# Contents

# 0 Introduction

Let $X$ be a pinched Hadamard manifold, and let Isom($X$) be the group of isometries of $X$. We may suppose that all sectional curvatures of $X$ are between $-a^2$ and $-1$. Let $H^n(a)$ be the Hadamard manifold with constant curvature $-a^2$; thus $H^n(1) = H^n$, the n-dimensional hyperbolic space.

In this dissertation, we extend many famous results of the groups of isometries of $H^3$ or $H^n$ to isometry groups of pinched Hadamard manifolds. If we take the upper-space $H^n$ with metric $ds = |dx|/x^n$ as a model for the n-dimensional hyperbolic space, then each isometry of $H^n$ is an extension of a Möbius transformation of $R^{n-1}$ to $H^n$ and vice versa. We know that there are some tools in the research of isometric groups of the hyperbolic space. For instance, every hyperbolic isometry can be represented by some matrix. In general, the isometry of $X$ does not have such explicit representation as that of hyperbolic spaces. Thus many methods in this dissertation are different from that in the hyperbolic space.

This dissertation is organized as follows: In the first chapter we recall some basic notation and basic results, which will be needed later on. In Chapter 2, we first obtain a classification of any subgroup of Isom($X$) according to its limit set.

*Let $G$ be a subgroup of Isom($X$). Then*

i). *The limit set $L(G) = \emptyset$ if and only if all elements of $G$ are elliptic and have a common fixed point in $X$.*

ii). *If $L(G) = \{x_0\}$, then $G$ is an infinite group and contains only elliptic and parabolic isometries and $x_0$ is the unique common fixed point of elements of $G$.*

iii). *$L(G) = \{x_0, y_0\}$ if and only if $G$ contains only loxodromic isometries with fixed points $x_0$ and $y_0$ and elliptic isometries which setwise stabilize the set $\{x_0, y_0\}$ and $G$ at least contains a loxodromic isometry.*

iv). *If $L(G)$ contains more than two points then $G$ contains a loxodromic element.*

At the same time, some criteria for loxodromicity and parabolicity of isometry of X are given. Then we describe all elementary groups in terms of their limit sets and orbits. Two classical results in the hyperbolic space are generalized. One (see [Be]) is that a subgroup is nonelementary if and only if it contains infinitely many loxodromic elements, and no two of which have a fixed points in common. The other (see [Gre]) is that the set of fixed points of loxodromic elements of a nonelementary subgroup is dense in its limit set.

It is known that discreteness is a fundamental notion in groups of isometries of the hyperbolic space. On the other hand, the limit of a discrete group sequence is important in various aspects of Kleinian group theory, in particular in research on group deformation spaces. Heinz Helling and Troels Jørgensen was the first to analyze and understand this phenomenon. In general, some well known facts in dimension three can not be automatically generalized to the case of the higher dimensions. For example, a classical result states that a nonelementary Möbius group is discrete if and only if all its two generator subgroups are. In the higher dimensional hyperbolic space $H^n$ $(n > 3)$, Abikoff and Hass in [AH] constructed a counterexample, that is, there is a nonelementary nondiscrete subgroup $\Gamma$ of $\mathrm{Isom}(H^n)$ that leaves a 2-plane in $H^4$ invariant and with the further property that every finitely generated subgroup of $\Gamma$ is discrete. For the limit of a sequence of discrete groups, Helling in [Hel] proved that if a sequence $\{G_m\}$ of $r$-generator discrete nonelementary subgroups in $\mathrm{SL}(2, \mathbb{R})$ converges algebraically to a group $G$ then $G$ is discrete nonelementary. Martin in [Ma1] constructed an example which said that for each $n \geq 4$, there is a sequence of finitely generated discrete nonelementary subgroups of $\mathrm{M\ddot{o}b}(n)$, such that the generators converge uniformly on $B^n$ to Möbius transformations which generate a nondiscrete group. Therefore, some restrictions in the research of the discreteness and convergence of groups of isometries of pinched Hadamard manifolds are necessary. Martin introduced the uniformly bounded torsion for a family of groups such that in a family of discrete groups with uniformly bounded torsion, the order of every elliptic element is bounded by some constant. Then Martin, under the condition of uniformly bounded torsion, generalized the discreteness theorem and convergence theorems of groups of isometries of three dimensional hyperbolic space $H^3$ to higher dimensions [Ma1] and also, more generally, to manifold with pinched curvatures [Ma2]. In Chapter 3 and Chapter 4, we introduce Condition A to weaken the assumption of uniformly bounded torsion so that in discrete groups the set of orders of elliptic elements may be unbounded. In particular, Condition A is naturally satisfied in groups of isometries of 3-dimensional hyperbolic spaces. Thus our results contain those classical results on discreteness and convergence. Our proofs are based on the facts obtained in Chapter 2 and the Margulis Lemma and so on. In

Chapter 3, we prove that for a nonelementary subgroup $G$ of $\mathrm{Isom}(X)$, if there exists a loxodromic subgroup $G_{\{x,y\}}$ which satisfies Condition A, then $G$ is discrete if and only if every two loxodromic generator subgroup is discrete. In Chapter 4, we generalize the three classical limit theorems, due to Helling and Jørgensen in hyperbolic case, to groups of isometries of pinched Hadamard manifolds under Condition A.

# 1 Notation and Definitions

For any Riemannian manifold $M$ we denote the Riemannian metric by $d$. A Hadamard manifold $H$ is a complete, simply connected Riemannian manifold of nonpositive curvature. A pinched Hadamard manifold $X$ is a Hadamard manifold of pinched negative curvature; i.e., all of the sectional curvatures $K(X)$ of $X$ satisfy

$$-a^2 \leq K(X) \leq -1,$$

where $a \neq 0$. In this paper, $X$ will always denote a pinched Hadamard manifold.

We denote by $\mathrm{Isom}(X)$ the group of all isometries on a pinched Hadamard manifold $X$. Let $\gamma \in \mathrm{Isom}(X)$ and define $d_\gamma(x) := d(x, \gamma x)$, the displacement function of $\gamma$ and $d_\gamma := \inf_{x \in X} d_\gamma(x)$.

The following is the classification of elements in $\mathrm{Isom}(X)$ [BGS, Ma2].

**Definition 1.1** $\gamma \in \mathrm{Isom(X)}$ *is called*
1) *elliptic, if* $d_\gamma = 0$ *and this minimum is achieved.*
2) *loxodromic, if* $d_\gamma > 0$ *and this minimum is achieved.*
3) *parabolic, if the minimum is not achieved.*

The term loxodromic we adopt from the papers [Bo1 and Bo2] is also called axial or hyperbolic by other authors.

Geodesics $c_1$ and $c_2$ of $X$ are asymptotes if $d(c_1 t, c_2 t) \leq a$ for some $a > 0$ and all $t \geq 0$. An equivalence class of asymptotes is a point at infinity for $X$, and $X(\infty)$ denotes the set of all points at infinity. For a geodesic $c : \mathbb{R} \to X$, let $c(\infty) \in X(\infty)$ be the corresponding class, and $c(-\infty)$ the class of the reversed geodesic $t \to c(-t)$. A pinched Hadamard manifold $X$ is a visibility manifold in the sense of [EO]. That is, any two distinct points $x, y \in \overline{X}$ may be joined by a unique geodesic, which we shall write $[x, y]$. We shall call $[x, y]$ a geodesic segment if both $x$ and $y$ lie in $X$, a geodesic ray tending to $y$ if $x \in X$ and $y \in X(\infty)$, or a bi-infinite geodesic if $x, y \in X(\infty)$. We shall always assume that geodesics are parametrized by the arc length. For $x \in X, z_1, z_2 \in \overline{X} = X \cup X(\infty), x \neq z_1, x \neq z_2$ we define $\measuredangle_x(z_1, z_2) := \measuredangle(\dot{c}_1(0), \dot{c}_2(0))$ where $c_i$ are the geodesics from $x$ to $z_i$. For $x \in X, z \in X(\infty), \varepsilon > 0$, let

$$C_x(z, \varepsilon) = \{y \in \overline{X} | y \neq x \quad \text{and} \quad \measuredangle_x(z, y) < \varepsilon\}.$$

The set $C_x(z,\varepsilon)$ is called the *cone* of vertex $x$ and *angle* $\varepsilon$. The *"cone topology"* on $\overline{X}$ is the topology generated by the open sets in $X$ and these cones. The induced topology on $X(\infty)$ is also called the *"sphere topology"*. It is well known from [BGS] that $S^{n-1}$ is homeomorphic to $X(\infty)$ and $\overline{X}$ is homeomorphic to a closed ball in the Euclidean $n$-space. Any isometry $g$ of $X$ extends to a homeomorphism of $\overline{X} = X \cup X(\infty)$, which we shall also denote by $g$. We shall write $\mathrm{fix}(g)$ for the set of fixed points of $g$ in $\overline{X}$.

We have known that a sequence $\{x_m\}$ in $X$ converges to $z \in X(\infty)$ if and only if for any fixed point $p \in X$: $d(p,x_m) \to \infty$ and $\measuredangle_p(z,x_m) \to 0$. For $\{x_m\} \subset X(\infty)$, we have that $x_m \to z$ as $m \to \infty$ if and only if $\measuredangle_p(x_m,z) \to 0$ as $m \to \infty$.

Any three non-collinear points of a Hadamard manifold $H$ determine a geodesic triangle and we have:

1) Law of Cosines: $c^2 \geq a^2 + b^2 - 2ab\cos\alpha$, where $a,b,c$ are the sides and $\alpha$ is the angle opposite $c$.

2) Angle sum: the sum of the interior angles of a triangle is at most $\pi$.

3) Double law of cosines: A geodesic triangle with sides $a,b,c$ and angles $\alpha,\beta$ and $\gamma$ opposite the corresponding sides satisfies $c \leq b\cos\alpha + a\cos\beta$.

We make use of the following consequence of the law of cosines: if $\{p_m\},\{q_m\}$ and $\{r_m\}$ are sequences in $H$ such that $d(p_m,q_m) \to \infty$ but $d(q_m,r_m) \leq A$ as $n \to \infty$, then $\measuredangle_{p_m}(q_m,r_m) \to 0$.

The following definition is the characterization of an individual element in $\mathrm{Isom}(X)$ [Bo1, Bo2].

**Definition 1.2** *$g \in \mathrm{Isom}(X)$ is of precisely one of the following types.*

0) *$g$ is the identity.*

1) *$g$ is elliptic. Thus $\mathrm{fix}(g)$ is a proper nonempty subspace of $\overline{X}$.*

2) *$g$ is parabolic. Thus, $\mathrm{fix}(g)$ consists of a single point $p \in X(\infty)$, and $g$ preserves setwise each horosphere about $p$.*

3) *$g$ is loxodromic. Thus, $\mathrm{fix}(g)=\{p,q\}$ where $p$ and $q$ are distinct points of $X(\infty)$. For all $x \in \overline{X} \setminus \{p,q\}$, we have $g^n(x) \to p$ and $g^{-n}(x) \to q$. We call $p$ the attracting fixed point of $g$, and $q$ the repelling fixed point of $g$. The bi-infinite geodesic $[p,q]$ is called the loxodromic axis. it is preserved setwise by $g$.*

The isometry $g \in \mathrm{Isom}(X)$ has $0,1,2$ or infinitely many fixed points in $X(\infty)$; $0$ for an elliptic isometry with an isolated fixed point in $X$, one for a parabolic isometry, two for a hyperbolic isometry or an elliptic isometry fixing a geodesic, infinitely many for an elliptic isometry.

Let $\mathrm{fix}(G) = \cap_{g\in G}\mathrm{fix}(g)$, that is, the set of all fixed points of $G$ in $\overline{X}$. We recall the definition of the elementary group [Bo2].

**Definition 1.3** *The group $G$ (possibly not discrete) is called elementary if either $\mathrm{fix}(G) \neq \emptyset$, or $G$ preserves setwise some bi-infinite geodesic in $\overline{X}$; Otherwise, we call $G$ non-elementary.*

For an elementary group $G$ there are three mutually exclusive classes.
(1) $\mathrm{fix}(G)$ is a nonempty subspace of $\overline{X}$;
(2) $\mathrm{fix}(G)$ consists of a single point in $X(\infty)$;
(3) $G$ has no fixed point in $X$, and preserves setwise a unique bi-infinite geodesic in $X$.

We have known that $\mathrm{Isom}(X)$ is a Lie group with the compact open topology. Let $G$ be a subgroup of $\mathrm{Isom}(X)$. The set of limit points of $G$, denoted by $L(G)$, is defined to be the intersection with $X(\infty)$ of the limit set of any orbit in $X$. That is,

$$L(G) = \{x \in X(\infty) \mid \exists g_m \in G \text{ with } g_m(p) \to x \text{ for some (hence any) point } p \in X\}$$

It is clear that the limit set $L(G)$ is closed in $X(\infty)$ and invariant under $G$. By definition the limit set $L(G)$ does not depend on the choice of the point $p \in X$. Let $\overline{G}$ be the closure of $G$ in $\mathrm{Isom}(X)$. It is easy to see that $L(\overline{G}) = L(G)$.

Other aspects of the theory of the Hadamard manifold and groups of its isometries can be found in [BGS], [Bo1 and 2], [CE], [EB], [EHS], [EO] and [Gro].

Lie groups arise in many natural ways as transformation groups of manifolds. The following concepts and theorems will be used in next chapter.

**Definition 1.4** *Let $M$ be a smooth manifold and $G$ a Lie group. A smooth map*

$$\mu : G \times M \to M,$$

*written by $\mu(g, x) = gx$, is said to be an action of $G$ on $M$, and $G$ is called a Lie transformation group on $M$, if*
(1) $g_1(g_2x) = (g_1g_2)x, \forall g_1, g_2 \in G$ *and* $\forall x \in M$;
(2) $I(x) = x, \forall x \in M$.

From Myers and Steenrod [MyS], if $M$ is a Riemannian manifold, the group $\mathrm{Isom}(M)$, with the compact open topology, is isomorphic, as a topological group, to a Lie group such that the natural action,

$$\mathrm{Isom}(M) \times M \to M$$

defined by $(g, x) \to g(x)$ is smooth (see [Co, Theorem 10.7.1]). Therefore $\mathrm{Isom}(M)$ is not only a Lie group, but also a Lie transformation group. Let $H'$ be a submanifold of $M$. When the topology on $M'$ induced by its $C^\infty$ structure is its topology as a subset of $M$, $M'$ is said to be a regular submanifold of $M$. The following closed subgroup theorem is well known [BC, Proposition 12.9.5] or [Co, Theorem 5.3.2].

**Theorem 1.5** (Closed subgroup theorem). *A closed subgroup $H$ of a Lie group $G$ either is a discrete subset of $G$ or can be given a unique structure as a regular submanifold of $G$ and is then a Lie subgroup of $G$.*

The following famous fixed point theorem was originally proved by É. Cartan (See [He, Ch.1, Theorem 13.5]).

**Theorem 1.6** (É. Cartan) *Let $M$ be a complete simply connected Riemannian manifold of negative curvature. Let $K$ be a compact Lie transformation group of $M$ whose elements are isometries of $M$. Then all members of $K$ have a common fixed point.*

The following definition of discrete convergence groups is based on that of Gehring and Martin [GM].

A group $\Gamma$ of homeomorphisms of $B^n$ is called a *discrete convergence group* if for every sequence $\{\gamma_j\}_{j\geq 1}$ of distinct elements of $\Gamma$, there is a subsequence $\{\gamma_{j_k}\}$ and points $x_0$ and $y_0$ in $S^{n-1}$ such that

$$\gamma_{j_k} \to x_0 \qquad \mathrm{c-uniformly\ \ in} \qquad B^n \setminus \{y_0\}$$

and

$$\gamma_{j_k}^{-1} \to y_0 \qquad \mathrm{c-uniformly\ \ in} \qquad B^n \setminus \{x_0\}.$$

The possibility of $x_0 = y_0$ may occur. In this case, each elliptic element is of finite order. Here the c-uniformly we mean locally uniformly. In [MS], Martin and Skora proved

**Theorem 1.7** *Let $X$ be a Hadamard manifold, such that $K(x) \leq k < 0$. If $G \subset \mathrm{Hom}(\overline{X})$ is a discrete group that acts as isometries on $X$, then $G$ is a discrete convergence group.*

# 2 Classification of Isometry Groups

In this chapter, we first discuss arbitrary subgroup $G$ of $\mathrm{Isom}(X)$ by dividing cardinality of the limit set $L(G)$ into $0, 1, 2$ and infinity and fixed points of the elements in $G$. From these discussions we obtain that any pure elliptic group must have a common fixed point in $X$ or on the boundary $X(\infty)$. In the mean time, some criteria of a loxodromic isometry or parabolic isometry are given. It is well known that a discrete group is elementary if and only if its limit set contains at most two elements. The result is not true for arbitrary group of isometries in general. We, in section 2, give some characterizations of an elementary group according to its limit sets and orbits. In section 3, we extend a well known result in 3-dimensional hyperbolic spaces [Be] to a pinched Hadamard manifold: A subgroup of $\mathrm{Isom}(X)$ is nonelementary if and only if it contains infinitely many loxodromic elements, no two of which have a fixed point in common. The result is one of main tools in the next two chapters in which we discuss discreteness and algebraic convergence of groups. Using the propositions of limit sets and locally uniformly convergence of sequences that we obtained, we generalized a result of an group of isometries in n-dimensional hyperbolic space [Gre]. In section 4, a result of Rosenberger [Ro1, 2] was generalized under some condition.

If curvature $K(X) \equiv -1$, then the Hadamard manifold $X$ may be viewed as a hyperbolic space $H^n$ or $B^n$. Thus many results we obtained contain the classical results in 3-dimensional hyperbolic spaces or $n$-dimensional spaces.

## 2.1 Limit Sets and Pure Elliptic Groups

We first consider the case that $L(G)$ is an empty set.

**Lemma 2.1** *Let $\overline{G}$ be the closure of $G$ in $\mathrm{Isom}(X)$. Then $L(G) = \emptyset$ if and only if $\overline{G}$ is compact.*

**Proof.** Let $\overline{G}$ be compact. We will show that $L(G) = \emptyset$. It is not hard to see that $L(\overline{G}) = L(G)$. Suppose, for contradiction, that the limit set is not empty. Assume that $\overline{G}$ is finite. Then it is easy to see that the limit set of $G$ is the empty set. This contradicts the supposition that $L(\overline{G}) \neq \emptyset$. If $\overline{G}$ is an infinite group, then since $L(G) \neq \emptyset$, there is a point $x_0 \in X(\infty)$ and a infinite sequence $\{g_j\}$ such that $\lim\limits_{j\to\infty} g_j(x) = x_0$, for some $x \in X$. It follows from the compactness of $\overline{G}$ that for every infinite sequence in $\overline{G}$ there exists at least a convergent subsequence. Therefore there is a subsequence $\{g_{j_i}\}$ of $\{g_j\}$ such that $\lim\limits_{i\to\infty} g_{j_i} = g \in \overline{G}$. Thus $\lim\limits_{i\to\infty} g_{j_i}(x) = g(x) \in X$.This is a contradiction. So $L(G)$ is empty.

We next prove the necessity. Assume that $\overline{G}$ is discrete. Then it is finite and so is compact. Now let $\overline{G}$ be not discrete. Since $\overline{G}$ is a subgroup of Lie transformation group Isom(X), it is also a Lie transformation group. As the limit set of $\overline{G}$ is also empty, the orbit $\overline{G}(x)$ is bounded. Because $\overline{G}$ is closed, the orbit $\overline{G}(x)$ is also closed for any $x \in X$ . It is easy to see from [He, p56, Theorem 10.3] that each bounded closed subset of any complete Riemannian manifold is compact. Therefore $\overline{G}(x)$ is compact.

The isotropy group of $x$ is the set $\overline{G}_x = \{g \in \overline{G} | g(x) = x\}$. If $\{g_m\}_{m=1}^{\infty} \subset \overline{G}_x$ converges to $g \in$Isom$(X)$, then, by the continuity of the group action,

$$g(x) = \lim_{m\to\infty} g_m(x) = \lim_{m\to\infty} x = x.$$

Thus $\overline{G}_x$ is a closed subgroup of $\mathrm{Isom}_x(X)$ where

$$\mathrm{Isom}_x(X) = \{g \in \mathrm{Isom}(X) | g(x) = x\}.$$

By Theorem 2.5 in [He], $\mathrm{Isom}_x(X)$ is compact. Hence $\overline{G}_x$ is compact. It follows from [BC, Proposition 13.3.2] that $\overline{G}/\overline{G}_x$ is diffeomorphic to $\overline{G}(x)$. Then $\overline{G}/\overline{G}_x$ is compact. $\overline{G}$ is a fibre space with compact base ($\overline{G}/\overline{G}_x$) and compact fibre ($\overline{G}_x$). Therefore $\overline{G}$ is compact.

**Theorem 2.2** *Let $L(G) = \emptyset$. Then all isometries of $G$ are elliptic and have a common fixed point in $X$.*

**Proof.** Let $\overline{G}$ be the closure of $G$ in Isom$(X)$. Since $\overline{G}$ is a closed subgroup of Lie group Isom$(X)$, by Theorem 1.5, $\overline{G}$ is either a discrete subgroup or a Lie subgroup of Isom$(X)$.

Firstly, let $\overline{G}$ be a Lie subgroup. We have known that Isom$(X)$ is a Lie transformation group. Since any Lie subgroup of a Lie transformation group is also a Lie transformation group [BC, Proposition 13.1.1], $\overline{G}$ is a Lie transformation group. By $L(G) = \emptyset$ and Lemma 2.1, it follows that $\overline{G}$ is compact. According to Theorem 1.6, we have that the isometries in $\overline{G}$ have a common fixed point in $X$.

Secondly, let $\overline{G}$ be a discrete subgroup. We assert that $\overline{G}$ is a finite subgroup. Assume that this is false. Since $\overline{G}$ is discrete, by Theorem 1.7 it is a convergence group. Then, according to the definition of convergence group, there is a infinite sequence $\{g_j\}$ in $\overline{G}$ and $x_0, y_0 \in X(\infty)$ such that

$$\lim_{j\to\infty} g_j(x) = x_0 \qquad \text{and} \qquad \lim_{j\to\infty} g_j^{-1}(x) = y_0$$

c-uniformly in $\overline{X} \setminus \{y_0\}$ and $\overline{X} \setminus \{x_0\}$, respectively. Therefore $x_0 \in L(G)$. We have thereby reached a contradiction. Thus $G$ is finite. In this case, it is well known that the isometries in $\overline{G}$ have a common fixed point in $X$.

The upper argument of proof of the theorem implies

**Corollary 2.3** *Let $G$ be a discrete group. Then $L(G) = \emptyset$ if and only if $G$ is a finite group.*

For $L(G)$ contains only one point we have

**Theorem 2.4** *Assume $L(G) = \{x_0\}$. Then $G$ is an infinite group and contains only elliptic and parabolic isometries and $x_0$ is the unique common fixed point of elements of $G$. If $G$ is discrete, then it definitely contains a parabolic isometry.*

**Proof.** Since $L(G)$ is invariant under $G$, $x_0$ is a common fixed point of all isometries in $G$. Notice that loxodromic fixed points are limit points by Definition 1.2, every element of $G$ is either parabolic or elliptic isometry. On the other hand, $G$ is definitely not finite; otherwise the limit set of $G$ is an empty set. We will show that $x_0$ is the unique fixed point of $G$. Firstly, assume that $G$ contains a parabolic isometry. Then $x_0$ is certainly the unique fixed point of $G$. Secondly, let $G$ be a pure elliptic group. Suppose, for contradiction, that there exists another point $y_0 \neq x_0$ which is also a fixed point of $G$. In this case, the elements of $G$ stabilize pointwise the bi-infinite geodesic $[x_0, y_0]$. Hence there is a point $z \in X$ fixed by all isometries in $G$. This is a contradiction, because, according to the definition of limit set, there is a sequence of distinct isometries $g_j \in G$ such that $\lim_{j\to\infty} g_j(z) = x_0$ for any $z \in X$. Thus the fixed point $x_0$ is unique.

If $G$ is discrete, then $G$ is a parabolic group and thus contains parabolic isometries, see [Bo2, Proposition 4.2].

The following lemma gives a criterion for a loxodromic isometry.

**Lemma 2.5** *Let $U$ be a cone in $\overline{X}$ containing $x_0 \in X(\infty)$. If there is an isometry $g$ of $X$ such that $g(\overline{U}) \subset U$, then $g$ is a loxodromic isometry with a fixed point in $U$ and the other in $\overline{X} \setminus \overline{U}$.*

**Proof.** We claim that the isometry $g$ is not of finite order $m$. Since otherwise we would have $\overline{U} = g^m(\overline{U}) \subset U$. Since $\overline{U}$ is a topological $n$-cell (see Corollary 2.12 in [EO]), we may apply the Brouwer fixed point theorem. Then $g$ has a fixed point in $U$. Next the hypotheses imply that $g^{-1}(\overline{V}) \subset V$ where $V = \overline{X} \backslash \overline{U}$. Using the Brouwer fixed point theorem again, we have that $g^{-1}$, and hence $g$, has another fixed point in $V$ . So $g$ has at least two fixed points in $\overline{X}$ and thus is loxodromic or elliptic of infinite order. We will show that the isometry $g$ can not be elliptic of infinite order. Suppose, for contradiction, that $g$ is elliptic with fixed points $x_0 \in U$ and $y_0 \in V$. Let $\sigma$ be the geodesic in $\overline{X}$ passing through $x_0$ and $y_0$. Then $g$ stabilizes pointwise the geodesic $\sigma$ and thus fixes the point $\partial\overline{U} \cap \sigma$. This contradicts the assumption that $g(\overline{U}) \subset U$. Therefore $g$ has exactly two fixed points in $X(\infty)$ and thus is loxodromic.

**Corollary 2.6** *Let $x_0$ and $y_0$ be distinct points in $X(\infty)$. If there is a sequence $\{g_j\}$ of isometries of $X$ such that $\lim_{j\to\infty} g_j(x) = x_0$, c-uniformly in $\overline{X} \setminus \{y_0\}$, then $g_j$ are loxodromic for all sufficiently large $j$. For any neighbourhood $U$ of $x_0$, $g_j$ has a fixed point in $U$ for all large $j$.*

**Proof.** Let $U$ be a neighbourhood of $x_0$ such that $y_0 \notin U$. We may assume that $U$ is a cone. Since $\lim_{j\to\infty} g_j(x) = x_0$, c-uniformly in $\overline{X} \setminus \{y_0\}$, it follows that $g_j(\overline{U}) \subset U$ for all large $j$. Our result then can be obtained from Lemma 2.5.

The following lemma is well known in the case of discrete convergence group acting on $\overline{R}^n$ [GM, Lemma 4.1]. We have known that the discrete group of isometries of a pinched Hadamard manifold $X$ is a convergence group acting on $\overline{X}$ (Theorem 1.7). Using similar method in [GM, Lemma 4.1] for discrete convergence group, it is easy to see that the following lemma under the condition that $G$ is a discrete group of isometries is also true. In general case (the group probably non-discrete) we have

**Theorem 2.7** *Let $L(G)$ be a non-empty set. Given $x_0 \in L(G)$, there is a sequence of isometries $g_j \in G$ and $y_0 \in L(G)$ such that*

$$\lim_{j\to\infty} g_j(x) = x_0 \qquad \text{and} \qquad \lim_{j\to\infty} g_j^{-1}(x) = y_0$$

*c-uniformly in $\overline{X} \setminus \{y_0\}$ and $\overline{X} \setminus \{x_0\}$, respectively.*

**Proof.** Given $x_0 \in L(G)$. By the definition of limit set, we may select an infinite sequence $\{g_j\}$ in $G$ and $x \in X$ such that

$$\lim_{j\to\infty} g_j(x) = x_0.$$

By passing to a subsequence of $\{g_j\}$, we may assume that $x$ is not the fixed point of each $g_j$. Since $d(x, g_j(x)) = d(x, g_j^{-1}(x))$ and $g_j(x) \to x_0$, it follows that $\{g_j^{-1}(x)\}$ is unbounded and so has an accumulation point in $X(\infty)$. By passing subsequence, again denoted by $g_j^{-1}$, we have that $g_j^{-1}(x) \to y_0$ as $j \to \infty$ for some $y_0 \in X(\infty)$. Then we have $y_0 \in L(G)$.

By the law of cosines, if $x_m \to x_0 \in X(\infty)$ and $d(x_m, y_m)$ is bounded, where $x_m, y_m \in X$, then $y_m \to x_0$. Thus for any $x \in X$ we have $g_j(x) \to x_0$ and $g_j^{-1} \to y_0$ as $j \to \infty$. We will show that for any $p \in X(\infty)$ and $p \neq y_0$, $g_j(p) \to x_0$ as $j \to \infty$. Let $\sigma : (-\infty, +\infty) \to X$ be a geodesic with $\sigma(-\infty) = y_0$ and $\sigma(+\infty) = p$. If we choose $x$ to lie on $\sigma$, then

$$\measuredangle_x(g_j^{-1}(p), y_0) \geq \pi - \measuredangle_x(g_j^{-1}(x), p).$$

Since $\measuredangle_x(g_j^{-1}(x), p) + \measuredangle_{g_j^{-1}(x)}(x, p) \leq \pi$, (cf. [EO, p52]), it follows that

$$\measuredangle_x(g_j(x), g_j(p)) = \measuredangle_{g_j^{-1}(x)}(x, p) \leq \pi - \measuredangle_x(g_j^{-1}(x), p) \leq \measuredangle_x(g_j^{-1}(x), y_0) \to 0.$$

Thus $g_j(p) \to x_0$. According to the same reason mentioned above, we have $g_j^{-1}(p) \to y_0$ if $p \neq x_0$.

Now let $K$ be any compact set of $\overline{X} \setminus \{y_0\}$. For any neighbourhood $U$ of $x_0$ and $x \in K$, there exists a neighbourhood $V_x$ of $x$ such that $g_j(V_x) \subset U$ for all sufficiently large $j$, since $g_j(x) \to x_0$ as $j \to \infty$ and $g_j : \overline{X} \to \overline{X}$ is a homeomorphism. According to the compactness of $K$, we have $g_j(K) \subset U$ for $j$ sufficiently large. Therefore $g_j(x) \to x_0$ as $j \to \infty$ c-uniformly in $\overline{X} \setminus \{y_0\}$. Similarly, we have $g_j^{-1}(x) \to y_0$ as $j \to \infty$ c-uniformly in $\overline{X} \setminus \{x_0\}$.

**Corollary 2.8** *Suppose that there are a sequence of isometries $g_j$ of $X$ and $x_0 \in X(\infty)$ such that $\lim_{j\to\infty} g_j(x) = x_0$ for some $x \in X$. Then there exists $y_0 \in X(\infty)$ and a subsequence $j_k$ such that $\lim_{k\to\infty} g_{j_k}(x) = x_0$ and $\lim_{k\to\infty} g_{j_k}^{-1}(x) = y_0$ c-uniformly in $\overline{X} \setminus \{y_0\}$ and $\overline{X} \setminus \{x_0\}$, respectively.*

We will show that if $L(G)$ contain at least two points, then $G$ definitely contains a loxodromic element. For $L(G)$ contains exactly two points, we have the generalization of an group of isometries of hyperbolic space.

**Theorem 2.9** *Let $G$ be a subgroup of Isom($X$). $L(G) = \{x_0, y_0\}$ if and only if $G$ contains only loxodromic isometries with fixed points $x_0$ and $y_0$ and elliptic isometries which setwise stabilize the set $\{x_0, y_0\}$ and $G$ at least contains a loxodromic isometry. In addition, if $G$ is discrete, then $G$ contains at most finitely many elliptic isometries which fix $x_0$ and $y_0$.*

**Proof.** Let $L(G) = \{x_0, y_0\}$. We first show that there definitely exists a loxodromic isometry in $G$. Let $U$ and $V$ be neighbourhoods of $x_0$ and $y_0$, respectively, so that $y_0 \notin U$ and $x_0 \notin V$. We may assume that $U$ and $V$ are cones in $\overline{X}$ containing $x_0, y_0 \in X(\infty)$, respectively. By Corollary 2.12 in [EO], $\overline{U}$ and $\overline{V}$ are topological n-cells. By Theorem 2.7 there is a point $z_0 \in L(G) = \{x_0, y_0\}$ and a sequence $\{g_j\}$ of isometries in $G$ such that

$$\lim_{j\to\infty} g_j(x) = x_0 \qquad \text{and} \qquad \lim_{j\to\infty} g_j^{-1}(x) = z_0$$

c-uniformly in $\overline{X} \setminus \{z_0\}$ and $\overline{X} \setminus \{x_0\}$, respectively.

If $z_0 = y_0$, then for sufficiently large $j$ we have $g_j(\overline{U}) \subset U$. Lemma 2.5 implies that $g_j$ is loxodromic. If $z_0 = x_0$, using Theorem 2.7 again, then there exist $w_0 \in L(G)$ and a sequence $h_j \in G$ such that

$$\lim_{j\to\infty} h_j(x) = y_0 \qquad \text{and} \qquad \lim_{j\to\infty} h_j^{-1}(x) = w_0$$

c-uniformly in $\overline{X} \setminus \{w_0\}$ and $\overline{X} \setminus \{y_0\}$, respectively.

Similarly, if $w_0 = x_0$, then $h_j$ is loxodromic for sufficiently large $j$. Now let $w_0 = y_0$. Then for disjoint neighbourhoods $U$ and $V$ of $x_0$ and $y_0$, respectively, we have that $g_j^{\pm}(\overline{V}) \subset U$ and $h_j^{\pm}(\overline{U}) \subset V$ by the c-uniformly. Thus

$$g_j h_j(\overline{U}) \subset g_j(V) \subset U \qquad \text{and} \qquad (g_j h_j)^{-1}(\overline{V}) \subset h_j^{-1}(U) \subset V.$$

Then by Lemma 2.5 it follows that $g_j h_j$ is loxodromic for some sufficiently large $j$. Thus there definitely exists a loxodromic isometry in $G$.

We next show that every isometry stabilizes setwise the loxodromic fixed points. Since $L(G)$ is invariant under $G$, for every $g \in G$ we have

$$g(x_0) = x_0 \qquad \text{and} \qquad g(y_0) = y_0; \tag{2.1}$$

or

$$g(x_0) = y_0 \qquad \text{and} \qquad g(y_0) = x_0. \tag{2.2}$$

Since every isometry which interchanges two distinct points in $X(\infty)$ must be elliptic isometry, all loxodromic isometries in $G$ fix $x_0$ and $y_0$ and the elliptic isometries either fix or interchange the points $x_0, y_0$. On the other hand, $G$ does not contain parabolic isometries because no such isometry satisfies (2.1) or (2.2)

For the sufficiency, since $G$ contains at least a loxodromic element by the hypothesis, and the fixed points of loxodromic isometries in $G$ are limit points, it follows that $\{x_0, y_0\} \subset L(G)$. For any $z_0 \in L(G)$, by Theorem 2.7 there exist a point $w_0 \in L(G)$ and a sequence of isometries $g_j \in G$ such that $g_j(x) \to z_0$ and $g_j^{-1}(x) \to w_0$ c-uniformly in $\overline{X} \setminus \{w_0\}$ and $\overline{X} \setminus \{z_0\}$, respectively. Since the elements stabilize setwise the set $\{x_0, y_0\}$, by passing the subsequence of $\{g_j\}$ if necessary, it follows that $z_0 = \lim_{j\to\infty} g_j(x_0) \in \{x_0, y_0\}$ if $w_0 \neq x_0$, or $z_0 = \lim_{j\to\infty} g_j(y_0) \in \{x_0, y_0\}$ if $w_0 \neq y_0$. So $z_0 \in \{x_0, y_0\}$ and thus $L(G) = \{x_0, y_0\}$.

If $G$ is discrete, then it is a convergence group by Theorem 1.7. By means of the same arguments as for convergence groups, we know that there are at most finitely many elliptic isometries in $G$ which fix $x_0$ and $y_0$.

The argument in the proof of Theorem 2.9 implies that

**Corollary 2.10** *$L(G)$ consists of at least two points if and only if $G$ contains a loxodromic isometry.*

Now we can generalize Theorem 2.2 as follows.

**Theorem 2.11** *The limit set $L(G) = \emptyset$ if and only if all isometries of $G$ are elliptic and have a common fixed point in $X$.*

**Proof.** We only prove the sufficiency. It is easy to see from Corollary 2.10 that $L(G)$ either is empty or contains exactly one point. In the later case, by Theorem 2.4, the elements of $G$ have a unique common fixed point and the point lies on $X(\infty)$. This contradicts the hypothesis that all isometries of $G$ have a common fixed point in $X$. The result follows.

Let $g$ be a loxodromic isometry of $X$, and $x_0, y_0$ be the attractive and repelling fixed points of $g$, respectively. Then $g$ generates a discrete group and $g^j(x) \to x_0, g^{-j}(x) \to y_0$ as $j \to \infty$ for any $x \in X$. On the other hand, we do not know if an arbitrary parabolic isometry generates a discrete group [EO, p82]. Let $g$ be a parabolic isometry with a unique fixed point $x_0 \in X(\infty)$. If $g$ generates a discrete group, then for any $p \in X$ the sequences $\{g^m(p)\}$ and $\{g^{-m}(p)\}$ converge to $x_0$ as $m \to \infty$. In the following we consider an arbitrary parabolic or loxodromic isometry and local uniformly convergence.

**Theorem 2.12** *Let* $g \in \mathrm{Isom}(X)$.

i). *$g$ is loxodromic with fixed points $x_0$ and $y_0$ if and only if*

$$\lim_{m\to\infty} g^m(x) = x_0 \qquad \textit{and} \qquad \lim_{m\to\infty} g^{-m}(x) = y_0 \tag{2.3}$$

*c-uniformly in $\overline{X} \setminus \{y_0\}$ and $\overline{X} \setminus \{x_0\}$, respectively;*

ii). *$g$ is parabolic with fixed point $x_0$ if and only if there is an integer subsequence $\{m_j\}$ such that*

$$\lim_{j\to\infty} g^{m_j}(x) = x_0 \qquad \textit{and} \qquad \lim_{j\to\infty} g^{-m_j}(x) = x_0 \tag{2.4}$$

*c-uniformly in $\overline{X} \setminus \{x_0\}$.*

**Proof**. i). It is easy to see that if $g$ is loxodromic, then $g$ generates an infinite cyclic discrete group or a discrete convergence group. The result then follows from [GM, Theorem 6.2]. Conversely, if (2.3) holds, then $\langle g\rangle$ is discrete with $\mathrm{ord}(g) = \infty$. It is obvious that the limit set of $L(\langle g\rangle)$ of $g$ is equal to $\{x_0, y_0\}$ and so $g\{x_0, y_0\} = \{x_0, y_0\}$. Thus $g$ is loxodromic or elliptic. From (2.3) we know that $g$ must be loxodromic.

ii). Let $\lim_{j\to\infty} g^{\pm m_j}(x) = x_0$, c-uniformly in $\overline{X} \setminus \{x_0\}$. Then for all $x \in X$, the set $\{g^m(x) | m \in Z\}$ is unbounded. Therefore $g$ is not an elliptic isometry by Lemma 6.3 in [BGS]. According to the result of i), $g$ is not loxodromic isometry, either. So $g$ is parabolic. Conversely, let $g$ be a parabolic isometry.

Each orbit $A_p = \{g^l(p) : l \in Z\}$ has accumulation points in $X(\infty)$ for all $p \in X$, since it is unbounded. Furthermore, any accumulation point in $X(\infty)$ of the set $A_p$ is a fixed point of $g$ (Lemma 6.2 in [EO] or [Eb]). Thus $x_0$ is the unique accumulation point in $X(\infty)$ of the set $A_p$. Therefore there is a subsequence $\{m_j\}$ such that $g^{\pm m_j}(p) \to x_0$ as $j \to \infty$. From Corollary 2.8 the proof follows.

The above theorem implies the following criterion that characterizes loxodromic elements in a group of isometries of pinched Hadamard manifolds.

**Corollary 2.13** *Suppose that $g$ is an isometry of $X$. Then $g$ is loxodromic if and only if there are a cone $U$ of $\overline{X}$ and an integer $j$ such that $g^j(\overline{U}) \subset U$.*

**Proof.** By Lemma 2.5, if $g^j(\overline{U}) \subset U$, then $g^j$ , and thus $g$ is loxodromic. On the other hand, if $g$ is loxodromic, then, by Theorem 2.12, for a sufficiently large $j$ we have $g^j(\overline{U}) \subset U$.

Let $H^n$ be the $n$-dimensional hyperbolic space, and $G$ the pure torsion subgroup of $\mathrm{Isom}(H^n)$. If $n = 3$, then $G$ has a common fixed point in $H^3$. From [Wa2], we know that, for $n > 3$, the elements of $G$ have a common fixed point either in $H^n$ or on the boundary of $H^n$. Furthermore, the example ($n = 5$) in [Wa] shows that there is a pure elliptic group of isometries which has a common fixed point on the boundary but there is not any common fixed point in $H^n$. The following theorem extends a result of Waterman to the group of isometries of a pinched Hadamard manifold.

**Theorem 2.14** *Let $G$ be a subgroup of* $\mathrm{Isom}(X)$ *which contains only elliptic isometries and the identity. Then one of the following must occur:*

(1) *The isometries of $G$ have a common fixed point in $X$.*

(2) *The isometries of $G$ have a unique fixed point on $X(\infty)$.*

**Proof.** By Corollary 2.10, the limit set of $G$ either is empty or contains a single point. If $L(G) = \emptyset$, then Theorem 2.2 implies that $G$ has a common fixed point in $X$. If $L(G) = \{x_0\}$, then $x_0$ is a unique common fixed point of $G$ by Theorem 2.4.

**Corollary 2.15** *Let $G$ be a subgroup of* $\mathrm{Isom}(X)$. *Then the following statements are equivalent.*

(1) *$G$ is a discrete and pure elliptic group.*
(2) *$G$ is a finite group.*
(3) *$G$ is discrete and $L(G) = \emptyset$.*

**proof.** (1)$\Rightarrow$(2). If $L(G)$ consists of more than two points, then $G$ contains a loxodromic isometry. Therefore $\mathrm{Card}(L(G)) \leq 1$. If $L(G) = \emptyset$, then the discreteness of $G$ and Corollary 2.11 imply that $G$ is finite. If $L(G)$ contains only a point, it follows from Theorem 2.4 that $G$ contains a parabolic isometry. It is impossible because $G$ is a pure elliptic group. Thus $G$ is finite.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) is obvious.

## 2.2 Elementary Groups

In this section we pay our attention on elementary groups and investigate the relation among elementary groups, limit sets and orbits. A characteristic of an elementary group is given.

**Lemma 2.16** *Let $G$ be a subgroup of $\mathrm{Isom}(X)$ containing only parabolic and elliptic isometries and containing at least one parabolic isometry. Then $L(G)$ contains only the fixed point of the parabolic isometry.*

**Proof.** Let $g \in G$ be a parabolic isometry, and $x_0$ the fixed point of $g$. We have known that the unique fixed point of $g$ is a limit point of some subsequence in $\langle g \rangle$. We have that $x_0 \in L(G)$. It follows from Theorem 2.9 and Corollary 2.10 that $L(G) = \{x_0\}$.

The following lemma shows that the limit set of $G$ contains either $0, 1, 2$ points or infinitely many points.

**Lemma 2.17** *Let $G$ be a subgroup of $\mathrm{Isom}(X)$. If $L(G)$ contains at least three points, then it contains infinitely many points.*

**Proof.** Suppose that $L(G)$ contains only finitely many points, say $L(G) = \{x_1, x_2, \cdots, x_m\}, x_i \in X(\infty)$ and $i = 1, 2, \cdots, m; \quad m \geq 3$. By Corollary 2.10, there is a loxodromic $g$ in $G$. We have known that $L(G)$ is $G$-invariant. Thus some power of $g$ fixes every point of $L(G)$. That is,

$g^l(x_i) = x_i, (i = 1, 2, \cdots, m)$, which is a contradiction since $g^l$ is also a loxodromic isometry. Thus $L(G)$ contains infinitely many points.

We now investigate the equivalent conditions of an elementary group. The following fact is basic.

**Theorem 2.18** *Suppose that $G$ is a subgroup of* $\mathrm{Isom}(X)$. *Then the following statements are equivalent.*

(1) *$G$ is an elementary group.*

(2) *$L(G)$ contains either at most two points or infinitely many points with fix(G)=$\{x_0\}$, where $x_0 \in X(\infty)$.*

(3) *$G$ has a finite orbit in $\overline{X}$.*

**Proof.** We first prove that (1)$\Rightarrow$ (2). Let $G$ be elementary. By the definition of elementary group, $G$ may be classified into one of the three mutually different cases: 1). $\mathrm{fix}(G)$ is non-empty, complete geodesic subspace of $X$, 2). $\mathrm{fix}(G)$ consists of a single point of $X(\infty)$, 3). $\mathrm{fix}(G) = \emptyset$ in $X$ and $G$ preserves setwise a unique bi-infinite geodesic in $X$. In case 1), it is enough to see that $G$ is a pure elliptic group because only elliptic isometries have fixed points in $X$. By Corollary 2.10, $L(G)$ consists of at most one point. In case 3), we will show that $G$ must contain a loxodromic isometry. Suppose, for contradiction, every element of $G$ is either elliptic or parabolic. If $G$ contains parabolic isometries, then $L(G)$ contains and only contains a point in $X(\infty)$ by Lemma 2.16, and $G$ has a unique fixed point in $X(\infty)$. This contradicts case (3). Assume that $G$ does not contain parabolic isometries, Then it is a pure elliptic group. By means of Theorem 2.14, we have that $\mathrm{fix}(G)$ contains at least a point in $X$ or a single point on $X(\infty)$, a contradiction. Now let $g \in G$ be a loxodromic isometry with fixed points $x_0, y_0$ in $X(\infty)$. Since loxodromic fixed points are limit points, we have $\{x_0, y_0\} \subset L(G)$. By case (3), we know that the unique bi-infinite geodesic $[x_0, y_0]$ is invariant under the isometries in $G$. Thus we have that $L(G)$ contains only $x_0$ and $y_0$. In case 2), If $G$ does not contain loxodromic isometries, then it is clear that $\mathrm{Card}(\mathrm{fix}(G)) \leq 1$. Now let $G$ contain loxodromic isometries. According to Corollary 2.10 and the argument in case 3), we know that $L(G)$ contains at least three points, and then it contains infinitely many points with $\mathrm{fix}(G) = \{x_0\}$ by Lemma 2.17, where $x_0$ is one of fixed points of the loxodromic isometry.

Secondly, we show that (2) $\Rightarrow$ (3). If Card$(L(G)) \leq 1$, then, by Theorems 2.2 and 2.4, there is a point $x_0 \in \overline{X}$ such that $G(x_0) = \{x_0\}$. If $L(G) = \{x_0, y_0\}$, then , by Theorem 2.9, $G(x_0) = \{x_0, y_0\}$. It is trivial for the case that $L(G)$ contains infinitely many points with fix$(G)$=$\{x_0\}$. Thus $G$ has a finite orbit in $\overline{X}$.

At last, we show that (3) $\Rightarrow$ (1). Let $G$ have a finite $G-$orbit in $\overline{X}$. Then we may suppose that $G(x) = \{x, x_1, \cdots, x_m\}$ with $x, x_i \in \overline{X}$. We will analyze the orbit by classifying it according to the cases that $G(x)$ consists one point, two points, and more than two points.

(i). If $G(x) = \{x\}$, then $x$ is a fixed point of the elements of $G$ and so $G$ is elementary.

(ii). If $G(x) = \{x, x_1\}$, then it is not hard to see that $G\{x, x_1\} = \{x, x_1\}$. If $G$ contains a loxodromic isometry $g$, then $x, x_1 \in X(\infty)$ are the fixed points of $g$. Therefore fix$(G) = \emptyset$ and every isometry of $G$ fixes a unique axis and thus $G$ is elementary. If $G$ does not contain any loxodromic isometry of $X$, then $G$ is elementary by Theorem 2.14 and Lemma 2.16.

(iii). Let $G(x) = \{x, x_1, \cdots, x_m\}$ with $m \geq 2$. In this case, we will show that $G$ is a pure elliptic group. Suppose that, on the contraries, $G$ contains a loxodromic or parabolic isometry $g$. Then $x \in X(\infty)$ is a fixed point of $g$ and so $g(x_i) = x_i$ for all $x_i$. This contradicts that $g$ has at most two fixed points. Thus $G$ contains only elliptic isometries and the identity. By Theorem 2.14, $G$ has either a fixed point in $X$ or a unique fixed point on $X(\infty)$ and so is elementary.

From the argument of Theorem 2.18, we know that if Card$(L(G)) \leq 2$, then $G$ is elementary. But the converse is not true in general. Furthermore, if we assume that $G$ is discrete , then we can strengthen Theorem 2.18 as follows.

**Corollary 2.19** *Let $G$ be a discrete subgroup of Isom($X$). Then $G$ is elementary if and only if Card$(L(G)) \leq 2$.*

**Proof.** If $L(G)$ contains infinitely many points, then there definitely exists a loxodromic element in $G$ . The argument in Theorem 2.18 implies that it only remains to show that fix$(G) = \{x_0\}$ and $G$ contains a loxodromic element can not appear. Assume that $g \in G$ is a loxodromic isometry with fixed points $x_0$ and $y_0$. Recall from [EO, proposition 6.8] that for any $h \in G$ with $h(x_0) = x_0$, if $g$ and $h$ generate a discrete group, than $h(y_0) = y_0$. Thus we have $\{x_0, y_0\} \subset$fix$(G)$, contradicting to the assumption that fix$(G) = \{x_0\}$ . The result follows.

## 2.3 Nonelementary Groups

It is well known that a non-elementary group in SL$(2, C)$ definitely contains infinitely many loxodromic elements and no two of which share a common fixed point [Be]. In the following we extend the well-known result to groups of isometries of a pinched Hadamard manifold. The result is the key to further generalizations of many classical results of discreteness and convergence of groups of isometries of three dimensional hyperbolic space.

**Theorem 2.20** *Let $G$ be a subgroup of* Isom$(X)$*. Then $G$ is non-elementary if and only if $G$ contains infinitely many loxodromic isometries, no two of which have a common fixed point.*

**Proof.** We first show the necessity. Let $G$ be non-elementary. It is not hard to see from Theorem 2.18 that the limit set of $G$ contains infinitely many points. Corollary 2.10 implies that $G$ contains a loxodromic isometry. Now let $h$ be a loxodromic isometry in $G$. In the following we will find infinitely many loxodromic isometries in $G$ by means of the element $h$ so that no two of them have a common fixed point.

By the definition of non-elementary, there exists an isometry $g$ in $G$ such that fix$(g)\cap$fix$(h)=\emptyset$ and $g$ does not exchange the fixed points of $h$. Let $x_0$ and $y_0$ are two distinct fixed points of $h$ in $X(\infty)$. Then one of the following three cases may occurs:

(1). $g\{x_0, y_0\} \cap \{x_0, y_0\} = \emptyset$.

(2). $g(x_0) = y_0$ and $g(y_0) \neq x_0$.

(3). $g(x_0) \neq y_0$ and $g(y_0) = x_0$.

In the first case, let $\gamma = g^{-1}hg$, then $\gamma$ is loxodromic. Thus $\gamma$ and $g$ have distinct fixed points and $\gamma(x_0) \neq y_0$, $\gamma(y_0) \neq x_0$ since otherwise, $x_0$ or $y_0$ is the fixed point of $g$ or $x_0, y_0$ are exchanged by $g$. Now we define $\gamma_k := h^k\gamma h^{-k}$ for $k = 0, 1, 2, \cdots$. Therefore all of elements $\gamma_k$ are loxodromic . we will show that there is a subsequence of $\{\gamma_k\}$ such that no two of distinct elements have a fixed point in common. Actually, let $p$ and $q$ are the fixed points of $\gamma$, and so the set of fixed points of $\gamma_k$ is $\{h^k(p), h^k(q)\}$. We first show that for $m \neq l$, $h^m(p) \neq h^l(p)$ and $h^m(q) \neq h^l(q)$. Suppose it is false. That is to say, $h^m(p) = h^l(p)$ or $h^m(q) = h^l(q)$. It is not hard to see that $p$ or $q$ is a fixed point of $h$, since $h$ and its any powers $h^k$ have the same fixed points, a contradiction. Thus two possible cases may occur for $h^k(p)$ and $h^k(q)$:

i). $h^k(p) \neq q$ and $h^k(q) \neq p$ for all non-negative integers $k$ ;

ii). there exists some integer $s > 0$ such that $h^s(p) = q$ or $h^s(q) = p$.

In case i), we assert that no two of isometries in $\{\gamma_k\}$ share a fixed point. In fact, suppose that for $m \neq l$, $\gamma_m$ and $\gamma_l$ have a common fixed point, say $h^m(p) = h^l(q)$. Then $h^{m-l}(p) = q$, which contradicts the case i).

In case ii), there is a positive integer $s$ such that $h^s(p) = q$ or $h^s(q) = p$. Then we take a subsequence $\{\gamma_{k(s+1)}\}$. We will show that $h^{k(s+1)}(p) \neq q$ and $h^{k(s+1)}(q) \neq p$ for each $k = 0, 1, 2, \cdots$. To see this let us suppose, for the purpose of contradiction, that $h^{k(s+1)}(p) = q$ or $h^{k(s+1)}(q) = p$. Without loss of generality, assume $h^s(p) = q$ in case ii). Then $h^{k(s+1)-s}(q) = q$ or $h^{k(s+1)+s}(p) = p$, and so $p$ or $q$ is a fixed point of $h$ since $k(s+1) - s \neq 0$ and $k(s+1) + s \neq 0$. This is a contradiction as $h$ and $\gamma$ have no fixed point in common. According to case i), $\{\gamma_{k(s+1)}\}$ is our desired.

In the second case, for the given loxodromic isometry $h$ we can deduce that either $ghg\{x_0, y_0\} \cap \{x_0, y_0\} = \emptyset$ or $gh^2g\{x_0, y_0\} \cap \{x_0, y_0\} = \emptyset$. We replace $g$ in case (1) by $ghg$ or $gh^2g$ and then prove case (2) by using the same method in case (1).

Case (3) can be proved in the same way as case (2).

We now prove the sufficiency. Since $G$ contains infinitely many loxodromic isometries which have different fixed points, we have that $L(G)$ is an infinite set and $\text{fix}(G) = \emptyset$. It is easy to see from Theorem 2.18 that $G$ is not elementary. This completes the proof.

Let $\text{Card}(L(G)) \geq 2$. For any points $x_1, x_2 \in L(G)$ and their neighbourhoods $U_1, U_2$ in $X(\infty)$, if there is a loxodromic isometry in $G$ such that its one fixed point in $U_1$ and the other in $U_2$, then we say that the fixed points of the loxodromic isometries of $G$ are dense in $L(G) \times L(G)$.

In 1962, Leon Greenberg proved that

**Theorem** (Greenberg). *Let $G$ be a subgroup of Isom($H^n$). If $G$ contains loxodromic elements, and the elements of $G$ do not have a common fixed point on the boundary of $H^n$, then the fixed points of the loxodromic elements of $G$ are dense in $L(G) \times L(G)$.*

In 1973, Eberlein and O$'$Neill (see [EO]) proved that for Visibility manifold $X$ and properly discontinuous group $G$ of Isom(X), the above theorem is true.

In 1987, Gehring and Martin in [GM] proved that discrete convergence group $G$ also has the property.

For any subgroup of Isom($X$), using the above properties about limit set, we obtain the following generalization.

**Theorem 2.21** *If $G$ is nonelementary, or $L(G)$ contains exactly two points, then the fixed points of the loxodromic isometries of $G$ are dense in $L(G) \times L(G)$.*

**Proof.** It is easy to see that $L(G)$ contains at least two points by Theorems 2.20 .

Firstly, let $L(G)$ contains only two points, $L(G) = \{x_0, y_0\}$ say. Then, by Theorem 2.9, $G$ contains at least a loxodromic isometry and all loxodromic isometries in $G$ fix $x_0$ and $y_0$ and thus the theorem is established.

Secondly, suppose that $G$ is nonelementary. Then there exist infinitely many loxodromic isometries in $G$, no two of which have a fixed point in common and $\mathrm{Card}(L(G)) > 2$ by Theorem 2.20. Let $x_0, y_0 \in L(G)$ and $U_1, U_2$ be respectively disjoint neighbourhoods in $X(\infty)$ of these points. By Theorem 2.7, there are $w_0, z_0 \in L(G)$ and sequences $\{g_j\}, \{h_j\}$ in $G$ such that

$$\lim_{j\to\infty} g_j(x) = x_0 \qquad \text{and} \qquad \lim_{j\to\infty} g_j^{-1}(x) = w_0 \tag{2.5}$$

c-uniformly in $\overline{X} \setminus \{w_0\}$ and $\overline{X} \setminus \{x_0\}$, respectively, and

$$\lim_{j\to\infty} h_j(x) = y_0 \qquad \text{and} \qquad \lim_{j\to\infty} h_j^{-1}(x) = z_0 \tag{2.6}$$

c-uniformly in $\overline{X} \setminus \{z_0\}$ and $\overline{X} \setminus \{y_0\}$, respectively.

We now consider $g_j$ and $h_j$ according to the positions of $x_0, w_0, y_0$ and $z_0$:

1). $w_0 = y_0$ or $z_0 = x_0$;

2). $x_0 = w_0$ and $y_0 = z_0$;

3). $x_0, w_0, y_0$ and $z_0$ are mutually different;

4). $x_0 = w_0$ and $y_0 \neq z_0$ or $x_0 \neq w_0$ and $y_0 = z_0$;

5). $x_0 \neq w_0$, $y_0 \neq z_0$ and $w_0 = z_0$

For case 1), the result directly follows from Corollary 2.6.

In case 2), for some large $j$, $g_j h_j$ is our desired by the same argument of Theorem 2.9.

In case 3), without loss of generality, we may assume that $w_0 \notin U$ and $z_0 \notin V$. Let $K$ and $W$ be disjoint neighbourhoods of $w_0$ and $z_0$, respectively, so that $W \cap U = \emptyset$ and $K \cap V = \emptyset$. Then we have that $g_j(K) \subset U$, $g_j^{-1}(\overline{V}) \subset W$ and $h_j(W) \subset V$, $h_j(\overline{U}) \subset K$ by (2.5) and (2.6). It is enough to deduce that $g_j h_j^{-1}(\overline{U}) \subset U$ and $(g_j h_j^{-1})^{-1}(\overline{V}) \subset V$. The result follows from Lemma 2.5.

In case 4), assume $x_0 = w_0$ in case 3). Using the same argument of case 3), $g_j h_j^{-1}$ is loxodromic with a fixed point in $U$ and the other in $V$ for some large $j$.

In case 5), by virtue of Theorem 2.20 there exists a loxodromic element $f$ in $G$ such that $\mathrm{fix}(f) \cap \{x_0, y_0\} = \emptyset$. Replacing $f$ by $f^{-1}$ , if necessary, we may assume $f(x_0) \neq z_0$. Then

$$\lim_{j\to\infty} g_j f(x) = x_0 \qquad \text{and} \qquad \lim_{j\to\infty} (g_j f)^{-1}(x) = f^{-1}(z_0)$$

c-uniformly in $\overline{X} \setminus \{f^{-1}(z_0)\}$ and $\overline{X} \setminus \{x_0\}$, respectively. Since $x_0 \neq f^{-1}(z_0)$, it follows by Corollary 2.6 that $g_j f$ is loxodromic and one fixed point in $U$ . Assume $f^{-1}(z_0) = y_0$. Then the result follows from Corollary 2.6. Otherwise $x_0, f^{-1}(z_0), y_0$ and $z_0$ are mutually distinct. Then case 3) may apply. Thus the theorem is established.

Let $G$ be a non-elementary subgroup of $\mathrm{Isom}(X)$ and let $L_0(G)$ denote the set of points fixed by some loxodromic element in $G$. Theorems 2.21 and 2.20 imply that

**Corollary 2.22** *If $G \subset \mathrm{Isom}(X)$ is a nonelementary group, then the limit set $L(G)$ is the closure of $L_0(G)$ in $X(\infty)$.*

The following theorem establishes the version of the theorem [Be] in the three dimensional space.

**Theorem 2.23** *For any non-elementary subgroup $G \subset \mathrm{Isom}(X)$, the limit set $L(G)$ is the smallest non-empty $G$-invariant closed subset of $X(\infty)$. In addition, $L(G)$ is a perfect set and is therefore uncountable.*

**Proof.** It is easy to see from the definition of limit set and Theorem 2.20 that $L(G)$ is non-empty $G$-invariant closed subset. We will show that $L(G)$ is smallest as follow.

Let $K$ be any non-empty $G$-invariant closed subset of $X(\infty)$. As $G$ is non-elementary, every orbit of $G$ is infinite by Theorem 2.18, thus $K$ is infinite. Let $z \in L_0(G)$. Then there is some loxodromic element $g \in G$ such that $g(z) = z$. Without loss of generality, we may assume that $z$ is the attractive fixed point of $g$. By Theorem 2.12, we have $\lim_{j \to \infty} g^j(x) = z$, c-uniformly in $\overline{X} \setminus \{y\}$, where $y$ is the repelling fixed point of $g$. Thus there is a point $w \in K$ such that $\lim_{j \to \infty} g^j(w) = z$. Since $K$ is closed $G$-invariant, it follows that $\{g^j(w\} \subset K$ and so $z \in K$. This implies that $L_0(G) \subset K$. Thus $L(G) \subset K$.

Theorems 2.21 and 2.20 enough show that $L(G)$ has no isolate points. Since $L(G)$ is closed and has no isolated points, it follows that $L(G)$ is perfect. Being a non-empty perfect set, $L(G)$ is uncountable.

## 2.4 Minimal Generating System

Let $G$ be any group. We call a cardinal number $r$ the rank $r(G)$ of $G$ if $G$ can be generated by a generating system $Y$ with cardinal number $r$ but not by a generating system $Z$ with cardinal number $s$ less than $r$. Let $r(G)$ be the rank of $G$.

We call a generating system $Y$ of $G$ a minimal generating system (M.G.S.) of $G$ if $Y$ has the cardinal number $r(G)$.

In 1984, Rosenberger [Ro1] proved that any nonelementary subgroup $G \subset SL(2, R)$ can be generated by a minimal generating system which contains only hyperbolic matrices. Furthermore, in 1988, he [Ro2] proved that if $G \subset SL(2, C)$, then the minimal generating system contains either only hyperbolic matrices or only loxodromic matrices. For the group of isometries of a pinch Hadamard manifold $X$ we give some similar results as follows. First we have

**Lemma 2.24** *Let $g$ be a loxodromic isometry of $X$ with attracting fixed point $x_0$ and a repelling fixed point $y_0$. If there is an isometry $h$ of $X$ such that $h(x_0) \neq y_0$ (or $h(y_0) \neq x_0$), then $g^m h$ (or $hg^{-m}$) are loxodromic for all sufficiently large $m$. In addition, if $h$ and $g$ have no common fixed points, then $g$ and $g^m h$ have no common fixed points.*

**Proof.** Assume $h(x_0) \neq y_0$. Since $g$ is loxodromic, by Theorem 2.12, it follows that $\lim\limits_{m\to\infty} g^m h(x) = x_0$, c-uniformly in $\overline{X} \setminus \{h^{-1}(y_0)\}$. By Corollary 2.6 and $x_0 \neq h^{-1}(y_0)$, we have that $g^m h$ are loxodromic for all sufficiently large $m$. it is easy to see that if $g$ and $g^m h$ have a common fixed point, then so do $g$ and $h$.

**Proposition 2.25** *Let $G$ be a nonelementary subgroup and $Y$ its minimal generating system. If $Y$ contains a loxodromic element, then $G$ can be generated by a minimal generated system $Z$ which contains only loxodromic isometries.*

**Proof** Let $g$ be a loxodromic element in $Y$ with an attractive fixed point $x_0$ and a repulsive fixed point $y_0$. By the definition of nonelementary groups, there is some element $h \in Y$ which does not leave $\{x_0, y_0\}$ invariant and two cases arise:

(1). $\{x_0, y_0\}$ and $\{hx_0, hy_0\}$ are disjoint;

(2). $\{x_0, y_0\}$ and $\{hx_0, hy_0\}$ have exactly one element in common.

In either case, without loss of generality, we may assume that $hx_0 \neq y_0$. By Lemma 2.24, it follows that $g^m h$ is loxodromic for some $m$, and $g^m h$ and $h$ have no common fixed points. Replace $h$ by $g^m h$ to get a minimal generating system $Y_1$ which contains two loxodromic elements without common fixed points. We denote the loxodromic $g^m h$ by $h$ in $Y_1$. Let $f \in Y \setminus \{g, h\}$ be any element. If all $f$ are loxodromic, then the lemma is complete. So we assume that $f$ is not loxodromic. Therefore $f$ has one of the following three mutually exclusive cases:

1). $f(x_0) = x_0$ and $f(y_0) = y_0$

2). $f(x_0) \neq y_0$ or $f(y_0) \neq x_0$

3). $f(x_0) = y_0$ and $f(y_0) = x_0$

In case 1), we have $gf(x_0) = x_0$ and $gf(y_0) = y_0$. Since $gf$ translates the unique geodesic which connects $x_0$ and $y_0$, it is enough to see that $gf$ is loxodromic. Replace $f$ by $gf$ to get a M.G.S. If $f$ satisfies case 2), then Lemma 2.24 implies that $g^m f$ or $fg^{-m}$ is loxodromic for some large $m$, and so replace $f$ by loxodromic $g^m$ or $fg^{-m}$. If case 3) is hold, then $hf(x_0) = h(y_0) \neq y_0$. Thus $g^m hf$ is loxodromic by Lemma 2.24 again. Replace $f$ by $g^m hf$ for some large $m$. Combining the replacements mentioned above, we get a desired M.G.S. $Z$.

**Proposition 2.26** *If a* M.G.S. $Y$ *of a non-elementary subgroup* $G \subset$Isom$(X)$ *contains a parabolic element, then there is a* M.G.S. $Z$ *contains only loxodromic elements.*

**Proof** Let $g \in Y$ be a parabolic isometry with a unique fixed point $\alpha \in X(\infty)$. Since $G$ is non-elementary, there exists an element $h \in Y$ such that $h\alpha \neq \alpha$. By Theorem 2.12 we have

$$\lim_{j\to\infty} g^{m_j} h\alpha = \alpha \quad \text{and} \quad \lim_{j\to\infty} (g^{m_j} h)^{-1}\alpha = h^{-1}\alpha$$

c-uniformly in $\overline{X} \setminus \{h^{-1}(\alpha)\}$ and $X \setminus \{\alpha\}$, respectively. From Corollary 2.6 and $\alpha \neq h^{-1}(\alpha)$ it is easy to see that $g^{m_j}h$ is loxodromic for some large $j$. Replace $h$ by $g^{m_j}h$ and apply Proposition 2.25 to get a desired M.G.S $Z$.

# 3 Discreteness

In this chapter, we extend some classical results concerning discreteness of a group of isometries in three dimensional hyperbolic space, $H^3$, to the group of isometries of a pinched Hadamard manifold.

## 3.1 Discreteness Criteria

We have known that the discreteness of groups of isometries of hyperbolic space is a fundamental notion and has been investigated by many authors. In two dimensional Möbius groups, there is a famous criterion of discreteness given by Jørgensen:

**Theorem 3.1** *A non-elementary group $G$ of $SL(2, \mathbb{C})$ is discrete if and only if every two-generator subgroup of $G$ is discrete.*

In 1990, Isachenko [Is] strengthened Theorem 3.1 as follows: A nonelementary subgroup of $\mathrm{SL}(2, \mathbb{C})$ is discrete if and only if every subgroup generated by two loxodromic elements is discrete. In 1989 and 1993, Martin [Ma1] and Waterman [Wa1] extended Theorem 3.1 for a finitely generated nonelementary subgroup $G$ to higher dimensional hyperbolic space. If the dimension of a hyperbolic space is larger than three, Abikoff and Haas in [AH] constructed a nonelementary nondiscrete subgroup $\Gamma$ of $\mathrm{Isom}(H^n)$ with the property that every finitely generated subgroup of $\Gamma$ is discrete. Thus it is necessary to add some condition to $G$. In 1993, Martin [Ma2] further extended Theorem 3.1 to negatively curved group under the condition of bounded torsion.

Let $\{G_i\}$ be a family of groups. We say that $\{G_i\}$ has *uniformly bounded torsion* if there is an integer $N$ such that for all $i$ and $g \in G_i$, either $\mathrm{ord}(g)=\infty$ or $\mathrm{ord}(g)\leq N$.

We introduce the following Condition A to weaken the condition of uniformly bounded torsion.

**Condition A** *Let $\{G_i\}$ be a family of subgroups of $\mathrm{Isom}(X)$. We say that $\{G_i\}$ has Condition A if there exists no sequences $\{g_j\}$ of distinct elements of finite order such that $\mathrm{Card}(\mathrm{fix}(g_j)) \geq 3$ in $X(\infty)$ and $g_j \to I$, where*

$g_j \in \cup_{i=1}^{\infty} G_i$.

Remark: In upper definition, if $G_i = G$ for all $i \in \mathbb{N}$, we say that the group $G$ of Isom$(X)$ has Condition A.

There are three cases that improve the uniformly bounded torsion;

1) Uniformly bounded torsion group certainly satisfies Condition A. It is clear that the group with Condition A may be not a uniformly bounded torsion group.

2) Let $g_i \in G_i$ and $g_i \to I$. If $\{G_i\}$ has uniformly bounded torsion, then $g_i$ must not be elliptic [BGS, p126]; But if $\{G_i\}$ has Condition A, then $g_i$ may be elliptic of infinite order or elliptic of finite order with Card(fix$(g_i)$) $\leq 2$.

3) Let $g_i \in G_i$ and $\{g_i\}$ converge to a non-elliptic element. If $\{G_i\}$ has uniformly bounded torsion, then $g_i$ is not elliptic [Ma2]. However, if $\{G_i\}$ has Condition A, then $g_i$ maybe elliptic.

From the cases mentioned above, it is clear that in discussing the discreteness and convergence of $\{G_i\}$ with Condition A, we will face some difficult problems about elliptic elements. Thus our methods of proofs are different from those of Martin's.

Since any element in Isom$(H^3)$ has at most two fixed points on the boundary of $H^3$, Condition A is naturally satisfied in 3-dimensional hyperbolic spaces. Thus our results (including those in next chapter) completely coincide with the classical cases in three dimensional hyperbolic spaces.

For a loxodromic Möbius transformation $g$, there is a neighbourhood of $g$ such that each element in the neighbourhood is not elliptic. We will show that for each loxodromic isometry in Isom$(X)$, there is a neighbourhood such that all isometries in it are loxodromic. This gives another tool for the study of discreteness and convergence.

**Lemma 3.2** *Let $\{g_j\}$ be a sequence of isometries of $X$. If $g = \lim_{j\to\infty} g_j$ is loxodromic, then eventually all the elements $g_j$ are loxodromic.*

**Proof**. Let fix$(g)$ =$\{p, q\}$, where $p \neq q$. Then by Theorem 2.12 we have

$$\lim_{m\to\infty} g^m(x) = p \qquad \text{and} \qquad \lim_{m\to\infty} g^{-m}(x) = q$$

c-uniformly in $\overline{X} \setminus \{q\}$ and $\overline{X} \setminus \{p\}$, respectively. Therefore there are neighbourhoods $U, V$ of $p$ with $V \subset U$ and $q \notin \overline{U}$ such that $g^m(\overline{U}) \subset V$ for some sufficiently large $m$. We may let $U = C_z(p, \varepsilon)$ and $V = C_z(p, \varepsilon/2)$ be cones containing $p$ for some vertex $z \in X$. Thus for any $x \in \overline{U}$ we have $\measuredangle_z(g^m(x), p) < \varepsilon/2$. As $g_j$ converges to $g$, it follows that $\lim_{j\to\infty} g_j^m(x) = g^m(x)$ uniformly in $\overline{X}$. Therefore there is an integer $N$ such that $\measuredangle_z(g_j^m(x)) < \varepsilon/2$ for any $x \in \overline{U}$ and all $j > N$. Thus we have

$$\measuredangle_z(g_j^m(x), p) \leq \measuredangle_z(g_j^m(x), g^m(x)) + \measuredangle_z(g^m(x), p) < \varepsilon/2 + \varepsilon/2.$$

It implies that $g_j^m(\overline{U}) \subset U$. By Lemma 2.5, it follows that $g_j^m$ is loxodromic and so $g_j$ is loxodromic when $j > N$. The proof of lemma is complete.

**Lemma 3.3** *Let $\{g_j\}$ be a sequence of distinct isometries of $X$ converging to the identity and let $x_j$ and $y_j$ be sequences in $\overline{X}$ that converge to distinct points $x_0$ and $y_0$ in $X(\infty)$, respectively. Then there are at most finitely many $g_j$ interchanging the two distinct points $x_j, y_j$.*

**Proof.** Without loss of generality, we may assume that $g_j(x_j) = y_j$ and $g_j(y_j) = x_j$ for all $j$. For any $\varepsilon > 0$, since $g_j$ converges to the identity, we have that $\lim_{j\to\infty} g_j(x) = x$ uniformly in $\overline{X}$ and then $\measuredangle_p(g_j(x), x) < \varepsilon/2$ for some $p \in X$ and any $x \in \overline{V}$. Thus $\measuredangle_p(g_j(x_j), x_j) < \varepsilon/2$. On the other hand, the limit $x_j \to x_0$ as $j \to \infty$ implies that $\measuredangle_p(x_j, x_0) < \varepsilon/2$ for sufficiently large $j$. It follows that

$$\measuredangle_p(g_j(x_j), x_0) \leq \measuredangle_p(g_j(x_j), x_j) + \measuredangle_p(x_j, x_0) < \varepsilon.$$

Thus $\lim_{j\to\infty} g_j(x_j) = x_0$. it contradicts the assumption that $\lim_{j\to\infty} g_j(x_j) = \lim_{j\to\infty} y_j = y_0$. The proof of lemma is completed.

We next recall some basic concepts about nilpotent groups.

A group $G$ is called to be *nilpotent* if, for some $m$, all $m$-fold commutators in $G$ are trivial. The smallest such $m$ is called the *class of G.* We define normal subgroups, $\Gamma_i$, of $G$ inductively as follows.

$$\Gamma_0 = \{I\}, \quad \Gamma_{i+1} = \{g \in G | ghg^{-1}h^{-1} \in \Gamma_i \text{ for all } h \in G\}.$$

This gives the *upper central series*

$$\{I\} = \Gamma_0 \lhd \Gamma_1 \lhd \Gamma_2 \lhd \cdots \lhd \Gamma_m = G,$$

where $m = \text{class}G$. It is easy to see that any subgroup of a nilpotent group is nilpotent.

A very important tool in the study of discrete groups acting on Hadamard manifolds is the following Margulis Lemma. This result can be found in [BGS, p101] or [Bo2, p565].

**Lemma 3.4** (Margulis Lemma) *Given $n \in N$ there are constant $\mu = \mu(n) > 0$ and $I(n) \in N$ with the following property: let $X$ be an $n$-dimensional Hadamard manifold which satisfies the curvature condition $-1 \leq K \leq 0$ and let $\Gamma$ be a discrete group of isometries acting on $X$. For $x \in X$ let $\Gamma_\mu(x) := \langle\{\gamma \in \Gamma | d_\gamma(x) \leq \mu\}\rangle$ be the subgroup generated by the elements $\gamma$ with $d_\gamma(x) \leq \mu$. Then $\Gamma_\mu(x)$ is virtually nilpotent, thus it contains a nilpotent subgroup of finite index. The index is bounded by $I(n)$.*

**Proposition 3.5** [Bo2, Proposition 3.1.1] *Any virtually nilpotent group of isometries of $X$ is elementary.*

Let $G$ be a group of isometries of $X$. i). We call $G$ *loxodromic* if it contains a loxodromic element and preserves setwise its axis. ii). We call $G$ *parabolic* if $\text{fix}(G)$ consists of a single point $x_0$ in $X(\infty)$, and if $G$ preserves setwise some (and hence every) horosphere about $x_0$.

It is obviously that for a loxodromic group, either $\text{fix}(G) = \{x, y\}$, where $x$ and $y$ are the end points of the loxodromic axis, or else there is some element of $G$ which swaps $x$ and $y$, in which case $\text{fix}(G) = \emptyset$. Let $G_{\{x,y\}} := \{g \in G : g\{x, y\} = \{x, y\}\}$. If $G_{\{x,y\}}$ contains a loxodromic isometry, then it is a loxodromic subgroup of $G$ and $x, y$ are the loxodromic fixed points. Thus in a non-elementary group there are infinitely many distinct loxodromic subgroups.

**Theorem 3.6** *Suppose that $G$ is a non-elementary subgroup of* $\text{Isom}(X)$ *and contains a loxodromic subgroup $G_{\{x,y\}}$ with Condition A. Then $G$ is discrete if and only if every two loxodromic generator subgroup is discrete.*

**Proof.** The necessity is obvious. We now prove the sufficiency. Assume that $G$ is non-discrete and every two loxodromic generator subgroup is discrete. Then there is a sequence $\{g_j\}$ in $G$ such that $g_j \to I$ as $j \to \infty$. In the following we prove that it is impossible.

By hypothesis that $G_{\{x,y\}}$ is loxodromic, we let $h$ in $G_{\{x,y\}}$ be a loxodromic element with fixed points $x, y$. As $\lim\limits_{j\to\infty} g_j h = h$, according to Lemma 3.2, it follows that $g_jh$ is loxodromic for sufficiently large $j$. Therefore $\langle g_jh, h\rangle = \langle g_j, h\rangle$ is discrete by our assumption. Since $g_j \to I$ as $j \to \infty$, it follows that $h^i g_j h^{-i} \to I$ for any integer $i$. Let $\mu = \mu(n)$ be a Margulis constant. we may get

$$d_{h^i g_j h^{-i}}(x) \leq \mu$$

where $i = 0, 1, \cdots, p+1$ and $p =$dim(fix$(g_j)$). If $g$ is parabolic or loxodromic, then set $p = 0$. It is easy to see that $\langle h^i g_j h^{-1} : i = 0, 1, \cdots, p+1\rangle$ is a subgroup of group $\langle g_j, h\rangle_\mu(x)$. Since any subgroup of the virtually nilpotent group is also virtually nilpotent. By Margulis Lemma [BGS, p101] $\langle g_j, h\rangle_\mu(x)$ is a virtually nilpotent group. So $\langle h^i g_j h^{-i} : i = 0, 1, \ldots, p+1\rangle$ is virtually nilpotent also and thus is elementary by Proposition 3.4. It follows from Lemmas 2.2 and 2.3 in [Ma2] that $\langle g_j, h\rangle$ is elementary. According to the classification of discrete elementary group, there are two cases of the group:

1). $g_j$ is loxodromic or elliptic and stabilizes pointwise the set of fixed points of $h$.

2). $g_j$ is elliptic and interchanges the two fixed points of $h$.

By means of Lemma 3.3 and $g_j \to I$, it is impossible that $g_j$ interchanges the fixed points of $h$ for sufficiently large $j$. Then only first case may occur and thus $g_j \in G_{\{x,y\}}$.

As $G$ is non-elementary, by Theorem 2.20 there exists another loxodromic isometry $f$ such that fix$(h)\cap$fix$(f) = \emptyset$. The reason mentioned above implies that $\langle g_j, f\rangle$ is also discrete elementary for sufficiently large $j$. Then $g_j$ stabilizes pointwise the set of fixed points of $f$. Since $\langle g_j, h\rangle$ is discrete, it follows that $g_j$ is not infinite order of elliptic isometry. In addition, since $G_{\{x,y\}}$ has Condition $A$ and $g_j \in G_{\{x,y\}}$, we have Card(fix$(g_j)$)= 2 for large $j$. Therefore fix$(f)$=fix$(h)$. This contradicts that $f$ and $h$ have not a common fixed point. Thus $G$ is discrete.

The following corollary is a generalization of the result of Isachenko [Is]. In the 3-dimensional hyperbolic space $H^3$, the Condition A is naturally satisfied. Thus, in $X = H^3$, our result completely coincides with that of Isachenko.

**Corollary 3.7** *A non-elementary subgroup $G$ of* Isom$(X)$ *with Condition $A$ is discrete if and only if every two loxodromic generator subgroup is discrete.*

**Proof.** Since $G$ is non-elementary, it follows from Theorem 2.20 that there definitely exists a loxodromic isometry $g$ in $G$. Let $x_0, y_0$ be the fixed points of $g$. If $G$ has Condition A, it is easy to see that $G_{\{x_0,y_0\}}$ has Condition A also. Thus Theorem 1 can be applied.

For a subgroup $G$ of $\mathrm{Isom}(X)$, it is not hard to see that if every two generator subgroup of $G$ is discrete, then arbitrary two loxodromic generator subgroup is discrete. Thus the following corollary improves the result of Martin [Ma2, Corollary 3.4] in group of isometries of pinched Hadamard manifold and that of [Ma1, Theorem 5.6] in n-dimensional Möbius group. In addition, in the three dimensional hyperbolic space, $H^3$, the corollary coincides with Theorem 3.1 [Jø].

**Corollary 3.8** *A non-elementary subgroup of* $\mathrm{Isom}(X)$ *with Condition A is discrete if and only if every two generator subgroup is discrete.*

## 3.2 A Generalization of a Jørgensen's Problem

The following well known result is due to Jørgensen in [Jø2].

**Proposition 3.9** *Every nonelementary and nondiscrete subgroup of* $SL(2, \mathbb{R})$ *has a subgroup generated by two elements, which is non-elementary and non-discrete.*

In his paper, Jørgensen pointed out that although $SL(2, \mathbb{C})$ has slightly more complicated subgroups, one might still ask whether a similar proposition could be true. Abikoff and Haas[AH] in 1990 and Waterman[Wa, Theorem 13] in 1993 generalized the proposition to n-dimensional hyperbolic isometric groups and Möbius transformation groups under some condition, respectively.

**Theorem 3.10** ([AH, Wa1]) *Let* $\Gamma$ *be a nonelementary group of isometries of* $H^n$ *(or Möbius transformations of* $\overline{\mathbb{R}}^{n-1}$*) with a sequence of distinct parabolic or loxodromic elements* $g_m \to I$*. Then* $\Gamma$ *contains a non-discrete, non-elementary two generator subgroup.*

The theorem is an important progress in $n$-dimensional isometric groups and Möbius groups. We will generalize the problem to the group of isometries of pinched Hadamard manifolds. On the other hand, the sequence $\{g_m\}$ possibly is the elliptic sequence in our next theorem.

**Theorem 3.11** *Assume that $G \subset \mathrm{Isom}(X)$ is a non-elementary subgroup and there is a sequence of distinct isometries $\{g_j\}$ in $G$ with $\mathrm{Card}(\mathrm{fix}(g_j)) \leq 2$ in $X(\infty)$, which converges to $I$. Then $G$ contains a non-discrete and non-elementary two loxodromic generator subgroup.*

**Proof.** Since $G$ is non-elementary, by Theorem 2.20 there exist loxodromic isometries $h_1, h_2$ and $h_3$ in $G$, no two of which share a common fixed points. We first show that $\langle g_j, h_k\rangle$ $(k = 1, 2, 3)$ is either discrete or non-elementary for all large $j$. Assume, without loss of generality, that $\langle g_j, h_1\rangle$ is discrete. Then we will prove that $\langle g_j, h_1\rangle$ is elementary. The hypothesis that $g_j \to I$ implies that $h_1 g_j h_1^{-1} \to I$ as $j \to \infty$. Then, for all sufficiently large $j$, we have

$$\sum_{i=0}^{p} d_{h_1^i g_j h_1^{-i}}(x) \leq \mu$$

where $\mu = \mu(n)$ is the Margulis constant and $p = \dim(\mathrm{fix}(g_j))$. By Margulis Lemma and above-mentioned inequality, it follows that

$$\langle h_1^i g_j h_1^{-i} : i = 0, 1, 2, \cdots, p+1\rangle$$

is virtually nilpotent and then is elementary. According to Lemma 2.3 in [Ma2] and $g_j \to I$ as $j \to \infty$, it is easy to see that $\langle g_j, h_1\rangle$ is elementary.

Next, we show that at least one of the three groups $\langle g_j, h_k\rangle$ $(k = 1, 2, 3)$ is nonelementary. Suppose, for contradiction, that the three groups are elementary. Since all the three isometries $h_k$ are loxodromic, it follows by Theorem 2.18 that either $L(\langle g_j, h_k\rangle)$ contains exactly two points or $\mathrm{fix}(\langle g_j, h_k\rangle) = \{x_0\}$, where $x_0 \in X(\infty)$. In former case, by Theorem 2.9 we have that $g_j$ fixes or interchanges the fixed points of $h_k$. Since $g_j$ converges to $I$, it follows from Lemma 3.3 that $g_j$ must stabilizes pointwise the fixed points of $h_k$. By condition that $\mathrm{Card}(\mathrm{fix}(g_j)) \leq 2$ in $X(\infty)$ we have $g_j$ have exactly two fixed points in $X(\infty)$. In later case, it is easy to see by Theorem 2.4 that $g_j$ fixes one of the two fixed points of $h_k$ and has only two fixed points in $X(\infty)$. Thus it is enough to see that at least two of $h_1, h_2, h_3$ have a fixed point in common. This contradicts that no two of $h_k$ $(k = 1, 2, 3)$ share a common fixed point.

At last, without loss of generality, we let $\langle g_j, h_1\rangle$ is non-elementary. By above-mentioned argument, it is non-discrete. Since $g_j h_1 \to h_1$ as $j \to \infty$, by Lemma 3.2 we have that $g_j h_1$ is loxodromic for sufficiently large $j$. Thus the group $\langle g_j, h_1\rangle = \langle g_j h_1, h_1\rangle$ is generated by two loxodromic isometries.

This completes the proof of this theorem.

The following corollary not only establishes Jørgensen's problem and slightly strengthens it with replacing two generator subgroup by two loxodromic generator subgroup.

**Corollary 3.12** *Every non-elementary and non-discrete subgroup of* $\mathrm{Isom}(H^3)$ *has a subgroup generated by two loxodromic elements, which is non-elementary and non-discrete.*

**Proof.** Since every isometry $g \in G$ has at most two fixed points and at least one fixed point on the boundary $\overline{\mathbb{R}}^2$ of $H^3$, it follows that $1 \leq \mathrm{Card}(\mathrm{fix}(g)) \leq 2$ in $\overline{\mathbb{R}}^2$. Thus Theorem 3.11 in the hyperbolic space case implies the corollary.

# 4 Convergence and Algebraic Convergence

William Thurston wrote in his famous lecture notes ([Th, Chapter 9.1]) that it is important for us to develop an understanding of the geometry of deformations of a given discrete group. A qualitative understanding can be attained most concretely by considering limits of sequences of groups. The situation is complicated by the fact that there is more than one reasonable sense in which a group can be the limit of a sequence of discrete groups. However, the limit may not be understood in the same sense. Heinz Helling and Troels Jørgensen were the first one to analyze and understand this phenomenon. In this chapter, we generalize the limit theorem and algebraic convergence theorems of Helling's [Hel] and Jørgensen's [Jø1 and 2] to groups of isometries of pinched Hadamard manifolds under Condition A. It is not hard to see that some assumption is necessary. We recall that even in the hyperbolic case, for dimensions greater than three, the limit of finitely generated discrete nonelementary groups may not be discrete (even if the limit is assumed to be nonelementary) [Ma1]. The problem arises as there may be elliptics of high order converging to an irrational rotation about a codimension two or large subspace stabilized by a nonelementary discrete subgroup. Thus Martin introduce the notion of uniformly bounded torsion. That is, in a sequence of discrete groups with uniformly bounded torsion, the order of every elliptic element is bounded by some constant. We use Condition A to weaken the assumption of uniformly bounded torsion such that the set of the orders of elliptic elements may be unbounded.

## 4.1 Convergence Theorem

The following convergence theorem is a generalization of Theorem 3.3 in [Ma2], and in three dimensional hyperbolic spaces it coincides with Proposition 1 in [Jø1].

**Theorem 4.1** *Let* $\Gamma$ *be a non-elementary subgroup of* $\mathrm{Isom}(X)$ *and* $\psi_m$ *a sequence of mappings from* $\Gamma$ *into* $\mathrm{Isom}(X)$*, such that for* $m = 1, 2, \cdots$*, the group* $\psi_m(\Gamma)$ *is discrete. If* $\{\psi_m(\Gamma)\}$ *has Condition A and* $\psi_m(\gamma) \to \gamma (m \to \infty)$ *for each* $\gamma \in \Gamma$*, then* $\Gamma$ *is discrete.*

**Proof.** Suppose that $\Gamma$ is not discrete. Then there is a sequence of elements $\{\gamma_j\}$ in $\Gamma$ such that $\gamma_j \to I$ as $j \to \infty$. From the hypothesis and

Theorem 2.20 there are two loxodromic isometries $h_1$ and $h_2$ in $\Gamma$ which have no common fixed points. Then we have $\psi_m(h_i) \to h_i$ as $m \to \infty$ with $i = 1, 2$ and $\psi_m(\gamma_j) \to I$ as $m, j \to \infty$. It follows by continuity that

$$d_{\psi_m^k(h_i)\psi_m(\gamma_j)\psi_m^{-k}(h_i)}(x) \leq \mu$$

$(k = 0, 1, \cdots, p+1)$ for sufficiently large $m$ and $j$. As $\psi_m(\Gamma)$ is discrete, by Margulis Lemma, the group

$$\langle \psi_m(\gamma_j), \psi_m(h_i)\psi_m(\gamma_j)\psi_m^{-1}(h_i), \ldots, \psi_m^{p+1}(h_i)\psi_m(\gamma_j)\psi_m^{-(p+1)}(h_i) \rangle$$

is virtually nilpotent and so is elementary [Bo, Proposition3.11]. According to [Mar, Lemmas 2.2 and 2.3], $\langle \psi_m(\gamma_j), \psi_m(h_i) \rangle$ is discrete and elementary. Since $\psi_m(\Gamma)$ is discrete, every elliptic isometry in $\psi_m(\Gamma)$ is of finite order. By the Condition $A$ and $\psi_m(\gamma_j) \to I$ as $m, j \to \infty$, for sufficiently large $m$ and $j$, $\psi_m(\gamma_j)$ is parabolic, loxodromic and elliptic of finite order with $\mathrm{Card}(\mathrm{fix}(\psi_m(\gamma_j))) \leq 2$ in $X(\infty)$. As $h_i$ is loxodromic and $\psi_m(h_i) \to h_i$ as $m \to \infty$, $\psi_m(h_i)$ is loxodromic for all sufficiently large $m$ by Lemma 3.2. Thus $\psi_m(\gamma_j)$ either stabilizes pointwise the set of fixed points of $\psi_m(h_i)$ or interchanges the two fixed points of $\psi_m(h_i)$. Since $\psi_m(\gamma_j) \to I$, by Lemma 3.3, we have known that it is impossible that $\psi_m(\gamma_j)$ interchange the fixed points of $\psi_m(h_i)$ for larger $m$ and $j$. Therefore $\mathrm{Card}(\mathrm{fix}(\psi_m(\gamma_j))) = 2$ in $X(\infty)$ and thus $\psi_m(h_1)$ and $\psi_m(h_2)$ share the two common fixed points. Taking the limit, it implies that $h_1$ and $h_2$ both have the same fixed points. This is a contradiction.

## 4.2 Algebraic Convergence

**Lemma 4.2** *Suppose that $\{g_j\}$ is a sequence of loxodromic isometries of $X$ which converges to a loxodromic isometry $g$ of $X$. Then for the fixed points $x_0$, $y_0$ of $g$, there exist the fixed points $x_j$, $y_j$ of $g_j$ such that $x_j \to x_0$, $y_j \to y_0$ as $j \to \infty$.*

**proof.** For any $\varepsilon > 0$, choose a neighbourhood of $x_0$ be a cone $U = C_z(x_0, \varepsilon)$. Since $g$ is loxodromic, by Theorem 2.12, we have $\lim\limits_{m\to\infty} g^m(x) = x_0$, c-uniformly in $\overline{X} \setminus \{y_0\}$. So there is an integer $m$ such that $g^m(\overline{U}) \subset V$, where $V = C_z(x_0, \varepsilon/2)$. That is, for any $x \in \overline{U}$,

$$\measuredangle_z(g^m(x), x_0) < \varepsilon/2.$$

According to the assumption that $\lim_{j\to\infty} g_j = g$, it follows that $g_j^m(x) \to g^m(x)$ as $j \to \infty$ uniformly in $\overline{X}$. Then there is an integer $N$ such that

$$\measuredangle_z(g_j^m(x),\ g^m(x)) < \varepsilon/2$$

for any $x \in \overline{U}$ and all $j > N$. Thus we have

$$\measuredangle_z(g_j^m(x),\ x_0) \leq \measuredangle_z(g_j^m(x),\ g^m(x)) + \measuredangle_z(g^m(x),\ x_0) < \varepsilon.$$

It implies that $g_j^m(\overline{U}) \subset U$ and so $g_j^m$ $(j > N)$, and hence $g_j$, is loxodromic with a fixed point in $U$ by Corollary 2.13. Let the fixed point of $g_j$ in $U$ be $x_j$. Then $x_j$ converges to $x_0$. Similarly, we may prove that the other fixed point $y_j$ of $g_j$ converges to $y_0$.

**Definition 4.3** *Let $\{G_m\}$ be a sequence of subgroups of* Isom$(X)$ *each with the same finite number of generators, say $\{g_{m,1}, g_{m,2}, \ldots, g_{m,r}\}$. If for each $i = 1, 2, \ldots, r$ there is an isometry $g_i$ such that*

$$g_{m,i} \to g_i \quad \textit{as} \quad m \to \infty$$

*then we say that the groups $G_m$ converges algebraically to the group $G = \langle g_1, g_2, \ldots, g_r \rangle$.*

In 1972, Helling [Hel] proved that if a sequence of $r$-generator discrete nonelementary groups $G_m$ in SL(2,$\mathbb{R}$) converges algebraically to a group $G$ then $G$ is also discrete nonelementary. In 1977, Jørgensen[Jø2] extended the convergence property to SL(2, $\mathbb{C}$). For high dimensional Möbius groups and groups of isometries of a pinched Hadamard manifold, Martin[Ma1 and Ma2] obtained analogous property under the condition that $\{G_m\}$ has uniformly bounded torsion. We have the following algebraic convergence results.

**Theorem 4.4** *Let $\{G_m\} = \{\langle g_{m,1}, g_{m,2}, \ldots, g_{m,r} \rangle\}$ be a sequence of discrete and non-elementary subgroups of* Isom$(X)$ *which algebraically converges to $G = \langle g_1,\ g_2, \ldots, g_r \rangle$. If there exists $h_m \in G_m$ which is not of finite order and converges to $h \in G$, then the following is true.*

*(1). $h$ is loxodromic or parabolic;*

*(2). If $\{G_m\}$ has Condition A, then $G$ is discrete non-elementary.*

(3). *If $\{G_m\}$ has Condition A and $G$ is finitely presented, then there exists a number $m_0 \in \mathbb{N}$ such that for each $m \geq m_0$, the mapping*

$$\psi_m : g_i \longrightarrow g_{m,i} \quad (i = 1, 2, \cdots, r)$$

*can be extended to a homomorphism of $G$ onto $G_m$.*

**Proof.** (1). We have known that isometries of infinity order in discrete subgroups of $\mathrm{Isom}(X)$ are loxodromic or parabolic. Then all $h_m$ are parabolic or loxodromic. We suppose, for contradiction, that $h$ is neither loxodromic nor parabolic. Then $h$ is elliptic or the identity. Let $\langle h \rangle$ be the subgroup generated by $h$. If $\langle h \rangle$ is discrete, then $h$ is of finite order. If $\langle h \rangle$ is non-discrete, then there is a sequence of distinct integers $k_j$ such that $\lim_{j \to \infty} h^{k_j} = I$. Hence there is an integer $k$ such that

$$0 \leq max\{d_{h^k}(x), d_{g_i^{-1}h^k g_i}(x) : i = 1, 2, \ldots, r\} < \mu,$$

where $\mu$ is the Margulis constant. Since $\lim_{m \to \infty} h_m = h$ and $\lim_{m \to \infty} g_{m,i} = g_i$, we have

$$0 \leq max\{d_{h_m^k}(x), d_{g_{m,i}^{-1}h_m^k g_{m,i}}(x) : i = 1, 2, \ldots, r\} < \mu$$

for all sufficiently large $m$. It is easy to see that $\langle h_m^k, g_{m,i}^{-1}h_m^k g_{m,i} \rangle$, being a subgroup of $G_m$, is discrete. By Margulis Lemma, it is virtually nilpotent and thus elementary for each $i$. Since $h_j$ is not of finite order, it follows from Lemma 2.2 of [Ma2] that $\langle h_m^k, g_{m,i} \rangle$ is elementary. By means of the classification of discrete and elementary groups, it is not hard to see that if $h_m$ is parabolic ,then $g_{m,i}$ stabilize the fixed point of $h_m$, and if $h_m$ is loxodromic, then $g_{m,i}$ stabilize the set of fixed points of $h_m$ for all $i = 1, 2, \ldots, r$ which says that $G_m$ is elementary. This contradicts the assumption that $G_m$ is a non-elementary group. Thus $h$ is loxodromic or parabolic.

(2). We now show that $G$ is discrete. If not, there is a sequence $\{\gamma_n\}$ in $G$ such that $\gamma_n \to I$ as $n \to \infty$. Our aim is to reach a contradiction.

Let $\gamma_{m,n}$ in $G_m$ be corresponding word. Therefore $\lim_{m \to \infty} \gamma_{m,n} = \gamma_n$. By passing to a subsequence of $\{\gamma_{m,n}\}$ we may obtain that $\gamma_{m_i,n_i} \to I$ as $i \to \infty$. Since $G_m$ is discrete and $\{G_m\}$ has Condition A, it follows that $\gamma_{m_i,n_i}$ is not of elliptic isometry of infinite order and Card(fix($\gamma_{m_i,n_i}$))$\leq 2$ in $X(\infty)$ for sufficiently large $i$. For the sake of convenience, we denote $\gamma_{m_i,n_i}$ by $\gamma_{m,n}$ below. Then for sufficiently large $m$ and $n$, we have

$$d_{h_m^i\gamma_{m,n}h_m^{-i}}(x) \;+\; d_{(g_{m,j}^{-1}h_m g_{m,j})^i\gamma_{m,n}(g_{m,j}^{-1}h_m g_{m,j})^{-i}}(x) + \\ + d_{(g_{m,j}h_m g_{m,j}^{-1})^i\gamma_{m,n}(g_{m,j}h_m g_{m,j}^{-1})^{-i}}(x) < \mu \qquad (4.7)$$

where $i = 0, 1, 2, \ldots, p$. Being the subgroups of discrete group $G_m$, then $\langle\gamma_{m,n}, h_m^i\gamma_{m,n}h_m^{-i} : i = 1, 2, \ldots, p\rangle$, $\langle\gamma_{m,n}, (g_{m,j}^{-1}h_m g_{m,j})^i\gamma_{m,n}(g_{m,j}^{-1}h_m g_{m,j})^{-i} : i = 1, 2 \ldots, p\rangle$ and $\langle\gamma_{m,n}, (g_{m,j}h_m g_{m,j}^{-1})^i\gamma_{m,n}(g_{m,j}h_m g_{m,j}^{-1})^{-i} : i = 1, 2, \ldots, p\rangle$ are discrete and so elementary by Margulis Lemma and [Bo2, Proposition 3.1.1]. Thus all the three subgroups

$$\langle\gamma_{m,n}, h_m\rangle,\ \langle\gamma_{m,n}, g_{m,j}h_m g_{m,j}^{-1}\rangle \text{ and } \langle\gamma_{m,n}, g_{m,j}^{-1}h_m g_{m,j}\rangle$$

are elementary. According to the assumption, $h_m$ are parabolic or loxodromic. Our purpose is to show that $G_m$ is elementary and then obtains a contradiction.

We first let $h_m$ be parabolic with fixed point $p$. Then, by the classification of discrete and elementary groups, $\gamma_{m,n}$ is parabolic or elliptic and fixes the fixed point of $h_m, g_{m,j}^{-1}h_m g_{m,j}$ and $g_{m,j}h_m g_{m,j}^{-1}$. For parabolic $\gamma_{m,n}$, we have that $h_m$ and $g_{m,j}^{-1}h_m g_{m,j}$ share a fixed point; that is, $h_m(p) = g_{m,j}^{-1}h_m g_{m,j}(p) = p$. So $h_m$ and $g_{m,j}$ have a common fixed point. For elliptic $\gamma_{m,n}$, it is easy to see that it has exactly two fixed points, say $x_0$ and $y_0$, and $\gamma_{m,n}$ and each of the three parabolic isometries have a common fixed point. We let $x_0 = p$. If $x_0$ is the fixed point of $g_{m,j}^{-1}h_m g_{m,j}$ or $g_{m,j}h_m g_{m,j}^{-1}$, then $h_m$ and $g_{m,j}$ have a common fixed point. Otherwise, $g_{m,j}^{-1}h_m g_{m,j}(y_0) = g_{m,j}h_m g_{m,j}^{-1}(y_0) = y_0$. Then $g_{m,j}(y_0)$ and $g_{m,j}^{-1}(y_0)$ are the fixed points of $h_m$. So $g_{m,j}$ stabilizes the set $\{x_0, y_0\}$, which means that $G_m$ is elementary.

We next let $h_m$ be loxodromic with fixed points $x_m$ and $y_m$. Then$\langle\gamma_{m,n}, h_m\rangle$ is loxodromic group, and so $\gamma_{m,n}$ is loxodromic or elliptic and stabilizes the set of fixed points of $h_m$. That is

$$\gamma_{m,n}(x_m) = x_m \qquad \text{and} \qquad \gamma_{m,n}(y_m) = y_m$$

or

$$\gamma_{m,n}(x_m) = y_m \qquad \text{and} \qquad \gamma_{m,n}(y_m) = x_m$$

It follows from Lemma 3.3 and Lemma 4.2 that the second case is not held for sufficiently large $n, m$ since $\gamma_{m,n} \to I$. For the first case, as $\langle\gamma_{m,n}, g_{m,j}h_m g_{m,j}^{-1}\rangle$ is also elementary, $\gamma_{m,n}$ stabilizes pointwise $\{g_{m,j}(x_m), g_{m,j}(y_m)\}$, the set of fixed points of $g_{m,j}h_m g_{m,j}^{-1}$. Thus $g_{m,j}\{x_m, y_m\} = \{x_m, y_m\}$. According to above-mentioned reason, we have $G\{x_m, y_m\} = \{x_m, y_m\}$, and thus $G_m$ is elementary. This is a contradiction.

Finally, we show that $G$ is non-elementary. For contradiction, we suppose that $G$ is elementary. Since $h_m$ is not elliptic, by the case 1), $h$ is not elliptic, either. If $h$ is loxodromic, then $g_j h g_j^{-1} = f_j$ is also loxodromic for each $j$ and they generate a discrete elementary group. Then, by Proposition 6.9 in [EO], there are integers $s$ and $t$ such that $f_j^t = h^s$ or $f_j^{-t} h^s = I$. It follows from algebraic convergence that $\lim_{m\to\infty} f_{m,j}^{-t} h_m^s = I$ where $f_{m,j} = g_{m,j} h_m g_{m,j}^{-1}$. For large $m$ we have that $\langle h_m, f_{m,j}^{-t} h_m^s \rangle$ is elementary by Margulis Lemma. It is easy to see that $f_{m,j}$ and $h_m$ have the same common fixed points, and so $g_{m,j}$ stabilizes the set of fixed points of $h_m$. It follows that $G_m$ is an elementary group, a contradiction. If $h$ is parabolic, then $\langle g_j h g_j^{-1}, h\rangle$ is a discrete parabolic group. By [Bo1,Theorem 4.1] or [Bo2, Proposition 4.2], a discrete parabolic group is finitely generated and virtually nilpotent. Let $f_j = g_j h g_j^{-1}$ for each $j$. Therefore there are integers $s$ and $t$ such that $f_j^s$ and $h^t$ generate a nilpotent group. Thus there is a number $d$ such that any $d$-fold commutator of $f_j^s$ and $h^t$ is trivial. Let $\gamma = [\quad , \quad]_d = I$ be any $d$-fold commutator and $\gamma_m$ be corresponding word in $\langle f_{m,j}, h_m \rangle$, where $f_{m,j} = g_{m,j} h_m g_{m,j}^{-1}$ . Thus $\gamma_m \to \gamma = I$. For each $d$-fold commutator $\gamma$ of $f_j^s$ and $h^t$, the corresponding word $\gamma_m$ is $d$-fold commutator of $f_{m,j}^s$ and $h_m^t$. It is clear that there are only finitely many $d$-fold commutators of $f_{m,j}^s$ and $h_m^t$. If $\gamma_m = I$ except finitely many $m$, then every $d$-fold commutator of $f_{m,j}^s$ and $h_m^t$ is trivial. Now the commutator identity

$$[a, bc] = [a, b] \centerdot [b, [a, c]] \centerdot [a, c]$$

implies that any $d$-fold commutator in the group generated by $f_{m,j}^s$ and $h_m^t$ is trivial, since $d$-fold commutators of products can be rewritten as (much longer) products of at least $d$-fold commutators of the generators $f_{m,j}^s, h_m^t$, see [UK, p28]. Therefore $\langle f_{m,j}^s, h_m^t \rangle$ is nilpotent and so is elementary. It is obvious that this is impossible. The remained problem is that there are distinct isometries $\gamma_m$ converging to $\gamma = I$. Similar to the proof of the above-mentioned discreteness it lead to that $\langle f_{m,j}^s, h_m^t \rangle$ is elementary and so $G_m$ is elementary, a contradiction.

(3). There is only one possibility of extending $\psi_m$ to a homomorphism, namely for each finite sequence $(s_i, t_i)$, where $s_i \in \{1, 2, \cdots, r\}$ and $t_i \in \mathbb{Z}$, the mapping should have

$$\prod_{i=1}^{N} g_{s_i}^{t_i} \to \prod_{i=1}^{N} g_{m,s_i}^{t_i}.$$

A necessary and sufficient condition for this correspondence to define a mapping of $G$ is that

$$\prod_{i=1}^{N} g_{s_i}^{t_i} = I \Rightarrow \prod_{i=1}^{N} g_{m,s_i}^{t_i} = I.$$

If the mapping exits, then it is a homomorphism of $G$ onto $G_m$ by extending $\psi_m$.

Since $G$ is finitely presented, we may let $R_1, R_2, \cdots, R_k$ be a basis for the relations in $G$. That is, $R_i = \prod_{j=1}^{N} g_{s_i(j)}^{t_i(j)}$. Let $R_{m,i} = \prod_{j=1}^{N} g_{m,s_i(j)}^{t_i(j)}$ be the corresponding words in $G_m$. Therefore

$$\lim_{m\to\infty} R_{m,i} = R_i = I.$$

We next show that $R_{m,i} = I$ for all sufficiently large $m$. For the purpose of contradiction, without loss of generality, assume that $R_{m,i} \neq I$ for all $m$ and $i$. We have already known that $G$ is discrete and nonelementary by 2). Then there are two loxodromic elements $h_1, h_2 \in G$ without common fixed points. Let $h_{m,1}, h_{m,2}$ be words corresponding to $h_1, h_2$ in $G_m$, respectively. Hence we have

$$\lim_{m\to\infty} h_{m,k} = h_k \quad (k = 1, 2)$$

It follows form Lemma 3.2 that $h_{m,k}$ are loxodromic for all sufficiently large $m$. Margulis Lemma and Lemma 2.3 in [Mar2] imply that $\langle R_{m,i}, h_{m,k}\rangle$ $(k = 1, 2)$ are elementary for large $m$. Then $R_{m,i}$ stabilize setwise the fixed point set of loxodromic elements $h_{m,k}$. As $R_{m,i} \to I$, it is enough to show that $R_{m,i}$ does not interchange the two fixed points of $h_{m,k}$. Since $\{G_m\}$ satisfies Condition A, it follows that Card(fix($R_{m,i}$))= 2. Thus $h_{m,1}$ and $h_{m,2}$ have the two same common fixed points. Taking the limit, Lemma 3.2 implies that $h_1$ and $h_2$ both have the same fixed points. This is a contradiction. The proof of theorem is completed.

From this theorem (1), it is not hard to deduce that

**Corollary 4.5** *For each $m \in \mathbb{N}$, let $G_m = \langle g_1, g_2, \cdots, g_r \rangle$ be a discrete and nonelementary subgroup of* $\mathrm{Isom}(X)$. *Suppose that $\{G_m\}$ converges algebraically to a group $G = \langle g_1, g_2, \cdots, g_r \rangle$ in* $\mathrm{Isom}(X)$. *If all generators $g_{m,i}$ are not elliptic, then all generators $g_i = \lim_{m\to\infty} g_{m,i}$ are loxodromic or parabolic.*

This corollary generalizes the following result of groups of hyperbolic isometries (or Möbius groups) obtained by Apanasov [Ap1] or [Ap2, Theorem 6.46] to groups of isometries of pinched Hadamard manifolds. Furthermore, the condition and result of Apanasov's theorem are improved. In fact, if all non-elliptic generators $g_{m,i}$ $(i = 1, 2, \cdots, r)$ have no common fixed points, then $\{G_m\}$ is nonelementary. The converse is incorrect. On the other hand, the statement that all the generators $g_i \neq I$ is replaced by the statement that the generators $g_j$ are loxodromic or parabolic.

**Theorem 4.6** (Apanasov [Ap1 and 2]) *Let a sequence of discrete groups $G_m = \langle g_{1,m}, \cdots, g_{k,m} \rangle \subset$ Möb$(n)$, $n \geq 2$, $\quad 2 \leq k \leq \infty$, be such that all the $g_{j,m}$ have no common fixed points and are not elliptic. If $G = \langle g_1, \cdots, g_k \rangle$ is the algebraic limit of the groups $G_m$, then all the generators $g_j = \lim_{m\to\infty} g_{j,m}$ are different from the identity one.*

**Remark.** We have known that, in three dimensional space, Condition A is naturally satisfied and if $G_m$ is nonelementary discrete and $\{G_m\}$ converges algebraically to $G$, then by Lemma 4 in [Jø2] there is a subsequence $\{g_{i_j}\}$ converges to $g \in G$, where $g_{i_j} \in G_{i_j}$. Hence, in 3-dimensional space, the assumptions of Condition A and of which there is a sequence of isometries in $G_m$ converges to an element in $G$ are satisfied. Thus, in this case, our theorem coincides with the theorem in [Jø2].

In order to prove our following convergence theorem we first define for an isometry $\gamma$ on a Hadamard manifold $X$ the *rotation* of $\gamma$ in $x \in X$ as

$$r_\gamma(x) := \max_{w \in T_x X} \sphericalangle(w, P_{\gamma(x),x} \circ \gamma_{*_x} w).$$

Here $\gamma_{*_x} : T_x X \to T_{\gamma(x)} X$ is the differential and $P_{\gamma(x),x} : T_{\gamma(x)} X \to T_x X$ is the parallel transport along the unique geodesic from $\gamma(x)$ to $x$. We then define the *norm of $\gamma$ at $x$:* $n_\gamma(x) := \max\{(r_\gamma(x), \quad 8d_\gamma(x)\}$.

We quote the following result about periodic isometry [BGS, §12.3, p126].

**Lemma 4.7** *If $X$ satisfies $-1 \leq K \leq 0$ and $\gamma$ is an isometry of finite order $m \geq 2$, then $n_\gamma(x) = max(8d_\gamma(x), r_\gamma(x)) \geq \frac{\pi}{m}$ for all $x \in X$.*

The following result is a generalization of theorems of Chuckrow, Marden and Jøgensen in 3-dimensional hyperbolic space, see [Jø1]. In $n$-dimensional hyperbolic case, Weilenberg ([We, Theorem 3]) and Martin ([Ma1, Theorem 5.10]) have the theorem under the condition that $\Gamma_0$ is torsion free and finitely generated, respectively. Martin also has the version of the following theorem in the case that $\{\Gamma_m\}$ has uniformly bounded torsion. We use Condition A to weaken the assumption of uniformly bounded torsion and in 3-dimensional hyperbolic case the theorem coincides with the result of Jørgensen's.

**Theorem 4.8** *Suppose that $\Gamma_0$ is a discrete non-elementary subgroup of* Isom$(X)$. *For each $m \in N$, let $\varphi_m$ be an isomorphism of $\Gamma_0$ onto a discrete group $\Gamma_m$. If $\{\Gamma_m\}$ has Condition A and for each $\gamma \in \Gamma_0$,*

$$\varphi(\gamma) = \lim_{m \to \infty} \varphi_m(\gamma)$$

*is an isometry of $X$, then the group $\Gamma_\infty$ consisting of $\varphi(\gamma), \gamma \in \Gamma_0$ is discrete and non-elementary, furthermore $\varphi$ is an isomorphism of $\Gamma_0$ onto $\Gamma_\infty$.*

**Proof.** We first show that the mapping $\varphi : \Gamma_0 \to \Gamma_\infty$ is an isomorphism. It is obviously that $\varphi$ is a homomorphism onto. We now only need to prove that the kernel of $\varphi$ is trivial. That is, for each non-identity isometry $\gamma$ in $\Gamma_0$, we have $\varphi(\gamma) \neq I$.

Let $\gamma \in \Gamma_0 \setminus \{I\}$. By assumption, $\varphi_m : \Gamma_0 \to \Gamma_m$ is an isomorphism. It is clear that if $\gamma$ is of finite order $k$, then $\varphi_m(\gamma)$ is of finite order $k$ for all $m$. Lemma 4.7 implies that $\varphi_m(\gamma)$ dose not so close the identity. Thus $\varphi_m(\gamma) \to \varphi(\gamma) \neq I$.

If $\gamma$ is of infinite order, then it parabolic or loxodromic since a discrete group does not have elliptic isometry of infinite order. As $\Gamma_0$ is discrete non-elementary, there is a loxodromic isometry $\beta \in \Gamma_0$ such that the subgroup generated by $\gamma, \beta$ is discrete and non-elementary. It is clear that $\varphi_m(\gamma)$ and $\varphi_m(\beta)$ are parabolic or loxodromic for all $m$, since $\varphi_m$ is an isomorphism. Being a subgroup of $\Gamma_m$, we know that $\langle \varphi_m(\gamma), \varphi_m(\beta) \rangle$ is discrete. We will show that $\langle \varphi_m(\gamma), \varphi_m(\beta) \rangle$ is non-elementary. In this case $\varphi(\gamma)$ is not the identity by Theorem 4.4.

Suppose, for contradiction, that $\langle \varphi_m(\gamma), \varphi_m(\beta)\rangle$ is elementary. If $\varphi_m(\gamma)$ is loxodromic, then there are integers $s, t$ such that $\varphi_m^s(\gamma) = \varphi_m^t(\beta)$ by Proposition 6.9 in [EO]. As $\varphi_m$ is an isomorphism, we have that $\gamma^s = \beta^t$. Thus $\langle \gamma, \beta\rangle$ is elementary, a contradiction. If $\varphi_m(\gamma)$ is parabolic, then $\langle \varphi_m(\gamma), \varphi_m(\beta)\rangle$ is a discrete parabolic group and so is a virtually nilpotent subgroup by [Bo1, Theorem 4.1]. Thus there exist a $d$ and $s, t$ such that any $d$-fold commutator of $\varphi_m^s(\gamma)$ and $\varphi_m^t(\beta)$ is trivial. It is obviously that any $d$- fold commutator of $\gamma^s$ and $\beta^t$ is trivial and so $\langle \gamma^s, \beta^t\rangle$ is elementary. As $\beta$ is loxodromic, $\gamma$ and $\beta$ have a common fixed point. This leads to that $\langle \gamma, \beta\rangle$ is elementary, a contradiction.

Finally, we show that $\Gamma_\infty$ is discrete non-elementary. Since $\Gamma_0$ is discrete non-elementary, there is a loxodromic isometry $\gamma \in \Gamma_0$. We may construct a subgroup $\langle \varphi_m(\gamma), \varphi_m(\beta)\rangle$ for all $m$ such that it is discrete non-elementary and the generators are parabolic or loxodromic. By means of Theorem 4.4, $\langle \varphi(\gamma), \varphi(\beta)\rangle$ certainly contains loxodromic isometries and is discrete non-elementary. The fact that $\Gamma_\infty$ contains a non-elementary subgroup implies that $\Gamma_\infty$ is non-elementary. Define a sequence of isomorphism $\psi_m : \Gamma_\infty \to \Gamma_m$ by

$$\psi_m(\gamma) = \varphi_m \varphi^{-1}(\gamma), \gamma \in \Gamma_\infty$$

Therefore, for every $\gamma \in \Gamma_\infty$, we have $\gamma = \lim_{m\to\infty} \psi_m(\gamma)$. By Theorem 4.1, $\Gamma_\infty$ is discrete.

# 5 References

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