Nonlinear stability of stationary solutions for Surface Diffusion with triple junction

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1 Surface Diffusion

The surface diffusion flow is a geometric evolution law in which

$$V=-\Delta_\Gamma \kappa \quad \xrightarrow{\text{for curves}} \quad V=-\kappa_{ss}$$

- $\Gamma=\{\Gamma_t\}_{t\geq 0}$ is an evolving hypersurface,
- V is the velocity in normal direction of Γ,
- κ is the sum of the principal curvatures of the surface,
- Δ_Γ is the Laplace-Beltrami operator of the surface Γ and s is arc length parameter.

1.1 Interesting geometrical features

Let Γ be a compact, closed and embedded hypersurface in $\mathbb{R}^n$

- The motion driven by surface diffusion is area decreasing and volume preserving
- This motion is the fastest way to decrease the area with the constraint that the volume is preserved with respect to (H^{-1}) inner product

1.2 Equilibria (Stationary solutions)

Clearly surfaces with constant mean curvature are equilibria.
In the compact, closed and embedded case the spheres are the only equilibria.

1.3 Question (Stability)

A natural question to ask is whether these stationary solutions are stable under the flow.
This question has been **answered** positive by

- Elliott and Garcke for circles in the plane. [1]
- Escher, Mayer and Simonett for spheres in higher dimensions [2]

Now in general, the surfaces will meet an outer boundary or they might intersect at triple junction!

2 Surface Diffusion with triple junction

We study the following problem: Take Ω to be a ball, consider three evolving curves lying in Ω, fulfilling the surface diffusion equation along each Γ_i, being perpendicular to the outer boundary and have a common intersection at a triple junction with 120 angle condition. More precisely,

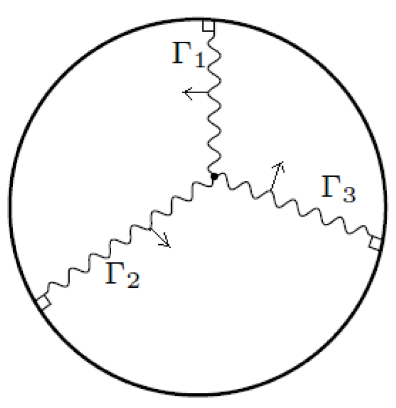
$$V^i=-\kappa^i_{ss}, \quad \text{along each } \Gamma_i. \tag{1}$$

At the triple junction

$$\begin{cases} \sphericalangle(\Gamma^1(t),\Gamma^2(t))=120, \quad \sphericalangle(\Gamma^2(t),\Gamma^3(t))=120, \quad \sphericalangle(\Gamma^3(t),\Gamma^1(t))=120, \\ \kappa^1+\kappa^2+\kappa^3=0, \\ \nabla_s\kappa^1\cdot n_{\partial\Gamma^1}=\nabla_s\kappa^2\cdot n_{\partial\Gamma^2}=\nabla_s\kappa^3\cdot n_{\partial\Gamma^3}. \end{cases}$$

At the outer boundary

$$\begin{cases} \sphericalangle(\Gamma^i(t),\partial\Omega)=\frac{\pi}{2}, \\ \nabla_s\kappa^i\cdot n_{\partial\Gamma^i}=0. \end{cases}$$

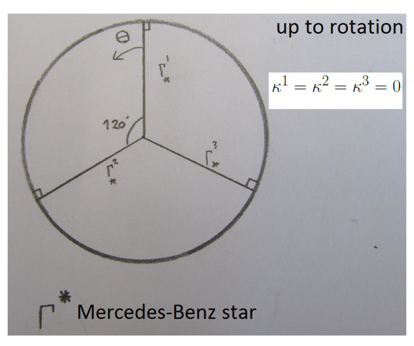


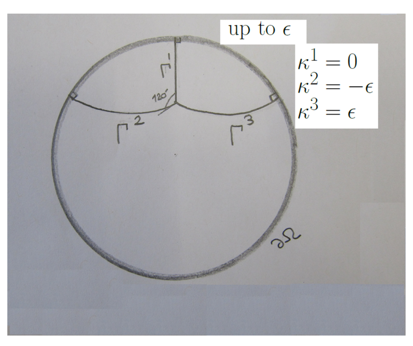
2.1 Geometric properties of the flow

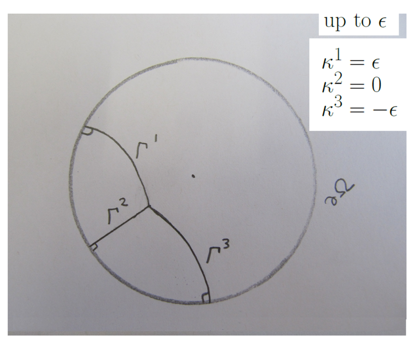
The flow decreases the total length and preserve the enclosed areas. And it is again a H^{-1} gradient flow for the total area functional.

2.2 Manifold of equilibria

Curves with constant curvature which satisfy the b.cs are equilibria. Let M denote the set of all equilibria.



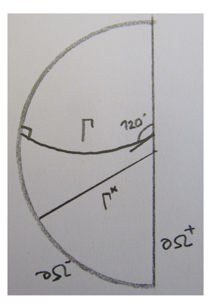




The Mercedes star Γ_* is stationary. We now prove the existence of a second solution which is curved. By symmetry we can reduce the problem to prove the following.

Theorem 2.1. *let Γ be a curve described by u for the parameterization in section 2.3 then there exists a neighborhood of Γ_* such that for every small constant ϵ there exist the unique solution of the following problem*

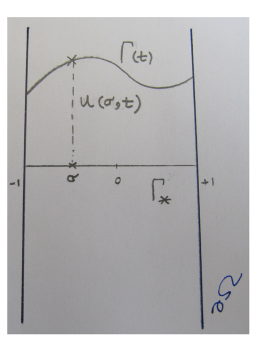
$$\kappa=\epsilon, \quad \sphericalangle(\Gamma,\partial\Omega_-)-\frac{\pi}{2}=0, \quad \sphericalangle(\Gamma,\partial\Omega_+)-120=0. \tag{2}$$



Proof. Linearization of the left hand side of (2) around Γ_* and prove that it is invertible operator and then apply inverse function theorem. □

2.3 Parametrization, PDE formulation [3]

Geometric evolution laws (free boundary problem) $\xrightarrow{\text{Parametrization}}$ PDE (fixed domain)



$\Gamma\equiv$ Graph of function $u:[-1,1]\to\mathbb{R}$
$\Gamma_*\rightsquigarrow u\equiv 0, \qquad V(\Gamma(t))\rightsquigarrow \partial_t u, \qquad \kappa(\Gamma(t))\rightsquigarrow u_{\sigma\sigma}.$
Of course dealing with triple junction may involve more work, see [3].

Anyhow you get the following nonlocal PDE after parametrization around Γ_*

$$\begin{cases} \partial_t u(t)=A(u(t))u(t)+F(u(t)) & \text{in } \Omega, \\ B_j(u(t))=0 & \text{on } \partial\Omega, \quad j\in\{0,1,2,3\}. \end{cases} \tag{3}$$

where

$$[A(u)v](\sigma)=a(\sigma,u,u_\sigma,...,u^3_\sigma,u(x_0))v^4_\sigma,$$
$$[F(u)](\sigma)=f\left(\sigma,u,u_\sigma,u(x_0),...,u^3_\sigma(x_0)\right)u^4_\sigma(x_0)+L\left(\sigma,u,u_\sigma,...,u^3_\sigma,u(x_0)\right),$$
$$B_j(u)=g_j\left(\sigma,u,u_\sigma,...,u^j_\sigma,u(x_0)\right).$$

Here $x_0=0$ (triple junction) , $\Omega=(0,1)$ and $u(\sigma,t)=(u_1(\sigma,t),u_2(\sigma,t),u_3(\sigma,t))$.

2.4 Linearization around stationary solution Γ_* [3]

The linearization of (3) at $u\equiv 0$ reads as follows, where $i=1,2,3$.

$$u^i_t=-u^i_{\sigma\sigma\sigma\sigma} \qquad \sigma\in(0,1),$$

at $\sigma=0$ (triple junction)

$$\begin{cases} u^1+u^2+u^3=0 \\ u^1_\sigma=u^2_\sigma=u^3_\sigma, \\ u^1_{\sigma\sigma}+u^2_{\sigma\sigma}+u^3_{\sigma\sigma}=0, \\ u^1_{\sigma\sigma\sigma}=u^2_{\sigma\sigma\sigma}=u^3_{\sigma\sigma\sigma}. \end{cases}$$

at $\sigma=1$ (outer boundary)

$$\begin{cases} -u^i_\sigma+u^i=0, \\ u^i_{\sigma\sigma\sigma}=0. \end{cases}$$

3 Stability of the Mercedes star

Now for proving the stability of stationary solutions we use the following theorem [4].

3.1 Generalized principle of linearized stability

Theorem 3.1. ***(Prüss, Simonett, and Zacher)*** *suppose that the linearized operator A has the property of maximal regularity and suppose the stationary solution u_* is normally stable, i.e. assume that*
(i) near u_* the set of equilibria M is a manifold of dimension m,
(ii) $T_{u_*}M=N(A)$,
(iii) 0 is a semi-simple eigenvalue of A, i.e. $N(A_0)\oplus R(A_0)=X$
(iv) $\sigma(A)\setminus\{0\}\subset\mathbb{C}_+$.

Then u_* is stable and solution starting nearby exist globally and converge to some point on the manifold of equilibria.

3.2 Main Theorem

Theorem 3.2. ***(work in progress)*** *A stationary solution having the form of a Mercedes star is stable under the flow (1) and solution starting nearby exist globally and converge to some point on the manifold of equilibria.*

Proof. **We first show that u_* is normally stable.**
Let us call the corresponding linearized operator as A_0 so by calculating $N(A_0)$ we get

$$N(A_0)=\left\{(\sigma,\sigma,\sigma)\,,(0,\,1-\sigma^2,\,\sigma^2-1)\,,(1-\sigma^2,\,0,\,\sigma^2-1)\right\}$$

Statement (i) follows from Theorem 2.1. In general, $T_{u_*}M\subset N(A_0)$ and now by proving $\dim N(A)=\dim T_{u_*}M=3$ we get (ii).

Let us prove (iii) since we have

$$\left(\exists\, P:X\to N(A_0) \text{ s.t } PA_0=A_0P=0\right)\to N(A_0)=N(A_0^2)\xrightarrow{\text{compact resolvent}} N(A_0)\oplus R(A_0)=X$$

We only need to prove the existence of such a projection P which is done by using the (H^{-1}) inner product and symmetry of the operator A_0.
The proof of (iv) was already done in [3].

In order to deal with nonlocality term $u^4_\sigma(x_0)$ we use parabolic hölder settings $C^{1+\frac{\alpha}{4},4+\alpha}$.

Now let us prove Maximal Regularity Based on the results of Solonnikov [5], it is enough to show normally ellipticity and Lopatinskii- Shapiro condition for A_0 and this was done by using energy methods.

Local well-posedness Now by applying fixed point argument in the space

$$Y=\left\{u\in C^{1+\frac{\alpha}{4},4+\alpha}([0,T]\times\bar{\Omega}): \quad u(0,.)=u_0, \quad \|u-u_*\|_{C^{1+\frac{\alpha}{4},4+\alpha}([0,T]\times\bar{\Omega})}\leq R\right\}$$

and do linearization around stationary solution rather than initial data and finally by choosing $\|u_0-u_*\|$ and T small enough and R suitably large we proved local existence and uniqueness.

Missing part In [4] they employ the L_p-setting so we are trying to extend their approach to cover a parabolic hölder settings too. □

4 References

[1] C.M . Elliott and H. Garcke, *Existence results for diffusive surface motion laws,* Adv. Math. Sci. Appl. 7 (1997), 465–488.
[2] Escher, J., Mayer, U., and Simonett, G., *The surface diffusion flow for immersed hypersurfaces,* SIAM J. Math. Anal. 29, no. 6 (1998), 1419–1433.
[3] Garcke H., Ito K., Kohsaka Y., *Surface diffusion with triple junctions: A stability criterion for stationary solutions,* Adv. Diff. Equ. 15, no.(5-6) (2010), p.437-472.
[4] J. Prüss, G. Simonett, and R. Zacher, *On convergence of solutions to equilibria for quasilinear parabolic problems, J. Differential Equations,* 246 (2009), 3902-3931.
[5] V.A. Solonnikov, *Boundary value problems in physics,* Proceedings of the Steklov Institute of Mathematics, Vol. LXXXIII (1965).