is closed under the operations of taking submodules, quotients, finite direct sums and under tensoring with finite-dimensional $\mathfrak{G}$-modules.*

2. The modules $M_{\mathcal{P}}(V)$ and $L_{\mathcal{P}}(V)$ are objects of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ for any simple $V \in \mathcal{L}$.

3. If V is a simple module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$, then $V \simeq L_{\mathcal{P}}(\hat{V})$ for some simple $\hat{V} \in \mathcal{L}$.

Proof. Statement (1) is obvious. To prove (2), it is enough to show that $M_{\mathcal{P}}(V)$ as an $\mathfrak{A}$-module decomposes into a direct sum of $\mathfrak{A}$-submodules in $\mathcal{L}$. This follows from the fact that $M_{\mathcal{P}}(W) \simeq U(\mathfrak{N}_-) \otimes W$ as a vector space by Proposition 3.2.1 and this isomorphism carries over the decomposition of $U(\mathfrak{N}_-)$ as a direct sum of finite-dimensional α-modules with respect to the adjoint action. We conclude that $M_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ and also $L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$.

Let V be a simple module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Since V is $\mathfrak{N}$-finite and $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable, there exists a non-zero element $v \in V$ such that $\mathfrak{N}v = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{H}_\alpha$ and some $\lambda \in \mathfrak{H}_{\mathfrak{A}}^*$. Put $V_\lambda = U(\alpha)v$. Then $\mathfrak{N}w = 0$ for any $w \in V_\lambda$ implying that V_λ is a simple α-module and $V \simeq L_{\mathcal{P}}(V_\lambda)$ by Proposition 3.2.1. This completes the proof. $\square$

10.2 Projectives in $\mathcal{O}(\mathcal{P}, \mathcal{L})$

Proposition 10.2.1. *Let $\mathcal{L}$ be admissible and let V be a projective module in $\mathcal{L}$. Fix a non-negative integer k. Then the module*

$$P(V, k) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} \left((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes V\right)$$

admits a (possibly infinite) filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_s = P(V, k)$$

with components indexed by a linearly ordered set I, such that for any $0 < i \in I$ the module

$$V_i / \sum_{k<i} V_k$$

is isomorphic to $M_{\mathcal{P}}(W_i)$ for some projective $W_i \in \mathcal{L}$.

Proof. Since V is a $\mathfrak{H}_{\mathfrak{A}}$-weight module, so is $P(V, k)$. Moreover, since k is finite, among all the weights of $P(V, k)$ there exists a maximal, say λ, with respect to the natural order. Consider a $\mathfrak{A}$-module $P(V, k)_\lambda$. The PBW theorem guarantees that the $U(\mathfrak{G})$–submodule generated by $P(V, k)_\lambda$ in $P(V, k)$ is $U(\mathfrak{N}(\mathfrak{A}))$-free. Since $U(\mathfrak{G})$ is a direct sum of finite-dimensional $\mathfrak{A}$-modules under the adjoint action, it follows that $P(V, k)_\lambda$ is isomorphic to $V \otimes F$ as a $\mathfrak{A}$-module for some finite-dimensional module F. Since $\mathcal{L}$ is admissible we can decompose $V \otimes F$ into a direct sum of modules in $\mathcal{L}$. Further, since tensoring with a finite-dimensional module is an exact functor we conclude that $V \otimes F = \oplus_t X(t)$ and each $X(t)$ is projective in $\mathcal{L}$. Since λ is a maximal weight, it follows that all $M_{\mathcal{P}}(X(t))$ are submodules in $P(V, k)$ and so we can construct the first step of our filtration. Now one just proceeds by induction completing the proof. $\square$

The filtration obtained in Proposition 10.2.1 will be called the *standard filtration.* For a given standard filtration of V we will denote by $[V : M_{\mathcal{P}}(V)]$ the number (finite or infinite) of i such that $M_{\mathcal{P}}(W_i)$ is isomorphic to $M_{\mathcal{P}}(V)$. The module $P(V, k)$ constructed above need not to be projective in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ in general. In order to construct some projective modules in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ we have to assume that $\mathcal{O}(\mathcal{P}, \mathcal{L})$ has certain properties.

A direct summand (or block) $\mathcal{O}_i$ of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ is said to be *quasi-finite* (resp. *finite*) if the set of highest weights (with respect to $\mathfrak{H}_{\mathfrak{A}}$) of all simple modules in $\mathcal{O}_i$ is finite (resp. $\mathcal{O}_i$

contains only finitely many simple objects up to isomorphism). We will say that $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a *quasi block decomposition* if $\mathcal{O}(\mathcal{P},\mathcal{L})$ decomposes into a direct sum of quasi-finite full subcategories. For example, from Corollary 3.3.3 it follows, that $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a quasi block decomposition if $\mathcal{L} = \oplus_{\chi\in Z^*(\alpha)}\mathcal{L}_\chi$, where $\mathcal{L}_\theta$ consists of those $M \in \mathcal{L}$ such that $(z-\chi(z))^k m = 0$ fo all $z \in Z(\mathfrak{G})$, $m \in M$ and some $k \in \mathbb{N}$. By the same argument, if each $\mathcal{L}_\theta$ has only finitely many non-isomorphic simple modules, then $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a block decomposition.

Theorem 10.2.1. *Suppose that $\mathcal{O}_i$ is a quasi-finite block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ and V is an indecomposable projective in $\mathcal{L}$ such that $M_{\mathcal{P}}(V) \in \mathcal{O}_i$. Then for k big enough, the $\mathcal{O}_i$–projection of the module $P(V,k)$ is projective in $\mathcal{O}(\mathcal{P},\mathcal{L})$.*

Proof. Let λ be an $\mathfrak{H}_{\mathfrak{A}}$-weight of V. Since $\mathcal{O}_i$ is quasi-finite, there exist a positive integer N such that for any $M \in \mathcal{O}_i$ holds $\mathfrak{N}^{(N)}M_\lambda = 0$. Let $k > N$. From the construction of $P(V,k)$ it follows that there is a canonical isomorphism between $\mathrm{Hom}_{\mathfrak{G}}(P(V,k),M)$ and $\mathrm{Hom}_{\mathfrak{A}'}(V,M_i)$ for any $M \in \mathcal{O}(\mathcal{P},\mathcal{L})$ (here M_i denotes the direct summand of M lying in $\mathcal{O}_i$). Since V is projective in $\mathcal{L}$ we conclude that the direct summand of $P(V,k)$ lying in $\mathcal{O}_i$ is projective in $\mathcal{O}(\mathcal{P},\mathcal{L})$, as stated. ☐

Corollary 10.2.1. *Suppose that $\mathcal{L}$ has enough projective modules (i.e. any module is a quotient of a projective module) and $\mathcal{O}_i$ is a quasi-finite block of $\mathcal{O}(\mathcal{P},\mathcal{L})$, then*

1. *$\mathcal{O}_i$ has enough projective modules;*
2. *Every projective in $\mathcal{O}_i$ has a standard filtration;*
3. *There is a one-to-one correspondence between the simple objects in $\mathcal{O}_i$ and the indecomposable projective objects in $\mathcal{O}_i$.*

Proof. The second statement follows form Proposition 10.2.1. The first and the third ones follow from Theorem 10.2.1 using standard arguments. ☐

Corollary 10.2.2. *Suppose that $\mathcal{L}$ has enough projective modules and $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a quasi block decomposition, then*

1. *$\mathcal{O}(\mathcal{P},\mathcal{L})$ has enough projective modules;*
2. *Every projective in $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a standard filtration;*
3. *There is a one-to-one correspondence between the simple objects in $\mathcal{O}(\mathcal{P},\mathcal{L})$ and the indecomposable projective modules in $\mathcal{O}(\mathcal{P},\mathcal{L})$.*

10.3 Blocks of $\mathcal{O}(\mathcal{P},\mathcal{L})$

Theorem 10.3.1. *Suppose that $\mathcal{L}$ has enough projective modules, $\mathcal{O}_i$ is a finite block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ and any object in $\mathcal{O}_i$ has finite length. Then $\mathcal{O}_i$ is isomorphic to the module category of a finite-dimensional algebra.*

Proof. Consider the endomorphism algebra of the sum of projective covers of all simple modules in $\mathcal{O}_i$. $\square$

Corollary 10.3.1. *Suppose that $\mathcal{L}$ has enough projective modules, $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a block decomposition and any object in $\mathcal{O}(\mathcal{P},\mathcal{L})$ has finite length. Then each block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ is isomorphic to the module category of a finite-dimensional algebra.*

Now we discuss which finite-dimensional algebras can appear in Corollary 10.3.1. For this we need some abstract notation. Let A be a finite dimensional algebra. A two-sided ideal J in A is called *projectively stratifying* if J is generated (as a two-sided ideal) by a primitive idempotent and J is projective as a left A-module. The algebra A is called *projectively stratified* if there exists an ordering $e_1, \dots, e_n$ of the equivalence classes of primitive idempotents of A such that for each l the idempotent e_l generates a projectively stratifying ideal in the quotient algebra $A/ < e_1, \dots, e_{l-1} >$.

This is equivalent to requiring that each projective module P has a filtration of the following form: $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = P$ where M_{l+1}/M_l is a direct sum of copies of the module $(A/ < e_1, \dots, e_l >) \cdot e_{l+1}$ which is projective over the quotient algebra $A/ < e_1, \dots, e_l >$. Any projectively stratified algebra is a stratifying endomorphism algebra in the sense of Cline, Parshall and Scott [21]. Any quasi-hereditary algebra is projectively stratified. A projectively stratified algebra A is quasi-hereditary if and only if all the rings E_l are semisimple if and only if A has finite global dimension (see [21]).

Theorem 10.3.2. *Assume that Λ is a sum of module categories of projectively stratified algebras. Then any finite block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ (if it exists), such that all its objects have finite length, also is the module category of a projectively stratified algebra.*

Proof. By Proposition 10.1.1 there is a natural bijection between simple objects in $\mathcal{L}$ and $\mathcal{O}(\mathcal{P},\mathcal{L})$. The induction process can glue several blocks of $\mathcal{L}$ together into one block of $\mathcal{O}(\mathcal{P},\mathcal{L})$. Assume that a finite block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ is given and call it $\mathcal{O}_i$. Fix the direct summand $\mathcal{L}_i$ of $\mathcal{L}$ (in general, this is a product of several blocks) such that the above bijection restricts to a bijection between $\mathcal{L}_i$-simples and $\mathcal{O}_i$-simples.

The functors occuring in the construction of projective objects in $\mathcal{O}(\mathcal{P},\mathcal{L})$ are exact and hence transport filtrations from $\mathcal{L}$ to $\mathcal{O}(\mathcal{P},\mathcal{L})$. Start with a module V which contains (up to an isomorphism) at least one copy of each isomorphism class of each non-isomorphic indecomposable projective in $\mathcal{L}_i$. Then the tensor product $(U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes V$ is again projective (in $\mathcal{L}$) and maps onto all projectives in $\mathcal{L}_i$. Hence it contains at least one copy of each isomorphism class of indecomposable projective modules in $\mathcal{L}_j$. A filtration of this module, as in the definition of projectively stratified algebra yields a similar filtration of the induced $U(\mathfrak{G})$-module. Since the number of isomorphism classes of indecomposable projectives in the block $\mathcal{O}_i$ equals the number of indecomposable projectives in $\mathcal{L}_j$, the resulting filtration has the correct length. $\square$

The following Corollary from Theorem 10.3.2 is obvious.

Corollary 10.3.2. *Under the conditions of Theorem 10.3.2 the following are equivalent for a finite block $\mathcal{O}_i$ of $\mathcal{O}(\mathcal{P},\mathcal{L})$:*

1. *The block $\mathcal{O}_i$ is equivalent to the module category of a quasi-hereditary algebra.*

2. *The block $\mathcal{O}_i$ has finite global dimension.*

3. *For any simple $L(V) \in \mathcal{O}_i$ the module V is projective in $\mathcal{L}$.*

10.4 Reciprocities in $\mathcal{O}(\mathcal{P}, \mathcal{L})$

In this Section we give two different generalizations of BGG reciprocity for $\mathcal{O}(\mathcal{P}, \mathcal{L})$. The first is based on dualities on projectively stratified algebras and the second is based on some numerical properties of simple modules in $\mathcal{L}$. We also disscuss a connection between them. The first analogue of BGG reciprocity is an abstract property of a projectively stratified algebra.

Theorem 10.4.1. *Let A be a projectively stratified algebra over an algebraically closed field k. Assume that A has a duality (i.e. a contravariant exact equivalence, which preserves simple objects). Assume also that each projective A–module has a filtration by "Verma modules" $M(i)$ (indexed by i in I, the set of isomorphism classes of indecomposable projective A–modules) satisfying $(M(i) : L(i)) = 1$ and $(M(i) : L(j)) \neq 0$ implies $j \leqslant i$. Denote by $l(i)$ the k–dimension of $End_A(P(i))$ which equals $[P(i) : M(i)]$. Then for all $i, j \in I$ there is a BGG-reciprocity*

$$[P(i) : M(j)] = l(j)(M(j) : L(i)).$$

Note that the only properties of Verma modules needed here are the ones mentioned in the assumptions. No universality is needed.

Proof. We proceed by induction along the filtration of A which makes it a projectively stratified algebra. Let j be a maximal index. Write $P(i) = Ae$ and $P(j) = Af$ for some primitive idempotents e and f. By the choice of j the trace ideal AfA is projective as a left module. We have $Ae \cap AfA = (Af)^l$ for some l which can be computed as $l = dim_k \operatorname{Hom}_A(Af, Ae)/l(j)$. By the condition on Verma modules, all occurences of $M(j)$ in a filtration of A are inside the ideal AfA. Hence $[P(i) : M(j)] = l \cdot l(j) = dim_k \operatorname{Hom}_A(Af, Ae) = \dim_k(fAe)$. Applying the duality on A we get $\dim_k(fAe) = \dim_k(eAf) = (P(j) : L(i))$. Again by the defining condition on Verma modules we have $(P(j) : L(i)) = l(j) \cdot (M(j) : L(i))$. $\square$

We note, that if all $l(i) = 1$ we will obtain a quasi-hereditary algebra and the classical BGG-reciprocity. We remark that all the results above are also true if the notion of a simple $\mathfrak{G}$-module and a simple object in $\mathcal{L}$ do not coincide.

The second analogue of the BGG reciprocity is the following Theorem.

Theorem 10.4.2. *Suppose that $\mathcal{L}$ has a block decomposition, $\mathcal{L}$ has enough projective modules, $\mathcal{O}(\mathcal{P}, \mathcal{L})$ has a block decomposition and any object in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ has finite length. Moreover, assume that simple objects of each block of $\mathcal{L}$ are indexed by a poset (depending* *on a block) such that for any indecomposable projectives P and Q in $\mathfrak{L}$ there are non-zero homomorphisms from P to Q only if the corresponding simple quotients $s(P)$ and $s(Q)$ satisfy $L(P) \leqslant L(Q)$. Suppose also that for any simple modules X and Y in $\mathfrak{L}$ there exists a constant $\mathfrak{i}(X,Y)$ such that for any finite-dimensional α-module F holds $((F\otimes X):Y) = \mathfrak{i}(X,Y)((F\otimes Y):X)$. (We will call this condition the* **duality condition**.*) Then for any two simple modules V and W in $\mathfrak{L}$ holds*

$$[P(L(V)) : M_{\mathcal{P}}(W)] = \mathfrak{i}(V,W)\mathfrak{l}(\tilde{V})(M_{\mathcal{P}}(W) : L(V)),$$

where $P(L(X))$ is the projective cover of $L(X)$ in $\mathcal{O}(\mathcal{P},\mathfrak{L})$ and $\mathfrak{l}(\tilde{X})$ is the length of the projective cover $\tilde{X}$ of X in $\mathfrak{L}$.

Proof. By Theorem 10.3.2 each block of $\mathcal{O}(\mathcal{P},\mathfrak{L})$ corresponds to a projectively stratified finite-dimensional algebra. First we note that $\dim\mathrm{Hom}(P(L(V)),M) = (M : L(V))$ for any module $M\in\mathcal{O}(\mathcal{P},\mathfrak{L})$. Thus we have only to show that

$$[P(L(V)) : M_{\mathcal{P}}(W)] = \mathfrak{i}(V,W)\mathfrak{l}(\tilde{V})\dim\mathrm{Hom}(P(L(V)),M_{\mathcal{P}}(W))$$

for any two simple modules V and W in $\mathfrak{L}$. Fix a block $\mathcal{O}_j$. Clearly, we need to check our equality inside $\mathcal{O}_j$ only, so we can assume that $L(V)$ and $L(W)$ belong to $\mathcal{O}_j$. Let k be big enough. Let $P(V,k)_j$ be the direct summand of $P(V,k)$ in $\mathcal{O}_j$. Then

$$P(V,k)_j = \sum_{L(K)\in\mathrm{Irr}(\mathcal{O}_j)} n_K(V)P(L(K))$$

and

$$n_K(V) = \dim\mathrm{Hom}_{\mathfrak{G}}(P(V,k)_j, L(K)) = \dim\mathrm{Hom}_{\mathfrak{A}'}(\tilde{V}, L(K)).$$

In particular $n_K(V) = 0$ if $V \not< K$ with respect to the order induced from $\mathfrak{H}_\alpha$ and $n_V(V) = 1$. This allows us to proceed by induction. From the linearity of our formula (in the induction step) we obtain that it is enough to prove that $[P(V,k)_j : M_{\mathcal{P}}(W)] = \mathfrak{i}(V,W)\mathfrak{l}(\tilde{V})\dim\mathrm{Hom}(P(V,k)_j, M_{\mathcal{P}}(W))$. Further it is clear that we only have to check that $[P(V,k) : M_{\mathcal{P}}(W)] = \mathfrak{i}(V,W)\mathfrak{l}(\tilde{V})\dim\mathrm{Hom}(P(V,k), M_{\mathcal{P}}(W))$. Clearly, from the construction of $P(V,k)$ it follows that there exists a finite-dimensional α-module F such that $[P(V,k) : M_{\mathcal{P}}(W)] = ((F\otimes\tilde{V}) : W) = \mathfrak{l}(\tilde{V})((F\otimes V) : W)$. On the other hand $\dim\mathrm{Hom}(P(V,k), M_{\mathcal{P}}(W)) = \dim\mathrm{Hom}_{\alpha\oplus\mathfrak{H}_\alpha}(\tilde{V}, M_{\mathcal{P}}(W)) = \dim\mathrm{Hom}_{\mathfrak{A}}(\tilde{V}, F\otimes W) = ((F\otimes W) : V)$ by the projectivity of $\tilde{V}$. Application of the duality condition for $\mathfrak{L}$ completes the proof. $\square$

Comparing Theorem 10.4.1 with Theorem 10.4.2 one obtains the following result characterizing the behavior of simple modules.

Corollary 10.4.1. *Assume that $\mathfrak{L}$ has a block decomposition with enough projective modules and a unique simple module in each block. Assume also that there is only finitely many simples in $\mathfrak{L}$ having the same central character and $\mathcal{O}(\mathcal{P},\mathfrak{L})$ has a duality (for some $\mathfrak{G}$ and $\mathcal{P}$). If for any two simple X and Y in $\mathfrak{L}$ with projective covers $P(X)$ and $P(Y)$ respectively there exists a constant $\mathfrak{i}(X,Y)$ such that for any finite-dimensional α-module F holds $((F\otimes X):Y) = \mathfrak{i}(X,Y)((F\otimes Y):X)$ then $\mathfrak{i}(X,Y) = \mathfrak{l}(P(Y))/\mathfrak{l}(P(X))$.*

Proof. It is easy to see that all the conditions of both Theorem 10.4.1 and Theorem 10.4.2 are satisfied, hence the statement follows by comparing the two reciprocity formulae. $\square$

10.5 Gelfand-Zetlin example

In this Section we present an example of an admissible category $\mathcal{L}$ consisting of generic GZ-modules. Assume that $\mathfrak{A}$ is $gl(n,\mathbb{C})$ (or $sl(n,\mathbb{C})$). We retain the notation from Chapter 6. Let $[\,l\,]$ be a tableau satisfying the conditions

- $l_{k,i} - l_{k,j} \notin \mathbb{Z}$ for all $k < n$ and all $i \neq j$;
- $l_{k,i} - l_{k+1,j} \notin \mathbb{Z}$ for all $k < n$ and all i, j,

and let $V([\,l\,])$ be the corresponding generic GZ module. It follows directly from Lemma 7.3.1 that the category $\mathcal{L} = \mathcal{L}([\,l\,])$ of all subquotients of $V([\,l\,]) \otimes F$, where F runs through the set of all finite-dimensional $\mathfrak{A}$-modules, is admissible. From the arguments, presented in Section 10.2, we easily obtain that the corrsponding $\mathcal{O}(\mathcal{P},\mathcal{L})$ has a block decomposition and, from Theorem 10.3.2, it follows that each finite block of $\mathcal{O}(\mathcal{P},\mathcal{L})$ is a module category over a projectively stratified algebra.

There is a natural duality on $\mathcal{O}(\mathcal{P},\mathcal{L})$, which corrspondts to σ and defined in a standard way (see, for example, [49] or [44]). Hence, we have an abstract BGG reciprocity on $\mathcal{O}(\mathcal{P},\mathcal{L})$:

$$\begin{aligned}
&[P(L(V([\,l(1)\,]))) : M(V([\,l(2)\,]))] = \\
&\qquad [P(L(V([\,l(2)\,]))) : M(V([\,l(2)\,]))] \cdot (M(V([\,l(2)\,])) : L(V([\,l(1)\,]))).
\end{aligned}$$

We note that, from Section 7.3, it follows that each indecomposable block of $\mathcal{L}$ has a unique simple module. In particular, this means that $[P(L(V([\,l(2)\,]))) : M(V([\,l(2)\,]))]$ equals the length $\mathfrak{l}(\hat{V}([\,l(2)\,]))$ of the projective cover $\hat{V}([\,l(2)\,])$ of $V([\,l(2)\,])$ in $\mathcal{L}$. Hence, we have

$$[P(L(V([\,l(1)\,]))) : M(V([\,l(2)\,]))] = \mathfrak{l}(\hat{V}([\,l(2)\,]))(M(V([\,l(2)\,])) : L(V([\,l(1)\,]))).$$

To prove the second BGG reciprocity we need the following result.

Lemma 10.5.1. *Let $\mathfrak{G}$ be a simple finite-dimensional complex Lie algebra, $\mathfrak{H}$ be its Cartan subalgebra and W be the Weyl group. For $\lambda, \mu \in \mathfrak{H}^*$ set $\lambda \sim \mu$ if and only if $\lambda \in W \cdot \mu$ (here $w \cdot \mu = w(\mu+\rho) - \rho$ is the standard dot-action of W). Then for any $\lambda, \mu \in \mathfrak{H}^*$ and any simple finite-dimensional $\mathfrak{G}$-module F holds*

$$|W\lambda| \sum_{\nu:\nu+\lambda\sim\mu} \dim F_\nu = |W\mu| \sum_{\nu:\nu+\mu\sim\lambda} \dim F_\nu.$$

Proof. Let W_λ (resp. W_μ) be the subgroup of W stabilizing λ (resp. μ). Then we can rewrite our equality in the form

$$|W_\mu| \sum_{\nu:\nu+\lambda\sim\mu} \dim F_\nu = |W_\lambda| \sum_{\nu:\nu+\mu\sim\lambda} \dim F_\nu.$$

Let $w_1, w_2, \dots, w_k$ be all the elements of W and define ν_i by $\nu_i + \lambda = w_i \cdot \mu$. We can rewrite the last equality as $w_i^{-1} \cdot \lambda = \mu - w_i^{-1}(\nu_i)$. Since $\dim F_\xi = \dim F_{w(\xi)}$ for all $\xi \in \mathfrak{H}^*$ and $w \in W$ we conclude that both the left and the right hand sides of the desired equality coincide with

$$\sum_{i=1}^{n} \dim F_{\nu_i}$$

and the lemma follows. □

Combining Lemma 10.5.1 and Lemma 7.3.1, we obtain that for two simple modules $V([\,l(1)\,])$ and $V([\,l(2)\,])$ from $\mathcal{L}$ having central characters χ_{μ_1} and χ_{μ_2} respectively holds

$$\mathfrak{i}(V([\,l(1)\,]), V([\,l(2)\,])) = |W(\mu_2)|/|W(\mu_1)|.$$

Thus all conditions of Theorem 10.4.2 are also satisfied and the corresponding analogue of BGG reciprocity holds. Since we have both reciprocities, we can apply Corollary 10.4.1 and obtain for $[\,l(1)\,]$, $[\,l(2)\,]$, μ_1, μ_2 as above

$$\mathfrak{i}(V([\,l(1)\,]), V([\,l(2)\,])) = \mathfrak{l}(P(V([\,l(2)\,])))/\mathfrak{l}(P(V([\,l(1)\,]))) = |W(\mu_2)|/|W(\mu_1)|.$$

If we fix $[\,l(1)\,]$ such that $l(1)_{n,i} = 0$ for all i, we will have $|W(\mu_1)| = 1$ and $\mathfrak{l}(P(V([\,l(1)\,])))$. Hence $\mathfrak{l}(P(V([\,l(2)\,]))) = |W(\mu_2)|$. We can now combine this in the following statement:

Theorem 10.5.1. *1. For any simple generic $V([\,l\,])$, having the central character χ_μ, the length of the projective cover of $V([\,l\,])$ in $\mathcal{L} = \mathcal{L}([\,l\,])$ equals $|W(\mu)|$.*

2. The BGG reciprocity in $\mathcal{O}(\mathcal{P}, \mathcal{L})$, $\mathcal{L} = \mathcal{L}([\,l\,])$ has the following form:

$$[P(L(V([\,l(1)\,]))) : M(V([\,l(2)\,]))] = |W(\mu)|(M(V([\,l(2)\,])) : L(V([\,l(1)\,]))),$$

where χ_μ is the central character of $L([\,l(2)\,])$.

10.6 $sl(2,\mathbb{C})$ case: a functor E

During Sections 10.6 – 10.8 we study, in detail, the category $\mathcal{O}(\mathcal{P}, \mathcal{L})$ in the case $\mathfrak{A} \simeq sl(2,\mathbb{C})$ and $\mathcal{L}$ is associated with a module $V(a, b)$, $a, b \in \mathbb{C}$ (see Section 4.3). This is exactly the case which corresponds to GVMs $M(\lambda, p)$, studied in Chapter 5. We begin by describing the category $\mathcal{O}(\mathcal{P}, \mathcal{L})$ more precisely. We retain the notation from Chapter 5.

Call a weight $\mathfrak{A}$-module V with finite-dimensional weight spaces *admissible*, provided f acts bijectively on V. By definition, any $V(a, b)$ is admissible. Let $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(V(a, b))$

denote the full subcategory of the category of $\mathfrak{A}$-modules, which consists of all admissible submodules and all admissible quotients of all modules having the form $V(a,b)\otimes F$, where F is a finite-dimensional $\mathfrak{A}$-module. One can easily see that $\tilde{\mathcal{L}}$ inherits the abelian structure from the category of all $\mathfrak{A}$-modules, i.e. $\tilde{\mathcal{L}}$ is closed under taking kernels and cokernels of homomorphisms. Moreover, $\tilde{\mathcal{L}}$ is closed under taking finite direct sums. One can extend $\tilde{\mathcal{L}}$ to the corresponding admissible category $\mathcal{L} = \mathcal{L}(V(a,b))$ of $\mathfrak{A}'$-modules. Set $\mathcal{O}(\mathcal{P},\mathcal{L}) = \mathcal{O}(\mathcal{P},\mathcal{L}(V(a,b)))$.

Since any $V(a,b)$ is a generic GZ module over $\mathfrak{A}$, we know from Section 10.5 that $\mathcal{L}$ has a block decomposition with a unique simple in each block and that the length of any projective in $\mathcal{L}$ is either 1 or 2. It is easy to see that this length equals 2 if and only if b is the square of a positive integer. Moreover, it follows from Section 4.4 that $V(a,(\sqrt{b}+i)^2)$, $i\in\mathbb{Z}$ is a complete list of simple objects in $\mathcal{L}(V(a,b))$. First of all, we establish an equivalence of certain categories $\mathcal{O}(\mathcal{P},\mathcal{L})$.

Theorem 10.6.1. *The categories $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a_1,b)))$ and $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a_2,b)))$ are (blockwise) equivalent (i.e. they are independent on a).*

Proof. To prove this we will use Mathieu's localization (see Section 4.5). We can assume that $a_1\neq a_2$. Since $\mathbb{C}$ is one-dimensional over itself, there exists $x\in\mathbb{C}$ such that $a_1 = a_2 + x\alpha$. Moreover, $x\notin\mathbb{Z}$ according to our assumption. By the definition of $\mathcal{O}(\mathcal{P},\mathcal{L})$, f acts bijectively on any module $V\in\mathcal{O}(\mathcal{P},\mathcal{L})$. Thus any V can be trivially extended to a U_f-module.

Now suppose that M is a U_f-module and $0\neq v\in M$ such that $H_\alpha v = av$ for some $a\in\mathbb{C}$. We know, from Lemma 4.5.1, that $\theta_y(H_\alpha)v = (a+2y)v$ for any $y\in\mathbb{C}$. From this, it follows immediately that the twist by θ_{-x} (resp. θ_x) is a well-defined functor from $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a_1,b)))$ to $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a_2,b)))$ (resp. from $\mathcal{O}(P,\mathcal{L}(V(a_2,b)))$ to $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a_1,b))))$. Since the composition of θ_x and θ_{-x} is an identity, we easily conclude that these fuctors are mutually inverse. The block version follows immediately. This completes the proof. $\square$

According to Theorem 10.6.1, the properties of the category $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a,b)))$ do not depend on a. Recall that for a fixed b there exists at least one $V(a,b)$ which is not simple. More precisely, if b is a square of an integer, such $V(a,b)$ is unique and if b is not a square of an integer, there are precisely two non-isomorphic non-simple modules $V(a',b)$ and $V(a'',b)$. Let $V(\mathfrak{l},b)$ be a non-simple module. The aim of this Section is to define and investigate a functor from $\mathcal{O}(\mathcal{P},\mathcal{L}(V(\mathfrak{l},b)))$ to $\mathcal{O}$.

For $M\in\mathcal{O}(\mathcal{P},\mathcal{L}) = \mathcal{O}(\mathcal{P},\mathcal{L}(V(\mathfrak{l},b)))$ denote by $E(M)$ the space of locally e-finite elements of M. Since e is locally ad-nilpotent, $E(M)$ is a $\mathfrak{G}$-submodule of M. We know, that any module in $\mathcal{O}(\mathcal{P},\mathcal{L})$ has finite length as a $\mathfrak{G}$-module. From this one deduces, that $E(M)\in\mathcal{O}$. We define E by restriction on homomorphism $\varphi: M\to N$ to $E(\varphi): E(M)\to E(N)$. Thus E is a well-defined functor from $\mathcal{O}(\mathcal{P},\mathcal{L}) = \mathcal{O}(\mathcal{P},\mathcal{L}(V(\mathfrak{l},b)))$ to $\mathcal{O}$.

Theorem 10.6.2. *Let $M\in\mathcal{O}(\mathcal{P},\mathcal{L})$.*

1. *$E(M) = 0$ if and only if $M = 0$.*

2. *For any* $\mu \in \operatorname{supp}(M)$ *and for any* $k \in \mathbb{N}$ *big enough holds* $\dim(M_{\mu-k\alpha}) = \dim(E(M)_{\mu-k\alpha})$.

3. *For any finite-dimensional* $\mathfrak{G}$*-module* F *holds* $E(M \otimes F) = E(M) \otimes F$.

4. E *sends projectives from* $\mathcal{O}(\mathcal{P}, \mathcal{L})$ *to projectives in* $\mathcal{O}$.

5. E *sends indecomposable modules to indecomposable modules.*

6. *For* $f : M \to N$ *holds* $E(f) = 0$ *if and only if* $f = 0$.

7. *If* M *is a simple object in* $\mathcal{O}(\mathcal{P}, \mathcal{L})$, *then* $E(M)$ *has a unique simple highest weight submodule, say* $L(\hat{E}(M))$, $\hat{E}(L) \in \mathfrak{H}^*$. *Moreover, for any* $M \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ *and any simple* $L \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ *holds* $(M : L) = (E(M) : L(\hat{E}(L)))$.

8. *Let* $P(L)$ *be the projective cover of a simple* $L \in \mathcal{O}(\mathcal{P}, \mathcal{L})$. *Then* $E(P(L)) = P(\hat{E}(L))$.

9. E *is a full functor.*

Proof. We will give the scheme for the proof. The reader can consult [45] for more details.

The functor E is defined on $\mathfrak{A}$-modules, so statements (1), (2), (3) and (7) follow from trivial $sl(2, \mathbb{C})$ computation. Now (4) follows from (3) and Section 10.2. (6) is obvious.

Prove (5). Assume that $M \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ is indecomposable, but $E(M) = N_1 \oplus N_2$ is a non-trivial decomposition in $\mathcal{O}$. Let $M_i = v \in M \mid X_{-\alpha}^k v \in N_i$, for some $k \in \mathbb{N}$, $i = 1, 2$. It is easy to see that $M = M_1 \oplus M_2$; moreover, both M_i, $i = 1, 2$ are $\mathfrak{G}$-submodules in M. This contradicts our assumption. This proves (5). Now (8) follows from (4), (5) and (7). To complete our proof we have to prove only (9).

Clearly, it is enough to show that, for any $M, N \in \mathcal{O}(\mathcal{P}, \mathcal{L})$, holds

$$\dim \operatorname{Hom}_{\mathcal{O}(\mathcal{P},\mathcal{L})}(M, N) = \dim \operatorname{Hom}_{\mathcal{O}}(E(M), E(N)).$$

From (7) and (8) it follows that this is true if M is a projective module. Assume that M is indecomposable. Let $P(M)$ be a projective cover of M. It is enough to prove that for any $f : E(M) \to E(N)$ there is ψ in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ such that $f = E(\psi)$. We have that $E(P(M))$ is a projective cover of $E(M)$. Let $z : E(P(M)) \to E(M)$ be a canonical epimorphism. Since $P(M)$ is projective, there exists an epimorphism $x : P(M) \to M$ and a homomorphism $y : P(M) \to N$ such that $z = E(x)$ and $f \circ z = E(y)$. For $m \in M$ set $\psi(m) = y \circ x^{-1}(m)$. We have to show that this is a well-defined map. But $\ker z \subset \ker f \circ z$, hence $\ker x \subset \ker y$ since $X_{-\alpha}$ acts bijectively on $P(M)$ and E acts on homomorphisms by restriction. This means that ψ is well-defined. Since both x and y are $\mathfrak{G}$-morphisms we deduce that ψ is also a $\mathfrak{G}$-morphism. Clearly, $E(\psi) = f$, since E is just restriction. This completes the proof of our theorem. ☐

Corollary 10.6.1. $\mathcal{O}(\mathcal{P}, \mathcal{L})$ *is equivalent to a full subcategory of* $\mathcal{O}$. *Moreover, the image of a block of* $\mathcal{O}(\mathcal{P}, \mathcal{L})$ *is contained in a block of* $\mathcal{O}$.

10.7 $sl(2,\mathbb{C})$ case: Soergel's Theorems

Using Theorem 10.6.2 it is possible to obtain an analogue of Soergel's combinatorial description of algebras, arising from $\mathcal{O}(\mathcal{P},\mathcal{L})$. Here we retain the notation from Section 2.5. Consider the principal block, $\mathcal{O}_{triv} = \mathcal{O}(\chi_{\lambda_0})$, of $\mathcal{O}$ and the big projective module $P(w_0(\lambda_0))$ in it. Suppose that b is the square of an integer. Denote by $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$ the direct summand of $\mathcal{O}(\mathcal{P},\mathcal{L}(V(\mathfrak{l},\gamma)))$ which has a non-trivial image in $\mathcal{O}_{triv}$ under E. Since E acts blockwise and $\mathcal{O}_{triv}$ is indecomposable, such indecomposable $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$ is unique. Let L be a simple object in $\mathcal{O}(\mathcal{P},\mathcal{L})$ such that $\hat{E}(L) = w_0(\lambda_0)$. L exists, since f acts injectively on $L(w_0(\lambda_0)) = M(w_0(\lambda_0))$. Call the projective module $P(L)$ in $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$ the *big projective module*. We remark that $P(L)$ can be characterized as the unique indecomposable projective in $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$, such that any GVM from this principal block occurs as a subquotient in a standard filtration of $P(L)$. The following Theorem is a direct analogue of the first part of Theorem 2.5.4.

Theorem 10.7.1. $\mathrm{End}_{\mathcal{O}(\mathcal{P},\mathcal{L})}(P(L)) \simeq \mathrm{End}_{\mathcal{O}}(P(w_0 \cdot 0))$. *In fact,* $\mathrm{End}_{\mathcal{O}(\mathcal{P},\mathcal{L})}(P(L))$ *is the coinvariant algebra.*

Proof. Follows from Theorem 10.6.2 and [107, Endomorphismensatz 3]. □

Using Theorem 10.6.1, one can generalize this result to arbitrary $\mathcal{O}(\mathcal{P},\mathcal{L}(V(a,b)))$, where b is the square of an integer. The next result generalizes Soergel's double centralizer Theorem (second part of Theorem 2.5.4).

Theorem 10.7.2. *Let B denote the (projectively stratified finite-dimensional) algebra associated with $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$. Then B is isomorphic to the endomorphism algebra of the big projective module, viewed as a module over its endomorphism ring.*

It is more convenient to prove this theorem in an "abstract" setting which we are going to introduce now. Let A (resp. B) denote the algebra associated with the principal block of $\mathcal{O}$ (resp. $\mathcal{O}(\mathcal{P},\mathcal{L})$). We recall, that according to Theorem 10.6.2 and Corollary 10.6.1, B is a (matrix) subalgebra of A. Let e be the primitive idempotent of A such that Ae is the big projective module in $\mathcal{O}_{triv}$. Then Be is the big projective module in $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$ and $C = eAe = eBe$ is the coinvariant algebra, which is the endomorphism algebra of Ae and Be. Let $T = \mathrm{Hom}_A(Ae, \_)$ denote Soergel's functor ([107]). Recall, that by Soergel's Theorem ([107, Struktursatz 2]), for any $M \in \mathcal{O}_{triv}$ and any projective $Q \in \mathcal{O}_{triv}$, holds

$$\mathrm{Hom}_A(M,Q) \simeq \mathrm{Hom}_{C=eAe}(T(M),T(Q)).$$

Proof of Theorem 10.7.2. We start from $B = \mathrm{Hom}_B(B,B)$. Applying Theorem 10.6.2, we have $\mathrm{Hom}_B(B,B) \simeq \mathrm{Hom}_A(E(B),E(B))$. Now applying the mentioned result by Soergel we obtain $\mathrm{Hom}_A(E(B),E(B)) \simeq \mathrm{Hom}_{eAe}(T(E(B)),T(E(B)))$. We know from Theorem 10.7.1, that $eAe = eBe$. Recall that $E(Be) = Ae$, hence $T(E(B)) = \mathrm{Hom}_A(Ae,E(B)) = \mathrm{Hom}_A(E(Be),E(B)) \simeq \mathrm{Hom}_B(Be,B) = eB$. Finally,

$$\mathrm{Hom}_{eAe}(T(E(B)),T(E(B))) \simeq \mathrm{Hom}_{eBe}(eB,eB).$$

Now we note that B is a matrix subalgebra of A and we can apply duality on A to the last endomorphism ring, obtaining $\mathrm{Hom}_{eBe}(eB, eB) \simeq \mathrm{Hom}_{eBe}(Be, Be)$, which completes the proof. $\square$

We note that in the proof above, we can also apply the canonical duality on B, mentioned in [50, 44].

10.8 $sl(2,\mathbb{C})$ case: tilting modules

In this Section we develop the theory of tilting module for $\mathcal{O}(\mathcal{P},\mathcal{L}) = \mathcal{O}(\mathcal{P},\mathcal{L}(V(\mathfrak{l},b)))$. As a preliminary result we have to investigate a family of tilting modules in $\mathcal{O}$. There is a general result about tilting modules for stratified algebras ([1]), but we are going to use a more symmetric definition of tilting modules for $\mathcal{O}(\mathcal{P},\mathcal{L})$; we will have to do some preliminary work. Retain the notation from Section 2.5.

Assume that $\lambda \in \mathfrak{H}^*$ is such that $s_\alpha(\lambda) = \lambda + k\alpha$ for some $k \in \mathbb{N}$. Consider the indecomposable projective module $P(\lambda) \in \mathcal{O}$. Clearly, there exists a Verma flag $P(\lambda) = P_0 \supset P_1 \supset P_2 \supset \dots$ of $P(\lambda)$ such that $P_0/P_1 \simeq M(\lambda)$ and $P_1/P_2 \simeq M(s_\alpha(\lambda))$. Set $\hat{P}(\lambda) = P(\lambda)/P_2$. Then $\hat{P}(\lambda)$ has a Verma flag with $M(\lambda)$ on the top and $M(s_\alpha(\lambda))$ on the bottom. Define a class $K(\alpha)$ of modules in $\mathcal{O}$ as follows: if $\lambda \in \mathfrak{H}^*$ is such that $s_\alpha(\lambda) - \lambda \notin (\mathbb{Z}\alpha \setminus \{0\})$ then $K(\alpha)$ contains $M(\lambda)$; in the other case, $K(\alpha)$ contains $\hat{P}(\lambda)$ if $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ and $K(\alpha)$ contains $\hat{P}(s_\alpha(\lambda))$ if $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$.

Denote by $O_1(\alpha)$ (resp. $O_2(\alpha)$) the full subcategory of $\mathcal{O}$, containing all modules which admit a filtration with subquotients from $K(\alpha)$ (resp. with subquotients of the form M^*, $M \in K(\alpha)$). Since any module in $K(\alpha)$ has a Verma flag, we have $O_1(\alpha) \subset O_1$, $O_2(\alpha) \subset O_2$ and $\mathcal{O}(K(\alpha)) = O_1(\alpha) \cap O_2(\alpha) \subset O_1 \cap O_2$. Hence any module in $\mathcal{O}(K(\alpha))$ (if there is any) is a tilting module. So to determine $\mathcal{O}(K(\alpha))$ we have to find out which indecomposable tilting modules belong to it.

Lemma 10.8.1. *For any $M \in K(\alpha)$ (resp. M such that $M^* \in K(\alpha)$) and any finite-dimensional $\mathfrak{G}$-module F the module $F \otimes M$ belongs to $O_1(\alpha)$ (resp. $O_2(\alpha)$).*

Proof. Follows from the exactness of $F \otimes \_$ by standard arguments combined with the observation that we are considering objects which are induced from projective objects. $\square$

Proposition 10.8.1. *$T(\lambda) \in \mathcal{O}(K(\alpha))$ if and only if either $s_\alpha(\lambda) - \lambda \notin \mathbb{Z}\alpha \setminus \{0\}$ or $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$.*

Proof. If $s_\alpha(\lambda) - \lambda \notin (\mathbb{Z}\alpha \setminus \{0\})$ then, according to the definition of $K(\alpha)$, any Verma module (resp. dual Verma module), occuring in the Verma flag (resp. dual Verma flag) of $T(\lambda)$ belongs to $K(\alpha)$ (resp. is of the form M^* for some $M \in K(\alpha)$). Hence $T(\lambda) \in \mathcal{O}(K(\alpha))$.

Recall that any increasing Verma flag of $T(\lambda)$ starts with $M(\lambda)$. From the definition of $K(\alpha)$ it follows that for any $\lambda \in \mathfrak{H}^*$ such that $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ there are no modules in $K(\alpha)$ such that their increasing Verma flag starts with $M(\lambda)$. Since any filtration with quotients from $K(\alpha)$ can be extended to a Verma flag, we obtain that in the case $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ the module $T(\lambda)$ cannot belong to $K(\alpha)$.

So we only have to prove that $T(\lambda) \in K(\alpha)$ in case $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$. This will follow easily if we recall the inductive construction of tilting modules via tensoring with finite-dimensional modules. Suppose that λ is such that $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$ and $M(s_\alpha(\lambda))$ is simple. Then $T(\lambda) \simeq \hat{P}(s_\alpha(\lambda))$, by the construction of $\hat{P}(s_\alpha(\lambda))$ and hence $T(\lambda) \in O_1(\alpha)$. But $T(\lambda)$ is also self-dual as a tilting module in the category $\mathcal{O}$, hence $T(\lambda) \in O_2(\alpha)$. Finally, $T(\lambda) \in \mathcal{O}(K(\alpha))$.

Now we note, that from Lemma 10.8.1 it follows that $\mathcal{O}(K(\alpha))$ is stable under tensoring with finite-dimensional modules. In particular, it means that if we fix λ as in the previous paragraph, then $T(\lambda) \otimes F$ belongs to $\mathcal{O}(K(\alpha))$ for any finite-dimensional $\mathfrak{G}$-module F. To complete the proof we only need to recall that any $T(\mu)$ with $\mu - s_\alpha(\mu) \in \mathbb{N}\alpha$ occurs as a direct summand in $T(\lambda) \otimes F$ for some λ as in the previous paragraph and some finite-dimensional F ([23]). $\square$

The modules in $\mathcal{O}(K(\alpha))$ will be called *strong tilting modules*. Later we will see that they are closely related to tilting modules in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Since λ_0 satisfies the conditions of Proposition 10.8.1, we have.

Corollary 10.8.1. *The big projective module is a strong tilting module.*

Let $\mathcal{L} = \mathcal{L}(V(\mathfrak{l}, b))$, b the square of an integer. In order to introduce the notion of a tilting module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$, we need a natural duality on $\mathcal{O}(\mathcal{P}, \mathcal{L})$. This can be easily done, using σ for $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(a, b)))$, in the case when $V(a, b)$ is a simple α-module. The same direct construction for the case $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(\mathfrak{l}, b)))$ does not work, because dualization does not preserve the bijectivity of the action of f. In fact, e acts bijectively on the dual module. There are two ways to solve this problem. The first way is to fix a non-integer x and to define a duality * on $\mathcal{O}(\mathcal{P}, \mathcal{L})$ as the composition of θ_x, the natural duality on $\theta_x(\mathcal{O}(\mathcal{P}, \mathcal{L}))$, which can be constructed via σ (here everything works since both e and f act bijectively on $\theta_x(\mathcal{O}(\mathcal{P}, \mathcal{L}))$), and θ_{-x}. The second way is to compose σ with the natural automorphism of $\mathfrak{G}$ corresponding to the simple reflection s_α. We choose the second way and from now on for $M \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ we will denote by M^* the corresponding dual module.

Let $\mathcal{G}(\Delta)$ (resp. $\mathcal{G}(\nabla)$) denote the full subcategory of $\mathcal{O}(\mathcal{P}, \mathcal{L})$ which consists of all modules having a standard filtration, i.e. a filtration, whose subquotients are isomorphic to $M_\mathcal{P}(W)$, with W being a projective in $\mathcal{L}$ (resp. a dual standard filtration, i.e. a filtration whose subquotients are isomorphic to $M_\mathcal{P}(W)^*$, W projective in $\mathcal{L}$). A module, $M \in \mathcal{O}(\mathcal{P}, \mathcal{L})$, will be called a *tilting module* if $M \in \mathcal{G}(\Delta) \cap \mathcal{G}(\nabla)$.

So far, we do not know if there is any tilting module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. The aim of this section is to describe all tilting module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. We recall that our definition of tilting module does not coincide with the general definition in [1]. The difference is in the definition of $\mathcal{G}(\nabla)$. In [1], the existence of a filtration is required, whose subquotients are isomorphic to $M_\mathcal{P}(W)^*$, where W is simple in $\mathcal{L}$. Our condition is more restrictive. Taking into account the uniqueness of characteristic tilting modules for standardly stratified algebras (this class includes, in particular, projectively stratified algebras) in [1], we only have to show that for any simple $L = L_\mathcal{P}(W) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ there exists an indecomposable tilting module $T(L) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ such that the standard filtration of $T(L)$ starts with $M_\mathcal{P}(W)$.

Lemma 10.8.2. *For any $M \in \mathcal{G}(\Delta)$ (resp. $M \in \mathcal{G}(\bigtriangledown)$) and any submodule N occuring in a standard filtration (resp. dual standard filtration) of M holds $E(N) \subset E(M)$ and $E(M/N) \simeq E(M)/E(N)$.*

Proof. Follows from the definition of E and the fact that $M \simeq N \oplus (M/N)$ as a $\mathfrak{A}$-module. □

Lemma 10.8.3. *Let T be a tilting module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Then $E(T)$ is a strong tilting module in $\mathcal{O}$.*

Proof. From the definition of $K(\alpha)$, it follows immediately that for any projective $W \in \mathcal{L}$ holds $E(M_{\mathcal{P}}(W)) \in K(\alpha)$. Now, by Lemma 10.8.2, the standard (resp. dual standard) filtration of T is sent by E to a filtration whose subquotients are elements of $K(\alpha)$ (resp. with subquotients, dual to modules in $K(\alpha)$). This completes the proof. □

Lemma 10.8.4. *For any strong tilting module $T' \in \mathcal{O}$, there exists a tilting module $T \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ such that $E(T) \simeq T'$.*

Proof. Clearly, it is enough to prove this statement for indecomposable T', so we can suppose that $T' = T(\lambda)$. First, assume that $M(s_\alpha(\lambda))$ is a simple $\mathfrak{G}$-module. Clearly, $M(\lambda)$ belongs to the image of E, hence $M(\lambda) = E(M_{\mathcal{P}}(W))$ for some simple object $W \in \mathcal{L}$. Let W' be the projective cover of W. From the definition of $K(\alpha)$ one immediately obtains $T(\lambda) = E(M_{\mathcal{P}}(W'))$. Now the statement follows from Theorem 10.6.2, the inductive construction of strong tilting modules, and the remark that tilting modules in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ are self-dual. □

Theorem 10.8.1. *For any simple object $L = L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ there exists exactly one indecomposable tilting module $T(L) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ such that the standard filtration of $T(L)$ starts with $M_{\mathcal{P}}(W')$, where W' is a projective cover of W in $\mathcal{L}$. The set $T(L)$, where L runs through simple modules in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ is a complete set of indecomposable tilting modules in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Any tilting module is a finite direct sum of indecomposable tilting modules.*

Proof. Existence follows from Lemma 10.8.4. The rest follows from [1, 2.1 and 2.2]. □

We have already proved the existence of tilting modules in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Now we are going to determine the formal character of a tilting module. Clearly, it is sufficient to do this for an indecomposable module $T(L)$, where L is a simple module in $\mathcal{O}(\mathcal{P}, \mathcal{L})$. Further, by the definition, $T(\lambda)$ has a standard filtration, hence it has a filtration by $M_{\mathcal{P}}(W)$, where $W \in \mathcal{L}$ is a simple object. Since $M_{\mathcal{P}}(W)$ is an extension of two Verma modules (with respect to a different basis in $\mathfrak{G}$), its character is known. So the problem is to determine the multiplicities $[T(L) : M_{\mathcal{P}}(W)]$. We solve this problem by reducing it to the recently solved analogous problem for $\mathcal{O}$ (see [110]).

Theorem 10.8.2. *Let W_1 and W_2 be simple objects in $\mathcal{L}$. Denote by $l(W_2')$ the length of the projective cover W_2' of W_2 in $\mathcal{L}$. Then*

$$[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2)] = l(W_2')[E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2))].$$

Proof. Set $m = [T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2)]$. Then $m = l(W_2')[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2')]$ and by Lemma 10.8.2, $[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2')] = [E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2'))]$. If $l(W_2') = 1$ then $W_2 = W_2'$ and we are done. Otherwise, it follows, from the definition of $K(\alpha)$, that the number of Verma modules in a Verma flag of $E(M_{\mathcal{P}}(W_2'))$ equals 2; moreover, $[E(M_{\mathcal{P}}(W_2')) : E(M_{\mathcal{P}}(W_2))] = 1$. This completes the proof. □

According to Lemma 3.3.1, $E(T(L_{\mathcal{P}}(W_1)))$ is a strong tilting module. In particular, it is a tilting module in $\mathcal{O}$. Furthermore, $E(M_{\mathcal{P}}(W_2))$ is a Verma module in $\mathcal{O}$, hence, the multiplicity $[E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2))]$ can be computed by Soergel's Theorem, [110, Theorem 5.12 and Theorem 6.7]. We also note, that again, applying the Mathieu's functors θ_x, one extends the above results to an arbitrary category $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(l,\gamma)))$.

Finally, if one looks at the proof of [110, Theorem 2.1], one sees that it implies another interesting result for the principal block $\mathcal{O}(\mathcal{P}, \mathcal{L})_{triv}$ of $\mathcal{O}(\mathcal{P}, \mathcal{L})$. We keep the notation from [110]. Let $S = S_\delta$ denote the semi-regular bimodule, associated with a semi-infinite character δ. As it was shown in [110], the functor $S \otimes_{U(\mathfrak{G})} \_$ maps an indecomposable projective $P(w(\lambda))$, $\lambda \in P^{++}$, $w \in W$, into an indecomposable tilting module $T(ww_0(\lambda)) \in \mathcal{O}$, where w_0 is the longest element of the Weyl group. Comparing Theorem 10.6.2, (8) with the definition of strong tilting module we see that for any indecomposable projective module $P(L) \in \mathcal{O}(\mathcal{P}, \mathcal{L})$ the module $S \otimes_{U(\mathfrak{G})} E(P(L))$ is an indecomposable strong tilting module. If we recall Lemmas 10.8.3 and 10.8.4 and the fact that $S \otimes_{U(\mathfrak{G})} \_$ is an equivalence of certain categories ([110, Section 2]) preserving short exact sequences, we obtain the following result:

Theorem 10.8.3. *The projectively stratified algebra of $\mathcal{O}(\mathcal{P}, \mathcal{L})_{triv}$ is its own Ringel dual (see [101, 77] for detail).*

11 Other properties of $L(\lambda, p)$

In this Chapter we generalize three classical facts about Verma modules, to GVMs $M(\lambda, p)$. These are analogues of the Kazhdan-Lusztig Theorem, the Demazure character formula, and Schubert filtrations. The corresponding results were obtained in [45, 75]. We will consider only modules $M(\lambda, p)$ as in Chapter 5, so we keep the notation from that Chapter and from Chapter 10.

11.1 Multiplicities of simples in a GVM

An analogue of the Kazhdan-Lustig Theorem is an immediate corollary from the seventh statement of Theorem 10.6.2.

Theorem 11.1.1. *Let $V(a', b')$ and $V(a'', b'')$ be simple objects in $\mathcal{O}(\mathcal{P}, \mathcal{L}(V(\mathfrak{l}, b)))$. Then*

$$(M_{\mathcal{P}}(V(a', b')) : L_{\mathcal{P}}(V(a'', b''))) = (M(\hat{E}(L_{\mathcal{P}}(V(a', b')))) : L(\hat{E}(L_{\mathcal{P}}(V(a'', b''))))).$$

Proof. From Theorem 10.6.2 it follows immediately that

$$(M_{\mathcal{P}}(V(a', b')) : L_{\mathcal{P}}(V(a'', b''))) = (E(M_{\mathcal{P}}(V(a', b'))) : L(\hat{E}(L_{\mathcal{P}}(V(a'', b''))))).$$

Clearly, $E(M_{\mathcal{P}}(V(a', b')))$ is a Verma module; moreover

$$(E(M_{\mathcal{P}}(V(a', b'))) : L(\hat{E}(L_{\mathcal{P}}(V(a'', b''))))) = (M(\hat{E}(L_{\mathcal{P}}(V(a', b')))) : L(\hat{E}(L_{\mathcal{P}}(V(a'', b''))))),$$

since $E(M_{\mathcal{P}}(V(a', b')))/M(\hat{E}(L_{\mathcal{P}}(V(a', b'))))$ is a direct sum of finite-dimensional $\mathfrak{A}$-modules. $\square$

Theorem 11.1.1 reduces the problem of detemining the multiplicities in $\mathcal{O}(\mathcal{P}, \mathcal{L})$ to the corresponding problem in $\mathcal{O}$, which is solved in [5, 18, 107].

11.2 Demazure formula

In this Section we extend the Weyl character formula, obtained in Section 9.4 to the Demazure character formula (see [2, 26, 115] for the classical case). Here we keep the notation from Chapter 9.

For $\beta \in \hat{\Delta}$ set

$$d_\beta = (1 - e^\beta)^{-1}(1 - e^\beta s_\beta).$$

Let

$$T = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha}\right)\left(\prod_{\beta \in -K \setminus \{\alpha\}} (1 - e^{-\beta})^{-1}\right).$$

Let W^α be the Weyl group generated by s_β, $\beta \in \pi \setminus \{\alpha\}$. We recall that with respect to π ($\pi \setminus \{\alpha\}$) there is a standard length function defined on W (W^α).

Theorem 11.2.1. *Let $(\lambda, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$. Then there exists an element $a(\lambda, p) \in \mathfrak{H}^*$ such that for any reduced decomposition, $w = s_{\beta_1} s_{\beta_2} \dots s_{\beta_k}$, of the longest element $w \in W^\alpha$ the following equality holds:*

$$\mathrm{ch}(L(\lambda, p)) = T e^{a(\lambda,p)} \left(d_{\beta_1} d_{\beta_2} \dots d_{\beta_k} e^{\lambda - \rho - a(\lambda,p)} \right).$$

Proof. By virtue of Theorem 9.4.1, there exists $a(\lambda, p) \in \mathfrak{H}^*$ such that $\mathrm{ch}(L(\lambda, p)) = T e^{a(\lambda,p)} Q$, where

$$Q = \left(\sum_{w \in W(\alpha)} (-1)^{l(w)} e^{w(\lambda - \rho - a(\lambda,p) + \rho_\alpha)} \right) \left(\sum_{w \in W(\alpha)} (-1)^{l(w)} e^{w(\rho_\alpha)} \right)^{-1}.$$

On the other hand, by the Weyl charcter formula ([28, Theorem 7.5.9]), Q can be considered as the character of a simple finite-dimensional module over the Lie algebra associated with $\hat{\Delta}$. From the Demazure character formula [115, Theorem 2.5.3], we obtain that

$$Q = d_{\beta_1} d_{\beta_2} \dots d_{\beta_k} e^{\lambda - \rho - a(\lambda,p)}$$

for any reduced decomposition, $w = s_{\beta_1} s_{\beta_2} \dots s_{\beta_k}$, of the longest element $w \in W^\alpha$. This completes the proof. □

11.3 Schubert filtrations

The classical Schubert filtration of a simple finite-dimensional module is associated with the Demazure character formula ([115]). In this Section we construct a filtration of $L(\lambda, p)$, $(\lambda, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$ by $U(\mathcal{P})$-modules, associated with the formula obtained in Theorem 11.2.1. The reader can consult [75] for any missing technical details. We start with some $sl(3, \mathbb{C})$-theory.

Let $\mathfrak{G} = sl(3, \mathbb{C})$, $\pi = \{\alpha, \beta\}$, $\hat{\Delta} = \{\beta\}$, $\mathcal{P} = \mathfrak{H} \oplus \mathfrak{N}_+ \oplus \mathfrak{G}_{-\alpha}$. For a weight element v of weight μ in a weight $\mathfrak{G}$-module V, such that $\mathfrak{c} v = t^2 v$, we set

$$X_+^v = \frac{1}{2}(\mu(H_\alpha) + 1 - t) X_\beta - X_{-\alpha} X_{\alpha+\beta},$$
$$X_-^v = \frac{1}{2}(\mu(H_\alpha) + 1 + t) X_\beta - X_{-\alpha} X_{\alpha+\beta}.$$

Lemma 11.3.1. *Let V be a simple weight $\mathfrak{G}$-module and $v \in V$ a non-zero element of weight μ such that $\mathfrak{c} v = t^2 v$, for some $t \in \mathbb{C}$. Then $\mathfrak{c}(X_+ v) = (t+1)^2 v$ and $\mathfrak{c}(X_- v) = (t-1)^2 v$.*

Proof. Consider the elements $v_1 = X_\beta v$ and $v_2 = X_{-\alpha} X_{\alpha+\beta} v$ in V and write down the results of the action of $\mathfrak{c}$ on them. By direct calculation, we obtain that the vector space generated by v_1, v_2 is invariant under $\mathfrak{c}$ and that the matrix of the transformation $\mathfrak{c}$, written in the basis v_1, v_2, has the following form:

$$\begin{pmatrix} t^2 - 2(\mu(H_\alpha) + 1) + 1 & t^2 - (\mu(H_\alpha) + 1)^2 \\ 4 & t^2 + 2(\mu(H_\alpha) + 1) + 1 \end{pmatrix}.$$

One can check that the eigenvalues of this matrix are $(t+1)^2$, $(t-1)^2$ and the corresponding eigenvectors are

$$v_+ = \left(\frac{1}{2}(\mu(H_\alpha)+1-t)X_\beta - X_{-\alpha}X_{\alpha+\beta}\right)v,$$
$$v_- = \left(\frac{1}{2}(\mu(H_\alpha)+1+t)X_\beta - X_{-\alpha}X_{\alpha+\beta}\right)v.$$

This completes our proof. $\square$

Now we are back to the original case.

Theorem 11.3.1. *Let $(\lambda,p)\in P(\alpha)^{++}\cup P(\alpha)^+$. There is an element $a(\lambda,p)\in\mathfrak{H}^*$ such that for any reduced decomposition, $w=s_{\beta_1}s_{\beta_2}\dots s_{\beta_k}$, of the longest element $w\in W^\alpha$ there exists a (canonical) filtration*

$$L(\lambda,p)=L_1\supset L_2\supset\cdots\supset L_k\supset L_{k+1}=0$$

of $L(\lambda,p)$ (viewed as a $\mathcal{P}$-module) by the $\mathcal{P}$-modules L_j, $j=1,2,\dots,k$ such that

$$\mathrm{ch}(L_j)=Te^{a(\lambda,p)}\left(d_{\beta_j}\dots d_{\beta_k}e^{\lambda-\rho-a(\lambda,p)}\right).$$

Proof. To simplify our notation, we set $L=L(\lambda,p)$. First we use Mathieu's localization to shift our module $L(\lambda,p)$ to $\mathcal{O}(\mathcal{P},\mathcal{L}(V(\mathfrak{l},p^2)))$. We fix $x\in\mathbb{C}$ such that the $\mathfrak{G}$-module $L'=\theta_x(L)$ is not α-stratified and hence is a (non-split) extension of a highest weight module $L(1)$ and a module $L(2)$ which is a highest weight module with respect to the basis $s_\alpha(\pi)$. It might happened that $L(1)$ is not uniquely defined. In this case from $sl(2,\mathbb{C})$-theory, it follows that $L(1)$ can be chosen in two different ways (as $L(1)^1$ or $L(1)^2$) with $L(1)^1\subset L(1)^2$. We choose $L(1)=L(1)^2$. So we have that $L(1)$ is a submodule in L' and $L(2)\simeq L'/L(1)$.

Let μ be the highest weight of $L(1)$. Then $\mu=\lambda-\rho+x\alpha$ by Lemma 4.5.1. Since $(\lambda,p)\in P(\alpha)^{++}\cup P(\alpha)^+$, we obtain that $p\in\mathbb{Z}$ and $p\neq 0$, hence $\mu(H_\alpha)=|p|-1$. From this, it follows that the unique simple quotient of $L(1)$ is finite-dimensional (or, equivalently, μ is integral dominant).

Let $\mathfrak{G}(\alpha)$ denote the Lie algebra corresponding to $\pm\hat{\Delta}$. Clearly, $\mathfrak{G}(\alpha)$ is a subalgebra of $\mathfrak{G}$. Consider the $\mathfrak{G}(\alpha)$-module $M=U(\mathfrak{G}(\alpha))L_\mu$. One can see that M is a simple finite-dimensional $\mathfrak{G}(\alpha)$-module.

Consider a reduced decomposition, $w=s_{\beta_1}s_{\beta_2}\dots s_{\beta_k}$, of the longest element $w\in W^\alpha$. According to the classical Demazure character formula and the classical Schubert filtration one can consider a Schubert filtration

$$M=M_1\supset M_2\supset\cdots\supset M_k\supset M_{k+1}=0$$

of M corresponding to the above decomposition.

Let v_j denote a (canonical) generator of M_j, for $j\in\{1,2,\dots,k\}$. First of all we have to look at the module $L(1)$ a bit closer. Clearly, $L(1)$ is obtained from the Verma module

$M(\mu+\rho)$ by factoring out all non-trivial submodules of the form $M(g(\mu+\rho))$, $g \in W^\alpha$. Consider a unital subalgebra, $\hat{K}$, of $U(\mathfrak{N}_-)$ generated by all elements $X_{-\gamma}$, $\gamma \in K$. The original module $M(\mu+\rho)$ is $\mathfrak{N}_-$-free. Any canonical generator of $M(g(\mu+\rho))$, $g \in W^\alpha$ is obtained from a canonical generator of $M(\mu+\rho)$ as a linear combination of elements from $U(\mathfrak{G}(\alpha))$. We note that $\pm\alpha \notin \hat{\Delta}$. Since any γ from K contains α in a decomposition with respect to π, it follows that for any non-zero $v \in M$ and any non-zero $u \in \hat{K}$, $uv \neq 0$.

Set $\widehat{M(1)} = \hat{K}v_1$ and $\widehat{L(1)} = U(D)\widehat{M(1)}$. This module can be extended to a normal $\mathcal{P}$-submodule $\widehat{L_1}$ in L' such that $\widehat{L_1} \cap L(1) = \widehat{L(1)}$. Let $L_1 = \theta_{-x}(\widehat{L_1})$. From the construction of L_1 it follows that

$$\operatorname{ch} L_1 = Te^{a(\lambda,p)}\left(d_{\beta_1}\dots d_{\beta_k}e^{\lambda-\rho-a(\lambda,p)}\right)$$

and thus $L_1 = L$.

Let $l \in \mathbb{N}$ be such that $X^l_{\beta_1}v_1 = av_2$ for $a \in \mathbb{C}$, $a \neq 0$. If $\{\beta_1 \pm \alpha\} \cap \Delta = \varnothing$ then we can define $\widehat{M(2)} = X^l_{\beta_1}\widehat{M(1)}$ and $\widehat{L(2)} = U(D)\widehat{M(2)}$. In the same way as above, the last module can be extended to a normal D-submodule $\widehat{L_2}$ in L' such that $\widehat{L_2} \cap L(1) = \widehat{L(2)}$. We set $L_2 = \theta_{-x}(\widehat{L_2})$. One can see that $\widehat{L(2)} \cap M = M_2$. Since $\hat{K}M$ is free over $\hat{K}$, it follows immediately that

$$\operatorname{ch} L_2 = Te^{a(\lambda,p)}\left(d_{\beta_2}\dots d_{\beta_k}e^{\lambda-\rho-a(\lambda,p)}\right)$$

and the module L_2 is constructed.

Finally, suppose that $\{\beta_1\pm\alpha\}\cap\Delta$ is non-empty. Then the Lie subalgebra in $\mathfrak{G}$ generated by $X_{\pm\alpha}$ and $X_{\pm\beta_1}$ is isomorphic to $sl(3,\mathbb{C})$. Set $\beta = \beta_1$. Using the notation from the previous section consider the elements $(X^{v_1}_+)^l v_1$ and $(X^{v_1}_-)^l v_1$. Clearly, exactly one of these elements is non zero and thus coincides with v_2 up to a non-zero complex constant. Set $\varepsilon = +\ (-)$ if $X^l_+ v_1 \neq 0$ ($X^l_- v_1 \neq 0$). Consider a vector-space $\widehat{M(2)}$ generated by $(X^v_\varepsilon)^l v$, where v runs through the set of all weight elements in $\widehat{M(1)}$ which are eigenvectors with respect to c. Set $\widehat{L(2)} = U(\mathcal{P})\widehat{M(2)}$. Using Lemma 11.3.1 one can show that the character of this vector-space coincides with the character of the vector-space $\hat{K}M_2$. In the same way as above, module $\widehat{L(2)}$ can be extended to a normal $\mathcal{P}$-submodule $\widehat{L_2}$ in L' such that $\widehat{L_2} \cap L(1) = \widehat{L(2)}$ and we set $L_2 = \theta_{-x}(\widehat{L_2})$. One can see that $\widehat{L(2)} \cap M = M_2$. Since $\hat{K}M$ is free over $\hat{K}$, it follows immediately that

$$\operatorname{ch} L_2 = Te^{a(\lambda,p)}\left(d_{\beta_2}\dots d_{\beta_k}e^{\lambda-\rho-a(\lambda,p)}\right)$$

and the module L_2 is constructed.

Now one can easily construct the modules L_i, $i \geqslant 3$ in the same way as above. Clearly, they are defined in a canonical way. □

We have to note, that in the case of the algebra $gl(n,\mathbb{C})$ (or $sl(n,\mathbb{C})$), the proof can be simplified by considering L as a Gelfand-Zetlin module. In this case the statement of the last theorem follows immediately from the geometric description of the GZ support of L (see Section 7.4).

12 Extended $sl(3,\mathbb{C})$ example

In this Chapter we illustrate all the results obtained above for the case of the Lie algebra $sl(3,\mathbb{C})$. In order to simplify the presentation we will recall all notation and adjust them to our fixed situation.

So we set $\mathfrak{G} = sl(3,\mathbb{C})$ – the Lie algebra of 3×3 complex matricies with zero trace. Denoting by $e_{i,j}$ the matrix units we have the standard basis $e_{1,2}$, $e_{2,3}$, $e_{1,3}$, $e_{2,1}$, $e_{3,2}$, $e_{3,1}$, $e_{1,1}-e_{2,2}$, $e_{2,2}-e_{3,3}$. We fix the Cartan subalgebra $\mathfrak{H}$, which consists of diagonal matrices. Then the root system Δ of $\mathfrak{G}$ is of type A_2, hence we can choose the basis $\pi = \{\alpha,\beta\}$ in Δ, corresponding to the roots $e_{1,2}$ and $e_{2,3}$ and then $\Delta_+ = \{\alpha,\beta,\alpha+\beta\}$. We fix the Weyl-Chevalley basis $X_\alpha = e_{1,2}$, $X_\beta = e_{2,3}$, $X_{\alpha+\beta} = e_{1,3}$, $X_{-\alpha} = e_{2,1}$, $X_{-\beta} = e_{3,2}$, $X_{-\alpha-\beta} = e_{3,1}$, $H_\alpha = e_{1,1}-e_{2,2}$, $H_\beta = e_{2,2}-e_{3,3}$. The Weyl grop W is generated by s_α and s_β and is isomorphic to the symmetric group S_3. We also have $\rho = \alpha+\beta$. For $\lambda\in\mathfrak{H}^*$ we will write $\lambda = (\lambda(H_\alpha),\lambda(H_\beta))$. Hence $\alpha = (2,-1)$, $\beta = (-1,2)$ and $\rho = \alpha+\beta = (1,1)$. Under this choice, $\mathfrak{N}_+$ (resp. $\mathfrak{N}_-$) is the subalgebra of strictly upper-triangular (resp. strictly lower-triangular) matrices and $\mathfrak{B}$ is the subalgebra of upper-triangular matrices.

Let $\mathcal{P}$ be a non-trivial parabolic subalgebra of $\mathfrak{G}$ containing $\mathfrak{B}$ (i.e $\mathcal{P}\neq\mathfrak{B}$ and $\mathcal{P}\neq\mathfrak{G}$). Using the symmetry of the Dynkin diagram we can assume, that $\mathcal{P} = \mathfrak{B}\oplus\mathfrak{G}_{-\alpha}$. Hence, we have $\mathfrak{A}\simeq sl(2,\mathbb{C})$ is generated by $\mathfrak{G}_{\pm\alpha}$ and has a basis $X_{-\alpha}$, X_α, H_α. Then $\mathfrak{A}' = \mathfrak{A}\oplus\mathfrak{H}(\mathfrak{A})$, where $\mathfrak{H}(\mathfrak{A})$ is one dimensional with generator $H_\alpha+2H_\beta$. Further $\pi(\mathfrak{A}) = \{\alpha\}$, $\Delta(\mathfrak{A})_+ = \{\alpha\}$, $W(\mathfrak{A})$ is generated by s_α and is isomorphic to the symmetric group S_2. Then $\mathfrak{N} = \mathfrak{G}_\beta\oplus\mathfrak{G}_{\alpha+\beta}$, $\mathfrak{N}(\mathfrak{A}) = \mathfrak{G}_{-\beta}\oplus\mathfrak{G}_{-\alpha-\beta}$, $\rho_\alpha = (1,-1/2)$, $\rho^\alpha = (0,3/2)$. We also have $\mathfrak{c} = (H_\alpha+1)+4X_{-\alpha}X_\alpha$.

12.1 Modules $M(\lambda,p)$ and $L(\lambda,p)$

Let $V(a,b)$, $a,b\in\mathbb{C}$, be the $\mathfrak{A}$-module with basis v_i, $i\in\mathbb{Z}$ and the following action of generators: $X_{-\alpha}v_i = v_{i-1}$, $H_\alpha v_i = (a+2i)v_i$, $X_\alpha v_i = (b-(a+2i+1)^2)/4v_{i+1}$. Choose $\lambda\in\mathfrak{H}^*$ such that $(\lambda-\rho)(H_\alpha) = a$, i.e $\lambda = (\lambda_1,\lambda_2)$ and $\lambda_1 = a+1$, $\lambda_2\in\mathbb{C}$. Set $p^2 = b$. Then the module $M(\lambda,p)$, which we will denote by $M(\lambda_1,\lambda_2,p)$, is, by definition,

$$M(\lambda,p) = U(\mathfrak{G})\bigotimes_{U(\mathcal{P})} V(a,b),$$

where $X_\beta V(a,b) = X_{\alpha+\beta}V(a,b) = 0$ and $(H_\alpha+2H_\beta)v = (\lambda_1+2\lambda_2-2)v$ for any $v\in V(a,b)$. Inside $\mathcal{K}^\alpha$ the module $M(\lambda_1,\lambda_2,p)$ has a unique simple quotient $L(\lambda_1,\lambda_2,p) = L(\lambda,p)$.

First of all we will obtain information about the support of $M(\lambda_1,\lambda_2,p)$. Clearly, $\lambda-\rho\in\mathrm{supp}(M(\lambda_1,\lambda_2,p))$. Since $X_{-\alpha}$ acts bijectively on $M(\lambda_1,\lambda_2,p)$, we obtain that, together with any μ, $\mathrm{supp}(M(\lambda_1,\lambda_2,p))$ contains also $\mu+k\alpha$ for all $k\in\mathbb{Z}$. Hence $\lambda-\rho+k\alpha\in\mathrm{supp}(M(\lambda_1,\lambda_2,p))$. Further, $M(\lambda_1,\lambda_2,p)$ is $U(\mathfrak{N}(\mathfrak{A})) = U(\mathfrak{G}_{-\beta}\oplus\mathfrak{G}_{-\alpha-\beta})$-free and we obtain

$$\mathrm{supp}(M(\lambda_1,\lambda_2,p)) = \{\lambda-\rho+k\alpha-m\beta\,|\,k\in\mathbb{Z}, m\in\mathbb{Z}_+\}.$$

In Section 9.2, we have seen that there exists $i\in\mathbb{Z}$ such that the $U(\langle X_{\pm\beta}\rangle)$-module $U(\langle X_{\pm\beta}\rangle)v_i$ is simple infinite-dimensional. Hence $\mathrm{supp}(L(\lambda_1,\lambda_2,p)) = \mathrm{supp}(M(\lambda_1,\lambda_2,p))$.

Finally, in this Section, we also calculte the dimensions of the weight spaces in $M(\lambda_1, \lambda_2, p)$. For $k \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$, we have that $\dim(M(\lambda_1, \lambda_2, p)_{\lambda-\rho+k\alpha-m\beta})$ equals the number of decompositions of $m\beta$ as $x\alpha + y\beta + z(\alpha+\beta)$, where $x \in \mathbb{Z}$ and $y, z \in \mathbb{Z}_+$. Clearly, this last equals $m+1$. Hence

$$\dim(M(\lambda_1, \lambda_2, p)_{\lambda-\rho+k\alpha-m\beta}) = m+1.$$

12.2 Inclusions and multiplicities

We can view $(\lambda_1, \lambda_2, p)$ as an element of $\mathbb{C}^3 = \Omega$. According to Theorem 5.3.1, $M(\mu_1, \mu_2, q)$ is a submodule in $M(\lambda_1, \lambda_2, q)$ if and only if $\mu = \lambda - m\beta + k\alpha$, $m \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, and either $m = 0$ and $q = \pm p$ or $m = n^{\pm}(\lambda, p) = (\lambda_1 + 2\lambda_2 \pm p)/2$ and $q^2 = (p \mp n^{\pm}(\lambda, p))^2$. Hence, for $n \in \mathbb{N}$, we can consider the hyperplanes $H(n, \pm) = \{(\lambda_1, \lambda_2, p) \mid \lambda_1 + 2\lambda_2 \pm p - 2n = 0\}$. Let H be the union of all $H(n, \pm)$, $n \in \mathbb{N}$. Then we have, that $M(\lambda_1, \lambda_2, p)$ is a simple object in $\mathcal{K}^\alpha$ if and only if $(\lambda_1, \lambda_2, p) \in \Omega \setminus H$. In this case $L(\lambda_1, \lambda_2, p) = M(\lambda_1, \lambda_2, p)$.

From Theorem 5.3.1, it follows also that, for $(\lambda_1, \lambda_2, p) \in H(n, \pm)$, one has $M(\lambda_1 + n, \lambda_2 - 2n, p \mp n) \subset M(\lambda_1, \lambda_2, p)$. In particular, the maximal (in $\mathcal{K}^\alpha$) submodule in $M(\lambda_1, \lambda_2, p)$ coincides with some $M(\lambda_1+n, \lambda_2-2n, p\mp n)$. Moreover, using Theorem 5.3.1, it is easy to describe this maximal submodule precisely. For $(\lambda_1, \lambda_2, p) \in H(n_1, +) \cap H(n_2, -)$, $p > 0$ we always have $n_1 > n_2$ and hence the maximal submodule in $M(\lambda_1, \lambda_2, p)$ coincides with $M(\lambda_1 + n_2, \lambda_2 - 2n_2, p + n_2)$. In other cases, it coincides with the unique Verma submodule in $M(\lambda_1 + n_2, \lambda_2 - 2n_2, p + n_2)$.

Now let $M(\lambda_1, \lambda_2, p)$ be a non-simple (in $\mathcal{K}^\alpha$) GVM and $M(\lambda_1 + n, \lambda_2 - 2n, p \mp n)$ be its maximal submodule. Then we have $L(\lambda_1, \lambda_2, p) = M(\lambda_1, \lambda_2, p)/M(\lambda_1 + n, \lambda_2 - 2n, p \mp n)$ and, hence, we can compute the dimensions of the weight spaces in $L(\lambda_1, \lambda_2, p)$:

$$\dim(L(\lambda_1, \lambda_2, p)_{\lambda-\rho+k\alpha-m\beta}) = \begin{cases} m+1, & m < n \\ n, & m \geqslant n \end{cases}.$$

In our situation the multiplicities of simple subquotients in a GVM can be also obtained directly from Theorem 5.3.1. In fact, we have that $(M(\lambda_1, \lambda_2, p) : L(\mu_1, \mu_2, q)) = 1$ if and only if $M(\mu_1, \mu_2, q) \subset M(\lambda_1, \lambda_2, p)$. In the other case, the corresponding multiplicity is just 0.

12.3 BGG resolution

According to Section 12.2, the maximal submodule (in $\mathcal{K}^\alpha$) of $M(\lambda_1, \lambda_2, p)$ is either 0 (in this case $M(\lambda_1, \lambda_2, p) = L(\lambda_1, \lambda_2, p)$) or coincides with $M(\mu_1, \mu_2, q)$ for some $(\mu_1, \mu_2, q) \in \Omega$. In the first case, we have a trivial resolution

$$0 \to M(\lambda_1, \lambda_2, p) \to L(\lambda_1, \lambda_2, p) \to 0.$$

In the second case, we also have a straightforward resolution of $L(\lambda_1, \lambda_2, p)$ by GVMs:

$$0 \to M(\mu_1, \mu_2, q) \to M(\lambda_1, \lambda_2, p) \to L(\lambda_1, \lambda_2, p) \to 0.$$

So, it is not necessary to apply Theorem 9.1.2 and Corollary 9.2.1. However, we have that W^+ is generated by s_β ad isomorphic to S_2. Further,

$$P(\alpha)^{++} = \bigcup_{m,n\in\mathbb{N}} \left(H(n,+)\cap H(m,-)\right), \quad P(\alpha)^{+} = s_\alpha s_\beta P(\alpha)^{++}.$$

And for $(\lambda_1,\lambda_2,p)\in P(\alpha)^{++}\cup P(\alpha)^{+}$ we have

$$0\to M(s_\beta(\lambda_1,\lambda_2,p))\to M(\lambda_1,\lambda_2,p)\to L(\lambda_1,\lambda_2,p)\to 0.$$

This is the BGG resolution of $L(\lambda_1,\lambda_2,p)$.

12.4 Weyl formula

The character formula for any $L(\lambda_1,\lambda_2,p)$, $(\lambda_1,\lambda_2,p)\in\Omega$, can be easily obtained from the resolutions given in the previous Section. Nevertheless, we will calculate $\mathrm{ch}(L(\lambda_1,\lambda_2,p))$ for $(\lambda_1,\lambda_2,p)\in P(\alpha)^{++}\cup P(\alpha)^{+}$ using Theorem 9.4.1. We have $K=\{-\alpha-\beta\}$, $\rho'=(-1/2,1)$, and hence, for any element $(\lambda_1,\lambda_2,p)\in P(\alpha)^{++}\cup P(\alpha)^{+}$, there exists an element $a(\lambda_1,\lambda_2,p)\in\mathfrak{H}^*$ such that

$$\mathrm{ch}(L(\lambda_1,\lambda_2,p)) = \frac{1}{1-e^{-\alpha-\beta}}\times\frac{e^{\lambda+\rho'}-e^{s_\beta(\lambda+a(\lambda_1,\lambda_2,p)+\rho')-a(\lambda_1,\lambda_2,p)}}{e^{\rho'}-e^{s_\beta(\rho')}}\times\sum_{i=-\infty}^{+\infty}e^{i\alpha}.$$

We can rewrite this as

$$\mathrm{ch}(L(\lambda_1,\lambda_2,p)) = \left(\sum_{i=-\infty}^{+\infty}e^{i\alpha}\right)\times\left(\sum_{i=0}^{\infty}e^{i(-\alpha-\beta)}\right)\times\left(e^{\lambda-\rho}+e^{\lambda-\rho-\beta}+\cdots+e^{\lambda-\rho-(N-1)\beta}\right),$$

where $N=(\beta,(\lambda_1,\lambda_2,p))_r$ according to the choice of $a(\lambda_1,\lambda_2,p)$.

12.5 Tableaux realization

The module $M(\lambda_1,\lambda_2,p)$ does not always have a tableaux realization. Sometimes it is convenient to change the basis π of the root system, in order to obtain a tableaux realization of this module. In this Section, we construct two series of GVMs over $sl(3,\mathbb{C})$ $(gl(3,\mathbb{C}))$ with their tableaux realization. We note that we will obtain GVMs with respect to different choices of π and $\pi(\mathfrak{A})$. We retain the notation from Section 6.1 and Section 7.4.

As the first case, consider a tableau $[\,l\,]\in\mathbb{C}^6$, which satisfies the following conditions:

- $l_{2,1}-l_{2,2}\notin\mathbb{Z}$;
- $l_{3,i}=l_{2,i}$, $i=1,2$.

Consider the set $B([\,l\,])$, which consists of all tableaux $[\,t\,]$ such that

- $t_{3,i} = l_{3,i}$, $i = 1, 2, 3$;
- $l_{2,i} - t_{2,i} \in \mathbb{Z}_+$, $i = 1, 2$;
- $l_{1,1} - t_{1,1} \in \mathbb{Z}$.

Then all the tableaux from $B([\,l\,])$ are good and the set $B([\,l\,])$ is closed under GZ formulae. So, the GZ formulae define on the vectorspace $M^1([\,l\,])$, with the basis $B([\,l\,])$, the structure of a $\mathfrak{G}$-module. It follows from the GZ fomulae that $M^1([\,l\,])$ is a GVM and is α-stratified if and only if $l_{1,1} - l_{2,i} \notin \mathbb{Z}$, $i = 1, 2$. In the α-stratified situation the space of all semiprimitive generators of $M^1([\,l\,])$ is generated by $[\,l\,] + i[\,\delta^{1,1}\,]$, $i \in \mathbb{Z}$.

As the second case, consider a tableau $[\,l\,] \in \mathbb{C}^6$, which satisfies the following conditions:

- $l_{2,1} - l_{2,2} \notin \mathbb{Z}$;
- $l_{3,1} = l_{2,1} = l_{1,1}$.

Consider the set $B([\,l\,])$, which contains all tableaux $[\,t\,]$ such that

- $t_{3,i} = l_{3,i}$, $i = 1, 2, 3$;
- $l_{2,2} - t_{2,2} \in \mathbb{Z}$;
- $l_{i,1} - t_{i,1} \in \mathbb{Z}_+$, $i = 1, 2$;
- $t_{2,1} \geqslant t_{1,1}$.

Then all the tableaux from $B([\,l\,])$ are good and the set $B([\,l\,])$ is closed under GZ formulae. So, the GZ formulae define on the vectorspace $M^2([\,l\,])$, with the basis $B([\,l\,])$, the structure of a $\mathfrak{G}$-module. It follows from the GZ fomulae that $M^2([\,l\,])$ is a GVM and is β-stratified if and only if $l_{3,i} - l_{2,2} \notin \mathbb{Z}$, $i = 2, 3$. In the β-stratified situation the space of all semiprimitive generators of $M^2([\,l\,])$ is generated by $[\,l\,] + i[\,\delta^{2,2}\,]$, $i \in \mathbb{Z}$.

12.6 Schubert filtration

Let $(\lambda_1, \lambda_2, p) \in P(\alpha)^{++} \cup P(\alpha)^{+}$. Since W^+ consists of two elements, the Schubert filtration of $L(\lambda_1, \lambda_2, p)$ will contain only two terms $L(\lambda_1, \lambda_2, p) = L_1 \supset L_0$, so we have only to construct the first term L_0. According to Sections 11.2 and 11.3, L_0 should be a $\mathcal{P}$-module, $\mathrm{supp}(L_0) = \mathrm{supp}(L(\lambda_1, \lambda_2, p))$, and all non-trivial weight spaces of L_0 should be one-dimensional.

The construction of L_0 is quite transparent. We will not use the tricks used in the proof of Theorem 11.3.1 and will construct L_0 directly. Let $M(\mu_1, \mu_2, q)$ be the maximal submodule in $M(\lambda_1, \lambda_2, p)$. Then either $q = p + n$ or $q = p - n$ for some $n \in \mathbb{N}$. Consider the $\mathfrak{A}$-module $V(a, p^2)$ from the definition of $M(\lambda_1, \lambda_2, p)$. Clearly, $1 \otimes V(a, p^2) \subset L_0$. Now consider the second $\mathfrak{A}$-level, $M_1 = \oplus_{k \in \mathbb{Z}} M(\lambda_1, \lambda_2, p)_{\lambda - \rho - \beta + k\alpha}$. As a $\mathfrak{A}$-module, M_1 is isomorphic to $F_2 \otimes V(a, p^2)$, where F_2 is a simple two-dimensional $\mathfrak{A}$-module. Hence, the subquotients of M_2 are $V(a,(p\pm 1)^2)$, counted with their multiplicities. If $p=0$, then M_2 is indecomposable and we add to L_0 the submodule $V(a,1)$. Clearly, this gives us a $\mathcal{P}$-module. If $q=p+n$ (resp. $q=p-n$) and $p\neq 0$, then we can add both $V(a,(p\pm 1)^2)$ obtaining a new $\mathcal{P}$-module. We choose $p-1$ if $q=p+n$ and $p+1$ if $q=p-n$. Now we continue the procedure above, applying it to the chosen $V(a,(p\pm 1)^2)$. As a limit we will obtain the desired $\mathcal{P}$-module L_0 having the necessary dimensions of the weight spaces. It is also clear that the above conditions uniquely define L_0 in $M(\lambda_1,\lambda_2,p)/M(\mu_1,\mu_2,q)$.

12.7 Category $\mathcal{O}(\mathcal{P},\mathcal{L})$

To describe the category $\mathcal{O}(\mathcal{P},\mathcal{L})$ for our case, we first study the category $\mathcal{L}(V(a,b))$ of $\mathfrak{A}$-modules. According to the equivalence given by Mathieu's functor (Theorem 10.6.1), it is enough to consider the case of a non-simple $V(a,b)$, i.e the case $V(\mathfrak{l},b)$, $(\mathfrak{l}+1)^2=b$.

If b is not the square of an integer, the module $V(\mathfrak{l},b)\otimes F$ decomposes into a direct sum of $V(\mathfrak{l},b_i)$, $i=1,2,\dots,\dim(F)$ for any simple finite-dimesional $\mathfrak{A}$-module F. Hence all indecomposable object of $\mathcal{L}(V(a,b))$ are simple and have the form $V(\mathfrak{l},(\sqrt{b}+i)^2)$, $i\in\mathbb{Z}$. This last follows directly from Kostant's Theorem.

Now suppose that b is the square of an integer. Then the simple objects in $\mathcal{L}(V(\mathfrak{l},b))$ are $V(\mathfrak{l},i^2)$, $i\in\mathbb{Z}$. Hence $\mathcal{L}(V(\mathfrak{l},b))=\mathcal{L}(V(\mathfrak{l},0))$. Moreover, if $i>0$, then $\mathcal{L}(V(a,b))$ also contains a length two selfextension $\hat{V}(\mathfrak{l},i^2)$ of $V(\mathfrak{l},i^2)$, which occurs as a direct summand in $V(\mathfrak{l},0)\otimes F$ for any simple F, such that $\dim(F)>i$. Moreover, from Kostant's Theorem, it follows that $V(\mathfrak{l},i^2)$, $i\in\mathbb{Z}_+$ and $\hat{V}(\mathfrak{l},i^2)$, $i\in\mathbb{N}$ exhaust all indecomposable modules in $\mathcal{L}(V(\mathfrak{l},0))$. For $i\in\mathbb{N}$ we have a natural non-split extension

$$0\to V(\mathfrak{l},i^2)\to\hat{V}(\mathfrak{l},i^2)\to V(\mathfrak{l},i^2)\to 0.$$

Applying the functor E we obtain the following sequence.

$$0\to E(V(\mathfrak{l},i^2))\to E(\hat{V}(\mathfrak{l},i^2))\to E(V(\mathfrak{l},i^2))\to 0.$$

Clearly, $E(V(\mathfrak{l},i^2))$ is a Verma module with the central character i^2, hence $E(V(\mathfrak{l},i^2))=M(i)$, $i\in\mathbb{Z}_+$. Further, $\hat{V}(\mathfrak{l},i^2)$ is projective in $\mathcal{L}(V(\mathfrak{l},0))$, hence $E(\hat{V}(\mathfrak{l},i^2))$ is also projective and thus $E(\hat{V}(\mathfrak{l},i^2))=P(-i)$, $i\in\mathbb{N}$, is the big projective module. Now we can rewrite the above sequence as

$$0\to M(i)\to P(-i)\to M(i)\to 0.$$

This sequence is not exact in $\mathcal{O}$, but it is exact in the image of E. This shows that the image of E does not inherit the abelian structure from $\mathcal{O}$. We also recall that $M(i)$, $i\in\mathbb{N}$ is projective in $\mathcal{O}$. By virtue of the last exact sequence, it is no longer projective in the image of E. It's projective cover there coincides with $P(-i)$.

Now we are ready to turn to $sl(3,\mathbb{C})$. We will restrict our consideration to the principal block $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$, where $\mathcal{L}=\mathcal{L}(V(1,0))$. We know that $\mathcal{O}(\mathcal{P},\mathcal{L})_{triv}$ is a full subcategory of $\mathcal{O}_{triv}$, so first we recall the stucture of $\mathcal{O}_{triv}$. The Verma modules in $\mathcal{O}_{triv}$ are $M(\alpha+$

$\beta)$, $M(\alpha)$, $M(\beta)$, $M(-\alpha)$, $M(-\beta)$, $M(-\alpha-\beta)$. Thus the simple modules in $\mathcal{O}_{triv}$ are $L(\alpha+\beta)$, $L(\alpha)$, $L(\beta)$, $L(-\alpha)$, $L(-\beta)$, $L(-\alpha-\beta)$ and the projective modules in $\mathcal{O}_{triv}$ are $P(\alpha+\beta)$, $P(\alpha)$, $P(\beta)$, $P(-\alpha)$, $P(-\beta)$, $P(-\alpha-\beta)$. We know that $L(\alpha+\beta)$ is the trivial (one-dimensional) $\mathfrak{G}$-module, $M(\alpha+\beta)=P(\alpha+\beta)$ is the *big Verma module*, $L(-\alpha-\beta)=M(-\alpha-\beta)$ is the unique socle in any Verma module in $\mathcal{O}_{triv}$ and $P(-\alpha-\beta)$ is the *big projective module*. Set $I=\{\alpha+\beta,\alpha,\beta,-\alpha,-\beta,-\alpha-\beta\}$. Since $M(-\alpha-\beta)$ is the unique socle in any Verma module from $\mathcal{O}_{triv}$, from BGG reciprocity we obtain $[P(-\alpha-\beta):M(\lambda)]=1$, for any $\lambda\in I$. We also recall that $s_\alpha(\alpha+\beta)=\beta$, $s_\beta(\alpha+\beta)=\alpha$, $s_\beta s_\alpha(\alpha+\beta)=-\beta$, $s_\alpha s_\beta(\alpha+\beta)=-\alpha$, $s_\alpha s_\beta s_\alpha(\alpha+\beta)=s_\beta s_\alpha s_\beta(\alpha+\beta)=-\alpha-\beta$.

Now we can describe the image $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ of E. For this we have to recall our $sl(2,\mathbb{C})$ description of $\mathcal{L}$. According to this description, $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ contains only those Verma modules whose highest weight generators generate a non-simple $\mathfrak{A}$-module. Hence, $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ contains only $M(\alpha+\beta)$, $M(\alpha)$, $M(-\beta)$. The corresponding simple objects in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ are the unique quotients of $M(\alpha+\beta)$ (resp. $M(\alpha)$, resp. $M(-\beta)$), which are the extensions of $L(\alpha+\beta)$ and $L(\beta)$ (resp $L(\alpha)$ and $L(-\alpha)$, resp. $L(-\beta)$ and $L(-\alpha-\beta)$). We note, that the mentioned extension of $L(-\beta)$ and $L(-\alpha-\beta)$ is in fact $M(-\beta)$. According to our description of projective modules (Theorem 10.6.2), $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ contains $P(\beta)$, $P(-\alpha)$ and $P(-\alpha-\beta)$. As in the $sl(2,\mathbb{C})$ case, the abelian structure on $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ comes from $\mathcal{O}(\mathcal{P},\mathcal{L})$ and is different from that on $\mathcal{O}$. For example, in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ we have a natural exact sequence

$$0\to M(\alpha+\beta)\to P(\beta)\to M(\alpha+\beta)\to 0,$$

which is not exact in $\mathcal{O}$.

In $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ we also have the standard modules $S(\beta)$, $S(-\alpha)$ and $S(-\alpha-\beta)$, which are induced from the projective modules in $\mathcal{L}$. They can be presented as self-extensions of Verma modules in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$:

$$0\to M(\alpha+\beta)\to S(\beta)\to M(\alpha+\beta)\to 0,$$

$$0\to M(\alpha)\to S(-\alpha)\to M(\alpha)\to 0,$$

$$0\to M(-\beta)\to S(-\alpha-\beta)\to M(-\beta)\to 0.$$

Now we can say something about the structure of projective modules. We know (Proposition 10.2.1) that any projective in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ has a standard filtration. In fact, $P(\beta)=S(\beta)$, $P(-\alpha)\supset S(\beta)$ and $P(-\alpha)/S(\beta)\simeq S(-\alpha)$ and finally, $P(-\alpha-\beta)\supset P(-\alpha)$ and $P(-\alpha-\beta)/P(-\alpha)\simeq S(-\alpha-\beta)$.

It is easy to see, that the projectively stratified finite-dimensional algebra, which corresponds to $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$, has infinite global dimension. Really, the minimal projective resolution of $M(\alpha+\beta)$ in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ has the form

$$\dots P(\beta)\to P(\beta)\to P(\beta)\to M(\alpha+\beta)\to 0,$$

and thus, is infinite. Here we also note that the big Verma module $M(\alpha+\beta)$, which is projective in $\mathcal{O}$, is no longer projective in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$; moreover, as we have seen, it has an infinite minimal projective resolution. We also note, that in contrast with classical $\mathcal{O}$, simple modules in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ can have self-extensions. For example, the standard module $S(-\alpha-\beta)$ is a self-extension of $M(-\beta)$ and the last one is simple in $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$.

Now a little bit about the tilting modules. The indecomposable tilting modules in $\mathcal{O}_{triv}$ are $T(\lambda)$, $\lambda \in I$. According to Lemma 10.8.4 and Proposition 10.8.1, $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ contains only $T(-\beta)$, $T(\alpha)$ and $T(\alpha+\beta)$. It is clear, that $T(-\beta) \simeq S(-\alpha-\beta)$ and $T(\alpha+\beta) \simeq P(-\alpha-\beta)$. According to Soergel's character fomula, $T(\alpha)$ has a standard filtration, $T(\alpha) \supset S(-\alpha)$ and $T(\alpha)/S(-\alpha) \simeq S(-\alpha-\beta)$.

Further, the functor $S \otimes_{U(\mathfrak{G})} \_$ sends $P(-\alpha-\beta)$ to itself $T(\alpha+\beta) \simeq P(-\alpha-\beta)$, $P(-\alpha)$ to $T(\alpha)$ and $P(\beta)$ to $T(-\beta)$. Hence, it produces the statement that the algebra of $E(\mathcal{O}(\mathcal{P},\mathcal{L})_{triv})$ coincides with its own Ringel dual.

References

[1] I.Agoston, D.Happel, E.Lukacs and L.Unger, Standardly stratified algebras and tilting, to appear.

[2] H.Andersen, Schubert varieties and Demazure character formula. Invent. Math. 5 (1985), 611 - 618

[3] A.O.Barut and R.Rachka, Theory of group representations and applications, PWN – Polish Scientific Publisher, Warszawa, 1977

[4] Bavula, V. V. Generalized Weyl algebras and their representations, (Russian) Algebra i Analiz 4 (1992), 75 - 97

[5] A.Beilinson, J.Bernstein, Localization de g-modules, C.R. Acad. Sc. Paris, 292 (1981), 15 - 18

[6] A.Beilinson, J.Bernstein, A proof of Jantzen conjecture, Adv. Sov. Math., 16 (1993), 1 - 50

[7] A.Beilinson, V.Ginzburg and W.Soergel, Koszul duality patterns in representation theory, J. AMS 9 (1996), 473 - 527

[8] J.Bernstein, Trace in categories. Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progr. Math., 92, Birkhaeuser Boston, Boston, MA, 1990, 417 - 423

[9] I.N.Bernstein, I.M.Gelfand, S.I.Gelfand, Differential operators on the base affine space and the study of $\mathfrak{G}$-modules, in:I.M.Gelfand, ed., Publ. of 1971 Summer School in Math., Janos Bolyai Math. Soc., Budapest, 21 - 64

[10] I.N.Bernstein, I.M.Gelfand, S.I.Gelfand, Structure of representations that are generated by vectors of higher weight, (Russian) Funckcional. Anal. i Prilozen. 5 (1971), 1 - 9

[11] I.N.Bernstein, I.M.Gelfand, S.I.Gelfand, A certain category of $\mathfrak{G}$-modules, (Russian) Funkcional. Anal. i Prilozen. 10 (1976), 1 - 8

[12] I.N.Bernstein, S.I.Gelfand, Tensor products of finite and infinite dimensional representations of semisimple Lie algebras, Compos. Math., 41 (1980), 245 - 285

[13] R.E.Block, Classification of the irreducible representations of $sl(2,\mathbb{C})$, Bull. Amer. Math. Soc. (N.S.), 1 (1979), no. 1, 247 - 250

[14] B.Boe, D.Collingwood, Multiplicity free categories of highest weight representations, Comm. Alg., 18 (1990), 947 - 1032

[15] B.Boe, D.Collingwood, Multiplicity free categories of highest weight representations II, Comm. Alg., 18 (1990), 1033 - 1070

[16] D.Britten, F.Lemire, A classification of simple Lie modules having one-dimensional weight space, Trans. AMS, 299 (1987), 683 - 697

[17] D.Britten, F.Lemire, V.Futorny, Simple A_2-modules with a finite-dimensional weight spaces, Comm. in alg., 23 (1995), 467 - 510

[18] J.L.Brylinski and M.Kashiwara, Kazhdan-Lusztig conjecture and Holonomic systems, Inv. Math., 64 (1981), 387 - 410

[19] L.Casian and D.Collingwood, The Kazhdan-Lusztig conjecture for generalized Verma modules, Math. Z., 195 (1987), 581 - 600

[20] E.Cline, B.Parshall, L.Scott, Finite dimensional algebras and highest weight categories, J. Reine Angew Math. 391 (1988), 85 - 99

[21] E.Cline, B.Parshall, L.Scott, Stratifying endomorphism algebras, Mem. Amer. Math. Soc. 124 (1996), n. 591.

[22] A.Coleman, V.Futorny, Stratified L-modules, J.Algebra, 163 (1994), 219 - 234

[23] D.Collingwood, R.Irving, A decomposition theorem for certain self-dual modules in the category $\mathcal{O}$, Duke Math. J., 58 (1989), 89 - 102

[24] D.Collingwood, B.Shelton, A duality theorem for extensiones of induced highest weight modules, Pac. J.Math., 146 (1990), 227 - 237

[25] A.Cylke, V.Futorny, S.Ovsienko, On the support of irreducible modules, SFB-343 Preprint 96-011, Bielefeld, 1996

[26] M.Demazure, Désingularisation des variétés de Schubert généralisées. Ann. Ec. Norm. Sup. 7 (1974), 53 - 88

[27] I.Dimitrov, O.Mathieu and I.Penkov, On the structure of weight modules, (to appear)

[28] J.Dixmier, Enveloping algebras, Graduate Studies in math, v. 11, AMS, 1996

[29] Yu.A.Drozd, Tame and wild matrix problems. (Russian) Matrix problems (Russian), pp. 104 - 114. Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1977

[30] Yu.A.Drozd, S.A.Ovsienko and V.M.Futorny, Harish-Chandra subalgebras and Gelfand-Zetlin modules, Math. and Phys. Sci., 424 (1994), 72 - 89

[31] Yu.A.Drozd, S.A.Ovsienko and V.M.Futorny, Gelfand-Zetlin modules over Lie algebra $sl(3)$, Cont. Math., 131 (1992), 23 - 29

[32] Yu.A.Drozd, S.A.Ovsienko and V.M.Futorny, Harish-Chandra S-homomorphism and $\mathfrak{G}$-modules generated by a semiprimitive element, Ukr. Math. J. (English translation), 42 (1990), 919 -924

[33] Yu.A.Drozd, S.A.Ovsienko and V.M.Futorny, On Gelfand-Zetlin modules. Proceedings of the Winter School on Geometry and Physics (1990). Rend. Circ. Mat. Palermo (2) Suppl. No. 26 (1991), 143 - 147

[34] Yu.A.Drozd, S.A.Ovsienko and V.M.Futorny, Irreducible weighted $sl(3)$-modules, (Russian) Funktsional. Anal. i Prilozhen., 23 (1989), 57 - 58

[35] T.Enright, B.Shelton, Decomposition in Categories of highest weight modules, J. Alg., 100 (1986), 380 - 402

[36] D.Fernando, Lie algebra modules with finite-dimensional weight spaces I, Trans. Amer. Math. Soc., 322 (1990), 757 - 781

[37] V.M.Futorny, A generalization of Verma modules, and irreducible representations of the Lie algebra $sl(3)$, (Russian) Ukrain. Mat. Zh. 38 (1986), 492 - 497

[38] V.M.Futorny, Weighted sl(3)-modules generated by semiprimitive elements, (Russian) Ukrain. Mat. Zh. 43 (1991), 281 - 285; translation in Ukrainian Math. J. 43 (1991), 250 - 254

[39] V.M.Futorny, Irreducible $sl(3)$-modules with infinite-dimensional weight subspaces, (Russian) Ukrain. Mat. Zh. 41 (1989), 1001 - 1004, translation in Ukrainian Math. J. 41 (1989), 856 - 859

[40] V.M.Futorny, The weight representations of semisimple finite-dimensional Lie algebras, Ph.D. Thesis, Kiev University, 1987

[41] V.M.Futorny, Root systems, representations and geometries, Preprint Ac.Sci. Ukraine Math. In-t, 8 (1990), 30 - 39

[42] V.M.Futorny, The graded representations of Affine Lie algebras, Suppl., Circolo Mathem. Palermo, 26 (1991), 155 - 161

[43] V.Futorny, Irreducible non-dense $A_1^{(1)}$-modules, Pac. J. Math., 172 (1996), 83 - 99

[44] V.Futorny, S.König and V.Mazorchuk, Categories of induced modules and projectively stratified algebras, Preprint 99-024, Bielefeld University

[45] V.Futorny, S.König and V.Mazorchuk, Combinatorial description of blocks in $\mathcal{O}(\mathcal{P},\mathcal{L})$ associated with $sl(2)$ induction, to appear

[46] V.Futorny, V.Mazorchuk, Structure of α-stratified modules for finite-dimentional Lie algebras I, J.Algebra, 183 (1996), 456 - 482

[47] V.Futorny, V.Mazorchuk, BGG-resolution for α-stratified modules over simply-laced finite-dimensional Lie algebras, J.Math. Kyoto Univ. 38 (1998), 229 - 240

[48] V.Futorny, V.Mazorchuk, Structure of α-stratified modules for finite-dimensional Lie algebras II. BGG-resolution in the simply-laced case. Vienna, Austria, Preprint ESI 324 (1996)

[49] V.Futorny, V.Mazorchuk, Highest weight categories of Lie algebra modules, to appear in J. Pure Appl. Alg.

[50] V.Futorny, D.Pollack, A new category of Lie algebra modules satisfying the BGG-reciprocity principle, Comm. in alg., 22 (1994), 213 - 227

[51] O.Gabber, A. Joseph, On the Bernstein – Gelfand – Gelfand resolution and the Duflo sum formula, Compositio Math., 434 (1981), 107 - 131

[52] A.M.Gavrilik and N.Z.Iorgov, q-deformed algebras $U_q(so_n)$ and their representations, Preprint

[53] H. Garland, J. Lepowsky, Lie algebra homology and the Macdonald-Kac formulas, Invent. Math. 34 (1976), 37 - 76

[54] I.M.Gelfand, M.L.Zetlin, Finite-dimensional representations of the group of unimodular matrices. (Russian) Doklady Akad. Nauk SSSR (N.S.), 71 (1950), 825 - 828

[55] I.M.Gelfand, M.L.Zetlin, Finite-dimensional representations of the group of orthogonal matrices. (Russian) Doklady Akad. Nauk SSSR (N.S.), 71 (1950), 1017 - 1020

[56] A.Gyoja, Further generalization of Generalized Verma modules, Publ. RIMS, KIyoto Univ., 29 (1993), 349 - 395

[57] J.Humphreys, Reflection groups and Coxeter groups, Cambrige University Press, 1990

[58] J.Humphreys, Highest weight modules for semisimple lie algebras, in: V.Dlab, P.Gabriel, ed., Representation Theory I, Lecture notes in Math. 831, Springer-Verlag, 1980, 72 - 103

[59] R.Irving, The socle filtration of a Verma module, Ann. scient. Ec. Norm. Sup., 4s. 21 (1988), 48 - 65

[60] R.Irving, BGG-algebras and BGG-reciprocity principle, J.Algebra 135 (1990), 363 - 380

[61] R.Irving, Singular blocks of the category $\mathcal{O}$, Math. Z., 204 (1990), 209 - 224

[62] R.Irving, Projective modules in the category $\mathcal{O}_S$: Self-duality, Trans. AMS., 291 (1985), 701 - 755

[63] R.Irving and B.Shelton, Loewy series and simple projective modules inthe category $\mathcal{O}_S$, Pac. J.Math., 132 (1988), 319 - 342

[64] J.Jantzen, Representations of algebraic groups, Pure and Appl. Math., vol.131, Acadamic Press, Orlando, Florida, USA, 1987

[65] M.Jimbo, Quantum R matrix related to the generalized Toda system: an algebraic approach, J.Math. Phys., 86 (1988), 335 - 358

[66] A.Joseph, Quantum Groups and Their Primitive Ideals, Springer-Verlag, New-York, 1994.

[67] A.Joseph The primitive spectrum of an enveloping algebra, Societe. Math. de France, Asterisque 173-174 (1989), 13 - 53

[68] V.G.Kac, Infinite dimensional Lie algebras, Third edition, Cambrige University Press, 1990

[69] V.G.Kac and D.A.Kazhdan, Structure of representations with highest weight of infinite-dimensional Lie algebra, Adv. Math. 34 (1979), 97 - 108

[70] D.Kazhdan, G.Lusztig, Representations of Coxeter groups and Hecke algebras, Inv. Math., 53 (1979), 165 - 184

[71] A.Khomenko, V.Mazorchuk, Generalized Verma modules over the Lie algebra of type G_2, Comm. Alg., 27 (1999), 777 - 783

[72] A.Khomenko, V.Mazorchuk, On multiplicities of simple subquotients in generalized Verma modules, Preprint 99-049, Bielefeld University

[73] A.Khomenko, V.Mazorchuk, On the determinant of Shapovalov form for Generalized Verma modules, to appear in J.Alg.

[74] A.Khomenko, V.Mazorchuk, A note on simplicity of generalized Verma modules, Preprint 99-049, Bielefeld University

[75] A.Khomenko, V.Mazorchuk, Schubert filtration for simple quotients of generalized Verma modules, Preprint 99-049, Bielefeld University

[76] A.Klimyk, K.Schmuedgen, Quantum groups and their representations. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997

[77] M.Klucznik ans S.König, Characteristic tilting modules over quasi-hereditary algebras, Course notes, available at http://www.mathematik.uni-bielefeld.de/~koenig/

[78] Kostant B. A formula for the multiplicity of a weight, Trans. Amer. Math. Soc., 93 (1959), 53 - 73

[79] Kostant B. On the tensor product of a finite and infinite dimensional representations, Journal of Func. analisis. 20 (1975), 257 - 285

[80] S.Kumar, G.Letzter, Shapovalov determinant for restricted and quantiyed restricted enveloping algebras, Pac. J. Math., 179 (1997), 123 - 161

[81] F.Lemire and J.Patera, Formal analytic continuation of Gelfand's finite dimensional representations of $gl(n, \mathbb{C})$, J.Math. Phys. 20 (1979), 820 - 829

[82] J.Lepowsky, A generalization of the Bernstein – Gelfand – Gelfand resolution, J.Algebra 49 (1977), 496 - 511

[83] J.Lepowsky, Generalized Verma modules, the Cartan-Helgason theorem and the Harish-Chandra homomorphism, J.Algebra 49 (1977), 470 - 495

[84] G.Lusztig, Quantum deformations of certain simple modules over envaloping algebras, Adv. Math. 70 (1988), 237 - 249

[85] G.Lusztig, Introduction to Quantum Groups, Birkhäuser, Boston, 1993

[86] O.Mathieu, Classification of irreducible weight modules, to appear.

[87] V.S.Mazorchuk, α-stratified modules over the Lie algebra $sl(n, \mathbb{C})$. (Russian) Ukrain. Mat. Zh. 45 (1993), 1215 - 1224; translation in Ukrainian Math. J. 45 (1993), 1360 - 1371

[88] V.Mazorchuk, The structure of generalized Verma modules, Ph.D. Thesis, Kyiv University, 1996.

[89] Mazorchuk V.S. On the structure of an α-stratified generalized Verma Module over Lie algebra $sl(n, \mathbb{C})$, Manuscripta Mathematica 88 (1995), 59 - 73

[90] V.Mazorchuk, Tableaux realization of generalized Verma modules, Can. J.Math. 50 (1998), 816 - 828

[91] V.Mazorchuk, On Gelfand-Zetlin modules over orthogonal Lie algebras, Preprint 98-106, Bielefeld University

[92] V.Mazorchuk, S.Ovsienko, The structure of generalized Verma modules induced from generic Gelfand-Zetlin Modules, preprint 97-007, Bielefeld University

[93] V.Mazorchuk and L.Turowska, On Gelfand-Zetlin modules over $U_q(gl_n)$, to appear

[94] A.I.Molev, A weight basis for representations of even orthogonal Lie algebras, Preprint math.RT/9902060

[95] A.I.Molev, A Basis for Representations of Symplectic Lie Algebras, Preprint math.QA/9804127

[96] R.V.Moody, A.Pianzola, Lie Algebras with Triangular Decomposition, CMS series, Wiley, New York, 1995

[97] S.Naito, The strong Bernstein – Gelfand – Gelfand resolution for generalized Kac-Moody algebras, I (The existence of the resolution), Publ. RIMS, KIyoto Univ., 29 (1993), 709 - 730

[98] U.Ottoson, A classicfication of the unitary irreducible representations of $SO_0(N, 1)$, Commun. math. Phys. 8 (1968), 228 - 244

[99] U.Ottoson, A classicfication of the unitary irreducible representations of $SU(N, 1)$, Commun. math. Phys. 10 (1968), 114 - 131

[100] S.A.Ovsienko, Some finitistic statements for Gelfand-Zetlin modules, to appear

[101] C.M.Ringel, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, Math. Z. 208 (1991), 209 - 223

[102] C.M.Ringel, The category of good modules over a quasi-hereditary algebra. Proceedings of the Sixth International Conference on Representations of Algebras (Ottawa, ON, 1992), 17 pp., Carleton-Ottawa Math. Lecture Note Ser., 14, Carleton Univ., Ottawa, ON, 1992

[103] A.Rocha-Caridi, Splitting criteria for $\mathfrak{G}$-modules induced from a parabolic and a Bernstein – Gelfand – Gelfand resolution of a finite-dimensional, irreducible $\mathfrak{G}$-module. Trans. Amer. Math. Soc., 262 (1980), 335 - 366

[104] A.Rocha-Caridi, N.Wallach, Highest weight modules over graded Lie algebras: resolutions, filtrations and character formulas, Trans. Amer. Math. Soc., 287 (1983), 133 - 161

[105] A.Rocha-Caridi, N.Wallach, Projective modules over graded Lie algebras I, Math.Z. 180 (1982), 151 - 177

[106] N.Shapovalov, On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra. Funct. Anal. Appl. 6 (1972), 65 - 70

[107] W.Soergel, Kategorie $\mathcal{O}$, perverse Garben und Moduln ueber den Koinvarianten zur Weylgruppe. (German) [Category $\mathcal{O}$, perverse sheaves and modules over the coinvariants for the Weyl group], J. Amer. Math. Soc. 3 (1990), 421 - 445

[108] W.Soergel, Équivalences de certaines catégories de $\mathfrak{G}$-modules, C.R. Acad. Sc. Paris, 4303 (1986), 725 - 728

[109] W.Soergel, The combinatorics of Harish-Chandra bimodules, J. reine angew. Math., 429 (1992), 49 - 74

[110] W.Soergel, Character formulas for tilting modules over Kac-Moody algebras, Repr. Theory 1 (1997), 115 - 132

[111] K.Ueno, T.Takebayashi and Y.Shibukawa, Gelfand-Zetlin Basis for $U_q(gl(N+1))$-modules, Letters Math. Phys. 18 (1989), 215 - 221

[112] K.Ueno, T.Takebayashi and Y.Shibukawa, Construction of Gelfand-Zetlin Basis for $U_q(gl(N+1))$-modules, Publ. RIMS Kyoto Univ., 26 (1990), 667 - 679

[113] Verma D.N. Structure of certain induced representations of complex semisimple Lie algebras, Bull. Amer. Math. Soc., 74 (1968), 160 - 166

[114] D.P.Zelobenko, Compact Lie groups and their representations. Translations of Mathematical Monographs, Vol. 40. American Mathematical Society, Providence, R.I., 1973

[115] D.Zelobenko, Representations of reductive Lie algebras, (in Russian), Moskow, "Nauka", 1994